A BAYESIAN APPROACH TO RELIABILITY ANALYSIS IN THE STRESS-STRENGTH MODEL WITH WEIGHTED EXPONENTIAL DISTRIBUTIONS CONSIDERING FUZZINESS

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Abstract

The paper considers a Bayesian approach to the analysis of reliability in a stress-strength model when both stress and strength follow a weighted exponential distribution. The main focus of the paper considers a situation when the available data incorporate fuzziness. The situations when stress and strength distributions have common shape parameters and also when they have different shape and scale parameters are entertained separately. The entire analysis is done using the Bayes paradigm using weak proper priors for the model parameters. Since the resulting posteriors are not available in analytically closed form, the paper uses the recourse of Markov chain Monte Carlo simulation technique. Finally, a numerical illustration is provided based on real data examples. The results are found to be satisfactory.

Keywords: Weighted exponential distribution, Scale and shape parameters, Stress-strength model, Fuzzy reliability, Method of moments, Bayes paradigm, Weak proper priors, Gibbs sampler, Metropolis algorithm, Adaptive Rejection Sampling.

1. Introduction

The exponential distribution is one of the most widely used continuous distributions in statistical reliability and life testing. To remold this distribution, a new class of exponential distribution called the weighted exponential (W_E) distribution was introduced by [16] to introduce a shape parameter to the exponential distribution. This distribution is used as an alternative to other positively skewed lifetime distributions such as gamma, Weibull and generalized exponential distributions where shapes of probability density functions (pdfs) almost resemble the corresponding shape of the W_E density. Truly speaking, both gamma and the generalised exponential distributions can be considered as different weighted versions of the exponential distribution. The cumulative distribution function (cdf) and the pdf of a random variable X having W_E distribution with a shape parameter $\theta > 0$ and a scale parameter $\nu > 0$ can be written as

$$F_X(x;\theta,\nu) = 1 + \Big(\frac{e^{-\nu x}(e^{-\theta\nu x} - \theta - 1)}{\theta}\Big); \quad x > 0, \tag{1}$$

$$f_X(x;\theta,\nu) = \Big(\frac{\theta+1}{\theta}\Big)\nu e^{-\nu x}(1 - e^{-\theta\nu x}); \quad x > 0 \tag{2}$$

and the corresponding hazard function of the model is given by

$$h_X(x) = \Big(\frac{\nu(\theta+1)(1-e^{-\theta\nu x})}{\theta+1-e^{-\theta\nu x}}\Big); \quad x > 0 \tag{3}$$

It may be noted that the W_E distribution approaches to an exponential distribution with scale parameter unity if ν=1 and the parameter θ approaches to ∞. Similarly, the distribution approaches to the gamma distribution with shape parameter 2 if the parameter θ goes to zero. Besides, the W_E distribution coincides with the generalized exponential distribution with shape parameter 2 when the parameter θ=1 (see, for example, [16]).

The term stress-strength was first introduced by [6] and its further concept was developed by [7]. The stress-strength model is not only used in reliability analysis, but also used in several other fields, such as psychology, engineering, and medical sciences, etc. In its simplest form, let us consider two random variables, X and Y, where Y represents the stress experienced by a component and X represents its strength to overcome the stress. The component works as long as its strength exceeds its stress and, accordingly, its reliability is given by $P(X > Y)$. Many authors have considered drawing inferences about stress-strength reliability by considering different distributions for X and Y and using both classical and Bayesian paradigms. Among the important classical developments, one can refer to [8] where the authors have considered this problem by using independent normal distributions for both stress and strength (see also [29]). [5] is another important reference where the authors developed two estimators of reliability in a multicomponent stress-strength model where both stress and strength follow exponential distributions. They studied the performance of the two estimators by means of a Monte Carlo simulation study for moderate sample sizes. [3] considered the problem of estimating reliability when both stress and strength are independent but not identically distributed Burr random variables. [17], on the other hand, used generalised exponential distribution with common scale but different shape parameters for both stress and strength but mostly confined to maximum likelihood (ML) estimation of parameters. The authors, however, used Weibull distribution with different scale but common shape parameters in their other work (see [18]). Besides, there are several other references on classical inferences to reliability in the stress-strength model using different distributions for both stress and strength. An exhaustive list of the same may not be possible due to space restrictions. A few other important references using classical paradigm worth mentioning include [24], [11], [1] and [21], etc.

Bayesian inferences to stress-strength models have also been considered by several authors, but not as frequently as classical inferences. An early important reference can be considered as [10] where the authors considered the estimation of the probability that one random variable exceeds the other random variable, assuming that the two random variables are exponentially distributed. [9] is another important reference in the Bayes paradigm where the authors considered estimation of reliability of a multicomponent stress-strength system where both stress and strength are exponentially distributed random variables. There are plenty of other references where Bayesian inferences are successfully developed either in a single component stress-strength model or a multicomponent stress-strength model assuming different probability distributions for both stress and strength. For instance, [22] considered Weibull distributions with equal scale parameters for both stress and strength. The work of [14] was theoretically sound, where the authors developed Bayes inferences for the stress-strength system using different non-informative priors, including the reference priors, and studied the propriety of corresponding posteriors. The authors also focussed on getting the class of non-informative matching priors by matching the coverage probabilities of the classical and the Bayesian intervals. [4] is another significant development based on stochastic process modelling where the authors visualized the stress-strength system as time-homogeneous Markov process. They finally developed a simulation based Bayesian inferences using both conjugate and non-informative priors. A few other Bayesian developments worth mentioning include [23], [34] and [13], etc.

In the stress-strength model, it often happens that the observed data are not precise or exact due to misdetection or mishandling either on the part of the experimenter or the nature of experimentation and, as a result, one has to take the recourse of fuzzy logic. It may be noted that the conventional study of reliability mostly relies on the exact data and assumptions but, in practice, data may be imprecise or compounded to some extent with uncertainties which may, among other things, occur due to subjective judgement of the reliability. Fuzzy reliability theory aims to account for such uncertainties using fuzzy logic while analysing the data. The theory involves fuzzy set theory, fuzzy logic and the notion of fuzzy probability. The concept of fuzzy set theory, introduced by [33], allows for the representation of uncertainty by defining membership functions that describe the degree to which an element belongs to a particular set. In the statistical analysis of reliability, this set may represent categories such as the 'high reliability', the 'medium reliability', and the 'low reliability'.

Fuzzy logic is used in numerous fields such as industrial automation, power engineering, environmental control, image processing and robotics, etc. Many researchers propose a different approach to prove fuzzy logic. With reference to the data obtained for analysing stress-strength reliability, [31] obtained conventional and fuzzy reliability by classical approach assuming a weighted exponential distribution for both stress and strength. As mentioned, the use of membership functions related to fuzzy logic is an important concept for the evaluation of fuzzy reliability. One can refer to [12] for a detailed description of the membership function and its relation to fuzzy logic.

The membership function takes different forms depending on the nature of the problem. Some common types of membership functions include triangular, trapezoidal, Gaussian, sigmoidal, among others. It may be noted that the choice of the membership function influences the behaviour and performance of a fuzzy logic system and, therefore, selecting an appropriate function is crucial for effective modelling and reasoning under uncertainty. Truly speaking, the membership function in the fuzzy set is the exposition of indicator functions in the classical setting. This paper also uses a specific form of fuzzy membership function to define reliability in a stress-strength relationship (see also [12]).

The remnants of this paper are systematized as follows. The next section considers the form of the likelihood and the reliability corresponding to W_E distribution in the conventional and the fuzzy layout. Section 3 provides the Bayesian model formulation for the considered model by specifying the proper but vague priors for the model parameters. The posteriors obtained in this section are analytically intractable and, therefore, the paper advocates the use of the Gibbs sampler algorithm for extracting samples from the posteriors and thereby drawing the sample-based inferences. It is, however, noted that some of the full conditionals are atypical from the viewpoint of sample generation and, therefore, these are managed by means of the Metropolis algorithm with an appropriately chosen kernel. A brief description of the two algorithms and their corresponding implementation details are also provided in Section 3. Section 4 gives a numerical illustration by considering a real data set. Finally, the findings of the paper are summarized in the conclusion section.

2. Reliability in a Stress - Strength Layout Assuming Weighted Exponential Distribution

To begin with, let us first focus on obtaining stress-strength reliability under both conventional and fuzzy logic assuming W_E distributions for both stress and strength. If X and Y denote the random variables corresponding to distributions representing strength and stress, the conventional stress-strength relationship suggests that the system is operational if $X > Y$. On the other hand, if Y surpasses X, the system fails. Let $f_X(x)$ and $f_Y(y)$ be the pdfs corresponding to two independent random strength and stress variables, then the expression for the conventional reliability can be written as

$$R = P(X > Y) = \iint_{x>y} f_x(x) f_y(y) dx dy. \tag{4}$$

Next, if $\Phi_{U(y)}$ is assumed to be the characteristic function for the set $U(y) = (x : x > y)$ (see also [12]), the same can be defined as

$$\Phi_{U(y)}(x) = \begin{cases} 1 & x > y \\ 0 & x \leq y. \end{cases} \tag{5}$$

Using the definition of the characteristic function $\Phi_{U(y)}$ given in (5), one can rewrite (4) as

$$R = P(X > Y) = \iint_{x>y} \Phi_{U(y)}(x) f_x(x) f_y(y) dx dy. \tag{6}$$

Let us next assume that both the random variables X and Y have the common shape parameters, that is, $X \sim W_E(\theta, \nu)$ and $Y \sim W_E(\theta, \eta)$. Under this assumption, the expression of reliability R, given in (4) for the stress-strength model, can be written as

$$R_1 = \Big(\frac{\theta+1}{\theta}\Big)^2 \nu\eta \left[\int_0^\infty \left(\int_y^\infty e^{-\nu x}(1-e^{-\theta\nu x}) dx \right) e^{-\eta y}(1-e^{-\theta\eta y}) dy \right], \tag{7}$$

which, on solving, reduces to

$$R_1 = \left(\frac{\eta^3(1+\theta) + 3\eta^2\nu}{\eta+\nu)(\eta+\theta\eta+\nu)(\eta+\nu+\theta\nu)} \right) + \left(\frac{\nu\eta^2(3\theta+\theta^2)}{(\eta+\nu)(\eta+\theta\eta+\nu)(\eta+\nu+\theta\nu)} \right). \tag{8}$$

It may be noted that the symbol R_1 is used for notational ease. Similarly, if we consider the W_E distributions for X and Y with different shape and scale parameters, that is, $X \sim W_E(\theta_1, \eta_1)$ and $Y \sim W_E(\theta_2, \eta_2)$, the reliability R_2 for the stress-strength model can be written as

$$R_2 = \Big(\frac{\theta_1+1}{\theta_1}\Big)\Big(\frac{\theta_2+1}{\theta_2}\Big)\eta_1\eta_2 \left[\int_0^\infty \left(\int_y^\infty e^{-\eta_1 x}(1-e^{-\theta_1\eta_1 x}) dx \right) e^{-\eta_2 y}(1-e^{-\theta_2\eta_2 y}) dy \right], \tag{9}$$

which, on solving, reduces to

$$\begin{aligned} R_2 =& \Big(\frac{\theta_1+1}{\theta_1}\Big)\Big(\frac{\theta_2+1}{\theta_2}\Big)\eta_1\eta_2^2\theta_2 \times \\ & \left[\left(\frac{1}{\eta_1(\eta_1+\eta_2)(\eta_1+(\eta_2(1+\theta_2)))} \right) - \right. \\ & \left. \left(\frac{1}{(1+\theta_1)(\eta_2+\eta_1(1+\theta_1))(\eta_1(1+\theta_1)+\eta_2(1+\theta_2))} \right) \right]. \end{aligned} \tag{10}$$

The traditional stress-strength reliability as defined in (6) simply considers in some way the distance between X and Y when X happens to be greater than Y. In the setup of fuzzy logic, however, the stress-strength reliability considers instead the distance X-Y when X is bigger than Y in the sense of fuzzy logic. The concept of fuzzy events is capable of being precisely specified within the framework of fuzzy sets. As such, one can define the membership function for the fuzzy event 'X exceeds Y fuzzily' as $\Psi_{U(y)}$, which captures the relevant fuzzy logic implications. For an increasing function d, the corresponding membership function can be defined as

$$\Psi_{U(y)}(x) = \begin{cases} d(x-y) & \text{if } x > y \\ 0 & \text{if } x \leq y. \end{cases} \tag{11}$$

In this scenario, the membership function in the fuzzy logic is equivalent to the characteristic function used in the traditional logic. Thus, for a constant $k > 0$, often known as fuzzy degree,

$\Psi_{U(y)}$ can be articulated by considering the fuzzy distance between the strength and the stress as under

$$\Psi_{U(y)}(x) = \begin{cases} 1 - e^{-k(x-y)} & \text{if } x > y \\ 0 & \text{if } x \le y. \end{cases} \tag{12}$$

As a result, the fuzzy stress-strength reliability can be written using the concept of fuzzy probability (see [32]) as

$$R_F = P(X \succ Y) = \iint_{x \succ y} \Psi_{U(y)}(x) f_x(x) f_y(y) dxdy \tag{13}$$

In fact, as the values of X-Y increase, the reliability of a stress-strength relationship enhances. Consequently, this insight could support a more nuanced interpretation. Also, it may be noted that as the value of the fuzzy degree k increases, the argument of the exponential term in (12) decreases, and therefore, the fuzzy reliability approaches towards conventional reliability.

Now coming back to the assumption that X$\sim$ $W_E(\theta, \nu)$ and Y$\sim$ $W_E(\theta, \eta)$, the fuzzy stress-strength reliability can be obtained as

$$R_{1F} = \left(\frac{\theta+1}{\theta}\right)^2 \nu\eta \left[\int_0^\infty \int_y^\infty \left(1 - e^{-k(x-y)}\right) \left(e^{-\nu x}(1 - e^{-\theta\nu x})\right) \left(e^{-\eta y}(1 - e^{-\theta\eta y})\right) dxdy \right], \tag{14}$$

which, on simplification, reduces to

$$\begin{aligned} R_{1F} = & \frac{k^2}{(k+\nu)(k+\nu+\theta\nu)} \times \\ & \left(\beta^2 \Big(\frac{(1+\theta)\eta + (3+\theta(3+\theta))\nu}{(\eta+\nu)(\eta+\theta\eta+\nu)(\eta+\theta+\theta\nu)} \Big)\right) + \\ & \left(\frac{k\eta^2(2+\theta)\nu((1+\theta)\eta + (2+\theta(2+\theta))\nu}{(k+\nu)(k+\nu+\theta\eta)(\eta+\nu+\theta\nu)} \right). \end{aligned} \tag{15}$$

It can also be noted that (15) can be written in the form of conventional reliability R_1 as

$$R_{1F} = \frac{k^2}{(k+\nu)(k+\nu+\theta\nu)} R_1 + \left(\frac{k\eta^2(2+\theta)\nu((1+\theta)\eta + (2+\theta(2+\theta))\nu}{(k+\nu)(k+\nu+\theta\eta)(\eta+\nu+\theta\nu)} \right). \tag{16}$$

Similarly, under the assumption that X$\sim$ $W_E(\theta_1,\eta_1)$ and Y$\sim$ $W_E(\theta_2,\eta_2)$, the fuzzy stress-strength reliability R_{2F} can be given as

$$R_{2F} = \left(\frac{\theta_1+1}{\theta_1}\right) \left(\frac{\theta_2+1}{\theta_2}\right) \eta_1\eta_2 \left[\int_0^\infty \int_y^\infty \left(1 - e^{-k(x-y)}\right) \left(e^{-\eta_1 x}(1 - e^{-\theta_1\eta_1 x}\right) \left(e^{-\eta_2 y}(1 - e^{-\theta_2\eta_2 y}\right) dxdy \right], \tag{17}$$

which, on simplification, reduces to

$$\begin{aligned} R_{2F} = & \Big(\frac{\theta_1+1}{\theta_1}\Big) \Big(\frac{\theta_2+1}{\theta_2}\Big) \eta_2^2 \theta_2 k \times \\ & \left[\left(\frac{1}{(\eta_1+\eta_2)(k+\eta_1)(\eta_1+\eta_2+\theta_2\eta_2)} \right) - \right. \\ & \left. \left(\frac{1}{(k+\eta_1(1+\theta_1))(1+\theta_1)(\eta_2+\eta_1(1+\theta_1))(\eta_1(1+\theta_1)+\eta_2+\theta_2\eta_2)} \right) \right]. \end{aligned} \tag{18}$$

Also, it can be seen that as k increases, both fuzzy reliabilities R_{1F} and R_{2F} converge to their conventional counterparts R_1 and R_2, respectively.

3. Bayesian Model Formulation

To begin with the Bayesian model formulation, we separately consider two cases: one where the shape parameters are the same for both the stress and strength random variables, and another where both the shape and scale parameters differ between the two random variables. The Bayesian model formulation for the two cases is given separately in subsections 3.1 and 3.2. These two cases will be referred to as Case 1 and Case 2 in the discussion that follows.

3.1. When the shape parameters are same for both stress and strength variables

Under the assumption that X follows $W_E(\theta, \nu)$ and Y follows independently $W_E(\theta, \eta)$, the joint pdf of X and Y can be written as

$$f_{X,Y}(x,y) = \Big(\frac{\theta+1}{\theta}\Big)\nu e^{-\nu x}(1-e^{-\theta\nu x})\Big(\frac{\theta+1}{\theta}\Big)\eta e^{-\eta y}(1-e^{-\theta\eta y}). \tag{19}$$

Now, let us assume that $x_1, x_2, \ldots, x_n$ are the observations corresponding to the strength variable X and $y_1, y_2, \ldots, y_n$ are the observations corresponding to the stress variable Y. Assuming the pdf given in (19), the corresponding likelihood function can be written as

$$L(\theta) = \Big(\frac{\theta+1}{\theta}\Big)^{n+m}\nu^n e^{-\nu\sum_1^n xi}\prod_1^n(1-e^{-\theta\nu xi})\eta^m e^{-\eta\sum_1^m yj}\prod_1^m(1-e^{-\theta\eta yj}). \tag{20}$$

For a Bayesian model formulation, we need to consider a prior for the shape parameter θ and the scale parameters ν and η. Since there is no *a priori* information available for θ, it is better to proceed with a weakly informative prior which may be taken as the uniform distribution in a large range. The same can be expressed as

$$\pi_1(\theta) = U(0, M) = \frac{1}{M}, \tag{21}$$

where M is the hyperparameter, usually taken to be large enough in order that the prior remains proper but vague. Similarly, for the parameters ν and η, one can consider independent gamma priors as given below.

$$\pi_2(\nu) = \frac{1}{\Gamma(a)}\frac{1}{b^a}\nu^{a-1}e^{-\nu/b}, \quad a>0, b>0 \tag{22}$$

$$\pi_3(\eta) = \frac{1}{\Gamma(c)}\frac{1}{d^c}\eta^{c-1}e^{-\eta/d}, \quad c>0, d>0 \tag{23}$$

where a, b, c, d are the hyperparameters, in a way that both (a, c) are the shape parameters and (b, d) are the scale parameters. A variety of strategies can be used to elicit the hyperparameters associated with the gamma priors. In the present paper, an attempt has been made to consider vague priors for both ν and η by taking sufficiently large variances of the considered gamma priors. To ensure large variance, one can obviously follow the strategy suggested by [26] and use a gamma prior that has a coefficient of variation at least 2.0 or larger. Since each of the considered gamma prior has two hyperparameters, one needs to formulate at least two equations to assess these two unknowns. Let us consider, for instance, the specification of hyperparameters of the prior given in (22). One can use the data-based information and accordingly consider the modal value of ν to be equal to its moment estimate. For formulating another equation, the coefficient of variation of ν can be taken as 2.0. It may be noted that the present paper considers the coefficient of variation as the ratio of the gamma standard deviation to its modal value. Thus, the two equations can be used to elicit the prior hyperparameters a and b associated with (22) that are presumably going to provide a vague prior. A similar strategy can be adopted to specify the hyperparameters associated with (23) resulting into another vague gamma prior for the parameter η. Following the specification of the priors, the joint prior can be expressed up to proportionality by combining the priors (21) to (23) as

$$\pi(\theta,\nu,\eta|x,y) \propto \frac{1}{M}\nu^{a-1}e^{-\nu/b}\eta^{c-1}e^{-\eta/d}. \tag{24}$$

The next step in the Bayesian model formulation is the specification of the posterior distribution and the same can be expressed up to proportionality by combining the prior in (24) with the likelihood in (20) using the Bayes theorem. That is

$$\begin{aligned} p(\theta,\nu,\eta|x,y) \propto & \Big(\frac{\theta+1}{\theta}\Big)^{n+m}\nu^{n}e^{-}\nu\sum_{1}^{n}xi\prod_{1}^{n}(1-e^{-\theta\nu xi})\times \\ & \eta^{m}e^{-\beta\sum_{1}^{m}yj}\prod_{1}^{m}(1-e^{-\theta\eta yj})\frac{1}{M}\nu^{a-1}e^{-\nu/b}\eta^{c-1}e^{-\eta/d}. \end{aligned} \tag{25}$$

Obviously, the posterior given in (25) appears to be analytically intractable and, therefore, we propose using the Gibbs sampler algorithm for extracting samples and drawing the sample-based inferences. The Gibbs sampler algorithm is an iterating scheme that often proceeds by drawing from different one-dimensional full conditionals that are also required to be specified up to proportionality only. The full conditional in each case is a conditional distribution of the specified variate presuming that all other variates are fixed known constants. The generation actually proceeds in a cyclic order, starting from one of the full conditionals and then successively iterating from all other full conditionals. The latest known values are used for the conditioning variates in each case. One can refer to [27] for details on the Gibbs sampler algorithm in the case of lifetime distributions (see also [28] and the references cited therein).

Looking on (25), the full conditionals for different variates involved in the joint posterior can be expressed up to proportionality as

$$p(\theta|x,y,\nu,\eta) \propto \Big(\frac{\theta+1}{\theta}\Big)^{n+m}\prod_{1}^{n}(1-e^{-\theta\nu xi})\prod_{1}^{m}(1-e^{-\theta\eta yj}), \tag{26}$$

$$p(\nu|x,y,\theta,\eta) \propto \nu^{n+a-1}e^{-\nu(1/b+\sum_{1}^{n}xi)}\prod_{1}^{n}(1-e^{-\theta\nu xi}), \tag{27}$$

$$p(\eta|x,y,\theta,\nu) \propto \eta^{m+c-1}e^{-\eta(1/d+\sum_{1}^{m}yj)}\prod_{1}^{m}(1-e^{-\theta\eta yj}). \tag{28}$$

It can be easily seen that the full conditionals given in (27) and (28) are concave on a logarithmic scale and, therefore, samples can be easily drawn from the two full conditionals using the procedure for drawing samples from logconcave densities. One such procedure is the adaptive rejection sampling method initially proposed by [15] that has been proved to be quite flexible and rich in a variety of cases (see also [27]). The form given in (26) is difficult from the viewpoint of direct sample generation and, therefore, one can use the Metropolis algorithm. A brief description of the same is given in the subsection 3.3.

3.2. When the shape and the scale parameters are different for both stress and strength variables

Under the assumption that X follows $W_E(\theta_1,\eta_1)$ and Y follows independently $W_E(\theta_2,\eta_2)$, the joint pdf of X and Y can be written as:

$$f_{X,Y}(x,y) = \Big(\frac{\theta_1+1}{\theta_1}\Big)\eta_1 e^{-\eta_1 x}(1-e^{-\theta_1\eta_1 x})\Big(\frac{\theta_2+1}{\theta_2}\Big)\eta_2 e^{-\eta_2 y}(1-e^{-\theta_2\eta_2 y}). \tag{29}$$

Now, let us assume that $x_1, x_2, \ldots, x_n$ are the observations corresponding to the strength variable X and $y_1, y_2, \ldots, y_n$ are the observations corresponding to the stress variable Y, then the likelihood function for the observations assuming the pdf (26) can be written as

$$L(\theta) = \Big(\frac{\theta_1+1}{\theta_1}\Big)^n \Big(\frac{\theta_2+1}{\theta_2}\Big)^m \eta_1^n e^{-\eta_1 \sum_1^n xi} \times \prod_1^n (1-e^{-\theta_1\eta_1 xi}) \eta_2^m e^{-\eta_2 \sum_1^m yj} \prod_1^m (1-e^{-\theta_2\eta_2 yj}). \tag{30}$$

The specification of the prior distributions for the parameters is the next stage in Bayesian model formulation. Due to lack of any *a priori* information, the specification proceeds exactly the way it is given in the previous subsection. As such, the prior distributions for θ_1 and θ_2 can be taken as uniform distributions given as under.

$$\pi_1(\theta_1) \propto U(0,V) = \frac{1}{V}, \tag{31}$$

$$\pi_2(\theta_2) \propto U(0,W) = \frac{1}{W}, \tag{32}$$

where, once again, the prior hyperparameters V and W can be taken large enough so that the priors remain more or less vague. Similarly, the priors for the scale parameters η_1 and η_2 can be taken to be gamma distributions with hyperparameters (q,r) and (s,t), respectively. These are given as

$$\pi_3(\eta_1) = \frac{1}{\Gamma(q)} \frac{1}{r^q} \eta_1^{q-1} e^{-\eta_1/r}, \quad r>0, q>0 \tag{33}$$

$$\pi_4(\eta_2) = \frac{1}{\Gamma(s)} \frac{1}{t^s} \eta_2^{s-1} e^{-\eta_2/t}, \quad s>0, t>0. \tag{34}$$

The elicitation of prior hyperparameters in (33) and (34) can proceed exactly in a way that is described in the previous subsection. The ultimate objective is to specify two equations to solve for these hyperparameters in a way that priors remain more or less vague.

The independent priors given in (31) to (34) can be multiplied to get the joint priors of the parameters and then the same can be finally combined with the likelihood function (30) via the Bayes theorem to get the joint posterior up to proportionality. This joint posterior can be written as

$$p(\theta_1,\theta_2,\eta_1,\eta_2|x,y) \propto \Big(\frac{\theta_1+1}{\theta_1}\Big)^n \Big(\frac{\theta_2+1}{\theta_2}\Big)^m \eta_1^n e^{-\eta_1 \sum_1^n xi} \times \prod_1^n (1-e^{-\theta_1\eta_1 xi}) \eta_2^m e^{-\eta_2 \sum_1^m yj} \prod_1^m (1-e^{-\theta_2\eta_2 yj}) \times \frac{1}{V}\frac{1}{W} \eta_1^{q-1} e^{-\eta_1/r} \eta_2^{s-1} e^{-\eta_2/t}. \tag{35}$$

The posterior given in (35) is also analytically intractable and, therefore, we shall look first at the possibility of applying the Gibbs sampler algorithm. For the purpose of implementation, let us first write the full conditionals for the three variates θ_1, θ_2 and θ_3 as

$$p(\theta_1|x,y,\theta_2,\eta_1,\eta_2) \propto \Big(\frac{\theta_1+1}{\theta_1}\Big)^n \prod_1^n (1-e^{-\theta_1\eta_1 xi}), \tag{36}$$

$$p(\theta_2|x,y,\theta_1,\eta_1,\eta_2) \propto \Big(\frac{\theta_2+1}{\theta_2}\Big)^m \prod_1^m (1-e^{-\theta_2\eta_2 yj}), \tag{37}$$

$$p(\eta_1|x,y,\theta_1,\theta_2,\eta_2) \propto \eta_1^{n+q-1} e^{-\eta_1(1/r+\sum_1^n xi)} \prod_1^n (1-e^{-\theta_1\eta_1 xi}), \tag{38}$$

$$p(\eta_2|x, y, \theta_1, \theta_2, \eta_1) \propto \eta_2^{m+s-1} e^{-\eta_2(1/s+\Sigma_1^m yj)} \prod_1^m (1 - e^{-\theta_2\eta_2 yj}), \quad (39)$$

respectively. As in the previous subsection, one can easily check that both (38) and (39) are concave on logarithmic scale and, therefore, adaptive rejection sampling of [15] can be used for corresponding sample generations. Both (36) and (37) do not offer any standard form for generating samples and, therefore, one can apply the Metropolis algorithm with an appropriately selected candidate generating density for generating the corresponding samples from (36) and (37).

3.3. A brief description of the Metropolis algorithm

As already mentioned, the two posteriors considered in the paper are analytically intractable, and an option is to draw sample-based inferences using the Gibbs sampler algorithm. The Gibbs sampler algorithm is used mainly for its straightforwardness and routine implementation. However, some of the full conditionals corresponding to the two posteriors considered in the paper were found to be difficult from the viewpoint of sample generation and thereby leaving for the scope of the Metropolis algorithm. This subsection briefly describes the Metropolis algorithm as a tool for sample generation from one-dimensional full conditionals.

The Metropolis algorithm is also a Markovian updating scheme to generate samples from a typically available density, often specified up to proportionality, using a symmetric Markov kernel. Although the algorithm is generally used for high-dimensional densities, given up to proportionality, it has been successfully employed for one-dimensional scenarios as well (see, for example, [26]. To briefly describe the algorithm for one-dimensional scenario, let us assume that p(θ|.) is the intended density of theta given fixed values of several other quantities (.). Let us consider a symmetric Markovian kernel to be used as a candidate generating density. Next, assume that the current state of the chain is θ and, given θ, the chain considers the next proposed realization θ' simulated from the candidate generating density. We shall, however, incorporate a further randomization and accept θ' with probability $\zeta(\theta, \theta') = \min\left(\frac{p(\theta|.)}{p(\theta'|.)}, 1\right)$. In case the proposed realization is not accepted, the chain remains at the value θ.

Various suggestions can be found in the literature on the selection of appropriate candidate generating densities. The choice is not difficult since we are mostly concerned with one-dimensional candidate generating density for simulating the intended one-dimensional full conditional. A few suggested choices could be a normal density with appropriately chosen mean and standard deviation, t-distribution, uniform density, etc. One can refer to [20] and [27], among others, for further details. The present paper considers a univariate normal kernel as the candidate generating density in each case where the corresponding full conditional cannot be generated easily. To start the process, we considered the initial value for the mean of the normal kernel as the moment estimate of the corresponding parameter and the standard deviation as δ times the square root of the Hessian evaluated at the moment estimator. The choice of moment estimator is certainly an approximation, and it cannot be considered as the most appropriate choice, though it has the advantage of being easy to evaluate for both the considered cases under the assumption of W_E model. It may also be noted that the choice of the initial values suggested above was upgraded after every 500 iterations using the generated sample output. Also, δ as defined above is the scaling constant, normally recommended in the range 0.5-1.0, to reduce the number of rejections in the iterating chain (see, for example, [27]).

Another important issue in the successful implementation of the Metropolis algorithm is monitoring convergence. The present paper considers the same based on ergodic averages obtained from a single long run of the chain. Once the convergence is observed, the simulating chain can be further continued and equidistant observations can be picked up to form the desired size of posterior samples. It may be noted that proper distancing among the generating variates will relegate the serial correlation to a bare minimum. These sample observations can then be used to form features of interest of the concerned posteriors (see also [28]).

4. Numerical Illustration

For numerical illustration, let us consider a real dataset that consists of breaking strength of jute fibre at various gauge lengths. The dataset was originally introduced by [30] and later on used by [25], [19] and [31] for estimating the reliability and other characteristics using different lifetime distributions. Here, we analyse the breaking strength of jute fibre at 10 mm and 20 mm for X and Y, respectively. The dataset is given below for a quick reference.

Data X: 693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16, 671.49, 183.16, 257.44, 729.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 753.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, 177.25.

Data Y: 71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 682.66, 45.58, 578.62, 756.70, 594.29, 166.49,94.72, 707.36, 765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55.

The datasets as given above were authenticated by [31] to have come from W_E distributions. The authors obtained classical p-values based on the Kolmogorov–Smirnov goodness of fit test, and they noted that the W_E model provides a very good fit to the two datasets.

To provide the complete Bayes analysis on the above datasets, let us begin by specifying the prior hyperparameters as described in subsection 3.1 and 3.2. The hyperparameters M, V, and W associated with the uniform distributions in the two cases were fixed at 50.0 each. It could be presumed that the value 50.0 was large enough to make the uniform priors quite weak although a number of other large values in the range 20.0 to 100.00 were also considered for each of these hyperparameters and no significant differences were observed in the results. Similarly, for the hyperparameters associated with the scale parameters in (22), (23), (33), and (34), we used the strategy suggested in the 3.1 and 3.2 and accordingly obtained the values of associated hyperparameters as ($a = 1.64, b = 0.05$), ($c = 2.62, d = 0.0025$), ($q = 2.62, r = 1.87$) and ($s = 2.62, t = 0.0025$).

Once the prior distributions were specified, the next task was to use the Gibbs sampler algorithm as discussed in Section 3 to simulate samples from each of the two posteriors corresponding to the two considered models given in (19) and (29). Of course, some of the full conditionals such as (26), (36) and (37) were simulated using the Metropolis algorithm (see subsection 3.3) with scaling constant δ empiically determined as 0.5 in each case. The implementation of the algorithm proceeded by simulating a single long run of chain using the details as given in Section 3. As mentioned, the initial values for the mean and standard deviation of the normal kernel were taken respectively as the corresponding moment estimates and the square root of the Hessian-based approximation evaluated at the moment estimates. The moment estimates of the parameters for the two considered cases (see also [2]) are reported in Table 1 although we needed these estimates for θ, θ_1 and θ_2 only (see also Section 3).

The convergence based on the ergodic averages corresponding to different parameters for Case 1 and Case 2 was obtained approximately at around 20K and 30K iterations, respectively. The computation time in order to achieve the convergence in running the Gibbs sampler algorithm with intermediate Metropolis steps as discussed earlier was approximately 4.83 seconds and 7.67 seconds, respectively, for Case 1 and Case 2 when the program was run on an i5 processor. Once the convergence was achieved, a sample of size 1K was obtained separately from the two posteriors corresponding to each of the parameters, ensuring that the serial correlation among the generating variates remains almost negligible. This was done by picking up every 10th observation from the generating chain.

Once the posterior samples are obtained, any feature of interest for the posterior distribution can be easily assessed which otherwise is not possible through non-sample based approaches. Table 2 shows some of the estimated posterior characteristics reported in the form of estimated posterior mean, median, mode, and highest posterior density interval with coverage probability

0.95 (0.95 HPD) for the parameters of each of the two considered models. It is obvious from the results that the estimated central values of the two posteriors, especially the posterior mode, are quite close to the moment estimates in most cases, giving the impression that our elicited priors are almost weakly informative and do not have any significant influence on the results (see also Tables 1-2). The estimated posterior characteristics also provide an impression that the corresponding posterior densities in each case are almost symmetric around the estimated central values except for the variates θ_1 and θ_2 which are slightly positively skewed, although the marginal density estimates are not shown in the paper. Besides, Table 2 also shows the estimated 0.95 HPD for the different parameters in the two cases. These values convey some information regarding the posterior variability that appears highest for θ in Case 1 and highest for both θ_1 and θ_2 in Case 2.

We also tried to examine the reason for delayed convergence in the single long run of iterating chain and the reason can be attributed to some extent to the high values of the estimated correlations between different combinations of variates. The estimated correlations were found to be -0.557, -0.507, -0.309, -0.408, respectively, between the pairs (θ, ν), (θ, η), (θ_1, η_1), (θ_2, η_2).

Finally, Table 3 shows the estimated values of conventional and fuzzy reliabilities for the two cases considered in the paper. These values are obtained simply by substituting the posterior modes of different parameters in the expressions of reliability given in (8), (10), (15) and (20). The estimates are extended in Table 4 by increasing the values of k. It is evident that, by increasing the value of k, the fuzzy reliability approaches towards the conventional reliability, which appears to be an obvious result.

Table 1: *Moment estimates of the parameters for Case 1 and Case 2*

Cases	Parameters	MM estimates
Case 1	θ	1.838
	ν	0.003
	η	0.003
Case 2	θ_1	2.010
	η_1	0.004
	θ_2	2.060
	η_2	0.003

Table 2: *Some of the estimated posterior summaries for the parameters of W_E distribution for Case 1 and Case 2*

Reliability		
Cases	Conventional reliability	Fuzzy reliability
Case 1	$R_1 = 0.535$	$R_{1F} = 0.528$
Case 2	$R_2 = 0.618$	$R_{2F} = 0.537$

Table 3: *Estimated conventional and fuzzy reliabilities for Case 1 and Case 2 when k=1*

Cases	Parameters	Mean	Median	Mode	0.95 HPD intervals
Case 1	θ	1.740	1.722	1.770	(0.451, 3.158)
	ν	0.003	0.003	0.003	(0.002, 0.004)
	η	0.003	0.003	0.003	(0.002, 0.004)
Case 2	θ_1	2.050	1.995	1.893	(1.202, 2.903)
	η_1	0.004	0.004	0.004	(0.002, 0.005)
	θ_2	3.136	3.080	2.820	(1.762, 4.451)
	η_2	0.004	0.004	0.004	(0.002, 0.005)

Table 4: *Estimated conventional and fuzzy reliabilities for Case 1 and Case 2 for different choices of k*

Cases	k	Conventional reliability	Fuzzy reliability
Case 1	1	0.535	0.528
	50	0.535	0.535
	100	0.535	0.535
Case 2	1	0.618	0.537
	50	0.618	0.538
	100	0.618	0.610

5. Conclusion

The paper provides the Bayes analysis of conventional and fuzzy reliabilities for two independent stress-strength variables when both stress and strength follow weighted exponential distributions. Two cases are explored. First, when both stress and strength have common shape parameters and, second, when both have different shape and scale parameters. Using the proper but weak priors for the parameters, the paper obtains sample based inferences based on the Gibbs sampler algorithm with intermediate Metropolis steps for some of the full conditionals. A numerical illustration is provided based on a real dataset consisting of the breaking strength of jute fibre at different gauge lengths. It is shown that the inferences are mostly affected by the data and, as expected, the priors have negligible effect on the inferences. Also, the numerical illustration effectively demonstrates that the fuzzy degree k significantly influences the fuzzy reliability in the sense that as the fuzzy degree k increases, the difference between fuzzy and conventional reliabilities diminishes, a finding that appears natural in this case.

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