A BATHTUB ADDITIVE FAILURE RATE MODEL FOR ACTUARIAL RELIABILITY

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Abstract

The actuarial approach to hardware reliability plays an important role in providing balanced avenue and fair play in dealing with multi-unit systems and dual failures occurring in such complex settings. In statistical reliability perspective, the additive failure rate methodology is believed to adequately adapt to the actuarial approach for effective system reliability. In line with this hypothetical subject, this paper introduces a lifetime model named Hybrid Weibull-Exponential Power model, based on the later phenomenon vis-à-vis the former approach. The failure rate (FR) of the proposed model displayed suitable compound trends such as increasing-decreasing and various complex forms of bathtub-curve, with and without the lingering useful life period. The consistency of its quantile function in producing effective samples was validated by simulation study, despite its nonclosed form. Other reliability properties of the model were also provided. The paper employs the maximum likelihood estimation procedure for estimating the unknown parameters of model, due to its reputable robustness in the aspect of applied statistics. The adequacy of the proposed estimator was evaluated by means of simulation study, where, negligible biases and declining MSEs were observed. The real-life performance of the proposed model in relation to three similar-structured models was weighed using bathtub failure times, where, least values of information criteria and p-values of goodness-of-fit statistics were obtained for the model. Highest loglikelihood value of the model was also observed. The additive failure rate model proposed in this paper allows for actuarial reliability analysis of compound systems, due to its outstanding performance on the dataset. The MLEs used for estimating the model parameters were proven consistent since insignificant biases and declining MSEs were obtained in all cases. Also, on the premise of the results obtained for the information criteria, the p-values for the goodness-of-fit statistics and the loglikelihood value, we infer that the new established model can better be employed over the compared models, for the dataset. Subsequently, effective system reliability with similar datasets can be achieved using the model. Succinct guide on the recent developments of the emerging family of additive FR models and their application in multi-unit system modelling, is also given.

Keywords: actuarial approach, statistical reliability, hybrid, bathtub failure rate, maximum likelihood estimation, multi-unit systems, simulation study

I. Introduction

In the industry of determining the lifetime behavior of engineered units, relevant information, such as the key phases gone through by a unit during its serviceable life, are often analyzed by reliability scientists via appropriate theoretical tools. The tools, predominantly, include classical probability models such as normal, lognormal, exponential, gamma, Rayleigh and, notably, the Weibull model [38, 40, 42]. The vital information derived, help in making essential decisions such as device warranty, optimal mode and time of replacement/or maintenance and so on.

However, with the persistent emergence of modern systems, more complex data with nonmonotonic FR behavior like the bathtub also keep recurring. Consequently, tentative decisions are believed to be made as a result of truncated information derived by employing the classical distributions for the analyses. This is because, majority of the familiar models fall flat to adequately capture non-monotone failure times (FTs), including the legend bathtub feature [39]. For example, as in the case of a single Weibull model [50], hence, leading to undesirable conclusions. The bathtub FR titled considering its form, consists of the three known consecutive stages, hypothetically shaping into a bathtub manner. First, the burn-in period, with usual high failure tendency, due to unidentified imperfections along the unit's construction process that can lead to infant mortality. Upon surviving this phase, the FR decreases and balances for some time in the second phase. This is the chance failure period or simply, useful life of the unit. As the unit is subjected to aging, the barely increasing FR begin to heighten. Thus, activates the third, wear-out period, which concludes the life cycle of the unit. Refer to Rausand and Holyland [40]. Bathtub FTs are vastly discussed in the reliability literature. These include the 36 times to first failure of 500MW generators [17]; the failure and running times of sample of 30 units [31]; 18 FTs of electronic devices [48], the early FTs of cable joints [46], the FTs of 24 DRP systems [32], the rates of failure and survivors for dielectrically insulated armature bars of a generator [18], to mention few.

To address the under-fitting problem of the classical models in representing complex FTs like the bathtub, many advanced models with desirable features were put forward by reliability statisticians to serve as better tools for which purpose classical models were used, thus, incorporating the traditional models' attributes. For instance, Lai et al. [21], proposed the Modified Weibull model for explaining bathtub FRs. It generalizes the Weibull and type I extreme value models respectively. Carrasco et al. [10], defined 4-parameter Generalized Modified Weibull model for lifetime modeling of monotone and non-monotone FRs. It extends the modified Weibull model and further having the generalized Rayleigh and extreme value distributions as special cases. Haupt and Schäbe [20], proposed a bathtub-curved FR model, however, infers not on the behavior of the infant mortality of the FR. Further, it offers not much workability to be adopted [10]. Sylwia [45] introduced a shape parameter to the Makeham's distribution which was strictly increasing. Thus, made its FRF shaping into the bathtub, hence, named it Makeham's modified distribution (MMD). Although, the author iterated on the MMD's applicability in burn-in procedure planning and preventive maintenance of non-repairable appliances, the MMD represents not well the constant region of the bathtub curve. Additionally, Alessa et al. [7] considered the bathtub FR characterization of two sets of FTs in [1] and [31] respectively, to examine the hypothetical contrast between their proposed modified exponential-Weibull (MEW) model and five other Weibull extensions. Subsequently, the MEW model was found enhanced over the rest of the models in terms of amply explaining the FTs.

Another important distribution in this regard, related to Weibull is proposed by Chen [12], with two parameters. However, inadequate in different scenarios, as it lacks a scale parameter. So, as an extension to [12], Xie *et al.* [22], proposed the Weibull extension model by adding a scale parameter to the Chen [12] model, thus, becomes flexible and plausible to practitioners. Correspondingly, Chaubey and Zang [11] extended the Chen [12] models with increasing or bathtub FRF. Similarly, various proposals focusing on combining the Chen [12] model or its extensions, with other classical models, more so, those of the Weibull family, have been put forward for characterizing the bathtub FTs. For instance, the Additive Chen-Weibull (ACW) model introduced by Thach and Bris [47] to describe bathtub FTs, it was tested on the two bench mark FTs [1, 31], in relation to the additive Weibull (AddW) and other Weibull-modified models. The results showed that the ACW provides better fit for the FTs in [1], while the New Modified Weibull model showed better fit for the FTs in [31], seconded by the ACW and AMW models respectively. Mendez-Gonzalez *et al.* [35] proposed and related the Chen-Perks (CP) distribution to several other models including AddW, based on three case studies reported by Wang *et al.* [48], Aarset [1] and Sylwia [45] respectively. The models performed well in fitting the FTs of [45], while, the CP outperformed the rest of the models for representing the FTs in [1]. However, many AFR models fail to fit bathtub FTs well, perhaps due to the role of the Chen model in the constituting models. Although it can have bathtub FR, but may provide not a fairly good representation of the bathtub curve, this gap may be attributed to it lack of scale parameter, posed Abba *et al*. [3].

One of the very rare two parameter lifetime models with possible bathtub FRs for complex system reliability, is the Exponential-power model. It was introduced, with Weibull type parameterization, by Smith and Bain [44], for modelling wear-out failures. It has U-shaped form of bathtub FR [26, 29]. Moreover, Choy and Walker [13] extended the family while Barriga *et al.* [9] annexed a shape parameter to the model and proposed a complementary version, to handle variety of non-monotone FRs. Abba *et al.* [4] came-up with flexible exponential power-Gompertz (FEPG4) by methodizing the two component models for complex reliability modelling.

- Relevance of Additive Failure Rate models in Actuarial reliability

The actuarial approach to hardware reliability entails the analysis of systems of several units (say $u_i, i = 1, \dots, n$). Whereby, the strength of the units and the entire data regarding the loads of operation, all combined, are described by the distributional probability, $\mathbb{G}(t)$, of the time-to-failure, T [40]. Although modelling the strength and the load is implicit, however, reliability function, FR and other reliability properties of the component units can be determined via the distributional probability, $\mathbb{G}(t)$. Where appropriate, the relations, $R(t) = 1 - \mathbb{G}(t)$ and $\hbar(t) = \frac{\mathbb{g}(t)}{R(t)}$ can simply be adopted, where, $R(t)$, $\mathbb{g}(t)$ and $\hbar(t)$ are defined in Table 1.

In recent times, statistical reliabilists offer considerable score of advanced models which are largely hybrid in structure, for modelling the bathtub and other complex FTs. These models are mostly formulated based on multi-unit systems assumption [2, 3, 41, 47, 50, etc.], i.e several components assembled in a system (mostly in series), in order to function in one direction for achieving one goal. Thus, points to their potential appropriateness for effective actuarial reliability analysis of hardware systems. In this direction, the additive-wise methodology of linear combination of two or more FRs of certain probability models, with the intent of reliability studies is gaining more attention in reliability theory. Admittedly, researchers devoted greatly towards introducing more distributions based on the additive phenomenon, particularly in the last couple of years, see for instance, [3], [5], [6], [47] and the references within. These hybrid models are usually termed as additive FR (AFR) models (for the compositional manner of their FRFs). They are formulated through their FRFs, which is obtained as a summation of two or more FRFs or reliability functions (RFn) of desirable classical probability distributions. The resultant compound models form another family of probability distributions, with members such as, the AddW of Xie and Lai [50], exponential-log logistic additive FR model of Rosaiah *et al.* [41], the AFR models derived by Abid and Hassan [6];; additive Perks (AddP) of Méndez-González *et al.* [33];; flexible exponential-power Gompertz of Abba *et al.* [4]; additive Gompertz-Weibull of Abba and Wang [2]; additive Chen (AddC) of Méndez-González *et al*. [34]; flexible Dhillon-Weibull Competing Risk model of Abba *et al.* [5], etc. These models are also known as multi-risk or competing risk distributions, due to their multi-risk representation ability. The distributions have lately become increasingly appealing to studies, for their versatility in modelling bathtub and other complex FTs.

Albeit, the AFR methodology can be regarded as old approach, see Refs. [16], [36] and [19]. However, its resurgence with more applications in reliability and survival analysis can be traced back nearly three decades after the establishment of Additive Weibull model [50]. In motivation of this additive methodology, Rosaiah *et al.* [41] and, Abid and Hassan [6] explained the fact that the reliability probabilities of such compound systems are either evaluated by the system's failure mode as a whole, such as

$$R(t) = \exp\left\{-\int_0^t \hbar(u)du\right\}, \tag{1}$$

or via the independent reliability probabilities of the units that define the whole system using their respective FRs. Formally described in section II. The AFR distributions are mostly applied to the modelling of complicated FTs with two or more failure mechanisms, or to the system reliability with serially independent subunits. A series system consists of serially installed units (say, 1, 2, …, r). Thus, subsequent units are considered a failure by the failure of the first unit in the installation line. Its brief illustration is given below.

Figure 1: *An instance of a series system*

Notwithstanding, great portion of the modern hybrid models presented as sensitive to the bathtub FTs, fail at some point, to represent effectively such FTs. Refer, for instance, to Abba and Wang [2], on the inability of notable bathtub distributions in representing the bathtub-curved times of early failures of cable joints [46]. The more realistic ones were able to fairly characterize the bathtub curve, however, overlook the worthwhile lingering life stage. For instance, the flexible additive Weibull model [25] and Xgamma-Burr XII [37] have not been able to sufficiently contain the constant and/or end-of-life regions of some bathtub FTs. Thus, the process of estimating and representing devices' FTs through these models may be further from the reality. Notwithstanding, a handful of distributions [43], [33] and [5], put forward in this regard perform satisfactorily well. However, these models may perhaps be outnumbered by the unprecedented overflow of the FR problems involving some variants of the bathtub and other non-monotone trends.

Thus, given above importunate concern, particularly, regarding the AFR models pointed out earlier constituting of Chen as submodel, specifically, the ACW that motivates this study. For the nonexistence of scale parameter in the Chen distribution, these models may not be well fit for properly monitoring the spread of bathtub datasets - where, the value of the scale parameter controls the dispersion a distribution. Hitherto, the quest for a much more satisfying lifetime model in this regard remains incessant. Also, given the importance of the actuarial approach to system reliability, there is need for simplified methodologies in its regard.

Accordingly, considering the AFR criterion, this study aims to establish a FT model for describing monotone and non-monotone FTs of non-dependent dual failures. It is also, purposeful, giving regard to the multi-unit assumption of the actuarial reliability analysis of series systems of independent sub-units. The proposed Hybrid Weibull Exponential-Power (HWEP) model can be deemed an alternate modification to the ACW model by swapping the Exponential-power for the Chen model accordingly.

For achieving this study, we adhered to these key motivations, (i) proposing an AFR model with the ample capability to explain the complex FTs of technical units based on actuarial phenomenon, (ii) running Monte Carlo trials to assess the consistency of the quantile function of the model, (iv) estimating the parameters of the model using MLE technique, and also, checking the consistency of the MLEs (v) testing the applicability of the model in real-life on bathtub FT data and evaluating its competitive performance using the information criteria.

Subsequent to the overview and pertinence put forward in section I, the research hierarchy goes as this: section II explains the AFR method for compounding statistical models for system reliability. Swift discussion on the parent models used as the baseline for the constructing the new model. Then, the proposed model was introduced with some of its reliability attributes, including its quantile function. The symbolizations utilized in the whole text are interpreted in this part as well. Results of the simulation studies and reliability relevance testing for the model were presented in section III. Section IV contains general discussions and section V concludes the study.

II. Methods

Succinct explanation on the methodology adopted, the underlying statistical models utilized as the baseline and the proposed model are all given in this section.

I. Additive Failure Rate Approach

Suppose that, random variables $T_{i;\,2\leq i\leq k}$ are the FTs of finite independent subunits of a system. Alternatively, set the random variables $T_{i;\,2\leq i\leq k}$ as FTs of multiple independent failure modes (FMs) suffered by a system. In both scenarios, suppose also that each of the FTs follow a specific probability model with FRF, $h_i(t)$ and RFn, $R_i(t)$. Then, by the AFR approach discussed earlier, we build a hybrid statistical model for system reliability via the sum of baseline models' FRF as:

$$h(t) = \sum_{i} h_i(t)\,, i = 2,..,k. \tag{2}$$

Then again, the product of the underlying models' RFn can be considered as:

$$R(t) = \prod_{i} R_i(t) = \exp\left\{-\sum_{i}\int_{0}^{t} h_i(u)du\right\}, i = 2,..,k. \tag{3}$$

Refer also to Murthy *et al.* [38], Thach and Bris [47], Mohammad *et al.* [37], Abba *et al.* [5], etc.

II. The Baseline Models

Concise introductions based on this study's portion of interest regarding the Weibull as well as exponential-power models are given accordingly as follows.

- The Weibull model

The well-known Weibull distribution by Weibull [48] can have its FRF and RFn respectively as,

$$h(t;\alpha,\gamma) = \alpha\gamma^{\alpha}\, t^{\alpha-1}, t > 0. \tag{4}$$

$$R(t;\alpha,\gamma) = e^{-(\gamma t)^{\alpha}}, t \geq 0;\ \alpha,\gamma > 0. \tag{5}$$

However, its FRF can only discretely assume increasing, decreasing or constant behavior [23, 47]. In consequence, numerous studies considered remodeling the distribution. Refer to Johnson *et al*. [23] for more elaboration. Also, Murthy *et al*. [38] gave wider, whereas, Lai and Xie [27] offered brief details on the notable developments of the Weibull models due then.

- The Exponential-power model

The exponential-power model is a life-casting model first studied by Smith and Bain [44] with the possibility of a bathtub or U-shaped FRF at its shape's value, $\tau < 1$ [26, 29]. Thus, making it one of the very rare natural two-parameter models with such attribute. The model is useful in case of reliability modeling of a unit that fails quickly during its wear-out phase, after being stable for some period [44]. It has its FRF and RFn respectively as:

$$h(t;\tau,\nu) = \tau\nu^{\tau}t^{\tau-1}e^{(\nu t)^{\tau}}, \tag{6}$$

$$R(t;\tau,\nu) = e^{1-e^{(\nu t)^{\tau}}}, t > 0;\ \tau\nu > 0. \tag{7}$$

Notable developments of the exponential-power model were given in section I.

- Symbolizations

Across this research work, we adhered to these symbolizations below.

Table 1: *Symbolizations utilized in the study*

$\hbar$	Number of units subjected to life test, also, number of FMs
T_i	i^{th} unit uptime until failure, $1 \leq i \leq \hbar$
$\mathbb{G}(\cdot)$	Cumulative distribution function (CDF) of T_i
$\mathbb{g}(\cdot)$	Probability density function (PDF) of T_i
$\hbar(\cdot)$	$\frac{\mathbb{g}(\cdot)}{R(\cdot)}$ FRF of T_i
$R(\cdot)$	$= 1 - \mathbb{G}(\cdot)$, reliability function (RFn) of T_i
$W(\alpha,\gamma)$	Weibull distribution with α,γ as parameters
$EP(\nu,\tau)$	Exponential power model with ν,τ as parameters

III. The Proposed HWEP Model and its Properties

This subsection describes the proposed model, its reliability and other distributional properties.

- Model specification

Define $T_i, (i = 1,2)$ as FTs resulting from $(\hbar = 2)$ independent forms of failure suffered by either of two sub-units of a system at a time. Next, set T as the overall FT of the system which can be interpreted as the first unit-specific time of failure arrived by the system. Otherwise, T may be taken as the first mode-specific failure arrived by a unit. Thus, $T = min\{T_1, T_2\}$. Further, assume that $T_1 \sim W(\alpha,\gamma)$ with FRF and RFn in Eq. (4) and Eq. (5) while, $T_2 \sim EP(\nu,\tau)$ with FRF and RFn in Eq. (6) and Eq. (7), respectively. Where W and EP stand as corresponding acronyms for Weibull and exponential-power models. That is to say, the proposed model, termed as Hybrid Weibull-Exponential Power (HWEP), describes the FTs or FMs of the subunits. In either case, the components' FRs are assumed to abide by Eq. (4) and Eq. (6) respectively. Thus, reference to Eq. (2), if $T \sim HWEP(\alpha,\gamma,\nu,\tau)$, then, we obtain the corresponding FRF, $\hbar(t;\alpha,\gamma,\nu,\tau)$ of T as:

$$\hbar(t;\alpha,\gamma,\nu,\tau) = \hbar_W(t;\alpha,\gamma) + \hbar_{EP}(t;\tau,\nu),$$
$$= \alpha\gamma^\alpha t^{\alpha-1} + \tau\nu^\tau t^{\tau-1} e^{(\nu t)^\tau},\ t > 0, \tag{8}$$

with $\boldsymbol{\xi} = (\alpha,\gamma,\nu,\tau)'$ denoting the HWEP's vector of parameters, among which $\alpha,\tau > 0$ and also $\gamma,\nu \geq 0$, stand dually as corresponding shape and scale.

The probability of a unit to execute its set function for a quoted time interval under usual (or preset) settings, determines its reliability [38]. The corresponding RFn, $R(t)$, is defined as:

$$R(t) = e^{-H(t)}, \tag{9}$$

where, $H(t)$, is the cumulative FR. It is obtained from Eq. (8) as:

$$H(t) = \int_0^t \hbar(u)du = (\gamma t)^\alpha + e^{(\nu t)^\tau} - 1. \tag{10}$$

Thus, putting Eq. (10) into Eq. (9) yields the following:

$$R(t) = \exp\{-(\gamma t)^\alpha - e^{(\nu t)^\tau} + 1\}, t > 0. \tag{11}$$

Equally, Eq. (11) can easily be attained via Eq. (3) using Eq. (5) and Eq. (7) respectively. Moreover, the associated CDF, $\mathbb{G}(t)$, of T is obtained from Eq. (11) as:

$$\mathbb{G}(t) = 1 - \exp\{-(\gamma t)^\alpha - e^{(\nu t)^\tau} + 1\},\ t > 0. \tag{12}$$

While, by differentiating the CDF in Eq. (12), with respect to t, we obtain the PDF, $\mathbb{g}(t)$, as:

$$\mathbb{g}(t) = \left[\alpha\gamma^\alpha t^{\alpha-1} + \tau\nu(\nu t)^{\tau-1} e^{(\nu t)^\tau}\right] \exp\{-\left[(\gamma t)^\alpha + e^{(\nu t)^\tau} - 1\right]\}, t > 0. \tag{13}$$

- FR attributes of the HWEP model

In reliability, the FR of a unit (say T), is the probability of failure at some time t, conditional on the unit's survival until then [28, 8]. Expressed mathematically as $Pr(t < T \leq t + \Delta t | T > t)$. It is quite significant; it explicitly interprets the quantity of system's failure within some time frame.

The FR of the proposed HWEP model Figure 2, exhibits monotonously decreasing and increasing trends, as shown in panel (a) and (b) respectively. It also possesses variety of bathtub shapes as in panel (c) and (d). Notably, the much-needed shape of the bathtub with the extended functional life span is uniquely observed in (c), for relatively higher-ordered values of α, and τ between 0.35 and 0.75. The FR sometimes reveal an increasing-decreasing, V-pattern, as seen in (b).

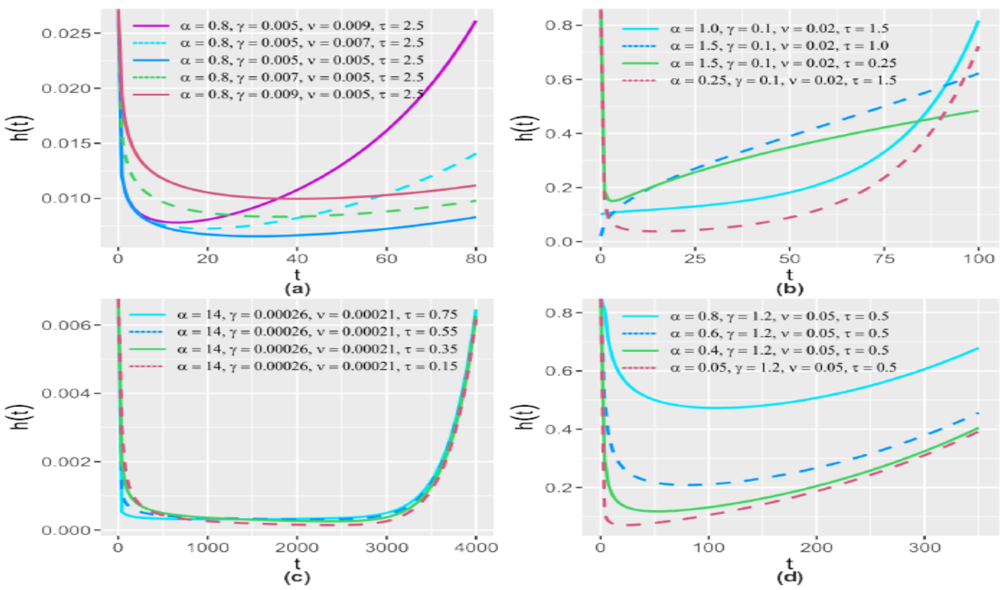

Figure 2: *Different FR Curves of the proposed HWEP model.*

- Mean Residual Life (MRL) and Mean Time-to-Failure (MTTF)

Let $X \sim HWEP$, then, at time t, the conditional statement $X - t|X > t$, gives the $X's$ residual life. Thus, MRL is the FT expected, of a system which has lived through some time t, expressed as:

$$M(t) = \mathbb{E}[X - t|X > t] = \frac{1}{R(t)} \int_0^{+\infty} R(x+t)dx,$$

$$= \frac{1}{R(t)} \int_0^{+\infty} e^{-(\gamma(x+t))^{\alpha} - e^{(\nu(x+t))^{\tau}} + 1} dx,$$

By series expansion of $e^{-e^{(\nu(x+t))^{\tau}}}$, we have

$$= \frac{1}{R(t)} \sum_{\ell,\ell=0}^{+\infty} \frac{(-1)^{\ell} \ell^{\ell} (\nu)^{\ell\tau} e^{1}}{\ell!\, \ell!} \int_0^{+\infty} (x+t)^{\ell\tau} e^{-(\gamma(x+t))^{\alpha}} dx. \quad (14)$$

We define $y = (\gamma(x+t))^{\alpha}$, so that, $(x+t)^{\ell\tau+1} = \gamma^{-(\ell\tau+1)}((\gamma(x+t))^{\alpha})^{\frac{\ell\tau+1}{\alpha}}$. Thus,

$$M(t) = \frac{1}{\alpha R(t)} \sum_{\ell,\ell=0}^{+\infty} \frac{(-1)^{\ell} (\ell)^{\ell} (\nu)^{\ell\tau} e^{1}}{\ell!\, \ell!\, \gamma^{\ell\tau+1}} \Gamma\left(\frac{\ell\tau+1}{\alpha}\right). \quad (15)$$

Where, $\Gamma(.)$, represents the gamma incomplete function, $\int_0^{+\infty} y^{\frac{\ell\tau+1}{\alpha}-1} e^{-y} dy$.

The reciprocal relationship between the FR and MRL functions can be observed in the curves in Figure 3 (a) and (b) respectively, under the same parameter settings.

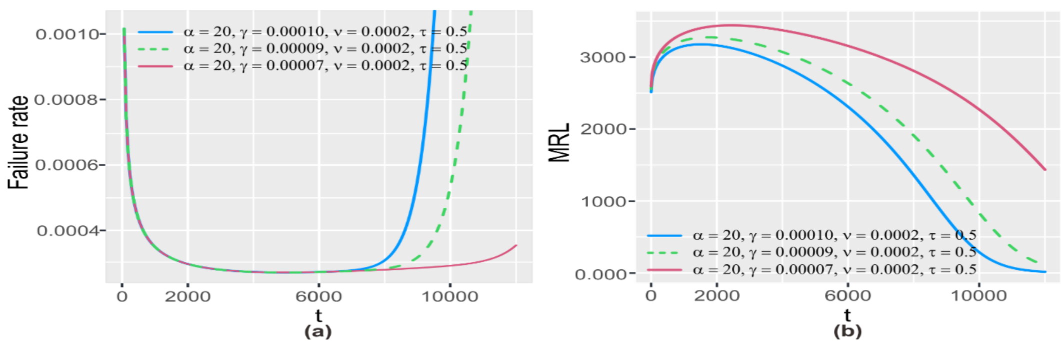

Figure 3: *FR and MRL curves of the proposed HWEP model*

Next, MTTF refers to the mean operational time of a system ahead of its failure. For HWEP unit, it is given as:

$$MTTF = \mathbb{E}[T] = \int_0^{+\infty} R(t)dt = \int_0^{+\infty} e^{-(\gamma t)^{\alpha} - e^{(\nu t)^{\tau}} + 1} dt.$$

Reference to Eq. (14)-(15), take $y = (\gamma t)^{\alpha}$ and $t^{\ell\tau+1} = \gamma^{-(\ell\tau+1)}((\gamma t)^{\alpha})^{\frac{\ell\tau+1}{\alpha}}$. Thus,

$$= \frac{1}{\alpha} \sum_{\mathscr{k},\ell=0}^{+\infty} \frac{(-1)^{\mathscr{k}}(\mathscr{k})^{\ell}(\nu)^{\ell\tau}e^{1}}{\mathscr{k}!\,\ell!\,\gamma^{\ell\tau+1}} \Gamma\left(\frac{\ell\tau+1}{\alpha}\right). \tag{16}$$

- Quantile, median and mode

The function of the proposed HWEP's quantile to be utilized for drawing random samples out of the model, including its median and mode, are highlighted in the present subsection.

Based on the CDF in Eq. (12), the HWEP model's quantile function, t_q, $q \sim U(0,1)$, is the actual value satisfying the non-linear equation below:

$$1 - ln(1-q) = (\gamma t)^{\alpha} + e^{(\nu t)^{\tau}}. \tag{17}$$

However, t_q in Eq. (17) may have no simple actual satisfying value. Therefore, it can numerically be attained out of the statement be low:

$$1 - ln(1-q) = \left(\gamma t_{q_j}\right)^{\alpha} + e^{-\left(\nu t_{q_j}\right)^{\tau}}, 0 < q_j < 1. \tag{18}$$

Below steps are required to randomly sample from the HWEP($\boldsymbol{\xi}$) based on the foregoing statement of equality:

1. Out of Uniform $(0,1)$, generate q_j, $j = 1,2, \dots, n$.
2. Assume some specific values of the HWEP parameters, and then apply the simulated q_j from above to solve for t_{q_j} in Eq. (18).

We conduct a simple simulation study to check the consistency of the quantile function in generating efficient samples. Three respective samples of size $n = 200, 1000$ and 10000 were used, each at parameter values of $\boldsymbol{\xi} = (\alpha = 14, \gamma = 0.00026, \nu = 0.00021, \tau = 0.75)'$. See section IV. From Eq. (18), the quantile (t_{q_j}) value evaluating $q = 1/2$, returns the median of the HWEP distribution.

While, the PDF of the random variable $T \sim HWEP$, $\mathbb{g}(t) = \mathscr{h}(t)R(t)$, reduces to mode of the distribution at its maximum, given the corresponding value of t. Thus, the mode is obtained as a solution to:

$$\mathscr{h}'(t)R(t) + \mathscr{h}(t)R'(t) = 0. \tag{19}$$

Where, $\mathscr{h}'(t)$ and $R'(t)$ are the corresponding first order derivatives of the FRF and RFn of the HWEP model, given as:

$$\mathscr{h}'(t) = \tau^2\nu^{\tau+1}(\nu t)^{\tau-1}t^{\tau-1}e^{-(\nu t)^{\tau}} + (\alpha-1)\alpha\gamma^{\alpha}t^{\alpha-2} + (\tau-1)\tau\nu^{\tau}t^{\tau-2}e^{-(\nu t)^{\tau}} \tag{20}$$

$$R'(t) = \left[-\alpha\gamma^{\alpha}t^{\alpha-1} - \tau\nu^{\tau}t^{\tau-1}e^{(\nu t)^{\tau}}\right]e^{-(\gamma t)^{\alpha} - e^{(\nu t)^{\tau}} + 1}. \tag{21}$$

For the complexity nature of the derivatives in Eqs. (20) and (21), Eq. (19) appears to have no convenient rational solution, rather, we opt for a suitable numerical approach.

IV. Estimation of the Proposed HWEP Model Parameters

This section presents the maximum likelihood estimation of the proposed HWEP model's unknown parameters using the maximum likelihood procedure.

Suppose the size n random sample, $t_1, t_2, \dots, t_n$, be the observed values from the HWEP model with unknown parameter vector $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$. Then, we start by defining the likelihood function from the PDF in Eq. (13) as:

$$\ell(T|\alpha,\gamma,\nu,\tau) = \prod_{i=1}^{n}\left[\alpha\gamma^{\alpha}\,t_i^{\alpha-1} + \tau\nu^{\tau}t_i^{\tau-1}e^{(\nu t_i)^{\tau}}\right]\exp\left\{-\sum_{i=1}^{n}\left[(\gamma t_i)^{\alpha} + e^{(\nu t_i)^{\tau}} - 1\right]\right\}. \tag{22}$$

Defining the natural logarithm of (22), yields,

$$\ell(\boldsymbol{\xi}) = \sum_{i=1}^{n} ln\left[\alpha\gamma^{\alpha}\, t_i^{\alpha-1} + \tau\nu^{\tau} t_i^{\tau-1} e^{(\nu t_i)^{\tau}}\right] - \sum_{i=1}^{n}\left[(\gamma t_i)^{\alpha} + e^{(\nu t_i)^{\tau}} - 1\right]. \quad (23)$$

Now, to get the MLEs, for the unknown statistics of HWEP model, $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$, the function of the log-likelihood, $\ell(\boldsymbol{\xi})$, is maximized by partially determining the first order derivatives of Eq. (23) apropos of each of the statistics, α, γ, ν *and* τ. In that respect, the components of the vector, $\mathcal{M}(\boldsymbol{\xi})$, are obtained as:

$$\mathcal{M}(\boldsymbol{\xi}) = \begin{pmatrix} \mathcal{M}_{\alpha}(\boldsymbol{\xi}) \\ \mathcal{M}_{\gamma}(\boldsymbol{\xi}) \\ \mathcal{M}_{\nu}(\boldsymbol{\xi}) \\ \mathcal{M}_{\tau}(\boldsymbol{\xi}) \end{pmatrix}$$

i.e,

$$\mathcal{M}_{\alpha}(\boldsymbol{\xi}) = \frac{\partial \ell(\boldsymbol{\xi})}{\partial \alpha} = \sum_{i=1}^{n} \frac{\alpha(\alpha-1)\gamma^{\alpha-1}\, t_i^{\alpha-2}}{\alpha\gamma^{\alpha}\, t_i^{\alpha-1} + \tau\nu^{\tau} t_i^{\tau-1} e^{(\nu t_i)^{\tau}}} - \sum_{i=1}^{n} \alpha(\gamma t_i)^{\alpha-1} = 0,$$

$$\mathcal{M}_{\gamma}(\boldsymbol{\xi}) = \frac{\partial \ell(\boldsymbol{\xi})}{\partial \gamma} = \sum_{i=1}^{n} \frac{\alpha^2\gamma^{\alpha-1}\, t_i^{\alpha-1}}{\alpha\gamma^{\alpha}\, t_i^{\alpha-1} + \tau\nu^{\tau} t_i^{\tau-1} e^{(\nu t_i)^{\tau}}} - \sum_{i=1}^{n} \alpha(\gamma t_i)^{\alpha-1} t_i = 0,$$

$$\mathcal{M}_{\nu}(\boldsymbol{\xi}) = \frac{\partial \ell(\boldsymbol{\xi})}{\partial \nu} = \sum_{i=1}^{n} \frac{\tau^2(\tau-1) t_i^{2} (\nu t_i)^{\tau-1} e^{(\nu t_i)^{\tau}}}{\alpha\gamma^{\alpha}\, t_i^{\alpha-1} + \tau\nu^{\tau} t_i^{\tau-1} e^{(\nu t_i)^{\tau}}} - \sum_{i=1}^{n} \tau t_i (\nu t_i)^{\tau-1} e^{(\nu t_i)^{\tau}} = 0,$$

$$\mathcal{M}_{\tau}(\boldsymbol{\xi}) = \frac{\partial \ell(\boldsymbol{\xi})}{\partial \tau} = \sum_{i=1}^{n} \frac{\tau^2\nu^{\tau-1}(\tau-1)(\nu t_i)^{\tau-1} t_i^{\tau-2} e^{(\nu t_i)^{\tau}}}{\alpha\gamma^{\alpha}\, t_i^{\alpha-1} + \tau\nu^{\tau} t_i^{\tau-1} e^{(\nu t_i)^{\tau}}} - \sum_{i=1}^{n} \tau(\nu t_i)^{\tau-1} e^{(\nu t_i)^{\tau}} = 0.$$

Next, working out the above together produces the MLEs, $\hat{\boldsymbol{\xi}} = (\hat{\alpha}, \hat{\gamma}, \hat{\nu}, \hat{\tau})'$. However, not easily manageable analytically. Alternatively, we adopt a numerical approach to optimize the log-likelihood function in Eq. (23) using maxlik package in R software, reference to [47], [5] among others. Notwithstanding, the MLEs of $R(t)$ and $\hbar(t)$ can be attained by the invariance property as:

$$\hat{R}(t) = \exp\left\{-(\hat{\gamma} t)^{\hat{\alpha}} - e^{(\hat{\nu} t)^{\hat{\tau}}} + 1\right\} \text{ and } \hat{\hbar}(t) = \hat{\alpha}\hat{\gamma}^{\hat{\alpha}} t^{\hat{\alpha}-1} + \hat{\tau}\hat{\nu}^{\hat{\tau}} t^{\hat{\tau}-1} e^{-(\hat{\nu} t)^{\hat{\tau}}}.$$

- Simulation Experiment for Evaluating the MLEs

For evaluating the consistency of the proposed MLEs of HWEP parameters, we adequately utilized the R function, inverseCDF(p, CDF, ...) from HDInterval, and the nlminb packages respectively. Where p is an n-size vector of probabilities q_j, j = 1, 2, …, n and the dots (…) denote the parameters to be passed to the CDF. For instance, for n = 1000, $\boldsymbol{\xi} = (\alpha = 14, \gamma = 0.00026, \nu = 0.00021, \tau = 0.75)'$. The simulation steps are: (i) pick some starting values for $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$ and K sample size; (ii) since the exact solution for Eq. (18) does not exist, then, use it to sample from the HWEP model using the R function above; (iii) use the samples generated in (ii) for optimizing the log-likelihood function in Eq. (23) and work out the estimates of $\boldsymbol{\xi}$; (iv) redo steps (i)-(iii) for n = 1000 rounds. Then, calculate the mean estimates, biases and mean squared errors (MSEs) of $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$. The results are presented in section III.

V. Reliability Relevance Proposed HWEP Model

This section demonstrates the outstanding relevance of the theoretical HWEP model in reliability context, in comparison with three AFR models based on real-life bathtub dataset presented in Table 3. The compared models by [47], [24] and [3] are respectively given in Table 2. In RStudio, the models were fitted to the dataset and the MLEs were successfully obtained by adequately utilizing Nelder Mead and BFGS optimization algorithms. The loglikelihood (ℓ) values and parametric Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (AICc), and non-parametric goodness-of-fit tests Kolmogorov-Smirnov (KS), Anderson-Darling (AD), Cramér-von Mises (CVM), are respectively used.

Table 2: *AFR Models Used for Comparison*

Models	FRF ($\hbar(t)$)
Additive Chen-Weibull (ACW)	$\lambda\gamma x^{\gamma-1}e^{x^{\gamma}}+\alpha\beta(\alpha x)^{\beta-1}$
Improved new modified Weibull (INMW)	$\alpha\theta(\alpha t)^{\theta-1}+\beta(\gamma+\lambda t)t^{\gamma-1}e^{\lambda t}$
Flexible additive Chen-Gompertz (FACG)	$\alpha\gamma x^{\gamma-1}e^{x^{\gamma}}+\lambda e^{\lambda x-\theta}$

- Time to First Failures (TTFFs) of 500 Megawatt (MW) Generators

The bathtub FR data in Table 3 comprises 36 TTFFs (1000s of hour) of 500MW generators [17]. It was recently applied [52], to evaluate the MLEs of q-Weibull model with other bathtub models using an optimization algorithm called Adaptive Hybrid Artificial Bee Colony.

Table 3: *TTFFs of 500MW Generators (in 1000s of Hours)*

0.058	0.070	0.090	0.105	0.113	0.121	0.153	0.159	0.224
0.421	0.570	0.596	0.618	0.834	1.019	1.104	1.497	2.027
2.234	2.372	2.433	2.505	2.690	2.877	2.879	3.166	3.455
3.551	4.378	4.872	5.085	5.272	5.341	8.952	9.188	11.399

III. Results

I. Simulation Study

This outcomes from the simulation experiment for evaluating the MLEs of HWEP parameters are given below, including the biases and MSEs for three distinct preset parameter values.

Table 4: *Simulation Results for evaluating MLEs*

	(α = 0.6, γ = 0.2, ν = 1.0, τ = 5.0)				(α = 0.9, γ = 0.6, ν = 0.7, τ = 0.8)				(α = 0.9, γ = 0.6, ν = 0.7, τ = 0.8)			
	Bias				Bias				Bias			
K	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\nu}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\nu}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\nu}$	$\hat{\tau}$
20	-0.0125	0.0046	-6e-04	0.0287	-0.0398	0.0319	0.0051	0.0215	-0.0015	0.0020	0.0089	0.0128
30	0.0010	0.0004	-2e-04	-0.0004	-0.0350	0.0288	0.0026	0.0190	0.0072	-0.0020	0.0094	0.0134
50	0.0000	0.0000	0.0000	0.0000	-0.0290	0.0260	-0.0010	0.0159	-0.0006	-0.0023	0.0012	0.0120
100	0.0000	0.0000	0.0000	0.0000	-0.0180	0.0177	-0.0008	0.0136	-0.0017	-0.0051	0.0088	0.0091
300	0.0000	0.0000	0.0000	0.0000	-0.0020	0.0020	-0.0002	0.0020	0.0037	0.0041	0.0023	0.0142
	MSE				MSE				MSE			
20	6e-04	1e-04	1e-04	0.0571	0.0020	0.0019	2e-04	0.0020	0.0020	0.0024	0.0023	0.0021
30	0.0000	0.0000	0.0000	0.0000	0.0017	0.0017	1e-04	0.0017	0.0022	0.0021	0.0020	0.0019
50	0.0000	0.0000	0.0000	0.0000	0.0014	0.0014	1e-04	0.0014	0.0022	0.0022	0.0018	0.0019
100	0.0000	0.0000	0.0000	0.0000	0.0009	0.0009	0.0000	0.0009	0.0020	0.0022	0.0017	0.0018
300	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0001	0.0019	0.0021	0.0016	0.0016

II. Inferences for the TTFFs of 500MW Generators

Table 5 contains values of the MLEs along with the comparison metrics for the proposed and the existing models compared. The Standard Deviations (SDs) and Asymptotic Confidence Intervals (ACIs) for the MLEs were presented in Table 6.

Table 5: *MLEs and the Comparison Metrics (p-values in parentheses) for TTFFs Dataset*

Model	Optimized parameters	ℓ	AIC	BIC	AICc	KS	AD	CVM
HWEP	$\hat{\alpha}$ = 1177.8, $\hat{\gamma}$ = 0.0877, $\hat{\nu}$ = 0.2002, $\hat{\tau}$ = 0.5861	-63.95	135.9084	142.2424	137.1987	0.0901 (0.9069)	0.4472 (0.2651)	0.0579 (0.3935)
ACW	$\hat{\alpha}$ = 0.4209, $\hat{\beta}$ = 0.7775, $\hat{\gamma}$ = 1.0702, $\hat{\lambda}$ = 1.7e-6	-67.50	142.9993	149.3333	144.2896	0.1146 (0.6893)	0.5206 (0.1735)	0.0790 (0.2067)
FACG	$\hat{\gamma}$ = 0.4196, $\hat{\alpha}$ = 0.2815, $\hat{\theta}$ = 709.82, $\hat{\lambda}$ = 62.266	-65.21	138.4264	144.7605	139.7168	0.0948 (0.8728)	0.4747 (0.2267)	0.0581 (0.3907)
INMW	$\hat{\alpha}$ = 0.4243, $\hat{\beta}$= 8.7e-12, $\hat{\gamma}$ = 0.3212, $\hat{\theta}$ = 0.7880, $\hat{\lambda}$ = 2.1648	-67.33	145.6036	153.5212	122.651	0.1098 (0.7373)	0.50043 (0.1953)	0.0742 (0.2393)

Table 6: *MLEs with 95% ACIs for the HWEP Parameters for TTFFs Dataset*

	MLE		
Parameters	Estimates	SDs	95% ACI
$\hat{\alpha}$	1177.8	11.860	[1154.5, 1201.0]
$\hat{\gamma}$	0.0877	7.4e-5	[0.0876, 0.0879]
$\hat{\nu}$	0.2002	0.0361	[0.1294, 0.2710]
$\hat{\tau}$	0.5861	0.0844	[0.4208, 0.7514]
MTTF	0.3459	0.0639	[0.2305, 0.4802]

IV. Discussion

I. Simulation Study

In Figure 4, the nonclosed quantile function in Eq. (18) is shown consistent in generating effective samples out of the proposed HWEP model. We can observe that the samples adequately replicate the bathtub FR pattern of the model as in Figure 2(c). Also, with successive increase in sample size (left to right), more data points (in the histogram) converge to the model density curve.

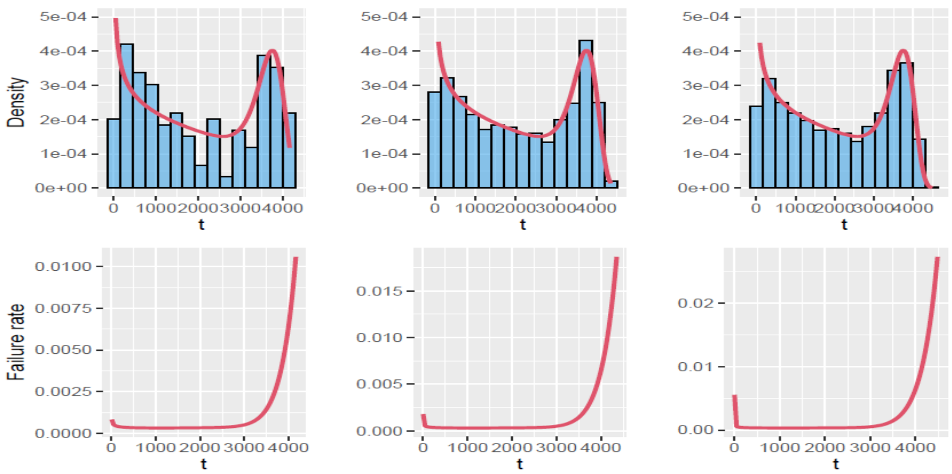

Figure 4: *Histograms of simulated density and FR curves*

While in Table 4, five different sample sizes and three distinct parameter settings were used for the MLEs performance evaluation. Negligible biases can be observed, and, with the increase in sample sizes, the biases and MSEs tend to zero. Thus, indicates estimators' consistency.

II. Relevance of the Proposed HWEP Model in System Reliability

We can see from Table 5 that, the established HWEP model outperformed the singled-out models for having the highest log-likelihood value of -63.95 and p-values of KS, AD and CVM statistics of 0.9069, 0.2651 and 0.3935 respectively. It also has the least values of AIC, BIC and AICc of 135.9084, 142.2424 and 137.1987 respectively, save for the INMW model which has the lower AICc value of 122.251. Thus, indicate better performance and fit for the dataset. This is shown graphically in Figure 5. While in Table 6, we can see that the ACIs properly contain their respective estimates, indicating satisfactorily significant estimates.

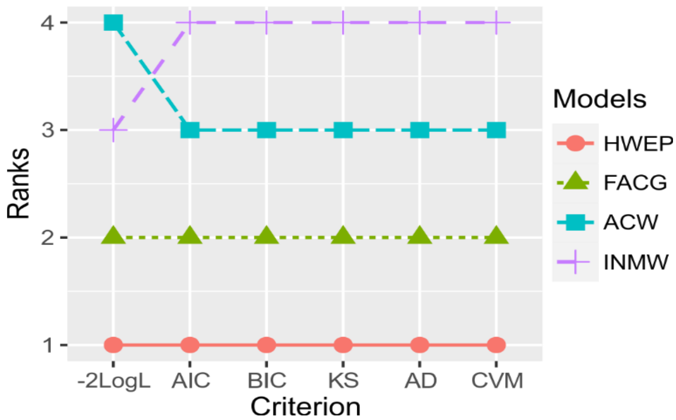

Figure 5: *Line plots showing measures' ranking for the trained models on TTFFs dataset*.

V. Conclusion

This study establishes a hybrid Weibull Exponential power model via the additive failure rate approach. Its structural formation and the significance of its reliability and statistical properties point to its relevance in actuarial reliability of multi-unit systems with bathtub, V-shaped and monotone FRs. Its parameters were estimated using MLE technique and the MLEs were shown consistent by simulation study. The outstanding performance of the model was demonstrated on real-life dataset, justified by the information criteria and goodness-of-fit statistics. Further efforts will consider Bayesian analysis and discussions on other reliability properties of the model.

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Declaration of Conflict of Interest

The Authors declare that there is no conflict of interest

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