# A NEW GENERALIZATION OF PARETO DISTRIBUTION:PROPERTIES AND APPLICATIONS

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**Abstract**

*This manuscript introduces a new extension of the Pareto distribution using the SMP transformation technique, known as SMP Pareto distribution (SMPP). The SMP technique is named after (Shamshad, Murtiza, Parvaiz) who pioneered this approach to enhance the flexibility and applicability of statistical models. This new distribution provides a better fit for data than many existing models. The shapes of the density function exhibit great flexibility. It can support different hazard shapes, such as increasing, decreasing and constant shapes. Various statistical properties of the proposed distribution such as moments, quantile function, entropy and order statistics were presented. The maximum likelihood technique is used to estimate the model parameters. An extensive simulation study is carried out to illustrate the behavior of MLEs. In addition, two real-world datasets are analyzed to showcase the applicability of the proposed approach. The results indicate that the SMP Pareto distribution (SMPP) is more flexible and offers a superior fit for describing data compared to several existing forms of the Pareto distribution.*

**Keywords:** SMP technique, Pareto distribution, Order statistics, Maximum likelihhood estimation.

## 1. Introduction

The field of probability distributions is fundamental in the area of statistics. A probability distribution is a mathematical function that tells us the likelihood of different outcomes in an experiment. Many distributions have been studied extensively in the literature for analyzing real-life data. These models play a crucial role across various fields, including physics, medicine, business management, engineering, and food science. In addressing complex real-world problems where data doesn't adhere to classical or standard models, developing new generalized probability models that offer greater flexibility for accurately modeling such data becomes essential. Over the years, statisticians have focused extensively on improving classical distributions by introducing additional parameters through generators or by combining existing distributions, making them more adaptable for analyzing empirical data. A methodology for introducing a new parameter to existing distributions was developed by [16], [5] introduced the T-X family of continuous distributions, where the PDF of the Beta distribution was replaced with the PDF of any continuous random variable. Alpha power transformation technique was proposed by [15]. The Rayleigh distribution was extended using the sine-G family of distributions by [17].

More recently,[20] proposed a novel transformation for generating probability distributions by introducing an extra parameter to continuous distributions, known as the SMP transformation technique. The idea was introduced to enhance the flexibility of the baseline distribution. The inverted exponential distribution was extended using the SMP transformation technique by [2], while the Kumaraswamy distribution was extended by [12] using the same transformation.

The Cumulative Distribution Function (CDF) of the SMP method is given as:

$$G_{SMP}(x) = \begin{cases} \frac{e^{\log(\alpha)\overline{F}(x)} - \alpha}{1-\alpha}, & \alpha \neq 1, \alpha > 0 \\ F(x), & \alpha = 1 \end{cases} \tag{1}$$

where $\overline{F}(x) = 1 - F(x)$

and the corresponding Probability Density Function (PDF) is obtained as follows:

$$g_{SMP}(x) = \begin{cases} \frac{e^{\log(\alpha)\overline{F}(x)} \log(\alpha) f(x)}{\alpha-1}, & \alpha \neq 1, \alpha > 0 \\ f(x) \quad , & \alpha = 1 \end{cases} \tag{2}$$

The Pareto distribution is named after the Italian economist Vilferdo Pareto (1848-1923) a well-known probability model for modeling and forecasting many socioeconomic elements. The Pareto distribution is often used to fit heavy-tailed phenomena. It has many applications in actuarial science, survival analysis, economics, reliability analysis and engineering.

The Probability Density Fuction (PDF) given by [11] is defined as

$$f(x) = \frac{\beta}{x^{\beta+1}}; \quad x \geq 1, \beta > 0, \tag{3}$$

and corresponding Cumulative Distribution Function (CDF) as follows:

$$F(x) = 1 - \frac{1}{x^{\beta}}; \quad x \geq 1, \beta > 0. \tag{4}$$

Where $\beta$ is a scale parameter and is greater than 0.

In the literature, various extensions of Pareto distributions have been proposed. Alpha power pareto distribution was studied by [11]. Cubic transmuted pareto distribution was obtained by [19]. The beta pareto distribution was proposed by [4]. On a new generalization of pareto distribution was studied by [3]. The Alpha power transformation generalized pareto distribution were discussed by [6]. Sine exponential pareto distribution was studied by [1]. Topp leone exponentiated pareto distribution was obtained by [7].

In this study, we create and investigate a novel extension of Pareto distribution using the SMP transformation technique. The proposed distribution is named as SMP Pareto (SMPP) distribution. The primary motivation of developing SMPP distribution are described via:

- The proposed model is highly efficient and adaptable for incorporating a new parameter to generalize the existing distribution.
- The proposed model offers better fits compared to other competitive models.
- The proposed model can be used to model diverse nature of datasets.
- The proposed model offers more flexible shapes in terms of hazard and density plots.

The rest of this paper is organised as follows: Section 2 defines the PDF and CDF of the proposed model, i.e., SMPP Distribution. Section 3 discusses the reliability measures of the model. In Section 4, we derive various mathematical properties. Section 5 investigates order statistics. The maximum likelihood estimation of the model parameters is addressed in Section 6. Section 7 presents the simulation study, while Section 8 discusses the applicability of the model. Finally, concluding remarks are provided in Section 9.

## 2. SMPP Distribution

A random variable X is said to follow SMPP distribution with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$, if its CDF is given as

$$G_{SMPP}(x;\alpha,\beta) = \begin{cases} \frac{e^{\frac{\log\alpha}{x^\beta}}-\alpha}{1-\alpha}, & \alpha \neq 1, \alpha > 0 \\ 1-\frac{1}{x^\beta} & , \quad \alpha = 1 \end{cases} \tag{5}$$

The corresponding PDF of SMPP distribution is given as

$$g_{SMPP}(x;\alpha,\beta) = \begin{cases} \frac{\log(\alpha)}{\alpha-1} \cdot \frac{\beta}{x^{\beta+1}} \cdot e^{\frac{\log(\alpha)}{x^\beta}}, & \alpha \neq 1, \alpha > 0 \\ \frac{\beta}{x^{\beta+1}} & , \quad \alpha = 1 \end{cases} \tag{6}$$

Figure (1) displays the PDF plot of the SMPP distribution for different parameter values. It demonstrates that the proposed distribution is unimodal, decreasing, and right-skewed in nature.

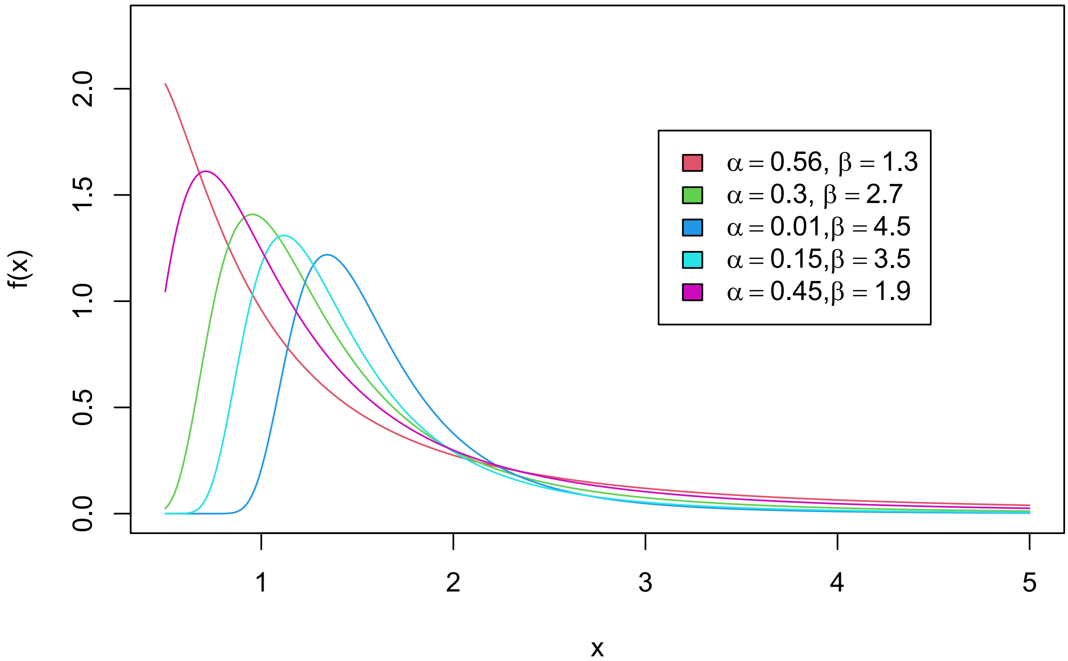


**Figure 1:** *PDF plot for SMPP distribution*

## 3. Reliability Analysis Of SMPP Distribution

In this section, we derive the expressions for the reliability function, hazard rate, reverse hazard function and Mills ratio for the SMPP distribution.

### 3.1. Reliability function

The reliability function represents the probability that an item does not fail before a given time, say x. For the SMPP distribution, it is expressed as

$$R(x;\alpha,\beta) = 1 - G_{SMPP}(x;\alpha,\beta) = \frac{1-e^{\frac{\log\alpha}{x^\beta}}}{1-\alpha}, \quad \alpha \neq 1 \tag{7}$$

### 3.2. Hazard Rate

The hazard rate function is an important measure used to characterize a life phenomenon. It is given by

$$h(x;\alpha,\beta) = \frac{\frac{\log\alpha}{\alpha-1}\cdot\frac{\beta}{x^{\beta+1}}\cdot e^{\frac{\log\alpha}{x^{\beta}}}}{e^{\frac{\log\alpha}{x^{\beta}}}-1} \tag{8}$$

Figure (2) illustrates the hazard rate graphs of the SMPP distribution for various parameter combinations. It reveals that the hazard function can exhibit unimodal, increasing, decreasing, or constant shapes depending on the parameter values.

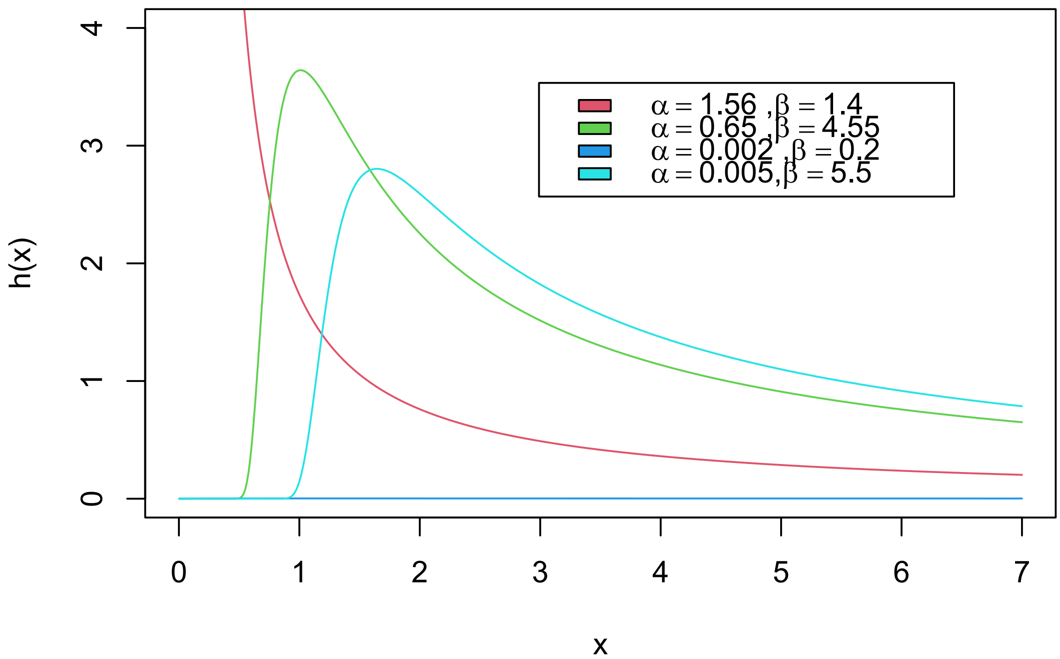


**Figure 2:** *Hazard plot for SMPP distribution*

### 3.3. Reverse Hazard Rate

The expression for the reverse hazard rate of the SMPP distribution is given as

$$h_r(x;\alpha,\beta) = \frac{(\log\alpha)e^{\frac{(\log\alpha)}{x^{\beta}}}\frac{\beta}{x^{\beta+1}}}{\alpha-e^{\frac{\log\alpha}{x^{\beta}}}} \tag{9}$$

### 3.4. Mills Ratio

The mills ratio for the SMPP is defined as

$$M.R = \frac{e^{\frac{\log\alpha}{x^{\beta}}}-\alpha}{1-e^{\frac{\log\alpha}{\beta}}} \tag{10}$$

### 3.5. Quantile function

**Theorem 1.** If $X \sim SMPP(\alpha, \beta)$ distribution, then the quantile function of X is given as

$$x = \left[\frac{\log \alpha}{\log \{u(1-\alpha)+\alpha\}}\right]^{\frac{1}{\beta}} \tag{11}$$

where u is a uniform random variable, $0 < u < 1$
**Proof.** Let $G_{SMPP}(x; \alpha, \beta) = u$. The quantile function of SMPP distribution can be obtained as follows.

$$\Rightarrow \frac{e^{\frac{log\alpha}{\beta}} - \alpha}{1-\alpha} = u$$

$$\Rightarrow e^{\frac{\log \alpha}{x^\beta}} = u(1-\alpha)+\alpha$$

Taking logarithm on both sides and simplifying further, we obtain the required quantile function as

$$x = \left[\frac{\log \alpha}{\log \{u(1-\alpha)+\alpha\}}\right]^{\frac{1}{\beta}} \tag{12}$$

Where u follows a uniform (0,1) distribution. The $q^{th}$ quantile function of SMPP distribution is expressed as:

$$x_q = \left[\frac{\log \alpha}{\log \{u(1-\alpha)+\alpha\}}\right]^{\frac{1}{\beta}}$$

The median can be obtained as

$$x_{0.5} = \left[\frac{\log \alpha}{\log \{0.5(1-\alpha)+\alpha\}}\right]^{\frac{1}{\beta}}$$

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## 4. Statistical properties of SMPP distribution

Some of the statistical properties of SMPP will be discussed in this section.

### 4.1. Moments

The $r^{th}$ moment for SMPP distribution can be obtained as

$$\mu_r{}' = E(x^r) = \int_1^\infty x^r g(x; \alpha, \beta)\, dx$$

$$E(x^r) = \beta \frac{\log \alpha}{\alpha - 1} \int_1^\infty x^{-\beta-1+r} e^{\frac{\log \alpha}{x^\beta}}$$

using expansion $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$

$$\Rightarrow \mu_r{}' = \frac{\beta}{\alpha-1} \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!} \left[\frac{1}{\beta+\beta_j - r}\right] \tag{13}$$

### 4.2. Mode

The mode of the distribution is determined by solving the following equation

$$\log g(x) = \log \left(\frac{\log \alpha}{\alpha - 1} \cdot \frac{\beta}{x^{\beta+1}} \cdot e^{\frac{\log \alpha}{x^\beta}}\right) \tag{14}$$

differentating above equation w.r.t $x$ and equate to zero we get,

$$x = \left(\frac{\beta \log \alpha}{\beta + 1}\right)^{\frac{1}{\beta}}$$

### 4.3. Moment Generating Function of SMPP

**Theorem 2.** Let $X \sim$ SMPP$(\alpha, \beta)$, then the moment generating function, $M_X(t)$ is,

$$M_X(t) = \frac{\beta}{\alpha - 1} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!} \frac{(\log \alpha)^{j+1}}{j!} \frac{1}{(\beta + \beta_j - r)} \tag{15}$$

**Proof.** The moment-generating function of SMPP distribution is given as

$$M_X(t) = \int_1^{\infty} e^{tx} g(x; \alpha, \beta)\, dx$$

$$M_X(t) = \int_1^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \dots\right) g(x; \alpha, \beta)\, dx$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_1^{\infty} x^r g(x; \alpha, \beta)\, dx$$

$$\Rightarrow M_X(t) = \frac{\beta}{\alpha - 1} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!} \frac{(\log \alpha)^{j+1}}{j!} \frac{1}{(\beta + \beta_j - r)} \tag{16}$$

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### 4.4. Characteristic Function of SMPP distribution

**Theorem 3.** Let $X \sim$ SMPP$(\alpha, \beta)$, then the characteristic function, $\phi_X(t)$ is given as

$$\phi_X(t) = \frac{\beta}{\alpha - 1} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{(it)^r}{r!} \frac{(\log \alpha)^{j+1}}{j!} \frac{1}{(\beta + \beta_j - r)}$$

**Proof.** The characteristic function of SMPP distribution is given as

$$\phi_X(t) = \int_1^{\infty} e^{itx} g(x; \alpha, \beta)\, dx$$

$$\phi_X(t) = \int_1^{\infty} \left(1 + itx + \frac{(itx)^2}{2!} + \dots\right) (x; \alpha, \beta)\, dx$$

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_1^{\infty} x^r g(x; \alpha, \beta)\, dx$$

$$\Rightarrow \phi_X(t) = \frac{\beta}{\alpha - 1} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{(it)^r}{r!} \frac{(\log \alpha)^{j+1}}{j!} \frac{1}{(\beta + \beta_j - r)} \tag{17}$$

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**Lemma 1.** Let us suppose that a random variable X follows SMPP $(\alpha, \beta)$ with PDF given in equation (6) and let $I_r(t) = \int_1^t x^r g(x; \alpha, \beta) dx$ denote the $r^{th}$ incomplete moment, then we have

$$I_r(t) = \frac{\beta}{\alpha - 1} \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!} \left[\frac{t^{r-\beta-\beta_j} - 1}{r - \beta - \beta_j}\right] \tag{18}$$

**Proof.**Using the PDF of SMPP given in equation (6), we have

$$I_r(t) = \frac{\beta}{\alpha - 1} \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!} \int_1^t x^{r-\beta-1-\beta_j}\, dx \tag{19}$$

$$I_r(t) = \frac{\beta}{\alpha - 1} \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!} \left[\frac{t^{(r-\beta-\beta_j)} - 1}{r - \beta - \beta_j}\right] \tag{20}$$

## 4.5. Mean Residual Life (MRL)

The MRL, say $\mu(t)$, of SMPP distribution can be determined as follows:

$$\mu(t) = \frac{1}{R(t)}\left[E(x) - \int_1^t xf(x;\alpha,\beta)\,dx\right] - t$$

$$\mu(t) = \frac{1-\alpha}{1-e^{\frac{\log\alpha}{x^\beta}}}\frac{\beta}{\alpha-1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{j!}\left[\frac{1}{\beta+\beta_j-1} - \frac{t^{1-\beta-\beta_j}}{1-\beta-\beta_j}\right] - t$$

## 4.6. Mean Waiting Time (MWT)

The MWT of $x$, say $\bar{\mu}(t)$,is determined by

$$\overline{\mu}(t) = t - \frac{1}{F(t)}\int_1^t xf(x;\alpha,\beta)\,dx$$

$$\overline{\mu}(t) = t - \frac{1-\alpha}{e^{\frac{log\alpha}{x^\beta}} - \alpha}\cdot\frac{\beta}{\alpha-1}\cdot\sum_{j=0}^{\infty}\frac{(log\alpha)^{j+1}}{j!}\left[\frac{t^{1-\beta-\beta_j}-1}{1-\beta-\beta_j}\right]$$

## 4.7. Renyi Entropy

The entropy of a random variable is defined as the measure of uncertainty. The Renyi entropy is defined as,

$$I_v = \frac{1}{1-v}\log\int_1^\infty g^v(x)\,dx$$

Using PDF given in Equation (6), we have

$$I_v = \frac{1}{1-v}\log\left(\frac{\beta\log\alpha}{\alpha-1}\right)^v\int_1^\infty\sum_{j=0}^{\infty}\frac{v^j(\log\alpha)^j}{j!x^{\beta_j}}\frac{1}{x^{(\beta+1)v}}\,dx$$

$$I_v = \frac{1}{1-v}\log\left(\frac{\beta\log\alpha}{\alpha-1}\right)^v\sum_{j=0}^{\infty}\frac{v^j(\log\alpha)^j}{j!}\frac{1}{(\beta_v+\beta_j+v-1)}$$

which is required expression of Renyi entropy for SMPP distribution.

## 4.8. Stress Strength Reliability

Suppose $X_1$ and $X_2$ be two continous random variables, where $X_1 \sim$ SMPP$(\alpha_1,\beta)$ and $X_2 \sim$ SMPP$(\alpha_2,\beta)$, then the stress-strength parameter,say R, is defined as

$$SSR = \int_{-\infty}^{\infty} f_1(x)F_2(x)\,dx. \tag{21}$$

Using Equations (5) and (6) in Equation (21), the stress-strength parameter R, can be obtained as

$$SSR = \frac{\beta\log\alpha_1}{(\alpha_1-1)(1-\alpha_2)}\int_1^\infty e^{\frac{(\log\alpha_1)}{x^\beta}}\left(e^{\frac{(\log\alpha_2)}{x^\beta}} - \alpha_2\right)dx$$

$$SSR = \frac{\beta\log\alpha_1}{(\alpha_1-1)(1-\alpha_2)}\sum_{j=0}^{\infty}\frac{(\log\alpha_1)^j}{j!}\left[\sum_{k=0}^{\infty}\frac{(\log\alpha_2)^k}{k!}\frac{1}{\beta_j+\beta_k-1} - \frac{\alpha_2}{\beta_j-1}\right]$$

## 5. Order Statistics

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ be the random sample of size n and let $X_{r:n}$ denote the $r^{th}$ order statistics, then the PDF of $X_{r:n}$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} (1 - F(x))^{n-r} g(x) \tag{22}$$

Using equation (5) we have,

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \left( \frac{e^{\frac{(log\alpha)}{x^\beta}} - \alpha}{1-\alpha} \right)^{r-1} \left( \frac{1 - e^{\frac{(log\alpha)}{x^\beta}}}{1-\alpha} \right)^{n-r} g(x)$$

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!(1-\alpha)^{n-1}} \left( e^{\frac{(log\alpha)}{x^\beta}} - \alpha \right)^{r-1} \left( 1 - e^{\frac{(log\alpha)}{x^\beta}} \right)^{n-r} \times \left( \frac{\log(\alpha)}{\alpha - 1} \frac{\beta}{x^{\beta+1}} e^{\frac{log\alpha}{x^\beta}} \right) \tag{23}$$

The expressions for the PDF of the smallest (minimum) and largest (maximum) order statistics of the SMPP distribution are obtained by setting $r = 1$ and $r = n$, respectively, in the above equation.

## 6. Estimation of Parameters

### 6.1. Maximum Likelihood Estimation

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n having PDF in equation (6). Then the likelihood function is given by

$$L = \prod_{i=1}^{n} \frac{\log\alpha}{\alpha - 1} \cdot \frac{\beta}{x_i^{\beta+1}} \cdot e^{\frac{\log(\alpha)}{x_i^\beta}}$$

The log likelihood function is given by;

$$\log L(x : \alpha, \beta) = -n \log(\alpha - 1) + n \log(\log\alpha) + n \log\beta - \sum_{i=1}^{n} (\beta + 1) log x_i + (\log\alpha) \sum_{i=1}^{n} x_i^{-\beta}$$

The MLEs of $\alpha, \beta$ are obtained by partially differentiating above equation with respect to model parameters and equating to zero, we have

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n}{\alpha - 1} + \frac{n}{\alpha \log\alpha} + \frac{1}{\alpha} \sum_{i=1}^{n} x_i^{-\beta} = 0 \tag{24}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log(x_i) - (\log\alpha) \sum_{i=1}^{n} (\log x_i \cdot x_i^{-\beta}) = 0 \tag{25}$$

As the above equations are not in closed form, we will use the Newton-Raphson method and employ R software to solve the equations and estimate the parameters.

## 7. Simulation Study

This section presents a simulation study conducted using R software to examine the behaviour of the MLEs. Random samples of sizes n=25, 75, 150, 300, 500 are generated using the quantile function, with the process repeated 1000 times. Two different parameter combinations, $(\alpha, \beta)$=(0.25, 0.94) and (0.50, 0.35) are considered. The average values of the MLEs, along with their biases and mean square errors (MSEs), are summarised in Tables 1 and 2. The results indicate that the estimates are consistent and converge closely to the true parameter values across the given sample sizes. Additionally, the MSE and bias consistently decrease as the sample size increases in all cases.

**Table 1:** *MLE, Bias, and MSE for the parameters $\alpha$ and $\beta$*

| Sample size | Parameters | | MLE | | Bias | | MSE | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | $\beta$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| 25 | 0.25 | 0.94 | 0.42332 | 1.00330 | 0.33034 | 0.22074 | 0.63895 | 0.07852 |
| 75 | | | 0.29609 | 0.96187 | 0.15788 | 0.12391 | 0.06768 | 0.02465 |
| 150 | | | 0.27572 | 0.94916 | 0.10828 | 0.08893 | 0.02251 | 0.01197 |
| 300 | | | 0.26238 | 0.94300 | 0.07090 | 0.05965 | 0.00896 | 0.00565 |
| 500 | | | 0.25658 | 0.94241 | 0.05176 | 0.04546 | 0.00449 | 0.00319 |

**Table 2:** *MLE, Bias, and MSE for the parameters $\alpha$ and $\beta$*

| Sample size | Parameters | | MLE | | Bias | | MSE | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | $\beta$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| 25 | 0.50 | 0.35 | 2.16491 | 0.38053 | 1.99298 | 0.09651 | 0.57886 | 0.01555 |
| 75 | | | 0.61870 | 0.35945 | 0.34101 | 0.05291 | 0.40309 | 0.00453 |
| 150 | | | 0.55311 | 0.35700 | 0.22113 | 0.03808 | 0.10438 | 0.00232 |
| 300 | | | 0.52724 | 0.35187 | 0.14309 | 0.02502 | 0.03608 | 0.00098 |
| 500 | | | 0.51296 | 0.35105 | 0.11196 | 0.02005 | 0.02309 | 0.00066 |

## 8. Application

The application of the SMPP distribution is demonstrated using two datasets to highlight its importance, usefulness, and flexibility compared to other existing distributions.The performance of the SMPP distribution was compared to the Generalized pareto distribution given by [18] , Exponentiated Generalized pareto distribution by [14] ,Inverse Pareto distribution by [9] and basic Pareto distribution by [13] with the following pdfs.

- Generalized Pareto Distribution

$$\frac{1}{\beta}\left(1+\frac{\alpha x}{\beta}\right)^{-\left(\frac{1}{\alpha}+1\right)}$$

- Exponentiated Generalized Pareto Distribution(ExGPD)

$$\frac{e^x}{\beta}\left(1+\frac{ae^x}{\beta}\right)^{-\left(\frac{1}{\alpha}+1\right)}$$

- Inverse Pareto Distribution

$$\frac{\alpha\beta x^{\alpha-1}}{(\beta+x)^{\alpha+1}}$$

- Basic Pareto Distribution(BP)

$$f(x)=\frac{\beta}{x^{\beta+1}}$$

The goodness-of-fit test is used to assess the performance of the SMPP distribution. The evaluation criteria include the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (AICC), and -2 log-likelihood (-2LL), along with the results of the Kolmogorov-Smirnov (KS) test and the p- value. In general, the model is to be considered the best for which these goodness-of-fit statistics have the least and the p-value is greater.

**Data Set 1. Patients receiving an analgesic.** The data set is taken [8] which consists of 20 observations of patients receiving an analgesic. The values are as follows.

$$1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7,$$
$$4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0$$

**Data set 2. Precipitation data (in inches)** The second data set consists of 25 observations for the precipitation (in inches) from Jug Bridge, Maryl and previosly used by [10] and is given as follows.

$$1.01, 1.11, 1.13, 1.15, 1.16, 1.17, 1.17, 1.20, 1.52, 1.54, 1.54, 1.57, 1.64$$
$$1.73, 1.79, 2.09, 2.09, 2.57, 2.752.93, 3.19, 3.54, 3.57, 5.11, 5.62$$

**Table 3:** *Comparison of SMPP and competitive models for Data set 1*

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $-2ll$ | AIC | BIC | AICC | K-S | p-value |
|---|---|---|---|---|---|---|---|---|
| SMPP | 0.00326 | 3.94075 | 30.71710 | 34.71076 | 36.70222 | 35.41666 | 0.10530 | 0.97940 IPP |
| 57.43931 | 0.01096 | 65.45431 | 69.45431 | 71.44578 | 70.1602 | 0.38689 | 0.00502 | |
| GPD | 0.00100 | 1.89999 | 65.67433 | 69.64433 | 71.66579 | 70.3801 | 0.43951 | 0.00088 |
| EGP | 0.14408 | 8.02475 | 53.06459 | 57.06459 | 59.05606 | 57.07704 | 0.30554 | 0.04779 |
| BP | | 1.69706 | 42.41429 | 44.41429 | 45.41000 | 45.12018 | 0.28505 | 0.07754 |

**Table 4:** *Comparison of SMPP and competitive models for Data set 2*

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $-2ll$ | AIC | BIC | AICC | K-S | p-value |
|---|---|---|---|---|---|---|---|---|
| SMPP | 0.29724 | 2.02042 | 57.07541 | 61.07541 | 63.43152 | 61.64684 | 0.14161 | 0.72170 |
| IPP | 53.11791 | 0.01120 | 84.42617 | 88.42617 | 90.78228 | 88.9976 | 0.29780 | 0.02833 |
| GPP | 0.00100 | 2.18124 | 85.43527 | 89.43527 | 91.79138 | 90.0067 | 0.37063 | 0.00273 |
| EPD | 0.73112 | 8.50710 | 81.15739 | 85.15739 | 87.51350 | 85.72882 | 0.25156 | 0.09590 |
| BP | | 1.54771 | 58.04799 | 65.04799 | 66.22605 | 65.61942 | 0.14360 | 0.60540 |

The results presented in Tables 3 and 4 demonstrate that the AIC, BIC, AICC, and -2LL values are lower for the SMPP distribution compared to the other fitted distributions. Consequently, the SMPP outperforms both the competing models and the base model. The promising performance of the proposed distribution is further illustrated in Figures 3 and 4.

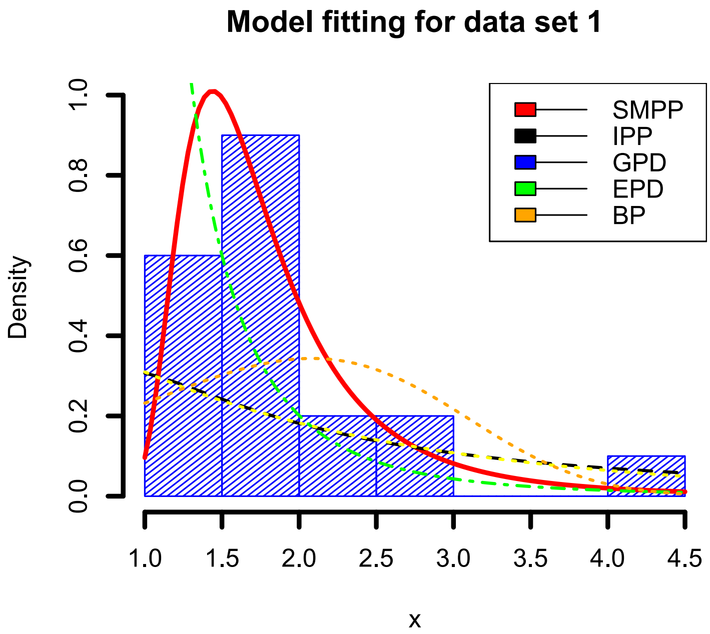


**Figure 3:** *Fitted density plots for data set 1*

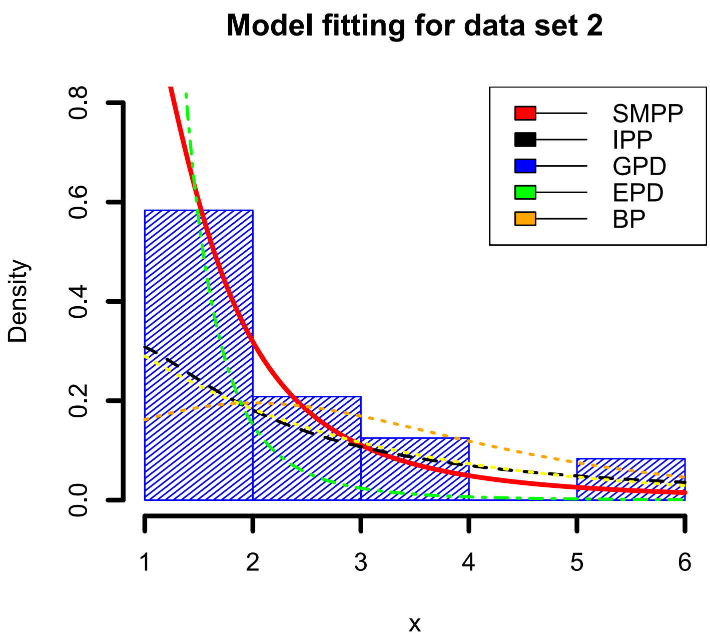


**Figure 4:** *Fitted density plots for data set 2*

## 9. Conclusion

In this paper, we propose a new model called the SMPP distribution, which extends the basic Pareto distribution. The applied transformation generalizes the standard distribution to enhance its flexibility for modeling real-world data. The key statistical properties of the proposed model have been derived. Its hazard rate function demonstrates greater flexibility and accommodates complex forms. The parameters are estimated using the maximum likelihood estimation (MLE) method, which is shown to be accurate and consistent through simulation analysis. Compared to other competing distributions, the proposed model has the potential to provide a superior fit for real-world applications. We believe this approach will find extensive use in the field of statistics.

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