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MOMENTS OF GENERALIZED ORDER STATISTICS FROM DOUBLY TRUNCATED MODIFIED MAKEHAM DISTRIBUTION AND ITS CHARACTERIZAION

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Abstract

The Makeham distribution has been extensively used in modeling actuarial data and helps as a fundamental life distribution. It describes a mortality failure law in which the hazard rate is composed of two components: a non-aging failure distribution and an aging failure distribution, the latter characterized by an exponentially increasing failure rate. The concept of generalized order statistics (gos), introduced by Kamps in 1995, provides a unified model for combining several models of ordered random variables, including order statistics, record values, progressive type II right-censored order statistics and more. The development of recurrence relations for the moments of gos is an important field of research due to the model's accessibility and flexibility. These recurrence relations simplify the computation of higher-order moments and establish the associations between moments of different orders. In this article, we have derived the recurrence relations of gos for single and product moments from the doubly truncated modified Makeham distribution and have also reduced these relations for specific ordered random schemes, such as order statistics, $k-$th record values and sequential order statistics for selected different values of parameters. In the final section, we have presented a characterization theorem based on the recurrence relations of the moments. This theorem has identified the doubly truncated modified Makeham distribution through its moment structure within the gos framework. This shows how important these relationships are for setting this distribution apart from other probability models.

Keywords: Modified Makeham distribution, generalized order statistics, order statistics, record values, recurrence relations and characterization.

1. Introduction

The moment properties of generalized order statistics (gos) for various distributions, as well as the recurrence relations between them, have recently attracted a lot of attention. Many authors have studied the moments of gos for several distributions. Kamps [1] proposed the concept of gos. The gos offers a unified approach to various models of ordered random variables such as upper record values, order statistics, $k-$th upper record values, progressively Type II censoring order statistics, sequential order statistics and Pfeifer records. These models have valuable applications, especially in the field of reliability theory. Khan *et al*. [2] obtained the moments properties of gos and also characterized the gos from the Lindley-Weibull distribution. Kumar and Dey [3] derived exact algebraic explicit expressions and also obtained recurrence relations for the single, product, and conditional moments of gos based on the extended exponential distribution. Athar *et al*. [4] studied moment properties of gos from the modified weighted Rayleigh distribution and also discussed the special cases of *gos* such as order statistics, record values, and progressive type II right censored order statistics. Furthermore, the results for the characterization are presented, which are based on the conditional expectation and moment properties. Khan and Khan [5] obtained the recurrence relation of the generalized upper record values for single and product moments and also characterized for single moments by using conditional expectations and recurrence relation from the modified Makeham distribution.

Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed (iid) random variables (rv) with absolutely continuous distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called *gos* if their joint *pdf* is given by

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i)\Big)[1 - F(x_n)]^{k-1} f(x_n) \tag{1}$$

on the cone $F^{-1}(0) \leq x_1 \ldots \leq x_n \leq F^{-1}(1)$ of $\mathbb{R}^n$.

Choosing the parameters appropriately (see, Cramer [20]), we get

Table 1: *Variants of the generalized order statistics*

	$\gamma_n = k$	γ_r	m_r
(i) Sequential order statistics	α_n	$(n - r + 1)\alpha_r$	$(\gamma_r - \gamma_{r+1} - 1)$
(ii) Ordinary order statistics	1	$n - r + 1$	0
(iii) Record values	1	1	-1
(iv) Progressively type II censored order statistics	$R_n + 1$	$n - r + 1 + \sum_{j=r}^{n} R_j$	R_r
(v) Pfeifer's record values	β_n	β_r	$\beta_r - \beta_{r+1} - 1$

In view of (1) with $m_i = m$, $i = 1, 2, \ldots n - 1$, the *pdf* of $r - th$ *gos* $X(r, n, m, k)$ is

$$f_r(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)) \tag{2}$$

and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$, is

$$f_{r,s}(x, y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} [\bar{F}(x)]^m g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(x) f(y), \quad \alpha \leq x < y \leq \beta, \tag{3}$$

where

$$\bar{F}(x) = 1 - F(x)$$

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} &, \ m \neq -1 \\ -\log(1-x) &, \ m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \ \ x \in [0,1).$$

The Makeham distribution has been widely used to fit actuarial data and is an essential life distribution (see Marshall and Olkin, [6]). As per Scollnik [7], the Makeham distribution of mortality describes a failure law, wherein the hazard rate is a combination of a non-aging failure distribution and an aging failure distribution characterized by an exponential increase in failure rates.
A random variable X is said to have modified Makeham distribution if its pdf is of the form

$$f_1(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} \exp\left[\left(\frac{x}{a}\right)^b\right] \exp\left[1 - \exp\left(\frac{x}{a}\right)^b\right], \ \ x \geq 0, \ \ a > 0, \ \ b > 0,$$

with the corresponding df

$$F_1(x) = 1 - \exp\left[1 - \exp\left(\frac{x}{a}\right)^b\right], \ \ x \geq 0, \ \ a > 0, \ \ b > 0,$$

where a and b are the scale and shape parameters respectively.

We assume without loss of generality, that $a = 1$, in which case the pdf of modified Makeham distribution becomes

$$f_1(x) = b\ x^{b-1}\ e^{x^b} e^{(1-e^{x^b})}, \ \ x \geq 0, \ \ b > 0, \tag{4}$$

and the corresponding df

$$F_1(x) = 1 - e^{(1-e^{x^b})}, \ \ x \geq 0, \ \ b > 0, \tag{5}$$

A truncated distribution is a probability distribution restricted to a particular range or interval. Truncation arises when the values over a specified range are omitted, and the values that remain are re-normalized to establish a proper probability distribution. Truncated distributions have wide applications in statistics, economics, engineering, and many other fields where data is naturally bounded. Here, two truncated points are given P_1 and Q_1.

Now,

$$\int_0^{Q_1} f_1(x)dx = Q \quad \text{and} \quad \int_0^{P_1} f_1(x)dx = P,$$

then the truncated pdf is given by

$$f(x) = \frac{1}{P-Q}\ b\ x^{b-1}\ e^{x^b} e^{(1-e^{x^b})}, \ \ Q_1 \leq x \leq P_1 \tag{6}$$

and corresponding truncated df $F(x)$ is

$$\bar{F}(x) = -P_2 + \frac{1}{b}\, x^{1-b}\, e^{-x^b} f(x), \tag{7}$$

where,

$$P_2 = \frac{1-P}{P-Q}. \tag{8}$$

Modified Makeham distribution is a distribution whose hazard function is bathtub for certain parameter values. This is highly desirable in the modeling of the lifetime of technical devices. furthermore, the skewness of the Modified Makeham distribution shifts from positive to negative as the shape parameter increases from zero to infinite. This distribution has been frequently used to describe human mortality and to establish actuarial tables. The basic characteristics of this distribution have been discussed in detail by Kosznik-Biernacka [8]. For detailed survey on characteristics and application of this distribution one may refer to Kosznik-Biernacka ([8], [9], [10]).

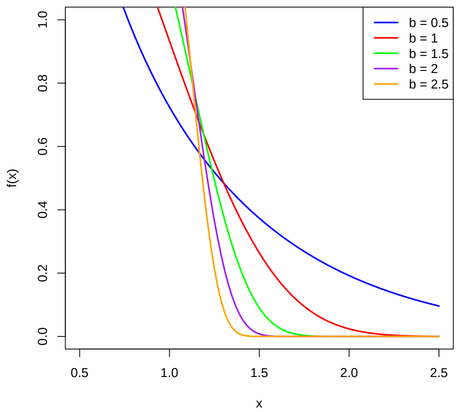

Figure 1: *pdfs of the DTMM distribution for selected values of parameter b.*

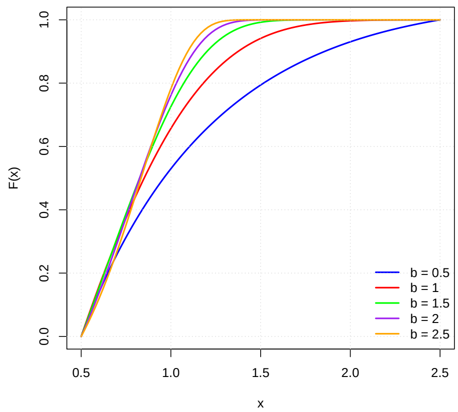

Figure 2: *cdfs of the DTMM distribution for selected values of parameter b.*

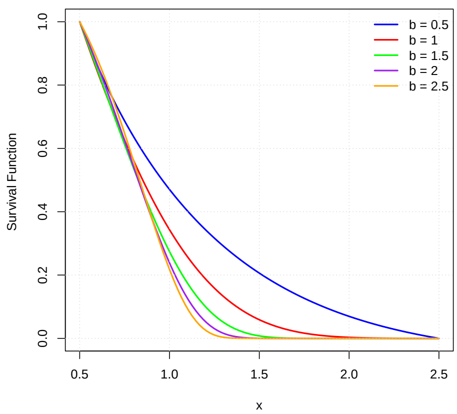

Figure 3: *Survival functions of the DTMM distribution for selected values of parameter b.*

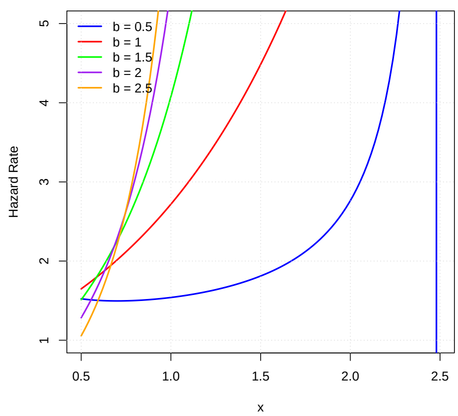

Figure 4: *Hazard rates of the DTMM distribution for selected values of parameter b.*

Recurrence relations for moments of *gos* for some specific as well as for general class of distribution are investigated by several authors in literature. Athar and Islam [16] and Anwar *et al*. [15] derived the recurrence relations for single and product moments of generalized order statistics for some general class of distributions whereas Faizan and Athar [22] obtained explicit expressions for exact moments of generalized order statistics. Bakar et al. [30] have derived recurrence relations for the single and product moments of *gos* based on the Lindley Pareto distribution. Khan *et al*. [29] obtained moments of generalized order statistics for Erlang truncated exponential distribution. Further, Athar *et al*. [17] obtained relations for moments of generalized order statistics from Marshall-Olkin extended Weibull distribution and Athar *et al*. [18] studied the expectation identities of Pareto distribution based on generalized order statistics. Minimol and Thomas [32] discussed some properties of Makeham distribution using generalized record values while Athar *et al*. [19] obtained some recurrence relations for moments of *gos* from doubly truncated Makeham distribution. Ismail and Abu-Youssef [24] derived the general recurrence relations for single and product moments of order statistics for doubly truncated modified Makeham ($DTMM$) distribution and characterized the modified Makeham distribution based on the properties of order statistics. For some additional results, one may refer to Cramer and Kamps [21], Kamps and Cramer [26], Pawlas and Szynal [33], Ahmad and Fawzy [14], Ahmad ([12], [13]), Mahmoud and Al-Nagar [31] among others.

The characterization of probability distributions plays a vital role in statistical studies as it allows us to understand and utilize several distributions effectively. The distributions can be characterized using different statistical properties such as moments, truncated moments, reliability functions, order statistics, record values, characteristic function, etc. Each of these techniques uses distinct statistical properties to uniquely identify or characterize specific probability distributions. This allows statisticians and researchers to model data effectively, understand underlying processes, and make informed decisions based on statistical analyses. Many authors have explored the characterization of probability distributions through truncated moments, conditional expectation, and recurrence relation. Khan and Abu-Salih [34] explored the characterization of probability distribution through conditional expectation of order statistics. Bakar *et al*. [38] studied the characterization of two generalized distributions based on conditional expectation of order statistics. Kilany [35] studied the characterization of the Lindley distribution using truncated moments of order statistics. The characterization results of two general classes of distributions, the Haseeb generalized Lindley and the Haseeb generalized inverse Lindley distributions, have been explored by Athar and Amir [37]. Athar *et al*. [36] studied the characterization of several generalized continuous probability distribution through doubly truncated moments.

The next part of the paper is structured as follows. In Section 2 and 3, we have obtained recurrence relations for single and product moments of generalized order statistics from doubly truncated modified Makeham distribution. Further, the results are deduced for order statistics and record values as a particular cases of generalized order statistics. In Section 4, we have established a theorem characterizing doubly truncated modified Makeham distribution through conditional expectation of generalized order statistics and the whole study of this paper concludes in the Section 5.

2. Single Moment

In this Section, we establish the recurrence relations of *gos* from doubly truncated modified Makeham distribution. Moreover, we drive the expression of the moments of order statistics and record value .

Lemma 2.1: For $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots,$

(i) $E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right]$

$$= \frac{C_{r-2}}{(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}\big(F(x)\big) dx. \tag{9}$$

$$\text{(ii)}\quad E\big[X^j(r-1,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big]$$

$$= -\frac{(m+1)C_{r-2}}{\gamma_1 (r-2)!}\, j \int_{Q_1}^{P_1} x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}\big(F(x)\big) dx. \tag{10}$$

$$\text{(iii)}\quad E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big]$$

$$= \frac{C_{r-1}}{\gamma_1 (r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}\big(F(x)\big) dx. \tag{11}$$

Proof: We have by Athar and Islam [16],

$$E\big[\xi\{X(r,n,m,k)\}\big] - E\big[\xi\{X(r-1,n,m,k)\}\big]$$

$$= \frac{C_{r-2}}{(r-1)!} \int_{\alpha}^{\beta} \xi'(x) \big[\bar{F}(x)\big]^{\gamma_r} g_m^{r-1}\big(F(x)\big) dx, \tag{12}$$

where $\xi(x)$ is a Borel measurable function of $x \in (\alpha, \beta)$.

Therefore, lemma [2] can be established by letting $\xi(x) = x^j$ in the (12). Relations (10) and (11) can be seen on the lines of (9).

Theorem 2.1: For the given modified Makeham distribution and $n \in N,\ m \in \mathbb{R},\ 2 \le r \le n,\ j = 1, 2, \ldots,$

(i) $E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n,m,k)\big]$

$$= -P_2\, K\big\{E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)]\big\}$$

$$+ \frac{j}{\gamma_r} \frac{1}{b} E[\psi\{X(r,n,m,k)\}]. \tag{13}$$

(ii) $E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big]$

$$= -P_2\, K \frac{\gamma_r}{\gamma_1} \big\{E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)]\big\}$$

$$+ \frac{j}{\gamma_1} \frac{1}{b} E[\psi\{X(r,n,m,k)\}], \tag{14}$$

where

$$\psi(x) = x^{j-b} e^{-x^b},\ K = \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1} \Big(\frac{\gamma_i}{\gamma_i^{(n-1)} + m} \Big),$$

$$C_{r-2}^{(n-1,k+m)} = \prod_{i=1}^{r-1} \gamma_i^{(n-1,k+m)} = \prod_{i=1}^{r-1} [(k+m) + (n-1-i)(m+1)]$$

and $\gamma_r^{(n-1)} = k + (n-1-r)(m+1).$

Proof: In view of (7) and (9), we have

$$E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n,m,k)\big]$$

$$= \frac{C_{r-2}}{(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r - 1} \Big\{ -P_2 + \frac{1}{b}\, x^{1-b}\, e^{-x^b} f(x) \Big\} g_m^{r-1}\big(F(x)\big) dx$$

$$= -P_2\, \frac{C_{r-2}}{(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}\big(F(x)\big) dx$$

$$+ \frac{1}{b} \frac{C_{r-2}}{(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-b} e^{-x^b} \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}\big(F(x)\big) f(x) dx$$

$$= -P_2 \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} \left\{ \frac{C_{r-2}^{(n-1,k+m)}}{(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1} \left[\bar{F}(x)\right]^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}\big(F(x)\big) dx \right\}$$

$$+ \frac{1}{b} \frac{j}{\gamma_r} \frac{C_{r-1}}{(r-1)!} \int_{Q_1}^{P_1} \psi(x) \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}\big(F(x)\big) f(x) dx,$$

as $\gamma_r - 1 = \gamma_r^{(n-1,k+m)} = (k+m) + [(n-1) - r](m+1)$.

Therefore,

$$E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right]$$
$$= -P_2\, K\big\{ E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)] \big\}$$
$$+ \frac{j}{\gamma_r} \frac{1}{b} E[\psi\{X(r,n,m,k)\}].$$

Hence (13) proved.

Now consider the following identity ($c.f.$ Lemma 3.1 of Kamps, [25], pp-231),

$$[k + (n-r-1)(m+1)] E\left[X^j(r,n,m,k)\right] + r(m+1) E\left[X^j(r+1,n,m,k)\right]$$
$$= [k + (n-1)(m+1)] E\left[X^j(r,n-1,m,k)\right]. \tag{15}$$

or

$$\gamma_r\, E\left[X^j(r-1,n,m,k)\right] + (r-1)(m+1) E\left[X^j(r,n,m,k)\right]$$
$$= \gamma_1\, E\left[X^j(r-1,n-1,m,k)\right]. \tag{16}$$

Now, from (13) and (16), we obtain

$$E\left[X^j(r,n,m,k)\right] + \frac{(r-1)(m+1)}{\gamma_r} E\left[X^j(r,n,m,k)\right] - \frac{\gamma_1}{\gamma_r} E\left[X^j(r-1,n-1,m,k)\right]$$
$$= -P_2\, K\big\{ E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)] \big\}$$
$$+ \frac{j}{\gamma_r} \frac{1}{b} E[\psi\{X(r,n,m,k)\}], \tag{17}$$

which, upon rearrangement, yields (14).

Remark 2.1: At $P = 1,\ Q = 0$, we get the relation for non-truncated case

$$E\left[X^j(r,n,m,k)\right] = E\left[X^j(r-1,n-1,m,k)\right] + \frac{j}{\gamma_1} \frac{1}{b} E[\psi\{X(r,n,m,k)\}].$$

Remark 2.2: Recurrence relation for single moments of order statistics ($m = 0,\ k = 1$) is

$$E(X^j_{r:n}) - E(X^j_{r-1:n-1}) = -P_2\{E(X^j_{r:n-1}) - E(X^j_{r-1:n-1})\} + \frac{j}{nb}E[\psi(X_{r:n})]$$

or

$$E(X^j_{r:n}) = -P_2 E(X^j_{r:n-1}) + Q_2 E(X^j_{r-1:n-1}) + \frac{j}{nb}E[\psi(X_{r:n})].$$

By convention, we use $X_{n:n-1} = P_1$ and $X_{0:n-1} = Q_1$.

Remark 2.3: For $k-th$ record statistic $(m = -1)$, recurrence relation for single moments reduces as

$$E(X^{(k)}_{U(r)})^j - E(X^{(k)}_{U(r-1)})^j = -P_2\left(\frac{k}{k-1}\right)^{r-1}\left\{E(X^{(k-1)}_{U(r)})^j - E(X^{(k-1)}_{U(r-1)})^j\right\} + \frac{j}{bk}E[\psi(X^{(k)}_{U(r)}].$$

Similarly, the recurrence relation for single moments of order statistics with non-integral sample size for $m = 0,\ k = \alpha - n + 1,\ \alpha \in \mathbb{R}_+$ and for sequential order statistics for $m = \alpha - 1,\ k = \alpha$ may be obtained.

Theorem 2.2: For the given modified Makeham distribution and $n \in N,\ m \in \mathbb{R},\ 2 \le r \le n,\ j = 1, 2, \ldots,$

$$\left(\frac{\gamma_1}{\gamma_1 - \gamma_r}\right)\Big\{E\big[X^j(r-1,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big]\Big\}$$
$$= P_2\ K\Big\{E\big[X^j(r,n-1,m,k+m)\big] - E\big[X^j(r-1,n-1,m,k+m)\big]\Big\} + \frac{j}{\gamma_r}\frac{1}{b}E\big[\psi\{X(r,n,m,k)\}\big], \tag{18}$$

Proof: By subtracting (13) from (14), (18) can be established.

Theorem 2.3: For the given modified Makeham distribution and $n \in N,\ m \in \mathbb{R},\ 2 \le r \le n,\ j = 1, 2, \ldots,$

$$E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big]$$
$$= \frac{(P-Q)}{\gamma_1}\frac{j}{b}\ K^* E[\phi\{X(r,n,m,k+1)\}] \tag{19}$$
$$= \frac{j}{\gamma_1}\frac{1}{b}\Big[E[\phi\{X(r,n,m,k)\}] - (1-P)E[\psi\{X(r,n,m,k)\}]\Big], \tag{20}$$

where

$$K^* = \frac{C_{r-1}}{C^{(k+1)}_{r-1}} = \prod_{i=1}^{r}\left(\frac{\gamma_i}{\gamma_i + 1}\right), \quad C^{(k+1)}_{r-1} = \prod_{i=1}^{r}[(k+1) + (n-i)(m+1)]$$

and $\qquad \phi(x) = \psi(x)\ e^{-\left\{1-e^{x^b}\right\}}.$

Proof: On using (7) in (11), we get

$$E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big]$$

$$= \frac{C_{r-1}}{\gamma_1(r-1)!}\, j\int_{Q_1}^{P_1} x^{j-1}[\bar{F}(x)]^{\gamma_r} \left\{\frac{(P-Q)}{b}\ x^{1-b}e^{-x^b}e^{-\left\{1-e^{x^b}\right\}} f(x)\right\} g_m^{r-1}\big(F(x)\big)dx$$

$$= \frac{(P-Q)}{\gamma_1}\frac{1}{b}\frac{C_{r-1}}{C_{r-1}^{(k+1)}}\, j \left\{\frac{C_{r-1}^{(k+1)}}{(r-1)!}\int_{Q_1}^{P_1} \phi(x)[\bar{F}(x)]^{\gamma_r^{(k+1)}-1} g_m^{r-1}\big(F(x)\big)f(x)dx\right\},$$

where $\gamma_r^{(k+1)} = (k+1)+(n-r)(m+1)$ and hence the theorem.

To prove (20), note that

$$\frac{\bar{F}(x)}{f(x)} = \frac{1}{b}\ x^{1-b}e^{-x^b} - \frac{(1-P)}{b}\ x^{1-b}e^{-x^b}e^{-\left\{1-e^{x^b}\right\}}$$

and the result follows from (11).

3. Product Moments

Theorem 3.1: For the given modified Makeham distribution and $1 \le r < s \le n-1,\ m \in \mathbb{R},\ n \ge 2$ and $i = 1,2,\ldots;\ j = 1,2,\ldots$

$$E[X^i(r,n,m,k).X^j(s,n,m,k)] - E[X^i(r,n,m,k).X^j(s-1,n,m,k)]$$

$$= -P_2\ K_1 \Big\{E[X^i(r,n-1,m,k+m).X^j(s,n-1,m,k+m)]$$

$$-E[X^i(r,n-1,m,k+m).X^j(s-1,n-1,m,k+m)]\Big\}$$

$$+\frac{j}{\gamma_s}\frac{1}{b}E\big[\psi\{X(r,n,m,k),X(s,n,m,k)\}\big], \tag{21}$$

where

$$\psi\{X(r,n,m,k),X(s,n,m,k)\} = x^i y^{j-b}e^{-y^b}$$

$$K_1 = \frac{C_{s-2}}{C_{s-2}^{(n-1,k+m)}} = \prod_{i=1}^{s-1}\Big(\frac{\gamma_i}{\gamma_i^{(n-1)}+m}\Big).$$

Proof: In view of Athar and Islam [16], we have

$$E[X^i(r,n,m,k).X^j(s,n,m,k)] - E[X^i(r,n,m,k).X^j(s-1,n,m,k)]$$

$$= \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} j\int_{Q_1}^{P_1}\int_x^{P_1} x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}\big(F(x)\big)$$
$$\times[h_m\big(F(y)\big)-h_m\big(F(x)\big)]^{s-r-1}[\bar{F}(y)]^{\gamma_s}dy\ dx. \tag{22}$$

Now using (7) in (22), we get

$$= \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} j\int_{Q_1}^{P_1}\int_x^{P_1} x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}\big(F(x)\big)$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}\Big\{-P_2 + \frac{1}{b}\, y^{1-b}\, e^{-y^b} f(y)\Big\}dy\, dx.$$

Hence (21) can be established after noting that $\gamma_s - 1 = \gamma_s^{(n-1,k+m)},\ C_{s-1} = \gamma_s C_{s-2}$.

Remark 3.1: At $P = 1,\ Q = 0$, we get the relation for non-truncated case

$$E[X^i(r,n,m,k).X^j(s,n,m,k)] = E[X^i(r,n,m,k).X^j(s-1,n,m,k)] + \frac{j}{\gamma_s}\frac{1}{b}E\big[\psi\{X(r,n,m,k).X(s,n,m,k)\}\big], \tag{23}$$

Remark 3.2: Recurrence relation between product moments of order statistics ($m = 0,\ k = 1$) is

$$E\big(X^i_{r:n}X^j_{s:n}\big) = E\big(X^i_{r:n}X^j_{s-1:n}\big) - P_2\,\frac{n}{n-s+1}\Big\{E\big(X^i_{r:n-1}X^j_{s:n-1}\big) - E\big(X^i_{r:n-1}X^j_{s-1:n-1}\big)\Big\} + \frac{j}{b(n-s+1)}E\big[\psi(X_{r:n}, X_{s:n})\big]. \tag{24}$$

Remark 3.3: Recurrence relation for product moments of $k-th$ record value (at $m = -1$) is given as

$$E[(X^{(k)}_{U(r)})^i(X^{(k)}_{U(s)})^j] - E[(X^{(k)}_{U(r)})^i(X^{(k)}_{U(s-1)})^j]$$

$$= -P_2\Big(\frac{k}{k-1}\Big)^{s-1}\Big\{E[(X^{(k-1)}_{U(r)})^i(X^{(k-1)}_{U(s)})^j] - E[(X^{(k-1)}_{U(r)})^i(X^{(k-1)}_{U(s-1)})^j]\Big\} + \frac{j}{bk}E\big[\psi(X^{(k)}_{U(r)}, X^{(k)}_{U(s)})\big],$$

where $(X^{(k)}_{U(r)})^i$ is $i-$th moment of $k-$th upper record value.

4. Characterization

Let $X(r,n,m,k),\ r = 1,2,\ldots$ be gos, then the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, in view of (2) and (3) is

$$f_{s|r}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[\bar{F}(x)]^{m-\gamma_r+1} \times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y) \tag{25}$$

Theorem 4.1: Let X be an absolutely continuous rv with $df\ F(x)$ and $pdf\ f(x)$ with $F(x) < 1$, for all $x \in (0,\infty)$. Then for two consecutive values r and $r+1,\ 2 \le r+1 < s \le n$,

$$E[e^{\{X(s,n,m,k)\}^b}|X(l,n,m,k) = x] = g_{s|r}(x) = e^{x^b} + \sum_{j=l}^{s-1}\frac{1}{\gamma_{j+1}}, \quad l = r, r+1, \text{ and } \gamma_{j+1} \neq 0 \tag{26}$$

if and only if

$$F(x) = 1 - e^{(1-e^{x^b})}, \quad x \geq 0, \ b > 0,. \tag{27}$$

Proof: To prove necessary part, for $s \geq r+1$,

$$E\left[e^{\{X(s,n,m,k)\}^b} | X(r,n,m,k) = x\right] \tag{28}$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \tag{29}$$

$$\times \int_x^\infty e^{y^b} \left[1 - \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1} \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_s - 1} \frac{f(y)}{\bar{F}(x)} dy. \tag{30}$$

Set

$$u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{e^{\{1-e^{y^b}\}}}{e^{\{1-e^{x^b}\}}}, \quad \text{which implies,} \quad e^{y^b} = e^{x^b} - \ln u.$$

Then the right hand side of (27) reduces to

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_0^1 \left[e^{x^b} - \ln u\right] \left(1 - u^{m+1}\right)^{s-r-1} u^{\gamma_s - 1} du$$

Let $u^{m+1} = t$, then we get

$$E\left[e^{\{X(s,n,m,k)\}^b} | X(r,n,m,k) = x\right]$$

$$= e^{x^b} - \frac{\prod_{j=r+1}^s \gamma_j}{(s-r-1)!(m+1)^{s-r+1}} \int_0^1 \ln t \ t^{\frac{\gamma_s}{m+1} - 1} (1-t)^{s-r-1} \, dt$$

$$= e^{x^b} - \frac{\prod_{j=r+1}^s \gamma_j}{(s-r-1)!(m+1)^{s-r+1}} \, \mathrm{B}\left(\frac{\gamma_s}{m+1}, s-r\right) \left[\psi\left(\frac{\gamma_s}{m+1}\right) - \psi\left(\frac{\gamma_r}{m+1}\right)\right],$$

where,

$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$,

$$\psi(x-n) - \psi(x) = -\sum_{k=1}^n \frac{1}{x-k} \tag{31}$$

[*c.f.* Gradshteyn and Ryzhik, [23], pp-540, 905]

and $\mathrm{B}(a,b)$ is the complete beta function.

Therefore,

$$E[e^{\{X(s,n,m,k)\}^b} | X(r,n,m,k) = x] = e^{x^b} + \sum_{j=r}^{s-1} \frac{1}{\gamma_{j+1}}.$$

To prove sufficiency part, we have

$$g_{s|r}(x) = E[e^{\{X(s,n,m,k)\}^b} | X(l,n,m,k) = x].$$

Therefore,

$$g_{s|r+1}(x) - g_{s|r}(x) = -\frac{1}{\gamma_{r+1}},$$

and

$$g'_{s|r}(x) = b\ x^{b-1} e^{x^b}.$$

Now, in view of Khan et al. [27], we get

$$\frac{f(x)}{\bar{F}(x)} = -\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{\left[g_{s|r+1}(x) - g_{s|r}(x)\right]}$$
$$= b\ x^{b-1} e^{x^b}.$$

Implying that

$$\bar{F}(x) = e^{(1-e^{x^b})}.$$

Remark 2.1: At $s = r + 1$ in (26), we have

$$E\left[e^{\{X(r+1,n,m,k)\}^b} | X(r,n,m,k) = x\right] = e^{x^b} + \frac{1}{\gamma_{r+1}}.$$

Remark 2.2: For order statistics (at $m = 0, k = 1$), we have

$$E[e^{\{X_{s:n}\}^b} | X_{r:n} = x] = e^{x^b} + \sum_{j=r}^{s-1} \frac{1}{n-j}.$$

Further, at $s = r + 1$, we get

$$E[e^{\{X_{r+1:n}\}^b} | X_{r:n} = x] = e^{x^b} + \frac{1}{n-r}.$$

5. Conclusion

In this research, we have studied the modified Makeham distribution and have derived the doubly truncated distribution using this distribution, which restricts the random variable to a specific interval $[P_1, Q_1]$. Above investigations demonstrate the recurrence relations for the moments of *gos* from doubly truncated modified Mekeham distribution. Since recurrence relations reduce the amount of direct computation and hence reduce the time and labor. Therefore the relations under consideration may be useful in computing the moments of any order of doubly truncated modified Mekeham distribution. Further, characterization of the distribution under consideration through conditional expectation of *gos* is also presented.

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