

ANALYSIS OF $MAP_1^I, PH_2^O / PH_1^I, PH_2^O / 1$ RETRIAL INVENTORY QUEUE, TWO WAY COMMUNICATION, (S, S) REPLENISHMENT POLICY, NEGATIVE ARRIVAL, WORKING BREAKDOWN AND REPAIR

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Abstract

The retrial inventory queueing model with two-way communication, the (s, S) replenishment strategy, negative arrival, working breakdown, and repair are all topics that are discussed in this paper. Our assumption is that the arrival is a Markovian arrival process and that a server is the entity that is responsible for providing phase type services. When there is a positive inventory and the server is idle, the customer who arrives is instantly attended to their needs. In the event that this instance does not occur, the customer who is arriving will be moved to orbit, and a retrial customer from orbit will join afterwards if the server is idle and has positive inventory. When the inventory level is positive, the server will make outgoing calls that follow phase type distribution during the idle period. If the inventory level is zero, the server will continue to stay idle. A negative arrival may occur while the server is providing service. Since the server is experiencing a breakdown, the consumer will experience slow service at this time. After the slow service ends, the server will immediately begin the repair procedure, which follows a phase-type distribution. The policy of (s, S) is utilised to replenish the items. By employing the matrix analytic method, we are able to derive the steady state probability vector. Additionally, we address the busy period, performance measurements, and cost analysis, in addition to providing some numerical examples.

Keywords: Retrial inventory queue, Two way communication, Matrix analytic method, (s, S) Replenishment policy, Negative arrival, Working breakdown, Repair

1. INTRODUCTION

Recent advancements in computer networking and communications technology have led to a growing interest in inventory retrial queueing systems. The level of attention on this topic has been growing due to its increasing relevance. Individual customers in a queueing inventory model are each given a product from the inventory once the service has been completed. This adaptable class of point processes on the real line, which are closely connected to finite-state Markov processes, was initially introduced by Neuts [18] in their work. Among the many point processes, there are a few that stand out as specific situations. These include renewal processes of phase type, Markov-modulated Poisson processes, and some semi-Markov point processes. Chakravarthy [10] offered a comprehensive explanation of the phase-type distribution as well as Markovian arrival process, which included a number of correlated and non-correlated arrival

types. Neuts [20] conducted an investigation into the techniques of matrix-analytic queueing theory.

Falin and Templeton [11] produced the first monograph on retrial queues, which offers a thorough explanation of fundamental techniques and findings. They have produced a number of analytical findings and talked about retrial queues for both single and multiserver systems. Artalejo [3, 4] have produced a bibliography on retrial queues that may be used as an index to find other books and articles about retrial queues. A survey of retrial queue theory and applications was presented by Tuan Phung-Duc [23]. In addition to providing a summary of retrial models that emerge from actual circumstances, he has also identified several unresolved issues and exciting avenues for further study.

There is a queueing order-demand inventory system, and according to Melikov and Molchanov [16], the most effective approach for reordering products is to adhere to the guidelines that are described. Berman et al. [9] conducted research on an inventory control system that was designed for a service center. This system was designed to utilize a single inventory item for each service that was performed. During their evaluation, Yadavalli et al. [25] used the assumption that requests are a component of the renewal process and that reorders are easily available without any difficulty. The inventory system included both a waiting room that was open for an endless amount of time and a service station. When it comes to service stations, it is generally acknowledged that inventory and queueing models have not been unsuccessful the majority of the time.

As a matter of fact, we frequently encounter situations in which there is a possibility of service station failures occurring. A server interruption inventory retrial queueing system was discussed by Krishnamoorthy et al. [14]. They incorporated (s, S) replenishment policy into their model. Both the lead time and the service time are determined by an exponential distribution, but the arrival time is determined by a Poisson distribution. This policy is distinguished by the fact that the lead time and service time are both determined by an exponential distribution. Ushakumari [24] conducted an investigation of the retrial inventory queue system, which was characterized by server failure. Ayyappan and Arulmozhi [6] have discussed retrial queueing inventory system with working vacation, flush out, balking, breakdown, and repair, as well as a constant retrial rate and collision with orbital customers. Additionally, Ayyappan and Meena [7] address the concept of a preemptive priority queue, which includes clients who are impatient, have a single vacation, and are in need of repair.

Artalejo and Phung-Duc [2] discussed the steady state behaviour of an $M/G/1$ retrial queue with two-way communication for a single channel. Ayyappan and Thilagavathy [8] is discussed standby server rendering service due to primary server is breakdowns with constant retrial rate and collision with orbit. The study conducted by Nazarov et al. [17] found that an investigation was performed on a single-server Markovian retrial queue with a finite number of sources and two-way communication. The exponential distribution takes into account not only the service times for receiving calls but also the service times for outgoing calls. Somasundaram et al. [5] presents a discussion on a single server constant retrial two-way communication queueing architecture. This architecture is designed to ensure that the server continues to provide service even after a breakdown, although at a slower rate, rather than entirely shutting down.

The research conducted by Karthick et al. [13] involved the implementation of a single channel queue that included server breakdown and repair, as well as differentiated vacation. Rakesh Kumar and Bhavneet Singh Soodan [21] did a study on a queueing model that had correlated inputs and reneging. They did a transient analysis on this model. Haghighi et al. [12] had discussed in detail about the impatient customers. Agassi Melikov et al. [1] studied single server models of queueing inventory systems with catastrophes in the warehouse part and negative customers in service facility.

2. MODEL DESCRIPTION

In our discussion, we consider a single server retrial inventory queueing model, where the incoming call arrivals follow MAP with representation of matrices (D_0, D_1) of order m_1 , where D_0 represents no arrival occurs and D_1 represents an arrival occurs to the system. The incoming call arrival gets service immediately only when the server is idle with positive inventory level, if not the arriving customer will join the orbit of infinite capacity. After service completion, the incoming call arrival leaves the system and server may provides the service if any incoming call arrival or retrial customer enters, otherwise the server is idle. During the idle period with the positive inventory level, the server provides the service if any incoming call arrival is entered otherwise will make outgoing calls which follows phase type distribution with representation (α, U) of order m_2 . The incoming and outgoing calls service times follow phase type distributions with representation (β_1, T_1) of order n_1 and (β_2, T_2) of order n_2 respectively. During the service time of incoming and out going arrivals, the server may struck with break down like power fluctuations, hanging the system, network issues, etc. During this breakdown period, the server provides service with slow rate to that customer. The working break down (i.e., slow service) of the server for both incoming and outgoing calls follow phase type distribution with representation $(\beta_1, \theta_1 T_1)$ of order n_1 and $(\beta_2, \theta_1 T_2)$ of order n_2 respectively. After slow service completion, the customer leaves the system and the server shall go to repair process which follows phase type distribution with representation (γ, R) of order l . After repair completion, the server is idle. During the idle period with the positive inventory level, the server provides the service if any incoming call arrival is entered otherwise will make outgoing calls. The negative arrival causes the incoming call arrival or retrial customer who is currently receiving service or slow service from the server to be disturbed and removed. Due to the negative arrival the service cannot be completed of the incoming call arrival also negative arrival disturbs only customer and his service. The inter negative arrival times follow exponential distribution with constant rate λ^- for every negative arrival. Also the inter retrial times follow exponential distribution with constant rate ψ for every retrial customer from orbit of infinite capacity and also the τ be the breakdown parameter during incoming and out call arrivals follow exponential distribution. To replenish the items (s, S) policy is applied and the duration of replenishment follows exponential distribution with parameter σ .

3. MATRIX GENERATIONS OF QBD PROCESS

We are going to discuss this part, which comprises the notation that forms the basis of the Quasi Birth and Death (QBD) process in our model.

- \otimes – Any two matrices with different order are multiplied by using a Kronecker product, this part of work was done by the research of Steeb and Hardy [22].
- \oplus – Any two different order matrices can be taken for sum by using a Kronecker sum.
- I_k – The identity matrix has k dimensions.
- e – The column vector's each of its elements 1 is taken part of the appropriate dimension as we need.
- e_k – The column vector for every k element is 1.
- The arrival rate of incoming call arrivals is represented by λ and described as $\lambda = \pi D_1 e$.
- The service rate for incoming call arrivals represented by μ_1 and described as $\mu_1 = [\beta_1(-T_1)^{-1}e_{n_1}]^{-1}$.
- The service rate for outgoing call arrivals represented by μ_2 and described as $\mu_2 = [\beta_2(-T_2)^{-1}e_{n_2}]^{-1}$.
- The working breakdown service rate for incoming call arrivals represented by $\theta_1\mu_1$ and described as $\theta_1\mu_1 = [\beta_1(-\theta_1 T_1)^{-1}e_{n_1}]^{-1}$.
- The working breakdown service rate for outgoing call arrivals represented by $\theta_2\mu_2$ and described as $\theta_2\mu_2 = [\beta_2(-\theta_2 T_2)^{-1}e_{n_2}]^{-1}$.
- Repair rate of server is δ , where $\delta = [\gamma(-R)^{-1}e_l]^{-1}$.

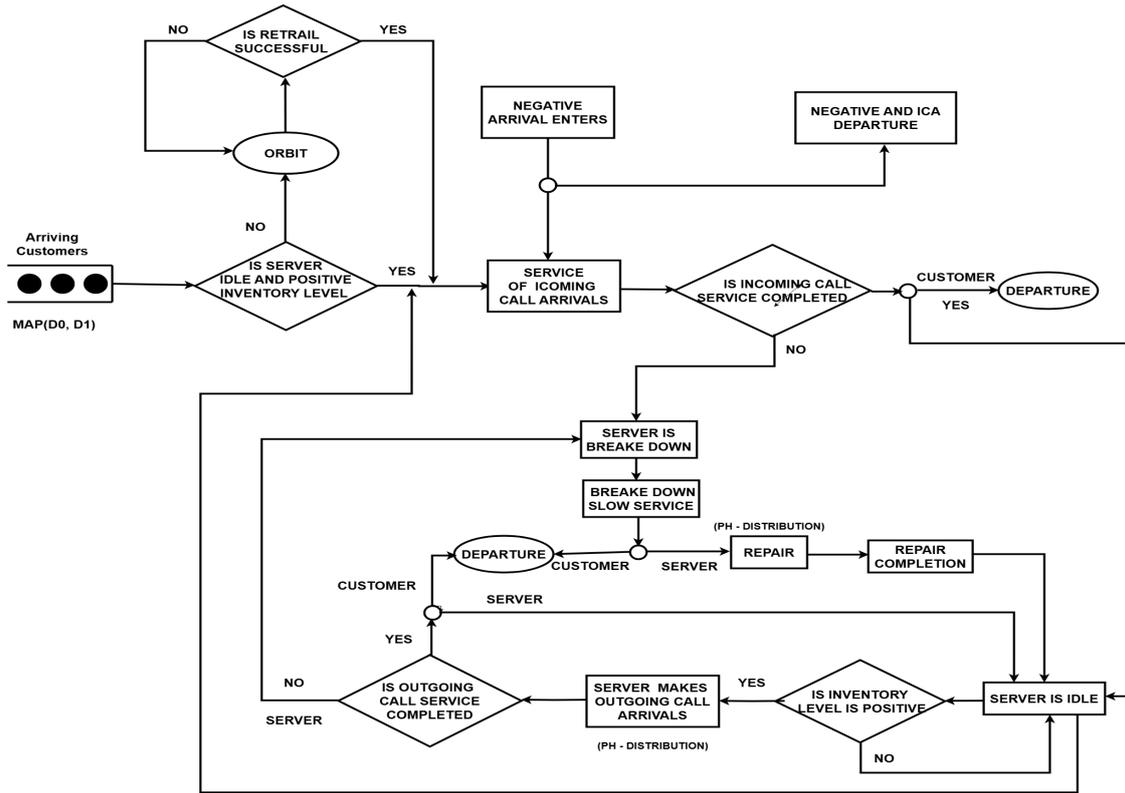


Figure 1: A pictorial representation of the model

- Rate at which server makes the outgoing call, where $\mu_3 = [\alpha(-U)^{-1}e_{m_2}]^{-1}$.
- Let $N(t)$ be number of customers in the orbit of infinite capacity at time t .
- Let $Y(t)$ be the level of server status at time t . Where

$$Y(t) = \begin{cases} 0, & \text{server being an idle also making outgoing calls when positive inventory,} \\ 1, & \text{server is offering service for incoming call arrivals,} \\ 2, & \text{server is offering service for outgoing call arrivals,} \\ 3, & \text{server is offering service during breakdown time for incoming call arrivals,} \\ 4, & \text{server is offering service during breakdown time for outgoing call arrivals,} \\ 5, & \text{server is under repair process.} \end{cases}$$

- Let $I(t)$ be the level of inventory items at time t .
- $J_1(t)$ be the incoming call service by the server at time t .
- $J_2(t)$ be the outgoing call service by the server at time t .
- $M_1(t)$ be the incoming call arrivals at time t .
- $M_2(t)$ be the incoming call arrivals at time t .

Let $\{N(t), Y(t), I(t), J_1(t), J_2(t), M_1(t), M_2(t) : t \geq 0\}$ represents a CTMC with state-level independent QBD processes whose state space is as follows:

$$\Omega = \bigcup_{i=0}^{\infty} l(i), \text{ where}$$

$$l(i) = \{(i, 0, 0, k_1) : 0 \leq k_1 \leq m_1\} \cup \{(i, 0, j, k_1, k_2) : 1 \leq j \leq S, 1 \leq k_1 \leq m_1, 1 \leq k_2 \leq m_2\} \\ \cup \{(i, u, j, r_1, k_1) : u = 1, 3, 1 \leq j \leq S, 1 \leq r_1 \leq n_1, 1 \leq k_1 \leq m_1\} \\ \cup \{(i, u, j, r_2, k_1) : u = 2, 4, 1 \leq j \leq S, 1 \leq r_2 \leq n_2, 1 \leq k_1 \leq m_1\} \\ \cup \{(i, 5, j, k_1) : 0 \leq j \leq S, 1 \leq k_1 \leq m_1\}.$$

The infinitesimal matrix generator of the QBD process is written as follows,

$$Q = \begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & 0 & 0 & \dots \\ V_2 & V_1 & V_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & V_2 & V_1 & V_0 & 0 & 0 & 0 & \dots \\ 0 & 0 & V_2 & V_1 & V_0 & 0 & 0 & \dots \\ 0 & 0 & 0 & V_2 & V_1 & V_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

In the following, the ingress of the matrix Q structures are presented.

$$A_{00} = \begin{bmatrix} A_{00}^{11} & A_{00}^{12} & A_{00}^{13} & 0 & 0 & 0 \\ A_{00}^{21} & A_{00}^{22} & 0 & A_{00}^{24} & 0 & 0 \\ A_{00}^{31} & 0 & A_{00}^{33} & 0 & A_{00}^{35} & 0 \\ 0 & 0 & 0 & A_{00}^{44} & 0 & A_{00}^{46} \\ 0 & 0 & 0 & 0 & A_{00}^{55} & A_{00}^{56} \\ A_{00}^{61} & 0 & 0 & 0 & 0 & A_{00}^{66} \end{bmatrix}, \quad A_{00}^{11} = \begin{bmatrix} G_1 & 0 & \dots & 0 & 0 & \dots & G_2 \\ 0 & G_3 & \dots & 0 & 0 & \dots & G_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & G_3 & 0 & \dots & G_4 \\ 0 & 0 & \dots & 0 & G_5 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & G_5 \end{bmatrix},$$

$$G_1 = D_0 - \sigma I_{m_1}, \quad G_2 = \alpha \otimes \sigma I_{m_1}, \quad G_3 = U \oplus D_0 - \sigma I_{2m_1}, \quad G_4 = \sigma I_{2m_1}, \quad G_5 = U \oplus D_0.$$

$$A_{00}^{12} = \begin{bmatrix} 0 \\ I_S \otimes e_{m_2} \otimes \beta_1 \otimes D_1 \end{bmatrix}, \quad A_{00}^{13} = \begin{bmatrix} 0 \\ I_S \otimes U^0 \otimes \beta_2 \otimes I_{m_1} \end{bmatrix}, \quad A_{00}^{21} = \begin{bmatrix} G_6 & G_7 & \dots & 0 & 0 \\ 0 & G_8 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & G_8 & G_7 \end{bmatrix},$$

$$G_6 = T_1^0 \otimes I_{m_1}, \quad G_7 = e_{n_1} \otimes \alpha \otimes \lambda^- I_{m_1}, \quad G_8 = T_1^0 \otimes \alpha \otimes I_{m_1}.$$

$$A_{00}^{22} = \begin{bmatrix} G_9 & \dots & 0 & 0 & \dots & G_{10} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & G_9 & 0 & \dots & G_{10} \\ 0 & \dots & 0 & G_{11} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & G_{11} \end{bmatrix}, \quad A_{00}^{33} = \begin{bmatrix} G_{12} & \dots & 0 & 0 & \dots & G_{13} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & G_{12} & 0 & \dots & G_{13} \\ 0 & \dots & 0 & G_{14} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & G_{14} \end{bmatrix}$$

$$A_{00}^{24} = [I_S \otimes \tau I_{n_1 m_1}], \quad A_{00}^{31} = \begin{bmatrix} T_2^0 I_{m_1} & 0 \\ 0 & I_{S-1} \otimes T_2^0 \alpha \otimes I_{m_1} \end{bmatrix}, \quad A_{00}^{35} = [I_S \otimes \tau I_{n_2 m_1}],$$

$$A_{00}^{44} = \begin{bmatrix} G_{15} & \dots & 0 & 0 & \dots & G_{10} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & G_{15} & 0 & \dots & G_{10} \\ 0 & \dots & 0 & G_{16} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & G_{16} \end{bmatrix}, \quad A_{00}^{46} = \begin{bmatrix} G_{17} & G_{18} & 0 & \dots & 0 & 0 \\ 0 & G_{17} & G_{18} & \dots & 0 & 0 \\ 0 & 0 & G_{17} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & G_{17} & G_{18} \end{bmatrix},$$

$$G_9 = T_1 \oplus D_0 - (\tau + \sigma + \lambda^-) I_{n_1 m_1}, \quad G_{10} = \sigma I_{m_1 n_1}, \quad G_{11} = T_1 \oplus D_0 - (\tau + \lambda^-) I_{n_1 m_1}, \\ G_{12} = T_2 \oplus D_0 - (\tau + \sigma) I_{n_2 m_1}, \quad G_{14} = T_2 \oplus D_0 - \tau I_{n_2 m_1}, \quad G_{15} = \theta_1 T_1 \oplus D_0 - (\sigma + \lambda^-) I_{n_1 m_1}, \\ G_{13} = \sigma I_{n_2 m_1}, \quad G_{16} = \theta_1 T_1 \oplus D_0 - \lambda^- I_{n_1 m_1}, \quad G_{17} = \theta_1 T_1^0 \otimes \gamma \otimes I_{m_1}, \quad G_{18} = e_{n_1} \otimes \gamma \otimes \lambda^- I_{m_1}$$

$$A_{00}^{55} = \begin{bmatrix} G_{19} & \dots & 0 & 0 & \dots & G_{13} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & G_{19} & 0 & \dots & G_{13} \\ 0 & \dots & 0 & G_{20} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & G_{20} \end{bmatrix}, \quad A_{00}^{66} = \begin{bmatrix} G_{21} & \dots & 0 & 0 & \dots & G_{22} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & G_{21} & 0 & \dots & G_{22} \\ 0 & \dots & 0 & G_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & G_{23} \end{bmatrix},$$

$$G_{19} = \theta_2 T_2 \oplus D_0 - \sigma I_{n_2 m_1}, \quad G_{20} = \theta_2 T_2 \oplus D_0, \quad G_{21} = R \oplus D_0 - \sigma I_{m_1}, \quad G_{22} = \sigma I_{m_1}$$

$$G_{23} = R \oplus D_0, \quad A_{00}^{56} = [I_S \otimes \theta_2 T_2^0 \otimes \gamma I_{m_1} \quad 0], \quad A_{00}^{61} = \begin{bmatrix} R^0 \otimes I_{m_1} & 0 \\ 0 & R^0 \otimes \alpha I_{m_1} \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} A_{01}^{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{01}^{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{01}^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{01}^{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{01}^{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{01}^{66} \end{bmatrix}, \quad A_{01}^{11} = [D_1], \quad A_{01}^{22} = [I_S \otimes I_{n_1} \otimes D_1] = A_{01}^{44},$$

$$A_{01}^{33} = [I_S \otimes I_{n_2} \otimes D_1] = A_{01}^{55}, \quad A_{01}^{66} = [I_{(S+1)} \otimes I_l \otimes D_1],$$

$$V_0 = A_{01}, \quad V_2 = \begin{bmatrix} 0 & V_2^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{where } V_2^{12} = \begin{bmatrix} 0 \\ \beta_1 \otimes e_{m_2} \otimes \psi I_{m_1} \end{bmatrix}.$$

$$V_1 = \begin{bmatrix} V_1^{11} & V_1^{12} & V_1^{13} & 0 & 0 & 0 \\ V_1^{21} & V_1^{22} & 0 & V_1^{24} & 0 & 0 \\ V_1^{31} & 0 & V_1^{33} & 0 & V_1^{35} & 0 \\ 0 & 0 & 0 & V_1^{44} & 0 & V_1^{46} \\ 0 & 0 & 0 & 0 & V_1^{55} & V_1^{56} \\ V_1^{61} & 0 & 0 & 0 & 0 & V_1^{66} \end{bmatrix}, \quad V_1^{11} = \begin{bmatrix} G_1 & 0 & \dots & 0 & 0 & \dots & G_2 \\ 0 & G_{24} & \dots & 0 & 0 & \dots & G_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & G_{24} & 0 & \dots & G_4 \\ 0 & 0 & \dots & 0 & G_{25} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & G_{25} \end{bmatrix},$$

$$G_{24} = U \oplus D_0 - (\psi + \sigma) I_{m_2 m_1}, \quad G_{25} = U \oplus D_0 - \psi I_{m_2 m_1}.$$

$$V_1^{12} = V_{00}^{12}, \quad V_1^{13} = V_{00}^{13}, \quad V_1^{21} = V_{00}^{21}, \quad V_1^{22} = V_{00}^{22}, \quad V_1^{24} = V_{00}^{24}, \quad V_1^{31} = V_{00}^{31}, \quad V_1^{33} = V_{00}^{33}, \\ V_1^{35} = V_{00}^{35}, \quad V_1^{44} = V_{00}^{44}, \quad V_1^{46} = V_{00}^{46}, \quad V_1^{55} = V_{00}^{55}, \quad V_1^{56} = V_{00}^{56}, \quad V_1^{61} = V_{00}^{61}, \quad V_1^{66} = V_{00}^{66}.$$

4. STATIONARY ANALYSIS

To justify of our model, we ensure the stability of the system under certain conditions as follows:

4.1. Criteria of system stability

Let us assume that the matrix V as $V = V_0 + V_1 + V_2$ is an irreducible infinitesimal generator matrix with dimension of $\{m_1 + Sm_1 m_2 + Sm_1 n_1 + Sm_1 n_2 + Sm_1 n_1 + Sm_1 n_2 + (S + 1)m_1 l\}$.

$$V = \begin{bmatrix} V^{11} & V^{12} & V^{13} & 0 & 0 & 0 \\ V^{21} & V^{22} & 0 & V^{24} & 0 & 0 \\ V^{31} & 0 & V^{33} & 0 & V^{35} & 0 \\ 0 & 0 & 0 & V^{44} & 0 & V^{46} \\ 0 & 0 & 0 & 0 & V^{55} & V^{56} \\ V^{61} & 0 & 0 & 0 & 0 & V^{66} \end{bmatrix}, \quad \text{where } V^{11} = V_1^{11} + V_0^{11}, \quad V^{12} = V_2^{12} + V_1^{12}, \quad V^{13} = V_1^{13},$$

$$V^{21} = V_1^{21} + V_0^{21}, \quad V^{22} = V_1^{22} + V_0^{22}, \quad V^{24} = V_1^{24}, \quad V^{31} = V_1^{31}, \quad V^{33} = V_1^{33} + V_0^{33}, \quad V^{35} = V_1^{35},$$

$$V^{44} = V_1^{44} + V_0^{44}, \quad V^{46} = V_1^{46}, \quad V^{55} = V_1^{55} + V_0^{55}, \quad V^{56} = V_1^{56}, \quad V^{61} = V_1^{61}, \quad V^{66} = V_1^{66} + V_0^{66}.$$

Let ζ be the invariant probability vector can be partitioned as $(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)$ with the dimensions $\{m_1 + Sm_1m_2, Sm_1n_1, Sm_1n_2, Sm_1n_1, Sm_1n_2, (S + 1)m_1l\}$ of the matrix V will be obtained the values such that it must be satisfied these conditions $\zeta V = 0$ and $\zeta e = 1$.

$$\begin{aligned} \text{i.e., } \zeta_0 V^{11} + \zeta_1 V^{21} + \zeta_2 V^{31} + \zeta_5 V^{61} &= 0, \\ \zeta_0 V^{12} + \zeta_1 V^{22} &= 0, \\ \zeta_0 V^{13} + \zeta_2 V^{33} &= 0, \\ \zeta_1 V^{24} + \zeta_3 V^{44} &= 0, \\ \zeta_2 V^{35} + \zeta_4 V^{55} &= 0, \\ \zeta_3 V^{46} + \zeta_4 V^{56} + \zeta_5 V^{66} &= 0, \end{aligned}$$

along with normalising condition that,

$$\zeta_0 e_{m_1} + \sum_{j=1}^S \zeta_0 e_{m_1 m_2} + \sum_{j=1}^S \zeta_1 e_{m_1 n_1} + \sum_{j=1}^S \zeta_2 e_{m_1 n_2} + \sum_{j=1}^S \zeta_3 e_{m_1 n_1} + \sum_{j=1}^S \zeta_4 e_{m_1 n_2} + \sum_{j=0}^S \zeta_5 e_{m_1 l} = 1.$$

Obviously, to examine the stability of our QBD model, it must satisfy condition $\zeta V_0 e < \zeta V_2 e$.

$$\left. \begin{aligned} \zeta_0 D_1 e_{m_1} + \sum_{j=1}^S \zeta_1 [e_{n_1} \otimes D_1 e_{m_1}] \\ \text{i.e., } + \sum_{j=1}^S \zeta_2 [e_{n_2} \otimes D_1 e_{m_1}] + \sum_{j=1}^S \zeta_3 [e_{n_1} \otimes D_1 e_{m_1}] \\ + \sum_{j=1}^S \zeta_4 [e_{n_2} \otimes D_1 e_{m_1}] + \sum_{j=0}^S \zeta_5 [e_l \otimes D_1 e_{m_1}] \end{aligned} \right\} < \sum_{j=1}^S \zeta_0 [\beta_1 \otimes e_{m_2} \otimes \psi I_{m_1}]$$

4.2. Analysis of Stationary Probability Vector

Assuming that x represents the steady-state probability vector, it is possible to divide it as (x_0, x_1, x_2, \dots) with each of its dimensions are $\{m_1 + Sm_2m_1, Sn_1m_1, Sn_2m_1, Sn_1m_1, Sn_2m_1, (S + 1)lm_1\}$ of the matrix Q will be obtained the values such that it must be satisfied these conditions $xQ = 0$ and $x e = 1$.

After the stability criterion has been met, we are able to determine the probability vector for the steady state by applying the equations that are listed below.

$$x_i = x_1 R^{i-1}, \quad i \geq 2.$$

The smallest non-negative solution R , which is based on Neuts [20] work, is able to satisfy the matrix quadratic equation $R^2 V_2 + R V_1 + V_0 = 0$. The rate matrix is obtained by solving the matrix quadratic equation. The order of the rate matrix R is determined by the equation $\{m_1 + Sm_2m_1 + Sn_1m_1 + Sn_2m_1 + Sn_1m_1 + Sn_2m_1 + (S + 1)lm_1\}$, and it satisfies the constraint $R V_2 e = V_0 e$.

It is possible to ascertain the subvector x_0 by employing the formulae that are presented below.

$$\begin{aligned} x_0 A_{00} + x_1 V_2 &= 0 \\ x_0 A_{01} + x_1 (V_1 + R V_2) &= 0 \end{aligned}$$

$$\text{along with normalising condition, } x_0 e + x_1 (I - R)^{-1} e = 1.$$

In accordance with the findings of Latouche and Ramaswami [15], it is possible to generate the R matrix by analytical means by utilising significant phases in the logarithmic reduction process.

5. BUSY PERIOD ANALYSIS

- For the purpose of computing an active period, the interval that occurs between clients using the void system and the system returning to its void state after the initial interval can

be utilised. There is a transition from level one to level zero. As a consequence of this, the busy cycle is comprised of the initial return to level 0, in addition to the minimum of one journey to a state at any other level.

- We give an overview of the fundamental epoch before moving on to the time being active. As the QBD procedure moves from level i to level $i - 1$, it takes into consideration the initial transition period $i \geq 2$.
- It is necessary to examine the boundary state possibilities that correspond to $i = 0, 1$ independently. For every level i with $i \geq 1$, there are $\{m_1 + Sm_2m_1 + Sn_1m_1 + Sn_2m_1 + Sn_1m_1 + Sn_2m_1 + (S + 1)lm_1\}$ states that correspond and need to be taken into account. This means that the state (i, j) at level i implies that the j^{th} state at level i is stated. This occurs when the states are organised in lexicographic order.
- The variable $\tilde{L}(w, s)$ indicates a change in j 's passage to the left and arrival to the state (i, j') . Conditional probability for the QBD procedure starts at time $t = 0$ in the state (i, j) , and it advances to level $i - 1$, but not earlier time w . It is necessary to first define the joint transform by Neuts [19].

$$\tilde{L}_{jj'}(w, s) = \sum_{i=1}^{\infty} w^i \int_0^{\infty} e^{-sx} dL_{jj'}(u, x); |w| \leq 1, Re(s) \geq 0,$$

when the matrix is displayed as $\tilde{L}(w, s) = \tilde{L}_{jj'}(w, s)$, the equation is satisfied by the previously defined matrix $\tilde{L}(w, s)$

$$\tilde{L}(w, s) = w(sI - V_1)^{-1}V_2 + (sI - V_1)^{-1}V_0\tilde{L}^2(w, s).$$

- Without taking into account the boundary states, the initial travel time would be computed with the matrix $L = L_{jj'} = \tilde{L}(1, 0)$. Matrix L can be found using the results if matrix R is previously known.

$$L = -(V_1 + RV_2)^{-1}V_2.$$

In the absence of such, it is possible to calculate the L matrix values by applying the idea of a logarithmic reduction process.

$$\begin{aligned} \tilde{L}^{(1,0)}(w, s) &= w(sI - V_1)^{-1}V_2 + (sI - V_1)^{-1}V_0\tilde{L}(w, s)\tilde{L}^{(1,0)}(w, s), \\ \tilde{L}^{(0,0)}(w, s) &= (sI - A_{00})^{-1}A_{01}\tilde{L}(w, s). \end{aligned}$$

According to the stochastic nature of B, $\tilde{L}^{(0,0)}(1, 0)$ and $\tilde{L}^{(1,0)}(1, 0)$, the matrices are employed in the computation of the ensuing scenarios. The moments that can be determined are as follows:

$$\begin{aligned} \vec{\mathfrak{L}}_1 &= -\frac{\partial \tilde{L}(w, s)}{\partial s} \Big|_{w=1, s=0} e = -[V_1 + V_0(I + L)]^{-1}e, \\ \vec{\mathfrak{L}}_2 &= \frac{\partial \tilde{L}(w, s)}{\partial z} \Big|_{w=1, s=0} e = -[V_1 + V_0(I + L)]^{-1}V_2e, \\ \vec{\mathfrak{L}}_1^{(1,0)} &= -\frac{\partial \tilde{L}^{(1,0)}(w, s)}{\partial s} \Big|_{w=1, s=0} e = -[V_1 + V_0L]^{-1}(V_0\vec{\mathfrak{L}}_1 + e), \\ \vec{\mathfrak{L}}_2^{(1,0)} &= \frac{\partial \tilde{L}^{(1,0)}(w, s)}{\partial z} \Big|_{w=1, s=0} e = -[V_1 + V_0L]^{-1}(V_0\vec{\mathfrak{L}}_2 + V_2e), \\ \vec{\mathfrak{L}}_1^{(0,0)} &= \frac{\partial \tilde{L}^{(0,0)}(w, s)}{\partial s} \Big|_{w=1, s=0} e = -A_{00}^{-1}[A_{01}\vec{\mathfrak{L}}_1^{(1,0)} + e], \\ \vec{\mathfrak{L}}_2^{(0,0)} &= \frac{\partial \tilde{L}^{(0,0)}(w, s)}{\partial z} \Big|_{w=1, s=0} e = -A_{00}^{-1}[A_{01}\vec{\mathfrak{L}}_2^{(1,0)}]. \end{aligned}$$

6. SYSTEM PERFORMANCE MEASURES

- Probability of a server makes outgoing call arrivals:

$$P_{SMOCA} = \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{k_2=0}^{m_2} \sum_{k_1=1}^{m_1} x_{i0jk_2k_1}$$

- Probability of a server is busy with incoming call arrivals:

$$P_{SBICA} = \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} x_{i1jr_1k_1}$$

- Probability of a server is busy with outgoing call arrivals:

$$P_{SBOCA} = \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} x_{i2jr_2k_1}$$

- Probability of a server provides slow service during breakdown for incoming call arrivals:

$$P_{SBSICA} = \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} x_{i3jr_1k_1}$$

- Probability of a server provides slow service during breakdown for outgoing call arrivals:

$$P_{SBSOCA} = \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} x_{i4jr_2k_1}$$

- The ratio of busy server:

$$P_{BS} = P_{SBICA} + P_{SBOCA} + P_{SBSICA} + P_{SBSOCA}$$

- Average count of customers in the orbit:

$$E_{orbit} = \sum_{i=0}^{\infty} ix_i e$$

- Average count of customers in the system:

$$E_{system} = E_{orbit} + P_{BS}$$

- Expected Reorder rate:

$$E_{RR} = \sum_{i=0}^{\infty} \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} x_{i1(s+1)r_1k_1} (T_1^0 \otimes I_{m_1} e) + \sum_{i=0}^{\infty} \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} x_{i2(s+1)r_2k_1} (T_2^0 \otimes I_{m_2} e) \\ + \sum_{i=0}^{\infty} \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} x_{i3(s+1)r_1k_1} (\theta_1^0 T_1^0 \otimes I_{m_1} e) + \sum_{i=0}^{\infty} \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} x_{i4(s+1)r_2k_1} (\theta_1^0 T_2^0 \otimes I_{m_2} e)$$

- Expected Inventory level:

$$E_{IL} = \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{k_2=1}^{m_2} \sum_{k_1=1}^{m_1} jx_{i0jk_2k_1} + \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} jx_{i1jr_1k_1} + \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} jx_{i2jr_2k_1} \\ + \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} jx_{i3jr_1k_1} + \sum_{i=0}^{\infty} \sum_{j=1}^S \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} jx_{i4jr_2k_1} + \sum_{i=0}^{\infty} \sum_{j=0}^S \sum_{k_1=1}^{m_1} jx_{i5jk_1}$$

7. TOTAL COST ANALYSIS

We discuss the cost of some performance measures of the system under consideration. Hence we introduce the total cost of system per unit time (TCS) is given below.

$$TCS = C_S E_{system} + C_H E_{IL} + C_R E_{RR}.$$

Where

- C_S – be the waiting cost of a customer in the system per unit time.
- C_H – be the carrying cost of inventory per unit time.
- C_R – be the setup cost per order.

8. NUMERICAL RESULTS

In the section that follows, we will be analysing the behaviour of the models by making use of both numerical and graphical demonstrations. The following three representations illustrate different aspects of the MAP, and each maintains a consistent mean value of 1 across all arrival processes. According to the research conducted by Chakravarthy [10], the three arrival value sets were used as input data in their literature.

- **A-ER(Arrival in Erlang):** $D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$
- **A-EX(Arrival in Exponential):** $D_0 = [-1], D_1 = [1],$
- **A-HE(Arrival in Hyper-exponential):** $D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix},$

The outgoing arrivals and services have three distinct phase-type distributions and the following notations S-ER, S-EX and S-HE respectively for exponential, Erlang and hyper-exponential cases under consideration.

- **S-ER(Erlang):** $\alpha = \beta_1 = \beta_2 = (1, 0), R = T_1 = T_2 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$
- **S-EX(Exponential):** $\alpha = \beta_1 = \beta_2 = [1], R = T_1 = T_2 = [1].$
- **S-HE(Hyper-exponential):** $\alpha = \beta_1 = \beta_2 = (0.8, 0.2), R = T_1 = T_2 = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$

8.1. Illustration – 1

We explored the effects of breakdown rate (τ), utilization factor (ρ) and number of consumers in the orbit (E_{orbit}). In order to attain system stability, we fix $\lambda = 2, \lambda^- = 1, \mu_1 = 12, \mu_2 = 14, \mu_3 = 1.5, \sigma = 3, s = 5, S = 12, \psi = 30, \theta_1 = 0.8, \theta_2 = 0.5, \tau = 1, \delta = 3, p = 0.4, q = 0.6.$

- We combine the arrival and service time categories in Tables 1 through 3 to investigate the breakdown rate (τ), the utilization factor (ρ) versus number of consumers in the orbit (E_{orbit}).
- When the breakdown rate (τ), utilization factor (ρ) increases, the number of consumers in the orbit (E_{orbit}) increases for various arrival and service distributions.
- When comparing arrival timings to those of all other arrivals, the number of consumers in the orbit, denoted by the symbol E_{orbit} , increases rapidly for hyper-exponential arrivals, but it increases slowly for Erlang arrivals. This is because the rate of breakdown (τ) is increasing. In a similar vein, when it comes to service durations, the (E_{orbit}) increases at a slower rate in Erlang services compared to hyper-exponential services..

8.2. Illustration – 2

We explored the effects of retrial rate(ψ) versus the number of consumers in the orbit (E_{orbit}). In order to attain system stability, we fix $\lambda = 2, \lambda^- = 1, \mu_1 = 12, \mu_2 = 14, \mu_3 = 1.5, \sigma = 3, s = 5, S = 12, \psi = 30, \theta_1 = 0.8, \theta_2 = 0.5, \tau = 1, \delta = 3, p = 0.4, q = 0.6.$

- We combine the arrival and service time categories in Figures 2 through 4 to investigate the retrial rate versus the number of consumers in the orbit.
- When the retrial rate (ψ) rises, the corresponding the number of consumers in the orbit (E_{orbit}) reduces.
- When the arrival times of hyper-exponential arrivals are compared to those of all other arrivals, the (E_{orbit}) decreases almost immediately, but the decrease for Erlang arrivals is more gradual. In a similar vein, when it comes to service durations, the (E_{orbit}) diminishes at a slower rate in Erlang services compared to hyper-exponential services.

8.3. Illustration – 3

We explored the effects of breakdown rate (τ) and service rate of the incoming calls (μ_1) versus the average number of consumers in the system (E_{system}). In order to attain system stability, we fix $\lambda = 2, \lambda^- = 1, \mu_1 = 12, \mu_2 = 14, \mu_3 = 1.5, \sigma = 3, s = 5, S = 12, \psi = 30, \theta_1 = 0.8, \theta_2 = 0.5, \tau = 1, \delta = 3, p = 0.4, q = 0.6$.

- We combine the arrival and service time categories to investigate the breakdown rate and service rate of the incoming calls (μ_1) versus the average number of consumers in the system, using Figures 5 through 13.
- When the breakdown rate (τ) and service rate of the incoming calls (μ_1) increases the corresponding average number of consumers in the system (E_{system}) decreases.
- When the arrival times of hyper-exponential arrivals are compared to those of all other arrivals, the E_{orbit} decreases rapidly, whereas the E_{orbit} decreases quite slowly for Erlang arrivals.

Table 1: Breakdown rate(τ) Vs ρ and E-orbit - EXP–A

τ	EXP–S		ERL–S		HEX–S	
	ρ	E-orbit	ρ	E-orbit	ρ	E-orbit
1.1	0.2454850	0.1982983	0.2486224	0.1619680	0.2264146	0.3956982
1.2	0.2503688	0.2093882	0.2537031	0.1710225	0.2303825	0.4162982
1.3	0.2551869	0.2203966	0.2587307	0.1800515	0.2342563	0.4363531
1.4	0.2599407	0.2313258	0.2637061	0.1890557	0.2380405	0.4558961
1.5	0.2646314	0.2421778	0.2686298	0.1980361	0.2417394	0.4749577
1.6	0.2692604	0.2529546	0.2735027	0.2069936	0.2453570	0.4935657
1.7	0.2738287	0.2636582	0.2783252	0.2159290	0.2488970	0.5117461
1.8	0.2783377	0.2742906	0.2830981	0.2248430	0.2523629	0.5295226
1.9	0.2827883	0.2848534	0.2878220	0.2337365	0.2557580	0.5469173
2.0	0.2871818	0.2953485	0.2924975	0.2426102	0.2590853	0.5639507

Table 2: Breakdown rate(τ) Vs ρ and E-orbit - ERL–A

τ	EXP–S		ERL–S		HEX–S	
	ρ	E-orbit	ρ	E-orbit	ρ	E-orbit
1.1	0.2340119	0.1240691	0.2372065	0.0933057	0.2145448	0.2950826
1.2	0.2387090	0.1319053	0.2421036	0.0992845	0.2183176	0.3115577
1.3	0.2433428	0.1396711	0.2469495	0.1052334	0.2220021	0.3275941
1.4	0.2479145	0.1473682	0.2517448	0.1111529	0.2256024	0.3432179
1.5	0.2524255	0.1549983	0.2564901	0.1170438	0.2291225	0.3584533
1.6	0.2568769	0.1625628	0.2611861	0.1229067	0.2325662	0.3733226
1.7	0.2612699	0.1700635	0.2658334	0.1287422	0.2359370	0.3878462
1.8	0.2656055	0.1775017	0.2704327	0.1345509	0.2392380	0.4020432
1.9	0.2698850	0.1848789	0.2749847	0.1403335	0.2424724	0.4159311
2.0	0.2741094	0.1921966	0.2794897	0.1460904	0.2456430	0.4295263

Table 3: Breakdown rate(τ) Vs ρ and E-orbit - HEX-S

τ	EXP-S		ERL-S		HEX-S	
	ρ	E-orbit	ρ	E-orbit	ρ	E-orbit
1.1	0.2609946	0.4154456	0.2639417	0.3535134	0.2427967	0.7141858
1.2	0.2661630	0.4386914	0.2693026	0.3736114	0.2470587	0.7497266
1.3	0.2712620	0.4619338	0.2746078	0.3938337	0.2512177	0.7844196
1.4	0.2762931	0.4851765	0.2798580	0.4141835	0.2552790	0.8183184
1.5	0.2812576	0.5084228	0.2850541	0.4346640	0.2592472	0.8514721
1.6	0.2861568	0.5316762	0.2901966	0.4552781	0.2631267	0.8839258
1.7	0.2909921	0.5549395	0.2952862	0.4760286	0.2669215	0.9157210
1.8	0.2957646	0.5782158	0.3003239	0.4969183	0.2706356	0.9468959
1.9	0.3004756	0.6015077	0.3053098	0.5179498	0.2742726	0.9774858
2.0	0.3051262	0.6248179	0.3102451	0.5391255	0.2778357	1.0075233

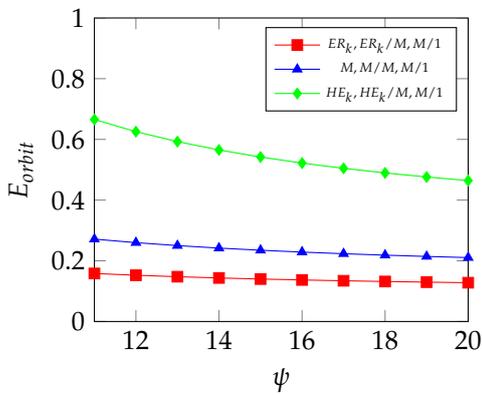


Figure 2: Retrial rate(ψ) Vs E_{orbit} - EXP-S

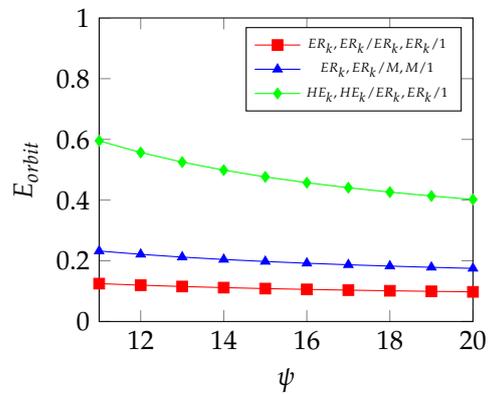


Figure 3: Retrial rate(ψ) Vs E_{orbit} - ER-S

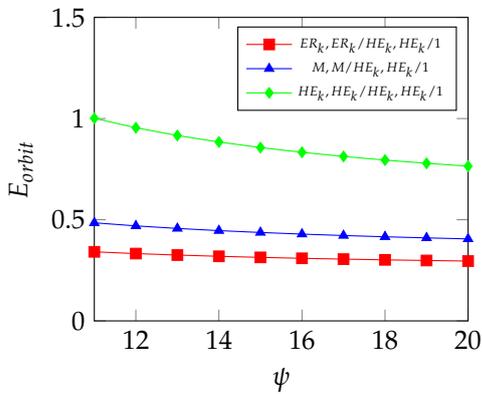


Figure 4: Retrial rate(ψ) Vs E_{orbit} - HEX-S

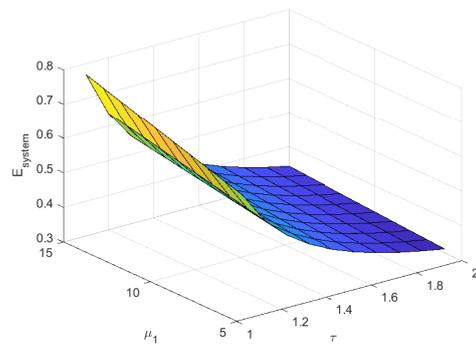


Figure 5: $ER_k, ER_k/ER_k, ER_k/1$

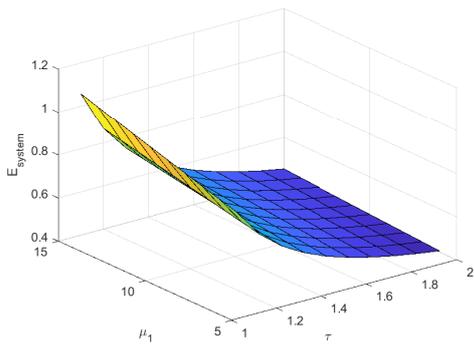


Figure 6: $M, M/ER_k, ER_k/1$

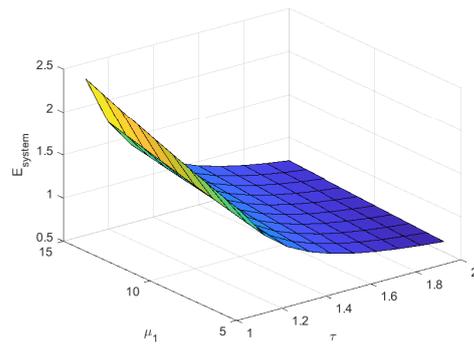


Figure 7: $HE_k, HE_k/ER_k, ER_k/1$

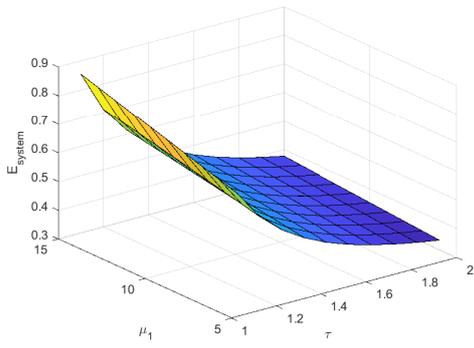


Figure 8: $ER_k, ER_k/M, M/1$

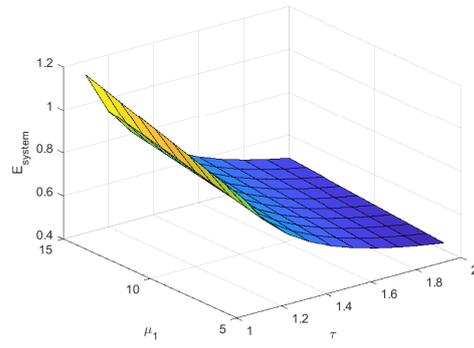


Figure 9: $M, M/M, M/1$

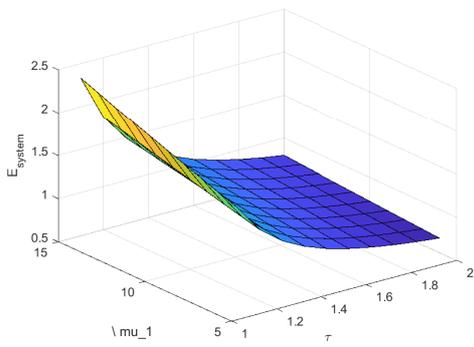


Figure 10: $HE_k, HE_k/M, M/1$

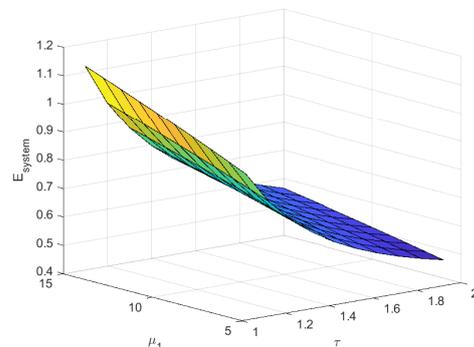


Figure 11: $ER_k, ER_k/HE_k, HE_k/1$

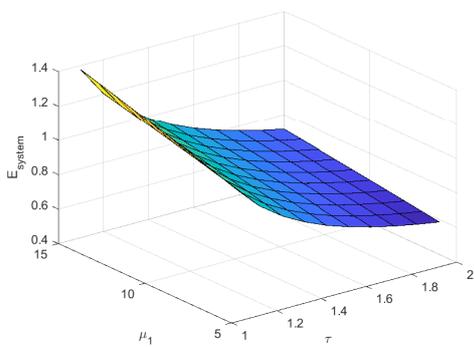


Figure 12: $M, M/HE_k, HE_k/1$

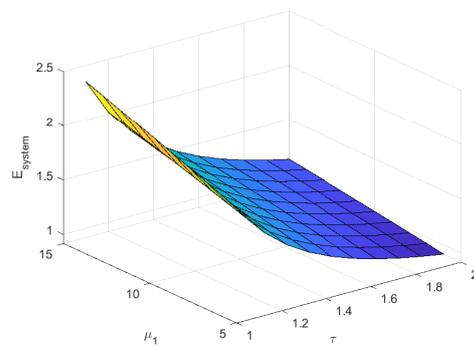


Figure 13: $HE_k, HE_k/HE_k, HE_k/1$

9. CONCLUSION

This article investigated with the study of a retrial inventory queueing system with Markovian Arrival Process along with two-way communication, a single-server, phase type distributed service time for both incoming and outgoing calls with (s, S) replenishment policy, negative arrival, working breakdown and phase type repair. The Stability condition of our system has been derived. Additionally, this inventory queueing system's busy period has been investigated and have performed the cost analysis. We have examined the effects of several settings on a few system measures using 2D and 3D graphs. Our work's primary benefit is that we have taken into account practically every fundamental issue or circumstance in the telecommunications system that is being studied. We have proposed a few strategies to lower the system's overall cost while maximizing server utilization. In future, the approach can be expanded to queueing models with a linear retrial policy.

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