FIXED POINT THEOREMS FOR SEQUENCE OF MAPPINGS IN TRICOMPLEX VALUED METRIC SPACE

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Abstract

Fixed point theory plays a crucial role in mathematical analysis, with profound applications in differential equations, optimization, and dynamic systems. Over the years, researchers have extended classical fixed point results to more complex structures, such as tricomplex valued metric spaces, to address problems in multidimensional and hypercomplex settings. Despite the growing interest in tricomplex variable spaces, existing literature on fixed point theorems within these frameworks remains limited by various constraints. The existing literature exhibits several limitations; therefore, in this article, we utilize the concepts of continuity and weak compatibility to establish fixed point results for a novel class of contraction mappings by employing a sequence of mappings within a tricomplex valued metric space. If two mappings commute at their point of coincidence, they are said to be weakly compatible. These results not only deepen the comprehension of fixed point theory but also open up new possibilities for its application in more intricate and varied mathematical contexts. As a result, our research progresses in the field, providing a solid foundation for future investigations and potential uses across multiple scientific and engineering domains.

Keywords: fixed point, complex valued metric space, tricomplex valued metric space.

1. Introduction

In 1922 [1] introduced us to the Banach contraction principle that paved the way for developing the metric fixed-point theory. Consequently, many researchers generalize these results in different ways [see [2], [3], [4], [5], [6], [7], [8], etc.]. In 2011 [9] introduced complex valued metric space and proved the existence of common fixed points of a pair of mappings that satisfy contractive conditions. Extended and improved the contraction condition of these results and gave applications by [10]. In 2013, [11] generalized complex-valued b-metric spaces, providing the fixed point theorem. In 2013 [12] used the EA property and proved common fixed point theorems in complex valued metric spaces. Rational contraction in complex valued metric space and used point-dependent control functions by [13]. In 2018 [14] introduced us to multicomplex spaces that show the direction to the researcher to generalization of multicomplex spaces. Introduced the notion of complex valued metric spaces [15]. Bicomplex metric space given by [16]. In 2022 [17] proved common fixed point theorems in tricomplex valued metric spaces. By using the control function [18] proved fixed point results in tricomplex valued metric space. In 2023 [19] worked on bicomplex metric space with rational contractions using control functions of one variable. [see also [20], [21] etc.].

In 1982 [22] introduced weak commutativity, a pair of self-mappings (P, Q) on a metric space (Δ, Ω) if $\Omega(PQ\mu, QP\mu) \leq \Omega(Q\mu, P\mu), \forall \mu \in \Delta$. In 1986 [23] introduced compatible, if (P, Q) is a pair of self-mappings on a metric space (Δ, Ω) such that $\lim_{n\to\infty} \Omega(P\mu_n, Q\mu_n) = 0$ whenever

$\lim_{n\to\infty} P\mu_n = \lim_{n\to\infty} Q\mu_n = l \in \Delta$. According to [23] a pair of mappings is said to be weakly compatible if they commute at their coincidence point. In this paper, we generalize these results in a new type of contraction mapping by using a sequence of mappings.

2. Preliminaries

In this paper, we denote the families of real, complex, bicomplex, and tricomplex numbers, by C^0, C^1, C^2 and C^3 respectively. In 2018 [14] introduced us multicomplex spaces and define bicomplex and tricomplex numbers as follows:
Bicomplex numbers:

$$\sigma = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 i_1 i_2,$$

where $\rho_1, \rho_1, \rho_1, \rho_1 \in C^0$ and independent units i_1 and i_2 such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$ the set of bicomplex numbers define as

$$C^2 = \{\sigma : \sigma = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 i_1 i_2, \rho_1, \rho_2, \rho_3, \rho_4 \in C^0\}$$

Or

$$C^2 = \{\sigma : \sigma = \Theta_1 + i_2 \Theta_2, \Theta_1, \Theta_2 \in C^1\}$$

where $\Theta_1 = \rho_1 + \rho_2 i_1$ and $\Theta_2 = \rho_3 + \rho_4 i_1$.
Tricomplex numbers:

$$\Gamma = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 j_1 + \rho_5 i_3 + \rho_6 j_2 + \rho_7 j_3 + \rho_8 i_4$$

Where $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8 \in C^0$ and $i_1, i_2, i_3, i_4, j_1, j_2$ and j_3 are independent units such that $i_1^2 = i_2^2 = i_3^2 = i_4^2 = -1, j_1^2 = j_2^2 = j_3^2 = 1, j_1 = i_1 i_2 = i_2 i_1, j_2 = i_1 i_3 = i_3 i_1$ and, $i_4 = i_1 i_3 = i_1 i_2 i_3$. the set of tricomplex numbers define as

$$C^3 = \{\Gamma : \Gamma = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 j_1 + \rho_5 i_3 + \rho_6 j_2 + \rho_7 j_3 + \rho_8 i_4, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8 \in C^0\}$$

$$C^3 = \{\Gamma : \Gamma = \gamma_1 + i_3 \gamma_2, \gamma_1, \gamma_2 \in C^2\}$$

where $\gamma_1 = \Theta_1 + \Theta_2 i_2 \in C^2$ and $\gamma_2 = \Theta_3 + \Theta_4 i_2 \in C^2$
Sum and Product of tricomplex numbers:
Let two tricomplex numbers Γ and Δ such that $\Gamma = \gamma_1 + i_3\gamma_2$ and $\Delta = \delta_1 + i_3 \delta_2$ then

$$\Gamma \pm \Delta = (\gamma_1 + i_3\gamma_2) \pm (\delta_1 + i_3\delta_2) = \gamma_1 \pm \delta_1 + i_3(\gamma_2 \pm \delta_2)$$

and

$$\Gamma.\Delta = (\gamma_1 + i_3\gamma_2)(\delta_1 + i_3\delta_2) = (\gamma_1\delta_1 - \gamma_2\delta_2) + i_3(\gamma_1\delta_2 + \gamma_2\delta_1)$$

In the families of tricomplex numbers C^3 have four idempotent elements $0, 1, \varphi_1 = \frac{1+j_3}{2}$ and $\varphi_2 = \frac{1-j_3}{2}$ such that $\varphi_1 + \varphi_2 = 1$ and $\varphi_1.\varphi_2 = 0$. Hence φ_1, φ_2 are not trivial. Every tricomplex numbers can be expressed as an union of φ_1, φ_2 for example

$$\Gamma = \gamma_1 + i_3\gamma_2 = (\gamma_1 - i_2\gamma_2)\varphi_1 + (\gamma_1 + i_2\gamma_2)\varphi_2$$

Here Γ is idempotent of the tricomplex numeral and $\Gamma_1 = (\gamma_1 - i_2\gamma_2)\varphi_1$ and $\Gamma_2 = (\gamma_1 + i_2\gamma_2)\varphi_2$ are the idempotent components of the bicomplex numeral Γ.
A tricomplex number $\Gamma = \gamma_1 + i_3\gamma_2$ is invertible if there exists $\Delta \in C^3$ such that $\Gamma\Delta = 1$. A tricomplex number $\Gamma = \gamma_1 + i_3\gamma_2$ is nonsingular if $|\gamma_1^2 + \gamma_2^2| \neq 0$ and singular if $|\gamma_1^2 + \gamma_2^2| = 0$.
Let $\Gamma = \gamma_1 + i_3\gamma_2$ and $\Delta = \delta_1 + i_3\delta_2$ be tricomplex numerals. Then the partial order relation $\leq_{i_3}$ on C^3 or $\Gamma \leq_{i_3} \Delta$ if the following axioms are satisfied.

1. $\gamma_1 = \delta_1, \gamma_2 = \delta_2$;
2. $\gamma_1 <_{i_3} \delta_1, \gamma_2 = \delta_2$;
3. $\gamma_1 = \delta_1, \gamma_2 <_{i_3} \delta_2$;
4. $\gamma_1 <_{i_3} \delta_1, \gamma_2 <_{i_3} \delta_2$;

If $\Gamma \lneqq_{i_3} \Delta$ implies $\Gamma <_{i_3} \Delta$ but $\Gamma \neq \Delta$
Having two tricomplex numerals $\Gamma, \Delta \in C^3$, the following conditions hold:

1. $\Gamma \leq_{i_3} \Delta$ if $||\Gamma|| \leq ||\Delta||$;
2. $||\Gamma + \Delta|| \leq ||\Gamma|| + ||\Delta||$;
3. $||\rho\Gamma|| = |\rho|||\Gamma||$, where $\rho \in C^0$
4. $||\Gamma\Delta|| \leq 2||\Gamma||||\Delta||$, If at least one of Γ, Δ nonsingular then equality holds.
5. $||\Gamma^{-1}|| = ||\Gamma||^{-1}$, if $||\Gamma|| \neq 0$
6. $||\frac{\Gamma}{\Delta}|| = \frac{||\Gamma||}{||\Delta||}$, if $||\Delta|| \neq 0$

Assume that $w_1, w_2 \in C^1$, where C^1 is the set of complex numbers that define a partial order $\leq$ on C^1 such that
$w_1 \leq w_2$ if and only if $Re(w_1) \leq Re(w_2)$ and $Im(w_1) \leq Im(w_2)$ that is $w_1 \leq w_2$. If any of the following is true [9]

1. $Re(w_1) = Re(w_2)$ and $Im(w_1) = Im(w_2)$;
2. $Re(w_1) < Re(w_2)$ and $Im(w_1) = Im(w_2)$;
3. $Re(w_1) = Re(w_2)$ and $Im(w_1) < Im(w_2)$;
4. $Re(w_1) < Re(w_2)$ and $Im(w_1) < Im(w_2)$;

we will write $w_1 \lneqq w_2$ if $w_1 \neq w_2$ and one of (2),(3), and (4) satisfiy. Take note of that

$$0 \leq w_1 \lneqq w_2 \Rightarrow |w_1| < |w_2|$$
$$w_1 \leq w_2 \text{ and } w_2 < w_3 \Rightarrow w_1 < w_3.$$

Let Δ be a set that is not empty. A mapping $\Omega : \Delta \times \Delta \to C^1$ is considered to be congruent with the following conditions [9]:

1. $0 \leq \Omega(\mu, \nu)$ for all $\mu, \nu \in \Delta$ and $\Omega(\mu, \nu) = 0$ if and only if $\mu = \nu$;
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$, for all $\mu, \nu \in \Delta$;
3. $\Omega(\mu, \nu) \leq \Omega(\mu, z) + \Omega(z, \nu)$ for all $\mu, \nu, z \in \Delta$.

In this case, the metric Ω is referred to as a complex valued metric on the set Δ, and the space (Δ, Ω) is referred to as a complex valued metric space.
Let $\Delta \neq 0$. A mapping $\Omega : \Delta \times \Delta \longrightarrow C^3$ is called a tricomplex-valued metric if the following conditions are satisfied for all $\mu, \nu \in \Delta$ [18]:

1. $\Omega(\mu, \nu) \geq_{i_3} 0$ and $\Omega(\mu, \nu) = 0$ if and only if $\mu = \nu$
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$
3. $\Omega(\mu, \nu) \leq_{i_3} \Omega(\mu, z) + \Omega(z, \nu)$, $\forall \mu, \nu, z \in \Delta$

Then (Δ, Ω) is called tricomplex-valued metric space.
Let C^3 is a tricomplex number and $\Omega : C^3 \times C^3 \longrightarrow C^3$ such that

$$\Omega(\rho_1, \rho_2) = |\mu_1 - \mu_2| + i_3|\nu_1 - \nu_2|$$

Where $\rho_1 = \mu_1 + i_3\nu_1$ and $\rho_2 = \mu_2 + i_3\nu_2$, then (C^3, Ω) is a tricomplex valued metric spaces. Let (Δ, Ω) is a tricomplex-valued metric space then [17]

1. A sequence $\{\nu_n\}$ is a Cauchy sequence if for all $\epsilon \in C^3$ and $\epsilon >_{i_3} 0$, there exist an integer N such that $\Omega(\nu_n, \nu_m) <_{i_3} \epsilon, \forall n, m > N$
2. A sequence $\{\nu_n\}$ is converges to if for all $\epsilon \in C^3$ and $\epsilon >_{i_3} 0$ there exist an integer N such that $\Omega(\nu_n, \nu) <_{i_3} \epsilon, \forall n > N$
3. tricomplex valued metric space (Δ, Ω) is complete if every Cauchy sequence in Δ converges in Δ.

Lemma 1. A sequence $\{\nu_n\}$ in a tricomplex-valued metric space (Δ, Ω) converges to ν if and only if $||\Omega(\nu_n, \nu)|| \longrightarrow 0$ as limit $\nu \longrightarrow \infty$ [18].

Lemma 2. A sequence $\{\nu_n\}$ in a tricomplex-valued metric space (Δ, Ω) is Cauchy if and only if $||\Omega(\nu_n, \nu_{n+m})|| \longrightarrow 0$ as limit $\nu \longrightarrow \infty$ [18].

3. Main Results

Theorem 1. Let (Δ, Ω) be a tricomplex valued metric space and P, Q be the self mappings and A^n and B^n are self mappings in Δ, such that for all $\mu, \nu \in \Delta$

1. $P(\Delta) \subset B^n(\Delta)$ and $Q(\Delta) \subset A^n(\Delta)$;
2. (A^n, P) and (B^n, Q) are commuting Pairs;
3. (A^n, P) and (B^n, Q) are commuting;
4. If $\Omega(Q\nu, A^n\mu) + \Omega(P\mu, B^n\nu) + \Omega(B^n\nu, A^n\mu) \neq 0$ then the following holds

$$\begin{aligned}
\Omega(P\mu, Q\nu) \leq_{i_3} \delta_1 &\left\{ \frac{\Omega(A^n\mu, P\mu) + \Omega(B^n\nu, Q\nu)\Omega(A^n\mu, P\mu)}{1 + \Omega(P\mu, Q\nu)} \right\} \\
&+\delta_2 \max\{\Omega(A^n\mu, B^n\nu), \Omega(A^n\mu, P\mu), \Omega(B^n\nu, P\mu)\} \\
&+\delta_3 \{\Omega(B^n\nu, Q\nu) + \Omega(Q\nu, A^n\mu) + \Omega(P\mu, B^n\nu)\} \\
&+\delta_4 \left\{ \frac{\Omega(Q\nu, B^n\nu)\Omega(P\mu, A^n\mu)}{\Omega(Q\nu, A^n\mu) + \Omega(P\mu, B^n\nu) + \Omega(B^n\nu, A^n\mu)} \right\} \\
&+\delta_5 \max\{\Omega(A^n\mu, P\mu), \Omega(B^n\nu, Q\mu), \Omega(A^n\mu, B^n\nu), \Omega(P\mu, Q\mu)\}
\end{aligned}$$

Where $\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + \delta_5 < 1$. For all $\mu, \nu \in \Delta$. Then A^n, B^n, P and Q have a unique common fixed point.

Proof. Let $\mu_0 \in \Delta$, since $P(\Delta) \subset B^n(\Delta)$ and $Q(\Delta) \subset A^n(\Delta)$ define for all $n \geq 0$ the sequence

$\{\nu_n\} \in \Delta$ by $\nu_{n+1} = P\mu_n = B^n\mu_{n+1}$ and $\nu_{n+2} = Q\mu_{n+1} = A^n\mu_{n+2}$ then,

$$\begin{aligned}
\Omega(\nu_{n+1}, \nu_{n+2}) &= \Omega(P\mu_n, Q\mu_{n+1}) \\
\preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(A^n\mu_n, P\mu_n) + \Omega(B^n\mu_{n+1}, Q\mu_{n+1})\Omega(A^n\mu_n, P\mu_n)}{1 + \Omega(P\mu_n, Q\mu_{n+1})} \right\} \\
&+\delta_2 \max\{\Omega(A^n\mu_n, B^n\mu_{n+1}), \Omega(A^n\mu_n, P\mu_n), \Omega(B^n\mu_{n+1}, P\mu_n)\} \\
&+\delta_3 \{\Omega(B^n\mu_{n+1}, Q\mu_{n+1}) + \Omega(Q\mu_{n+1}, A^n\mu_n) + \Omega(P\mu_n, B^n\mu_{n+1})\} \\
&+\delta_4 \left\{ \frac{\Omega(Q\mu_{n+1}, B^n\mu_{n+1})\Omega(P\mu_n, A^n\mu_n)}{\Omega(Q\mu_{n+1}, A^n\mu_n) + \Omega(P\mu_n, B^n\mu_{n+1}) + \Omega(B^n\mu_{n+1}, A^n\mu_n)} \right\} \\
&+\delta_5 \max\{\Omega(A^n\mu_n, P\mu_n), \Omega(B^n\mu_{n+1}, Q\mu_n), \Omega(A^n\mu_n, B^n\mu_{n+1}), \Omega(P\mu_n, Q\mu_n)\} \\
\preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(\nu_n, \nu_{n+1}) + \Omega(\nu_{n+1}, \nu_{n+2})\Omega(\nu_n, \nu_{n+1})}{1 + \Omega(\nu_{n+1}, \nu_{n+2})} \right\} \\
&+\delta_2 \max\{\Omega(\nu_n, \nu_{n+1}), \Omega(\nu_n, \nu_{n+1}), \Omega(\nu_{n+1}, \nu_{n+1})\} \\
&+\delta_3 \{\Omega(\nu_{n+1}, \nu_{n+2}) + \Omega(\nu_{n+2}, \nu_n) + \Omega(\nu_{n+1}, \nu_{n+1})\} \\
&+\delta_4 \left\{ \frac{\Omega(\nu_{n+2}, \nu_{n+1})\Omega(\nu_{n+1}, \nu_n)}{\Omega(\nu_{n+2}, \nu_n) + \Omega(\nu_{n+1}, \nu_{n+1}) + \Omega(\nu_{n+1}, \nu_n)} \right\} \\
&+\delta_5 \max\{\Omega(\nu_n, \nu_{n+1}), \Omega(\nu_{n+1}, \nu_{n+1}), \Omega(\nu_n, \nu_{n+1}), \Omega(\nu_{n+1}, \nu_{n+1})\}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow ||\Omega(\nu_{n+1}, \nu_{n+2})|| &\leq |\delta_1|||\Omega(\nu_n, \nu_{n+1})|| + |\delta_2|||\Omega(\nu_n, \nu_{n+1})|| + |\delta_3|||\Omega(\nu_n, \nu_{n+1})|| \\
&+|\delta_4|||\Omega(\nu_n, \nu_{n+1})|| + |\delta_5|||\Omega(\nu_n, \nu_{n+1})|| \\
&\leq |(\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5)|||\Omega(\nu_n, \nu_{n+1})||
\end{aligned}$$

Now, $||\Omega(\nu_{n+1}, \nu_{n+2})|| \leq |(\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5)|||\Omega(\nu_n, \nu_{n+1})||$
Similarly,

$$\begin{aligned}
\Omega(\nu_{n+3}, \nu_{n+4}) &= \Omega(P\mu_{n+2}, Q\mu_{n+3}) \\
\preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(A^n\mu_{n+2}, P\mu_{n+2}) + \Omega(B^n\mu_{n+3}, Q\mu_{n+3})\Omega(A^n\mu_{n+2}, P\mu_{n+2})}{1 + \Omega(P\mu_{n+2}, Q\mu_{n+3})} \right\} \\
&+\delta_2 \max\{\Omega(A^n\mu_{n+2}, B^n\mu_{n+3}), \Omega(A^n\mu_{n+2}, P\mu_{n+2}), \Omega(B^n\mu_{n+3}, P\mu_{n+2})\} \\
&+\delta_3 \{\Omega(B^n\mu_{n+3}, Q\mu_{n+3}) + \Omega(Q\mu_{n+3}, A^n\mu_{n+2}) + \Omega(P\mu_{n+2}, B^n\mu_{n+3})\} \\
&+\delta_4 \left\{ \frac{\Omega(Q\mu_{n+3}, B^n\mu_{n+3})\Omega(P\mu_{n+2}, A^n\mu_{n+2})}{\Omega(Q\mu_{n+3}, A^n\mu_{n+2}) + \Omega(P\mu_{n+2}, B^n\mu_{n+3}) + \Omega(B^n\mu_{n+3}, A^n\mu_{n+2})} \right\} \\
&+\delta_5 \max\{\Omega(A^n\mu_{n+2}, P\mu_{n+2}), \Omega(B^n\mu_{n+3}, Q\mu_{n+2}), \Omega(A^n\mu_{n+2}, B^n\mu_{n+3}), \Omega(P\mu_{n+2}, Q\mu_{n+2})\} \\
\preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(\nu_{n+2}, \nu_{n+3}) + \Omega(\nu_{n+3}, \nu_{n+4})\Omega(\nu_{n+2}, \nu_{n+3})}{1 + \Omega(\nu_{n+3}, \nu_{n+4})} \right\} \\
&+\delta_2 \max\{\Omega(\nu_{n+2}, \nu_{n+3}), \Omega(\nu_{n+2}, \nu_{n+3}), \Omega(\nu_{n+3}, \nu_{n+3})\} \\
&+\delta_3 \{\Omega(\nu_{n+3}, \nu_{n+4}) + \Omega(\nu_{n+4}, \nu_{n+2}) + \Omega(\nu_{n+3}, \nu_{n+3})\} \\
&+\delta_4 \left\{ \frac{\Omega(\nu_{n+4}, \nu_{n+3})\Omega(\nu_{n+3}, \nu_{n+2})}{\Omega(\nu_{n+4}, \nu_{n+2}) + \Omega(\nu_{n+3}, \nu_{n+3}) + \Omega(\nu_{n+3}, \nu_{n+2})} \right\} \\
&+\delta_5 \max\{\Omega(\nu_{n+2}, \nu_{n+3}), \Omega(\nu_{n+3}, \nu_{n+3}), \Omega(\nu_{n+2}, \nu_{n+3}), \Omega(\nu_{n+3}, \nu_{n+3})\}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow ||\Omega(\nu_{n+3}, \nu_{n+4})|| &\leq \delta_1||\Omega(\nu_{n+2}, \nu_{n+3})|| + \delta_2||\Omega(\nu_{n+2}, \nu_{n+3})|| + \delta_3||\Omega(\nu_{n+2}, \nu_{n+3})|| \\
&+\delta_4||\Omega(\nu_{n+2}, \nu_{n+3})|| + \delta_5||\Omega(\nu_{n+2}, \nu_{n+3})|| \\
&\leq (\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5)||\Omega(\nu_{n+2}, \nu_{n+3})||
\end{aligned}$$

Now, $||\Omega(\nu_{n+3}, \nu_{n+4})|| \leq (\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5)||\Omega(\nu_{n+2}, \nu_{n+3})||$
Preceding in this way we get

$||\Omega(\nu_n, \nu_{n+1})|| \leq (\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5)||\Omega(\nu_{n-1}, \nu_n)||$
$\implies ||\Omega(\nu_n, \nu_{n+1})|| \leq \delta||\Omega(\nu_{n-1}, \nu_n)||$
Where $(\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5) < 1$

$$\begin{aligned} ||\Omega(\nu_n, \nu_{n+1})|| &\leq \delta||\Omega(\nu_{n-1}, \nu_n)|| \\ &\leq \delta^2||\Omega(\nu_{n-2}, \nu_{n-1})|| \\ &\leq \delta^3||\Omega(\nu_{n-3}, \nu_{n-2})|| \\ &\ldots \leq \delta^n||\Omega(\nu_0, \nu_1)|| \end{aligned}$$

Now $\forall k > n$, we have,

$$||\Omega(\nu_k, \nu_n)|| \leq \delta^n||\Omega(\nu_0, \nu_1)|| + \delta^{n-1}||\Omega(\nu_0, \nu_1)|| + \delta^{n-2}||\Omega(\nu_0, \nu_1)|| + \ldots + \delta^{k-1}||\Omega(\nu_0, \nu_1)||$$

$\implies ||\Omega(\nu_k, \nu_n)|| \leq (\delta^n + \delta^{n-1} + \delta^{n-2} + \delta^{k-1})||\Omega(\nu_0, \nu_1)||$
$\implies ||\Omega(\nu_k, \nu_n)|| \leq \frac{\delta^n}{1-\delta}||\Omega(\nu_0, \nu_1)||$
as $\delta < 1$ then $\lim_{n,m\to\infty} ||\Omega(\nu_k, \nu_n)|| = 0$
Hence $\{\nu_n\}$ is a Cauchy sequence. Since Δ is complete then ν_n and its subsequences $\{P\mu_n\}, \{B^n\mu_{n+1}\}, \{Q\mu_{n+1}\}$ and $\{A^n\mu_{n+2}\}$ also converges to ω
$\lim_{n\to\infty} \nu_n = \lim_{n\to\infty} P\mu_n = \lim_{n\to\infty} B^n\mu_{n+1} = \lim_{n\to\infty} Q\mu_{n+1} = \lim_{n\to\infty} A^n\mu_{n+2} = \omega$
Let there exist $\omega* \in \Delta$ such that $\nu_n \to \omega*$ as $n \to \infty$, and $P\omega* = A^n\omega* = B^n\omega* = Q\omega* = \omega$
(A^n, P) and (B^n, Q) are weakly compatible, then they commute at their coincidence point.
Therefore
$P\omega = PA^n\omega* = A^nP\omega* = A^n\omega$ and $B^n\omega = B^nQ\omega* = QB^n\omega* = Q\omega$
Now we show that $P\omega = Q\omega$ then we have

$$\begin{aligned} \Omega(P\omega, Q\mu_{n+1}) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\mu_{n+1}, Q\mu_{n+1})\Omega(A^n\omega, P\omega)}{1 + \Omega(P\omega, Q\mu_{n+1})} \right\} \\ &+\delta_2 \max\{\Omega(A^n\omega, B^n\mu_{n+1}), \Omega(A^n\omega, P\omega), \Omega(B^n\mu_{n+1}, P\omega)\} \\ &+\delta_3 \{\Omega(B^n\mu_{n+1}, Q\mu_{n+1}) + \Omega(Q\mu_{n+1}, A^n\omega) + \Omega(P\omega, B^n\mu_{n+1})\} \\ &+\delta_4 \left\{ \frac{\Omega(P\mu_{n+1}, B^n\mu_{n+1})\Omega(P\omega, A^n\omega)}{\Omega(Q\mu_{n+1}, A^n\omega) + \Omega(P\omega, B^n\mu_{n+1}) + \Omega(B^n\mu_{n+1}, A^n\omega)} \right\} \\ &+\delta_5 \max\{\Omega(A^n\omega, P\omega), \Omega(B^n\mu_{n+1}, Q\omega), \Omega(A^n\omega, B^n\mu_{n+1}), \Omega(P\omega, Q\omega)\} \end{aligned}$$

$$\begin{aligned} \implies \Omega(P\omega, \nu_{n+2}) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(P\omega, P\omega) + \Omega(\nu_{n+1}, \nu_{n+2})\Omega(P\omega, P\omega)}{1 + \Omega(P\omega, \nu_{n+2})} \right\} \\ &+\delta_2 \max\{\Omega(P\omega, \nu_{n+1}), \Omega(P\omega, P\omega), \Omega(\nu_{n+1}, P\omega)\} \\ &+\delta_3 \{\Omega(\nu_{n+1}, \nu_{n+2}) + \Omega(\nu_{n+2}, P\omega) + \Omega(P\omega, \nu_{n+1})\} \\ &+\delta_4 \left\{ \frac{\Omega(\nu_{n+2}, \nu_{n+1})\Omega(P\omega, P\omega)}{\Omega(\nu_{n+2}, P\omega) + \Omega(P\omega, \nu_{n+1}) + \Omega(\nu_{n+1}, P\omega)} \right\} \\ &+\delta_5 \max\{\Omega(P\omega, P\omega), \Omega(\nu_{n+1}, P\omega), \Omega(P\omega, \nu_{n+1}), \Omega(P\omega, P\omega)\} \end{aligned}$$

If we take $n \to \infty$, then we have

$$\begin{aligned} \implies \Omega(P\omega, \omega) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(P\omega, P\omega) + \Omega(\omega, \omega)\Omega(P\omega, P\omega)}{1 + \Omega(P\omega, \omega)} \right\} \\ &+\delta_2 \max\{\Omega(P\omega, \omega), \Omega(P\omega, P\omega), \Omega(\omega, P\omega)\} \\ &+\delta_3 \{\Omega(\omega, \omega) + \Omega(\omega, P\omega) + \Omega(P\omega, \omega)\} \\ &+\delta_4 \left\{ \frac{\Omega(\omega, \omega)\Omega(P\omega, P\omega)}{\Omega(\omega, P\omega) + \Omega(P\omega, \omega) + \Omega(\omega, P\omega)} \right\} \\ &+\delta_5 \max\{\Omega(P\omega, P\omega), \Omega(\omega, P\omega), \Omega(P\omega, \omega), \Omega(P\omega, P\omega)\} \end{aligned}$$

$\Longrightarrow ||\Omega(P\omega,\omega)|| \leq \delta_2||\Omega(P\omega,\omega)|| + 2\delta_3||\Omega(P\omega,\omega)|| + \delta_5||\Omega(P\omega,\omega)||$
$\Longrightarrow ||\Omega(P\omega,\omega)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(P\omega,\omega)||$
$\Longrightarrow ||\Omega(P\omega,\omega)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(P\omega,\omega)||$
This is contradiction that $(\delta_2 + 2\delta_3 + \delta_5) < 1$ therefore $P\omega = \omega$ and $A^n\omega = P\omega \Rightarrow A^n\omega = \omega$. Now we will prove that $Q\omega = \omega$, putting $\mu = \nu = \omega$ then we have

$$\begin{aligned}\Omega(P\omega, Q\omega) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\omega, Q\omega)\Omega(A^n\omega, P\omega)}{1+\Omega(P\omega, Q\omega)}\right\}\\ &+\delta_2 \max\{\Omega(A^n\omega, B^n\omega), \Omega(A^n\omega, P\omega), \Omega(B^n\omega, P\omega)\}\\ &+\delta_3\{\Omega(B^n\omega, Q\omega) + \Omega(Q\omega, A^n\omega) + \Omega(P\omega, B^n\omega)\}\\ &+\delta_4\left\{\frac{\Omega(Q\omega, B^n\omega)\Omega(P\omega, A^n\omega)}{\Omega(Q\omega, A^n\omega) + \Omega(P\omega, B^n\omega) + \Omega(B^n\omega, A^n\omega)}\right\}\\ &+\delta_5 \max\{\Omega(A^n\omega, P\omega), \Omega(B^n\omega, Q\omega), \Omega(A^n\omega, B^n\omega), \Omega(P\omega, Q\omega)\}\end{aligned}$$

Taking $n \to \infty$ we get

$$\begin{aligned}\Omega(\omega, Q\omega) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(\omega,\omega) + \Omega(Q\omega, Q\omega)\Omega(\omega,\omega)}{1+\Omega(\omega, Q\omega)}\right\}\\ &+\delta_2 \max\{\Omega(\omega, Q\omega), \Omega(\omega,\omega), \Omega(Q\omega,\omega)\}\\ &+\delta_3\{\Omega(Q\omega, Q\omega) + \Omega(Q\omega,\omega) + \Omega(\omega, Q\omega)\}\\ &+\delta_4\left\{\frac{\Omega(Q\omega, Q\omega)\Omega(\omega,\omega)}{\Omega(Q\omega,\omega) + \Omega(\omega, Q\omega) + \Omega(Q\omega,\omega)}\right\}\\ &+\delta_5 \max\{\Omega(\omega,\omega), \Omega(Q\omega, Q\omega), \Omega(\omega, Q\omega), \Omega(\omega, Q\omega)\}\end{aligned}$$

$\Longrightarrow \Omega(\omega, Q\omega) \leq_{i_3} \delta_2\Omega(\omega, Q\omega) + 2\delta_3\Omega(\omega, Q\omega) + \delta_5\Omega(\omega, Q\omega)$
$\Longrightarrow ||\Omega(\omega, Q\omega)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(\omega, Q\omega)||$
Then
$\Longrightarrow ||\Omega(\omega, Q\omega)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(\omega, Q\omega)||$ Which is again a contradiction. Therefore $Q\omega = \omega$ and $B^n\omega = Q\omega \Longrightarrow B^n\omega = \omega$.

Uniqueness of fixed point

Let ω_0 be another fixed point of P, Q, A^n, B^n then we have

$$\begin{aligned}\Omega(P\omega, Q\omega_0) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\omega_0, Q\omega_0)\Omega(A^n\omega, P\omega)}{1+\Omega(P\omega, Q\omega_0)}\right\}\\ &+\delta_2 \max\{\Omega(A^n\omega, B^n\omega_0), \Omega(A^n\omega, P\omega), \Omega(B^n\omega_0, P\omega)\}\\ &+\delta_3\{\Omega(B^n\omega_0, Q\omega_0) + \Omega(Q\omega_0, A^n\omega) + \Omega(P\omega, B^n\omega_0)\}\\ &+\delta_4\left\{\frac{\Omega(Q\omega_0, B^n\omega_0)\Omega(P\omega, A^n\omega)}{\Omega(Q\omega_0, A^n\omega) + \Omega(P\omega, B^n\omega_0) + \Omega(B^n\omega_0, A^n\omega)}\right\}\\ &+\delta_5 \max\{\Omega(A^n\omega, P\omega), \Omega(B^n\omega_0, Q\omega), \Omega(A^n\omega, B^n\omega_0), \Omega(P\omega, Q\omega)\}\end{aligned}$$

Taking limit $n \to \infty$

$$\begin{aligned}\Omega(\omega, \omega_0) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(\omega,\omega) + \Omega(\omega_0,\omega_0)\Omega(\omega,\omega)}{1+\Omega(\omega,\omega_0)}\right\}\\ &+\delta_2 \max\{\Omega(\omega,\omega_0), \Omega(\omega,\omega), \Omega(\omega_0,\omega)\}\\ &+\delta_3\{\Omega(\omega_0,\omega_0) + \Omega(\omega_0,\omega) + \Omega(\omega,\omega_0)\}\\ &+\delta_4\left\{\frac{\Omega(\omega_0,\omega_0)\Omega(\omega,\omega)}{\Omega(\omega_0,\omega) + \Omega(\omega,\omega_0) + \Omega(\omega_0,\omega)}\right\}\\ &+\delta_5 \max\{\Omega(\omega,\omega), \Omega(\omega_0,\omega), \Omega(\omega,\omega_0), \Omega(\omega,\omega)\}\end{aligned}$$

$\Longrightarrow ||\Omega(\omega,\omega_0)|| \leq \delta_2||\Omega(\omega,\omega_0)|| + 2\delta_3||\Omega(\omega,\omega_0)|| + \delta_5||\Omega(\omega,\omega_0)||$
$\Longrightarrow ||\Omega(\omega,\omega_0)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(\omega,\omega_0)||$
$\Longrightarrow ||\Omega(\omega,\omega_0)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(\omega,\omega_0)||$
Which is again contradiction therefore $\omega = \omega_0$ is a unique common fixed point of P, Q, A^n and B^n ■

Theorem 2. Let (Δ, Ω) be a tricomplex valued metric space and P, Q be the self mappings and A^n and B^n are self mappings in Δ, such that for all $\mu, \nu \in \Delta$

1. $P(\Delta) \subset B^n(\Delta)$ and $Q(\Delta) \subset A^n(\Delta)$;
2. (A^n, P) are compatible, A^n or P is continuous and (B^n, Q) are weakly compatible;
3. (B^n, Q) are compatible, B^n or Q is continuous and (A^n, P) are weakly compatible;
4. For all $\mu, \nu \in \Delta$ and $\mu \neq \nu$ we have

$$\begin{aligned}\Omega(P\mu, Q\nu) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\mu, P\mu) + \Omega(B^n\nu, Q\nu)\Omega(A^n\mu, P\mu)}{1+\Omega(P\mu, Q\nu)}\right\}\\ &+\delta_2 \max\{\Omega(A^n\mu, B^n\nu), \Omega(A^n\mu, P\mu), \Omega(B^n\nu, P\mu)\}\\ &+\delta_3\{\Omega(B^n\nu, Q\nu) + \Omega(Q\nu, A^n\mu) + \Omega(P\mu, B^n\nu)\}\\ &+\delta_4\left\{\frac{\Omega(Q\nu, B^n\nu)\Omega(P\mu, A^n\mu)}{\Omega(Q\nu, A^n\mu) + \Omega(P\mu, B^n\nu) + \Omega(B^n\nu, A^n\mu)}\right\}\\ &+\delta_5 \max\{\Omega(A^n\mu, P\mu), \Omega(B^n\nu, Q\mu), \Omega(A^n\mu, B^n\nu), \Omega(P\mu, Q\mu)\}\end{aligned}$$

Where $\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + \delta_5 < 1$. For all $\mu, \nu \in \Delta$. Then A^n, B^n, P and Q have a unique common fixed point.

Proof. By theorem (1) $\{\nu_n\}$ is a Cauchy sequence and Δ is complete, then sequence $\{\nu_n\}$ converge to ω and subsequences $\{P\mu_n\}$, $\{B^n\mu_{n+1}\}$, $\{Q\mu_{n+1}\}$ and $\{A^n\mu_{n+2}\}$ also converges to ω.

$$\Longrightarrow \lim_{n\to\infty} \nu_n = \lim_{n\to\infty} P\mu_n = \lim_{n\to\infty} B^n\mu_{n+1} = \lim_{n\to\infty} A^n\mu_{n+2} = \lim_{n\to\infty} Q\mu_{n+1} = \omega. \tag{1}$$

Since (A^n, P) are compatible, let P is continuous then we have

$$\lim_{n\to\infty} A^n P\mu_{n+2} = \lim_{n\to\infty} PA^n\mu_{n+2} = P\omega \tag{2}$$

Now putting $\mu = \mu_{n+2}, \nu = \mu_{n+1}$ we have

$$\begin{aligned}\Omega(P\mu_{n+2}, Q\mu_{n+1}) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\mu_{n+2}, P\mu_{n+2}) + \Omega(B^n\mu_{n+1}, Q\mu_{n+1})\Omega(A^n\mu_{n+2}, P\mu_{n+2})}{1+\Omega(P\mu_{n+2}, Q\mu_{n+1})}\right\}\\ &+\delta_2 \max\{\Omega(A^n\mu_{n+2}, B^n\mu_{n+1}), \Omega(A^n\mu_{n+2}, P\mu_{n+2}), \Omega(B^n\mu_{n+1}, P\mu_{n+2})\}\\ &+\delta_3\{\Omega(B^n\mu_{n+1}, Q\mu_{n+1}) + \Omega(Q\mu_{n+1}, A^n\mu_{n+2}) + \Omega(P\mu_{n+2}, B^n\mu_{n+1})\}\\ &+\delta_4\left\{\frac{\Omega(Q\mu_{n+1}, B^n\mu_{n+1})\Omega(P\mu_{n+2}, A^n\mu_{n+2})}{\Omega(Q\mu_{n+1}, A^n\mu_{n+2}) + \Omega(P\mu, B^n\mu_{n+1}) + \Omega(B^n\mu_{n+1}, A^n\mu_{n+2})}\right\}\\ &+\delta_5 \max\{\Omega(A^n\mu_{n+2}, P\mu_{n+2}), \Omega(B^n\mu_{n+1}, Q\mu_{n+2}), \Omega(A^n\mu_{n+2}, B^n\mu_{n+1}), \Omega(P\mu_{n+2}, Q\mu_{n+2})\}\end{aligned}$$

Taking limit $n \to \infty$ and using (1) and (2), we get

$$\begin{aligned}\Longrightarrow \Omega(P\omega, \omega) \leq_{i_3} \delta_1 &\left\{\frac{\Omega(\omega,\omega) + \Omega(\omega,\omega)\Omega(\omega,\omega)}{1+\Omega(\omega,\omega)}\right\}\\ &+\delta_2 \max\{\Omega(\omega,\omega), \Omega(\omega,\omega), \Omega(\omega,\omega)\}\\ &+\delta_3\{\Omega(\omega,\omega) + \Omega(\omega,\omega) + \Omega(\omega,\omega)\}\\ &+\delta_4\left\{\frac{\Omega(\omega,\omega)\Omega(\omega,\omega)}{\Omega(\omega,\omega) + \Omega(\omega,\omega) + \Omega(\omega,\omega)}\right\}\\ &+\delta_5 \max\{\Omega(\omega,\omega), \Omega(\omega,\omega), \Omega(\omega,\omega), \Omega(\omega,\omega)\}\end{aligned}$$

$\Longrightarrow ||\Omega(P\omega,\omega)|| \leq 0$
$\Longrightarrow ||\Omega(P\omega,\omega)|| \leq 0$. Therefore $P\omega = \omega$.
Now again putting $\mu = \omega$ and $\nu = \mu_{n+1}$ we have

$$\begin{aligned}\Omega(P\omega, Q\mu_{n+1}) \preceq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\mu_{n+1}, Q\mu_{n+1})\Omega(A^n\omega, P\omega)}{1+\Omega(P\omega, Q\mu_{n+1})}\right\}\\ &+\delta_2 \max\{\Omega(A^n\omega, B^n\mu_{n+1}), \Omega(A^n\omega, P\omega), \Omega(B^n\mu_{n+1}, P\omega)\}\\ &+\delta_3\{\Omega(B^n\mu_{n+1}, Q\mu_{n+1}) + \Omega(Q\mu_{n+1}, A^n\omega) + \Omega(P\omega, B^n\mu_{n+1})\}\\ &+\delta_4\left\{\frac{\Omega(Q\mu_{n+1}, B^n\mu_{n+1})\Omega(P\omega, A^n\omega)}{\Omega(Q\mu_{n+1}, A^n\omega) + \Omega(P\omega, B^n\mu_{n+1}) + \Omega(B^n\mu_{n+1}, A^n\omega)}\right\}\\ &+\delta_5 \max\{\Omega(A^n\omega, P\omega), \Omega(B^n\mu_{n+1}, Q\omega), \Omega(A^n\omega, B^n\mu_{n+1}), \Omega(P\omega, Q\omega)\}\end{aligned}$$

Let $n \to \infty$ and using (1) and (2), we have

$$\begin{aligned}\Longrightarrow \Omega(\omega,\omega) \preceq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\omega, \omega) + \Omega(\omega,\omega)\Omega(A^n\omega, P\omega)}{1+\Omega(\omega,\omega)}\right\}\\ &+\delta_2 \max\{\Omega(A^n\omega,\omega), \Omega(A^n\omega,\omega), \Omega(\omega,\omega)\}\\ &+\delta_3\{\Omega(\omega,\omega) + \Omega(\omega, A^n\omega) + \Omega(\omega,\omega)\}\\ &+\delta_4\left\{\frac{\Omega(\omega,\omega)\Omega(\omega, A^n\omega)}{\Omega(\omega, A^n\omega) + \Omega(\omega,\omega) + \Omega(\omega, A^n\omega)}\right\}\\ &+\delta_5 \max\{\Omega(A^n\omega,\omega), \Omega(\omega,\omega), \Omega(A^n\omega,\omega), \Omega(\omega,\omega)\}\end{aligned}$$

$\Longrightarrow (\delta_1+\delta_2+\delta_3+\delta_5)||\Omega(A^n\omega,\omega)|| \geq 0$
$\Longrightarrow ||\Omega(A^n\omega,\omega)|| \geq 0$. Therefore $A^n\omega = \omega$
Since $P(\Delta) \subset B^n(\Delta)$, there exist $\omega* \in \Delta$ such that $P\omega = B^n\omega*$. Now to prove $B^n\omega* = Q\omega*$ putting $\mu = \omega$ and $\nu = \omega*$ we get

$$\begin{aligned}\Omega(P\omega, Q\omega*) \preceq_{i_3} \delta_1 &\left\{\frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\omega*, Q\omega*)\Omega(A^n\omega, P\omega)}{1+\Omega(P\omega, Q\omega*)}\right\}\\ &+\delta_2 \max\{\Omega(A^n\omega, B^n\omega*), \Omega(A^n\omega, P\omega), \Omega(B^n\omega*, P\omega)\}\\ &+\delta_3\{\Omega(B^n\omega*, Q\omega*) + \Omega(Q\omega*, A^n\omega) + \Omega(P\omega, B^n\omega*)\}\\ &+\delta_4\left\{\frac{\Omega(Q\omega*, B^n\omega*)\Omega(P\omega, A^n\omega)}{\Omega(Q\omega*, A^n\omega) + \Omega(P\omega, B^n\omega*) + \Omega(B^n\omega*, A^n\omega)}\right\}\\ &+\delta_5 \max\{\Omega(A^n\omega, P\omega), \Omega(B^n\omega*, Q\omega), \Omega(A^n\omega, B^n\omega*), \Omega(P\omega, Q\omega)\}\end{aligned}$$

Here given that $P\omega = \omega = B^n\omega$. Then we get

$$\begin{aligned}\Omega(\omega, Q\omega*) \preceq_{i_3} \delta_1 &\left\{\frac{\Omega(\omega,\omega) + \Omega(\omega, Q\omega*)\Omega(\omega,\omega)}{1+\Omega(\omega, Q\omega*)}\right\}\\ &+\delta_2 \max\{\Omega(\omega,\omega), \Omega(\omega,\omega), \Omega(\omega,\omega)\}\\ &+\delta_3\{\Omega(\omega, Q\omega*) + \Omega(Q\omega*, \omega) + \Omega(\omega,\omega)\}\\ &+\delta_4\left\{\frac{\Omega(Q\omega*, \omega)\Omega(\omega,\omega)}{\Omega(Q\omega*, \omega) + \Omega(\omega,\omega) + \Omega(\omega,\omega)}\right\}\\ &+\delta_5 \max\{\Omega(\omega,\omega), \Omega(\omega, Q\omega), \Omega(\omega,\omega), \Omega(\omega, Q\omega)\}\end{aligned}$$

$\Longrightarrow ||\Omega(\omega, Q\omega*)|| \leq 2\delta_3||\Omega(\omega, Q\omega*)|| + \delta_5||\Omega(\omega, Q\omega*)||$
$\Longrightarrow ||\Omega(\omega, Q\omega*)|| \leq (2\delta_3+\delta_5)||\Omega(\omega, Q\omega*)||$
$\Longrightarrow Q\omega* = \omega$
Since $B^n\omega*$ and Q are weakly compatible then we have $B^n\omega = B^nQ\omega* = Q\omega$. Therefore ω is a

coincidence point of B^n and P. Now to prove $Q\omega = \omega$, putting $\mu = \omega$ and $\nu = \omega$ we get

$$\begin{aligned}\Omega(P\omega, Q\omega) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\omega, Q\omega)\Omega(A^n\omega, P\omega)}{1 + \Omega(P\omega, Q\omega)} \right\} \\ &+\delta_2 \max\left\{\Omega(A^n\omega, B^n\omega), \Omega(A^n\omega, P\omega), \Omega(B^n\omega, P\omega)\right\} \\ &+\delta_3 \left\{\Omega(B^n\omega, Q\omega) + \Omega(Q\omega, A^n\omega) + \Omega(P\omega, B^n\omega)\right\} \\ &+\delta_4 \left\{ \frac{\Omega(Q\omega, B^n\omega)\Omega(P\omega, A^n\omega)}{\Omega(Q\omega, A^n\omega) + \Omega(P\omega, B^n\omega) + \Omega(B^n\omega, A^n\omega)} \right\} \\ &+\delta_5 \max\left\{\Omega(A^n\omega, P\omega), \Omega(B^n\omega, Q\omega), \Omega(A^n\omega, B^n\omega), \Omega(P\omega, Q\omega)\right\}\end{aligned}$$

$$\begin{aligned}\implies \Omega(\omega, Q\omega) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(\omega, \omega) + \Omega(\omega, Q\omega)\Omega(\omega, \omega)}{1 + \Omega(\omega, Q\omega)} \right\} \\ &+\delta_2 \max\left\{\Omega(\omega, \omega), \Omega(\omega, \omega), \Omega(\omega, \omega)\right\} \\ &+\delta_3 \left\{\Omega(\omega, Q\omega) + \Omega(Q\omega, \omega) + \Omega(\omega, \omega)\right\} \\ &+\delta_4 \left\{ \frac{\Omega(Q\omega, \omega)\Omega(\omega, \omega)}{\Omega(Q\omega, \omega) + \Omega(\omega, \omega) + \Omega(\omega, \omega)} \right\} \\ &+\delta_5 \max\left\{\Omega(\omega, \omega), \Omega(\omega, Q\omega), \Omega(\omega, \omega), \Omega(\omega, Q\omega)\right\}\end{aligned}$$

$\implies ||\Omega(\omega, Q\omega)|| \leq 2\delta_3||\Omega(\omega, Q\omega)|| + \delta_5||\Omega(\omega, Q\omega)||$
$\implies ||\Omega(\omega, Q\omega)|| \leq (2\delta_3 + \delta_5)||\Omega(\omega, Q\omega)||$
$\implies Q\omega = \omega$

Uniqueness of fixed point

Let ω_0 be another fixed point of P, Q, A^n, B^n then we have

$$\begin{aligned}\Omega(P\omega, Q\omega_0) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(A^n\omega, P\omega) + \Omega(B^n\omega_0, Q\omega_0)\Omega(A^n\omega, P\omega)}{1 + \Omega(P\omega, Q\omega_0)} \right\} \\ &+\delta_2 \max\left\{\Omega(A^n\omega, B^n\omega_0), \Omega(A^n\omega, P\omega), \Omega(B^n\omega_0, P\omega)\right\} \\ &+\delta_3 \left\{\Omega(B^n\omega_0, Q\omega_0) + \Omega(Q\omega_0, A^n\omega) + \Omega(P\omega, B^n\omega_0)\right\} \\ &+\delta_4 \left\{ \frac{\Omega(Q\omega_0, B^n\omega_0)\Omega(P\omega, A^n\omega)}{\Omega(Q\omega_0, A^n\omega) + \Omega(P\omega, B^n\omega_0) + \Omega(B^n\omega_0, A^n\omega)} \right\} \\ &+\delta_5 \max\left\{\Omega(A^n\omega, P\omega), \Omega(B^n\omega_0, Q\omega), \Omega(A^n\omega, B^n\omega_0), \Omega(P\omega, Q\omega)\right\}\end{aligned}$$

Taking limit $n \to \infty$

$$\begin{aligned}\Omega(\omega, \omega_0) \preceq_{i_3} \delta_1 &\left\{ \frac{\Omega(\omega, \omega) + \Omega(\omega_0, \omega_0)\Omega(\omega, \omega)}{1 + \Omega(\omega, \omega_0)} \right\} \\ &+\delta_2 \max\left\{\Omega(\omega, \omega_0), \Omega(\omega, \omega), \Omega(\omega_0, \omega)\right\} \\ &+\delta_3 \left\{\Omega(\omega_0, \omega_0) + \Omega(\omega_0, \omega) + \Omega(\omega, \omega_0)\right\} \\ &+\delta_4 \left\{ \frac{\Omega(\omega_0, \omega_0)\Omega(\omega, \omega)}{\Omega(\omega_0, \omega) + \Omega(\omega, \omega_0) + \Omega(\omega_0, \omega)} \right\} \\ &+\delta_5 \max\left\{\Omega(\omega, \omega), \Omega(\omega_0, \omega), \Omega(\omega, \omega_0), \Omega(\omega, \omega)\right\}\end{aligned}$$

$\implies ||\Omega(\omega, \omega_0)|| \leq \delta_2||\Omega(\omega, \omega_0)|| + 2\delta_3||\Omega(\omega, \omega_0)|| + \delta_5||\Omega(\omega, \omega_0)||$
$\implies ||\Omega(\omega, \omega_0)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(\omega, \omega_0)||$
$\implies ||\Omega(\omega, \omega_0)|| \leq (\delta_2 + 2\delta_3 + \delta_5)||\Omega(\omega, \omega_0)||$
Which is contradiction since $(\delta_2 + 2\delta_3 + \delta_5) < 1$ therefore $\omega = \omega_0$ is a unique common fixed point of P, Q, A^n and B^n ■

Corollary 1. Let (Δ, Ω) be a tricomplex valued metric space and P, Q, A, B are self mappings in Δ, such that for all $\mu, \nu \in \Delta$

1. $P(\Delta) \subset B(\Delta)$ and $Q(\Delta) \subset A(\Delta)$;
2. (A, P) and (B, Q) are commuting Pairs;
3. (A, P) and (B, Q) are commuting;
4. If $\Omega(Q\nu, A\mu) + \Omega(P\mu, B\nu) + \Omega(B\nu, A\mu) \neq 0$ then the following holds

$$\begin{aligned}\Omega(P\mu, Q\nu) \preceq_{i_3} \delta_1 &\left\{\frac{\Omega(A\mu, P\mu) + \Omega(B\nu, Q\nu)\Omega(A\mu, P\mu)}{1 + \Omega(P\mu, Q\nu)}\right\} \\ &+\delta_2 \max\{\Omega(A\mu, B\nu), \Omega(A\mu, P\mu), \Omega(B\nu, P\mu)\} \\ &+\delta_3 \{\Omega(B\nu, Q\nu) + \Omega(Q\nu, A\mu) + \Omega(P\mu, B\nu)\} \\ &+\delta_4 \left\{\frac{\Omega(Q\nu, B\nu)\Omega(P\mu, A\mu)}{\Omega(Q\nu, A\mu) + \Omega(P\mu, B\nu) + \Omega(B\nu, A\mu)}\right\} \\ &+\delta_5 \max\{\Omega(A\mu, P\mu), \Omega(B\nu, Q\mu), \Omega(A\mu, B\nu), \Omega(P\mu, Q\mu)\}\end{aligned}$$

Where $\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + \delta_5 < 1$. For all $\mu, \nu \in \Delta$. Then P, Q, A and B have a unique common fixed point.

Corollary 2. Let (Δ, Ω) be a tricomplex valued metric space and P, Q, A, B are self mappings in Δ, such that for all $\mu, \nu \in \Delta$

1. $P(\Delta) \subset B(\Delta)$ and $Q(\Delta) \subset A(\Delta)$;
2. (A, P) are compatible, A or P is continuous and (B, Q) are weakly compatible;
3. (B, Q) are compatible, B or Q is continuous and (A, P) are weakly compatible;
4. For all $\mu, \nu \in \Delta$ and $\mu \neq \nu$ we have

$$\begin{aligned}\Omega(P\mu, Q\nu) \preceq_{i_3} \delta_1 &\left\{\frac{\Omega(A\mu, P\mu) + \Omega(B\nu, Q\nu)\Omega(A\mu, P\mu)}{1 + \Omega(P\mu, Q\nu)}\right\} \\ &+\delta_2 \max\{\Omega(A\mu, B\nu), \Omega(A\mu, P\mu), \Omega(B\nu, P\mu)\} \\ &+\delta_3 \{\Omega(B\nu, Q\nu) + \Omega(Q\nu, A\mu) + \Omega(P\mu, B\nu)\} \\ &+\delta_4 \left\{\frac{\Omega(Q\nu, B\nu)\Omega(P\mu, A\mu)}{\Omega(Q\nu, A\mu) + \Omega(P\mu, B\nu) + \Omega(B\nu, A\mu)}\right\} \\ &+\delta_5 \max\{\Omega(A\mu, P\mu), \Omega(B\nu, Q\mu), \Omega(A\mu, B\nu), \Omega(P\mu, Q\mu)\}\end{aligned}$$

Where $\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + \delta_5 < 1$. For all $\mu, \nu \in \Delta$. Then P, Q, A and B have a unique common fixed point.

4. Conclusion

In this paper, we proved generalizations of fixed-point theorems in tricomplex-valued metric space for the sequence of mappings with rational type contraction and using commuting, compatible and weakly compatible pairs. our contributions have enhanced the field of tricomplex valued metric spaces.

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