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PROPERTIES AND APPLICATION OF ALPHA LOGARITHM TRANSFORMED SINE WEIBULL DISTRIBUTION

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Abstract

In this research, we introduced a new submodel of the alpha logarithm transform sine-X family of distributions, called the alpha logarithm transform sine-Weibull. We derived its statistical properties, including survival function, hazard function, moments, moment-generating function, and order statistics. We also estimated the parameters using the maximum likelihood method. The performance of the maximum likelihood estimators was evaluated in terms of mean, bias, and root mean squared errors through two simulation studies. Furthermore, the practicality of this family was demonstrated using COVID-19 mortality data from Saudi Arabia and milk output data sets, where the proposed model consistently provided better fits compared to other existing distributions.

Keywords: Alpha Logarithm Transformed Sine-X family of Distributions, Weibull Distribution, Properties, Simulation, Application

1. INTRODUCTION

Modeling lifetime data is an important concept in statistics, as many existing distributions do not fit the data well, which can lead to unreliable results in various fields, including engineering, economics, biological science, and mathematics (1). This limitation has attracted researchers to introduce more flexible and versatile distributions that can serve as substitutes for current baseline distributions (3). This approach provides a framework for introducing flexible distributions along with their characterization and properties, including entropy, quantile, moment, moment-generating function, survival function, hazard rate function, and order statistics. Estimation of parameters is a crucial aspect, involving the use of maximum likelihood and other estimation methods.

In some cases, existing distributions can handle the task of fitting real data sets; however, due to certain limitations, it may be necessary to modify or enhance these distributions to make them more versatile than in their initial form. (4) examines some existing distributions and potential improvements. For example, the Rayleigh-Pareto distribution was modified using the arcsine function to create the arcsine Rayleigh-Pareto distribution (7), while the sine-Weibull distribution was explored by (18), the gamma-extended Weibull by (9), and a new alpha-power cosine Weibull was introduced by (10), among others.

The Weibull distribution is a continuous probability distribution first discovered by (17). Although it was not widely recognized until the work of (12), it found real-life applications through the efforts of (13). The Weibull distribution is particularly applicable in situations where a skewed distribution is necessary to model data that require a heavy-tailed distribution. In such instances, the Weibull distribution is one of the appropriate options to provide accurate and reliable results.

A generalized family of distributions known as the alpha power transformation distribution was introduced by (14). This family increases the flexibility of existing models by adding a parameter. Consequently, the alpha power family and the sine-Weibull distribution can be combined to create the alpha logarithm transform sine-Weibull distribution.

The need to introduce a new probability distribution arises in various fields, including biological science, engineering, and economics, when current probability distributions do not adequately fit lifetime data that may be positively or negatively skewed. The current sine-Weibull distribution (18), which has only two parameters, may not be flexible enough to meet the requirements of standard distributions. Therefore, there is a need to extend it by adding another parameter using logarithmic transformation to enhance the flexibility of the existing sine-Weibull distribution.

2. Methods

In this section, we will discuss the proposed model, its validity check, mathematical properties, and parameter estimation.

2.1. Alpha Logarithm Transform-G Distribution (ALT)

The alpha logarithm transformed method can be obtained using the following procedure: let (G(x)) be a continuous distribution function with probability density function (g(x)). Then, (F(x)), the cumulative distribution function (CDF) of the alpha logarithm transformed family of distributions proposed by Pappas et al. (2012), takes the following form.

$$F_{ALT}(x) = 1 - \frac{log(\alpha - (\alpha - 1)G(x))}{log(\alpha)} \quad \alpha > 0 and \alpha \neq .1 \tag{1}$$

$$f_{ALT}(x) = \frac{(\alpha - 1)g(x)}{(\alpha - (\alpha - 1)G(x))log(\alpha)} \tag{2}$$

Where, G(x) is the baseline cumulative distribution function. g(x) is the corresponding probability density function and α is the shape parameter.

2.2. Weibull distribution

The weibull distribution is a two parameters continuous probability distribution introduced by If T has a weibull distribution (17), with CDF

$$F(x) = \left(1 - e^{-(\frac{x}{\lambda})^{\beta}}\right) \tag{3}$$

and PDF

$$f(x) = \frac{x^{\beta-1}}{\lambda^{\beta}} e^{-(\frac{x}{\lambda})^{\beta}} \left(1 - e^{-(\frac{x}{\lambda})^{\beta}}\right) \tag{4}$$

respectively, where $\beta > 0$ is the scale parameter and $\lambda > 0$ is the shape parameter. Weibull distribution was used by many researchers as a baseline distribution including sine weibull distribution by (18), Topp-leone weibull distribution by (19), and alpha power topp-leone weibull distribution by (20) among others.

2.3. Alpha Logarithm Transformed Sine Weibull Distribution (ALTSW-D)

Let x follows Alpha logarithm transformed sine weibull random variable with cumulative density function given as

$$F_{ALSW}(x,\alpha,\beta,\lambda) = 1 - \frac{log\left(\alpha - \bar{\alpha}Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)}{log(\alpha)} \tag{5}$$

And corresponding probability density function (pdf). Given as;

$$f_{ALSW}(x,\alpha,\beta,\lambda) = \frac{\bar{\alpha}\pi\lambda\beta x^{-\lambda x^{\beta}}Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2(\alpha-\bar{\alpha})log(\alpha)Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)} \tag{6}$$

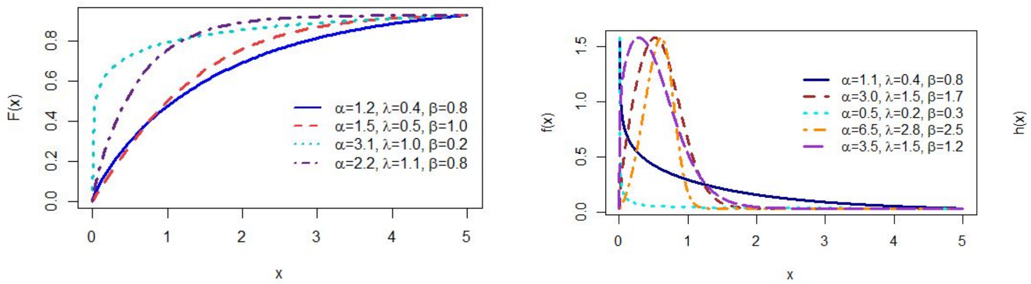

Figure 1: *CDF and PDF plot of ALTSW-D for some parameters*

Figures 1 display the PDF and CDF charts for the ALTSW-D. The risk shows growing and uni-model forms, and the PDF is right-skewed.

2.3.1 Linear representation of ALTSW-D

The probability density function of the proposed distribution is reduced as follows,

$$f_{ALSW}(x,\alpha,\beta,\lambda) = \frac{\bar{\alpha}\pi\lambda\beta x^{-\lambda x^{\beta}}e^{-\lambda x^{\beta}}Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2(\alpha-\bar{\alpha})log(\alpha)Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)} \tag{7}$$

Using power series expansion

$$e^{-\lambda x^{\beta}} = \sum_{i=0}^{\infty}\frac{(-1)^i\lambda x^{\beta i}}{i!} \tag{8}$$

Substituting equation (8) into (7), we have

$$f(x) = f_{ALSW}(x) = \sum_{i=0}^{\infty}\frac{\bar{\alpha}\pi\lambda^{i+1}\beta x^{\beta i+\beta-1}(-1)^i Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2i!(\alpha-\bar{\alpha})log(\alpha)Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)} \tag{9}$$

Let

$$\psi = \frac{\pi\beta\bar{\alpha}\lambda^{i+1}}{2i!log(\alpha)} \tag{10}$$

$$\Rightarrow f(x) = \sum_{i=0}^{\infty} \frac{\psi x^{\beta i+\beta-1 Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}}{\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)} \tag{11}$$

Also let

$$\gamma = \frac{x^{\beta i+\beta-1} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)} \tag{12}$$

$$\Rightarrow f(x) = \sum_{i=0}^{\infty} \gamma\psi \tag{13}$$

2.4. Mathematical Properties of ALTSW-D

In this part, some properties of the proposed distribution was derived including quantile function, survival function, hazard rate function, moment, moment generating function, order statistics and estimation of parameters using maximum likelihood method.

2.4.1 Quantile function of ALTSW-D

The quantile function is the inverse of cumulative density function use for simulation study. The quantile function of ALTSW-D were derived as follows Let

$$u = F_{ALSW}(x) = 1 - \frac{log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)}{log(\alpha)} \tag{14}$$

$$\Rightarrow u = 1 - \frac{log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)}{log(\alpha)} \tag{15}$$

$$\Rightarrow 1 - u = \frac{log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)}{log(\alpha)} \tag{16}$$

By cross multiply, we have

$$\Rightarrow (1-u) log(\alpha) = log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right) \tag{17}$$

$$\Rightarrow log(\alpha)^{1-u} = log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right) \tag{18}$$

Taking the exponential of both sides

$$\Rightarrow e^{log(\alpha)^{1-u}} = e^{log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)} \tag{19}$$

$$\Rightarrow (\alpha)^{1-u} = \alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right) \tag{20}$$

By collecting like terms, we have

$$\Rightarrow \alpha - (\alpha)^{1-u} = \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right) \tag{21}$$

Dividing equation (21) by $\bar{\alpha}$

$$\Rightarrow \frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}} = Sin\left(\frac{\pi(1 - e^{-\lambda x^{\beta}})}{2}\right) \tag{22}$$

$$\Rightarrow Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right) = \frac{\pi(1 - e^{-\lambda x^{\beta}})}{2} \tag{23}$$

Multiply both sides by $\frac{2}{\pi}$

$$\Rightarrow 1 - e^{-\lambda x^{\beta}} = \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right) \tag{24}$$

$$\Rightarrow e^{-\lambda x^{\beta}} = 1 - \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right) \tag{25}$$

Taking the natural log of both sides of equation (25), we have

$$\Rightarrow log e^{-\lambda x^{\beta}} = log\left(1 - \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right)\right) \tag{26}$$

$$\Rightarrow -\lambda x^{\beta} = log\left(1 - \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right)\right) \tag{27}$$

Also, divide both sides by $-\lambda$

$$\Rightarrow x^{\beta} = \frac{-1}{\lambda} log\left(1 - \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right)\right) \tag{28}$$

Taking power of $\frac{1}{\beta}$ to both sides of equation (28)

$$\Rightarrow (x^{\beta})^{\frac{1}{\beta}} = \left\{\frac{-1}{\lambda} log\left(1 - \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right)\right)\right\}^{\frac{1}{\beta}} \tag{29}$$

$$\Rightarrow x_u = \left\{\frac{-1}{\lambda} log\left(1 - \frac{2}{\pi} Arcsine\left(\frac{\alpha - (\alpha)^{1-u}}{\bar{\alpha}}\right)\right)\right\}^{\frac{1}{\beta}} \tag{30}$$

2.4.2 Survival and Hazard function of ALTSW-D

A random variable $X \sim ALTSW - D$ has the survival and hazard rate function given as;

$$S(x) = \frac{log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)}{log(\alpha)} \tag{31}$$

$$H(x) = \frac{\bar{\alpha}\pi\lambda\beta x^{-\lambda x^{\beta}} e^{-\lambda x^{\beta}} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right) log(\alpha)}{2(\alpha - \bar{\alpha}) log(\alpha) Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right) log\left(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)} \tag{32}$$

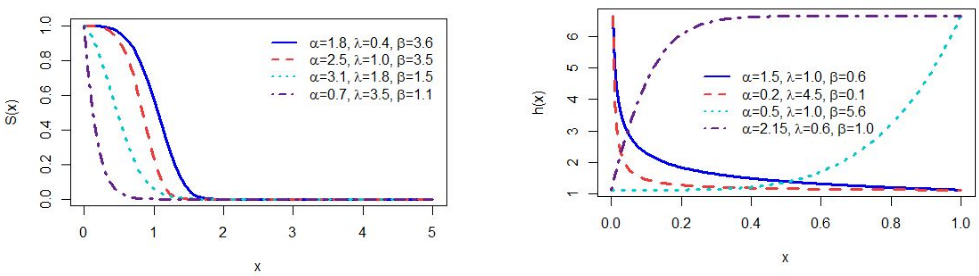

Figure 2: *Hazard rate and Survival plots of ALTSW-D for different values of parameters*

2.4.3 Renyi's Entropy of ALTSW-D

Renyi entropy forms the basis of the concept of generalized dimensions, and is given by;

$$I(x) = \frac{1}{1-\theta} log \int_{-\infty}^{\infty} f(x)^{\theta} dx \tag{33}$$

$$\Rightarrow I(x) = \frac{1}{1-\theta} log \int_{0}^{\infty} \left(\frac{\bar{\alpha}\pi\lambda\beta x^{\beta-1} e^{-\lambda x^{\beta}} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)) log(\alpha)} \right)^{\theta} dx \tag{34}$$

$$\Rightarrow I(x) = \frac{1}{1-\theta} log \int_{0}^{\infty} (\omega\epsilon)^{\theta} dx \tag{35}$$

$$\Rightarrow I(x) = \frac{1}{1-\theta} log\omega^{\theta} + \int_{0}^{\infty} (\epsilon)^{\theta} dx \tag{36}$$

$$\Rightarrow I(x) = \frac{1}{1-\theta} \left[\theta log\omega + \int_{0}^{\infty} (\epsilon)^{\theta} dx \right] \tag{37}$$

Where

$$\omega = \frac{\bar{\alpha}\pi\lambda\beta}{2(\alpha - \bar{\alpha}) log(\alpha)} \tag{38}$$

And

$$\epsilon = \frac{x^{\beta-1} e^{-\lambda x^{\beta}} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right))} \tag{39}$$

2.4.4 Moment of ALTSW-D

The r^{th} moment about the origin of the Proposed distribution "alpha logarithm transformed sine weibull distribution" (ALSW-D) is given by.

$$\mu^{r} = E(x^{r}) = \int_{-\infty}^{\infty} x^{r} f(x) dx \tag{40}$$

$$\mu^{r} = \int_{0}^{\infty} \frac{x^{r}\bar{\alpha}\pi\lambda\beta x^{\beta-1} e^{-\lambda x^{\beta}} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)) log(\alpha)} dx \tag{41}$$

Using power series expansion,

$$e^{-\lambda x^{\beta}} = \sum_{i=1}^{\infty} \frac{(-1)^{i} \lambda^{i} x^{\beta i}}{i!} \tag{42}$$

This implies that equation (41) become

$$\mu^{r} = \sum_{i=1}^{\infty} \int_{0}^{\infty} \frac{\bar{\alpha}\pi\lambda^{i+1}\beta x^{\beta i+\beta+r-1} (-1)^{i} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2i!(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)) log(\alpha)} dx \tag{43}$$

$$\mu^{r} = \sum_{i=1}^{\infty} \gamma\kappa \tag{44}$$

Where

$$\gamma = \int_{0}^{\infty} \frac{x^{\beta i+\beta+r-1} Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{(\alpha - \bar{\alpha} Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right))} dx \tag{45}$$

And

$$\kappa = \frac{\bar{\alpha}\pi\lambda^{i+1}\beta(-1)^i}{2i!log(\alpha)} \tag{46}$$

2.4.5 Moment Generating Function of ALTSW-D

In Statistics and Probability Theorem, the Moment Generating function is used to encodes the information about the moments of a random variable. and is given as;

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{47}$$

$$\Rightarrow M_x(t) = \int_0^{\infty} \frac{e^{tx}\bar{\alpha}\pi\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2(\alpha - \bar{\alpha}Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right))log(\alpha)}dx \tag{48}$$

But

$$e^{-\lambda x^{\beta}} = \sum_{j=1}^{\infty} \frac{(-1)^j\lambda^j x^{\beta j}}{j!} \tag{49}$$

And

$$e^{tx} = \sum_{p=1}^{\infty} \frac{t^p x^p}{p!} \tag{50}$$

Substituting (50) and (49), in to (48), we have

$$M_x(t) = \sum_{j=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\infty} \frac{\bar{\alpha}\pi\lambda^{j+1}\beta(-1)^j t^p x^{\beta+\beta j+p-1}Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{2j!p!\left(\alpha - \bar{\alpha}Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)log(\alpha)}dx \tag{51}$$

$$\Rightarrow M_x(t) = \sum_{j=1}^{\infty}\sum_{p=1}^{\infty} \tau\eta \tag{52}$$

Where

$$\tau = \int_0^{\infty} \frac{x^{\beta+\beta j+p-1}Cos\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)}{\left(\alpha - \bar{\alpha}Sin\left(\frac{\pi(1-e^{-\lambda x^{\beta}})}{2}\right)\right)}dx \tag{53}$$

And

$$\eta = \frac{\bar{\alpha}\pi\lambda^{j+1}\beta(-1)^j t^p}{2j!p!log(\alpha)} \tag{54}$$

2.4.6 Order Statistics of ALTSW-D

The order statistics of a random observation $X_{1,n} < X_{2,n} < ... < X_{n,n}$ from continues data with probability density function g(x) and cumulative distribution function G(x). The CDF and PDF of $X_{i,n}$ is defined as;

$$f_{(i,n)}(x) = \frac{n!}{(i-1)(n-i)} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{55}$$

Now using power series expansion:

$$[1-F(x)]^{n-1} = \sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^j \tag{56}$$

It implies that equation (55) become:

$$f_{(i,n)}(x) = \frac{n!}{(i-1)(n-i)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x)[F(x)]^{i+j-1} \tag{57}$$

$$f_{(i,n)}(x) = \frac{n!(-1)^j}{(i-1)(n-i-j)} \sum_{j=0}^{n-i} \binom{n-i}{j} f(x)[F(x)]^{i+j-1} \tag{58}$$

$$f_{i,n}(x) = \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{n!\bar{\alpha}\pi\lambda\beta(-1)^j x^{-\lambda x^\beta} Cos\left(\frac{\pi(1-e^{-\lambda x^\beta})}{2}\right)}{2(i-1)!(n-i-j)!j!\left(\alpha - \bar{\alpha}Sin\left(\frac{\pi(1-e^{-\lambda x^\beta})}{2}\right)\right) log(\alpha)} \times$$

$$\left[1 - \frac{log\left(\alpha - \bar{\alpha}Sin\left(\frac{\pi(1-e^{-\lambda x^\beta})}{2}\right)\right)}{log(\alpha)}\right]^{i+j-1} \tag{59}$$

2.5. Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$, be a random sample of size "n" from ALTSW distribution then the likelihood function L is given as;

$$l = logl(x/\bar{\theta}) = log \prod_{i=1}^{n} f(x/\bar{\theta}) \tag{60}$$

$$l(\bar{\theta}) = \prod_{i=1}^{n} \frac{\bar{\alpha}\pi\lambda\beta x^{-\lambda x^\beta} Cos\left(\frac{\pi(1-e^{-\lambda x^\beta})}{2}\right)}{2(\alpha - \bar{\alpha})log(\alpha)Sin\left(\frac{\pi(1-e^{-\lambda x^\beta})}{2}\right)} \tag{61}$$

$$= nlog\pi + nlog\lambda + nlog\beta - nlog2 - nlog(log\alpha) - nlog\alpha + 2nlog\bar{\alpha} + \sum_{i=1}^{\infty} logx^{\beta-1} -$$

$$\lambda \sum_{i=1}^{\infty} x^\beta + \sum_{i=1}^{\infty} logCos\frac{\pi}{2}(1 - e^{-\lambda x^\beta}) + \sum_{i=1}^{\infty} logSin\frac{\pi}{2}(1 - e^{-\lambda x^\beta}) \tag{62}$$

$$\frac{\delta l}{\delta \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{\infty} \frac{\pi x^\beta e^{-\lambda x^\beta} Sin\frac{\pi}{2}(1 - e^{-\lambda x^\beta})}{2Cos\frac{\pi}{2}(1 - e^{-\lambda x^\beta})} - \sum_{i=1}^{\infty} \frac{\pi x^\beta e^{-\lambda x^\beta} Cos\frac{\pi}{2}(1 - e^{-\lambda x^\beta})}{2Sin\frac{\pi}{2}(1 - e^{-\lambda x^\beta})} \tag{63}$$

$$\frac{\delta l}{\delta \beta} = \frac{n}{\beta} + \sum_{i=1}^{\infty} logx - \lambda\beta \sum_{i=1}^{\infty} x^{\beta-1} - \sum_{i=1}^{\infty} \frac{\pi\lambda x^\beta e^{-\lambda x^\beta} e^{-\lambda x^\beta} logxSin\frac{\pi}{2}(1-e^{-\lambda x^\beta})}{2Cos\frac{\pi}{2}(1-e^{-\lambda x^\beta})} -$$

$$\sum_{i=1}^{\infty} \frac{\pi\lambda x^\beta e^{-\lambda x^\beta} e^{-\lambda x^\beta} logxCos\frac{\pi}{2}(1 - e^{-\lambda x^\beta})}{2Sin\frac{\pi}{2}(1 - e^{-\lambda x^\beta})} \tag{64}$$

$$\frac{\delta l}{\delta \alpha} = \frac{2n}{\bar{\alpha}} - \frac{n}{\alpha} - \frac{n}{(log\alpha)^2} \tag{65}$$

3. Results

This section will introduce the simulation study designed to examine the behavior and consistency of maximum likelihood estimation, as well as the application of real data to observe the performance of the proposed model and other competing models.

3.1. Simulation

The behavior of the maximum likelihood of ALTSW-D for certain parameter values in the first trial (i.e. $\alpha = 0.8$, $\lambda = 2.5$, $\omega = 3.4$, $\gamma = 4.5$) and second trial (i.e. $\alpha = 1.5$, $\lambda = 1.3$, $\omega = 3.8$,$\gamma = 2.8$) was investigated using a created finite sample of size n=20, 20, 100, 250, 500, and 1000. The random numbers for the ALTSW-D were generated using the quantile function. For 1000 repeats. The Means, Bias, and RMSE were then calculated. Table 1 presents the outcomes of the simulation. We concluded that the proposed model yields consistent results when predicting parameters for the mode based on the results of the Monte Carlo simulation.

Table 1: *Result of simulation for different values of parameters*

		$\alpha = 1.4$	$\beta = 1.1$	$\lambda = 0.7$	$\alpha = 0.8$	$\beta = 0.5$	$\lambda = 1.0$
n=20	Means	1.7016	1.2342	0.7151	0.9404	0.5764	1.0616
	Bias	1.7016	1.2342	0.7151	0.1404	0.0764	0.0616
	RMSE	1.7339	0.3272	0.2263	1.0495	0.1719	0.3239
n=50	Means	1.6060	1.1514	0.7073	0.9304	0.5284	1.0227
	Bias	0.2060	0.0514	0.0073	0.1304	0.0284	0.0227
	RMSE	1.2910	0.1794	0.1570	0.8533	0.0889	0.2198
n=100	Means	1.5276	1.1273	0.7015	0.8846	0.5154	1.0081
	Bias	0.1276	0.0273	0.0015	0.0846	0.0154	0.0081
	RMSE	0.9281	0.1224	0.1174	0.6152	0.0605	0.1597
n=250	Means	1.5386	1.1091	0.7044	0.9043	0.5056	1.0287
	Bias	0.1386	0.0091	0.0044	0.1043	0.0056	0.0087
	RMSE	0.7772	0.0809	0.0905	0.5350	0.0419	0.1331
n=500	Means	1.4982	1.1051	0.7013	0.8671	0.5024	1.0041
	Bias	0.0982	0.0051	0.0013	0.0671	0.0024	0.0041
	RMSE	0.6628	0.0627	0.0804	0.3994	0.0311	0.1029
n=1000	Means	1.4913	1.1000	0.7054	0.8476	0.5008	1.0051
	Bias	0.0913	0.0000	0.0054	0.0476	0.0008	0.00513
	RMSE	0.5219	0.0483	0.0640	0.3058	0.0238	0.0810

3.2. Application 1

This data set consists of the total milk output from the first calves of 107 SINDI race cows. The data was used in (15). The data set is included0.729, 0.2361, 0.514, 0.5232, 0.4438, 0.4143, 0.3821, 0.0168, 0.7261,0.4151, 0.5529, 0.535, 0.2303, 0.5744, 0.686, 0.6789, 0.675, 0.2356,0.5285, 0.6844, 0.4564, 0.5483, 0.7471, 0.48, 0.6196, 0.6012, 0.7804,0.065, 0.577, 0.4049, 0.6174, 0.3906, 0.5349, 0.4553, 0.4694, 0.3175,0.4823, 0.1479, 0.5853, 0.4371, 0.3188, 0.499, 0.3945, 0.5481,0.8147, 0.6488, 0.348, 0.4612, 0.7131, 0.5707, 0.3635, 0.5113,0.0671, 0.4741, 0.2681, 0.3627, 0.4517, 0.426, 0.6114, 0.6058,0.5878, 0.4365, 0.6907, 0.7629, 0.5912, 0.6927, 0.4576, 0.6707,0.7687, 0.4332, 0.5553, 0.1546, 0.8492, 0.3383, 0.3751, 0.3134,0.6768, 0.1525, 0.1167, 0.447, 0.5394, 0.0609, 0.515, 0.4752, 0.3323, 0.1131, 0.453, 0.3598, 0.622, 0.8781, 0.216, 0.4111, 0.5629, 0.3259,0.2747, 0.5627, 0.3406, 0.0776, 0.5447, 0.5941, 0.2605, 0.4675,0.3891, 0.6891, 0.0854, 0.3413, 0.6465

Table 2: *Information criteria measure for the fitted models using milk output data*

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{LL}$	AIC	CAIC	BIC	HQIC
WEW	100.00	10.03	0.19	71.59	82.58	173.15	174.29	179.91	175.59
BW	0.13	27.80	19.34	10.51	77.85	163.69	164.84	170.46	166.14
GEW	18.64	2.09	19.11	100.23	96.34	200.67	201.81	207.43	203.11
KwW	0.48	70.75	7.97	20.89	82.38	172.76	173.89	179.51	175.19
ALSW	1.00e+02	6.60e-03	2.81e+00	—	79.96	165.93	166.59	170.99	167.76

Table 3: *Goodness of fit measure for the fitted models using milk output data*

Distribution	KS	A	W	P-value
WEW	0.1085	0.5889	0.0793	0.7346
BW	0.0671	0.1010	0.0135	0.9937
GEW	0.1777	2.7989	0.4717	0.16
KwW	0.1078	0.5598	0.0750	0.7407
ALSW	0.0824	0.2391	0.0296	0.9485

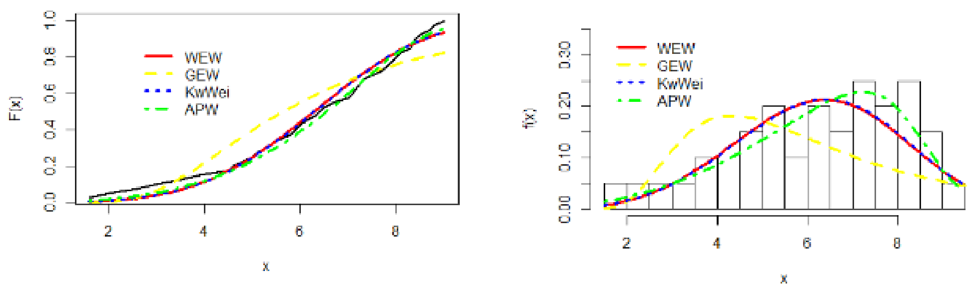

Figure 3: *Fitted PDF and CDF from first data sets*

3.3. Application 2

The first data set contains the COVID-19 mortality rate associated with Saudi Arabia during the time between July 6, 2021 and September 4, 2021. The data set contains 61 observations, and it has been used by (16). The data are as follows: 0.3086, 0.3283, 0.2865, 0.2450, 0.2852, 0.3251, 0.2636, 0.3236, 0.2824, 0.2817, 0.3012, 0.2603, 0.2997, 0.2393, 0.2785, 0.2778, 0.2375, 0.2963, 0.2167, 0.2752, 0.2353, 0.2347, 0.1951, 0.2140, 0.2329, 0.2711, 0.2126, 0.2314, 0.1924, 0.2113, 0.2683, 0.2487, 0.2674, 0.1716, 0.2666, 0.2091, 0.2278, 0.1706, 0.2271, 0.1890, 0.2077, 0.2452, 0.1319, 0.2259, 0.1504, 0.1879, 0.1689, 0.2063, 0.2249, 0.1686, 0.1310, 0.1497, 0.1309, 0.1495, 0.1121, 0.1121, 0.1307, 0.1120, 0.1306, 0.1492, 0.0932.

Table 4: *Information criteria measure for the fitted models using Covid-19 mortality data*

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{LL}$	AIC	CAIC	BIC	HQIC
WEW	100.00	9.45	0.12	79.92	-20.98	-33.95	-33.56	-23.26	-29.62
BW	-47.43	-47.04	-36.74	-43.09	-27.76	0.24	103.01	7.06	1.42
GEW	32.62	33.01	43.31	36.95	12.3076	21.42	0.06	0.61	78.93
KwW	0.58	78.31	4.45	2.84	-21.5449	-35.09	-34.69	-24.39	-30.76
ALSW	100.00	5.04	1.82	—	79.96	-48.93	-48.69	-40.91	-45.67

Table 5: *Goodness of fit measure for the fitted models using covid-19 mortality data*

Distribution	KS	A	W	P-value
WEW	0.0852	1.5891	0.2426	0.4184
BW	0.0832	0.5244	0.0829	0.4493
GEW	0.1872	7.2646	1.2791	0.0011
KwW	0.0834	1.4961	0.2276	0.4456
ALSW	0.0597	0.4988	0.0725	0.8396

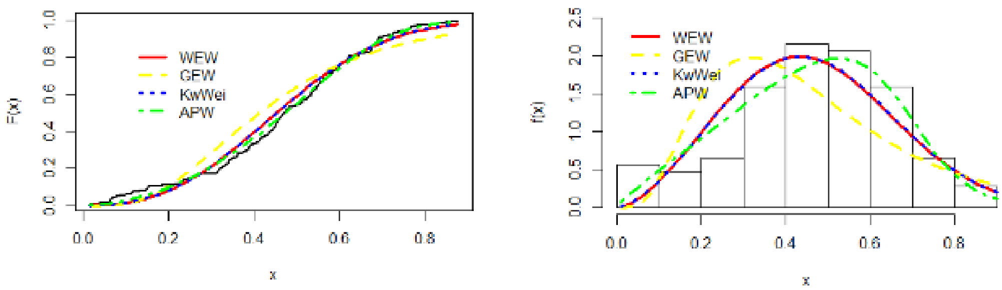

Figure 4: *Fitted PDF and CDF from second data sets*

3.4. Discussion

The analysis of the alpha logarithm transform sine Weibull distribution demonstrated its efficiency in modeling real-life data. This new distribution outperformed its competitors, as indicated by smaller values of information criteria (2), as shown in Tables 2 and 6. The simulation results in 1 indicated that the model's performance improves as the sample size increases in both the first and second trials. The fitted CDF and PDF plots from the first and second applications illustrate the flexibility of the proposed distribution compared to existing distributions.

Conclusion

This paper introduces a new distribution called the alpha power logarithm transform sine Weibull distribution. The mathematical properties were derived, and the parameter estimation of the new distribution was examined using the maximum likelihood method. The behavior of the maximum likelihood estimation was also investigated to assess consistency. In conclusion, we suggest that the proposed distribution performs better than existing distributions in terms of information criteria and goodness-of-fit tests.

Author's contribution

Abdulhameed Ado Osi: R-Coding and analysis of models performance
Olawoyin Olatunji Ishaq: Conceptualization and review
Bilal Alhassan Abdulsalam: typing and compilation
Usman Abubakar: Latex coding, manipulation, and results imputation.

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Conflict of interest:

The authors did not encounter any conflicts of interest during this work.

Ethical Consideration:

No ethical clearance is required before the commencement of this research.

Data availability statement:

The data used in this study are available in the manuscript.

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