A NEW EXTENSION OF TWO PARAMETER ARADHANA DISTRIBUTION WITH COMPREHENSIVE STATISTICAL PROPERTIES AND APPLICATIONS

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Abstract

In this study, we introduce a modified extension of standard two parameter Aradhana distribution called as the length biased two parameter Aradhana distribution. The length-biased distributions are a subclass of weighted distributions that adjust the probabilities based on lengths or sizes of observed phenomena, making them suitable for modelling certain practical scenarios. The length biased version is identified as a specific instance of a broader category of weighted distributions, which are widely used in statistical modelling when data collection mechanisms introduce some inherent bias. The distribution is thoroughly examined and analyzed with new structural properties and parameter estimation is performed using a robust and widely used technique of maximum likelihood estimation. The distribution's superiority and effectiveness are evaluated through a comparative analysis involving two real-world datasets. These datasets are used to highlight the practical applicability of the new distribution, demonstrate its flexibility and ability to fit real-world data better than existing models. This comprehensive approach highlights both theoretical advancements and real-world relevance, emphasizing the distribution's versatility and utility.

Keywords: length biased distribution, two parameter Aradhana distribution, reliability measures, order statistics

I. Introduction

In probability and statistics, the inception of weighted distributions plays a key role because the standard distributions may not provide the best fit from different data. Since then a situation arised to introduce an additional parameter to existing classical distribution which brings more flexibility in their nature. This additional parameter can be approached through several famous techniques i.e.

weighted technique. The idea of weighted distributions was suggested by Fisher [7] which later on modified by Rao [16] in unifying theory for problems where observations fall in non-experimental, non-replicated and non-random manner. The weighted distribution reduces to the length biased distribution especially if weight function considers only length of units of interest. The concept of length biased distribution was introduced by Cox [6] in renewal theory. Length biased sampling situation may occur in clinical trials, reliability theory and population studies were proper sampling frame is absent.

Many authors contributed extraordinarily to describe some length biased probability distributions along with applications. Ganaie et al. [8], Shanker and Prodhani [19], Alzoubi [4], Mustafa and Khan [13], Saghir et al. [18], Rashwan [15], Modi and Gill [11], Mudasir and Ahmad [12], Alidamat and Al-Omari [1], Ganaie and Rajagopalan [9], Al-Omari and Alsmairan [3], Almakhareez and Alzoubi [2], Chaito and Khamkong [5].

A two-parameter Aradhana distribution is a recently introduced new distribution which was proposed by Shanker et al. [20] which is a particular case of one parameter Aradhana distribution. It's important statistical properties have been described and derived. Further, its parameters are estimated by using the maximum likelihood estimation.

II. Length Biased Two Parameter Aradhana (LBTPA) Distribution

The probability density function of two parameter Aradhana distribution is given by

$$f(x;\theta,\alpha)=\frac{\theta^3}{\left(\theta^2\alpha^2+2\theta\alpha+2\right)}(\alpha+x)^2e^{-\theta x}\ ;\ x>0,\theta>0,\alpha>0 \tag{1}$$

and cumulative distribution function of two parameter Aradhana distribution is given by

$$F(x;\theta,\alpha)=1-\left(1+\frac{\theta x(\theta x+2\theta\alpha+2)}{\left(\theta^2\alpha^2+2\theta\alpha+2\right)}\right)e^{-\theta x}\ ;x>0,\theta>0,\alpha>0 \tag{2}$$

Consider the random variable X which represents non-negative condition with probability density function $f(x)$. Suppose its non-negative weight function $w(x)$, then probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

Here $w(x)$ is the weight function which is non - negative, then $E(w(x))=\int w(x)f(x)dx<\infty.$

In this paper, we have considered the weight function as $w(x)=x$ to obtain the length biased version of two parameter Aradhana distribution and then its probability density function is given by

$$f_l(x;\theta,\alpha)=\frac{x\,f(x;\theta,\alpha)}{E(x)} \tag{3}$$

Here $E(x)=\int_0^\infty x\,f(x;\theta,\alpha)\,dx$

$$E(x)=\frac{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}{\theta\left(\theta^2\alpha^2+2\theta\alpha+2\right)} \tag{4}$$

After substituting equation (1) and (4) in equation (3), we will get the probability density function of length biased two parameter Aradhana distribution which is given by

$$f_l(x;\theta,\alpha) = \frac{x\theta^4}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x} \tag{5}$$

The cumulative distribution function of length biased two parameter Aradhana distribution should be determined as

$$F_l(x;\theta,\alpha) = \int_0^x f_l(x;\theta,\alpha)dx$$

$$= \int_0^x \frac{x\theta^4}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x}dx \tag{6}$$

After simplification of equation (6), we will get the cumulative distribution function of length biased two parameter Aradhana distribution which is given by

$$F_l(x;\theta,\alpha) = \frac{1}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(2,\ \theta x) + \gamma(4,\ \theta x) + 2\alpha\theta\,\gamma(3,\ \theta x)\right) \tag{7}$$

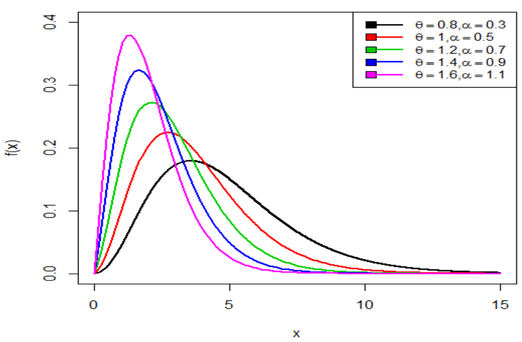

Figure 1: *Pdf plot of LBTPA distribution*

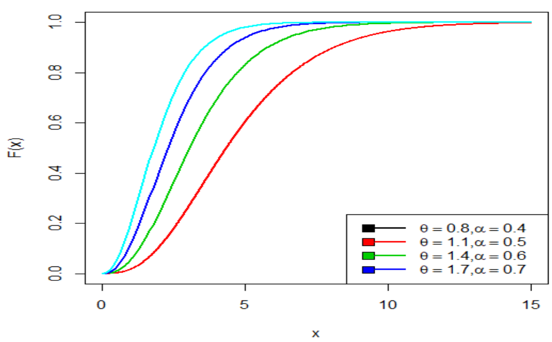

Figure 2: *Cdf plot of LBTPA distribution*

III. Reliability measures

In this section, here we have derived the reliability function, hazard rate function, reverse hazard rate function and mills ratio of length biased two parameter Aradhana distribution. The reliability function of length biased two parameter Aradhana distribution is given by

$$R(x) = 1 - F_l(x;\theta,\alpha)$$

$$R(x) = 1 - \frac{1}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(2,\ \theta x) + \gamma(4,\ \theta x) + 2\alpha\theta\,\gamma(3,\ \theta x)\right)$$

The hazard rate of length biased two parameter Aradhana distribution is given by

$$h(x) = \frac{f_l(x;\theta,\alpha)}{1 - F_l(x;\theta,\alpha)}$$

$$h(x)=\frac{x\theta^4(\alpha+x)^2 e^{-\theta x}}{(\alpha^2\theta^2+6+4\alpha\theta)-(\alpha^2\theta^2\gamma(2,\ \theta x)+\gamma(4,\ \theta x)+2\alpha\theta\gamma(3,\ \theta x))}$$

The reverse hazard rate function of length biased two parameter Aradhana distribution is given by

$$h_r(x)=\frac{f_l(x;\theta,\alpha)}{F_l(x;\theta,\alpha)}$$

$$h_r(x)=\frac{x\theta^4(\alpha+x)^2 e^{-\theta x}}{(\alpha^2\theta^2\gamma(2,\ \theta x)+\gamma(4,\ \theta x)+2\alpha\theta\gamma(3,\ \theta x))}$$

The Mills Ratio of length biased two parameter Aradhana distribution is given by

$$M.R=\frac{1}{h_r(x)}=\frac{(\alpha^2\theta^2\gamma(2,\ \theta x)+\gamma(4,\ \theta x)+2\alpha\theta\gamma(3,\ \theta x))}{x\theta^4(\alpha+x)^2 e^{-\theta x}}$$

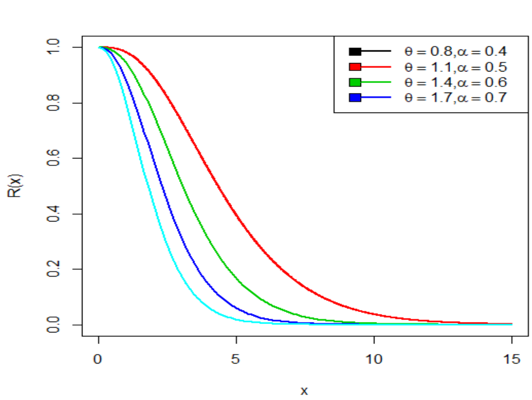

Figure 3:*Reliability plot of LBTPA distribution*

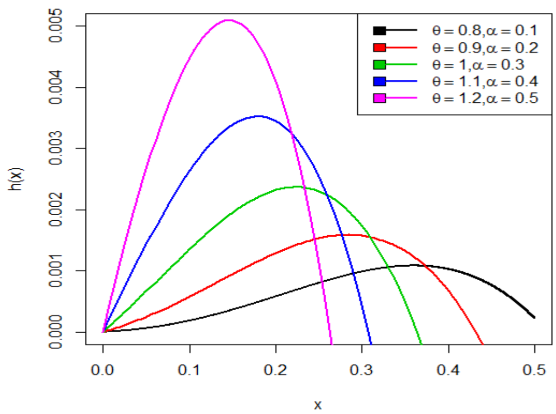

Figure 4:*Hazard plot of LBTPA distribution*

IV. Order Statistics

Order statistics is a basic concept in statistics which deals with arrangement of independent and identically distributed random variables. Consider the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ of a random sample $X_1, X_2, \ldots, X_n$ from a continuous distribution with probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$, then probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \tag{8}$$

After substituting equations (5) and (7) in equation (8), we will get the probability density function of r^{th} order statistics $X_{(r)}$ of length biased two parameter Aradhana distribution as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{x\theta^4(\alpha+x)^2 e^{-\theta x}}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}\right)\left(\frac{1}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(2,\theta x)+\gamma(4,\theta x)+2\alpha\theta\gamma(3,\theta x)\right)\right)^{r-1}$$

$$\times\left(1-\frac{1}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(2,\theta x)+\gamma(4,\theta x)+2\alpha\theta\gamma(3,\theta x)\right)\right)^{n-r}$$

For $r = n$, we will get the probability density function of higher order statistics $X_{(n)}$ of length biased two parameter Aradhana distribution as

$$f_{x(n)}(x) = \frac{n x\theta^4}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x}\left(\frac{1}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(2,\theta x)+\gamma(4,\theta x)+2\alpha\theta\gamma(3,\theta x)\right)\right)^{n-1}$$

For $r = 1$, we will get the probability density function of first order statistics $X_{(1)}$ of length biased two parameter Aradhana distribution as

$$f_{x(1)}(x) = \frac{n x\theta^4}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x}\left(1-\frac{1}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(2,\theta x)+\gamma(4,\theta x)+2\alpha\theta\gamma(3,\theta x)\right)\right)^{n-1}$$

V. Entropy

The term entropy is a fundamental concept which leads to measure the amount of uncertainty or disorder in a system.

I. Renyi Entropy

The Renyi entropy is a quantity which formulates the basis of concept of generalized dimensions. The Renyi entropy also generalizes several forms of entropy

$$R_T(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f_l^{\gamma}(x;\theta,\alpha)\,dx\right\}$$

$$= \frac{1}{1-\gamma}\log\left(\int_0^{\infty}\left(\frac{x\theta^4}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x}\right)^{\gamma}dx\right) \tag{9}$$

After the simplification of equation (9), we get

$$R_T(\gamma) = \frac{1}{1-\gamma}\log\left(\left(\frac{\theta^{3\gamma-k-1}}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)^{\gamma}}\right)\sum_{k=0}^{2\gamma}\binom{2\gamma}{k}\alpha^{2\gamma-k}\frac{\Gamma(\gamma+k+1)}{\gamma^{\gamma+k+1}}\right)$$

Obvious, the Tsallis entropy of proposed distribution will be determined by using the expression

$$T_s(\beta) = \frac{1}{\beta-1}\left(1-\int_0^{\infty} f_l^{\beta}(x;\theta,\alpha)\,dx\right)$$

$$= \frac{1}{\beta-1}\left(1-\int_0^{\infty}\left(\frac{x\theta^4}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x}\right)^{\beta} dx\right) \quad (10)$$

After the simplification of above equation (10), we get

$$T_s(\beta) = \frac{1}{\beta-1}\left(1-\left(\frac{\theta^{3\beta-i-1}}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)^{\beta}}\right)\sum_{i=0}^{2\beta}\binom{2\beta}{i}\alpha^{2\beta-i}\frac{\Gamma(\beta+i+1)}{\beta^{\beta+i+1}}\right)$$

VI. Statistical Properties

In this section, we have described and derived several structural properties of length biased two parameter Aradhana distribution those are

I. Moments

Consider random variable X constitutes the length biased two parameter Aradhana distribution, then r^{th} order moment of presented distribution will be determined as

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f_l(x;\theta,\alpha)\,dx$$

$$= \int_0^{\infty} x^r \frac{x\theta^4}{\left(\alpha^2\theta^2+6+4\alpha\theta\right)}(\alpha+x)^2 e^{-\theta x}dx \quad (11)$$

After the simplification of equation (11), we get

$$\mu_r' = E(X^r) = \frac{\left(\alpha^2\theta^2\Gamma(r+2)+\Gamma(r+4)+2\alpha\theta(r+3)\right)}{\theta^r\left(\alpha^2\theta^2+6+4\alpha\theta\right)} \quad (12)$$

Now substituting $r = 1$ and 2 in equation (12), we will get the first two moments of length biased two parameter Aradhana distribution

$$\mu_1' = \frac{\left(2\alpha^2\theta^2 + 24 + 12\alpha\theta\right)}{\theta\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}$$

$$\mu_2' = \frac{\left(6\alpha^2\theta^2 + 120 + 48\alpha\theta\right)}{\theta^2\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}$$

$$\text{Variance} = \frac{\left(2\alpha^4\theta^4 + 24\alpha^3\theta^3 + 108\alpha^2\theta^2 + 192\alpha\theta + 144\right)}{\theta^2\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)^2}$$

$$\text{S.D} = \frac{\sqrt{\left(2\alpha^4\theta^4 + 24\alpha^3\theta^3 + 108\alpha^2\theta^2 + 192\alpha\theta + 144\right)}}{\theta\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}$$

II. Harmonic Mean

The harmonic mean of executed distribution will be determined by applying following expression

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{x\theta^4}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}(\alpha + x)^2 e^{-\theta x}dx \tag{13}$$

After the simplification of equation (13), we get

$$H.M = \frac{\theta(\alpha^2\theta + 2 + 2\alpha\theta)}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}$$

III. Moment generating function and characteristic function

The moment generating function of proposed length biased two parameter Aradhana distribution will be determined as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty e^{tx} \frac{x\theta^4}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}(\alpha + x)^2 e^{-\theta x}dx \tag{14}$$

After the simplification of equation (14), we get

$$M_X(t) = \frac{\theta^4}{(\theta - t)^4\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}\left(\alpha^2(\theta - t)^2 + 6 + 4\alpha(\theta - t)\right)$$

Obviously, the characteristic function of length biased two parameter Aradhana distribution is

$$M_X(it) = \frac{\theta^4}{(\theta - it)^4 \left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}\left(\alpha^2(\theta - it)^2 + 6 + 4\alpha(\theta - it)\right)$$

VII. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves were applied in economics to reveal the graphical representation of distribution of income or wealth. The proposed curves are defined as

$$B(p) = \frac{1}{p\mu_1'}\int_0^q x f_l(x;\theta,\alpha)\,dx$$

$$L(p) = \frac{1}{\mu_1'}\int_0^q x f_l(x;\theta,\alpha)\,dx$$

$$\text{Here } \mu_1' = \frac{\left(2\alpha^2\theta^2 + 24 + 12\alpha\theta\right)}{\theta\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}$$

$$B(p) = \frac{\theta\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}{p\left(2\alpha^2\theta^2 + 24 + 12\alpha\theta\right)}\int_0^q x \frac{x\theta^4}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}(\alpha + x)^2 e^{-\theta x}dx \tag{15}$$

After the simplification of equation (15), we get

$$B(p) = \frac{1}{p\left(2\alpha^2\theta^2 + 24 + 12\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(3,\ \theta q) + \gamma(5,\ \theta q) + 2\alpha\theta\gamma(4,\ \theta q)\right)$$

$$L(p) = \frac{1}{\left(2\alpha^2\theta^2 + 24 + 12\alpha\theta\right)}\left(\alpha^2\theta^2\gamma(3,\ \theta q) + \gamma(5,\ \theta q) + 2\alpha\theta(4,\ \theta q)\right)$$

VIII. Maximum Likelihood Estimation

In this section, we estimate the parameters of length biased two parameter Aradhana distribution by using maximum likelihood estimation. Consider the random sample $X_1, X_2,\ldots,X_n$ of size n from the length biased two parameter Aradhana distribution, then the likelihood function can be defined as

$$L(x) = \prod_{i=1}^{n} f_l(x;\theta,\alpha)$$

$$= \prod_{i=1}^{n}\left(\frac{x_i\,\theta^4}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}(\alpha + x_i)^2 e^{-\theta x_i}\right)$$

$$= \frac{\theta^{4n}}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)^n}\prod_{i=1}^{n}\left(x_i(\alpha + x_i)^2 e^{-\theta x_i}\right)$$

The log likelihood function should be stated as

$$\log L = 4n\log\theta - n\log\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right) + \sum_{i=1}^{n}\log x_i + 2\sum_{i=1}^{n}\log(\alpha + x_i) - \theta\sum_{i=1}^{n}x_i \quad (16)$$

Now differentiating log-likelihood equation (16) with respect to parameters θ and α. The following normal equations must be established

$$\frac{\partial \log L}{\partial\theta} = \frac{4n}{\theta} - n\left(\frac{\left(2\alpha^2\theta + 4\alpha\right)}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}\right) - \sum_{i=1}^{n}x_i = 0$$

$$\frac{\partial \log L}{\partial\alpha} = -n\left(\frac{\left(2\alpha\theta^2 + 4\theta\right)}{\left(\alpha^2\theta^2 + 6 + 4\alpha\theta\right)}\right) + 2\sum_{i=1}^{n}\left(\frac{1}{(\alpha + x_i)}\right) = 0$$

The above likelihood equations are too complicated to solve it algebraically. Therefore, we use the numerical technique like R and wolfram mathematics for estimating the required parameters.

IX. Application

In this section, we have performed the goodness of fit of length biased two parameter Aradhana distribution by analyzed the two real lifetime data sets and then comparison has been developed in order to reveal that the length biased two parameter Aradhana distribution provides a better fit in comparison over two parameter Aradhana, Garima, Pranav, Pratibha, Akash and Uma distributions.
Data set 1:The following real data set given below which represents the waiting times (in minutes) before service of 100 bank customers that was examined and analyzed by Ghitany et al. [10] for fitting the lindley distribution and the data set is given as under
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5
Data set 2:The second real symmetric data set constitutes the failure times of windshields which was reported by Murthy et al. [14] and the real data set is given below as
0.04, 0.3, 0.31, 0.557, 0.943, 1.07, 1.124, 1.248, 1.281, 1.281, 1.303, 1.432, 1.48, 1.51, 1.51, 1.568, 1.615, 1.619, 1.652, 1.652, 1.757, 1.795, 1.866, 1.876, 1.899, 1.911, 1.912, 1.9141, 0.981, 2.010, 2.038, 2.085, 2.089, 2.097, 2.135, 2.154, 2.190, 2.194, 2.223, 2.224, 2.23, 2.3, 2.324, 2.349, 2.385, 2.481, 2.610, 2.625, 2.632, 2.646, 2.661, 2.688, 2.823, 2.89, 2.9, 2.934, 2.962, 2.964, 3, 3.1, 3.114, 3.117, 3.166, 3.344, 3.376, 3.385, 3.443, 3.467, 3.478, 3.578, 3.595, 3.699, 3.779, 3.924, 4.035, 4.121, 4.167, 4.240, 4.255, 4.278, 4.305, 4.376, 4.449, 4.485, 4.570, 4.602, 4.663, 4.694

To enumerate the model comparison criterions the unknown parameters are estimated thoroughly by applying R software. To predict the capability and effectiveness of length biased two parameter Aradhana distribution over two parameter Aradhana, Garima, Pranav, Pratibha, Akash and Uma distributions, we consider criterions like *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Corrected Akaike Information Criterion), Shannon's entropy *H(X)* and *-2logL*. It is quite obvious that the distribution is better if it has the smaller criterion values of *AIC, BIC, AICC, CAIC, H*(X) and *-2logL* in comparison over other compared distributions. The formulae for computing the criterion value are as given

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$CAIC = \frac{2kn}{n-k-1} - 2\log L, \qquad H(X) = -\frac{2\log L}{n}$$

Where n is sample size, k is the number of parameters in statistical model and $-2\log L$ is maximized value of log-likelihood function under the considered model.

Table 1: *Shows MLE and S.E of Data set 1 and Data set 2 of performed distributions*

Data set 1			Data set 2		
Distribution	MLE	S.E	Distribution	MLE	S.E
LBTPA	$\hat{\alpha} = 5.011230$ $\hat{\theta} = 2.028885$	$\hat{\alpha} = 1.103495$ $\hat{\theta} = 1.430256$	LBTPA	$\hat{\alpha} = 0.330304$ $\hat{\theta} = 1.444080$	$\hat{\alpha} = 0.239622$ $\hat{\theta} = 0.103652$
TPA	$\hat{\alpha} = 1.79978517$ $\hat{\theta} = 0.25969442$	$\hat{\alpha} = 1.04516631$ $\hat{\theta} = 0.02479628$	TPA	$\hat{\alpha} = 0.0946144$ $\hat{\theta} = 1.1265797$	$\hat{\alpha} = 0.088121$ $\hat{\theta} = 0.078480$
Garima	$\hat{\theta} = 0.15154412$	$\hat{\theta} = 0.01309641$	Garima	$\hat{\theta} = 0.5571774$	$\hat{\theta} = 0.050554$
Pranav	$\hat{\theta} = 0.4047845$	$\hat{\theta} = 0.0200951$	Pranav	$\hat{\theta} = 1.2094683$	$\hat{\theta} = 0.052114$
Pratibha	$\hat{\theta} = 0.28916795$	$\hat{\theta} = 0.01658711$	Pratibha	$\hat{\theta} = 0.8937320$	$\hat{\theta} = 0.050786$
Akash	$\hat{\theta} = 0.29527837$	$\hat{\theta} = 0.01683424$	Akash	$\hat{\theta} = 0.9320800$	$\hat{\theta} = 0.055209$
Uma	$\hat{\theta} = 0.39680518$	$\hat{\theta} = 0.01958624$	Uma	$\hat{\theta} = 1.2098061$	$\hat{\theta} = 0.059669$

Table 2: *shows comparison, performance and analysis of fitted distributions of data set 1*

Distribution	-2logL	AIC	BIC	AICC	CAIC	H(X)
LBTPA	634.6014	638.6014	643.8117	638.7251	638.7251	6.3460
TPA	637.5075	641.5075	646.7179	641.6312	641.6312	6.3750
Garima	649.6375	651.6375	654.2427	651.6783	651.6783	6.4963
Pranav	665.9148	667.9148	670.52	667.9556	667.9556	6.6591
Pratibha	640.3329	642.3329	644.9381	642.3737	642.3737	6.4033
Akash	641.9292	643.9292	646.5344	643.9700	643.9700	6.4192
Uma	659.1849	661.1849	663.7901	661.2257	661.2257	6.5918

Table 3: *shows comparison, performance and analysis of fitted distributions of data set 2*

Distribution	-2logL	AIC	BIC	AICC	CAIC	H(X)
LBTPA	282.5138	286.5138	291.4685	286.6549	286.6549	3.2103
TPA	286.6318	290.6318	295.5865	290.7729	290.7729	3.2571
Garima	331.4615	333.4615	335.9388	333.5080	333.5080	3.7666
Pranav	300.3776	302.3776	304.8549	302.4241	302.4241	3.4133
Pratibha	303.18	305.18	307.6573	305.2265	305.2265	3.4452
Akash	306.3466	308.3466	310.824	308.3931	308.3931	3.4812
Uma	298.3461	300.3461	302.8234	300.3926	300.3926	3.3902

It is quite clear and illustrated from the results given above in table 2 and table 3 that the length biased two parameter Aradhana distribution has smaller *AIC*, *BIC*, *AICC*, *CAIC*, *H(X)* and *-2logL* values in comparison over two parameter Aradhana, Garima, Pranav, Pratibha, Akash and Uma distributions which indicates that the length biased two parameter Aradhana distribution provides a better fit over two parameter Aradhana, Garima, Pranav, Pratibha, Akash and Uma distributions.

X. Conclusion

In this paper, we have incorporated a new extension of two parameter Aradhana distribution termed as length biased two parameter Aradhana distribution has been established. The formulated new distribution has been comprehensively introduced and explored by applying the length biased technique to two parameter Aradhana distribution. Some of its fundamental statistical features like moments, shape of behavior of pdf and cdf, variance and its mean, coefficient of variation, reliability function, hazard function, reverse hazard function, moment generating function, bonferroni and lorenz curves have been described. Further its order statistics and Renyi entropy has been discussed. Additionally its parameters are estimated by using maximum likelihood estimation. Furthermore, the predictability and potentiality of length biased two parameter Aradhana distribution has been examined and highlighted by applying the two real lifetime data sets and hence it is carefully explored from the result that length biased two parameter Aradhana distribution provides a quite satisfactory fit as compared over two parameter Aradhana, Garima, Pranav, Pratibha, Akash and Uma distributions.

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