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DISCOUNT PRICING POLICY WITH STOCHASTIC DEMAND AND PRESERVATION TECHNOLOGY FOR DETERIORATING ITEMS

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Abstract

Managing inventory for deteriorating items presents significant challenges, especially in environments where demand is stochastic and price-sensitive. Retailers face the dual pressure of minimizing losses from product deterioration while ensuring sufficient stock availability to meet customer demand. Additionally, backlogged shortages, if not managed strategically, can erode customer satisfaction and profitability. These complexities are further compounded by the need to strike a balance between promotional efforts, preservation investments, and pricing strategies to remain competitive in dynamic markets. To address these challenges, this study develops a comprehensive mathematical model aimed at optimizing replenishment strategies for deteriorating items. The model integrates preservation technologies and strategic price discounts to enhance product demand while accounting for the intricate relationship between market potential, stock levels, promotional efforts, and selling prices. The objective is to determine optimal pricing policies, order quantities, promotional expenditures, preservation costs, and schedules for shortages and replenishment, ultimately maximizing total profit per unit time. The findings reveal that total profit per unit time is concave concerning price, shortage duration, and inventory holding period. Numerical analyses demonstrate the efficacy of preservation technologies in mitigating deterioration rates, enabling retailers to adopt competitive pricing and stimulate sales. Additionally, the demand pattern index is shown to significantly influence inventory policies, while promotional campaigns play a critical role in augmenting product sales. Quantity discounts, tied to demand and order volumes, are identified as effective tools for reducing procurement costs and enhancing retailer profitability. These insights highlight the strategic importance of investing in preservation technologies, adopting dynamic pricing strategies, and implementing targeted promotional campaigns. Together, these approaches provide a robust framework for optimizing inventory management, increasing sales, and maximizing profitability in highly competitive markets.

Keywords: Inventory, Deterioration, Stochastic Optimal Control, Preservation Technology Investment, Quantity Discounts

1. Introduction

The effective management of inventory systems has long been acknowledged as a critical concern in business operations, particularly in industries where products are prone to deterioration over time. This challenge is especially prominent in the management of perishable goods such as pharmaceuticals, volatile liquids, and fresh produce. If left unaddressed, product deterioration can result in significant economic losses and a decline in customer service levels. Consequently, the formulation of sophisticated inventory models that account for deterioration dynamics has attracted substantial attention from researchers in operations research, computer science, and management studies.

In response to the complexities of real-world inventory systems, researchers have continually refined theoretical frameworks to reflect practical scenarios and improve decision-making processes. A key advancement in this domain has been the integration of preservation technologies, which aim to reduce the rate of product deterioration, thereby extending shelf life and minimizing economic losses. Simultaneously, the application of discount policies has emerged as an effective strategy for retailers to accelerate product turnover, enhance revenue streams, and maintain market competitiveness. Within this multidimensional context, the development of optimal inventory policies has become a focal point, requiring a careful combination of preservation technologies, pricing strategies, and promotional measures to achieve maximum profitability while ensuring satisfactory service levels.

Early contributions, such as the Economic Order Quantity (EOQ) model introduced by Harris, established the foundational principles of inventory management by balancing inventory holding and setup costs to determine optimal order quantities. While the EOQ model remains a cornerstone in inventory theory, contemporary research has advanced to address the complexities associated with deteriorating inventory systems. Extensions of the EOQ framework have incorporated diverse deterioration patterns, such as constant, Weibull, and time-dependent linear rates. Notable advancements in this area have been demonstrated in the works of several researchers, including [1], [2], and [3]. These studies have provided critical insights into the interaction between deterioration dynamics and inventory management strategies, offering a comprehensive basis for optimizing decision-making processes.

The integration of stochastic elements into inventory modeling has profoundly enriched theoretical frameworks, providing deeper insights into demand variability and its implications for optimal decision-making. Significant contributions to this domain include studies on inventory models characterized by constant deterioration rates, constant demand, and instantaneous replenishment, as examined by researchers such as [4]. These investigations have offered valuable perspectives on the dynamics of inventory systems operating under stochastic demand conditions. Expanding upon this foundation, subsequent research by [5] and [6] has focused on inventory models with fixed deterioration rates and exponentially time-fluctuating demand, emphasizing strategies to effectively manage uncertainties in both supply and demand dynamics.

In parallel, advancements in fuzzy inventory modeling have introduced novel approaches to addressing the ambiguity and uncertainty inherent in real-world inventory systems. The application of fuzzy concepts has led to the development of models that incorporate price-dependent demand and time-varying holding costs, as demonstrated in the works of [7] and [8]. These contributions present robust methodologies for making informed decisions in uncertain environments. Furthermore, models accounting for linearly time-dependent deterioration rates and partially backlogged shortages have been introduced by [9], highlighting the necessity of integrating sophisticated deterioration dynamics into inventory management frameworks.

Expanding beyond conventional inventory management approaches, recent research has increasingly emphasized multi-objective inventory optimization, particularly in scenarios involving incremental discounts and fuzzy decision-making frameworks. These methodologies have been instrumental in enhancing supplier selection and order allocation processes. Notable investigations, such as those conducted by [10], have explored strategies for minimizing purchasing risks while simultaneously maximizing total profit. This line of research underscores the significance of fostering sustainable buyer-supplier relationships, especially in competitive market environments.

Building on this extensive body of literature, the present study seeks to contribute to the field of stochastic optimal control in inventory modeling by introducing a comprehensive framework that integrates preservation technologies, pricing strategies, and promotional initiatives. By employing stochastic optimization techniques, the proposed model aims to identify optimal inventory policies that effectively balance economic objectives with service level requirements. This approach is designed to enhance both profitability and competitiveness in dynamic market conditions. Additionally, through extensive numerical analysis and sensitivity testing, the study systematically evaluates the impact of various parameters on decision variables and overall profitability, thereby offering actionable insights for both industry practitioners and academic researchers.

In the modern business landscape, promotional strategies have emerged as a pivotal mechanism for stimulating consumer demand, differentiating brands, and increasing product visibility. Traditional promotional schemes—including price discounts, gifts, and coupons—have evolved in parallel with technological advancements. The rise of digital platforms, such as social networking websites and blogs, has further transformed the promotional landscape, enabling highly targeted marketing campaigns. Consequently, the integration of promotional strategies into inventory management research has gained substantial relevance, underscoring their critical role in shaping demand dynamics and optimizing operational efficiency.

A well-structured discount policy plays a pivotal role in fostering mutually beneficial relationships between consumers and retailers, generating economic advantages for both parties within the business ecosystem. The efficacy of quantity discount schedules, categorized into all-unit and incremental discounts, has been extensively documented in academic literature. Foundational contributions, such as the seminal EOQ model, have demonstrated the minimization of discounted cash flows through the determination of optimal order quantities under varying reorder point scenarios, as explored by researchers including [11].

Moreover, the intricate relationship between pricing strategies and inventory management is exemplified through dynamic lot-sizing problems formulated under discounting schemes. Research in this domain, such as the study presented in [12], leverages forecast data to guide speculative stocking decisions, reinforcing the strategic implications of discount-driven inventory control. Further investigations into the quantity discount-pricing problem, including the work of [13], highlight the complexities of pricing decisions in price-dependent demand environments. Expanding upon this foundation, subsequent research by [14] incorporates supplier credit terms, providing valuable insights into order decision-making under diverse financial constraints.

The strategic interplay between temporary price discounts and inventory policies has also garnered significant scholarly attention. Investigations by researchers such as [15], [16], and [17] underscore the deliberate use of pricing incentives to maximize savings and profitability within inventory systems characterized by time-dependent demand and transient price fluctuations. These contributions emphasize the necessity of informed decision-making in dynamic pricing environments, illustrating the critical role of adaptive pricing strategies in enhancing operational efficiency and financial performance.

The emerging domain of joint pricing and preservation decision-making underscores the synergy between pricing strategies, inventory control, and preservation technologies. Notable advancements, including models integrating stochastic demand, pricing, and preservation decisions, illustrate the intricate trade-offs involved in optimizing total profit under uncertain market conditions, as demonstrated in the work of [18]. Acknowledging the stochastic nature of demand and pricing dynamics, contemporary research has increasingly focused on optimizing pricing and production policies within stochastic inventory systems. Game-theoretic approaches to retail and wholesale pricing optimization, as explored by [20], highlight the strategic considerations underpinning competitive market pricing decisions. Additionally, pioneering insights into pricing and production policy optimization under fuzzy environments have been provided by [21], reinforcing the importance of dynamic pricing strategies in mitigating the adverse effects of uncertainty on inventory performance.

A sustainable production framework has been developed to address machine inefficiencies, environmental impacts, and demand-price uncertainties, offering versatile strategies applicable across greenhouse gas-emitting industries. This framework effectively balances economic and environmental objectives, contributing to the advancement of sustainable industrial practices, as explored in studies such as [22]. Additionally, stochastic inventory models leveraging Particle Swarm Optimization (PSO) have demonstrated superior performance over deterministic models by effectively capturing market uncertainties. This approach enhances adaptive decision-making and risk mitigation within supply chains, reinforcing its applicability in complex and volatile market conditions, as highlighted in the research of [23].

Innovative approaches to inventory modeling have gained prominence, particularly in contexts characterized by seasonal sales and dynamic pricing. Research on stochastic demand and dynamic pricing strategies for seasonal sales has revealed that real-time pricing adjustments outperform traditional methods by improving inventory management and optimizing profit margins during demand fluctuations. This perspective is further substantiated by studies such as [24]. Similarly, an inventory model tailored for declining markets has integrated promotional efforts with pricing strategies to manage non-instantaneous deteriorating items. By dynamically adjusting pricing and replenishment schedules in response to time- and price-sensitive demand, this model has demonstrated its effectiveness in maximizing profitability, as examined in [25].

For deteriorating products, an advanced inventory management framework has been introduced, categorizing production life-cycle stages and incorporating rework processes to enhance product quality. This framework employs a hybrid approach combining Grey Wolf and Ant Colony Optimization algorithms, achieving significant cost reductions and improved profitability, as evidenced in studies like [26]. Furthermore, a conversion optimization model has been developed to determine the optimal timing, quantity, cost, and selling price for homogeneous and heterogeneous conversions. Particle Swarm Optimization has proven more effective than classical methods in maximizing profit by optimizing multiple variables, as demonstrated in research such as [27].

Comparative analyses of forecasting methods have also gained scholarly attention. A study evaluating Holt-Winters Exponential Smoothing (HWES) and ARIMA models concluded that ARIMA offers superior performance in minimizing lost sales and enhancing demand forecasting accuracy under varying economic conditions. These findings underscore the importance of advanced forecasting techniques in improving supply chain resilience and operational efficiency, as discussed in [28]. Additionally, an inventory management model integrating pricing fluctuations, advertising strategies, and preservation costs has demonstrated its potential to enhance supply chain efficiency while aligning inventory practices with sustainability goals in competitive markets, as explored in [29].

Finally, delayed deteriorating EOQ models have been developed to optimize inventory costs by accounting for time-varying demand and partial backlogging, thereby addressing critical challenges in managing inventory systems with deteriorating products ([30]).

Building on these scholarly advancements, this paper aims to address critical questions related to optimal pricing, discounting, and shortage management within stochastic inventory control systems. By examining the complexities of replenishment scheduling and order quantity determination under conditions of uncertain demand and dynamic promotional strategies, this study seeks to advance the discourse on stochastic inventory management. The findings are intended to provide actionable insights, thereby supporting both practitioners in decision-making and scholars in furthering theoretical developments in the field.

2. Notations & Assumption

Notations and assumption bearing usual tradition, utilized in the subsequent discussions, are laid down as follows:

Notations

Parameters:

$D(s)$ Non-linear discounted price and dependent demand function for products,

$I(t)$ On-hand inventory of product at time t,

s The selling price per unit,

θ The deterioration rate of defective products,

ξ The preservation technology cost per unit,

C_p Purchasing cost per unit,

C_h Holding cost per item per unit time,

C_b Backorder cost per unit per time unit,

C_o Ordering cost per order,

C_l The lost sale cost per unit,

ε The random variable of the demand function with $E(\varepsilon) = \mu$,

t_s The replenishment point, the duration of the shortage time period,

T The inventory period,

ρ The promotional effort, $\eta \geq 1$,

Q The ordering quantity per cycle,

$I_1(t)$ The level of negative inventory at time t where $0 \leq t \leq t_{(s)}$,

$I_2(t)$ The level of positive inventory at time t where $t_s \leq t \leq t_{(s)} + T$,

$\widehat{\Psi}$ The total profit.

3. Assumptions

1. The replenishment rate is infinite, but its order size is finite.
2. The basic demand function is $D(s) + \varepsilon$, where $D(s)$ is a decreasing and deterministic function of the selling price s, and ε is a non-negative and continuous random variable with $E(\varepsilon) = \mu$.
3. The promotional effort cost PC is considered as $PC = K(\rho - 1)^2 \left[\int_0^1 (D(s) + \varepsilon)dt\right]^\alpha$ where $K > 0$ and α is a constant.
4. Shortages occur at the beginning of the cycle in this model, and a fraction of the demand varying with waiting time is backlogged. Assume that the customer's impatient function is $B(x) = e^{-\delta x}$.
5. The demand function is influenced by promotional effort ρ and can be expressed by mark-up over the promotional effort.
6. The item deteriorates at a time-varying rate of deterioration θ, where $0 < \theta < 1$. Besides, there is no repair or replacement of deteriorated units during the replenishment cycle.
7. The proportion of reduced deterioration rate, $m(\xi) = 1 - e^{-\eta\xi}$, is a continuous, concave, and increasing function of capital investment, where η is a parameter and ξ is the preservation technology cost.

4. Mathematical Formulation

The inventory model commences with shortages and concludes without shortages, as illustrated graphically in Figure 1. Shortages are partially backlogged, with backlogged demand fulfilled at the replenishment point t_s, while the remaining inventory suffices to meet demand until time T. The depletion of inventory arises from the combined impact of demand and deterioration during the interval $[t_s, T]$, as depicted in Figure 1. Notably, preservation technology is employed to regulate the deterioration rate. The demand function is contingent on a non-linear discounted price, defined by:

$$D(s) = a(ds)^{-b_1}, \quad \text{where} \quad a,\, d,\, b_1 > 0.$$

With this understanding, the variation in inventory level at any instant $t \in [0, t_m]$ is governed by the following differential equation: The entire inventory system can be characterized by

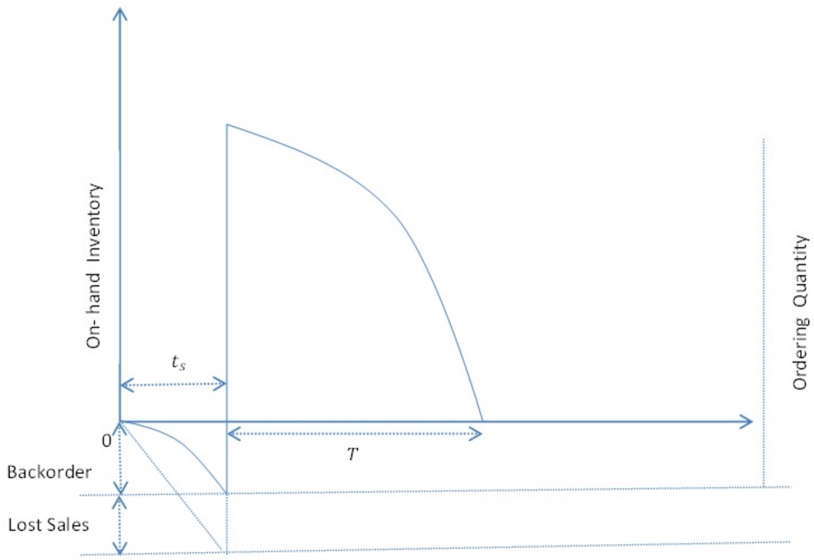

Figure 1: *Graphical Representation of the Inventory System*

leveraging the differential equation:

$$\frac{dI_1(t)}{dt} = -\rho(D(s) + \varepsilon)B(t - t_s), \quad 0 \le t \le t_s, \tag{1}$$

where $I_1(t)$ denotes the level of negative inventory, $B(t - t_s)$ represents the customer's impatient function capturing backlogged demand, and ε signifies the stochastic element of demand. This equation encapsulates the dynamic evolution of inventory levels during the shortage period.
with $I_1(t) = 0$, at $t = 0$

$$B(x) = e^{-\delta x}$$

Utilizing the conditions, the solutions of above differential equation

$$I_1(t) = -\frac{1}{\delta}\left[e^{t_s - t}(e^{\delta t} + 1)(D(s) + \varepsilon)\rho\right] \tag{2}$$

The change in inventory level at any instant of time $t \in [t_m, T]$, is represented by following differential equation

$$\frac{dI_2(t)}{dt} = -\theta(1 - m(\xi))I_2(t) - \rho(D(s) + \varepsilon), \quad 0 \le t \le t_m \tag{3}$$

with $I_2(T) = 0$ the solutions of above differential equation

$$I_2(t) = -\frac{1}{\theta\big(m(\xi)-1\big)}\left[e^{t_2\theta\big(m(\xi)-1\big)}\left\{e^{t_2\theta\big(m(\xi)-1\big)} + e^{t\theta\big(m(\xi)-1\big)}\right\}\rho(D(s)+\varepsilon)\right] \tag{4}$$

The lot sale quantity at time t, is

$$I_l(t) = (1 - e^{-(t-t_s)\delta})(D(s)+\varepsilon)\rho \tag{5}$$

The replenishment size with back logged is

$$Q = \big\{I_2(0 - I_1(t_s)\big\} \tag{6}$$

$$Q = \left[\frac{1}{\delta}\left[(e^{\delta t_s}+1) - \frac{1}{\theta\big(m(\xi)-1\big)}\left\{e^{T\theta\big(m(\xi)-1\big)}\left(e^{T\theta\big(m(\xi)-1\big)}+1\right)\right\}\right](D(s)+\varepsilon)\rho \tag{7}$$

The cost of lost sale during time interval $[0, t_s]$

$$\widehat{TLC} = E\left[C_l \int_0^{t_s} I_l(t)dt\right] \tag{8}$$

$$\widehat{TLC} = C_l\left(t_s + \frac{1-e^{t_s\delta}}{\delta}\right)(D(s)+\mu)\rho \tag{9}$$

The total cost of back logged for stock out during the time interval $[0, t_s]$

$$\widehat{TSC} = E\left[C_b \int_0^{t_s} -I_1(t)dt\right] \tag{10}$$

$$\widehat{TSC} = -\frac{C_h}{(\theta)^2\big(m(\xi)-1\big)^2}\left[e^{-t_2\theta\big(m(\xi)-1\big)}\left\{1 + e^{t_2\theta\big(m(\xi)-1\big)}\left(t_2\theta(m(\xi)-1)-1\right)\right\}\rho(D(s)+\varepsilon)\right] \tag{11}$$

The total holding cost during the time interval $[0, T]$

$$\widehat{THC} = E\left[C_h \int_0^{T} I_2(t)dt\right] \tag{12}$$

$$\widehat{THC} = \frac{C_h}{\delta^2}\left\{1 + e^{t_s\delta}(t_s\delta - 1)\right\}(D(s)+\mu)\rho \tag{13}$$

The total purchasing cost is

$$\widehat{TPC} = E(C_p.Q) = (D(s)+\mu)C_p\rho\left\{\frac{1-e^{t_s\delta}}{\delta} - \frac{1-e^{t_2\theta(m(\xi)-1)}}{\theta(m(\xi)-1)}\right\} \tag{14}$$

The total preservation technology cost is

$$\widehat{TPTC} = \xi(t_s + T) \tag{15}$$

The total promotional cost is

$$\widehat{T_pC} = E\left[K(\rho-1)^2\left\{\int_0^{t_s+T}(D(s)+\varepsilon)dt\right\}^{\alpha}\right] \tag{16}$$

$$\widehat{TPC} = K\bigg((t_s+T)(D(s)+\mu)\bigg)^{\alpha}(\rho-1)^2 \tag{17}$$

The total sales revenue is

$$\widehat{TSR} = E\left[s\left\{\rho(D(s)+\varepsilon)\int_0^T dt + \int_0^{t_s} B(t_s-t)dt\right\}\right] \tag{18}$$

$$\widehat{TSR} = s(D(s)+\mu)\rho\left(T + \frac{e^{t_s\delta}-1}{\delta}\right) \tag{19}$$

Now, the total profit

$$\widehat{\Psi} = \widehat{TSR} - \widehat{TC} \tag{20}$$

$$\begin{aligned}\widehat{\Psi}(t_s,T,s) =& s(D(s)+\mu)\rho\left(T+\frac{e^{t_s\delta}-1}{\delta}\right) - \Bigg[A + C_l\left(t_s + \frac{1-e^{t_s\delta}}{\delta}\right)(D(s)+\mu)\rho \\ &- \frac{C_h}{(\theta)^2\big(m(\xi)-1\big)^2}\left\{e^{-t_2\theta\left(m(\xi)-1\right)}\left\{1+e^{t_2\theta\left(m(\xi)-1\right)}\Big(T(m(\xi)-1)-1\Big)\right\}\rho(D(s)\right. \\ &\left.+\mu)\right\} + \frac{C_h}{\delta^2}\Big(1+e^{t_s\delta}(t_s\delta-1)\Big)(D(s)+\mu)\rho + (D(s)+\mu)C_p\rho\left\{\frac{1-e^{t_s\delta}}{\delta}\right. \\ &\left.- \frac{1-e^{T\theta(m(\xi)-1)}}{\theta(m(\xi)-1)}\right\} + \xi(t_s+T) + K\Big((t_s+T)(D(s)+\mu)\Big)^{\alpha}(\rho-1)^2\Bigg]\end{aligned} \tag{21}$$

In order to optimize the above non-linear optimization problem at first, the objective function is simplified and then some theorems are proposed. It is useful mentioning here that for a small value of x, the Taylor theorem (series) says that the exponential function has an approximation of $e^x \approx 1 + x + \frac{x^2}{2!}$. Using this result in Eq. (21), it is obtained the following:

$$\begin{aligned}\widehat{\Psi}(t_s,T,s) = \Bigg[& -4A - 4(t_s+T)\xi - 4K\big((t_s+T)(D(s)+\mu)\big)^{\alpha}(\rho-1)^2 + 2C_p t_s(2+t_s\delta)(D(s)+\mu)\rho \\ &+ 2C_l t_s^2\delta(D(s)+\mu)\rho - 2C_b t_s^2(1+t_s\delta)(D(s)+\mu)\rho + 2s\big(2(t_s+T)+t_s^2\delta\big)(D(s)+\mu)\rho \\ &- C_h T^2(D(p)+\mu)\rho\big(1-T\theta(1-m(\xi))\big)\Big\{2+T\theta(2+T\theta) \\ &+ T\theta m(\xi)\big(T\theta(m(\xi)-2)-2\big)\Big\} - C_p T(D(s)+\mu)\rho\big(T\theta(m(\xi)-1)+2\big)\big(2+T\theta(2 \\ &+ T\theta) + T\theta m(\xi)\big(T\theta(m(\xi)-2)-2\big)\Bigg]\end{aligned} \tag{22}$$

Therefore, average total profit per unit time is

$$\widehat{\Psi_a}(t_s,T,s) = \frac{\widehat{\Psi}(t_s,T,s)}{t_s+T} \tag{23}$$

$$\begin{aligned}\widehat{\Psi_a}(t_s,T,s) =& \frac{1}{4(t_s+T)}\Bigg[-4A - 4(t_s+T)\xi - 4K\big((t_s+T)(D(s)+\mu)\big)^{\alpha}(\rho-1)^2 + 2C_p t_s(2+t_s\delta)(D(s)+\mu)\rho \\ &+ 2C_l t_s^2\delta(D(s)+\mu)\rho - 2C_b t_s^2(1+t_s\delta)(D(s)+\mu)\rho + 2s\big(2(t_s+T)+t_s^2\delta\big)(D(s)+\mu)\rho \\ &- C_h T^2(D(p)+\mu)\rho\big(1-T\theta(1-m(\xi))\big)\Big\{2+T\theta(2+T\theta)+T\theta m(\xi)\big(T\theta(m(\xi)-2)-2\big)\Big\} \\ &- C_p T(D(s)+\mu)\rho\big(T\theta(m(\xi)-1)+2\big)\big(2+T\theta(2+T\theta)+T\theta m(\xi)\big(T\theta(m(\xi)-2)-2\big)\Bigg]\end{aligned} \tag{24}$$

5. **Solution Procedure**

In this study, we aim to determine the optimal values of the inventory cycle length , shortage period , and selling price that maximize the average total profit per unit time . The objective function, derived using Taylor series approximations. Now, we seek to solve the following constrained optimization problem:

$$\max_{t_s,T,s} \quad \widehat{\Psi_a}(t_s,T,s) \tag{25}$$

To ensure the existence of a global maximum for the optimization problem, we examine the concavity of the profit function. A function is concave if its Hessian matrix is negative definite, i.e., all its leading principal minors have alternating signs. We state the following theorem:

Theorem 5.1. *The profit function $\widehat{\Psi_a}(t_s, T, s)$ is concave with respect to inventory period T shortage time t_s and selling price s if the corresponding Hessian matrix H of expected profit function is negative definite. where*

$$\text{H=}\begin{pmatrix} \frac{\partial^2\widehat{\Psi_a}}{\partial T^2} & \frac{\partial^2\widehat{\Psi}}{\partial s\partial T} & \frac{\partial^2\widehat{\Psi}}{\partial T\partial t_s} \\ \frac{\partial^2\widehat{\Psi}}{\partial s\partial T} & \frac{\partial^2\widehat{\Psi}}{\partial s^2} & \frac{\partial^2\widehat{\Psi}}{\partial t_s\partial s} \\ \frac{\partial^2\widehat{\Psi}}{\partial T\partial t_s} & \frac{\partial^2\widehat{\Psi}}{\partial t_s\partial s} & \frac{\partial^2\widehat{\Psi}}{\partial {t_s}^2} \end{pmatrix}$$

Theorem 5.2. *When shortage time t_s is fixed, then the price s^* and the inventory period T^* obtained from Eq. (25) and Eq. (26) have absolute second order condition for profit objective maximization $\widehat{\Psi_a}$.*

At the value s^* and T^* obtained from the equating the necessary derivatives to zero:

$$H = \begin{bmatrix} \frac{\partial^2\widehat{\Psi_a}}{\partial s^2} & \frac{\partial^2\widehat{\Psi_a}}{\partial s\partial T} \\ \frac{\partial^2\widehat{\Psi_a}}{\partial T\partial s} & \frac{\partial^2\widehat{\Psi_a}}{\partial T^2} \end{bmatrix}$$

$$Det(H) = \frac{\partial^2\widehat{\Psi_a}}{\partial s^2} * \frac{\partial^2\widehat{\Psi_a}}{\partial T^2} - \left[\frac{\partial^2\widehat{\Psi_a}}{\partial s\partial T}\right]^2 > 0 \tag{26}$$

The profit function will be concave with respect to selling price s and inventory period T, if the $Det(H) > 0$ or in other word corresponding Hessian matrix of the profit function is negative definite, i.e. if all the eigenvalues of the Hessian matrix H are negative. We proved it with the help of Mathematica Software.

Theorem 5.3. *When the inventory period T is fixed, then the price s^* and the shortage time ${t_s}^*$ obtained from Eq.(25) and Eq.(27) have absolute second order condition for profit objective maximization $\widehat{\Psi_a}$.*

Theorem 5.4. *When the selling price s^* is fixed, then the inventory period T^* and the shortage time ${t_s}^*$ obtained from Eq.(25) and Eq.(26) have absolute second order condition for profit objective maximization $\widehat{\Psi}$.*

Theorem 5.5. *When shortage time t_s, selling price s and inventory period T are fixed, then the profit function $\widehat{\Psi_a}(t_s, T, s)$ is concave with respect to promotional effort ρ.*

Proof. The first and second order partial derivatives of the total profit function $\widehat{\Psi}(t_s, T, s)$ given by Eq. (22) with respect to ρ are given below:

$$\frac{\partial^2\widehat{\Psi_a}}{\partial \rho^2} = -\frac{2K\left[(t_s+T)\left(a(ds)^{-b_1}+\mu\right)\right]^{\alpha}}{t_s+T} < 0 \tag{27}$$

Hence, ρ^* is global optimal that optimizes the profit function $\widehat{\Psi_a}(t_s, T, s)$ for fixed values of selling price s, inventory cycle length T and shortage time t_s.

6. Numerical Example & Sensitivity Analysis

Using Mathematica, the example validates inventory management models by deriving optimal replenishment time, order quantity, and selling price. It demonstrates the framework's effectiveness in real-world scenarios, providing actionable insights for inventory optimization and strategic decision-making.

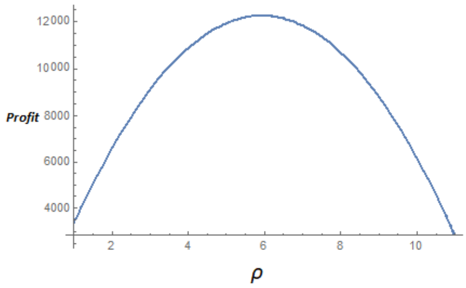

Figure 2: *Profit function w.r.t. promotional effort* ρ

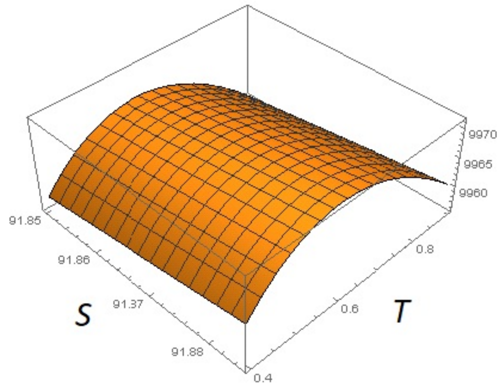

Figure 3: *Profit function w.r.t.* T *and* s

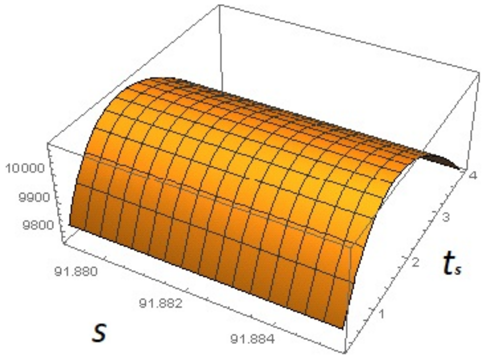

Figure 4: *Profit function w.r.t.* s *and* t_s

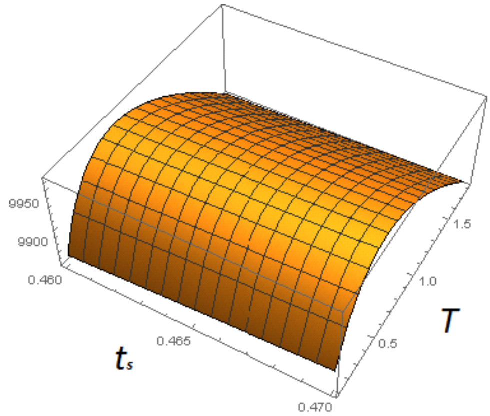

Figure 5: *Profit function w.r.t.* t_s *and* T

Example 1. *In this example, the demand function is given by* $D(s) = a(ds)^{-b_1}$*, where* a*,* d*, and* b_1 *are positive parameters. The system parameters are specified as follows: the ordering cost is* $A = 500$ *per cycle, the holding cost is* $C_h = 2.5$ *per unit per unit time, and the purchasing cost is* $C_p = 3$ *per unit. The backorder cost is* $C_b = 5$ *per unit, while the lost sale cost is* $C_l = 1.5$ *per unit, and the demand rate is* $\mu = 40$ *units per unit time. The coefficient for promotional effort is* $\rho = 3.4$*, and the customer impatience rate is* $\delta = 0.009$ *units per unit time. The deterioration rate is* $\theta = 0.02$ *per unit time, with a constant parameter* $\alpha = 1$*. Additional parameters include* $a = 100$*,* $d = 1$*, an adjustment factor* $a_1 = 0.001$*, and an elasticity parameter* $b_1 = 1.5$*. The cost of preservation technology is* $\xi = 1.2$ *per unit.*
To determine the optimal solutions, the following system of equations is solved:

$$\frac{\partial \widehat{\Psi_a}}{\partial s} = 0, \quad \frac{\partial \widehat{\Psi_a}}{\partial T} = 0, \quad and \quad \frac{\partial \widehat{\Psi_a}}{\partial t_s} = 0.$$

The optimal solutions are found to be $t_s^* = 0.4095$ *units of time,* $T^* = 1.4728$ *units of time, and* $s^* = 87$ *units. Correspondingly, the total average profit is calculated as:*

$$\widehat{\Psi_a}^* = 9102.46.$$

Through this example, we evaluate all eigenvalues of the Hessian matrix H, given by:

$$H = \begin{pmatrix} \frac{\partial^2 \widehat{\Psi_a}}{\partial T^2} & \frac{\partial^2 \widehat{\Psi_a}}{\partial s \partial T} & \frac{\partial^2 \widehat{\Psi_a}}{\partial T \partial t_s} \\ \frac{\partial^2 \widehat{\Psi_a}}{\partial s \partial T} & \frac{\partial^2 \widehat{\Psi_a}}{\partial s^2} & \frac{\partial^2 \widehat{\Psi_a}}{\partial t_s \partial s} \\ \frac{\partial^2 \widehat{\Psi_a}}{\partial T \partial t_s} & \frac{\partial^2 \widehat{\Psi_a}}{\partial t_s \partial s} & \frac{\partial^2 \widehat{\Psi_a}}{\partial {t_s}^2} \end{pmatrix}$$

The obtained eigenvalues -862.4011, -23.145, and -0.0000146 are negative, indicating that the profit function $\widehat{\Psi_a}(t_s, T, s)$ is concave with respect to t_s, T, and s.

Example 2. *Building on the dataset utilized in the previous example, a discount of* 30% *(*$d = 0.3$*) is applied to the selling price. Under these conditions, the optimal solutions are determined as follows: the optimal time for stock depletion is* $t_s^* = 0.4781$ *units of time, the replenishment cycle time is* $T^* = 1.4501$ *units of time, and the optimal stock level is* $s^* = 91.88$ *units. Consequently, the total average profit is calculated as:*

$$\widehat{\Psi_a}^* = 9906.30.$$

Furthermore, the eigenvalues of the Hessian matrix are determined as follows:

$$H = \begin{pmatrix} \frac{\partial^2 \widehat{\Psi_a}}{\partial T^2} & \frac{\partial^2 \widehat{\Psi_a}}{\partial s \partial T} & \frac{\partial^2 \widehat{\Psi_a}}{\partial T \partial t_s} \\ \frac{\partial^2 \widehat{\Psi_a}}{\partial s \partial T} & \frac{\partial^2 \widehat{\Psi_a}}{\partial s^2} & \frac{\partial^2 \widehat{\Psi_a}}{\partial t_s \partial s} \\ \frac{\partial^2 \widehat{\Psi_a}}{\partial T \partial t_s} & \frac{\partial^2 \widehat{\Psi_a}}{\partial t_s \partial s} & \frac{\partial^2 \widehat{\Psi_a}}{\partial {t_s}^2} \end{pmatrix}$$

The obtained eigenvalues -785.3681, -22.015231, and -0.0000146 are negative, indicating the concavity of the profit function $\widehat{\Psi_a}(t_s, T, s)$ with respect to t_s, T, and s.

Example 3. *Consider the inventory system characterized by the demand function* $D(s) = a(ds)^{-b_1}$*, where* $a, d, b_1 > 0$*. The following parametric values are used: the ordering cost is* $A = 600$ *per cycle, the holding cost is* $C_h = 1.5$ *per unit per unit time, and the purchasing cost is* $C_p = 3.2$ *per unit. The backorder cost is specified as* $C_b = 3$ *per unit, while the lost sale cost is* $C_l = 2$ *per unit, and the mean demand is* $\mu = 50$ *units. The promotional effort parameter is* $\rho = 2.4$*, and the impatience parameter is* $\delta = 0.009$ *units per unit time. The deterioration rate is* $\theta = 0.01$ *per unit time, and the constant parameter is* $\alpha = 1.2$*. Additional parameters include* $a = 200$*,* $d = 1.5$*,* $a_1 = 0.001$*,* $b_1 = 1.3$*, and the preservation technology cost* $\xi = 1.5$ *per unit.*
Solving the equations $\frac{\partial \widehat{\Psi_a}}{\partial s} = 0$*,* $\frac{\partial \widehat{\Psi_a}}{\partial T} = 0$*, and* $\frac{\partial \widehat{\Psi_a}}{\partial t_s} = 0$ *yields the optimal solutions* $t_s^* = 0.2314$ *units of time,* $T^* = 0.9251$ *units of time, and* $s^* = 105$ *units. Correspondingly, the total average profit is computed as:*

$$\widehat{\Psi_a}^* = 10027.253.$$

Table 1: *Effect of changes of parameters on optimal solution*

Parameters	Change	t_s^*	T^*	s^*	$\widehat{\Psi_a}^*$
a	100	0.4780	1.4501	91.88	9906.30
	101	0.4761	1.4571	91.98	9919.54
	104	0.4712	1.4821	92.27	9959.33
b_1	1.50	0.4780	1.4501	91.88	9906.30
	1.53	0.4809	1.4339	87.59	9311.17
	1.54	0.4813	1.4295	86.28	9128.50
δ	0.009	0.4780	1.4501	91.88	9906.30
	0.006	0.4551	1.5101	92.72	10000.6
	0.005	0.4410	1.5413	92.96	10021.6
a_1	0.001	0.4780	1.4501	91.88	9906.3
	0.200	0.4725	1.4599	91.97	9917.14
	0.300	0.4701	1.4639	92.01	9921.77
K	9	0.4780	1.4501	91.88	9906.300
	12	0.8012	0.5231	99.74	10500.84
	13	0.8625	0.3921	103.4	10807.00

Table 2: *Effect of Changes for Different Values of ρ and Computational Results for Normal Distribution of ε*

Parameter	Value	t_s^*	T^*	s^*	$\widehat{\Psi_a}^*$
Effect of Promotional Effort (ρ)					
	3.4	0.4780	1.4501	91.88	9906.3
	3.5	0.5321	1.3071	93.82	10400.20
ρ	3.6	0.5510	1.1391	95.17	10825.21
	3.7	0.6532	0.9451	95.88	10931.00
	3.8	0.7251	0.7091	96.25	11320.20
Computational Results for Normal Distribution of ε					
$\varepsilon \sim N(30, 1)$	-	0.5490	1.1586	88.50	7104.50
$\varepsilon \sim N(33, 1)$	-	0.5386	1.2041	89.00	7875.86
$\varepsilon \sim N(36, 1)$	-	0.5386	1.2825	89.91	8706.00
$\varepsilon \sim N(39, 1)$	-	0.4917	1.3974	91.26	9586.61
$\varepsilon \sim N(42, 1)$	-	0.4447	1.5898	93.51	10604.00

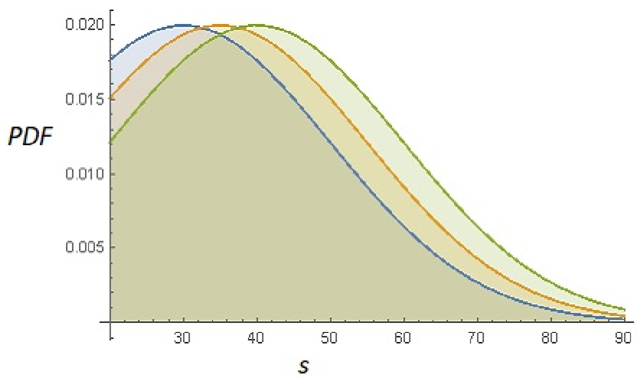

Figure 6: *Graphical representation of probability distribution function with price s*

7. Sensitivity Analysis

(i) From the analysis presented in Table 1, it is evident that an increase in the parameter a results in a corresponding increase in the optimal shortage time t_s^*, optimal inventory period T^*, optimal selling price s^*, and optimal profit $\widehat{\Psi_a}^*$. This trend signifies an enhancement in the market potential of the product, leading to improved optimal outcomes.

(ii) An escalation in the demand rate parameter b_1 leads to an increase in the optimal shortage time t_s^*. Consequently, the heightened shortage costs contribute to a decrease in the optimal profit. This phenomenon arises due to a reduction in demand resulting from the increased parameter b_1.

(iii) The observed results indicate that a decrease in the parameter δ corresponds to a decrease in the optimal shortage time t_s^*. This reduction in backlogging rates aids in minimizing total costs, consequently leading to an increase in optimal profit.

(iv) A rise in the parameter K correlates with a decrease in the optimal inventory period T^*, accompanied by increases in the optimal shortage time t_s^*, optimal selling price s^*, and optimal profit $\widehat{\Psi}^*$. This effect is attributed to the enhanced promotional efforts, resulting in heightened demand and subsequently increased profits.

The promotional effort parameter ρ exhibits a noteworthy influence on the optimal solution. As its value increases, there is a corresponding rise in the optimal shortage time t_s^*, optimal selling price s^*, and optimal profit $\widehat{\Psi}^*$, while the optimal replenishment period T^* decreases. The heightened promotional efforts lead to an augmented demand rate, resulting in a nonlinear increase in expected profit and subsequent decrease. This underscores the importance of judiciously implementing promotional strategies to ensure optimal outcomes.

Table 3 presents the anticipated results for the normal distribution parameters of a random variable. As the demand variable's value increases, so does the expected mean value. Correspondingly, the optimal selling price, preservation technology cost, total profit, and order quantity also increase, while the optimal shortage period decreases. This highlights the opportunity for retailers to escalate production and preservation technology costs in response to increased demand, thereby enhancing total profit and market potential.

Additionally, graphical analysis using three-dimensional (3D) plots for the profit function was conducted. Figures 3, 4, and 5 depict piecewise 3D plots of the profit function against two corresponding variables. The concave nature of the profit function concerning s and T (Figure 3), s and t_s (Figure 4), and T and t_s (Figure 5) is evident. Furthermore, Figure 2 illustrates the concave nature of the profit function with respect to promotional efforts ρ. Figure 6 showcases the probability distribution function with price s.

8. Conclusion

The proposed model offers inventory policies tailored for retailers to optimize their total profit within the current market landscape. Retailers strategically employ price discounts to stimulate product demand while leveraging preservation technology to mitigate deterioration rates. Promotional strategies are intermittently implemented to bolster sales, with market potential influenced by promotional efforts, inventory levels, and product pricing. A numerical example, coupled with sensitivity analysis, provides actionable managerial insights, elucidating the significance of promotional activities such as price discounts. Moreover, investing in preservation technology yields improved total profit, underscoring the pivotal role of prudent decision-making. Retailers and wholesalers alike stand to benefit from strategic promotional efforts, which amplify product demand and subsequently enhance total profit. Additionally, offering price discounts incentivizes larger order sizes, mutually benefiting retailers and customers.

Numerous avenues for future research exist, including extensions of this model to accommodate fuzzy environments, production policies, credit-linked demand, stock-dependent demand, and payment delay facilities. Furthermore, considerations such as supply disruptions, bargaining dynamics, campaign efforts, and return strategies warrant exploration. Additionally, the model's applicability can be extended to encompass multi-item EOQ models and other pertinent inventory management scenarios.

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Declaration: We declare that this research paper represents our own work and has not been submitted elsewhere for publication. All sources used have been properly cited.

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