

FUZZY DATA-DRIVEN ESTIMATION FOR DUAL WEIBULL POPULATIONS: EM AND BAYESIAN METHODS WITH GIBBS SAMPLING

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Abstract

In statistical analysis, accurately estimating parameters in the presence of uncertain and imprecise data is critical, particularly when dealing with complex, dual population models. Fuzzy data, which effectively represents real-world ambiguity, provides a framework for handling such uncertainties. While parameter estimation for single-population models with fuzzy data has been explored extensively, extending these methods to dual populations remains challenging. This study addresses this gap by developing estimation techniques for two Weibull populations that share a common shape parameter but differ in scale parameters, under fuzzy data conditions. We employ the Expectation-Maximization (EM) algorithm for Maximum Likelihood estimation and a Bayesian framework with TK approximation for parameter estimation. To refine our Bayesian estimates, we use Gibbs sampling to compute posterior densities. Through Monte Carlo simulations, for generated data as well as for a real dataset, we evaluate the accuracy and robustness of the proposed estimators, demonstrating their practical utility in applications where data imprecision is a significant factor. This research highlights the importance of robust methodologies for dual-population Weibull models, contributing to enhanced reliability in statistical analysis across diverse fields.

Keywords: Gibbs sampling; EM algorithm; TK approximation; Fuzzy data; Weibull distribution.

1. INTRODUCTION

Statistical estimation of parameters, under uncertain and imprecise conditions, becomes even more crucial in cases with complex multi-population models. Most real-world scenarios are characterized by uncertainty and vagueness, which cannot be adequately captured by classical data representations. For management of such uncertainties, fuzzy data provides a richer framework that allows researchers to make better inferences. At the same time, modeling through fuzzy sets enables one to manage data variability in a systematic and mathematically constructive manner so that statistical analysis could attain better precision.

Due to its suitability in modeling with limited precise data, the Weibull distribution has vital usages in reliability and survival analysis. For example, survival times of patients or the materials' failure times depend quite heavily on various external factors or incomplete information thereby making an imprecision such as these. Apart from classical literature, the model incorporates fuzzy data in a Weibull model so that such uncertainties are realistically represented. In this study, two Weibull populations have been considered with similar shape parameters hence suggesting similar survival or failure rates and distinct from each other with respect to scale parameters that provide different averages for these populations.

The Weibull distribution is commonly used in failure modeling, reliability determination, and life data analysis. Its introduction in various situations can be traced back to Waloddi Weibull, who presented it in 1951. The variety of applications for this distribution in biology, engineering, and weather forecasting is due to its adaptability and capability to model a variety of data types. The p.d.f of weibull distribuiton with scale parameter σ and shape parameter k is,

$$f(x; \sigma, k) = k\sigma x^{k-1} e^{-\sigma x^k}, \quad x > 0, \quad k > 0, \quad \sigma > 0.$$

By looking into some conclusions drawn by different scholars on the Weibull distribution. In 1977 [1], the maximum likelihood method was used to estimate parameters of Weibull distribution, and they also provided a method of calculating confidence intervals. Lai in 2011 [2] has given extensive explanations of various extensions, parameter estimation techniques, and basic Weibull distribution properties useful in modeling complex data sets. N. Balakrishnan and Kateri proposed a new method based on a simple and easy-to-apply graphical method that easily shows the existence and uniqueness of maximum likelihood estimators [14]. To the best of our knowledge, the only work in this direction that discusses the estimation of a common shape parameter of several Weibull populations is considered by Yang and Lin [15]. The estimation of parameters of various lifetime distributions including weibull distribution is considered by [16]. In 2009 [17] came to the conclusion that there is no analytical solution for the restricted MLE of the scale parameter for a given shape parameter, and he finally developed a new approach that is thought to be more effective and efficient at handling interval data than regular MLE methods, directly developed EM algorithms, as well as genetic algorithms.

Recently, the application of fuzzy sets in estimation theory has attracted considerable attention from researchers because it can simultaneously take care of uncertainties inherent in lifespan data. Such uncertainties are often of human error nature or can originate from technical failures or unforeseen circumstances, which could be easily modeled using fuzzy sets. The idea was formed in 1991 by Coppi et al. [4], who studied different applications of fuzzy techniques in statistical analysis. Building on this work, Denoeux [5] later investigated the use of the EM approach for maximum likelihood estimations based on fuzzy data. This was followed by significant advances, including Huang's contribution in 2006 [3], where a new method was devised to compute membership functions and reliability for multiparameter lifetime distributions. Pak et al. [6], [7] extended this work by conducting several experiments for the development of inferential methods for lifespan distributions based on fuzzy data, setting a deep base for forthcoming research. For one a recent study of, Ashok et al. [9] studied the parameter estimation of gamma distribution when the available information as imprecise data, while Khoolejani et al. [8] offered an estimate of the mean parameter of the exponential distribution based on Type-II censoring. The use of fuzzy clustering methods for estimating parameters has a much longer history. Like, researchers such as Bezdek and Dunn (1975), Gath and Geva (1989), and Yang (1994) [10] already tested these methods for normal distributions, which just emphasizes the flexibility of fuzzy logical application in statistical modeling. More recently, Basharat et al. [12] showed how fuzzy data may estimate parameters of a linear combination of two exponentially distributed random variables.

Given the complex task of estimating under fuzzy data conditions, the study employs several sophisticated statistical techniques, chief among them being the Expectation-Maximization (EM) algorithm for Maximum Likelihood Estimation and the Bayesian approach, which is further conditioned by the TK approximation. Further Bayesian estimates are achieved using Gibbs sampling to derive posterior densities so that robust and more reliable parameter estimates are achieved. Performance of the proposed methodologies is evaluated through Monte Carlo simulation demonstrating convincingly that they are capable of handling data imprecision. These methodologies are practical in very applicable areas, such as medical research, industrial reliability testing, and the modeling of the environmental conditions of uncertainty.

2. MAXIMUM LIKELIHOOD ESTIMATION

Suppose two independent random samples are taken from two Weibull populations with a common shape parameter k and possibly different scale parameters σ_1 and σ_2 , respectively. Specifically, let $\mathbf{X} = (X_1, X_2, \dots, X_m)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ be m and n random samples taken from two Weibull populations Weibull(σ_1, k) and Weibull(σ_2, k), respectively. Here, Weibull(σ_1, k) denotes the Weibull population with the probability density function,

$$f(x; \sigma_1, k) = k\sigma_1 x^{k-1} e^{-\sigma_1 x^k}, \quad x > 0, \quad k > 0, \quad \sigma_1 > 0$$

likewise, Weibull(σ_2, k) denotes the population having the probability density function

$$f(y; \sigma_2, k) = k\sigma_2 y^{k-1} e^{-\sigma_2 y^k}, \quad y > 0, \quad k > 0, \quad \sigma_2 > 0$$

the joint probability density function of \mathbf{X} and \mathbf{Y} is obtained as follows,

$$f(x, y; \sigma_1, \sigma_2) = k^{m+n} \sigma_1^m \sigma_2^n \prod_{i=1}^m x_i^{k-1} \prod_{j=1}^n y_j^{k-1} e^{-(\sigma_1 \sum_{i=1}^m x_i^k + \sigma_2 \sum_{j=1}^n y_j^k)}$$

taking into account the scenario where the information that is currently accessible regarding x and y are fuzzy data rather than crisp data, that is, $\tilde{X} = \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m$, $\tilde{Y} = \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n$ denotes its observations, where \tilde{x}_i, \tilde{y}_j are the fuzzy observed number for the random variable x_i, y_j the likelihood function based on fuzzy observations \tilde{X} and \tilde{Y} is defined as, with the membership function $\mu_{\tilde{x}_i}(x)$ and $\mu_{\tilde{x}_i}(x)\mu_{\tilde{y}_j}(y)$ as a function of θ . The observed likelihood function L_0 and observed log likelihood function L_* are as follows,

$$L_0 = f(\tilde{x}, \tilde{y}; k, \sigma_1, \sigma_2) = k^{m+n} \sigma_1^m \sigma_2^n \prod_{i=1}^m \int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu_{\tilde{x}_i}(x) dx \prod_{j=1}^n \int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu_{\tilde{y}_j}(y) dy$$

$$\begin{aligned} L_* &= (m+n) \log k + m \log \sigma_1 + n \log \sigma_2 + \sum_{i=1}^m \log \int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu_{\tilde{x}_i}(x) dx \\ &+ \sum_{j=1}^n \log \int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu_{\tilde{y}_j}(y) dy \end{aligned}$$

by setting the partial derivatives of the log-likelihood with respect to the parameters σ_1, σ_2 and k to zero, two resulting equations can be obtained for solving the parameters.

$$\frac{\partial L_*}{\partial \sigma_1} = \frac{m}{\sigma_1} - \frac{\sum \int x^{2k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\sum \int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}$$

$$\frac{\partial L_*}{\partial \sigma_2} = \frac{n}{\sigma_2} - \frac{\sum \int y^{2k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\sum \int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}$$

$$\begin{aligned} \frac{\partial L_*}{\partial k} &= \frac{m+n}{k} + \frac{\sum_{i=1}^m \int (x^{k-1} - \sigma_1 x^{2k-1}) \log x e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\sum_{i=1}^m \int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx} + \\ &\frac{\sum_{j=1}^n \int (y^{k-1} - \sigma_2 y^{2k-1}) \log y e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\sum_{j=1}^n \int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy} \end{aligned}$$

to obtain the estimation using the ML-method, it is tough to figure out the above nonlinear equations. In this case, an iterative numerical method called an EM-algorithm and the Newton-Raphson method is used to obtain the MLEs.

2.1. EM Algorithm

By employing the whole data log-likelihood function, the EM method continually tackles the problem of maximizing the observed data log-likelihood function. The Expectation phase and the Maximization step are the two steps that make up each iteration of the procedure. Iteratively computing the expectation of the observed data log-likelihood function using the full data log-likelihood function is part of the E-step. The following is the expression for the whole data log-likelihood function:

$$\log L = (m + n) \log k + m \log \sigma_1 + n \log \sigma_2 + (k - 1) \sum \log x_i + (k - 1) \sum \log y_j - \sigma_1 \sum x_i^k - \sigma_2 \sum y_j^k$$

by differentiating the above equation with respect to k , σ_1 and σ_2 simplifying we get,

$$\frac{m + n}{k} = \sigma_1 \sum x_i^k \log x + \sigma_2 \sum y_j^k \log y - \sum \log x - \sum \log y$$

$$\frac{m}{\sigma_1} = \sum x_i^k \quad \text{and} \quad \frac{n}{\sigma_2} = \sum y_j^k$$

the iterative process of the EM algorithm is as follows,

- Give initial values (calculated by sample mean and variance) of $\rho = \rho^{(0)}$ and $\eta = \eta^{(0)}$, set $h = 0$.
- In the $(h + 1)$ th iterative process, the EM approach aims to maximize the log-likelihood function $L_*(\theta)$ of the observed data by repeatedly processing the log-likelihood function of the entire data.
- **E-step:** The E-step requires the computation of conditional expectations utilizing the expression.

$$E_{1i} = E(\log x_i | \tilde{x}_i) = \frac{\int x_i^{k-1} \log x_i e^{-\sigma_1 x_i^k} \mu(x) dx}{\int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu(x) dx}$$

$$E_{2i} = E(\log y_j | \tilde{y}_j) = \frac{\int y_j^{k-1} \log y_j e^{-\sigma_2 y_j^k} \mu(y) dy}{\int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu(y) dy}$$

$$E_{3i} = E(x_i^k \log x_i | \tilde{x}_i) = \frac{\int x_i^{2k-1} \log x_i e^{-\sigma_1 x_i^k} \mu(x) dx}{\int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu(x) dx}$$

$$E_{4i} = E(y_j^k \log y_j | \tilde{y}_j) = \frac{\int y_j^{2k-1} \log y_j e^{-\sigma_2 y_j^k} \mu(y) dy}{\int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu(y) dy}$$

$$E_{5i} = E(x_i^k | \tilde{x}_i) = \frac{\int x_i^k e^{-\sigma_1 x_i^k} \mu(x) dx}{\int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu(x) dx}$$

$$E_{6i} = E(y_j^k | \tilde{y}_j) = \frac{\int y_j^k e^{-\sigma_2 y_j^k} \mu(y) dy}{\int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu(y) dy}$$

now, the non-linear equations will be replaced by,

$$\frac{m + n}{k} = \sigma_1 \sum E_{3i} + \sigma_2 \sum E_{4i} - \sum (E_{1i} - E_{2i})$$

$$\frac{m}{\sigma_1} = \sum E_{5i} \quad \text{and} \quad \frac{n}{\sigma_2} = \sum E_{6i}$$

- **M-step:**In the M-step after solving the above Non-linear eq's,

$$k^{(h+1)} = \frac{m+n}{\sigma_1^{(h+1)} \sum E_{3i} + \sigma_2^{(h+1)} \sum E_{4i} - \sum (E_{1i} - E_{2i})}$$

$$\sigma_1^{(h+1)} = \frac{m}{\sum E_{5i}} \quad \text{and} \quad \sigma_2^{(h+1)} = \frac{n}{\sum E_{6i}}$$

- **Convergence:** If convergence is achieved, the current values of $\sigma_1^{(h+1)}$, $\sigma_2^{(h+1)}$ and $k^{(h+1)}$ are the maximum likelihood estimators (MLEs) of σ_1 , σ_2 and k . Otherwise, repeat steps 2 and 3.

Once the parameters' MLEs have been established, the 95% asymptotic confidence intervals for the parameters may be computed as follows.

$$\hat{\theta} \pm Z_{\alpha/2} \sqrt{I(\theta)^{-1}}$$

the inverse of the fisher information matrix will be,

$$I(\sigma_1, \sigma_2, k)^{-1} = \begin{bmatrix} \frac{T_{22}T_{33} - T_{23}^2}{D} & \frac{-T_{31}T_{23}}{D} & \frac{T_{22}T_{31}}{D} \\ \frac{-T_{31}T_{23}}{D} & \frac{T_{11}T_{33} - T_{31}^2}{D} & \frac{-T_{11}T_{32}}{D} \\ \frac{T_{22}T_{31}}{D} & \frac{-T_{11}T_{32}}{D} & \frac{T_{11}T_{22}}{D} \end{bmatrix}$$

here,

$$D = T_{11}(T_{22}T_{33} - T_{32}^2) + T_{13}^2 T_{22}$$

now, we construct 95% asymptotic confidence intervals using the information matrix for the model parameters σ_1 , σ_2 and k are as follows:

$$\hat{\sigma}_{1ML} \pm 1.96 \sqrt{\frac{T_{22}T_{33} - T_{23}^2}{D}},$$

$$\hat{\sigma}_{2ML} \pm 1.96 \sqrt{\frac{T_{11}T_{33} - T_{31}^2}{D}},$$

$$\hat{k}_{2ML} \pm 1.96 \sqrt{\frac{T_{11}T_{22}}{D}}.$$

The Appendix contains all of the notations and calculations. For varying sample sizes, all of these findings are calculated quantitatively and displayed in Section 5.

3. BAYES ESTIMATION

Being an even powerful and viable alternative to the traditional statistical approaches, Bayesian perspective has received considerable attention for decades on statistical inference. In this section, the Bayesian estimate of the model-associated parameters is discussed. When the prior distribution of parameters is known beforehand, the Bayes estimator plays a crucial role. Assume that $\rho_1(k)$ is the shape parameter k prior density function. For the scale parameters σ_1 and σ_2 , let $\rho_2(\sigma_1)$ and $\rho_3(\sigma_2)$ be the gamma prior densities, respectively.

$$\rho_1(k) = \frac{b_1^{a_1}}{\Gamma(a_1)} k^{a_1-1} \exp(-kb_1), \quad k > 0$$

$$\rho_2(\sigma_1) = \frac{b_2^{a_2}}{\Gamma(a_2)} \sigma_1^{a_2-1} \exp(-\sigma_1 b_2), \quad \sigma_1 > 0$$

$$\rho_3(\sigma_2) = \frac{b_3^{a_3}}{\Gamma(a_3)} \sigma_2^{a_3-1} \exp(-\sigma_2 b_3), \quad \sigma_2 > 0$$

the joint probability density function of (k, σ_1, σ_2) based on the above priors is,

$$f(k, \sigma_1, \sigma_2) = (m + n) \log k + m \log \sigma_1 + n \log \sigma_2 + \sum_{i=1}^m \log \int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu_{x_i}(x) dx + \sum_{j=1}^n \log \int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu_{y_j}(y) dy \rho_1(k) \cdot \rho_2(\sigma_1) \cdot \rho_3(\sigma_2)$$

the joint posterior density function of $\theta = (k, \sigma_1, \sigma_2)$ given the data (X, Y) is,

$$f(\sigma_1, \sigma_2, k | x, y) = \frac{f(\sigma_1, \sigma_2, k, x, y)}{\int_0^\infty \int_0^\infty \int_0^\infty f(\sigma_1, \sigma_2, k, x, y) d\sigma_1 d\sigma_2 dk}$$

the Bayes estimator of $g(\sigma_1, \sigma_2, k)$ with respect to the squared error loss function is the mean of the posterior distribution and is given by the following.

$$\hat{g}_B(\sigma_1, \sigma_2, k) = \mathbb{E}[g(\sigma_1, \sigma_2, k) | x, y]$$

$$= \frac{\int_0^\infty \int_0^\infty \int_0^\infty g(\sigma_1, \sigma_2, k) f(\sigma_1, \sigma_2, k, x, y) d\sigma_1 d\sigma_2 dk}{\int_0^\infty \int_0^\infty \int_0^\infty f(\sigma_1, \sigma_2, k, x, y) d\sigma_1 d\sigma_2 dk}$$

analytical methods for finding the ratio of integrals in above equation are difficult to use. However, one might obtain a numerical quantity with certain approximations. In this paper, we employ Tierney and Kadane's TK approximation to estimate the Bayes estimators under squared error loss function.

3.1. TK approximation

In this part, we use Tierney and Kadane's [13] approximation approach to find the Bayes estimators for the common Shape parameter k and the other scale parameters σ_1, σ_2 . Tierney and Kadane's approximation simplifies the computation of the posterior expectation $E(u(\theta)|\text{data})$. The approximation is given by,

$$E(u(\theta) | \tilde{X}, \tilde{Y}) = \left[\frac{\det \Sigma^*}{\det \Sigma} \right]^{\frac{1}{2}} \exp\{k(L_*(\theta_*) - L_0(\theta_0))\},$$

where:

- $L_0(\theta_0)$: The log-likelihood function maximized over the parameter space θ ,
- $L_*(\theta_*) = L_0(\theta_0) + \frac{1}{k} \log u(\theta)$,
- Σ^* : Minus the inverse Hessian of $L_*(\theta)$ at the mode θ^* ,
- Σ : Minus the inverse Hessian of $L_0(\theta_0)$ at the mode θ^* .

$$L_0(\theta_0) = \frac{1}{k} [L(\theta) + \log v(\theta)] = \frac{1}{k} [L(k, \sigma_1, \sigma_2 | \tilde{x}) + \log v(\theta)]$$

$$L_*(\theta) = L_0(\theta_0) + \frac{1}{k} \log v(\theta)$$

In our case the approximation will be,

$$E(u(\theta) | \tilde{X}, \tilde{Y}) = \left[\frac{\det \Sigma^*}{\det \Sigma} \right]^{\frac{1}{2}} \exp\{L_*(\hat{k}_*, \hat{\sigma}_{1*}, \hat{\sigma}_{2*}) - L_0(\hat{k}_0, \hat{\sigma}_{10}, \hat{\sigma}_{20})\}.$$

The Hessians are calculated as the negative second derivatives of $L_*(k, \sigma_1, \sigma_2)$ and $L_0(k, \sigma_1, \sigma_2)$. Let the parameter vector be $\theta = (k, \sigma_1, \sigma_2)$.

- For $L_*(\hat{k}_*, \hat{\sigma}_{1*}, \hat{\sigma}_{2*})$:

$$\Sigma^* = - \begin{bmatrix} \frac{\partial^2 L_*}{\partial k^2} & \frac{\partial^2 L_*}{\partial k \partial \sigma_1} & \frac{\partial^2 L_*}{\partial k \partial \sigma_2} \\ \frac{\partial^2 L_*}{\partial \sigma_1 \partial k} & \frac{\partial^2 L_*}{\partial \sigma_1^2} & \frac{\partial^2 L_*}{\partial \sigma_1 \partial \sigma_2} \\ \frac{\partial^2 L_*}{\partial \sigma_2 \partial k} & \frac{\partial^2 L_*}{\partial \sigma_2 \partial \sigma_1} & \frac{\partial^2 L_*}{\partial \sigma_2^2} \end{bmatrix}^{-1} = (\Sigma_{ij}^*)$$

- For $L_0(\hat{k}_0, \hat{\sigma}_{10}, \hat{\sigma}_{20})$:

$$\Sigma = - \begin{bmatrix} \frac{\partial^2 L_0}{\partial k^2} & \frac{\partial^2 L_0}{\partial k \partial \sigma_1} & \frac{\partial^2 L_0}{\partial k \partial \sigma_2} \\ \frac{\partial^2 L_0}{\partial \sigma_1 \partial k} & \frac{\partial^2 L_0}{\partial \sigma_1^2} & \frac{\partial^2 L_0}{\partial \sigma_1 \partial \sigma_2} \\ \frac{\partial^2 L_0}{\partial \sigma_2 \partial k} & \frac{\partial^2 L_0}{\partial \sigma_2 \partial \sigma_1} & \frac{\partial^2 L_0}{\partial \sigma_2^2} \end{bmatrix}^{-1} = (\Sigma_{ij}).$$

It should be noted that the elements of L_0 and L_* are being evaluated at $(\hat{\sigma}_{10}, \hat{\sigma}_{20}, \hat{k}_0)$ and $(\hat{\sigma}_{1*}, \hat{\sigma}_{2*}, \hat{k}_*)$, respectively. Now it is simple to evaluate for the Bayes estimate of k . The same procedure can be used for obtaining the Bayes estimates for $u(\sigma_1, \sigma_2, k) = \sigma_1$ and $u(\sigma_1, \sigma_2, k) = \sigma_2$.

3.2. Gibbs sampling

Gibbs sampling is one of the most popular MCMC methods in Bayesian statistics, simplifying sampling from complex distributions, especially in high dimensions or with intractable forms. It constructs a Markov chain that converges to the desired posterior by iteratively sampling from conditional distributions.

This paper applies Gibbs sampling to estimate the parameters of the Weibull distribution, which is widely used in survival and reliability research. Gibbs sampling handles fuzzy data effectively by iteratively refining parameter estimates and demonstrates its flexibility in managing uncertainty within probabilistic models such as the Weibull distribution. The Gibbs sampling procedure can be outlined as follows:

- Assume that $X \sim \text{Weibull}(k, \sigma_1)$ and $Y \sim \text{Weibull}(k, \sigma_2)$. The joint density function is given by:

$$L_* = (m+n) \log k + m \log \sigma_1 + n \log \sigma_2 + \sum_{i=1}^m \log \int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu_{\tilde{x}_i}(x) dx + \sum_{j=1}^n \log \int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy$$

- Assume the following prior distributions for the parameters:

$$\begin{aligned} k &\sim \text{Gamma}(b_1, a_1) \\ \sigma_1 &\sim \text{Gamma}(b_2, a_2) \\ \sigma_2 &\sim \text{Gamma}(b_3, a_3) \end{aligned}$$

- Derive the full conditional posterior distributions for k, σ_1 , and σ_2 :

$$\begin{aligned} \pi(k | \sigma_1, \sigma_2, \tilde{X}, \tilde{Y}) &\propto L_*(k, \sigma_1, \sigma_2 | \tilde{X}, \tilde{Y}) \pi(k) \\ \pi(\sigma_1 | k, \tilde{X}) &\propto L_*(k, \sigma_1 | \tilde{X}) \pi(\sigma_1) \\ \pi(\sigma_2 | k, \tilde{Y}) &\propto L_*(k, \sigma_2 | \tilde{Y}) \pi(\sigma_2) \end{aligned}$$

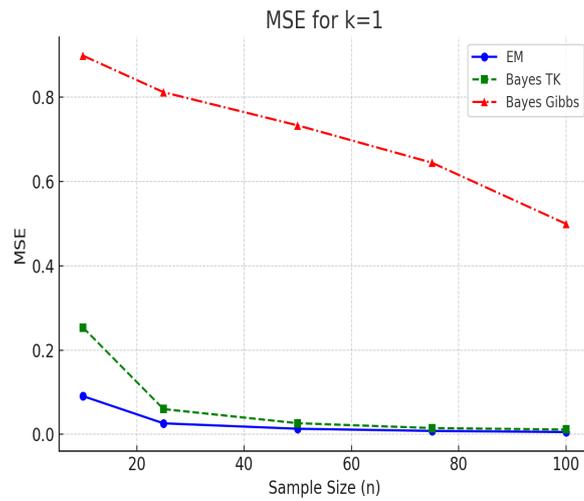
- Estimate the parameters iteratively using the following steps:
 1. Start with initial values for the parameters $k^{(0)}$, $\sigma_1^{(0)}$, and $\sigma_2^{(0)}$.
 2. For each iteration $t = 1, 2, \dots, T$.
 3. Sample $k^{(t+1)}$ from its conditional posterior distribution $\pi(k | \sigma_1^{(t)}, \sigma_2^{(t)}, \tilde{x}, \tilde{y})$.
 4. Sample $\sigma_1^{(t+1)}$ from its conditional posterior distribution $\pi(\sigma_1 | k^{(t+1)}, \tilde{x})$.
 5. Sample $\sigma_2^{(t+1)}$ from its conditional posterior distribution $\pi(\sigma_2 | k^{(t+1)}, \tilde{y})$.

The samples obtained can be used for posterior inference once the Gibbs sampler runs for sufficiently many iterations to guarantee convergence. This implies estimating posterior means, credible intervals, and other relevant statistics with respect to the parameters under consideration.

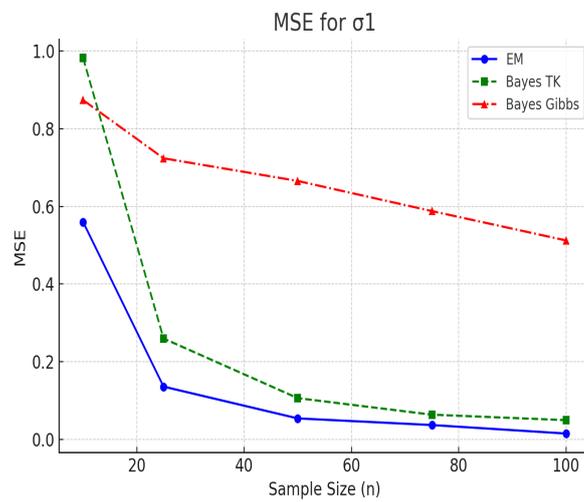
4. SIMULATION STUDY

This section presents experimental results regarding the evaluation of the ranged performance of different estimation methods on the basis of sample sizes. The estimation is aimed at the common shape parameter k and possibly distinct scale parameters σ_1 and σ_2 . These estimations were conducted using maximum likelihood (ML), implementing it via EM algorithm, and Bayesian estimation approaches with TK approximation and Gibbs sampling method. All the computations were carried out under R software version 4.4.1 along with the curve plots, supporting the analysis. Ten thousand random samples are drawn from two Weibull populations for each combination of parameter values and sample sizes for comparisons across the estimators. These samples form the fuzzy values using the triangular membership function. Three sets of parameter values are analyzed for a different sample size. A summary of the results is given as in Table 1 compares estimates and mean square errors (MSE) for various sample sizes using the EM method, Bayes (TK), and Bayes (Gibbs) methods, assuming parameter values $k = 1$, $\sigma_1 = 1.5$, and $\sigma_2 = 2$. Table 2 presents similar comparisons for parameter values $k = 1$, $\sigma_1 = 2$, and $\sigma_2 = 2.5$. Table 3 evaluates results for parameter values $k = 2$, $\sigma_1 = 3$, and $\sigma_2 = 4.5$. Moreover, Table 4 and Table 5 give asymptotic confidence intervals and the highest posterior density (HPD) intervals, arising from the Fisher information matrix and using Gibbs sampling, respectively. The results are summarized in several tables as follows:

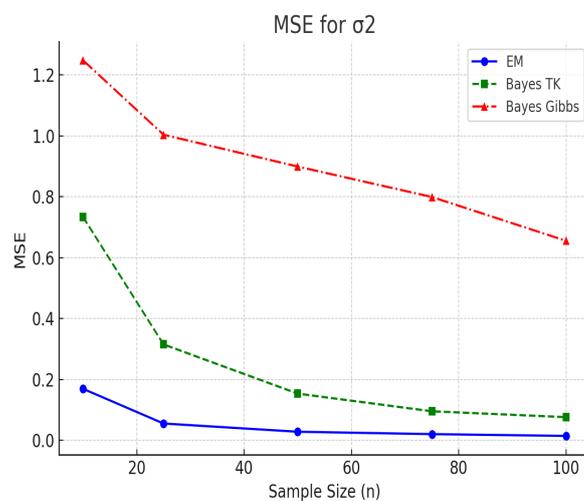
- The analytical results showed that increasing the sample size gave all estimators more accuracy as it led to considerable reductions in bias and mean squared error (MSE).
- As the sample numbers increased, the widths of the intervals decreased, indicating a higher degree of precisions, with all parameter estimates coming under their respective confidence intervals.
- The sample size, along with the selected confidence level, significantly shaped the confidence intervals. All other things being equal, a larger sample infinitesimally shrinks the confidence intervals at any fixed confidence level.
- Estimation accuracy was demonstrated by narrowing estimations of the confidence interval with increased sample size.
- The figures 1(a), 1(b) and 1(c) in 1 clearly indicates that, EM algorithm is the better one when compared to the remaining estimation methods.
- The Bayes (Gibbs) method exhibited relatively higher MSE values, particularly for small sample sizes, indicating that it requires larger datasets to achieve accuracy comparable to the EM algorithm.
- The confidence intervals derived from Gibbs sampling were generally wider than those obtained through asymptotic methods, suggesting greater uncertainty in Bayesian estimates, especially when dealing with smaller sample sizes.



(a)



(b)



(c)

Figure 1: MSE values of different parameters at various sample sizes

Table 1: Comparison of estimates and mean square errors (MSE) for various sample sizes using EM, Bayes(TK), and Bayes(Gibbs) methods for the parameters $k = 1$, $\sigma_1 = 1.5$, and $\sigma_2 = 2$.

(m,n)	Parameter	EM		Bayes(TK)		Bayes(Gibbs)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE
(10,10)	$k = 1$	1.0887	0.0908	1.1988	0.2534	1.3187	0.8988
	$\sigma_1 = 2$	2.2599	0.5599	2.2681	0.9824	2.4235	0.8739
	$\sigma_2 = 2.5$	2.4054	0.1689	2.2734	0.7331	3.0273	1.2482
(25,25)	$k = 1$	1.0359	0.0263	1.0741	0.0601	1.2944	0.8122
	$\sigma_1 = 2$	2.1085	0.1357	2.1485	0.2596	2.3804	0.7239
	$\sigma_2 = 2.5$	2.5110	0.0549	2.4208	0.3148	2.8977	1.0037
(50,50)	$k = 1$	1.0227	0.0134	1.0371	0.0266	1.2642	0.7334
	$\sigma_1 = 2$	2.0603	0.0538	2.0767	0.1060	2.3491	0.6655
	$\sigma_2 = 2.5$	2.5022	0.0279	2.4667	0.1532	2.7649	0.8992
(75,75)	$k = 1$	1.0123	0.0083	1.0223	0.0151	1.2183	0.6449
	$\sigma_1 = 2$	2.0422	0.0367	2.0478	0.0634	2.3197	0.5879
	$\sigma_2 = 2.5$	2.4994	0.0208	2.4686	0.0951	2.6772	0.7893
(100,100)	$k = 1$	1.0091	0.0057	1.0197	0.0115	1.1877	0.4998
	$\sigma_1 = 2$	2.0335	0.0256	2.0469	0.0493	2.2893	0.5123
	$\sigma_2 = 2.5$	2.4951	0.0142	2.4615	0.0759	2.4893	0.6553

Table 2: Comparison of estimates and mean square errors (MSE) for various sample sizes using EM, Bayes(TK), and Bayes(Gibbs) methods for the parameters $k = 1$, $\sigma_1 = 1.5$, and $\sigma_2 = 2$.

(m,n)	Parameter	EM		Bayes(TK)		Bayes(Gibbs)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE
(10,10)	$k = 1$	1.1237	0.1362	1.2134	0.2702	1.3743	0.2687
	$\sigma_1 = 1.5$	1.6968	0.2878	1.8441	0.6373	1.7234	0.4672
	$\sigma_2 = 2$	1.9992	0.1509	1.7990	0.4623	2.3216	0.6653
(25,25)	$k = 1$	1.0497	0.0297	1.0733	0.0641	1.3374	0.1763
	$\sigma_1 = 1.5$	1.5747	0.0802	1.6142	0.1338	1.6576	0.1563
	$\sigma_2 = 2$	1.9821	0.0555	1.9153	0.1954	2.2486	0.3573
(50,50)	$k = 1$	1.0226	0.0123	1.0352	0.0241	1.2498	0.1287
	$\sigma_1 = 1.5$	1.5369	0.0317	1.5599	0.0548	1.6143	0.1287
	$\sigma_2 = 2$	1.8992	0.0278	1.9624	0.0953	2.1459	0.1983
(75,75)	$k = 1$	1.0116	0.0079	1.0196	0.0152	1.2249	0.1092
	$\sigma_1 = 1.5$	1.5279	0.0212	1.5425	0.0373	1.5871	0.0928
	$\sigma_2 = 2$	1.9919	0.0192	1.9731	0.0645	2.1274	0.1761
(100,100)	$k = 1$	1.0099	0.0063	1.0144	0.0113	1.2005	0.1005
	$\sigma_1 = 1.5$	1.5192	0.0148	1.5239	0.0285	1.5628	0.0783
	$\sigma_2 = 2$	1.9993	0.0139	1.9858	0.0484	2.1108	0.1588

4.1. Example

This case study illustrates the production of light-emitting diodes (LEDs) with a focus on luminous intensities of LED sources in the context of the proposed methodologies. The $m=15$ and $n=15$

Table 3: Comparison of estimates and mean square errors (MSE) for various sample sizes using EM, Bayes(TK), and Bayes(Gibbs) methods for the parameters $k = 2$, $\sigma_1 = 3$, and $\sigma_2 = 4.5$.

(m,n)	Parameter	EM		Bayes(TK)		Bayes(Gibbs)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE
(10,10)	$k = 2$	2.1963	0.4355	2.4495	0.9702	2.7756	2.3379
	$\sigma_1 = 3$	3.4624	1.3698	3.6768	2.2658	3.8809	1.4407
	$\sigma_2 = 4.5$	4.5163	0.1874	4.0461	2.1738	5.1004	1.5981
(25,25)	$k = 2$	2.0751	0.1206	2.1566	0.2465	2.6672	1.9906
	$\sigma_1 = 3$	3.1559	0.2966	3.2257	0.5706	3.7603	1.4008
	$\sigma_2 = 4.5$	4.5001	0.0716	4.3524	0.9024	4.9982	1.5398
(50,50)	$k = 2$	2.04482	0.0545	2.0897	0.1015	2.5904	1.4877
	$\sigma_1 = 3$	3.0534	0.1189	3.1428	0.2382	3.6208	1.3779
	$\sigma_2 = 4.5$	4.4965	0.0344	4.3888	0.4258	4.8793	1.4095
(75,75)	$k = 2$	2.0192	0.0321	2.0593	0.0632	2.4795	1.0908
	$\sigma_1 = 3$	3.0519	0.0786	3.0922	0.1527	3.5003	1.3089
	$\sigma_2 = 4.5$	4.4952	0.0221	4.4219	0.2992	4.7566	1.1688
(100,100)	$k = 2$	2.0215	0.0261	2.0364	0.0443	2.3528	0.9038
	$\sigma_1 = 3$	3.0354	0.0584	3.0443	0.1019	3.3908	0.8997
	$\sigma_2 = 4.5$	4.4941	0.0157	4.4587	0.2163	4.6931	0.6733

Table 4: 95% Asymptotic and HPD intervals using Fisher information matrix and Gibbs sampling for the parameters $(k, \sigma_1, \sigma_2) = (1.5, 2, 2.5)$ at various sample sizes

(m,n)	Observed	Asymptotic	Gibbs
(10,10)	$k=1.5$	[1.1232, 3.4567]	[0.3456, 2.1234]
	$\sigma_1=2$	[0.9123, 4.1234]	[0.2678, 1.8765]
	$\sigma_2=2.5$	[1.2345, 5.2345]	[0.4123, 3.2345]
(25,25)	$k=1.5$	[1.4123, 3.2345]	[0.3890, 2.2345]
	$\sigma_1=2$	[1.3012, 3.8765]	[0.4567, 2.6789]
	$\sigma_2=2.5$	[1.6789, 4.5678]	[0.6012, 3.7890]
(50,50)	$k=1.5$	[1.5678, 2.9012]	[0.3567, 2.1123]
	$\sigma_1=2$	[1.4890, 3.3456]	[0.4234, 2.7890]
	$\sigma_2=2.5$	[1.8901, 4.1234]	[0.5678, 3.8901]
(75,75)	$k=1.5$	[1.6789, 2.7890]	[0.3678, 2.0345]
	$\sigma_1=2$	[1.6234, 3.2345]	[1.1345, 3.9234]
	$\sigma_2=2.5$	[2.0012, 3.9012]	[0.7890, 4.8123]
(100,100)	$k=1.5$	[1.7890, 2.6789]	[0.4012, 2.0012]
	$\sigma_1=2$	[1.7456, 3.1123]	[0.9123, 3.6789]
	$\sigma_2=2.5$	[2.1234, 3.7890]	[0.9234, 4.8901]

sample sizes used for the analysis represent two Weibull populations. These intensities were point estimates with lower and upper bounds as fuzzy data, due to the intrinsic heterogeneity in the luminous intensity data for each LEDs, as explained in [7]. On the other hand, the Kolmogorov-Smirnov (K-S) test was performed after the data were processed using the alpha-cut technique to confirm that the Weibull distribution was appropriate for the observed data. The two groups reflect typical sources of raw materials in manufacturing but different scenarios of use, and they are referred to as low-usage and high-usage scenarios. In the LED production case study, the

Table 5: 95% Asymptotic and HPD intervals using Fisher information matrix and Gibbs sampling for the parameters $(k, \sigma_1, \sigma_2) = (2, 3, 4.5)$ at various sample sizes

(m,n)	Observed	Asymptotic	Gibbs
(10,10)	$k=2$	[1.7892, 5.1234]	[0.5678, 3.2345]
	$\sigma_1=3$	[1.4567, 6.3456]	[0.7890, 4.5678]
	$\sigma_2=4.5$	[2.2345, 8.1234]	[0.8901, 5.7890]
(25,25)	$k=2$	[2.1234, 4.8901]	[0.6789, 3.5678]
	$\sigma_1=3$	[2.0012, 5.5678]	[0.8456, 4.8901]
	$\sigma_2=4.5$	[2.7890, 7.3456]	[1.0345, 6.1234]
(50,50)	$k=2$	[2.5678, 4.3456]	[0.7123, 3.2345]
	$\sigma_1=3$	[2.4345, 5.1123]	[0.9012, 4.5678]
	$\sigma_2=4.5$	[3.1234, 6.7890]	[1.1234, 5.8901]
(75,75)	$k=2$	[2.7890, 4.1234]	[0.7345, 3.0012]
	$\sigma_1=3$	[2.6789, 4.9012]	[1.2345, 4.3456]
	$\sigma_2=4.5$	[3.3456, 6.2345]	[1.4567, 5.5678]
(100,100)	$k=2$	[3.0012, 4.0012]	[0.8901, 2.8901]
	$\sigma_1=3$	[2.9345, 4.6789]	[1.4567, 3.7890]
	$\sigma_2=4.5$	[3.5678, 5.8901]	[1.7890, 5.1234]

Table 6: Comparison of Biases and Mean Square Errors (MSE) of Several Estimators for Real Data.

(m,n)	Parameter	EM		Bayes(TK)		Bayes(Gibbs)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE
(15,15)	k	2.0580	0.0441	1.8612	0.2241	1.9473	0.3158
	σ_1	1.1148	0.0195	0.8441	0.2193	0.7987	0.2371
	σ_2	1.5256	0.0024	1.4162	0.5798	1.3857	0.7496

most effective procedure for estimating unknown parameters (shape and scale parameters) for the two Weibull populations is the EM method. Although Bayesian methods are useful, they are not as precise, especially for small sample sizes like $m = 15$ and $n = 15$. The conclusions show how effectively different estimating methods deal with identification of the unknown characteristics linked with these populations.

5. CONCLUSION

The current study has concluded by showing the performance of estimation methods for multi-population Weibull models under imprecise data. Monte Carlo simulations were employed to explain the performance of Bayesian theory with TK approximation and expectations maximization for maximum likelihood estimation. As a result, the comparison of the mean squared error (MSE) values produces evidence that the EM algorithm is superiorly accurate and robust relative to alternatives. Such findings demonstrate the utility of the expectation-maximization (EM) algorithm when there are applications with imprecise or fuzzy data. The results showed that for all the three parameters k , σ_1 , and σ_2 , accuracy and precision, with regard to the EM method, beat TK and Gibbs sampling Bayesian methods. The EM method produces the most reliable estimates, with minimum MSE, therefore enhancing the reliability of the statistical analysis on multi-population models that have different applications in the real world.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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6. APPENDIX

The observed log likelihood function is,

$$L_* = (m + n) \log k + m \log \sigma_1 + n \log \sigma_2 + \sum_{i=1}^m \log \int x_i^{k-1} e^{-\sigma_1 x_i^k} \mu_{\tilde{x}_i}(x) dx + \sum_{j=1}^n \log \int y_j^{k-1} e^{-\sigma_2 y_j^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy$$

$$T_{11} = \frac{\partial^2 L_*}{\partial \sigma_1^2} = -\frac{m}{\sigma_1^2} + \sum \frac{\int x^{3k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx} - \sum \left(\frac{\int x^{2k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx} \right)^2$$

$$T_{12} = T_{21} = \frac{\partial^2 L_*}{\partial \sigma_1 \partial \sigma_2} = 0$$

$$T_{13} = T_{31} = \frac{\partial^2 L_*}{\partial \sigma_1 \partial k} = -\frac{\int (1 + x^k \sigma_1 + x^{2k-1}) e^{x^k} \log x \mu_{\tilde{x}_i}(x) dx}{\int x^{k-1} e^{\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx} + \frac{\int (x^{k-1} - \sigma_1 x^{2k-1}) \log x e^{\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx \cdot \int x^{2k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\left(\int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx \right)^2}$$

$$T_{22} = \frac{\partial^2 L_*}{\partial \sigma_2^2} = -\frac{n}{\sigma_2^2} + \sum \frac{\int y^{3k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy} - \sum \left(\frac{\int y^{2k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy} \right)^2$$

$$T_{23} = T_{32} = \frac{\partial^2 L_*}{\partial \sigma_2 \partial k} = -\frac{\int (1 + y^k \sigma_2 + y^{2k-1}) e^{y^k} \log y \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\int y^{k-1} e^{\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy} + \frac{\int (y^{k-1} - \sigma_2 y^{2k-1}) \log y e^{\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy \int y^{2k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\left(\int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy \right)^2}$$

$$T_{33} = \frac{\partial^2 L_*}{\partial k^2} = -\frac{m+n}{k^2} + \sum \frac{\int (x^{k-1} - x^{2k-1}) (1 - \sigma_1 x^k)^2 e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx} - \sum \left(\frac{\int (x^{k-1} - \sigma_1 x^{2k-1}) \log x e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx}{\int x^{k-1} e^{-\sigma_1 x^k} \mu_{\tilde{x}_i}(x) dx} \right)^2 + \sum \frac{\int (y^{k-1} - y^{2k-1}) (1 - \sigma_2 y^k)^2 e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy} - \sum \left(\frac{\int (y^{k-1} - \sigma_2 y^{2k-1}) \log y e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy}{\int y^{k-1} e^{-\sigma_2 y^k} \mu_{\tilde{x}_i}(x) \mu_{\tilde{y}_j}(y) dy} \right)^2.$$