FIXED POINT THEOREMS IN TRICOMPLEX VALUED CONTROLLED METRIC SPACES

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Abstract

Throughout the years, scholars have broadened traditional fixed point theories to include more intricate structures like tricomplex valued metric spaces, aiming to tackle issues in multidimensional and hyper-complex contexts. While there is a rising interest in tricomplex valued spaces, the current literature addressing fixed point theorems in these areas presents several limitations. The existing research highlights numerous shortcomings; thus, this article employs innovative types of contraction mappings via a control function to illustrate fixed point theorems in tricomplex valued controlled metric spaces. These findings enhance the understanding of fixed point theory and pave the way for new applications in more complex and diverse mathematical frameworks. Consequently, our study fosters progress in the field, establishing a strong basis for future research and potential uses across various scientific and engineering fields.

Keywords: fixed point, controlled metric spaces, complex valued metric space, tricomplex valued metric space, tricomplex valued controlled metric space.

1. Introduction

The concept of fixed points is essential in many theoretical and practical domains, such as variational and linear inequalities, approximation theory, nonlinear analysis, integral and differential equations and inclusions, dynamic systems theory, mathematics of fractals, and mathematical economics and modeling. The idea of Banach contraction was first introduced in a metric space by [1] in 1922. Consequently, various researchers have interpreted these findings in numerous ways [see, [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], etc.]. Complex valued metric spaces were first presented by [12] in 2011. In 2012, expanded on these results and provided applications by [13]. Complex valued b-metric spaces introduced by [14] in 2013. Multicomplex spaces were introduced by [15]. Following this, many researchers took an interest in this field, bicomplex valued metric spaces introduced in 2021 by [16], where they demonstrated fixed point theorems. Explored controlled metric spaces with bicomplex value and applied these concepts to fractional differential equations by [17]. they also introduced tricomplex valued metric spaces and proved common fixed point theorems [18]. Fixed point results in tricomplex valued metric spaces using control functions and provided practical applications as well by [19]. The concept of controlled metric spaces given by [20]. Utilizing rational contraction to establish fixed point theorems were pioneers by [21] . Recently, tricomplex valued controlled metric spaces introduced by [22]. Building on these results, we generalized a new type of contraction mapping and established fixed point theorems in tricomplex valued controlled metric spaces.

2. Preliminaries

In this article the families of real, complex, bicomplex, and tricomplex numbers, denoted by C^0, C^1, C^2 and C^3 respectively.

Let,

$$\omega = \varrho_1 + \varrho_2 i_1 + \varrho_3 i_2 + \varrho_4 i_1 i_2,$$

where $\varrho_1, \varrho_1, \varrho_1, \varrho_1 \in C^0$ and independent units i_1 and i_2 such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$ the set of bicomplex numbers define as

$$C^2 = \{\omega : \omega = \varrho_1 + \varrho_2 i_1 + \varrho_3 i_2 + \varrho_4 i_1 i_2, \varrho_1, \varrho_2, \varrho_3, \varrho_4 \in C^0\}$$

$$\text{Or } C^2 = \{\omega : \omega = \Theta_1 + i_2 \Theta_2, \Theta_1, \Theta_2 \in C^1\}$$

where $\Theta_1 = \varrho_1 + \varrho_2 i_1$ and $\Theta_2 = \varrho_3 + \varrho_4 i_1$.

$$\Pi = \varrho_1 + \varrho_2 i_1 + \varrho_3 i_2 + \varrho_4 j_1 + \varrho_5 i_3 + \varrho_6 j_2 + \varrho_7 j_3 + \varrho_8 i_4$$

Where $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_8 \in C^0$ and $i_1, i_2, i_3, i_4, j_1, j_2$ and j_3 are independent units such that $i_1^2 = i_2^2 = i_3^2 = i_4^2 = -1, j_1^2 = j_2^2 = j_3^2 = 1, j_1 = i_1 i_2 = i_2 i_1, j_2 = i_1 i_3 = i_3 i_1$ and, $i_4 = i_1 i_3 = i_1 i_2 i_3$. the set of tricomplex numbers define as

$$C^3 = \{\Pi : \Pi = \varrho_1 + \varrho_2 i_1 + \varrho_3 i_2 + \varrho_4 j_1 + \varrho_5 i_3 + \varrho_6 j_2 + \varrho_7 j_3 + \varrho_8 i_4, \varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_8 \in C^0\}$$

$$C^3 = \{\Pi : \Pi = \alpha_1 + i_3 \alpha_2, \alpha_1, \alpha_2 \in C^2\}$$

where $\alpha_1 = \Theta_1 + \Theta_2 i_2 \in C^2$ and $\alpha_2 = \Theta_3 + \Theta_4 i_2 \in C^2$
Let two tricomplex numbers Π and χ such that $\Pi = \alpha_1 + i_3 \alpha_2$ and $\chi = \beta_1 + i_3 \beta_2$ then

$$\Pi \pm \chi = (\alpha_1 + i_3 \alpha_2) \pm (\beta_1 + i_3 \beta_2) = \alpha_1 \pm \beta_1 + i_3 (\alpha_2 \pm \beta_2)$$
$$\text{and } \Pi.\chi = (\alpha_1 + i_3 \alpha_2)(\beta_1 + i_3 \beta_2) = (\alpha_1 \beta_1 - \alpha_2 \beta_2) + i_3 (\alpha_1 \beta_2 + \alpha_2 \beta_1)$$

In the families of tricomplex numbers C^3 have four idempotent elements $0, 1, \varphi_1 = \frac{1+j_3}{2}$ and $\varphi_2 = \frac{1-j_3}{2}$ such that $\varphi_1 + \varphi_2 = 1$ and $\varphi_1.\varphi_2 = 0$. Hence φ_1, φ_2 are not trivial. Every tricomplex numbers can be expressed as an union of φ_1, φ_2 for example

$$\Pi = \alpha_1 + i_3 \alpha_2 = (\alpha_1 - i_2 \alpha_2)\varphi_1 + (\alpha_1 + i_2 \alpha_2)\varphi_2$$

Here Π is idempotent of the tricomplex numeral and $\Pi_1 = (\alpha_1 - i_2 \alpha_2)\varphi_1$ and $\Pi_2 = (\alpha_1 + i_2 \alpha_2)\varphi_2$ are the idempotent components of the bicomplex numeral Π.
A tricomplex number $\Pi = \alpha_1 + i_3 \alpha_2$ is invertible if there exists $\chi \in C^3$ such that $\Pi \chi = 1$.
A tricomplex number $\Pi = \alpha_1 + i_3 \alpha_2$ is nonsingular if $|\alpha_1^2 + \alpha_2^2| \neq 0$ and singular if $|\alpha_1^2 + \alpha_2^2| = 0$.
Let $\Pi = \alpha_1 + i_3 \alpha_2$ and $\chi = \beta_1 + i_3 \beta_2$ be tricomplex numerals. Then the partial order relation $\leq_{i_3}$ on C^3 or $\Pi \leq_{i_3} \chi$ if the following axioms are satisfied.

1. $\alpha_1 = \beta_1, \alpha_2 = \beta_2$;

2. $\alpha_1 <_{i_3} \beta_1, \alpha_2 = \beta_2$;

3. $\alpha_1 = \beta_1, \alpha_2 <_{i_3} \beta_2$;

4. $\alpha_1 <_{i_3} \beta_1, \alpha_2 <_{i_3} \beta_2$;

If $\Pi \leq_{i_3} \chi$ implies $\Pi <_{i_3} \chi$ but $\Pi \neq \chi$
Having two tricomplex numerals $\Pi, \chi \in C^3$, the following conditions hold:

1. $\Pi \leq_{i_3} \chi$ if $||\Pi|| \leq ||\chi||$;

2. $||\Pi + \chi|| \leq ||\Pi|| + ||\chi||$;

3. $||\varrho\Pi|| = |\varrho|||\Pi||$, where $\varrho \in C^0$

4. $||\Pi\chi|| \leq 2||\Pi||||\chi||$, If at least one of Π, χ nonsingular then equality holds.

5. $||\Pi^{-1}|| = ||\Pi||^{-1}$, if $||\Pi|| \neq 0$

6. $||\frac{\Pi}{\chi}|| = \frac{||\Pi||}{||\chi||}$, if $||\chi|| \neq 0$

Let $\aleph \neq 0$ and $\phi : \aleph \times \aleph \longrightarrow [1, \infty)$, then the function $\tau : \aleph \times \aleph \longrightarrow [1, \infty)$ such that [20]

1. $\tau(\Phi, \Psi) \geq 0$ and $\tau(\Phi, \Psi) = 0 \Longleftrightarrow \Phi = \Psi$;

2. $\tau(\Phi, \Psi) = \tau(\Psi, \Phi)$;

3. $\tau(\Phi, \nu) \leq \phi(\Phi, \Psi)\tau(\Phi, \Psi) + \phi(\Phi, \nu)\phi(\Psi, \nu)$;

Then $(\aleph, \tau)$ is called a controlled metric space for all $\Phi, \Psi, \nu \in \aleph$. Let $\omega_1, \omega_2 \in C^1$, where C^1 is the set of complex numbers that define a partial order $\leq$ on C^1 such that [12], $\omega_1 \leq \omega_2$ if and only if $Re(\omega_1) \leq Re(\omega_2)$ and $Im(\omega_1) \leq Im(\omega_2)$ that is $\omega_1 \leq \omega_2$. If any of the following is true

1. $Re(\omega_1) = Re(\omega_2)$ and $Im(\omega_1) = Im(\omega_2)$;

2. $Re(\omega_1) < Re(\omega_2)$ and $Im(\omega_1) = Im(\omega_2)$;

3. $Re(\omega_1) = Re(\omega_2)$ and $Im(\omega_1) < Im(\omega_2)$;

4. $Re(\omega_1) < Re(\omega_2)$ and $Im(\omega_1) < Im(\omega_2)$;

we also write $\omega_1 \lesssim \omega_2$ if $\omega_1 \neq \omega_2$ and one of (2),(3), and (4) satisfiy.

$$0 \leq \omega_1 \lesssim \omega_2 \Rightarrow |\omega_1| < |\omega_2|$$
$$\omega_1 \leq \omega_2 \text{ and } \omega_2 < \omega_3 \Rightarrow \omega_1 < \omega_3.$$

Let $\aleph \neq 0$. A mapping $\tau : \aleph \times \aleph \rightarrow C^1$ such that [12],

1. $0 \leq \tau(\Phi, \Psi)$ for all $\Phi, \Psi \in \aleph$ and $\tau(\Phi, \Psi) = 0$ if and only if $\Phi = \Psi$;

2. $\tau(\Phi, \Psi) = \tau(\Psi, \Phi)$, for all $\Phi, \Psi \in \aleph$;

3. $\tau(\Phi, \Psi) \leq \tau(\Phi, \nu) + \tau(\nu, \Psi)$ for all $\Phi, \Psi, \nu \in \tau$.

Then $(\aleph, \tau)$ is called a complex valued metric space.
Suppose $\aleph \neq 0$. A mapping $\tau : \aleph \times \aleph \longrightarrow C^3$ is such that [22],

1. $\tau(\Phi, \Psi) \geq_{i_3} 0$ and $\tau(\Phi, \Psi) = 0 \Longleftrightarrow \Phi = \Psi$

2. $\tau(\Phi, \Psi) = \tau(\Psi, \Phi)$

3. $\tau(\Phi, \Psi) \leq_{i_3} \tau(\Phi, \nu) + \tau(\nu, \Psi), \forall \Phi, \Psi, \nu \in \aleph$

Then $(\aleph, \tau)$ is called a tricomplex valued metric space.
Suppose $\aleph$ is nonempty set and $\phi : \aleph \times \aleph \longrightarrow [1, \infty)$, then the function $\tau : \aleph \times \aleph \longrightarrow C^3$ such that [22],

1. $\tau(\Phi, \Psi) \geq_{i_3} 0$ and $\tau(\Phi, \Psi) = 0 \Longleftrightarrow \Phi = \Psi$;

2. $\tau(\Phi, \Psi) = \tau(\Psi, \Phi)$;

3. $\tau(\Phi, \nu) \leq_{i_3} \phi(\Phi, \Psi)\tau(\Phi, \Psi) + \phi(\Phi, \nu)\phi(\Psi, \nu)$;

Then $(\aleph, \tau)$ is called a tricomplex valued controlled metric space for all $\Phi, \Psi, \nu \in \aleph$. Let $(\aleph, \tau)$ be a tricomplex valued controlled metric space then,

1. A sequence $\{\Phi_n\} \in \aleph$ is convergent to $\Phi \in \aleph$, if $\tau(\Phi_n, \Phi) <_{i_3} l$ for all $l >_{i_3} 0$ where $l \in C^3$, $\forall n \geq N$ that is $\lim_{n \longrightarrow \infty} \Phi_n = \Phi$.

2. A sequence $\{\Phi_n\} \in \aleph$ is Cauchy sequence in $(\aleph, \tau)$, if $\tau(\Phi_n, \Phi_m) <_{i_3} l$ for all $l >_{i_3} 0$ where $l \in C^3$ and $\forall n, m \geq N$.

3. If every Cauchy sequence is convergent in $\aleph$ then it is complete tricomplex valued controlled metric space.

Lemma 1. Suppose $(\aleph, \tau)$ be a tricomplex valued controlled metric space then, a sequence $\{\Phi_n\} \in \aleph$ is Cauchy sequence, such that $\Phi_n \neq \Phi_m$ where $n \neq m$ then, $\{\Phi_n\} \in \aleph$ converges to at most one point [22].

Lemma 2. Suppose $(\aleph, \tau)$ be a tricomplex valued controlled metric space then, a function $\tau : \aleph \times \aleph \longrightarrow C^3$ is continuous with respect to the partial order "$\leq_{i_3}$" [22].

3. Main result

Theorem 1. Let (Ξ, τ) be a complete tricomplex controlled metric space. Let the continuous mapping $\aleph : \Xi \to \Xi$ be such that

$$\tau(\aleph\Phi, \aleph\Psi) \leq_{i_3} a\frac{\{1 + \tau(\Phi, \aleph\Phi)\}\,\tau(\Psi, \aleph\Psi)}{1 + \tau(\Phi, \Psi)} + b\tau(\Phi, \Psi) + c\tau(\Phi, \aleph\Phi) + d\tau(\Psi, \aleph\Psi) \qquad (1)$$

where $a, b, c, d \in [0, 1)$ with $a + b + c + d < 1$ and $\forall \Phi, \Psi \in \Xi$ for $\Phi_0 \in \Xi$, take $\Phi_n = \aleph^n \Phi_0$. Suppose that,

$$\max_{m \geq 1} \lim_{i_3 \to \infty} \frac{\rho(\Phi_{i_3+1}, \Phi_{i_3+2})\rho(\Phi_{i_3+1}, m)}{\rho(\Phi_{i_3}, \Phi_{i_3+1})} < \frac{1}{\kappa}$$

Assume that $\forall \Phi \in \Xi, \lim_{n \to \infty} \rho(\Phi_n, \Phi)$ and $\lim_{n \to \infty} \rho(\Phi, \Phi_n)$ exist and are finite. Then $\aleph$ has a unique fixed point.

Proof. Suppose that $\Phi_0 \in \Xi$. we construct $\{\Phi_n\}$ in Ξ by $\Phi_{n+1} = \aleph\Phi_n, \forall n \in \mathcal{N}$ such that $\Phi_{n+1} = \Phi_n$, then $\aleph\Phi_n = \Phi_n$. Hence the proof is finished. Now, we assume that $\Phi_{n+1} \neq \Phi_n, \forall n \in \mathbb{N}$. Thus, by (1), we get

$$\tau(\Phi_n, \Phi_{n+1}) = \tau(\aleph\Phi_{n-1}, \aleph\Phi_n)$$
$$\leq_{i_3} a\frac{\{1 + \tau(\Phi_{n-1}, \aleph\Phi_{n-1})\}\,\tau(\Phi_n, \aleph\Phi_n)}{1 + \tau(\Phi_{n-1}, \Phi_n)} + b\tau(\Phi_{n-1}, \Phi_n)$$
$$+ c\tau(\Phi_{n-1}, \aleph\Phi_{n-1}) + d\tau(\Phi_n, \aleph\Phi_n)$$
$$\leq_{i_3} a\frac{\{1 + \tau(\Phi_{n-1}, \Phi_n)\}\,\tau(\Phi_n, \Phi_{n+1})}{1 + \tau(\Phi_{n-1}, \Phi_n)} + b\tau(\Phi_{n-1}, \Phi_n)$$
$$+ c\tau(\Phi_{n-1}, \Phi_n) + d\tau(\Phi_n, \Phi_{n+1})$$
$$\leq_{i_3} a\tau(\Phi_n, \Phi_{n+1}) + b\tau(\Phi_{n-1}, \Phi_n) + c\tau(\Phi_{n-1}, \Phi_n) + d\tau(\Phi_n, \Phi_{n+1})$$

$$\implies \tau(\Phi_n, \Phi_{n+1}) \leq_{i_3} \frac{b+c}{1-a-d}\tau(\Phi_{n-1}, \Phi_n)$$

Since $\kappa = \frac{b+c}{1-a-d} < 1$, we have

$$\tau(\Phi_n, \Phi_{n+1}) \leq_{i_3} \kappa\tau(\Phi_{n-1}, \Phi_n)$$
$$\leq_{i_3} \kappa^2\tau(\Phi_{n-2}, \Phi_{n-1})$$
$$\vdots$$
$$\leq_{i_3} \kappa^n\tau(\Phi_0, \Phi_1)$$

For every $n, m \in \mathbb{N}$ and $n < m$, we have

$$
\begin{aligned}
\tau(\Phi_n, \Phi_m) \leq_{i_3} & \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \rho(\Phi_{n+1}, \Phi_m)\tau(\Phi_{n+1}, \Phi_m) \\
\leq_{i_3} & \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+1}, \Phi_{n+2})\tau(\Phi_{n+1}, \Phi_{n+2}) \\
& + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+2}, \Phi_m)\tau(\Phi_{n+2}, \Phi_m) \\
\leq_{i_3} & \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+1}, \Phi_{n+2})\tau(\Phi_{n+1}, \Phi_{n+2}) \\
& + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+2}, \Phi_m)\rho(\Phi_{n+2}, \Phi_{n+3}) + \tau(\Phi_{n+2}, \Phi_{n+3}) \\
& + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+2}, \Phi_m)\rho(\Phi_{n+3}, \Phi_m)\tau(\Phi_{n+3}, \Phi_m) \dots \\
\leq_{i_3} & \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\rho(\Phi_j, \Phi_m)\right)\rho(\Phi_i, \Phi_{i+1})\tau(\Phi_i, \Phi_{i+1}) \\
& + \prod_{i=n+1}^{m-1}\rho(\Phi_i, \Phi_m)\tau(\Phi_{m-1}, \Phi_m)
\end{aligned}
$$

$$
\begin{aligned}
\tau(\Phi_n, \Phi_m) \leq_{i_3} & \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\rho(\Phi_j, \Phi_m)\right)\rho(\Phi_i, \Phi_{i+1})\tau(\Phi_i, \Phi_{i+1}) \\
& + \left(\prod_{i=n+1}^{m-1}\rho(\Phi_i, \Phi_m)\right)\tau(\Phi_{m-1}, \Phi_m)\rho(\Phi_{m-1}, \Phi_m)\tau(\Phi_{m-1}, \Phi_m) \\
\leq_{i_3} & \rho(\Phi_n, \Phi_{n+1})\kappa^n\tau(\Phi_0, \Phi_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\rho(\Phi_j, \Phi_m)\right)\rho(\Phi_i, \Phi_{i+1})\kappa^i\tau(\Phi_0, \Phi_1) \\
& + \left(\prod_{i=n+1}^{m-1}\rho(\Phi_i, \Phi_m)\right)\kappa^{m-1}\tau(\Phi_0, \Phi_1)\rho(\Phi_{m-1}, \Phi_m) \\
= & \rho(\Phi_n, \Phi_{n+1})\kappa^n\tau(\Phi_0, \Phi_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\rho(\Phi_j, \Phi_m)\right)\rho(\Phi_i, \Phi_{i+1})\kappa^i\tau(\Phi_0, \Phi_1)
\end{aligned}
$$

Let

$$
\aleph_\theta = \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\rho(\Phi_j, \Phi_m)\right)\rho(\Phi_i, \Phi_{i+1})\kappa^i\tau(\Phi_0, \Phi_1)
$$

Then we get

$$
\tau(\Phi_n, \Phi_m) \leq \tau(\Phi_0, \Phi_1)\left[\kappa^n\rho(\Phi_n, \Phi_{n+1}) + (\aleph_{m-1} - \aleph_n)\right]
$$

applying $(\Phi, \Psi) \geq 1$ and ratio test, $\lim_{n\to\infty} \aleph_n$ exists. Thus $\{\aleph_n\}$ is a Cauchy sequence.

$$
\implies \lim_{n,m\to\infty} \tau(\Phi_n, \Phi_m) = 0
$$

Thus, $\{\Phi_n\}$ is a Cauchy sequence in the complete tricomplex valued metric space in (Ξ, τ). Then $\{\Phi_n\}$ converges to Φ^*. From (1), it has a unique limit and by (2) we get

$$
\Phi^* = \lim_{n\to\infty} \Phi_{n+1} = \lim_{n\to\infty} \aleph\Phi_n = \aleph(\lim_{n\to\infty} \Phi_n) = \aleph\Phi^*.
$$

For uniqueness, let Φ^*, Ψ^* both be fixed points of $\aleph$. Then from (1) we have

$$
\begin{aligned}
\tau(\Phi^*, \Psi^*) = & \tau(\aleph\Phi^*, \aleph\Psi^*) \\
\leq_{i_3} & a\frac{\{1 + \tau(\Phi^*, \aleph\Phi^*)\}\tau(\Psi^*, \aleph\Psi^*)}{1 + \tau(\Phi^*, \Psi^*)} + b\tau(\Phi^*, \Psi^*) \\
& + c\tau(\Phi^*, \aleph\Phi^*) + d\tau(\Psi^*, \aleph\Psi^*) \\
\leq_{i_3} & a\frac{\{1 + \tau(\Phi^*, \Phi^*)\}\tau(\Psi^*, \Psi^*)}{1 + \tau(\Phi^*, \Psi^*)} + b\tau(\Phi^*, \Psi^*) \\
& + c\tau(\Phi^*, \Phi^*) + d\tau(\Psi^*, \Psi^*)
\end{aligned}
$$

$$\implies \tau(\Phi^*, \Psi^*) = 0$$

Then $\aleph$ has a unique fixed point and $\Phi^* = \Psi^*$ ∎

Theorem 2. Let (Ξ, τ) be a complete tricomplex controlled metric space and $\sigma, \varsigma : \Xi \to \Xi$. If there exists mappings $\lambda, \delta : \Xi \to [0, 1)$ such that for all $\Phi, \Psi \in \Xi$,

1. $\lambda(\sigma\Phi) \leq \lambda(\Phi)$, $\delta(\sigma\Phi) \leq \delta(\Phi)$

2. $\lambda(\varsigma\Phi) \leq \lambda(\Phi)$ and $\delta(\varsigma\Phi) \leq \delta(\Phi)$

3. $(\lambda + \delta)\Phi < 1$

4.
$$\tau(\sigma\Phi, \varsigma\Psi) \leq_{i_3} \lambda(\Phi) \max \left\{ (\tau(\Phi, \Psi), \frac{\tau(\Phi, \sigma\Phi)\tau(\Psi, \varsigma\Psi)}{\tau(\Phi, \varsigma\Psi) + \tau(\Psi, \sigma\Phi) + \tau(\Phi, \Psi)} \right\}$$
$$+ \delta(\Phi) \left\{ \tau(\Phi, \sigma\Phi) + \tau(\Psi, \varsigma\Psi) \right\} \tag{2}$$

for all $\Phi, \Psi \in \Xi$ such that $\Phi \neq \Psi$, $\tau(\Phi, \varsigma\Psi) + \tau(\Psi, \sigma\Phi) + \tau(\Phi, \Psi) \neq 0$. Suppose that,

$$\max_{m \geq 1} \lim_{i_3 \to \infty} \frac{\rho(\Phi_{i_3+1}, \Phi_{i_3+2})\rho(\Phi_{i_3+1}, m)}{\rho(\Phi_{i_3}, \Phi_{i_3+1})} < \frac{1}{\kappa}$$

where $\Phi_{n+1} = \sigma\Phi_n$ and $\Phi_{n+2} = \varsigma\Phi_{n+1}$ for each $n \geq 0$. Assume that $\forall \Phi \in \lim_{i \to \infty} \rho(\Phi_n, \Phi)$ and $\lim_{i \to \infty} \rho(\Phi, \Phi_n)$ exist and are finite. Then, σ, ς has a unique common fixed point.

Proof. Again, we construct $\{\Phi_n\}$ in Ξ by $\Phi_{n+1} = \sigma\Phi_n$ and $\Phi_{n+2} = \varsigma\Phi_{n+1}$ and assume that $\Phi_{n+1} \neq \Phi_n, \forall n \in \mathbb{N}$. Thus, by (2), we get

$$\tau(\Phi_n, \Phi_{n+1}) = \tau(\sigma\Phi_{n-1}, \varsigma\Phi_n)$$
$$\leq_{i_3} \lambda(\Phi) \max \left\{ (\tau(\Phi_{n-1}, \Phi_n), \frac{\tau(\Phi_{n-1}, \sigma\Phi_{n-1})\tau(\Phi_n, \varsigma\Phi_n)}{\tau(\Phi_{n-1}, \varsigma\Phi_n) + \tau(\Phi_n, \sigma\Phi_{n-1}) + \tau(\Phi_{n-1}, \Phi_n)} \right\}$$
$$+ \delta(\Phi) \left\{ \tau(\Phi_{n-1}, \sigma\Phi_{n-1}) + \tau(\Phi_n, \varsigma\Phi_n) \right\}$$
$$\leq_{i_3} \lambda(\Phi) \max \left\{ (\tau(\Phi_{n-1}, \Phi_n), \frac{\tau(\Phi_{n-1}, \Phi_n)\tau(\Phi_n, \Phi_{n+1})}{(\Phi_n, \Phi_{n+1}) + \tau(\Phi_n, \Phi_n) + \tau(\Phi_{n-1}, \Phi_n)} \right\}$$
$$+ \delta(\Phi) \left\{ \tau(\Phi_{n-1}, \Phi_n) + \tau(\Phi_n, \Phi_{n+1}) \right\}$$
$$\leq_{i_3} \lambda(\Phi)\tau(\Phi_{n-1}, \Phi_n) + \delta(\Phi)\tau(\Phi_{n-1}, \Phi_{n+1})$$
$$\leq_{i_3} (\lambda(\Phi) + \delta(\Phi))\tau(\Phi_{n-1}, \Phi_n)$$

Since $\kappa = (\lambda(\Phi) + \delta(\Phi)) < 1$, we have

$$\implies \tau(\Phi_n, \Phi_{n+1}) \leq_{i_3} \kappa\tau(\Phi_{n-1}, \Phi_n)$$
$$\leq_{i_3} \kappa^2\tau(\Phi_{n-2}, \Phi_{n-1})$$
$$\vdots$$
$$\leq_{i_3} \kappa^2\tau(\Phi_0, \Phi_1)$$

For every $n, m \in \mathbb{N}$ and $n < m$, we have

$$
\begin{aligned}
\tau(\Phi_n, \Phi_m) \leq_{i_3} & \; \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \rho(\Phi_{n+1}, \Phi_m)\tau(\Phi_{n+1}, \Phi_m) \\
\leq_{i_3} & \; \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+1}, \Phi_{n+2})\tau(\Phi_{n+1}, \Phi_{n+2}) \\
& + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+2}, \Phi_m)\tau(\Phi_{n+2}, \Phi_m) \\
\leq_{i_3} & \; \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+1}, \Phi_{n+2})\tau(\Phi_{n+1}, \Phi_{n+2}) \\
& + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+2}, \Phi_m)\rho(\Phi_{n+2}, \Phi_{n+3}) + \tau(\Phi_{n+2}, \Phi_{n+3}) \\
& + \rho(\Phi_{n+1}, \Phi_m)\rho(\Phi_{n+2}, \Phi_m)\rho(\Phi_{n+3}, \Phi_m)\tau(\Phi_{n+3}, \Phi_m) \ldots \\
\leq_{i_3} & \; \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \rho(\Phi_j, \Phi_m) \right) \rho(\Phi_i, \Phi_{i+1})\tau(\Phi_i, \Phi_{i+1}) \\
& + \prod_{i=n+1}^{m-1} \rho(\Phi_i, \Phi_m)\tau(\Phi_{m-1}, \Phi_m)
\end{aligned}
$$

$$
\begin{aligned}
\tau(\Phi_n, \Phi_m) \leq_{i_3} & \; \rho(\Phi_n, \Phi_{n+1})\tau(\Phi_n, \Phi_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \rho(\Phi_j, \Phi_m) \right) \rho(\Phi_i, \Phi_{i+1})\tau(\Phi_i, \Phi_{i+1}) \\
& + \left(\prod_{i=n+1}^{m-1} \rho(\Phi_i, \Phi_m) \right) \tau(\Phi_{m-1}, \Phi_m)\rho(\Phi_{m-1}, \Phi_m)\tau(\Phi_{m-1}, \Phi_m) \\
\leq_{i_3} & \; \rho(\Phi_n, \Phi_{n+1})\kappa^n \tau(\Phi_0, \Phi_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \rho(\Phi_j, \Phi_m) \right) \rho(\Phi_i, \Phi_{i+1})\kappa^i \tau(\Phi_0, \Phi_1) \\
& + \left(\prod_{i=n+1}^{m-1} \rho(\Phi_i, \Phi_m) \right) \kappa^{m-1} \tau(\Phi_0, \Phi_1)\rho(\Phi_{m-1}, \Phi_m) \\
= & \; \rho(\Phi_n, \Phi_{n+1})\kappa^n \tau(\Phi_0, \Phi_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \rho(\Phi_j, \Phi_m) \right) \rho(\Phi_i, \Phi_{i+1})\kappa^i \tau(\Phi_0, \Phi_1)
\end{aligned}
$$

Let

$$
\Re_\theta^* = \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \rho(\Phi_j, \Phi_m) \right) \rho(\Phi_i, \Phi_{i+1})\kappa^i \tau(\Phi_0, \Phi_1)
$$

Then we get

$$
\tau(\Phi_n, \Phi_m) \leq_{i_3} \tau(\Phi_0, \Phi_1) \left[\kappa^n \rho(\Phi_n, \Phi_{n+1}) + (\Re_{m-1}^* - \Re_n^*) \right]
$$

applying $(\Phi, \Psi) \geq_{i_3} 1$ and ratio test, $\lim_{n \to \infty} \Re_n^*$ exists. Thus $\{\Re_n^*\}$ is a Cauchy sequence.

$$
\implies \lim_{n,m \to \infty} \tau(\Phi_n, \Phi_m) = 0
$$

Thus, $\{\Phi_n\}$ is a Cauchy sequence in the complete tricomplex valued metric space in (Ξ, τ). Then $\{\Phi_n\}$ converges to Φ_1. From (1), it has a unique limit and by (2) we get

$$
\begin{aligned}
\Phi_1 &= \lim_{n \to \infty} \Phi_{n+1} = \lim_{n \to \infty} \sigma\Phi_n = \sigma(\lim_{n \to \infty} \Phi_n) = \sigma\Phi_1. \\
\Phi_1 &= \lim_{n \to \infty} \Phi_{n+1} = \lim_{n \to \infty} \varsigma\Phi_n = \varsigma(\lim_{n \to \infty} \Phi_n) = \varsigma\Phi_1
\end{aligned}
$$

For uniqueness, let Φ_1, Ψ_1 both are fixed points of σ, ς. Then from (2) we have

$$
\begin{aligned}
\tau(\Phi_1, \Psi_1) = \tau(\sigma\Phi_1, \varsigma\Psi_1) \\
\leq_{i_3} & \; \lambda(\Phi) \max \left\{ (\tau(\Phi_1, \Psi_1), \frac{\tau(\Phi_1, \sigma\Phi_1)\tau(\Psi_1, \varsigma\Psi_1)}{\tau(\Phi_1, \varsigma\Psi_1) + \tau(\Psi_1, \sigma\Phi_1) + \tau(\Phi_1, \Psi_1)} \right\} \\
& + \delta(\Phi) \left\{ \tau(\Phi_1, \sigma\Phi_1) + \tau(\Psi_1, \varsigma\Psi_1) \right\} \\
\leq_{i_3} & \; \lambda(\Phi) \max \left\{ (\tau(\Phi_1, \Psi_1), \frac{\tau(\Phi_1, \Phi_1)\tau(\Psi_1, \Psi_1)}{\tau(\Phi_1, \Psi_1) + \tau(\Phi_1, \Psi_1) + \tau(\Phi_1, \Psi_1)} \right\} \\
& + \delta(\Phi) \left\{ \tau(\Phi_1, \Phi_1) + \tau(\Psi_1, \Psi_1) \right\}
\end{aligned}
$$

$$\implies \tau(\Phi_1, \Psi_1) = 0$$

Then σ, ς have a unique common fixed point and $\Phi_1 = \Psi_1$ ■

4. CONCLUSION

Following Price's initial description of multicomplex spaces, further investigations have been conducted by other researchers. Recently, Ramaswamy et al. [22] presented the concept of tricomplex valued controlled metric spaces. In this paper, we expand on these results by proposing a new type of contraction mapping, where we also establish fixed point theorems and make use of the control function within tricomplex valued controlled metric spaces.

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