BAYESIAN ESTIMATION OF POISSON-COMPOUNDED EXPONENTIAL TYPE DISTRIBUTION UNDER DIFFERENT LOSS FUNCTIONS

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Abstract

Poisson moment exponential distribution is an important distribution and has gained special attention recently. It plays role in various fields, mostly in actuarial sciences. Thus its parametric estimation becomes important thing to do. The classical approach using the maximum likelihood method is the most used way to estimate the parameters of a distribution. In this paper, we considered the Bayesian approach to estimate the parameter of the distribution using beta prior which is a conjugate prior. The Bayes estimate for the parameter is obtained under Squared Error Loss Function (SELF) which is a symmetric loss function, Weighted SELF (WSELF) and Entropy Loss Function (ELF). Through a simulation study, the comparison is made on the performance of Bayes estimate under these loss functions with respect to Bias and Mean Square Error (MSE).

Keywords: Bayesian, Loss function, Maximum likelihood estimator, Poisson moment exponential distribution, Simulation

1. Introduction

Bayesian statistics is still one of the most potent concepts in the opinion of many statisticians. Even years after Thomas Bayes first proposed the idea of the Bayes theorem in 1770, the significance of Bayesian statistics has not diminished. In Bayesian statistics, the unknown parameter is considered as the value of a random variable from a specified probability distribution with some prior knowledge about the parameter. Researchers, engineers, statisticians, and other applied scientists who utilise prediction techniques for a variety of objectives find it highly intriguing to estimate the lifespan of future samples based on an informative sample. One-sample and two-sample prediction problems, which are a particular form of the multiple-sample prediction issue, are the two categories into which the future prediction problem falls.

The application of Bayesian approaches in many statistical and non-statistical domains is gaining popularity. In 2020, Johannesson [12] proposed that at any rate, there is a disagreement between classical statistics and the Bayesian theory.

There are several benefits of Bayesian approaches over Frequentistic approach, few among them are listed as under:

- By adjusting prior distribution, Bayesian inference can prevent problems with model identification. Since frequentist inference lacks prior distributions, so it can lead to problems with model identification when employing any numerical approximation approach.

- In Bayesian approach, unknown parameters are treated as random variables, and the data is fixed. According to frequentist inference, the data are random and the unknown parameters are fixed. Estimates are made using both the current data and hypothetically repeated sampling of similar data. The proper answer is provided by the Bayesian technique in the sense that it bases its conclusions on actual data rather than test statistics across hypothetical samples that are unobserved (see [19]).
- In order to inform the current model, prior knowledge or the outcomes of a previous model can be utilised, and Bayesian inference permits informative priors. If β is the unknown parameter, then P(β | data) is estimated using Bayesian inference. On the other hand, Frequentist inference, calculates P(data | β).
- By including uncertainty into the probability model, Bayesian inference produces predictions that are closer to reality. Less realistic predictions are produced by frequentist inference as it does not account for the uncertainty of the parameter estimates.
- Probability intervals-quantile-based, maximum posterior density or, lowest posterior loss-are used in Bayesian inference to express the likelihood that β lies between two points. On the other hand, Confidence intervals, are used in frequentist inference, that must be interpreted with a probability of zero or one to indicate whether or not β is in the region. The frequentist is never certain whether β is in the region; all it can say is that, in the event that 100 repeat samples are taken in the future, 95 of them would be.

In the area of Bayesian estimation and future observation prediction for various models, some references are as follows: Migdadi [17] derived the Bayes estimators for the scale parameter of the discrete Rayleigh distribution. Kamari *et al.* [15] obtained a Bayesian analysis of the discrete Burr distribution. Ahmad [1] and Ahmad [2] studied the Bayesian estimation of different models including size biased Generalized Power Series and Zero-Inflated Generalized Power Series respectively under various loss functions. Ashour and Muiftah [3] introduced Bayesian estimation of the parameters of the discrete Weibull Type-I distribution. Chaiprasithikul and Duangsaphon [7] studied the Bayesian estimation for the discrete Weibull regression under type-I right censored data and showed that the Bayes estimator with informative priors for the parameters are more appropriate for the model in case of over dispersion. Halder *et al.* [11] introduced the Bayesian estimation of Power series model when the stress (X) and strength (Y) are from different members of the power series family. Also, Hegazy *et al.* [10] studied Bayesian inference to estimate the parameters of discrete Gompertz distribution. Hegazy *et al.* [10] showed that linear exponential loss function gives better results than squared error loss function. El-Morshedy *et al.* [18] also studied the Bayesian estimation of the parameters of discrete odd Weibull-G family of distributions with application to count data. Yahya *et al.* [23] proposed a Bayesian method for testing non-inferiority between two independent binomial proportions.

The objective of this study is to derive the Bayesian estimate for the unknown parameter of Poisson Moment Exponential (PME) distribution which is a one parameter discrete distribution given by Ahsan-ul-Haq [4] and its zero inflated version is given by Skinder *et al.* [20]. Here three different loss function namely-SELF, WSELF and ELF are used to produce the Bayes estimators. The theoretical insights produced in this work are illustrated with a numerical example in Section 6. The rest of the paper is organised as follows: Section 2 discusses about the Bayesian approach including prior and posterior distribution. Section 3 gives an idea about different loss functions. In section 4 and 5, Bayesian estimates of PME distribution and simulation analysis are provided respectively. Section 6 discusses few applications and section 7 concludes.

2. Bayesian Approach

Let's suppose β be the unknown parameter of interest, then the Bayesian approach would consider the parameter β as a random variable, and inference about β is derived from both prior assumptions about β and the data x. The posterior density of β given x is formed by combining the evidence with prior beliefs, and Bayesian inference is based on this posterior.

2.1. Prior Distribution

The prior distribution is an important part of Bayesian analysis. It represents the information about an unknown parameter which is mixed with the probability distribution of new data to get the posterior distribution, which in turn is used for future prediction. In certain cases, a valuable apriori knowledge on the probable values of parameter β exists. Then, an informative prior that affects the posterior distribution is defined. The prior density of β's functional form is frequently selected based on computational ease of use or analytical tractability. In order to set up a prior distribution, the following points are crucial:
1. what data is included in the previous distribution;
2. the characteristics of the resulting posterior distribution.

2.2. Posterior Distribution

The posterior distribution of the evidence x given β, represented as P(β | x), combines information from the current data x with conditional probability P(x | β) and a prior probability $\pi(\beta)$ captures previous ideas about β. Now, from conditional probability theorem, we have:

$$P(\beta|x) = \frac{P(\beta, x)}{P(x)} \tag{1}$$

If prior probability associated with β is denoted by $\pi(\beta)$, then we can also write as $P(\beta, x) = P(x|\beta)\pi(\beta)$. So, equation (1) becomes

$$P(\beta|x) = \frac{P(x|\beta)\pi(\beta)}{P(x)}$$

where $P(x)$ is the normalizing constant calculated a;

$$P(x) = \int P(x|\beta)\pi(\beta)d\beta$$

We can also write posterior distribution as:

$$\rho(\beta|x) = \frac{L(x, \beta)\pi(\beta)}{\int L(x, \beta)\pi(\beta)d\beta} \tag{2}$$

i.e., posterior $\propto$ likelihood $\times$ prior.

3. Loss Functions

Wald [22] was the first to use the term "loss function" . It is a measurement of the discrepancy between the data's observed values and those determined by using the adjustment function. This loss function is to be minimised during the model-fitting process. There are several loss functions and few among them are discussed as follows:

3.1. Squared Error Loss Function

The squared error loss function ($SELF$) proposed by Legendre [16] and Gauss [9] is defined by:

$$L(\phi(\beta), d) = (\phi(\beta) - d)^2$$

where $\phi(\beta)$ is a function of β and d is an action(estimator).

An alternative of this loss function is a weighted squared error loss function ($WSELF$) defined by:

$$L(\phi(\beta), d) = w(\beta)(\phi(\beta) - d)^2$$

where $w(\beta)$ is the weight associated with the estimator.

In case of squared error loss function, the Bayes estimator $\delta^{\pi}(x)$ of β associated with the prior distribution π is the posterior mean of β defined by:

$$\delta_s^{\pi}(x) = E_{\pi}(.|x)(\beta) = \int L(\beta, \delta(x))\pi(\beta|x)d\beta \quad , d = \delta(x)$$

Since, the Bayes estimator minimizes the posteriori loss, that is to say:

$$\rho(\pi, \delta) = E^{\pi(.|x)}(L(\beta, \delta(x)))$$

So, under the assumption of squared error loss, we have:

$$\rho(\pi, \delta) = E^{\pi(.|x)}(\beta - \delta(x))^2;$$
$$= E^{\pi(.|x)}(\beta)^2 - 2\delta(x)E^{\pi(.|x)}(\beta) + \delta(x)^2$$

It is evident that it is a polynomial second degree in $\delta(x)$. The minimum value will occur in $E^{\pi(.|x)}(\beta)$.

3.2. Entropy Loss Function

The entropy loss function (ELF) was introduced by Calabria and Pulcini [5] as a loss function that originates from the Linex loss function. It is described as follows:

$$L_e(\beta, d) \propto \left(\frac{d}{\beta}\right)^p - p\ln\left(\frac{d}{\beta}\right) - 1 \quad ; p > 0$$

When $d = \beta$, the entropy loss function is minimum. Under this loss function, the Bayes estimator of parameter β is given by:

$$\delta_e^{\pi}(x) = (E_{\beta}(\beta)^{-p})^{-1/p}$$

The Bayes estimator coincides with the Bayes estimator under weighted squared error loss function $\frac{(\beta-d)^2}{\beta}$ when $p = 1$.

The Bayes estimator coincides with the Bayes estimator under squared error loss function when $p = -1$.

4. Bayesian Estimation of PME Distribution

From Ahsan-ul-Haq [4], the pmf of PME distribution is given as:

$$P(X = x) = \frac{\beta^x(1 + x)}{(1 + \beta)^{(x+2)}} \quad ; x = 0, 1, 2, ... \quad ; \beta > 0 \tag{3}$$

The likelihood of equation (3) can be written as:

$$L(\beta) = \frac{\beta^{\sum x_i} \prod_{min}^{max}(1 + x_i)}{(1 + \beta)^{2n+\sum x_i}} \tag{4}$$

Differentiating equation (4) and equating to 0 we get the MLE of β as:

$$\hat{\beta} = \frac{\bar{x}}{2} \tag{5}$$

Now, instead of treating parameter β as constant, treat it as a random variable following beta prime distribution i.e.,

$$\beta \sim Beta(c, d) \tag{6}$$

where c and d are hyper parameters.

$$\therefore \pi(\beta) = \frac{\beta^{c-1}}{B(c,d)(1+\beta)^{c+d}} \tag{7}$$

where $B(c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}$

Now, using likelihood (4) and prior distribution (7), the Posterior distribution can also be obtained as:

$$\rho(\beta, x) = \frac{L(\beta)\pi(\beta)}{\int_0^\infty L(\beta)\pi(\beta)d\beta}$$

$$\rho(\beta|x) = \frac{\beta^{\sum_{i=1}^n x_i + c - 1}}{B(\sum_{i=1}^n x_i + c, 2n+d)(1+\beta)^{\sum_{i=1}^n x_i + 2n + c + d}} \tag{8}$$

Bayes estimate Under SELF

$$\delta_s^\pi(x) = \int_0^\infty \phi(\beta)\rho(\beta|x)d\beta$$

Let $\phi(\beta) = \beta^r$, then

$$\delta_s^\pi(x) = \frac{B(\sum x_i + c + 1, 2n + d - 1)}{B(\sum x_i + c, 2n + d)}$$

Bayes estimate Under WSELF

$$\delta_w^\pi(x) = \frac{\int_0^\infty w(\beta)\phi(\beta)\rho(\beta|x)d\beta}{\int_0^\infty w(\beta)\rho(\beta|x)d\beta}$$

let $w(\beta) = \beta^{-2}$, then

$$\delta_w^\pi(x) = \frac{B(\sum x_i + c - 1, 2n + d + 1)}{B(\sum x_i + c - 2, 2n + d + 2)}$$

Bayes estimate Under ELF

$$\delta_e^\pi(x) = [E_\beta(\beta)^{-p}]^{-1/p} = \left[\int_0^\infty \beta^{-p}\rho(\beta|x)d\beta\right]^{-1/p}$$

$$\Rightarrow \delta_e^\pi(x) = \left[\frac{B(\sum x_i + c, 2n + d)}{B(\sum x_i + c - p, 2n + d + p)}\right]^p \quad ; p > 0$$

5. Simulations: Comparing Classical and Bayesian Estimation Methods

Table 1: *The Bias, Var, and MSEs for different parameter values under classical and Bayesian approach*

		$\beta = 0.2$				$\beta = 0.28$			
n		MLE	SELF	WSELF	ELF	MLE	SELF	WSELF	ELF
10	Bias	-0.0209	0.1621	0.0121	-0.0283*	-0.0209	0.2648	-0.0900*	-0.0333
	Var	0.0102	0.0483	0.0032	0.0093	0.0122	0.0847	0.0101	0.0111
	MSE	0.0107	0.0745	0.0033*	0.0101	0.0127	0.1549	0.0182	0.0122*
35	Bias	-0.0037	0.1129	-0.0231*	-0.0065	0.0046	0.2314	-0.0172*	0.0006
	Var	0.0026	0.0104	0.0025	0.0025	0.0043	0.0282	0.0041	0.0042
	MSE	0.0026	0.0231	0.0029	0.0026*	0.0044	0.0818	0.0044	0.0042*
65	Bias	-0.0025	0.1018	-0.0130*	-0.0039	-0.0008	0.1997	-0.0126*	-0.0029
	Var	0.0019	0.0075	0.0018	0.0019	0.0025	0.0150	0.0024	0.0025
	MSE	0.0019	0.0179	0.0019	0.0019*	0.0024	0.0549	0.0026	0.0025*
150	Bias	0.0103	0.1165	0.0056*	0.0096	0.0027	0.1962	-0.0025*	0.0018
	Var	0.0009	0.0037	0.0009	0.0009	0.0015	0.0091	0.0014	0.0014
	MSE	0.0010	0.0173	0.0009*	0.0010	0.0015	0.0476	0.0014*	0.0014
270	Bias	0.0064	0.1049	0.0038*	0.0059	-0.0009	0.1826	-0.0038*	-0.0014
	Var	0.0005	0.0019	0.0005	0.0005	0.0008	0.0045	0.0008	0.0008
	MSE	0.0005	0.0129	0.0005*	0.0005	0.0008	0.0379	0.0008	0.0008*
		$\beta = 0.8$				$\beta = 1.6$			
n		MLE	SELF	WSELF	ELF	MLE	SELF	WSELF	ELF
10	Bias	0.0001	2.1553	-0.1182	-0.0381*	-0.0679	9.5929	-0.2527*	-0.1409
	Var	0.0575	2.2052	0.0475	0.0521	0.2454	75.1893	0.2028	0.2226
	MSE	0.0575	6.8503	0.0615	0.0536*	0.2500	167.2129	0.2667	0.2424*
35	Bias	-0.0166	1.8191	-0.0522*	-0.0276	-0.0197	9.0940	-0.0775	-0.0419
	Var	0.0212	0.7969	0.0200	0.0206	0.0369	7.6018	0.0349	0.0358
	MSE	0.0215	4.1060	0.0228	0.0214*	0.0373	90.3030	0.0409	0.0376
65	Bias	0.0155	1.9604	-0.0044*	0.0093	0.0085	9.5598	-0.0235*	-0.0038
	Var	0.0119	0.5139	0.0116	0.0117	0.0386	8.8758	0.0374	0.0379
	MSE	0.0122	4.3572	0.0116*	0.0118	0.0386	100.2657	0.0379*	0.0379
150	Bias	-0.0039	1.7981	-0.0125*	-0.0065	0.0023	9.3219	-0.0116*	-0.0029
	Var	0.0049	0.1795	0.0048	0.0048	0.0131	3.1097	0.0129	0.0129
	MSE	0.0049	3.4128	0.0049	0.0049*	0.0131	90.0093	0.0130	0.0129*
270	Bias	0.0116	1.8783	0.0068*	0.0101	0.0082	9.3329	-0.0013*	-0.0013
	Var	0.0022	0.0841	0.0022	0.0022	0.0122	2.5134	0.0113	0.0113
	MSE	0.0024	3.6120	0.0023*	0.0023	0.0122	89.6159	0.0113*	0.0113

The value with * is the best estimation in its row for all estimation methods.

From the Table 1, we can see that the bias and MSE is low in case of Bayesian statistics under different loss functions as compared to MLE in case of Classical statistics.

6. Applications

To demonstrate how the aforementioned findings might be applied in real-world scenarios, we fitted PME distribution to four data sets and carried out the goodness of fit test. The data sets in table 2, 3, 4, and 5 represent the number of fires that occur in Greece between 1st July and 1st August, 1998 (see [14]) recently used by Elwa *et al.* [8]; deaths due to horse kicks in the Prussian Army (see [21]); frequency of lost shoes at a museum gate (see[6]) and number of outbreaks of strike in vehicle manufacturing industry in U.K during 1948-95 (see [13]) respectively. For the selection of values of (c, d), there was no information available in the Bayes estimation other than the fact that they are real and positive numbers. As a result, we examined 16 combinations of values of $(c, d = 1, 2, 3, 4)$ and only those values were chosen for which the predicted Bayes frequencies were reasonably similar to the observed frequencies. In all the four data sets, it was observed that the predicted Bayes frequencies for $c = d = 1$ were quite similar to the observed frequencies. Table 2, 3, 4, and 5 shows the observed frequencies, expected frequencies, the values chi-square statistics along with the degrees of freedom $(d.f)$ and their respective p-values. These predicted frequencies are obtained by using MLE, and under beta prior, the Bayes frequencies are obtained using squared error loss function, weighted squared error loss function and entropy loss function.

Table 2: *Number of fires that occurred in Greece (see [14])*

No. of Fires	Observed frequency	Expected Frequency			
			Under beta prior		
		MLE	SELF	WSELF	ELF
0	16.00	9.90	9.90	10.00	9.90
1	13.00	14.20	14.10	14.30	14.20
2	14.00	15.20	15.20	15.30	15.30
3	9.00	14.50	14.50	14.60	14.60
4	11.00	13.00	13.00	13.10	13.00
5	13.00	11.20	11.20	11.20	11.20
6	8.00	9.40	9.40	9.30	9.40
7	4.00	7.70	7.70	7.60	7.70
8	9.00	6.20	6.20	6.10	6.20
9	6.00	4.90	4.90	4.90	4.90
10	3.00	3.90	3.90	3.80	3.90
11	4.00	3.00	3.00	3.00	3.00
12	6.00	2.40	2.40	2.30	2.30
13	4.00	1.80	1.80	1.80	1.80
14	1.00	1.40	1.40	1.40	1.40
15	1.00	1.10	1.10	1.00	1.10
16	1.00	0.80	0.80	0.80	0.80
$Degrees\ of\ Freedom$		9	9	9	9
χ^2		13.92	13.90	14.24	14.16
$p - value$		0.12530	0.12587	0.11415	0.11677
$\hat{\beta}$		2.52846	2.53252	2.50403	2.51822

Table 3: *Deaths due to horse-kicks in the Prussian Army (see [21])*

No. of Deaths	Observed frequency	Expected Frequency			
			Under beta prior		
		MLE	SELF	WSELF	ELF
0	109.00	117.40	117.00	118.20	117.60
1	65.00	54.90	55.00	54.70	54.90
2	22.00	19.20	19.40	19.00	19.20
3	3.00	6.00	6.10	5.90	6.00
4	1.00	1.80	1.80	1.70	1.70
$Degrees\ of\ Freedom$		2.00	2.00	2.00	2.00
χ^2		4.72	4.64	4.83	4.67
$p-value$		0.09448	0.09832	0.08917	0.09665
$\hat{\beta}$		0.305	0.3075	0.30099	0.30424

Table 4: *Frequency distribution of lost shoes at a Museum gate (see [6])*

No. of shoes lost	Observed frequency	Expected Frequency			
			Under beta prior		
		MLE	SELF	WSELF	ELF
0	109.00	107.30	106.90	108.00	107.50
1	52.00	57.40	57.50	57.30	57.40
2	26.00	23.00	23.20	22.80	23.00
3	10.00	8.20	8.30	8.00	8.20
4	3.00	2.70	2.80	2.70	2.70
$Degrees\ of\ Freedom$		2.00	2.00	2.00	2.00
χ^2		1.33	1.23	1.44	1.32
$p-value$		0.51407	0.54051	0.48602	0.51561
$\hat{\beta}$		0.365	0.3675	0.36069	0.36409

Table 5: *Number of outbreaks of Strike in Vehicle manufacturing Industry (see [13])*

No. of Outbreaks	Observed frequency	Expected Frequency			
			Under beta prior		
		MLE	SELF	WSELF	ELF
0	110.00	107.40	106.80	108.20	107.50
1	33.00	36.60	36.80	36.20	36.50
2	9.00	9.30	9.50	9.10	9.30
3	3.00	2.10	2.20	2.00	2.10
4	1.00	0.50	0.50	0.40	0.40
$Degrees\ of\ Freedom$		1.00	1.00	1.00	1.00
χ^2		0.52	0.54	0.51	0.52
$p-value$		0.47139	0.46214	0.47579	0.47264
$\hat{\beta}$		0.20513	0.20833	0.20064	0.20447

It is clear from Tables 2, 3, 4, and 5, that the Bayes estimator obtained under beta prior gives closed fit than MLE.

7. Conclusion

In this paper Bayes estimation of a discrete count model namely poisson moment exponential distribution is discussed. By using three different loss functions it has been observed that how Bayesian approach gains priority over classical approach. A simulation study suggests that the bias and mean square error is always low in case of Bayesian approach. Four data sets were also analysed and it has been found that the p-value is highest in case of bayesian approach. From the study, it can be inferred that Bayesian approach is more reliable than classical approach.

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