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A NEW MIN-MAX RANKED SET SAMPLING SCHEME WITH UNEQUAL SAMPLE SIZES

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Abstract

In this paper, a new type of ranked set sampling scheme with unequal sample sizes called the new min-max ranked set sampling scheme with unequal sample sizes, is proposed to expedite the sampling process and reduce costs by utilizing fewer data compared to the existing ranked set sampling plans with unequal sample sizes, such as the minimum ranked set sampling scheme with unequal samples. We study the parameter estimation for the exponential distribution using the maximum likelihood and Bayesian methods based on the new scheme. The Metropolis-Hastings strategy is implemented to derive the approximate Bayesian estimates of the parameter under two loss functions. A simulation study is conducted, from which the effectiveness of the new scheme can be observed compared to the simple random sampling scheme and the minimum ranked set sampling scheme with unequal samples. A real data set is also analyzed. Finally, The paper ends with some remarks.

Keywords: Bayesian estimation, exponential distribution, maximum likelihood estimation, minimum ranked set sampling scheme with unequal samples, simulation.

1. Introduction

In many practical scenarios, obtaining the complete observations of a population can be a time-consuming, costly, and sometimes impossible task. In such situations, it is crucial to employ sampling methods that reduce costs. The simplest and most common sampling method is the simple random sampling (SRS) plan, where sample units are randomly selected with equal chances from the population. McIntyre [26] proposed another sampling scheme, called the ranked set sampling (RSS) plan. McIntyre [26] established that this plan can improve the efficiency of the estimator of the population mean for typical unimodal distributions and applied this plan to pasture measurement. Since then, many modifications of the ranked set sampling scheme have been developed by various authors, for example, Muttlak [31] proposed the paired rank set sampling technique, Samawi et al. [42] suggested the extreme ranked set sampling scheme, Samavi [41] worked on the stratified ranked set sampling (SRSS), and Muttlak [32] introduced the median ranked set sampling plan. In more recent years, Bhoj [7] and Al-Odat and Al-Saleh [4] explored the concept of ranked set sampling schemes with unequal sample sizes. Later, Biradar and Santosha [8] proposed the idea of the maximum ranked set sampling with unequal samples (MaxRSSU), which is similar to the idea of [4]. Biradar and Santosha [9] introduced the extreme ranked set sampling plan with unequal sample sizes, which is equivalent to the minimax ranked set sampling strategy discussed by Al-Nasser and Al-Omari [3]. This

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technique has been utilized by Hanandeh and Al-Nasser [15] to develop a new Shewhart-type mean control chart.

In recent years, many studies have been conducted based on the RSS procedure and/or its modified versions. For example, Jeelani and Bouza [22] suggested a new form of ratio estimator using the linear combination of the median and the quartile deviation of the auxiliary variable under the RSS plan. Jeelani et al. [23] worked on estimating the population mean under the RSS plan using an improved ratio estimator. Joukar et al. [24] and Ramezani et al. [40] investigated the estimation of the parameters and the stress-strength parameter, respectively, for the exponential-Poisson distribution based on the SRS, RSS, and MaxRSSU plans. Abdallah and Al-Omari [1] explored the problem of estimating the cumulative distribution function (CDF) and the odds measure under the plan suggested by [4] or [8]. Biradar and Shivanna [10], discussed the Bayesian estimation for the exponential distribution based on the strategy introduced by [7]. See also Al-Omari and Bouza [5] for many older studies.

Qiu and Eftekharian [35] emphasized that the minimum ranked set sampling with unequal samples (MinRSSU) and procedures are two very useful modifications of the RSS procedure. In the MinRSSU plan, n unequal samples are drawn from the population, with the size of the i-th sample equal to i for $i = 1, \cdots, n$. It is assumed that the researcher is able to identify the first ordered units without actual measurement. Then, for each of these samples, the smallest ranked unit is chosen for measurement. This procedure can be repeated m times to arrive at $n_0 = m \times n$ measured values. Clearly, one main purpose of suggesting a new modification of the RSS plan could be to increase the efficiency of the estimators relative to the other sampling schemes especially the SRS plan. In this study, we propose a new sampling scheme with unequal samples, called the new min-max ranked set sampling scheme with unequal sample sizes (NMRSSU scheme) which requires fewer samples than the MinRSSU plan for $n > 2$, with the aim of finding a more efficient scheme than the SRS and MinRSSU plans. Note that n is the number of measured observations extracted from one cycle.

Perhaps, the exponential distribution is the most well-known lifetime distribution in the reliability discussions. Suppose that X follows an exponential distribution with parameter θ, then the probability density function (PDF) and CDF of X are given by

$$f(x;\theta) = \theta \exp(-\theta x), \qquad x > 0, \theta > 0, \tag{1}$$

and

$$F(x;\theta) = 1 - \exp(-\theta x), \qquad x > 0, \theta > 0,$$

respectively.

Many authors have considered the one-parameter exponential distribution as the base distribution of their studies, see for example [2, 6, 10, 14, 30]. We also have selected the exponential distribution as the base distribution in this paper and discuss the estimation problem of the parameter of this distribution based on the new scheme. The rest of the paper is outlined as follows: In Section 2, we describe the new scheme. The maximum likelihood estimation of the parameter of the exponential distribution based on the NMRSSU plan is explored in Section 3. Section 4 is devoted to the Bayesian estimation. As the Bayesian estimates of the parameter do not seem to possess explicit forms, we utilize the Metropolis-Hasting algorithm to approximate these estimates. A simulation study is conducted in Section 5. In Section 6, we analyze a real data example using the NMRSSU, MinRSSU, and SRS plans. Finally, some concluding remarks are presented in Section 7.

2. The new min-max ranked set sampling scheme with unequal samples

Suppose that we want to obtain $n_0 = m \times n$ observations with the help of a sampling plan. Let $k = \left[\frac{n}{2}\right] + 1$ where $[x]$ denotes the integer value of x. In the new min-max ranked set sampling scheme with unequal samples (NMRSSU) plan, we draw k simple random samples in each cycle, where the size of the i-sample is equal to i for $i = 1, \cdots, k$. The plan is described as follows:

Algorithm 1

Step 1: Select $k = \left[\frac{n}{2}\right] + 1$ simple random samples of sizes $1, 2, \cdots, k$, respectively.
Step 2: Order the elements of each sample through visual inspection or other alternative methods that do not involve actual measurement of the desired characteristics.
Step 3: If n is an even number, then select the largest ordered element from the first and second samples, and select the smallest and largest ordered elements from the remaining samples. Otherwise, for an odd value of n, select the first sample, as well as the smallest and largest ordered elements from the remaining samples.
Step 4: Repeat the above steps m times until the desired sample size $n_0 = m \times n$ is obtained.

Note that if n is an even number, then $k = \frac{n}{2} + 1$. In this case, the procedure for one cycle may be summarized as follows:

$$\begin{cases} \boxed{X_{(1:1)1}} & & & \\ X_{(1:2)2} & \boxed{X_{(2:2)2}} & & \\ \boxed{X_{(1:3)3}} & X_{(2:3)3} & \boxed{X_{(3:3)3}} & \\ \vdots & \vdots & \ddots & \\ \boxed{X_{(1:k)k}} & X_{(2:k)k} & \cdots & \boxed{X_{(k:k)k}} \end{cases}$$

Then, $\{X_{(1:1)1}, X_{(2:2)2}, (X_{(1:3)3}, X_{(3:3)3}), \ldots, (X_{(1:k)k} X_{(k:k)k})\}$, represents the sample extracted from one cycle.

Next, if n is odd, then we have $k = \frac{n+1}{2}$ and the procedure can be displayed as follows:

$$\begin{cases} \boxed{X_{(1:1)1}} & & & \\ \boxed{X_{(1:2)2}} & \boxed{X_{(2:2)2}} & & \\ \boxed{X_{(1:3)3}} & X_{(2:3)3} & \boxed{X_{(3:3)3}} & \\ \vdots & \vdots & \ddots & \\ \boxed{X_{(1:k)k}} & X_{(2:k)k} & \cdots & \boxed{X_{(k:k)k}} \end{cases}$$

So $\{X_{(1:1)1}, (X_{(1:2)2}, X_{(2:2)2}), \ldots, (X_{(1:k)k}, X_{(k:k)k})\}$ becomes the sample that is extracted from one cycle.

Since this procedure uses unequal samples and selects both the largest and smallest ordered units in most cases simultaneously, it involves fewer samples compared to the MinRSSU plan when $n > 2$. Suppose that T_{ji} and Z_{ji} denote the smallest observation and the largest observation in the i-th sample and j-th cycle, respectively, for $i = 1, \cdots, k$ and $j = 1, \cdots, m$. Then, in the NMRSSU plan, $\{Z_{j1}, Z_{j2}, (T_{j3}, Z_{j3}), \ldots, (T_{jk}, Z_{jk})\}$ represents the set of observations extracted from the j-th cycle when n is even and $\{Z_{j1}, (T_{j2}, Z_{j2}), (T_{j3}, Z_{j3}), \ldots, (T_{jk}, Z_{jk})\}$ represents the set of observations extracted from the j-th cycle when n is odd, for $j = 1, \cdots, m$. In other words, if we set $\boldsymbol{T}_e = (T_{13}, \cdots, T_{1k}, \cdots, T_{m3}, \cdots, T_{mk})$, $\boldsymbol{T}_o = (T_{12}, \cdots, T_{1k}, \cdots, T_{m2}, \cdots, T_{mk})$, and $\boldsymbol{Z} = (Z_{11}, \cdots, Z_{1k}, \cdots Z_{m1}, \cdots, Z_{mk})$, then $\{\boldsymbol{T}_e, \boldsymbol{Z}\}$ and $\{\boldsymbol{T}_o, \boldsymbol{Z}\}$ constitute the NMRSSU samples from the population for even and odd values of n, respectively. In what follows, we focus on the estimation of the parameter of the exponential distribution based on NMRSSU data.

3. Maximum likelihood estimation

In this section, we discuss the maximum likelihood (ML) estimation for the exponential distribution based on NMRSSU data. This section consists of two parts. In the first part, we assume that n, the number of measured observations extracted from one cycle, is an even number, and in the second part, we assume n is odd.

3.1. ML estimation when n is even

Assume $n_0 = m \times n$ and n is even. Let $\boldsymbol{t}_e = (t_{13}, \cdots, t_{1k}, \cdots, t_{m3}, \cdots, t_{mk})$ and $\boldsymbol{z} = (z_{11}, \cdots, z_{1k}, \cdots, z_{m1}, \cdots, z_{mk})$ be the observed sets of $\boldsymbol{T}_e$ and $\boldsymbol{Z}$, respectively, where $k = \frac{n}{2} + 1$. If the judgment ranking is perfect, then likelihood function of the parameter given $(\boldsymbol{t}_e, \boldsymbol{z})$ can be written as

$$\begin{aligned}\mathcal{L}_e(\theta|\boldsymbol{t}_e, \boldsymbol{z}) &= \prod_{j=1}^{m}\prod_{i=1}^{2} i\,[F(z_{ji})]^{i-1} f(z_{ji}) \times \prod_{j=1}^{m}\prod_{i=3}^{k} i(i-1)[F(z_{ji}) - F(t_{ji})]^{i-2} f(t_{ji}) f(z_{ji}) \\ &= \prod_{j=1}^{m} 2 f(z_{j1}) f(z_{j2}) F(z_{j2}) \times \prod_{j=1}^{m}\prod_{i=3}^{k} i(i-1)[F(z_{ji}) - F(t_{ji})]^{i-2} f(t_{ji}) f(z_{ji}). \qquad (2)\end{aligned}$$

Therefore, for the case of the exponential distribution with PDF (1), we have

$$\begin{aligned}\mathcal{L}_e(\theta|\boldsymbol{t}_e, \boldsymbol{z}) &= \left(2^m \prod_{j=1}^{m}\prod_{i=3}^{k} i(i-1)\right)\theta^{mn}\prod_{j=1}^{m}\left[1 - \mathrm{e}^{-\theta z_{j2}}\right] \\ &\times \exp\left\{-\theta\sum_{j=1}^{m}(z_{j1} + z_{j2}) - \theta\sum_{j=1}^{m}\sum_{i=3}^{k}(t_{ji} + z_{ji})\right\}\prod_{j=1}^{m}\prod_{i=3}^{k}\left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right]^{i-2}.\end{aligned}$$

The log-likelihood function of the parameter is then given by

$$\begin{aligned}\ell_e(\theta|\boldsymbol{t}_e, \boldsymbol{z}) &= \log\left(2^m \prod_{j=1}^{m}\prod_{i=3}^{k} i(i-1)\right) + mn\log(\theta) + \sum_{j=1}^{m}\log\left[1 - \mathrm{e}^{-\theta z_{j2}}\right] \\ &- \theta\sum_{j=1}^{m}(z_{j1} + z_{j2}) - \theta\sum_{j=1}^{m}\sum_{i=3}^{k}(t_{ji} + z_{ji}) + \sum_{j=1}^{m}\sum_{i=3}^{k}(i-2)\log\left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right].\end{aligned}$$

We set the derivative of the log-likelihood function with respect to (w.r.t.) θ equal to zero, and thus we get

$$\begin{aligned}\frac{\partial \ell_e(\theta|\boldsymbol{t}_e, \boldsymbol{z})}{\partial\theta} &= \frac{mn}{\theta} + \sum_{j=1}^{m}\frac{z_{j2}}{\mathrm{e}^{\theta z_{j2}} - 1} - \sum_{j=1}^{m}(z_{j1} + z_{j2}) - \sum_{j=1}^{m}\sum_{i=3}^{k}(t_{ji} + z_{ji}) \\ &+ \sum_{j=1}^{m}\sum_{i=3}^{k}(i-2)\frac{z_{ji}\mathrm{e}^{-\theta z_{ji}} - t_{ji}\mathrm{e}^{-\theta t_{ji}}}{\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}} = 0. \qquad (3)\end{aligned}$$

The ML estimate of θ may be obtained by solving the nonlinear equation (3). Here, we can use a numerical technique to solve this equation.

Let $\widehat{\theta}_M$ denote the ML estimator of θ. Then, under some reguarality conditions stated in Lehmann and Casella [25], $\widehat{\theta}_M$ may be asymptotically normal distributed with the mean θ and variance $\frac{1}{I_{\boldsymbol{T}_e,\boldsymbol{Z}}(\theta)}$, where $I_{\boldsymbol{T}_e,\boldsymbol{Z}}(\theta)$ is the expected Fisher information about θ based on $(\boldsymbol{T}_e, \boldsymbol{Z})$. In other words

$$I_{\boldsymbol{T}_e,\boldsymbol{Z}}(\theta) = -E\left(\frac{\partial^2 \ln f_{\boldsymbol{T}_e,\boldsymbol{Z}}(T_{13}, \cdots, T_{mk}, Z_{11}, \cdots Z_{mk})}{\partial\theta^2}\right),$$

where $f_{\boldsymbol{T}_e,\boldsymbol{Z}}(t_{13},\cdots,t_{mk},z_{11},\cdots z_{mk})$ is the joint PDF of $T_{13},\cdots,T_{1k},\cdots T_{m1},\cdots,T_{mk}$, $Z_{11},\cdots,Z_{1k},\cdots,Z_{m1},\cdots,Z_{mk}$, provided that the above expectation exists.

The second derivative of $\ell_e(\theta|\boldsymbol{t}_e,\boldsymbol{z})$ w.r.t. θ is given by

$$\frac{\partial^2\ell_e(\theta|\boldsymbol{t}_e,\boldsymbol{z})}{\partial\theta^2}=\frac{-mn}{\theta^2}-\sum_{j=1}^{m}\frac{z_{j2}^2 e^{\theta z_{j2}}}{\left(e^{\theta z_{j2}}-1\right)^2}-\sum_{j=1}^{m}\sum_{i=3}^{k}(i-2)\left(\frac{z_{ji}-t_{ji}}{e^{-\theta z_{ji}}-e^{-\theta t_{ji}}}\right)^2 e^{-\theta(t_{ji}+z_{ji})}.$$

Therefore, a $100(1-\gamma)\%$ two-sided equi-tailed asymptotic confidence interval (TE ACI) for θ may be given by

$$\left(\widehat{\theta}_M-\frac{z_{\frac{\gamma}{2}}}{\sqrt{\tilde{I}_e(\widehat{\theta}_M)}},\widehat{\theta}_M+\frac{z_{\frac{\gamma}{2}}}{\sqrt{\tilde{I}_e(\widehat{\theta}_M)}}\right),$$

where z_τ is the τ-th upper quantile of the standard normal distribution and

$$\tilde{I}_e(\widehat{\theta}_M)=-\left.\frac{\partial^2\ell(\theta\mid\boldsymbol{T}_e,\boldsymbol{Z})}{\partial\theta^2}\right|_{\theta=\widehat{\theta}_M}.$$

3.2. ML estimation when n is odd

Assume $n_0=m\times n$ and n is odd. Let $\boldsymbol{t}_o=(t_{12},\cdots,t_{1k},\cdots,t_{m2},\cdots,t_{mk})$ and $\boldsymbol{z}=(z_{11},\cdots,z_{1k},\cdots,z_{m1},\cdots,z_{mk})$ be the observed sets of $\boldsymbol{T}_o$ and $\boldsymbol{Z}$, respectively, where $k=\dfrac{n+1}{2}$. If the judgment ranking is perfect, then likelihood function of the parameter given $(\boldsymbol{t}_o,\boldsymbol{z})$ becomes

$$\mathcal{L}_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})=\prod_{j=1}^{m}f(z_{j1})\times\prod_{j=1}^{m}\prod_{i=2}^{k}i(i-1)[F(z_{ji})-F(t_{ji})]^{i-2}f(t_{ji})f(z_{ji}).$$

Consequently, for the case of the exponential distribution with PDF (1), we get

$$\mathcal{L}_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})=\theta^{mn}\left(\prod_{j=1}^{m}\prod_{i=2}^{k}i(i-1)\right)\prod_{j=1}^{m}\prod_{i=2}^{k}\left[e^{-\theta t_{ji}}-e^{-\theta z_{ji}}\right]^{i-2}$$
$$\times\exp\left\{-\theta\left(\sum_{j=1}^{m}z_{j1}+\sum_{j=1}^{m}\sum_{i=2}^{k}(t_{ji}+z_{ji})\right)\right\}.$$

The log-likelihood function of the parameter is then given by

$$\ell_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})=mn\log\theta+\log\left(\prod_{j=1}^{m}\prod_{i=2}^{k}i(i-1)\right)+\sum_{j=1}^{m}\sum_{i=2}^{k}(i-2)\log\left[e^{-\theta t_{ji}}-e^{-\theta z_{ji}}\right]$$
$$-\theta\left(\sum_{j=1}^{m}z_{j1}+\sum_{j=1}^{m}\sum_{i=2}^{k}(t_{ji}+z_{ji})\right)$$

Upon taking the derivative of $\ell_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})$ w.r.t. θ and setting the result equal to zero, we have

$$\frac{\partial\ell_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})}{\partial\theta}=\frac{mn}{\theta}+\sum_{j=1}^{m}\sum_{i=2}^{k}(i-2)\frac{z_{ji}e^{-\theta z_{ji}}-t_{ji}e^{-\theta t_{ji}}}{e^{-\theta t_{ji}}-e^{-\theta z_{ji}}}-\sum_{j=1}^{m}z_{j1}$$
$$-\sum_{j=1}^{m}\sum_{i=2}^{k}(t_{ji}+z_{ji})=0. \tag{4}$$

A numerical method may be implemented to solve (4).

The second derivative of the $\ell_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})$ w.r.t. θ is as follows

$$\frac{\partial^2\ell_o(\theta|\boldsymbol{t}_o,\boldsymbol{z})}{\partial\theta^2}=-\frac{mn}{\theta^2}-\sum_{j=1}^{m}\sum_{i=2}^{k}(i-2)\left(\frac{z_{ji}-t_{ji}}{e^{-\theta z_{ji}}-e^{-\theta t_{ji}}}\right)^2 e^{-\theta(t_{ji}+z_{ji})}.$$

Now, following a similar argument stated in Subsection 3.1, we may consider the following interval as the $100(1-\gamma)\%$ TE ACI for θ

$$\left(\widehat{\theta}_M - \frac{z_{\frac{\gamma}{2}}}{\sqrt{\tilde{I}_o\left(\widehat{\theta}_M\right)}}, \widehat{\theta}_M + \frac{z_{\frac{\gamma}{2}}}{\sqrt{\tilde{I}_o\left(\widehat{\theta}_M\right)}}\right),$$

where $\widehat{\theta}_M$ denotes the ML estimator of θ and

$$\tilde{I}_o\left(\widehat{\theta}_M\right) = -\left.\frac{\partial^2 \ell(\theta \mid \boldsymbol{T}_o, \boldsymbol{Z})}{\partial\theta^2}\right|_{\theta=\widehat{\theta}_M}.$$

4. Bayesian estimation

In this section, we focus on the Bayesian estimation of the parameter θ based on NMRSSU data. To achieve this, we utilize a gamma prior for the parameter with positive hyperparameters α and β, with the following PDF

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\,\theta^{\alpha-1}\,\mathrm{e}^{-\beta\theta}, \qquad \alpha > 0, \quad \beta > 0. \tag{5}$$

Additionally, to evaluate the accuracy of our estimates, we employ two different loss functions: the squared error loss (SEL) function and the linear-exponential loss (LEL) function. The SEL function assigns equal weights to underestimation and overestimation due to its symmetry, whereas the LEL function stands out as a widely favored asymmetric loss function. Let δ be an estimator of θ. Then, the SEL function is defined as $L_1(\theta,\delta) = (\delta-\theta)^2$, and the LEL function, introduced by [48] and discussed by [49], is defined as

$$L_2(\theta,\delta) = b\left(\mathrm{e}^{c(\delta-\theta)} - c(\delta-\theta) - 1\right),$$

where $c \neq 0$ and $b > 0$. We may take $b = 1$ without loss of generality. Positive values of c result in more serious consequences for overestimation than for underestimation, and vice versa; refer to [49].

4.1. Bayesian estimation when n is even

When n is even, from (2) and (5), the posterior density of θ given $(\boldsymbol{t}_e, \boldsymbol{z})$ is expressed as follows

$$\pi_e(\theta|\boldsymbol{t}_e,\boldsymbol{z}) = \frac{1}{D_1}\,\theta^{mn+\alpha-1}\left(\prod_{j=1}^{m}\left[1-\mathrm{e}^{-\theta z_{j2}}\right]\right)\left(\prod_{j=1}^{m}\prod_{i=3}^{k}\left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right]^{i-2}\right)$$
$$\times \exp\left\{-\theta\left(\beta + \sum_{j=1}^{m}(z_{j1}+z_{j2}) + \sum_{j=1}^{m}\sum_{i=3}^{k}(t_{ji}+z_{ji})\right)\right\},$$

where

$$D_1 = \int_0^\infty \theta^{mn+\alpha-1}\left(\prod_{j=1}^{m}\left[1-\mathrm{e}^{-\theta z_{j2}}\right]\right)\left(\prod_{j=1}^{m}\prod_{i=3}^{k}\left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right]^{i-2}\right)$$
$$\times \exp\left\{-\theta\left(\beta + \sum_{j=1}^{m}(z_{j1}+z_{j2}) + \sum_{j=1}^{m}\sum_{i=3}^{k}(t_{ji}+z_{ji})\right)\right\}\mathrm{d}\theta.$$

The Bayesian estimates of θ under the SEL and LEL functions are then given by

$$\widehat{\theta}^{e}_{SEL} = \frac{1}{D_1}\int_0^\infty \theta^{mn+\alpha}\left(\prod_{j=1}^{m}\left[1-\mathrm{e}^{-\theta z_{j2}}\right]\right)\left(\prod_{j=1}^{m}\prod_{i=3}^{k}\left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right]^{i-2}\right)$$
$$\times \exp\left\{-\theta\left(\beta + \sum_{j=1}^{m}(z_{j1}+z_{j2}) + \sum_{j=1}^{m}\sum_{i=3}^{k}(t_{ji}+z_{ji})\right)\right\}\mathrm{d}\theta, \tag{6}$$

and

$$\widehat{\theta}^{e}_{LEL} = \frac{-1}{c} \log \left(\frac{1}{D_1} \int_0^\infty \theta^{mn+\alpha-1} \left(\prod_{j=1}^{m} \left[1 - \mathrm{e}^{-\theta z_{j2}}\right] \right) \left(\prod_{j=1}^{m} \prod_{i=3}^{k} \left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right]^{i-2} \right) \right.$$
$$\left. \times \exp\left\{ -\theta \left(\beta + c + \sum_{j=1}^{m} (z_{j1} + z_{j2}) + \sum_{j=1}^{m} \sum_{i=3}^{k} (t_{ji} + z_{ji}) \right) \right\} \mathrm{d}\theta \right), \tag{7}$$

respectively, provided that the above integrals exist.

It seems that there exist no closed forms for the Bayes estimates (6) and (7). So we use the Metropolis-Hastings (MH) algorithm to approximate these estimates. The MH algorithm is extensively used in Bayesian inference. It was initially presented by [28] and later developed by [17]. Here, the MH method can be expressed using the following algorithm.

Algorithm 2

Step 1: Start with an initial guess $\theta_0 = \hat{\theta}_M$, where $\hat{\theta}_M$ is the ML estimate of θ and set $s = 1$.
Step 2: Given θ_{s-1}, generate θ^* from the truncated normal distribution, $N\left(\theta_{s-1}, \sigma^2\right) I_{\{\theta>0\}}$. Then, set $\theta_s = \theta^*$ with probability

$$P = \min\left\{ \frac{\pi_e\left(\theta^* \mid \boldsymbol{t}_e, \boldsymbol{z}\right) q\left(\theta_{s-1} \mid \theta^*\right)}{\pi_e\left(\theta_{s-1} \mid \boldsymbol{t}_e, \boldsymbol{z}\right) q\left(\theta^* \mid \theta_{s-1}\right)}, 1 \right\},$$

where $q(x|b)$ denotes the density of $N\left(b, \sigma^2\right) I_{\{x>0\}}$, otherwise set $\theta_s = \theta_{s-1}$.
Step 3: Set $s = s + 1$ and repeat Step 2, $S - 1$ times, where S is a large number. Let M be the burn-in period. Then, $\{\theta_{M+1}, \theta_{M+2}, \cdots, \theta_S\}$ is the desired generated sample.

Now, given the generated sample from the MH algorithm $\{\theta_{M+1}, \theta_{M+2}, \cdots, \theta_S\}$, the approximate Bayesian estimates for θ under the SEL and LEL functions are given by

$$\widetilde{\theta}_{SEL} = \frac{1}{S^*} \sum_{s=M+1}^{S} \theta_s, \qquad \text{and} \qquad \widetilde{\theta}_{LEL} = \frac{-1}{c} \ln\left(\frac{1}{S^*} \sum_{s=M+1}^{S} \mathrm{e}^{-c\theta_s} \right),$$

respectively, where $S^* = S - M$.

Now, let $\theta_{(1)} \leq \cdots \leq \theta_{(S^*)}$ represent the ordered values of the generated sample using the MH strategy $\{\theta_{M+1}, \cdots, \theta_S\}$. Then, the $100(1-\gamma)\%$ Chen and Shao shortest credible interval (CSSW CrI) for θ can be expressed as follows (see [12]):

$$\left(\theta_{(q)}, \theta_{(q+[(1-\gamma)S^*])} \right),$$

where q is chosen such that

$$\theta_{(q+[(1-\gamma)S^*])} - \theta_{(q)} = \min_{s \in \{1, \cdots, S^* - [(1-\gamma)S^*]\}} \theta_{(s+[(1-\gamma)S^*])} - \theta_{(s)}.$$

4.2. Bayesian estimation when n is odd

When n is odd, the posterior density of θ given $(\boldsymbol{t}_o, \boldsymbol{z})$ is expressed as follows

$$\pi_o(\theta|\boldsymbol{t}_o, \boldsymbol{z}) = \frac{1}{D_2} \theta^{mn+\alpha-1} \left(\prod_{j=1}^{m} \prod_{i=2}^{k} \left[\mathrm{e}^{-\theta t_{ji}} - \mathrm{e}^{-\theta z_{ji}}\right]^{i-2} \right)$$
$$\times \exp\left\{ -\theta \left(\beta + \sum_{j=1}^{m} z_{j1} + \sum_{j=1}^{m} \sum_{i=2}^{k} (t_{ji} + z_{ji}) \right) \right\},$$

where

$$D_2 = \int_0^\infty \theta^{mn+\alpha-1} \left(\prod_{j=1}^{m} \prod_{i=2}^{k} \left[e^{-\theta t_{ji}} - e^{-\theta z_{ji}} \right]^{i-2} \right) \times \exp\left\{ -\theta \left(\beta + \sum_{j=1}^{m} z_{j1} + \sum_{j=1}^{m} \sum_{i=2}^{k} (t_{ji} + z_{ji}) \right) \right\} d\theta.$$

The Bayesian estimates of θ under the SEL and LEL functions are then given by

$$\widehat{\theta}^o_{SEL} = \frac{1}{D_2} \int_0^\infty \theta^{mn+\alpha} \left(\prod_{j=1}^{m} \prod_{i=2}^{k} \left[e^{-\theta t_{ji}} - e^{-\theta z_{ji}} \right]^{i-2} \right) \times \exp\left\{ -\theta \left(\beta + \sum_{j=1}^{m} z_{j1} + \sum_{j=1}^{m} \sum_{i=2}^{k} (t_{ji} + z_{ji}) \right) \right\} d\theta, \tag{8}$$

and

$$\widehat{\theta}^o_{LEL} = \frac{-1}{c} \log \left(\frac{1}{D_2} \int_0^\infty \theta^{mn+\alpha-1} \left(\prod_{j=1}^{m} \prod_{i=2}^{k} \left[e^{-\theta t_{ji}} - e^{-\theta z_{ji}} \right]^{i-2} \right) \times \exp\left\{ -\theta \left(\beta + c + \sum_{j=1}^{m} z_{j1} + \sum_{j=1}^{m} \sum_{i=2}^{k} (t_{ji} + z_{ji}) \right) \right\} d\theta \right), \tag{9}$$

respectively, provided that the above integrals exist.

Now, based on a similar argument stated in Subsection 4.1, we can derive the approximate Bayesian estimates for θ under the SEL and LEL functions, as well as the $100(1-\gamma)\%$ CSSW CrI for θ.

5. A simulation study

In this section, we carry out a Monte Carlo simulation study with $N = 10000$ iterations to evaluate several classical and Bayesian estimators (approximate Bayesian estimators in the case of the NMRSSU plan) for the exponential parameter θ. This evaluation is based on the NMRSSU, MinRSSU, and SRS designs. We focus on the parameter values $\theta = 2, 3$, and 5, and chose sample sizes $n_0 = m \times n \in \{24, 33, 40, 45\}$, configured as follows: Sample 1 $\equiv 2 \times 12$, Sample 2 $\equiv 3 \times 11$, Sample 3 $\equiv 4 \times 10$, and Sample 4 $\equiv 5 \times 9$. For the Bayesian estimation, the hyperparameters are specified to be $\alpha = \beta = 0.01$, resulting in an approximate non-informative prior. The parameter of the LEL function is also taken to be $c = -0.5$ and 0.5. We calculated the ML estimates for θ. Moreover, we calculated the Bayesian estimates of θ based on the MinRSSU and SRS plans, whereas the Bayesian estimates for θ based on the NMRSSU plan have been approximated using the MH strategy. We take $S = 10000$, $M = 1000$, and $\sigma = 1$ for the MH method. The convergence of the MH Markov chains have been checked using Geweke's test (see [13]), Raftery and Lewiss diagnostic (see [38, 39]), and Heidelberger and Welch's convergence diagnostic (see [20]). Notice that [20] used or hinted the outcomes of [18, 19, 43, 44, 45].

Let $\widehat{\theta}$ be an estimator of θ and $\widehat{\theta}_i$ be the corresponding estimate obtained in the i-th iteration. Then, the estimated risk (ER) under the SEL function, the ER under the LEL function, and the estimated bias (bias for short) of $\widehat{\theta}$ are expressed as

$$\mathrm{ER}_S(\widehat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} (\widehat{\theta}_i - \theta)^2,$$

$$\mathrm{ER}_L(\widehat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} \left\{ e^{c(\widehat{\theta}_i - \theta)} - c(\widehat{\theta}_i - \theta) - 1 \right\},$$

and

$$Bias(\widehat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta),$$

respectively.

We compute the ER of each Bayesian estimator according to its own loss function, whereas all the ERs are calculated for the ML estimators. Note that under the NMRSSU plan, we compute the approximate Bayesian estimators. Besides, the 95% interval estimators are evaluated by means of the average width (AW) and coverage probability (CP) criteria. In the context of Bayesian interval estimation, we obtain exact 95% shortest width credible intervals for θ when the SRS and MinRSSU plans are used. The biases and ERs of the point estimators are reported in Tables 1–3 and the AWs and CPs of the 95% interval estimators are given in Table 4. From Tables 1–4, we may conclude the following points:

- The point estimators based on the NMRSSU scheme perform better than the corresponding estimators based on the SRS and MinRSSU schemes w.r.t. the criteria of the absolute value of bias (AVB) and ER (there is only one exception in Table 3, where the AVB of the estimator based on the MinRSSU method is lower than the corresponding metric based on the NMRSSU method). The point estimators based on the SRS and MinRSSU schemes perform rather close to each other w.r.t. the ER and AVB.
- The Bayesian estimators (approximate Bayesian estimators in the case of the NMRSSU plan) under the LEL function with $c = 0.5$ perform better than the ML estimators in the sense of ER, whereas the ML estimators perform better than the Bayesian estimators (approximate Bayesian estimators in the case of the NMRSSU plan) under the LEL function with $c = -0.5$ in this respect. The Bayesian estimators under the SEL function perform better than the ML estimators in the sense of ER when the MinRSSU and SRS plans are used, whereas the ML estimators perform better than the approximate Bayesian estimators under the SEL function in this respect when the NMRSSU method is used.
- The Bayesian estimators (approximate Bayesian estimators in the case of the NMRSSU plan) under the LEL function with $c = 0.5$ perform the best among the considered types of estimators in the sense of AVB. The next best estimators are the Bayesian estimators under the SEL function in terms of AVB when the MinRSSU and SRS plans are used, otherwise, the ML estimators are the second best estimators in this regard when the NMRSSU method is used. Moreover, the Bayesian estimators (approximate Bayesian estimators in the case of the NMRSSU plan) under the LEL function with $c = -0.5$ perform the worst among the considered types of estimators in the sense of AVB.
- The Bayesian estimators (approximate Bayesian estimators in the case of the NMRSSU plan) under the LEL function with $c = 0.5$ when $\theta = 5$, are the only estimators that possess negative bias values.
- In the context of interval estimation, the interval estimators under the NMRSSU plan exhibit smaller AWs compared to those under the SRS and MinRSSU plans. The interval estimators based on the SRS and MinRSSU schemes perform rather close to each other w.r.t. the AW.
- The interval estimators obtained using the Bayesian methods are, on average, shorter than the classical ones. All of the 95% interval estimators possess CPs close to the nominal value of 0.95.

Table 1: *The biases and ERs of the ML estimators of θ.*

	$\theta=2$			$\theta=3$			$\theta=5$		
Sample 1	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Bias	0.03594	0.08319	0.08761	0.05412	0.14136	0.13692	0.08994	0.22734	0.21653
ER_S	0.09242	0.20432	0.20449	0.21173	0.47043	0.46664	0.59507	1.28767	1.31819
$\mathrm{ER}_{L(c=+0.5)}$	0.01213	0.02879	0.02884	0.02878	0.07217	0.07157	0.08860	0.24051	0.25121
$\mathrm{ER}_{L(c=-0.5)}$	0.01113	0.02336	0.02337	0.02505	0.05158	0.05124	0.06826	0.13394	0.13654
Sample 2	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Bias	0.03356	0.05683	0.05970	0.05443	0.09213	0.09336	0.08215	0.14874	0.13977
ER_S	0.08043	0.13792	0.14269	0.17799	0.31682	0.31471	0.49783	0.89761	0.86688
$\mathrm{ER}_{L(c=+0.5)}$	0.01055	0.01872	0.01947	0.02406	0.04523	0.04512	0.07138	0.14566	0.13794
$\mathrm{ER}_{L(c=-0.5)}$	0.00968	0.01620	0.01669	0.02110	0.03627	0.03588	0.05804	0.09914	0.09636
Sample 3	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Bias	0.02659	0.05343	0.04966	0.03217	0.07605	0.07973	0.05974	0.12985	0.14091
ER_S	0.05842	0.11380	0.11415	0.13019	0.26774	0.26267	0.36281	0.72525	0.72594
$\mathrm{ER}_{L(c=+0.5)}$	0.00756	0.01523	0.01533	0.01711	0.03761	0.03687	0.05009	0.11306	0.11330
$\mathrm{ER}_{L(c=-0.5)}$	0.00711	0.01350	0.01350	0.01575	0.03096	0.03039	0.04304	0.08113	0.08089
Sample 4	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Bias	0.02671	0.04702	0.04299	0.04149	0.07150	0.06482	0.06574	0.12704	0.11919
ER_S	0.06430	0.09978	0.09933	0.14059	0.22931	0.22478	0.39452	0.64183	0.63012
$\mathrm{ER}_{L(c=+0.5)}$	0.00835	0.01330	0.01320	0.01862	0.03172	0.03098	0.05502	0.09709	0.09535
$\mathrm{ER}_{L(c=-0.5)}$	0.00780	0.01187	0.01184	0.01690	0.02673	0.02635	0.04660	0.07251	0.07116

Table 2: *The ER values for the estimators of θ obtained based on the Bayesian approach.*

	$\theta = 2$			$\theta = 3$			$\theta = 5$		
Sample 1	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
ER_S	0.09331	0.20357	0.20373	0.21346	0.46740	0.46365	0.59943	1.27265	1.30303
$\mathrm{ER}_{L(c=+0.5)}$	0.01143	0.02477	0.02476	0.02602	0.05630	0.05591	0.07294	0.14992	0.15611
$\mathrm{ER}_{L(c=-0.5)}$	0.01190	0.02615	0.02621	0.02739	0.06084	0.06036	0.07761	0.17097	0.17308
Sample 2	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
ER_S	0.08093	0.13757	0.14232	0.17905	0.31541	0.31330	0.49940	0.89038	0.85998
$\mathrm{ER}_{L(c=+0.5)}$	0.01000	0.01686	0.01750	0.02198	0.03819	0.03805	0.06036	0.10636	0.10152
$\mathrm{ER}_{L(c=-0.5)}$	0.01024	0.01755	0.01810	0.02283	0.04072	0.04033	0.06466	0.11719	0.11378
Sample 3	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
ER_S	0.05873	0.11355	0.11391	0.13070	0.26677	0.26171	0.36391	0.72038	0.72095
$\mathrm{ER}_{L(c=+0.5)}$	0.00726	0.01394	0.01406	0.01605	0.03277	0.03207	0.04427	0.08736	0.08720
$\mathrm{ER}_{L(c=-0.5)}$	0.00743	0.01446	0.01444	0.01664	0.03400	0.03345	0.04673	0.09347	0.09366
Sample 4	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
ER_S	0.06446	0.09959	0.09915	0.14094	0.22856	0.22406	0.39511	0.63792	0.62633
$\mathrm{ER}_{L(c=+0.5)}$	0.00798	0.01230	0.01223	0.01730	0.02804	0.02746	0.04807	0.07714	0.07597
$\mathrm{ER}_{L(c=-0.5)}$	0.00814	0.01261	0.01256	0.01795	0.02913	0.02863	0.05077	0.08259	0.08091

Table 3: *The bias values for the estimators of θ obtained based on the Bayesian approach.*

	$\theta = 2$			$\theta = 3$			$\theta = 5$		
Sample 1	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
SEL	0.04162	0.08217	0.08659	0.06167	0.13838	0.13395	0.10088	0.21764	0.20685
LEL ($c = +0.5$)	0.01891	0.03642	0.04066	0.01100	0.03600	0.03186	−0.03749	−0.05709	−0.06711
LEL ($c = -0.5$)	0.06515	0.13091	0.13553	0.11509	0.25102	0.24625	0.25201	0.53980	0.52818
Sample 2	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
SEL	0.03724	0.05613	0.05900	0.05913	0.09008	0.09131	0.08814	0.14201	0.13308
LEL ($c = +0.5$)	0.01803	0.02381	0.02656	0.01608	0.01783	0.01902	−0.02920	−0.05405	−0.06216
LEL ($c = -0.5$)	0.05701	0.08992	0.09292	0.10407	0.16731	0.16857	0.21424	0.36111	0.35117
Sample 3	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
SEL	0.02977	0.05286	0.04910	0.03614	0.07439	0.07807	0.06492	0.12439	0.13542
LEL ($c = +0.5$)	0.01535	0.02631	0.02264	0.00400	0.01529	0.01887	−0.02360	−0.03688	−0.02650
LEL ($c = -0.5$)	0.04452	0.08038	0.07653	0.06934	0.13678	0.14057	0.15836	0.30088	0.31265
Sample 4	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
SEL	0.02885	0.04652	0.04249	0.04398	0.07003	0.06337	0.06873	0.12221	0.11437
LEL ($c = +0.5$)	0.01346	0.02309	0.01915	0.00952	0.01768	0.01126	−0.02581	−0.02131	−0.02867
LEL ($c = -0.5$)	0.04460	0.07071	0.06659	0.07962	0.12494	0.11802	0.16872	0.27762	0.26925

Table 4: *The AWs and CPs of the 95% interval estimators of θ.*

$\boxed{\theta = 2}$	AW			CP		
Sample 1	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	1.17564	1.66657	1.67012	0.9507	0.9525	0.9533
Bayesian	1.16084	1.64387	1.64736	0.9487	0.9507	0.9512
Sample 2	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	1.08252	1.40986	1.40280	0.9521	0.9511	0.9537
Bayesian	1.06887	1.39590	1.38891	0.9499	0.9490	0.9529
Sample 3	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	0.93924	1.27394	1.27250	0.9536	0.9475	0.9543
Bayesian	0.92750	1.26354	1.26211	0.9514	0.9464	0.9532
Sample 4	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	0.97007	1.19379	1.19175	0.9496	0.9497	0.9511
Bayesian	0.95751	1.18513	1.18311	0.9466	0.9489	0.9495
$\boxed{\theta = 3}$	AW			CP		
Sample 1	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	1.76399	2.50650	2.49813	0.9506	0.9519	0.9555
Bayesian	1.74286	2.47122	2.46299	0.9471	0.9486	0.9536
Sample 2	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	1.62643	2.11099	2.11079	0.9538	0.9501	0.9509
Bayesian	1.60771	2.08942	2.08922	0.9502	0.9490	0.9494
Sample 3	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	1.40534	1.90801	1.91258	0.9503	0.9531	0.9543
Bayesian	1.39015	1.89193	1.89646	0.9471	0.9516	0.9515
Sample 4	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	1.45588	1.79344	1.79299	0.9512	0.9526	0.9502
Bayesian	1.43917	1.78001	1.77956	0.9484	0.9512	0.9484
$\boxed{\theta = 5}$	AW			CP		
Sample 1	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	2.93916	4.18449	4.16863	0.9508	0.9507	0.9499
Bayesian	2.90510	4.12182	4.10625	0.9486	0.9469	0.9466
Sample 2	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	2.70566	3.52110	3.53176	0.9501	0.9553	0.9514
Bayesian	2.67406	3.48287	3.49339	0.9485	0.9525	0.9498
Sample 3	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	2.34541	3.17217	3.19171	0.9521	0.9551	0.9509
Bayesian	2.32155	3.14379	3.16314	0.9495	0.9547	0.9489
Sample 4	NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
Classical	2.42441	2.98541	2.98560	0.9489	0.9507	0.9483
Bayesian	2.39914	2.96168	2.96187	0.9454	0.9499	0.9464

6. Real data example

In this section, we utilize real data concerning the intervals between failures of Boeing air conditioner systems in hours, originally reported by Proschan [36]. It is important to note that Proschan [36] omitted certain failure intervals due to major overhauls, see also [21]. Ahmadi et al. [2] considered a part of these data and emphasized that the exponential distribution is a suitable model for the considered data. Here, applying the bootstrapped version of the Kolmogorov-Smirnov test (see [11, 46, 47]), and using the strategy described in [29] for data sets with ties, we found out that the hypothesis that the above data follow the exponential distribution cannot be rejected at a significance level of 0.05. Thus, we may accept the assumption that the data follow an exponential distribution. We selected a sample of size $n_0 = m \times n = 3 \times 6 = 18$ from the specified real data, where m is the number of cycles and n is the sample size of one cycle. To implement the NMRSSU plan, we began by choosing $m = 3$ samples, each consisting of 10 observations, and treated each of these samples as a separate cycle. Each cycle was further segmented into $k = 4$ groups, organized by the number of observations as follows: the first group had one observation, the second group comprised two observations, and finally, the last group contained four observations. This arrangement allowed us to extract a total of $n = 6$ observations from each cycle, according to the specified plan. For the MinRSSU plan, we selected $m = 3$ samples, but here each sample included 21 observations. Similar to the previous plan, each sample was regarded as a cycle and was divided into $n = 6$ groups. The configuration for these groups was one observation in the first, two observations in the second, and finally, six observations in the last group. Consequently, we could extract $n = 6$ observations from every cycle as intended by the plan.

We utilized an approximate non-informative prior with hyperparameters $\alpha = \beta = 0.01$ for the parameter. We took $S = 10000$, $M = 1000$, and $\sigma = 0.1$ for the MH method. We calculated both point estimates and 95% interval estimates for θ using classical and Bayesian methods. Note that in the context of Bayesian interval estimation, we obtained exact 95% shortest width credible intervals for θ up to 5 decimal places when the SRS and MinRSSU plans were used. Tables 5 and 6 give the numerical results for point estimation and interval estimation, respectively.

Table 5: *The point estimates of θ.*

ML			Bayesian								
NMRSSU	MinRSSU	SRS	NMRSSU			MinRSSU			SRS		
			SEL	LEL ($c=+0.5$)	LEL ($c=-0.5$)	SEL	LEL ($c=+0.5$)	LEL ($c=-0.5$)	SEL	LEL ($c=+0.5$)	LEL ($c=-0.5$)
0.01007	0.01417	0.01228	0.01012	0.01012	0.01013	0.01418	0.01418	0.01418	0.01229	0.01228	0.01229

Table 6: *The* 0.95% *interval estimates of θ.*

Classical			Bayesian		
NMRSSU	MinRSSU	SRS	NMRSSU	MinRSSU	SRS
(0.00628 , 0.01386)	(0.00763 , 0.02072)	(0.00661 , 0.01795)	(0.00643 , 0.01385)	(0.00797 , 0.02085)	(0.00691 , 0.01806)

7. Concluding remarks

In this paper, we introduced a new RSS-type sampling plan with unequal sample sizes, called the new min-max ranked set sampling scheme with unequal sample sizes. The new scheme is applied to parameter estimation for the exponential distribution using the ML and Bayesian approaches. The findings of the simulation study highlight the effectiveness of the new scheme in comparison with the SRS and MinRSSU schemes. As a future study, we may work on some other new sampling plans with unequal sample sizes and explore their efficacy. All the numerical computations of this paper were done using the statistical software R [37] and the packages coda [33, 34], truncnorm [27], and nleqslv [16] therein.

Data Availability Statement

The data set used in this paper is explained in the manuscript.

Declaration of Conflicting Interests

The Authors declare that there is no conflict of interest.

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