ON CHARACTERIZATION OF 2KTH ORDER WEIGHTED MAXWELL BOLTZMANN DISTRIBUTION

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Abstract

Characterizing a probability distribution is essential in the fields of probability and statistics. This paper presents the characterization results of the 2Kth order weighted Maxwell Boltzmann distribution (KWMBD). The characterization results are based on the coefficient of variance, a straightforward relationship between two truncated moments, the conditional expectation of a specific function of the random variable, the hazard function, and the reverse hazard function. Notably, for these characterizations, the cumulative distribution function (CDF) does not need to have a closed form; instead, the CDF is derived from the solution of a first-order differential equation. This approach not only extends the applicability of the KWMBD but also establishes a crucial link between probability theory and differential equations. The resulting insights have significant implications for both theoretical research and practical applications in statistical modeling and analysis.

Keywords: Characterization, Truncated Moments, Hazard rate function, reverse hazard rate function.

I. Introduction

The 2Kth order weighted Maxwell Boltzmann distribution is a flexible continuous univariate probability distribution suitable for modeling datasets of decreasing increasing, increasing and constant behavior. The cumulative distribution function (CDF) and probability density function (PDF) of KWMBD is given by:

$$F(y,\alpha,k) = 1 - \frac{\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \qquad y>0,\alpha>0,k>-1.5. \tag{1}$$

$$f(y,\alpha,k) = \frac{y^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{y^2}{2\alpha^2}}}{2^{\left(k+\frac{1}{2}\right)}\Gamma(k+\frac{3}{2})}, \qquad y>0,\alpha>0,k>-1.5. \tag{2}$$

The hazard function, reverse hazard function of KWMBD are respectively given by

$$\lambda(y,\alpha,k)=\frac{y^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{y^2}{2\alpha^2}}}{2^{\left(k+\frac{1}{2}\right)}\left\{\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)\right\}},\qquad y>0,\alpha>0,k>-1.5. \tag{3}$$

$$r(y,\propto,k)=\frac{y^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{y^2}{2\alpha^2}}}{2^{\left(k+\frac{1}{2}\right)}\left\{\Gamma\left(k+\frac{3}{2}\right)-\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)\right\}},\quad y>0,\alpha>0,k>-1.5. \tag{4}$$

The structural properties like mean, variance, and coefficient of variance are respectively given as

$$E(y)=\sqrt{2}\alpha\frac{\Gamma\left(\frac{2k+4}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \tag{5}$$

$$var(y)=\frac{2\propto^2}{\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2}\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right] \tag{6}$$

$$CV=\frac{\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right]^{\frac{1}{2}}}{\Gamma\left(\frac{2k+4}{2}\right)} \tag{7}$$

II. Characterization

To develop a stochastic function for a specific problem, it is crucial to ensure that the function aligns with the theory of a particular underlying probability distribution. This necessitates the study of characterizations of the probability distribution. Characterization involves identifying certain distributional or statistical properties of a statistic that uniquely determine the associated probability distribution. Recently, the problem of characterizing a distribution has drawn considerable attention from researchers, leading to the development of various characterizations in multiple directions. For example, Glänzel [5] investigated characterizations of probability distributions by truncated moments, while Glänzel [6] delved into the consequences of such characterizations. Hamedani [7],[8] focused on characterizations of univariate continuous distributions. These characterizations can serve as a basis for parameter estimation of a probability distribution. Before applying a particular probability distribution model to fit real-world data, it is essential to confirm that the distribution meets the necessary requirements through its characterization.

Many scholars have used characterization theorems to describe both discrete and continuous probability distributions and to highlight the importance of characterizations in verifying the applicability and accuracy of probability distribution models in representing real-world data. Prominent examples include the characterization of the uniform distribution by Arslan et al. [1], the characterization of an exponential power life-testing distribution by Shakil et al. [10], the characterization of the Maxwell distribution based on truncated moments by Saghir et al. [11], the characterization and estimation of the Maxwell distribution by Dar et al. [4], the characterization of the new Weibull-X family of distributions by Ahmad et al. [2], and the characterization of the exponential distribution by Bakouch et al. [3].

In this article, KWMBD is characterized through coefficient of variance theorem, ratio of truncated moments, conditional expectation of a specific function, hazard rate function, and reverse hazard rate function.

I. Characterization of KWMBD through Coefficient of variance

In this subsection we characterize the KWMBD by statistical measure (coefficient of variation) (CV)). It is a measure of relative variability, calculated as the ratio of the standard deviation to the mean.

$$CV = \frac{\sigma}{\mu}$$

Where, σ and μ are respectively the mean and standard deviation.

Theorem I: Let $Y_1, Y_2, \ldots, Y_m$, be a independent random sample of size m drawn from KWMBD. Then, the square of sample coefficient of variation is asymptotically unbiased estimator of the square of population coefficient of variation.
Mathematically,

$$\lim_{m\to\infty} E\left(\frac{{S_m}^2}{\overline{Y_m}^2}\right) = \lim_{m\to\infty} E\left(\frac{S_m}{\overline{Y_m}}\right)^2 = \left(\frac{\sigma}{\mu}\right)^2,$$

Where $\overline{Y_m}$ and ${S_m}^2$ are respectively the mean and variance of the sample.
Proof: Since, the sample mean $(\overline{Y_m})$ is an unbiased estimator of population mean (μ) with variance $\frac{\sigma^2}{m}$. i.e, $E(Y_m) = \mu$, $var\,(Y_m) = \frac{\sigma^2}{m}$
Also, we have,

$$var\,(\overline{Y_m}) = E(\overline{Y_m}^2) - [E(\overline{Y_m})\,]^2$$

$$E(\overline{Y_m}^2) = var\,(\overline{Y_m}\,) + [E\overline{(Y_m)}\,]^2$$

Substituting (5) and (6), we get,

$$E\left(\overline{Y_m}^2\right) = \frac{2 \propto^2}{m\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2}\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) - (1-m)\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right] \tag{8}$$

Also, $E({S_m}^2) = \sigma^2$, therefore, from (6) we have,

$$E({S_m}^2) = \frac{2 \propto^2}{\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2}\left[\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right) - \left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2\right] \tag{9}$$

Since ,$Y_1, Y_2, \ldots, Y_m$ are independent, we can write,

$$E\left(\frac{{S_m}^2}{\overline{Y_m}^2}\right)=\frac{E\left({S_m}^2\right)}{E\left(\overline{Y_m}^2\right)}$$

Now, using (8) and (9) we obtain

$$E\left(\frac{{S_m}^2}{\overline{Y_m}^2}\right)=\frac{\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2}{\frac{1}{m}\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(1-\frac{1}{m}\right)\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2}$$

Applying $\lim_{m\to\infty}$ on both sides we get,

$$\lim_{m\to\infty}E\left(\frac{{S_m}^2}{\overline{Y_m}^2}\right)=\frac{\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2}{\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2}$$

$$\lim_{m\to\infty}E\left(\frac{{S_m}^2}{\overline{Y_m}^2}\right)=\left[\frac{\sqrt{\Gamma\left(\frac{2k+5}{2}\right)\Gamma\left(k+\frac{3}{2}\right)-\left(\Gamma\left(\frac{2k+4}{2}\right)\right)^2}}{\Gamma\left(\frac{2k+4}{2}\right)}\right]^2=\left(\frac{\sigma}{\mu}\right)^2$$

Hence the theorem.

II. Characterization based on ratio of truncated moments

This subsection explores the characterization of the KWMBD through the ratio of two truncated moments, utilizing Theorem II from Glänzel [5]. The result is valid even if the interval H is not closed, since the theorem's condition applies within the interior of H.

Theorem II: Let $(\Omega;F;P)$ be a given probability space, and let $H=[e,f]$ be an interval for some $e<f$, Where, $e=-\infty$ and $f=+\infty$ might as well be allowed. Also, let $Y:\Omega\to H$ be a continuous RV with the CDF $F(y)$ and $h_1(y)$ and $h_2(y)$ be two real functions defined on H and such that

$$E[h_1(y)|Y\geq y]=\vartheta(y)E[h_2(y)|Y\geq y]\qquad y\epsilon H$$

Is defined with some real function $\vartheta(y)$. Assume that $h_1(y)$, $h_2(y)\epsilon C^1(H)$, $\vartheta(y)\epsilon C^2(H)$, and $F(y)$ is a twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\vartheta(y)h_2(y)=h_1(y)$ has no real solution in the interior of H. Then, $F(y)$ is uniquely determined by the functions $h_1(y)$, $h_2(y)$, and $\vartheta(y)$ as follows:

$$F(y)=\int_0^y C\left|\frac{\vartheta'(u)}{\vartheta(u)h_1(u)-h_2(u)}\right|e^{-S(u)}\,du$$

Where the function $s(y)$ is a solution of the differential equation

$$S(y) = \frac{\vartheta'(y)h_1(y)}{\vartheta(y)h_1(y) - h_2(y)}$$

And C is a constant such that $\int_H dF(y) = 1$.

Preposition I: Let $Y: \Omega\ (0, \infty)$ be a continuous random variable and let $h_1(y) = y^{-(2k+1)}$ and $h_2(y) = h_1(y)e^{-\frac{y^2}{2\alpha^2}}$ for > 0. Then, the random variable has PDF $f(y)$ if and only if the function defined in Theorem II is of the form,

$$\vartheta(y) = \frac{1}{2}e^{-\frac{y^2}{2\alpha^2}}: \qquad y > 0 \tag{10}$$

Or satisfying differential equation

$$\frac{\vartheta'(y)h_1(y)}{\vartheta(y)h_1(y) - h_2(y)} = \frac{y}{\alpha^2}; \qquad y > 0 \tag{11}$$

Proof:
Necessity: Suppose the RV $Y \sim KWMBD(\alpha, k)$, with the CDF and PDF given by equations (1) and (2), respectively, then, we have,

$$[1 - F(y)]E[h_1(y)|Y \geq y] = qe^{-\frac{y^2}{2\alpha^2}}$$

And

$$[1 - F(y)]E[h_2(y)|Y \geq y] = \frac{q}{2}e^{-\frac{y^2}{\alpha^2}}$$

Where, q is constant.
Also, we have by theorem (II)

$$E[h_1(y)|Y \geq y] = E[h_2(y)|Y \geq y]\vartheta(y)$$

$$=> \quad \vartheta(y) = \frac{E[h_1(y)|Y \geq y]}{E[h_2(y)|Y \geq y]}$$

$$\vartheta(y) = \frac{1}{2}e^{-\frac{y^2}{2\alpha^2}}$$

We also have,

$$\vartheta(y)h_1(y) - h_2(y) = -\frac{1}{2}y^{-(2k+1)}e^{-\frac{y^2}{2\alpha^2}} = -\frac{1}{2}h_1(y)e^{-\frac{y^2}{2\alpha^2}}$$

$$=> \quad \frac{\vartheta'(y)h_1(y)}{\vartheta(y)h_1(y) - h_2(y)} = \frac{y}{\alpha^2}$$

Hence equation (11) holds

Sufficiency: If the function ϑ(y) satisfies the equation (11), then it follows that

$$S'(y) = \frac{\vartheta'(y)h_1(y)}{\vartheta(y)h_1(y) - h_2(y)} = \frac{y}{\alpha^2}$$

$$=> \qquad S(y) = \frac{y^2}{2\alpha^2}$$

Thus from theorem (II), $F(x)$ is uniquely determined by the functions $h_1(y)$, $h_2(y)$ and ϑ(y) as follows:

$$F(y) = \int_0^y C\left|\frac{\vartheta'(y)}{\vartheta(y)h_1(y) - h_2(y)}\right| e^{-S(y)}\,dy$$

$$F(y) = \int_0^y C\frac{y^{2k+2}}{\alpha^2} e^{-\frac{y^2}{2\alpha^2}}\,dy$$

Where, C is normalizing constant such that $\int_\Omega f(y)dy = 1$

$$C^{-1} = \alpha^{(1+2k)} 2^{(k+\frac{1}{2})}\Gamma(k + \frac{3}{2})$$

Thus,

$$F(y) = \int_0^y \frac{y^{2k+2}}{\alpha^{(2+2k)} 2^{(k+\frac{1}{2})}\Gamma(k+\frac{3}{2})} e^{-\frac{y^2}{2\alpha^2}}\,dy$$

This is distribution function of KWMBD, thus RV $Y \sim KWMBD(\alpha, k)$.

Collorary I: Let $Y: \Omega\ (0, \infty)$ be a continuous random variable and let $h_1(y)$ be as in above proportion. The PDF of Y is (2) iff there exits $h_2(y)$ and ϑ(y) defined in theorem (II) satisfying differential equation

$$\frac{\vartheta'(y)h_1(y)}{\vartheta(y)h_1(y) - h_2(y)} = \frac{y}{\alpha^2}; \qquad y > 0 \qquad (12)$$

After simplification, we get ,

$$\vartheta'(y) = \frac{y}{\alpha^2}\left(\vartheta(y) - h_1^{-1}(y)h_2(y)\right)$$

Upon integrating and multiplying both sides by $e^{-\frac{y^2}{2\alpha^2}}$. It is easy to show that the general solution of differential equation is

$$\vartheta'(y) = e^{-\frac{y^2}{2\alpha^2}}\left(-\int \frac{y}{2\alpha^2} e^{-\frac{y^2}{2\alpha^2}} h_1^{-1}(y)h_2(y) + M\right)$$

Where, M is a constant. Note that a set of functions satisfying the differential equation (12) is given in Proposition I with $M = 0$. However, it should also be noted that there are other triplets $(h_1; h_1;\ \vartheta)$ satisfying the conditions of Theorem (II).
Proof: Same as preposition (I)

III. Characterization based on the conditional expectation of certain function of the random variable

In this subsection, we utilize a single function ϑ of Y to describe the distribution of Y based on the truncated moments of $\vartheta(y)$. This Theorem III, which appeared in Hamedani's earlier work [9], is presented here to aid in characterizing the KWMBD.

Theorem III: Let $Y: \Omega \to (e, f)$ be a continuous random variable with CDF $F(\mathrm{y})$. Let $\vartheta(\mathrm{y})$ be a differentiable function on (e, f) with $\lim_{y \to e^+} \vartheta(\mathrm{y})$. Then for $\delta \neq 1$,

$$E[\vartheta(\mathrm{y})|Y \geq y] = \delta\vartheta(\mathrm{y}) \qquad y\epsilon(\,e, f)$$

If and only if

$$\vartheta(\mathrm{y}) = [1 - F(y)]^{\frac{1}{\delta}-1} \qquad y\epsilon(\,e, f)$$

Preposition II: Let $Y: \Omega\ (\,0, \infty)$ be a continuous random variable with CDF (1). Let $\vartheta(\mathrm{y})$ be a differentiable function on $(\,0, \infty)$ with $\lim_{y \to 0^+} \vartheta(\mathrm{y})$,then for $\delta = \frac{1}{2}$

$$E[\vartheta(\mathrm{y})|Y \geq y] = \delta\vartheta(\mathrm{y}) \qquad y\epsilon(\,0, \infty)$$

Iff

$$\vartheta(\mathrm{y}) = \frac{1}{2}\left[\frac{\Gamma\left(\mathrm{k} + \frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k + \frac{3}{2}\right)}\right]^{\frac{1}{\delta}-1} \tag{13}$$

Proof:
Necessity: Suppose the RV $Y \sim KWMBD(\alpha\,, k)$, with the CDF and PDF given by equations (1) and (2), respectively, then, we have,

$$E[\vartheta(\mathrm{y})|Y \geq y] = \int_y^\infty \vartheta(\mathrm{y}) \frac{f(y)}{P(Y \geq y)} dy$$

$$=> \qquad E[\vartheta(\mathrm{y})|Y \geq y] = \frac{1}{2}\int_y^\infty f(y)dy$$

$$=> \qquad E[\vartheta(\mathrm{y})|Y \geq y] = \frac{1}{2}\left[\frac{\Gamma\left(\mathrm{k} + \frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k + \frac{3}{2}\right)}\right]^{\frac{1}{\frac{1}{2}}-1}$$

$$=> \qquad \vartheta(\mathrm{y}) = \left[\frac{\Gamma\left(\mathrm{k}+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}\right]^{\frac{1}{\delta}-1}$$

Hence, the necessary condition.

Sufficiency: If the function $\vartheta(y)$ satisfies the equation (13), then it follows that

$$\vartheta(y) = \frac{1}{2}\left[\frac{\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}\right]^{\frac{1}{\delta}-1}$$

Also,

$$E[\vartheta(y)|Y \geq y] = \frac{1}{2}\left[\frac{\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}\right]^{\frac{1}{\frac{1}{2}}-1}$$

$$=> \quad [1-F(y)] = \left[\frac{\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}\right]$$

$$=> \quad F(y) = 1 - \frac{\Gamma\left(k+\frac{3}{2},\frac{y^2}{2\alpha^2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}$$

This is distribution function of KWMBD, thus RV $Y \sim KWMBD(\alpha\,, k)$.

IV. Characterization in terms of hazard function

The hazard function $\lambda(y)$ of a twice-differentiable distribution function $F(y)$ satisfies a specific first-order differential equation given in theorem IV. For many univariate continuous distributions, this is the sole differential equation involving the hazard function. In this subsection, we provide a detailed characterization of the KWMBD using the hazard function.

Theorem IV: The RV Y has continuous density function $f(y)$ if and only if its hazard function $\lambda(y)$ satisfies the differential equation

$$\frac{f'(y)}{f(y)} = \frac{\lambda'(y)}{\lambda(y)} - \lambda(y)$$

Preposition III: Let $Y: \Omega\ (\,0, \infty)$ be a continuous random variable. The PDF of Y is (2) if and only if its hazard function $\lambda(y)$ satisfies the differential equation

$$\lambda'(y) - \frac{2(k+1)}{y}\,\lambda(y) = -\frac{1}{q\alpha^2} y^{2k+3} e^{-\frac{y^2}{2\alpha^2}} \tag{14}$$

Proof:
Necessity: If has PDF (2), then, we have

$$\lambda'(y) = -\frac{1}{q\alpha^2} y^{2k+3} \boldsymbol{e}^{-\frac{y^2}{2\alpha^2}}\left[\mathbf{1} - \frac{\mathbf{2}(\boldsymbol{k}+\mathbf{1})\alpha^2}{y^2}\right]$$

Also,

$$\lambda'(y) - \frac{2(k+1)}{y}\,\lambda(y) = -\frac{1}{q\alpha^2} y^{2k+3} e^{-\frac{y^2}{2\alpha^2}}$$

Hence the necessary condition
Sufficiency: if its hazard function $\lambda(y)$ satisfies the differential equation (14)

$$y^{-(2k+2)}\left[\lambda'(y) - \frac{2(k+1)}{y}\,\lambda(y)\right] = -\frac{y}{q\alpha^2} e^{-\frac{y^2}{2\alpha^2}}$$

$$=> \quad \frac{d}{dy}\left(y^{-(2k+2)}\lambda(y)\right) = -\frac{1}{q}\frac{d}{dy} e^{-\frac{y^2}{2\alpha^2}}$$

$$=> \quad \lambda(y) = \frac{1}{q\alpha^2} y^{2k+2} e^{-\frac{y^2}{2\alpha^2}}$$

This is hazard rate function of KWMBD, thus RV $Y \sim KWMBD(\alpha\,, k)$.

V. Characterization in terms of the reverse hazard function

The reverse hazard function $r(y)$ of a twice-differentiable distribution function F is defined as $\frac{f(y)}{1-F(y)}$. In this subsection, we provide a characterization of the KWMBD using the reverse hazard function.

Preposition IV: Let $Y{:}\,\Omega\ (\,0,\infty)$ be a continuous random variable. The PDF of Y is (2) if and only if its reverse hazard function $r(y)$ satisfies the differential equation

$$r'(y) - \frac{2(k+1)}{y}\,r(y) = \frac{y^{2k+2}}{q\alpha^2}\frac{d}{dy}\left(\frac{e^{-\frac{y^2}{2\alpha^2}}}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}\right) \tag{15}$$

Proof:
Necessity: If has PDF (2), then, we have

$$r'(y) = \frac{y^{2k+2}}{q\alpha^2}\frac{d}{dy}\left(\frac{e^{-\frac{y^2}{2\alpha^2}}}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}\right) + \frac{1}{q\alpha^2}\left(\frac{e^{-\frac{y^2}{2\alpha^2}}}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}\right)(2k+1)y^{2k+2}$$

$$=> \quad r'(y) - \frac{2(k+1)}{y}\, r(y) = \frac{y^{2k+2}}{q\alpha^2}\frac{d}{dy}\left(\frac{e^{-\frac{y^2}{2\alpha^2}}}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}\right)$$

Hence, the necessary part.

Sufficiency: if its reverse hazard function $r(y)$ satisfies the differential equation (15)

$$y^{-(2k+2)}\left[r'(y) - \frac{2(k+1)}{y}\, r(y)\right] = \frac{1}{q\alpha^2}\frac{d}{dy}\left(\frac{e^{-\frac{y^2}{2\alpha^2}}}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}\right)$$

$$=> \quad \frac{d}{dy}\left(y^{-(2k+2)} r(y)\right) = \frac{1}{q\alpha^2}\frac{d}{dy}\left(\frac{e^{-\frac{y^2}{2\alpha^2}}}{\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)}\right)$$

$$=> \quad r(y) = \frac{y^{-(2k+2)} e^{-\frac{y^2}{2\alpha^2}}}{q\left(\Gamma\left(k+\frac{3}{2}\right) - \Gamma\left(\mathrm{k}+\frac{3}{2}, \frac{y^2}{2\alpha^2}\right)\right)}$$

This is reverse hazard rate function of KWMBD, thus RV $Y \sim KWMBD(\alpha\, , k)$.

III. Conclusion

In this article, we studied the characterization results of 2Kth order weighted Maxwell Boltzmann distribution (KWMBD), an extension of the traditional Maxwell-Boltzmann distribution. We have characterized KWMBD via the coefficient of variance, ratio of truncated moments, the conditional expectation of a specific function of the random variable, hazard rate functions, and reverse hazard rate functions. Our principal findings are presented in Preposition I, II, III and IV. We found that, for these characterization results, the cumulative distribution function (CDF) does not need to have a closed form, instead, the CDF is derived from the solution of a first-order differential equation. These findings establish a crucial link between probability theory and differential equations, thereby extending the applicability of the KWMBD.

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