CERTAIN STATIONARY POINT OUTCOMES FOR ASYMPTOTICALLY REGULAR MAPS IN INTERVAL-VALUED N-FUZZY METRIC SPACE

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Abstract

In this study, we introduce interval valued N-fuzzy metric space (IVNFM) and we verify certain popular fixed point results on the structure of interval valued N-fuzzy metric space through asymptotically regular mappings. Fixed point theorems are crucial in various mathematical and applied fields, providing solutions to equations where a function maps a point to itself. In this research, we focus on applying diverse contractive conditions specifically in the context of asymptotically regular mappings to achieve our objectives. By employing this type of mapping, we are able to verify and validate several well-established fixed point results within IVNFM. This approach not only supports the known results but also broadens their applicability, thereby enhancing the theoretical foundation of fixed point theorems in fuzzy metric spaces.

Keywords: Interval-valued N-fuzzy metric, common fixed point, asymptotically regular maps, altering distance function, commutative self mappings.

I. Introduction

The foundation of fuzzy mathematics is laid by [25] in 1965. In 1975, [10] proposed the idea of fuzzy metric space. [14] employed fuzzy metric space membership functions in 1983 to determine the fuzzy metric between two fuzzy sets; this approach differed from that suggested by [10]. Fuzzy metric space was reformulated in 1994 when [3] presented the modified concept of Continuous t-norm. Letter on, numerous authors and researchers generalized various type of metric spaces like fuzzy 2-metric space, D-metric space, G-metric space-metric space, Q(G)-metric space, D*-metric space M-metric spaces-metric space and have thoroughly and extensively examined a variety of issues pertaining to this space from a variety of perspectives, including compatible mapping, R-weakly computing mapping, Weak compatible mapping, CLR-Property, E.A. property, etc. and have produced new findings on fuzzy metric space. Based on an interval valued membership function, [25] first the idea of an interval-valued fuzzy set in 1975. By inspiring the concept of compatible maps, [6] presented the idea of compatible maps of type (A) in metric space and Banach space in 1993. Three different types of distances between two interval-valued fuzzy sets on real line R were employed by [11] in 2009. Using the concept of continuous interval-valued t-norm, it is possible to define interval-valued fuzzy metric space and describe the uncertainty of the distance between two points in a fuzzy metric space. In 2015, [12] introduced the new notion of

NM-fuzzy metric space, Pseudo N-fuzzy metric space and describes some of their properties and examples. In 2020 [5] proved some fixed point theorems for asymptotically regular maps in N-fuzzy b-metric space. [1] recently established some common fixed point solutions for mapping on interval-valued fuzzy metric space that is occasionally weakly compatible. This was demonstrated in 2023.

In this study, we provide solutions to equations where a function maps a point to itself. In this research, we focus on applying diverse contractive conditions specifically in the context of asymptotically regular mappings to achieve our objectives validate certain well-known fixed point results for asymptotically regular mapping in interval valued N-fuzzy metric space (IVFNM). Our results generalize and extend the results of the common fixed point theorem for intimate mapping that were published recently.

II. Basic Definitions

Definition 1[18] In a non empty set $\mho$, a mapping $R_n: \mho \to [I]$ is called an interval- valued fuzzy set on $\mho$. Collection of all interval-valued fuzzy sets on $\mho$ is denoted by $IVF(\mho)$. if $R_n \in IVF(\mho)$. , let $R_n(x) = [R_n^-, R_n^+]$, $R_n^-(x) \le R_n^+(x)$ for all $x \in \mho$, then the set $R_n^-: \mho \to [I]$ and $R_n^+: \mho \to [I]$ are called lower fuzzy set and upper fuzzy set of R_n and if $R^-(x) = R_n^+(x)$ then is called degenerate fuzzy set for all $x \in \mho$.

Definition 2[18]. A binary operation of the form is an interval valued t_{norm} is $*_I: [I] \times [I] \to [I]$ on [I] such that for all $\overline{u}, \overline{v}, \overline{w} \in [I]$ if satisfying following four conditions:

(1) Commutativity : $\overline{u}_{ivf} *_I \overline{v}_{ivf} = \overline{v}_{ivf} *_I \overline{u}_{ivf}$,
(2) Associativity: $[\overline{u}_{ivf} *_I \overline{v}_{ivf}] *_I \overline{w}_{ivf} = \overline{u}_{ivf} *_I [\overline{v}_{ivf} *_I \overline{w}_{ivf}]$,
(3) Monotonicity: $\overline{u}_{ivf} *_I \overline{v}_{ivf} \le \overline{u}_{ivf} *_I \overline{w}_{ivf}$, whenever $\overline{v}_{ivf} *_I \overline{w}_{ivf}$,
(4) Boundary condition: $\overline{u}_{ivf} *_I \overline{1} = \overline{u}_{ivf}$, $\overline{u}_{ivf} *_I \overline{0} = [u^-, u^+] *_I [0,1] = [0, u^+]$.

Note: Each interval valued t_{norm} satisfies some additional boundary conditions for all $\overline{u} \in [I]$:
$\overline{u} *_I \overline{0} = \overline{0} *_I \overline{u} = \overline{0}$,
$\overline{I} *_I \overline{u} = [0,1] *_I [u^-, u^+] = \overline{0}$,
$\overline{1} *_I \overline{u} = \overline{1}$.
Example: (a) $\overline{u} *_I \overline{v} = [u^-.v^-, u^+.v^+]$; (b) $\overline{u} *_I \overline{v} = [u^- \wedge v^-, u^+ \wedge v^+]$

Definition 3[18]. Let $\{\overline{u}_i\} = \{[u_i^-, u_i^+]\}, i \in \mathbb{N}^+$ be a sequence of interval numbers in [I], $\overline{u} = [u^-, u^+] \in [I]$, if $\lim_{i\to\infty} u_i^- = u^-$ and $\lim_{i\to\infty} u_i^+ = u^+$ then the sequence $\{\overline{u}_i\}$ is convergent to $\overline{u}$ and denoted by $\lim_{i\to\infty} \overline{u}_i = \overline{u}$.

Definition 4 [1]. An interval valued t_{norm} $*_I$ is continuous iff it is continuous in its first component, i.e. for each $\overline{v} \in [I]$, if $\lim_{i\to\infty} \overline{u}_i = \overline{u}$, then

$\lim_{i\to\infty}(\overline{u}_i *_I \overline{v}) = (\lim_{i\to\infty} \overline{u}_i *_I \overline{v}) = \overline{u} *_I \overline{v}$, Where $\{\overline{u}_i\} \subseteq [\mathbf{I}], \overline{u} \in [I]$.

Definition 5 [5]. A triplet $(\mathfrak{X}, M, *)$ is called fuzzy metric space (FMS) if $\mathfrak{X}$ is an arbitrary set, $*$ is a continuous t_{norm} on [0,∞] and M is a fuzzy set on $\mathfrak{X}^2 \times (0,\infty)$ satisfying the following conditions :

(1) $M(x, y, t) > 0$;
(2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, z, t_1 + t_2) \ge T(M(x, y, t_1), M(x, y, t_2))$; $\forall x, y, z \in \mathfrak{X}$ and $t_1, t_2 > 0$
(5) $M(x, y, t): [0,\infty) \to [0,1]$ is continuous;

Definition 6[5]. A triplet $(\mathfrak{X}, M, *)$ is called N-fuzzy metric space (N-FMS) if $\mathfrak{X}$ is an arbitrary set, $*$ is a continuous t_{norm} on [0,∞] and M is a fuzzy set on $\mathfrak{X}^3 \times (0,\infty)$ satisfying the following conditions:

(1) $M(x, y, z, t) > 0$;
(2) $M(x, y, z, t) = 1$ for all $t > 0$ iff $x = y = z$;
(3) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, x, a, t_1) * M(y, y, a, t_2) * M(z, z, a, t_3)$;
$\forall x, y, z \in \mathfrak{X}$ and $t_1, t_2, t_3 > 0$
(4) $M(x, y, z, *): (0, \infty) \to [0,1]$ is continuous;

Definition 7[5]. A triplet $(\mathfrak{X}, N_{IVFNM}, *_I)$ is called interval valued N-fuzzy metric space (IVFMS) if $\mathfrak{X}$ is an arbitrary set, $*_I$ is a continuous interval valued t_{norm} on [I] and N_{IVFNM} is a fuzzy set on $\mathfrak{X}^3 \times (0, \infty)$ satisfying the following conditions :

(1) $N_{IVFNS}(x, y, z, t_{norm}) > \bar{0}$;
(2) $N_{IVFNM}(x, y, z, t_{norm}) = \bar{1}$ for all $t_{norm} > 0$ iff $x = y = z$;
(3) $N_{IVFNM}(x, y, z, a, t_1 + t_2 + t_3) \geq N_{IVFNM}(x, x, a, t_1) *_I N_{IVFNM}(y, y, a, t_2) *_I N_{IVFNM}(z, z, a, t_3)$;
$\forall x, y, z \in \mathfrak{X}$ and $t_1, t_2, t_3 > 0$
(4) $N_{IVFNM}(x, y, z, *_I): [0, \infty] \to (I)$ is continuous;
(5) $\lim_{t\to\infty} N_{IVFNM}(x, y, z, t_{norm}) = \bar{1}$; $\forall x, y, z \in \mathfrak{X}, t_{norm} > 0$,

Note: Every metric can induce an interval valued N-fuzzy metric space.

Definition 8[1]. Let $(\mathfrak{X}, N_{IVFNM}, *_I)$ is an IVFNM

(a) If $\beta > t_{norm} > 0$ then $N_{IVFNM}(x, y, z, t_{norm}) \leq N_{IVFNM}(x, y, z, \beta)$ for $x, y, z \in \mathfrak{X}$.
(b) A sequence $\{x_n\}$ in $\mathfrak{X}$ is referred to as a Cauchy sequence if for all $\bar{\epsilon} > \bar{0}$ and $t_{norm} > 0$ then there exists a $n_0 \in \mathbb{N} \ni N_{IVFNM}(x, y, z, t_{norm}) > 1 - \bar{\epsilon}$, for all $x, y, z, \geq n_0$.
(c) If every Cauchy sequence is convergent in $(\mathfrak{X}, N_{IVFNM}, *_I)$ then it is complete IVFNM.

Definition 9[1]. A mapping $\gamma: [I] \to [I]$ is called an altering distance function if

(a) γ is strictly decreasing and left continuous.
(b) $\gamma(\alpha) = 0$ if and only if $\alpha = \bar{1}$ i.e. $\lim_{\gamma\to I^-} \gamma(I) = 0$.

Definition 10[1]. Let two self mapping $\mathfrak{K}$ and $\mathfrak{L}$ on IVFNM $(\mathfrak{X}., V_{IVFNM}, *_I)$ then compatible if $\lim_{n\to\infty} N_{IVFNM}(\mathfrak{K}\mathfrak{L}x_n, \mathfrak{K}\mathfrak{L}x_n, \mathfrak{L}\mathfrak{K}x_n, t_{norm}) = \bar{1}$ for all $t_{norm} > 0$ whenever a sequence $\{x_n\}$ in $\mathfrak{X}$ provided $\lim_{n\to\infty} \mathfrak{K}x_n = \lim_{n\to\infty} \mathfrak{L}x_n = u$, for all $u \in \mathfrak{X}$.

Definition 11[1]. Let two self mapping $\mathfrak{K}$ and $\mathfrak{L}$ on IVFNM $(\mathfrak{X}, \mathfrak{M}_{IVFMS}, *_I)$ and a sequence $\{x_n\}$ in $\mathfrak{X}$ then $\mathfrak{K}$ is called asymptotically regular at a point $z_0 \in \mathfrak{X}$
if $\lim_{n\to\infty} N_{IVFNM}(\mathfrak{K}^n(z_0), \mathfrak{K}^n(z_0), \mathfrak{K}^{n+1}(z_0), t_{norm}) = \bar{1}$, for all $t_{norm} > 0$
and also the sequence $\{x_n\}$ is called asymptotically regular with respect to the pair $(\mathfrak{K}, \mathfrak{L})$
if $\lim_{n\to\infty} N_{IVFNM}(\mathfrak{K}(z_0), \mathfrak{K}(z_0), \mathfrak{L}(z_0), t_{norm}) = \bar{1}$ for all $t_{norm} > 0$.

III. Main Results

***Theorem 1*:** Let $\mathfrak{F}: \mathcal{X} \to \mathcal{X}$ be a self mapping of a complete IVFNM $(\mathcal{X}, N_{IVFNM}, *_I)$ and γ be the altering distance function then satisfying the following conditions:

$$\begin{aligned}\gamma(N(\mathfrak{F}(\varrho), \mathfrak{F}(\varrho), \mathfrak{F}(\sigma), t_{norm}))\\ &\leq b_1\theta[\min\{\gamma(N(\varrho, \varrho, \mathfrak{F}(\varrho), t_{norm})), \gamma(N(\sigma, \sigma, \mathfrak{F}(\sigma), t_{norm}))\}]\\ &+ b_2\omega[\{\gamma(N(\varrho, \varrho, \mathfrak{F}(\varrho), t_{norm})).\gamma(N(\sigma, \sigma, \mathfrak{F}(\sigma), t_{norm}))\}]\\ &+ b_3[\{\gamma(N(\varrho, \varrho, \mathfrak{F}(\varrho), t_{norm})) + \gamma(N(\sigma, \sigma, \mathfrak{F}(\sigma), t_{norm}))\}]\\ &+ b_4[\{\gamma(N(\varrho, \varrho, \mathfrak{F}(\sigma), t_{norm})) + \gamma(N(\mathfrak{F}(\varrho), \mathfrak{F}(\varrho), \sigma, t_{norm}))\}]\\ &+ b_5\gamma(N(\varrho, \varrho, \sigma, t_{norm}))\end{aligned}$$

Where $\forall \varrho, \sigma \in \mathcal{X}$ and $t_{norm} > 0, b_i = b_i(\varrho, \sigma) \geq 0, i = 1,2 \ldots 5$ are such that for some arbitrarily fixed $\lambda_1 > 0, 0 < \lambda_2 < 1$,

$b_1 + b_2 \leq \lambda_1$ and $(b_3 + 2b_4 + b_5) \leq \lambda_2$ and $\theta, \omega: R^+ \to R^+$ are continuous functions at 0 and $\theta(0) = \omega(0) = 0$. If $\mathfrak{F}$ is asymptotically regular at some point $\varrho_0 \in \mathcal{X}$ then has a unique fixed point in $\mathcal{X}$.

Proof: Let a sequences $\{\varrho_n\}$ in $\mathcal{X}$ where $\varrho_0 \in \mathcal{X}$ and $\varrho_{n+1} = \mathfrak{F}(\varrho_n)$, for all $n \geq 0$,
Now if $n \geq 0$, $\varrho_{n+1} = \varrho_n$ then ϱ_n is a fixed point of $\mathfrak{F}$. suppose $\varrho_{n+1} \neq \varrho_n$ then to prove $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{X}$.
Suppose to the contrary $\exists, 0 < \epsilon < 1, t_{norm} > 0$ and two sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n > v_n > n$,

$$N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}) \leq 1{-}\epsilon,$$
$$N_{IVFNM}(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{v_{n-1}}, t_{norm}) > 1{-}\epsilon$$
$$N_{IVFNM}(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{v_n}, t_{norm}) > 1{-}\epsilon, \forall n \in \mathbb{N} \cup \{0\} \ldots\ldots (a)$$

Now we have

$$1{-}\epsilon \geq N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}) \geq N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{u_n-1}, t_{norm})$$
$$*_I\ N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{u_n-1}, t_{norm}) *_I N_{IVFNM}(\varrho_{v_n}, \varrho_{v_n}, \varrho_{u_n-1}, t_{norm})$$

$1{-}\epsilon \geq \lim_{n\to\infty} N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}) \geq (\bar{1} *_I \bar{1} *_I 1{-}\epsilon)$ (Since $\mathfrak{F}$ asymptotically regular at ϱ_0)

$\Rightarrow \lim_{n\to\infty} N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}) = 1{-}\epsilon$ …… (b)

Again,

$$N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n-1}, t_{norm}) \geq$$
$$N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}) *_I N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}) *_I N_{IVFNM}(\varrho_{v_{n-1}}, \varrho_{v_{n-1}}, \varrho_{v_n}, t_{norm})$$
$$\lim_{n\to\infty} N_{IVFNM}(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n-1}, t_{norm}) > 1{-}\epsilon \ \ \ldots\ldots(c)$$

By putting $\varrho = \varrho_{u_n-1}$, $\sigma = \varrho_{v_{n-1}}$ in theorem (3.1) then we have ,

$$\begin{aligned}
&\gamma\left(N\left(\mathfrak{F}(\varrho_{u_n-1}), \mathfrak{F}(\varrho_{u_n-1}), \mathfrak{F}(\varrho_{v_{n-1}}), t_{norm}\right)\right) \\
&\leq b_1\theta\left[\min\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \mathfrak{F}(\varrho_{u_n-1}), t_{norm}\right)\right), \gamma\left(N\left(\varrho_{v_n-1}, \varrho_{v_n-1}, \mathfrak{F}(\varrho_{v_n-1}), t_{norm}\right)\right)\right\}\right] \\
&+ b_2\omega\left[\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \mathfrak{F}(\varrho_{u_n-1}), t_{norm}\right)\right) . \gamma\left(N\left(\varrho_{v_n-1}, \varrho_{v_n-1}, \mathfrak{F}(\varrho_{v_n-1}), t_{norm}\right)\right)\right\}\right] \\
&+ b_3\left[\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \mathfrak{F}(\varrho_{u_n-1}), t_{norm}\right)\right) + \gamma\left(N\left(\varrho_{v_n-1}, \varrho_{v_n-1}, \mathfrak{F}(\varrho_{v_n-1}), t_{norm}\right)\right)\right\}\right] \\
&+ b_4\left[\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \mathfrak{F}(\varrho_{v_n-1}), t_{norm}\right)\right) + \gamma\left(N\left(\mathfrak{F}(\varrho_{u_n-1}), \mathfrak{F}(\varrho_{u_n-1}), \varrho_{v_{n-1}}, t_{norm}\right)\right)\right\}\right] \\
&+ b_5\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{v_n-1}, t_{norm}\right)\right)
\end{aligned}$$

Since $\mathfrak{F}$ asymptotically regular at ϱ_0.

$$\begin{aligned}
&\gamma\left(N\left(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n}, t_{norm}\right)\right) \\
&\leq b_1\theta\left[\min\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{v_n}, t_{norm}\right)\right), \gamma\left(N\left(\varrho_{v_{n-1}}, \varrho_{v_{n-1}}, \varrho_{v_n}, t_{norm}\right)\right)\right\}\right] \\
&+ b_2\omega\left[\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{u_n}, t_{norm}\right)\right) . \gamma\left(N\left(\varrho_{v_n-1}, \varrho_{v_n-1}, \varrho_{v_n}, t_{norm}\right)\right)\right\}\right] \\
&+ b_3\left[\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{u_n}, t_{norm}\right)\right) + \gamma\left(N\left(\varrho_{v_n-1}, \varrho_{v_n-1}, \varrho_{v_n}, t_{norm}\right)\right)\right\}\right] \\
&+ b_4\left[\left\{\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{v_n}, t_{norm}\right)\right) + \gamma\left(N\left(\varrho_{u_n}, \varrho_{u_n}, \varrho_{v_n-1}, t_{norm}\right)\right)\right\}\right] \\
&+ b_5\gamma\left(N\left(\varrho_{u_n-1}, \varrho_{u_n-1}, \varrho_{v_n-1}, t_{norm}\right)\right)
\end{aligned}$$

Letting $n \to \infty$ and applying (a), (b) and (c) and altering distance

$$\begin{aligned}
\gamma(1{-}\epsilon) &\leq b_1\theta[\min\{\gamma(\bar{1}), \gamma(\bar{1})\}] + b_2\omega[\{\gamma(\bar{1}).\gamma(\bar{1})\}] + b_3[\{\gamma(\bar{1}) + \gamma(\bar{1})\}] \\
&+ b_4[\{\gamma(1{-}\epsilon) + \gamma(1{-}\epsilon)\}] + b_5\gamma(1{-}\epsilon)
\end{aligned}$$
$$\gamma(1{-}\epsilon) \leq 2b_4\gamma(1{-}\epsilon) + b_5\gamma(1{-}\epsilon) < \gamma(1{-}\epsilon)$$

This is a contradiction.

Thus $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{X}$. since complete IVFNM $(\mathcal{X}, N_{IVFNM}, *_I)$ then $\exists$, $p \in \mathcal{X}$ such that $\varrho_n \to p$.

$$\begin{aligned}
&\gamma\big(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\mathfrak{F}(p),t_{norm})\big)\\
&\quad\le b_1\theta\big[\min\{\gamma\big(N(\varrho_n,\varrho_n,\mathfrak{F}(\varrho_n),t_{norm})\big),\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_2\omega\big[\{\gamma\big(N(\varrho_n,\varrho_n,\mathfrak{F}(\varrho_n),t_{norm})\big).\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_3\big[\{\gamma\big(N(\varrho_n,\varrho_n,\mathfrak{F}(\varrho_n),t_{norm})\big)+\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_4\big[\{\gamma\big(N(\varrho_n,\varrho_n,\mathfrak{F}(p),t_{norm})\big)+\gamma\big(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_5\gamma\big(N(p,p,\mathfrak{F}(\varrho_n),t_{norm})\big)
\end{aligned}$$

$$\begin{aligned}
&\gamma\big(N(\varrho_{n+1},\varrho_{n+1},\mathfrak{F}(p),t_{norm})\big)\\
&\quad\le b_1\theta\big[\min\{\gamma\big(N(\varrho_n,\varrho_n,\varrho_{n+1},t_{norm})\big),\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_2\omega\big[\{\gamma\big(N(\varrho_n,\varrho_n,\varrho_{n+1},t_{norm})\big).\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_3\big[\{\gamma\big(N(\varrho_n,\varrho_n,\varrho_{n+1},t_{norm})\big)+\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_4\big[\{\gamma\big(N(\varrho_n,\varrho_n,\mathfrak{F}(p),t_{norm})\big)+\gamma\big(N(\varrho_{n+1},\varrho_{n+1},\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_5\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)
\end{aligned}$$

Letting $n\to\infty$ then we have,

$$\begin{aligned}
&\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\\
&\quad\le b_3\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)+2b_4\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\\
&\quad+b_5\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)
\end{aligned}$$

$$\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\le(b_3+2b_4+b_5)\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)$$

$$\{1-(b_3+2b_4+b_5)\}\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\le 0.$$

$$\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\le 0,\quad[\text{since }\{1-(b_3+2b_4+b_5)\}<1]$$

$$\Rightarrow\lim_{n\to\infty}\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)=0\Rightarrow\mathfrak{F}(p)=p.$$

If u be the another fixed point of $\mathfrak{F}$ in $\mathcal{X}$. then

$$\begin{aligned}
&\gamma\big(N(\mathfrak{F}(u),\mathfrak{F}(u),\mathfrak{F}(p),t_{norm})\big)\\
&\quad\le b_1\theta\big[\min\{\gamma\big(N(u,u,\mathfrak{F}(u),t_{norm})\big),\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_2\omega\big[\{\gamma\big(N(u,u,\mathfrak{F}(u),t_{norm})\big).\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_3\big[\{\gamma\big(N(u,u,\mathfrak{F}(u),t_{norm})\big)+\gamma\big(N(p,p,\mathfrak{F}(p),t_{norm})\big)\}\big]\\
&\quad+b_4\big[\{\gamma\big(N(u,u,\mathfrak{F}(p),t_{norm})\big)+\gamma\big(N(\mathfrak{F}(u),\mathfrak{F}(u),p,t_{norm})\big)\}\big]+b_5\gamma\big(N(u,u,p,t_{norm})\big)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\gamma\big(N(u,u,p,t_{norm})\big)\\
&\quad\le b_4\big[\{\gamma\big(N(u,u,p,t_{norm})\big)+\gamma\big(N(u,u,p,t_{norm})\big)\}\big]+b_5\gamma\big(N(u,u,p,t_{norm})\big)\\
&\Rightarrow\gamma\big(N(u,u,p,t_{norm})\big)\le 2b_4\gamma\big(N(u,u,p,t_{norm})\big)+b_5\gamma\big(N(u,u,p,t_{norm})\big)\\
&\Rightarrow\gamma\big(N(u,u,p,t_{norm})\big)\le(2b_4+b_5)\gamma\big(N(u,u,p,t_{norm})\big)\\
&\Rightarrow\{1-(2b_4+b_5)\}\gamma\big(N(u,u,p,t_{norm})\big)\le 0.\\
&\Rightarrow\gamma\big(N(u,u,p,t_{norm})\big)=0.\\
&\Rightarrow u=p.
\end{aligned}$$

Example: Let IVFNM $(\mathcal{X},N_{IVFNM},*_I)$ and $\mathcal{X}=[0,1]$ and a interval valued N-fuzzy set on $\mathcal{X}^3\times(0,\infty)$ to [0,1] defined by $N_{IVFNM}(x,y,z,t_{norm})=\frac{t}{t+[|x-z|+|y-z|]}$, for all $x,y,z\in\mathcal{X}$ and $*_I$ is minimum t_{norm} define $\mathfrak{F}:\mathcal{X}\to\mathcal{X}$ by

$$\mathfrak{F}(x)=\begin{cases}\frac{x}{3}, & x\in[0,\frac{1}{2}]\\ \frac{1}{6}, & x\in[\frac{1}{2},1]\end{cases}\quad\text{and}\quad\gamma(\alpha)=1-\alpha,\ \alpha\in[0,1].$$

Let $\theta(u)=\sqrt{u}$ and $\omega(v)=v^2$, $u,v\in R^+$

Also let $b_1(\varrho,\sigma)=|\varrho-\sigma|$, $b_2(\varrho,\sigma)=|\varrho^2-\sigma^2|$, $b_3(\varrho,\sigma)=\begin{cases}\frac{1}{|x-y|}, & x\neq y\\ 0, & x=y\end{cases}$,

$b_4(\varrho,\sigma)=b_5(\varrho,\sigma)=\left|\frac{1-b_3(\varrho,\sigma)}{3}\right|$. Then all the conditions of theorem 3.1 verified. Hence $\mathfrak{F}$ has a unique fixed point.

Theorem 2: Let $\mathfrak{F}$, $\mathscr{L}:\mathcal{X}\to\mathcal{X}$ be a complete IVFNM $(\mathcal{X},N_{IVFNM},*_I)$ and γ be the altering distance function. If $\mathfrak{F}$, $\mathscr{L}$ is asymptotically regular at a point $\varrho_0\in\mathcal{X}$ both satisfy the inequality (1).

Moreover, if

$$\gamma(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\mathcal{L}(\sigma),t_{norm})) \leq \lambda[\gamma(N(\varrho,\varrho,\sigma,t_{norm}))+\gamma(N(\varrho,\varrho,\mathfrak{F}(\varrho),t_{norm}))+\gamma(N(\sigma,\sigma,\mathcal{L}(\sigma),t_{norm}))]$$

Where $\forall\ \varrho,\sigma\in\mathcal{X}$, $\lambda<1$.Then $\mathfrak{F}$ and $\mathscr{L}$ has a unique common fixed point in $\mathcal{X}$.

Proof: By above theorem both $\mathfrak{F}$ and $\mathscr{L}$ have unique fixed point say u and v respectively.then it's satisfied above inequality:

$$\gamma(N(\mathfrak{F}(u),\mathfrak{F}(u),\mathcal{L}(v),t_{norm})) \leq \lambda[\gamma(N(u,u,v,t_{norm}))+\gamma(N(u,u,\mathfrak{F}(u),t_{norm}))+\gamma(N(v,v,\mathcal{L}(v),t_{norm}))]$$

$$\gamma(N(u,u,v,t_{norm})) \leq \lambda[\gamma(N(u,u,v,t_{norm}))+\gamma(N(u,u,u,t_{norm}))+\gamma(N(v,v,v,t_{norm}))]$$

By definition of altering distance

$$(N(u,u,v,t_{norm})) \leq \lambda(N(u,u,v,t_{norm}))$$

$$(1-\lambda)(N(u,u,v,t_{norm})) \leq 0$$

$(N(u,u,v,t_{norm})) \leq 0$, (since $\lambda<1$)

$\Rightarrow u=v$. Hence $\mathfrak{F}$ and $\mathscr{L}$ have unique fixed point.

Theorem 3: Let $\mathfrak{F}:\mathcal{X}\to\mathcal{X}$ be a complete IVFNM $(\mathcal{X},N_{IVFNM},*_I)$ and γ be the altering distance function. If $\mathfrak{F}$ is asymptotically regular at a point $\varrho_0\in\mathcal{X}$. then satisfying the conditions:

$$\gamma(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\mathfrak{F}(\sigma),t_{norm}))$$
$$\leq \alpha\,\mathrm{Min}[\gamma(N(\varrho,\varrho,\sigma,t_{norm})),\gamma(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\varrho,t_{norm})),\gamma(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\sigma,t_{norm}))]$$
$$+\beta\,\mathrm{Min}[\gamma(N(\varrho,\varrho,\sigma,t_{norm})),\gamma(N(\mathfrak{F}(\sigma),\mathfrak{F}(\sigma),\sigma,t_{norm})),\gamma(N(\varrho,\varrho,\mathfrak{F}(\sigma),t_{norm}))]$$

Where $\forall\ \varrho,\sigma\in\mathcal{X},t_{norm}>0$, $\alpha,\beta>0$ such that $\alpha+\beta<1$ .Then $\mathfrak{F}$ have a unique common fixed point in $\mathcal{X}$.

Proof : Let a sequences $\{\varrho_n\}$ in $\mathcal{X}$ where $\varrho_0\in\mathcal{X}$ and $\varrho_{n+1}=\mathfrak{F}(\varrho_n)$, for all $n\geq 0$,

Now if $n\geq 0$, $\varrho_{n+1}=\varrho_n$ then ϱ_n is a fixed point of $\mathfrak{F}$. suppose $\varrho_{n+1}\neq\varrho_n$ then to prove $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{X}$.

Let $m,n\in\mathbb{N}\cup\{0\}$

$$\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_m),t_{norm}))$$
$$\leq \alpha\,\mathrm{Min}[\gamma(N(\varrho_n,\varrho_n,\varrho_m,t_{norm})),\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\varrho_n,t_{norm})),\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\varrho_m,t_{norm}))]$$
$$+\beta\,\mathrm{Min}[\gamma(N(\varrho_n,\varrho_n,\varrho_m,t_{norm})),\gamma(N(\mathfrak{F}(\varrho_m),\mathfrak{F}(\varrho_m),\varrho_m,t_{norm})),\gamma(N(\varrho_n,\varrho_n,\mathfrak{F}(\varrho_m),t_{norm}))]$$

Since $\mathfrak{F}$ is asymptotically regular at a point $\varrho_0\in\mathcal{X}$ taking $n,m\to\infty$ then

$$\lim_{n,m\to\infty}\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_m),t_{norm}))=0$$

and $\lim_{n,m\to\infty}(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_m),t_{norm}))=1$

Thus $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{X}$. since IVFNM $(\mathcal{X},N_{IVFNM},*_I)$ complete then $\exists$, $p\in\mathcal{X}$ such that $\varrho_n\to p$(say).

$$\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\mathfrak{F}(p),t_{norm}))$$
$$\leq \alpha\,\mathrm{Min}[\gamma(N(\varrho_n,\varrho_n,p,t_{norm})),\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),\varrho_n,t_{norm})),\gamma(N(\mathfrak{F}(\varrho_n),\mathfrak{F}(\varrho_n),p,t_{norm}))]$$
$$+\beta\,\mathrm{Min}[\gamma(N(\varrho_n,\varrho_n,p,t_{norm})),\gamma(N(\mathfrak{F}(p),\mathfrak{F}(p),p,t_{norm})),\gamma(N(\varrho_n,\varrho_n,\mathfrak{F}(p),t_{norm}))]$$

$$\gamma(N(\varrho_{n+1},\varrho_{n+1},\mathfrak{F}(p),t_{norm}))$$
$$\leq \alpha\,\mathrm{Min}[\gamma(N(\varrho_n,\varrho_n,p,t_{norm})),\gamma(N(\varrho_{n+1},\varrho_{n+1},\varrho_n,t_{norm})),\gamma(N(\varrho_{n+1},\varrho_{n+1},p,t_{norm}))]$$
$$+\beta\,\mathrm{Min}[\gamma(N(\varrho_n,\varrho_n,p,t_{norm})),\gamma(N(\mathfrak{F}(p),\mathfrak{F}(p),p,t_{norm})),\gamma(N(\varrho_n,\varrho_n,\mathfrak{F}(p),t_{norm}))]$$

$$\Rightarrow \lim_{n\to\infty}\gamma(N(\varrho_{n+1},\varrho_{n+1},\mathfrak{F}(p),t_{norm}))=0.$$

$$\Rightarrow \gamma(N(\mathfrak{F}(p),\mathfrak{F}(p),p,t_{norm}))=0.$$

$$\Rightarrow \mathfrak{F}(p)=p.$$

Hence p is the unique fixed point.

Theorem 4: Let a complete IVFNM $(\mathcal{X},N_{IVFNM},*_I)$ and γ be the altering distance function and $\mathfrak{F}$ and $\mathcal{L}$ be two commutative self mappings on $\mathcal{X}$ such that

$$\gamma\big(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\mathfrak{F}(\sigma),t_{norm})\big) \le \lambda_1\big[\gamma\big(N(\mathcal{L}(\varrho),\mathcal{L}(\varrho),\mathcal{L}(\sigma),t_{norm})\big)+\lambda_2\gamma\big(N(\mathcal{L}(\varrho),\mathcal{L}(\varrho),\mathfrak{F}(\sigma),t_{norm})\big) + \gamma\big(N(\mathcal{L}(\sigma),\mathcal{L}(\sigma),\mathfrak{F}(\sigma),t_{norm})\big)\big]$$

Where $\forall\ \varrho,\sigma \in \mathcal{X}$, $t_{norm} > 0$ and $\lambda_1:\mathbb{R}\to[I].\lambda_1 > 0,\ \lambda_2 < 1$.

Moreover, if

(i) $\mathfrak{F}, \mathscr{L}$ is asymptotically regular at a point $\varrho_0 \in \mathcal{X}$.

(ii) $\mathfrak{F}(\mathcal{X}) \subseteq \mathcal{L}(\mathcal{X})$.

(iii) $\mathfrak{F}(\mathcal{X})$ or $\mathcal{L}(\mathcal{X})$ is a complete subspace of $\mathcal{X}$.

Then $\mathfrak{F}$ and $\mathscr{L}$ has a unique common fixed point in $\mathcal{X}$.

Proof: Since $\mathfrak{F}(\mathcal{X}) \subseteq \mathcal{L}(\mathcal{X})$. Let a sequence $\{g_n\}$ in $\mathcal{X}$ where $g_0 \in \mathcal{X}$ and $g_{n+1} = \mathfrak{F}(\varrho_n) = \mathcal{L}(\varrho_{n+1})$, $n \in \mathbb{N}\cup\{0\}$, for all $n \ge 0$, and also $\mathfrak{F}, \mathscr{L}$ are asymptotically regular at a point $\varrho_0 \in \mathcal{X}$.

$$\lim_{n\to\infty}\gamma\big(N(g_n, g_n, g_{n+1}, t_{norm})\big) = 0$$

Then to prove $\{g_n\}$ is a Cauchy's sequence in $\mathcal{X}$.

Suppose to the contrary $\exists, 0<\epsilon<1, t_{norm} > 0$ and two sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n > v_n > n$,

$$N_{IVFNM}\big(\varrho_{u_n},\varrho_{u_n},\varrho_{v_n}t_{norm}\big) \le 1-\epsilon,$$
$$N_{IVFNM}\big(\varrho_{u_n-1},\varrho_{u_n-1},\varrho_{v_{n-1}}t_{norm}\big) > 1-\epsilon$$
$$N_{IVFNM}\big(\varrho_{u_n-1},\varrho_{u_n-1},\varrho_{v_n}t_{norm}\big) > 1-\epsilon,\ \forall n \in \mathbb{N}\cup\{0\}\ldots\ldots \text{(a)}$$

Following concept applied in theorem 3.4 then we can show that

$$\lim_{n\to\infty} N_{IVFNM}\big(\varrho_{u_n},\varrho_{u_n},\varrho_{v_n}t_{norm}\big) = 1-\epsilon,$$

Now

$$\gamma\Big(N\big(\mathfrak{F}(\varrho_{u_n}),\mathfrak{F}(\varrho_{u_n}),\mathfrak{F}(\varrho_{v_n}),t_{norm}\big)\Big) \le \lambda_1\Big[\gamma\Big(N\big(\mathcal{L}(\varrho_{u_n}),\mathcal{L}(\varrho_{u_n}),\mathcal{L}(\varrho_{v_n}),t_{norm}\big)\Big) + \lambda_2\gamma\Big(N\big(\mathcal{L}(\varrho_{u_n}),\mathcal{L}(\varrho_{u_n}),\mathfrak{F}(\varrho_{v_n}),t_{norm}\big)\Big) + \gamma\Big(N\big(\mathfrak{F}(\varrho_{v_n}),\mathfrak{F}(\varrho_{v_n}),\mathcal{L}(\varrho_{v_n}),t_{norm}\big)\Big)\Big]$$

$$\gamma\Big(N\big(g_{u_n+1}, g_{u_n+1}, g_{v_n+1}, t_{norm}\big)\Big) \le \lambda_1\Big[\gamma\Big(N\big(\mathcal{L}(\varrho_{u_n}),\mathcal{L}(\varrho_{u_n}),\mathcal{L}(\varrho_{v_n}),t_{norm}\big)\Big) + \lambda_2\gamma\Big(N\big(\mathcal{L}(\varrho_{u_n}),\mathcal{L}(\varrho_{u_n}),\mathfrak{F}(\varrho_{v_n}),t_{norm}\big)\Big) + \gamma\Big(N\big(\mathfrak{F}(\varrho_{v_n}),\mathfrak{F}(\varrho_{v_n}),\mathcal{L}(\varrho_{v_n}),t_{norm}\big)\Big)\Big]$$

Taking $n\to\infty$ and using above facts then we have

$\gamma(1-\epsilon) \le \lambda_1[\gamma(1-\epsilon)] < \gamma(1-\epsilon)$ a contradiction. Hence $\{g_n\}$ is a Cauchy's sequence in $\mathcal{X}$.

Since $\mathcal{L}(\mathcal{X})$ is a complete subspace of $\mathcal{X}$ then $\exists, h \in \mathcal{L}(\mathcal{X})$ such that $\lim_{n\to\infty} g_n = h$ and also for some $z \in \mathcal{X}$ then $\mathcal{L}(z) = h$.

Now

$$\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z), g_{n+1},t_{norm})\big) = \gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),\mathfrak{F}(g_n),t_{norm})\big) \le \lambda_1\big[\gamma\big(N(\mathcal{L}(z),\mathcal{L}(z),\mathcal{L}(g_n),t_{norm})\big)+\lambda_2\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),\mathcal{L}(z),t_{norm})\big) + \gamma\big(N(\mathfrak{F}(g_n),\mathfrak{F}(g_n),\mathcal{L}(g_n),t_{norm})\big)\big]$$

As $n\to\infty$

$$\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),h,t_{norm})\big) \le \lambda_1\big[\gamma\big(N(h,h,h,t_{norm})\big)+\lambda_2\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),h,t_{norm})\big)+\gamma\big(N(h,h,h,t_{norm})\big)\big]$$

$$\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),h,t_{norm})\big) \le \lambda_1\big[\lambda_2\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),h,t_{norm})\big)\big]$$

$\Rightarrow (1-\lambda_1\lambda_2)\gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),h,t_{norm})\big) \le 0$

$\Rightarrow \gamma\big(N(\mathfrak{F}(z),\mathfrak{F}(z),h,t_{norm})\big) = 0$

$\Rightarrow \mathfrak{F}(z) = h$

$\Rightarrow \mathfrak{F}(z) = h = \mathcal{L}(z)$. i.e. h is a coincident point of $\mathfrak{F}$ and $\mathcal{L}$.

Now put $\varrho = \mathfrak{F}(z)$ and $\sigma = z$

$$\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)$$
$$\leq \lambda_1\left[\gamma\left(N(\mathcal{L}(\mathfrak{F}(z)),\mathcal{L}(\mathfrak{F}(z)),\mathcal{L}(z),t_{norm})\right)\right.$$
$$\left.+\lambda_2\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathcal{L}(\mathfrak{F}(z)),t_{norm})\right)+\gamma\left(N(\mathfrak{F}(z),\mathfrak{F}(z),\mathcal{L}(z),t_{norm})\right)\right]$$

$$\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)$$
$$=\lambda_1\left[\gamma\left(N(\mathfrak{F}(\mathcal{L}(z)),\mathfrak{F}(\mathcal{L}(z)),\mathcal{L}(z),t_{norm})\right)\right.$$
$$\left.+\lambda_2\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathcal{L}(z)),t_{norm})\right)+\gamma\left(N(\mathfrak{F}(z),\mathfrak{F}(z),\mathcal{L}(z),t_{norm})\right)\right]$$
(since $\mathfrak{F}\mathcal{L}=\mathcal{L}\,\mathfrak{F}$)

$$\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)$$
$$=\lambda_1\left[\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)\right.$$
$$\left.+\lambda_2\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),t_{norm})\right)+\gamma\left(N(\mathfrak{F}(z),\mathfrak{F}(z),\mathfrak{F}(z),t_{norm})\right)\right]$$

$$\Rightarrow (1-\lambda_1)\gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)=0$$
$$\Rightarrow \gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)=0$$
$$\Rightarrow \gamma\left(N(\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(\mathfrak{F}(z)),\mathfrak{F}(z),t_{norm})\right)=0$$
$$\Rightarrow \mathfrak{F}(\mathfrak{F}(z))=\mathfrak{F}(z)$$

Hence $\mathfrak{F}(\mathfrak{F}(z))=\mathfrak{F}(z)$
In similar manner we can show $\mathcal{L}(\mathcal{L}(z))=\mathcal{L}(z)=h$.
Thus h is unique common fixed point of $\mathfrak{F}$ and $\mathscr{L}$ in $\mathcal{X}$.
Uniqueness easily can be shown.

Corollary: Let a complete IVFNM $(\mathcal{X}, N_{IVFNM}, *_I)$ and γ be the altering distance function and $\mathfrak{F}$ and $\mathcal{L}$ be two self mappings on $\mathcal{X}$ such that

$$\gamma\left(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\mathfrak{F}(\sigma),t_{norm})\right)$$
$$\leq \lambda_1\left[\gamma\left(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\mathcal{L}(\varrho),t_{norm})\right)+\gamma\left(N(\mathfrak{F}(\sigma),\mathfrak{F}(\sigma),\mathcal{L}(\sigma),t_{norm})\right)\right]$$
$$+\lambda_2\left[\gamma\left(N(\mathfrak{F}(\varrho),\mathfrak{F}(\varrho),\mathcal{L}(\sigma),t_{norm})\right)+\gamma\left(N(\mathcal{L}(\varrho),\mathcal{L}(\varrho),\mathcal{L}(\sigma),t_{norm})\right)\right]$$
$$+\lambda_3\gamma\left(N(\lambda_3(\sigma),\mathfrak{F}(\sigma),\mathcal{L}(\varrho),t_{norm})\right)$$

Where $\forall\ \varrho,\sigma\in\mathcal{X}$, $t_{norm}>0$ and $\lambda_1>0$, $\lambda_2,\lambda_3<1$, $\lambda_1+2\lambda_2+\lambda_3<1$.

Moreover, if

(iv) $(\mathfrak{F},\mathcal{L})$ is asymptotically regular at a point ϱ_0 and commutes with each others.

(v) $\mathfrak{F}(\mathcal{X})$ or $\mathcal{L}(\mathcal{X})$ is a complete subspace of $\mathcal{X}$ and Range($\mathfrak{F}$) $\subseteq$ Range($\mathcal{L}$).

Then $\mathfrak{F}$ and $\mathscr{L}$ has a unique common fixed point in $\mathcal{X}$.

Example: Let IVFNM $(\mathcal{X}, N_{IVFNM}, *_I)$ and $\mathcal{X}=\{1,2,3,4,5\}$ and a interval valued N-fuzzy set on $\mathcal{X}^3\times(0,\infty)$ defined by $N_{IVFNM}(x,y,z,t_{norm})=\frac{t}{t+[|x-z|+|y-z|]}$, for all $x,y,z\in\mathcal{X}$ and $t_{norm}>0$. then $(\mathcal{X}, N_{IVFNM}, *_I)$ is complete interval valued N-fuzzy space w.r.t t_{norm} T(a , b)= min(a ,b), a ,b ∈ [0,1]. Let $\gamma(\alpha)=1-\alpha$ and define $\mathfrak{F},\mathcal{L}:\mathcal{X}\to\mathcal{X}$ by $\mathfrak{F}(1)=\mathfrak{F}(2)=2,\mathfrak{F}(3)=\mathfrak{F}(4)=\mathfrak{F}(5)=1$ and $\mathcal{L}(1)=\mathcal{L}(2)=2,\mathcal{L}(3)=\mathcal{L}(4)=\mathcal{L}(5)=1$. then $\mathfrak{F}$ and $\mathscr{L}$ satisfy theorem 3.3 and $\mathfrak{F}$ and $\mathscr{L}$ has a unique common fixed point in $\mathcal{X}$.

IV. Conclusion

In this research, we verify certain well-known fixed point conclusions using asymptotically regular mappings in interval valued fuzzy N-metric space (IVFMN). Our results generalize and extend the results of the common fixed point theorem for interval valued fuzzy N-metric space (IVFMN), which were published recently. For future research, these findings can be applied to solve fuzzy differential equations and integral equations, which will be of great assistance to the researchers.

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