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LAKIBUL - G FAMILY OF DISTRIBUTIONS: SPECIAL MODELS, PROPERTIES AND APPLICATIONS

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Abstract

In this study, a new generated family of distributions is introduced, called the Lakibul - G Family of Distributions. The T-X family of distributions is used in the derivation of the proposed class. Three special models of the proposed family are derived, such as the Lakibul - Uniform, Lakibul-Kumaraswamy, and Lakibul-Rayleigh distributions. Some properties of the proposed special models such as moments, mean, variance, moment generating function, survival, and hazard functions are studied. The Maximum likelihood approach is used to estimate the parameters of proposed models. Results show the following: (i) for Lakibul-Uniform distribution, it is found that this proposed probability distribution produces better estimate for electronic dataset as compared with the Cubic Transmuted Uniform distribution; (ii) for Lakibul - Kumaraswamy distribution is found to give a good fit for the COVID-19 dataset compared with the Marshall-Olkin reduce Kies and the Transmuted Kumaraswamy distributions; and (iii) for the Lakibul - Rayleigh distribution, it is observed that this distribution provides a better fit for precipitation dataset as compared with the Exponentiated Transform Inverse Rayleigh and the Exponentiated Inverse Rayleigh distributions.

Keywords: Uniform distribution, T-X family of distributions, Rayleigh distribution, Kumaraswamy distribution.

1. Introduction

The probability distribution plays an important role in describing the behavior of real-life data. The Weibull, log-logistic, and log-normal distributions are popular classical probability distributions for modeling non-negative or lifetime data. For bimodal behavior, bathtub behavior, and other complex behaviors of the data, the classical distributions are no longer flexible to describe the said behaviors of the data. Because of that, the extension and generalization of any classical distributions have become an area of interest for some researchers.

Azzalini [3] proposed a skewed family of distributions to generate a distribution with an additional skewed parameter. Eugene et al. [7] introduced the Beta-G family of distributions using the beta distribution. Cordeiro and de Castro [6] developed a Kumaraswamy - G family as an alternative to the Beta-G family using Kumaraswamy distribution. Other identified family of distributions are the Transmuted - G family [16], the exponentiated generalized class of distributions [5], the generalized transmuted - G family of distributions [13] and the exponentiated generalized Kumaraswamy - G class [17].

Rahman et al. [14] proposed a cubic transmuted family of distributions by adding an additional parameter to the baseline distribution. In addition, an extended version of the uniform distribution called the cubic transmuted uniform distribution was derived, and some of its statistical properties such as its moments, mean, variance, moment generating function, characteristic function, mean absolute deviation, quantile function, median, reliability function, hazard function, and Shannon entropy were studied.

Lakibul and Tubo [11] established a new generated family of distributions called the T-extended Standard U-quadratic - G family of distributions (TeSU-G). In addition, a bimodal version of the Weibull distribution was derived using the TeSU-G family of distributions. An extension of the TeSU-G family of distributions can be found in the study by Lakibul and Tubo [10]. Furthermore, the bivariate version of the extended Standard U-quadratic distribution can be seen in the paper by Lakibul, Polestico, and Supe [12].

In this paper, the main objectives are the following: (i) to introduce the Lakibul-G family of distributions; (ii) to derive some special distributions to be derived such as the Lakibul-Uniform distribution, Lakibul-Kumaraswamy distribution and the Lakibul-Rayleigh distribution using the proposed class; (iii) to derive some properties of each of the proposed distributions such as moment, mean, variance, moment generating function, survival and hazard functions; (iv) to estimate the parameters of each of the proposed distributions using the maximum likelihood method; and (v) to apply the proposed distributions on real data and compare with some existing distributions.

The remainder of the paper is organized as follows. Lakibul - G family of distributions is introduced in Section 2. In Section 3, Lakibul - Uniform distribution is derived and its properties are studied. Lakibul-Kumaraswamy distribution and its properties are presented in Section 4. In Section 5, the Lakibul-Rayleigh distribution and its properties are introduced and derived. The applicability of each of the proposed distributions is discussed in Section 6. In Section 7, a concluding remark is presented.

2. Lakibul-G Family of Distributions

Alzaatreh et al. [2] introduced the T-X family of distributions to extend any continuous distribution to a generalized distribution. The cumulative distribution function of the special case of the T-X family of distribution is given by

$$F_{T-X}(x) = \int_0^{G(x)} f(t)dt, \tag{1}$$

where $G(x)$ is any baseline cumulative distribution function and $f(t)$ is any probability density function (pdf) with support on the interval $[0,1]$. Lakibul - G family of distributions is obtained by using

$$f(t) = 2(1-\lambda)(1-t) + 3\lambda(4t^2 - 4t + 1), 0 \leq \lambda \leq 1. \tag{2}$$

in (1), and the cumulative distribution function (cdf) of the proposed family of distribution is

$$F_L(x) = (2+\lambda)G(x) - (1+5\lambda)G^2(x) + 4\lambda G^3(x) \tag{3}$$

with corresponding probability density function given by

$$f_L(x) = g(x)[2 + \lambda - 2(1+5\lambda)G(x) + 12\lambda G^2(x)]$$

, where $g(x)$ is the probability density function associated with the cdf $G(x)$.

3. Lakibul-Uniform Distribution (LU) and Its Properties

Let X be a random variable follows a uniform distribution with cumulative distribution function given by

$$G_U(x) = x, x \in [0,1]. \tag{4}$$

The cdf of Lakibul-Uniform distribution (LU) is obtained by inserting (4) into (3) and is

$$F_{LU}(x) = (2+\lambda)x - (1+5\lambda)x^2 + 4\lambda x^3 \tag{5}$$

with corresponding pdf given by

$$f_{LU}(x) = 2+\lambda - 2(1+5\lambda)x + 12\lambda x^2, x \in [0,1], \lambda \in [0,1]. \tag{6}$$

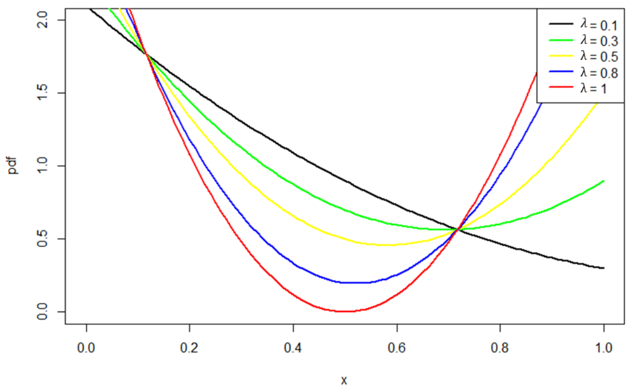

Figure 1: *pdf plots of the LU distribution for varied values of λ.*

Figure 1 shows some possible density shapes of the Lakibul-Uniform distribution. It reveals that the Lakibul-Uniform distribution can model a data with bathtub density shape.

3.1. Moments of LU

Theorem 1. The *rth* raw moment of LU distribution with density (6) and is

$$\mu'_r = \frac{r(3\lambda r + 2 + \lambda) + 6}{(r+1)(r+2)(r+3)}. \tag{7}$$

The mean and variance are, respectively, $\frac{\lambda+2}{6}$ and $\frac{22\lambda - 5\lambda^2 - 10}{180}$.

Proof. The *rth* raw moment is defined by

$$\begin{aligned}
\mu'_r &= E[X^r] \\
&= \int_0^\infty x^r f(x) dx \\
&= \int_0^1 x^r [2+\lambda - 2(1+5\lambda)x + 12\lambda x^2] dx \\
&= (2+\lambda)\int_0^1 x^r dx - 2(1+5\lambda)\int_0^1 x^{r+1} dx + 12\lambda \int_0^1 x^{r+2} dx \\
&= \frac{2+\lambda}{r+1} - \frac{2(1+5\lambda)}{r+2} + \frac{12\lambda}{r+3} \\
&= \frac{r(3\lambda r + 2 + \lambda) + 6}{(r+1)(r+2)(r+3)}.
\end{aligned}$$

The mean of LU distribution is obtained by taking $r = 1$ in (7) and is

$$\mu'_1 = \frac{\lambda+2}{6}.$$

The 2*nd* raw moment of LU distribution is obtained from (7) by taking $r = 2$ and is

$$\mu'_2 = \frac{7\lambda+5}{30}.$$

The variance of LU distribution is obtained as

$$\begin{aligned}\sigma^2 &= \mu_2' - (\mu_1')^2 \\ &= \frac{7\lambda+5}{30} - \left(\frac{\lambda+2}{6}\right)^2 \\ &= \frac{22\lambda - 5\lambda^2 - 10}{180}.\end{aligned}$$

3.2. Moment Generating Function of LU

Theorem 2. Let X follows the LU distribution then the moment generating function $M_X(t)$ is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r[r(3\lambda r + 2 + \lambda) + 6]}{r!(r+1)(r+2)(r+3)}, t \in \mathbb{R}.$$

Proof. By definition of moment generating function and pdf (6), we have

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^1 e^{tX} f_{LU}(x)dx.$$

Recall that $e^{tX} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r$ then we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'.$$

Hence,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r[r(3\lambda r + 2 + \lambda) + 6]}{r!(r+1)(r+2)(r+3)}.$$

3.3. Survival and Hazard Functions of LU

Let X be a random variable with cdf (5) and pdf (6) then the survival $S_{LU}(x)$ and hazard $h_{LU}(x)$ functions of LU distribution are respectively, given by

$$S_{LU}(x) = 1 - (2+\lambda)x - (1+5\lambda)x^2 + 4\lambda x^3$$

and

$$h_{LU}(x) = \frac{2+\lambda - 2(1+5\lambda)x + 12\lambda x^2}{1-(2+\lambda)x - (1+5\lambda)x^2 + 4\lambda x^3},$$

where $\lambda \in [0,1]$.

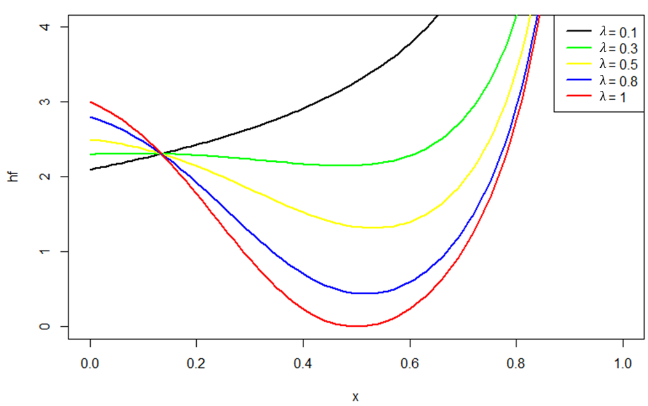

Figure 2: *hf plots of the LU distribution for varied values of* λ.

Figure 2 shows some possible shapes of the hazard function of the LU distribution. It reveals that the hazard function of the LU distribution can model the behavior of the bathtub hazard rate of the data.

3.4. Maximum Likelihood Estimates of LU

Let X_1, X_2,..., X_n be a random sample of size n from LU distribution, then the likelihood function is

$$L = \prod_{i=1}^{n} [2 + \lambda - 2(1 + 5\lambda)x_i + 12\lambda x_i^2].$$

The log-likelihood function is

$$l = \sum_{i=1}^{n} log[2 + \lambda - 2(1 + 5\lambda)x_i + 12\lambda x_i^2]. \tag{8}$$

Taking the derivative of (8) with respect to parameter λ then, we have

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \frac{1 - 10x_i + 12x_i^2}{2 + \lambda - 2(1 + 5\lambda)x_i + 12\lambda x_i^2}. \tag{9}$$

Equating (9) to 0, we can obtain the numerical maximum likelihood estimate of the LU distribution parameter.

4. Lakibul-Kumaraswamy Distribution (LK) and Its Properties

Let X be a random variable follows a Kumaraswamy distribution with cdf given by

$$G_K(x) = 1 - (1 - x^a)^b, x \in (0, 1), a, b > 0. \tag{10}$$

The cdf of Lakibul - Kumaraswamy distribution (LK) is obtained by inserting (10) into (3) and is

$$F_{LK}(x) = 1 - 3\lambda(1 - x^a)^b + (7\lambda - 1)(1 - x^a)^{2b} - 4\lambda(1 - x^a)^{3b} \tag{11}$$

with corresponding pdf given by

$$f_{LK}(x) = abx^{a-1}(1 - x^a)^{b-1} \left[3\lambda + 2(1 - 7\lambda)(1 - x^a)^b + 12\lambda(1 - x^a)^{2b}\right], \tag{12}$$

where $x \in (0, 1)$, a, $b > 0$ and $\lambda \in [0, 1]$.

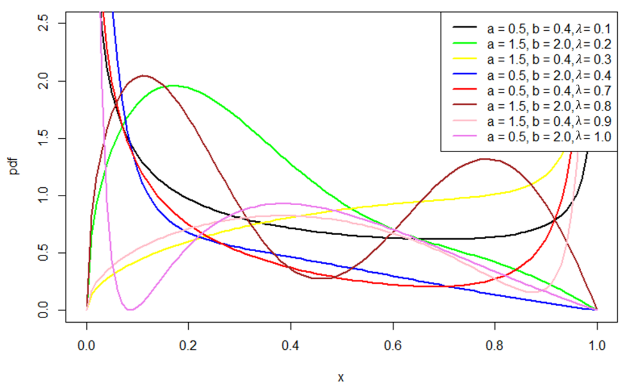

Figure 3: *pdf plots of the LK distribution for some values of a and b and varied values of λ.*

Figure 3 presents some possible density shapes of the LK distribution. It reveals that the LK distribution can model data with either of the following shapes: exponential, unimodal, bimodal, and bathtub density shapes.

4.1. Moments of LK

Theorem 3. The *rth* raw moment of LK distribution with density (12) and is

$$\mu'_r = 3b\lambda\mathbb{B}\left(\frac{r}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{r}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{r}{a}+1,3b\right). \qquad (13)$$

The mean and variance of LK distribution are respectively, given by

$$\mu = 3b\lambda\mathbb{B}\left(\frac{1}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{1}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{1}{a}+1,3b\right)$$

and

$$\begin{aligned}Var(X) =&3b\lambda\mathbb{B}\left(\frac{2}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{2}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{2}{a}+1,3b\right)\\ &-b^2\left(3b\lambda\mathbb{B}\left(\frac{1}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{1}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{1}{a}+1,3b\right)\right)^2.\end{aligned}$$

Proof. The *rth* raw moment is defined by

$$\begin{aligned}\mu'_r =&E[X^r]\\ =&\int_0^\infty x^r f(x)dx\\ =&\int_0^1 x^r abx^{a-1}(1-x^a)^{b-1}\left[3\lambda+2(1-7\lambda)(1-x^a)^b+12\lambda(1-x^a)^{2b}\right]dx\\ =&3\lambda ab\int_0^1 x^r x^{a-1}(1-x^a)^{b-1}dx+2(1-7\lambda)ab\int_0^1 x^r x^{a-1}(1-x^a)^{2b-1}dx\\ &+12\lambda ab\int_0^1 x^r x^{a-1}(1-x^a)^{3b-1}dx\\ =&3b\lambda\mathbb{B}\left(\frac{r}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{r}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{r}{a}+1,3b\right).\end{aligned}$$

The mean of LK distribution is obtained by taking $r=1$ in (13) and is

$$\mu=\mu'_1 = 3b\lambda\mathbb{B}\left(\frac{1}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{1}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{1}{a}+1,3b\right).$$

The 2nd raw moment of LK distribution is obtained by taking $r=2$ in (13) and is

$$\mu'_2 = 3b\lambda\mathbb{B}\left(\frac{2}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{2}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{2}{a}+1,3b\right).$$

The variance of LK distribution is obtained as

$$\begin{aligned}Var(X) =&\mu'_2-(\mu'_1)^2\\ =&3b\lambda\mathbb{B}\left(\frac{2}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{2}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{2}{a}+1,3b\right)\\ &-b^2\left(3b\lambda\mathbb{B}\left(\frac{1}{a}+1,b\right)+2b(1-7\lambda)\mathbb{B}\left(\frac{1}{a}+1,2b\right)+12b\lambda\mathbb{B}\left(\frac{1}{a}+1,3b\right)\right)^2.\end{aligned}$$

4.2. Moment Generating Function of LK

Theorem 4. Let X follows the LK distribution, then the moment generating function $M_X(t)$ is given by

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r b}{r!}\left[3\lambda\mathbb{B}\left(\frac{r}{a}+1,b\right)+2(1-7\lambda)\mathbb{B}\left(\frac{r}{a}+1,2b\right)+12\lambda\mathbb{B}\left(\frac{r}{a}+1,3b\right)\right], t\in\mathbb{R}.$$

The proof is similar to the proof of Theorem 2.

4.3. Survival and Hazard Functions of LK

Let X be a random variable with cdf (11) and pdf (12), then the survival $S_{LK}(x)$ and hazard $h_{LK}(x)$ functions of LK distribution are respectively, given by

$$S_{LK}(x) = 3\lambda(1-x^a)^b - (7\lambda-1)(1-x^a)^{2b} + 4\lambda(1-x^a)^{3b}$$

and

$$h_{LK}(x) = \frac{abx^{a-1}(1-x^a)^{b-1}\left[3\lambda + 2(1-7\lambda)(1-x^a)^b + 12\lambda(1-x^a)^{2b}\right]}{3\lambda(1-x^a)^b - (7\lambda-1)(1-x^a)^{2b} + 4\lambda(1-x^a)^{3b}},$$

where $x \in (0,1)$, a, $b > 0$ and $\lambda \in [0,1]$.

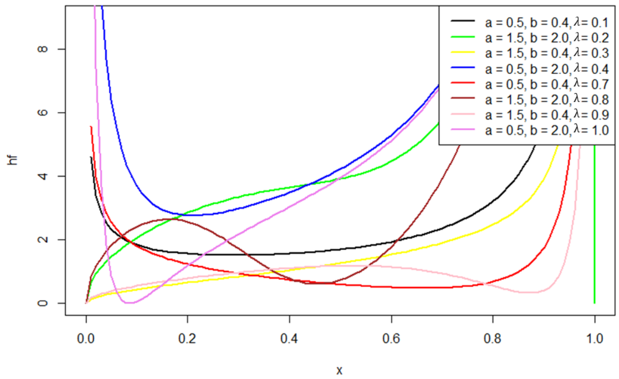

Figure 4: *hf plots of the LK distribution for some values of a and b and varied values of λ.*

Figure 4 shows some possible hazard rate behaviors of the LK distribution. It shows that the hazard rate of the LK distribution can model a data with bathtub, monotonic increasing, or non-monotonic increasing hazard behaviors.

4.4. Maximum Likelihood Estimates of LK

Let X_1, X_2,..., X_n be a random sample of size n from LK distribution then the likelihood function is

$$L = \prod_{i=1}^{n} abx_i^{a-1}(1-x_i^a)^{b-1}\left[3\lambda + 2(1-7\lambda)(1-x_i^a)^b + 12\lambda(1-x_i^a)^{2b}\right].$$

The log-likelihood function is

$$l = nlog(a) + nlog(b) + (a-1)\sum_{i=1}^{n} log(x_i) + (b-1)\sum_{i=1}^{n} log(1-x_i^a)$$
$$+\sum_{i=1}^{n} log\left[3\lambda + 2(1-7\lambda)(1-x_i^a)^b + 12\lambda(1-x_i^a)^{2b}\right]. \quad (14)$$

Taking the derivative of (14) with respect to the parameters a, b and λ then, we have

$$\frac{\partial l}{\partial a} = \frac{n}{a} - 2b\sum_{i=1}^{n}\frac{x_i^a(1-x_i^a)^{b-1}log(x_i)\left[1-7\lambda+12\lambda(1-x_i^a)^b\right]}{3\lambda+2(1-7\lambda)(1-x_i^a)^b+12\lambda(1-x_i^a)^{2b}}$$
$$+\sum_{i=1}^{n} log(x_i) - (b-1)\sum_{i=1}^{n}\frac{x_i^a log(x_i)}{1-x_i^a}, \quad (15)$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log(1-x_i^a) + 2\sum_{i=1}^{n}\frac{(1-x_i^a)^b log(1-x_i^a)\left[1-7\lambda+12\lambda(1-x_i^a)^b\right]}{3\lambda+2(1-7\lambda)(1-x_i^a)^b+12\lambda(1-x_i^a)^{2b}} \quad (16)$$

and

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \frac{3 - 14(1 - x_i^a)^b + 12(1 - x_i^a)^{2b}}{3\lambda + 2(1 - 7\lambda)(1 - x_i^a)^b + 12\lambda(1 - x_i^a)^{2b}}. \tag{17}$$

The numerical maximum likelihood estimates of the LK distribution parameters can be obtained by equating (15), (16) and (17) to 0, respectively.

5. Lakibul-Rayleigh Distribution (LR) and Its Properties

Let X be a random variable follows a Rayleigh distribution with cdf given by

$$G_R(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}, x > 0, \sigma > 0. \tag{18}$$

The cdf of Lakibul-Rayleigh distribution (LR) is derived by inserting (18) into (3) and is

$$F_{LR}(x) = 1 - 3\lambda e^{-\frac{x^2}{2\sigma^2}} + (7\lambda - 1)e^{-\frac{x^2}{\sigma^2}} - 4\lambda e^{-\frac{3x^2}{2\sigma^2}} \tag{19}$$

with corresponding pdf given by

$$f_{LR}(x) = \frac{xe^{-\frac{x^2}{2\sigma^2}}}{\sigma^2}\left[3\lambda - 2(7\lambda - 1)e^{-\frac{x^2}{2\sigma^2}} + 12\lambda e^{-\frac{x^2}{\sigma^2}}\right], \tag{20}$$

where $x > 0$, $\sigma > 0$ and $\lambda \in [0, 1]$.

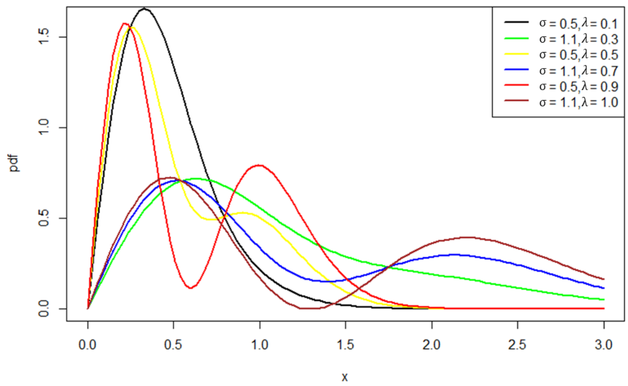

Figure 5: *pdf plots of the LR distribution for some values of σ and varied values of λ.*

Figure 5 presents some possible density shapes of the LR distribution. It reveals that the LR distribution can model data with right-tailed unimodal or bimodal behaviors.

5.1. Moments of LR

Theorem 5. Let X be a random variable that follows Lakibul-Rayleigh distribution, then the *rth* raw moment of LR distribution is given by

$$\mu_r' = \sigma^r \left[1 + \lambda\left(2^{\frac{r}{2}}3 + 2^{\frac{r}{2}+2}3^{-\frac{r}{2}} - 7\right)\right]\Gamma\left(\frac{r}{2} + 1\right). \tag{21}$$

The mean and variance of LR distribution are respectively, given by

$$\mu = \frac{\sigma\sqrt{\pi}}{2}\left[1 + \lambda\left(3\sqrt{2} + \frac{4\sqrt{2}}{\sqrt{3}} - 7\right)\right]$$

and

$$Var(X) = \sigma^2\left\{1+\frac{5\lambda}{3}-\frac{\pi}{4}\left[1+\lambda\left(3\sqrt{2}+\frac{4\sqrt{2}}{\sqrt{3}}-7\right)\right]^2\right\}.$$

Proof. The rth raw moment is defined by

$$\begin{aligned}
\mu_r' &= E[X^r] \\
&= \int_0^\infty x^r f(x)dx \\
&= \int_0^\infty x^r \frac{xe^{-\frac{x^2}{2\sigma^2}}}{\sigma^2}\left[3\lambda - 2(7\lambda-1)e^{-\frac{x^2}{2\sigma^2}} + 12\lambda e^{-\frac{x^2}{\sigma^2}}\right]dx \\
&= 3\lambda\int_0^\infty x^r\frac{xe^{-\frac{x^2}{2\sigma^2}}}{\sigma^2}dx - 2(7\lambda-1)\int_0^\infty x^r\frac{xe^{-\frac{x^2}{\sigma^2}}}{\sigma^2}dx + 12\lambda\int_0^\infty x^r\frac{xe^{-\frac{3x^2}{2\sigma^2}}}{\sigma^2}dx \\
&= 3\lambda 2^{\frac{r}{2}}\sigma^r\Gamma\left(\frac{r}{2}+1\right) - (7\lambda-1)\sigma^r\Gamma\left(\frac{r}{2}+1\right) + 2^{\frac{r}{2}+2}3^{-\frac{r}{2}}\lambda\sigma^2\Gamma\left(\frac{r}{2}+1\right) \\
&= \sigma^r\left[1+\lambda\left(2^{\frac{r}{2}}3 + 2^{\frac{r}{2}+2}3^{-\frac{r}{2}} - 7\right)\right]\Gamma\left(\frac{r}{2}+1\right).
\end{aligned}$$

The mean of LR distribution is obtained by taking $r = 1$ in (21) and is

$$\begin{aligned}
\mu = \mu_1' &= \sigma\left[1+\lambda\left(2^{\frac{1}{2}}3 + 2^{\frac{1}{2}+2}3^{-\frac{1}{2}} - 7\right)\right]\Gamma\left(\frac{1}{2}+1\right) \\
&= \sigma\left[1+\lambda\left(3\sqrt{2}+\frac{4\sqrt{2}}{\sqrt{3}}-7\right)\right]\Gamma\left(\frac{3}{2}\right) \\
&= \sigma\left[1+\lambda\left(3\sqrt{2}+\frac{4\sqrt{2}}{\sqrt{3}}-7\right)\right]\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{\sigma\sqrt{\pi}}{2}\left[1+\lambda\left(3\sqrt{2}+\frac{4\sqrt{2}}{\sqrt{3}}-7\right)\right].
\end{aligned}$$

The $2nd$ raw moment of LR distribution is obtained by taking $r = 2$ in (21) and is

$$\begin{aligned}
\mu_2' &= \sigma^2\left[1+\lambda\left(2^{\frac{2}{2}}3 + 2^{\frac{2}{2}+2}3^{-\frac{2}{2}} - 7\right)\right]\Gamma\left(\frac{2}{2}+1\right) \\
&= \frac{\sigma^2}{3}(3+5\lambda)\Gamma(2) \\
&= \frac{\sigma^2}{3}(3+5\lambda).
\end{aligned}$$

The variance of LR distribution is obtained as

$$\begin{aligned}
Var(X) &= \mu_2' - (\mu_1')^2 \\
&= \frac{\sigma^2}{3}(3+5\lambda) - \frac{\sigma^2\pi}{4}\left[1+\lambda\left(3\sqrt{2}+\frac{4\sqrt{2}}{\sqrt{3}}-7\right)\right]^2 \\
&= \sigma^2\left\{1+\frac{5\lambda}{3}-\frac{\pi}{4}\left[1+\lambda\left(3\sqrt{2}+\frac{4\sqrt{2}}{\sqrt{3}}-7\right)\right]^2\right\}.
\end{aligned}$$

5.2. Moment Generating Function of LR

Theorem 6. Let X follows the LR distribution, then the moment generating function $M_X(t)$ is given by

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r\sigma^r\left[1+\lambda\left(2^{\frac{r}{2}}3 + 2^{\frac{r}{2}+2}3^{-\frac{r}{2}} - 7\right)\right]}{r!}\Gamma\left(\frac{r}{2}+1\right), t\in\mathbb{R}.$$

The proof is similar to the proof of Theorem 2.

5.3. Survival and Hazard Functions of LR

Let X be a random variable with cdf (19) and pdf (20), then the survival $S_{LR}(x)$ and hazard $h_{LR}(x)$ functions of LR distribution are respectively, given by

$$S_{LR}(x) = 3\lambda e^{-\frac{x^2}{2\sigma^2}} - (7\lambda - 1)e^{-\frac{x^2}{\sigma^2}} + 4\lambda e^{-\frac{3x^2}{2\sigma^2}}$$

and

$$h_{LR}(x) = \frac{xe^{-\frac{x^2}{2\sigma^2}}\left[3\lambda - 2(7\lambda - 1)e^{-\frac{x^2}{2\sigma^2}} + 12\lambda e^{-\frac{x^2}{\sigma^2}}\right]}{\sigma^2\left[3\lambda e^{-\frac{x^2}{2\sigma^2}} - (7\lambda - 1)e^{-\frac{x^2}{\sigma^2}} + 4\lambda e^{-\frac{3x^2}{2\sigma^2}}\right]},$$

where $x > 0$, $\sigma > 0$ and $\lambda \in [0, 1]$.

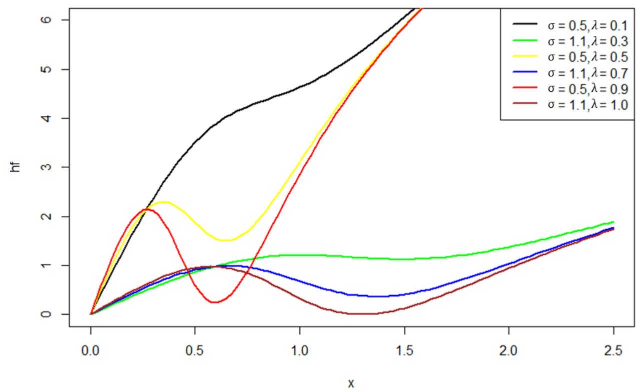

Figure 6: *hf plots of the LR distribution for some values of σ and varied values of λ.*

Figure 6 shows some possible shapes of the hazard rate function of the LR distribution. It reveals that the hazard rate function of the LR distribution can model data with monotonic or non-monotonic increasing hazard behaviors.

5.4. Maximum Likelihood Estimates of LR

Let X_1, X_2,..., X_n be a random sample of size n from LR distribution then the likelihood function is

$$L = \prod_{i=1}^{n} \frac{x_i e^{-\frac{x_i^2}{2\sigma^2}}}{\sigma^2}\left[3\lambda - 2(7\lambda - 1)e^{-\frac{x_i^2}{2\sigma^2}} + 12\lambda e^{-\frac{x_i^2}{\sigma^2}}\right].$$

The log-likelihood function is

$$l = \sum_{i=1}^{n} log(x_i) - \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} - 2nlog(\sigma) + \sum_{i=1}^{n} log\left[3\lambda - 2(7\lambda - 1)e^{-\frac{x_i^2}{2\sigma^2}} + 12\lambda e^{-\frac{x_i^2}{\sigma^2}}\right]. \quad (22)$$

Taking the derivative of (22) with respect to the parameters σ and λ then, we have

$$\frac{\partial l}{\partial \sigma} = \frac{1}{\sigma}\sum_{i=1}^{n} x_i^2 - \frac{2n}{\sigma} + \frac{2}{\sigma}\sum_{i=1}^{n} \frac{x_i^2 e^{-\frac{x_i^2}{2\sigma^2}}\left(1 - 7\lambda + 12\lambda e^{-\frac{x_i^2}{2\sigma^2}}\right)}{3\lambda - 2(7\lambda - 1)e^{-\frac{x_i^2}{2\sigma^2}} + 12\lambda e^{-\frac{x_i^2}{\sigma^2}}} \quad (23)$$

and

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \frac{3 - 14e^{-\frac{x_i^2}{2\sigma^2}} + 12e^{-\frac{x_i^2}{\sigma^2}}}{3\lambda - 2(7\lambda - 1)e^{-\frac{x_i^2}{2\sigma^2}} + 12\lambda e^{-\frac{x_i^2}{\sigma^2}}}. \tag{24}$$

Equating (23) and (24) to 0, respectively, we can obtain numerical maximum likelihood estimates of the LR distribution parameters.

6. Application

In this section, we apply the LU, LK and LR distributions on the real datasets and compare with some existing distributions, respectively. The applications of each of the proposed distributions are presented in the following subsections. In the analysis of data, we use a package "fitdistrplus" in R software. For comparison of the distributions, we use the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).

6.1. Application of LU Distribution

In this subsection, the LU distribution is applied to real data and compared with the Cubic Transmuted Uniform (CTU) distribution proposed by Rahman et al. [14]. The probability density function of the CTU distribution is given by

$$f_{CTU}(x) = 1 - \lambda + 6\lambda x - 6\lambda x^2, x \in [0,1], \lambda \in [-1,1].$$

The data set used in this application is about the lifetimes of 30 electronic devices in days. This set of data was used by Rahman et al. [14]. They fitted the CTU distribution to this dataset and compared it with the Kumaraswamy, Beta, and Transmuted Uniform distributions, and they found that the CTU distribution had the best fit for this dataset. The data set is given as follows: 0.020, 0.029, 0.034, 0.044, 0.057, 0.096, 0.106, 0.139, 0.156, 0.164, 0.167, 0.177, 0.250, 0.326, 0.406, 0.607, 0.650, 0.672, 0.676, 0.736, 0.817, 0.838, 0.910, 0.931, 0.946, 0.953, 0.961, 0.981, 0.982 and 0.990.

Table 1 lists the MLE estimates with the corresponding standard errors of the models fitted to the electronic data set. Table 2 indicates that the LU distribution gives a better fit for this dataset since it has the smallest AIC and BIC values compared to the CTU distribution. Moreover, the estimated pdf of the models fitted to the electronic dataset is presented in Figure 7, revealing that the LU distribution can mimic the behavior of the electronic dataset compared to the CTU distribution.

Table 1: *MLEs and Standard Error (SE) of the fitted models for the electronic dataset.*

Distribution	$\hat{\lambda}$	*SE*
LU	0.8239858	0.159316
CTU	-1	0.4436516

Table 2: *Numerical values of log-likelihood (logLike), AIC, and BIC of the fitted models for electronic dataset.*

Distribution	*logLike*	*AIC*	*BIC*
LU	**7.020164**	**-12.04033**	**-10.63913**
CTU	6.196299	-10.3926	-8.991401

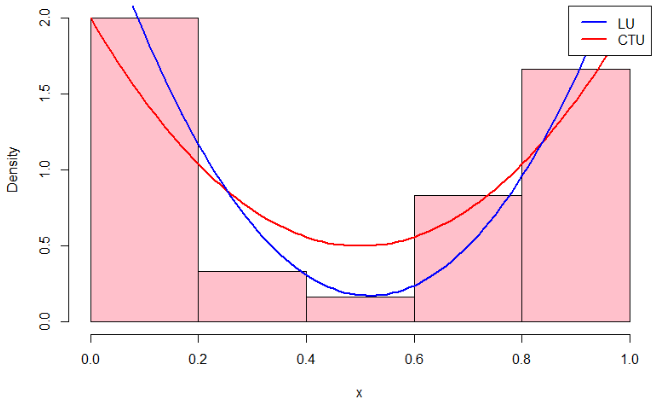

Figure 7: *Estimated pdf of the fitted models for electronic dataset.*

6.2. Application of LK Distribution

In this subsection, the LK distribution is applied to the COVID-19 dataset. This dataset covers the death rates due to COVID-19 infectious in Spain from 3 March to 7 May 2020. This dataset was used by Afify et al. [1] and fitted the Marshall-Olkin reduced Kies (MORKI) distribution, and compared to the Kumaraswamy, Beta, reduced Kies and G-beta distributions. They found that the MORKI distribution was the best fit for this dataset. The dataset is given as follows: 0.3330, 0.5000, 0.5000, 0.5714, 0.2500, 0.3469, 0.4839, 0.2105, 0.2311, 0.3127, 0.4800, 0.2749, 0.3625, 0.3922, 0.3414, 0.3711, 0.4288, 0.4077, 0.3939, 0.4076, 0.4079, 0.4408, 0.4046, 0.3836, 0.3545, 0.3275, 0.3162, 0.3150, 0.3053, 0.2930, 0.2790, 0.2685, 0.2588, 0.2492, 0.2481, 0.2453, 0.2355, 0.2285, 0.2241, 0.2193, 0.2162, 0.2153, 0.2129, 0.2098, 0.2037, 0.2066, 0.2087, 0.2038, 0.2029, 0.2023, 0.1993, 0.1962, 0.1711, 0.1678, 0.1646, 0.1629, 0.1613, 0.1544, 0.1510, 0.1484, 0.1465, 0.1453, 0.1436, 0.1420, 0.1396 and 0.1372.

In this application, we compare the LK distribution with the MORKI and transmuted Kumaraswamy distributions (TK) [9]. The probability density functions of the MORKI and TK are, respectively, given by

$$f_{MORKI}(x) = \frac{\varphi\delta x^{\delta-1}(1-x)^{-\delta-1}e^{-\left(\frac{x}{1-x}\right)^{\delta}}}{\left\{1-(1-\varphi)e^{-\left(\frac{x}{1-x}\right)^{\delta}}\right\}^2}, x \in (0,1), \varphi > 0, \delta > 0;$$

$$f_{TK}(x) = abx^{a-1}(1-x^a)^{b-1}\left[1-\lambda+2\lambda(1-x^a)^b\right], x \in (0,1), a > 0, b > 0, \lambda \in [-1,1].$$

Table 3 lists the MLE estimates of the models fitted to the COVID-19 dataset. Table 4 reveals that the LK distribution is chosen as the best model for this dataset compared to MORKI and TK since it has the lowest AIC and BIC statistics values. Furthermore, the plots of the fitted models in the COVID-19 dataset is presented in Figure 8 and shows that the LK distribution provides a good fit for this dataset.

Table 3: *MLEs of the fitted models for the second COVID-19 dataset.*

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\varphi}$	$\hat{\delta}$
LK	3.6039558	47.6246377	0.4579338		
MORKI				0.04649913	2.95909586
TK	2.884895	21.050565	0.479449		

Table 4: *Numerical values of the AIC and BIC of the models fitted to the COVID-19 dataset.*

Distribution	*AIC*	*BIC*
LK	**-111.994**	**-106.4251**
MORKI	−110.6726	−106.2933
TK	−105.86	−99.29104

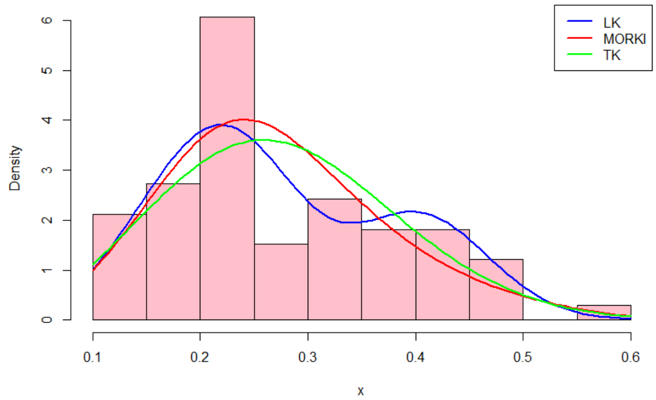

Figure 8: *Estimated pdf of the fitted models for the COVID-19 dataset.*

6.3. Application of LR Distribution

In this subsection, we apply the LR distribution on real data and compare with the following distributions:

- Exponentiated Transformed Inverse Rayleigh distribution (ETIR) [4]

$$f_{ETIR}(x) = \left(\frac{2\sigma^2}{e-1}\right)\frac{1}{x^3}e^{e^{-\left(\frac{\sigma}{x}\right)^2}}, x > 0, \sigma > 0;$$

- Exponentiated Inverse Rayleigh dsitribution (EIRD) [15]

$$f_{EIRD}(x) = \frac{2\alpha\sigma^2}{x^3}e^{-\left(\frac{\sigma}{x}\right)^2}\left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1}, x \geq 0, \sigma > 0, \alpha > 0.$$

The dataset used in this application is from the study of Hinkley [8]. This data consists of thirty successive observations of the March precipitation (in inches). The dataset is given as follows: 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9 and 2.05. This data was used by [4] and fitted the ETIR distribution, and compared with the exponentiated inverse Rayleigh, inverse Rayleigh, transmuted inverse Rayleigh and the Rayleigh distributions. They found that the ETIR was the best fit for this dataset.

The results of this analysis are given in Tables 5 and 6. Table 5 presents the MLE estimates of the models fitted to this dataset. Table 6 reveals that the LR distribution gives a better estimate for this dataset since it has the lowest values of AIC and BIC compared to the EIRD and ETIR distributions. In addition, the same result is observed in Figure 9.

Table 5: *MLEs of the fitted models for the Precipitation dataset.*

Distribution	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\alpha}$
LR	1.7814893	0.1136674	
$ETIR$	0.8293138		
$EIRD$	0.8287479		0.7315789

Table 6: *Numerical values of the AIC and BIC of the fitted models for the Precipitation dataset.*

Distribution	AIC	BIC
LR	**80.59769**	**83.40009**
$ETIR$	86.05263	87.45383
$EIRD$	90.40234	93.20474

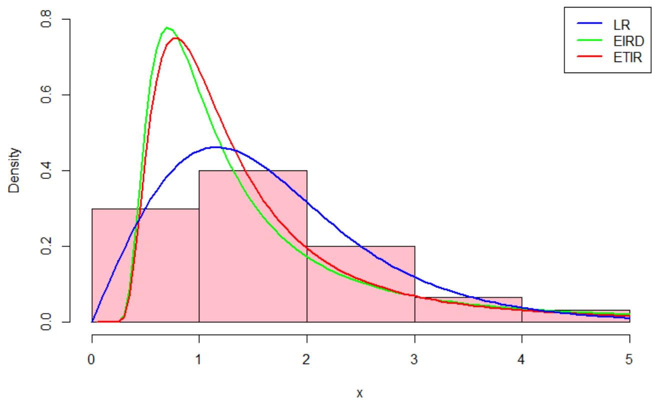

Figure 9: *Estimated pdf of the fitted models for the Precipitation dataset.*

7. Concluding Remarks

In this paper, a new generated family of distributions called the Lakibul-G Family of Distributions has been introduced. Three special distributions of the proposed class, such as the Lakibul-Uniform, Lakibul-Kumaraswamy, and Lakibul-Rayleigh distributions, were derived. Some properties of each of the proposed models, such as its moments, mean, variance, and moment generating function, were derived. The maximum likelihood method was used to estimate the parameters of each of the proposed distributions. Real data sets were used to examine the flexibility and applicability of the proposed distributions. For the Lakibul-Uniform distribution, it was found that this distribution provides a better estimate for the electronic dataset compared to the Cubic Transmuted Uniform distribution. Next, for the Lakibul - Kumaraswamy distribution, it was observed that the LK distribution produces a better fit for the COVID-19 dataset as compared with the Marshall - Olkin reduce Kies and the Transmuted Kumaraswamy distributions. Lastly, for the Lakibul - Rayleigh distribution, it was found that the LR distribution gives better estimates than the Exponentiated Transform Inverse Rayleigh and the Exponentiated Inverse Rayleigh distributions.

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