RELIABILITY MEASURES AND CLASSICAL AND BAYESIAN PARAMETRIZATION OF TWO NON-IDENTICAL UNITS SYSTEM MODEL WITH ON-LINE/OFF-LINE REPAIRS OF REPAIR MACHINE

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Abstract

The present paper deals with the reliability analysis of a system model which consists of two non identical units and a repair machine. One of the units initially, is in operative mode and other is kept in standby mode. Repair machine is initially in good condition, it may fails while repairing the failed units. In case of failure of repair machine, firstly it undergoes on-line repair and then to off-line repair if not repaired online. The failure time distributions of both the units and repair machine are assumed to be exponential while the repair time distributions are taken as general. By using regenerative point techniques, various measures of system effectiveness such as transition probabilities, MTSF, reliability, availability, busy period etc. have been obtained. Also, some of them have been studied graphically. A simulation study is also carried out to analyse the considered system model both in Classical and Bayesian setups. From the graphs and tables, various important conclusions are drawn.

Keywords: Reliability, Mean Time to System Failure, Busy Period Analysis, Regenerative Point Technique, Repair Machine, Bayesian Estimation, Maximum Likelihood Estimation.

1. Introduction

Reliability theory is a fundamental framework that plays a pivotal role in various fields, ranging from engineering and economics to healthcare and social sciences. The essence of reliability theory lies in assessing and ensuring the dependability and consistency of systems, processes, or components over time.At its core, reliability theory addresses the inherent uncertainty and variability in real-world systems, acknowledging that components may fail or degrade over time. The primary objective is to quantify and enhance the reliability of these systems, ultimately aiming for optimal functionality and durability. One key aspect of reliability theory is the study of failure mechanisms and their impact on overall system performance.In the real-world, the situations arise in many cases when the repair machine fails itself. Kumar et al. [5] studied reliability measures of a two unit system with Inspection and On-Line/Off-Line Repairs Using the Regenerative Point Technique. In this study they analysed a system model having two identical units with the inspection of each unit also with single repair facility which is always available with the system. Tyagi and Chaudhary [1] analysed a two non-identical unit parallel system subject to two types of failure and correlated life times. They used the concept of two non identical unit system with two types of repair. Also the units in this model are arranged in parallel configuration. Kumar and Gupta [3] studied performance measures of a two non-identical unit system model with repair and replacement policies. Kishan and Jain [4] carried a study on two non-identical unit standby system model with repair, inspection and post-repair from Classical and Bayesian viewpoints, considering both failure and repair time distribution as weibull. Also a Monte carlo simulation study is carried out. Lin et al.[6] adhered to the reliability analysis and preventive maintenance using Classical and Bayesian semi-parametric degradation approaches of a locomotive wheel-sets as a case study. Yue et al. [8] accomplished performance analysis and optimization of a machine repair problem with warm spares and two heterogeneous repairmen. Here first repairman is always available for serving the failed units, while the second repairman leaves for a vacation of random length when the number of failed units is less than N. Goyal et al.[2] studied reliability measures and profit exploration of windmill water-pumping systems incorporating warranty and two types of repair. Malik et al.[7] studied reliability and economic measures of a system with inspection for online repair and no repair activity in abnormal weather. They considered two reliability models in which unit fails completely via partial failure. In the present study, we analyze the reliability characteristics and estimate in Classical and Bayesian paradigms. The parameters of two non identical units system model with a repair machine. Here, one of the units initially is in operative mode and other is kept in standby mode. Repair machine is initially in good condition, it may fail while repairing the failed units. In case of failure of repair machine, firstly it undergoes on-line repair and then to off-line repair. The failure time distributions of both the units and repair machine are assumed to be exponential while the repair time distributions are taken as general. By using regenerative point techniques, various measures of system effectiveness such as transition probabilities, MTSF, reliability, availability, busy period etc. have been obtained. Some of them have been studied graphically. Also a simulation study is carried out to analyse the considered system model both in Classical and Bayesian setups. From the graphs and tables, various important conclusions are drawn.

2. System Description and Assumptions

The system comprises of two non identical units A and B and one Repair machine. Initially one of the units is in operative mode and other is kept in standby mode. Repair machine is used to repair the failed units. Initially, repair machine is assumed to be in good condition. There might be online and offline repair of repair machine. As soon as repair machine fails firstly it goes to online repair and then to offline repair, if online repair fails to fix problem. The failure time distributions of both the units and Repair machine are assumed to be exponential while the repair time distributions are taken general in nature. Once a component is repaired it is as good as new.

3. Notations and Symbols

α_1 :Failure rate of unit A — $F_1(.)$: Cdf of repair time of unit A
α_2 :Failure rate of unit B — $F_2(.)$ Cdf of repair time of unit B
α_3 :Failure rate of repair machine — $F_3(.)$: Cdf of online repair time of repair machine
β : Online to offline repair rate. — $F_4(.)$: Cdf of offline repair time of repair machine
A_o/B_o :Units are in operative mode — A_s/B_s :Units are in standby mode
A_r/B_r :Units are under repair — A_{wr}/B_{wr}:Units are waiting for repair
${}_{on}RM_r/{}_{off}RM_r$: Online/Offline repair of Repair machine.

3.1. Symbols for the states of the system

$S_0 = [A_o, B_s, RM]$ — $S_3 = [A_{wr}, B_o, {}_{on}RM_r]$
$S_1 = [A_r, B_o, RM]$ — $S_4 = [A_{wr}, B_o, {}_{off}RM_r]$
$S_2 = [A_r, B_{wr}, RM]$ — $S_5 = [A_{wr}, B_{wr}, {}_{on}RM_r]$
$S_7 = [A_o, B_r, RM]$ — $S_6 = [A_{wr}, B_{wr}, {}_{off}RM_r]$
$S_8 = [A_o, B_{wr}, {}_{on}RM_r]$ — $S_9 = [A_o, B_{wr}, {}_{off}RM_r]$

The transition diagram along with all transitions is shown in Figure 1

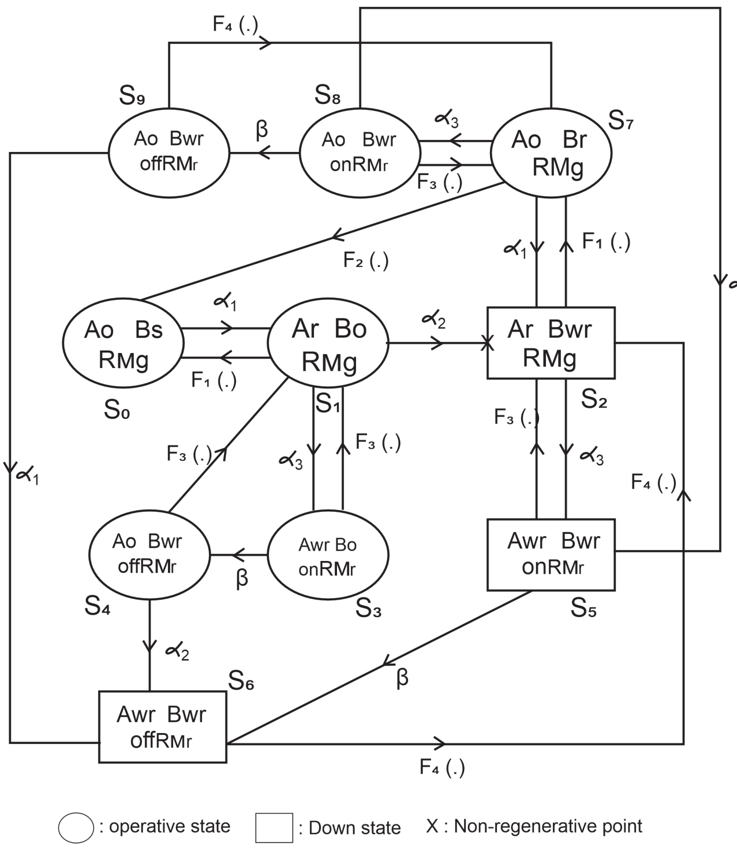

Figure 1: *Transition Diagram*

4. Transition Probabilities and Sojourn Times

The steady state transition probabilities are given by,

$$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) = \int q_{ij}(t)dt \quad p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t) \quad and \quad p_{ij}^{(k,l)} = \lim_{t\to\infty} Q_{ij}^{k,l}(t)$$

In particular we have

$$p_{10}(t) = \int e^{-\alpha_3 t} e^{-\alpha_2 t} dF_1(t) \quad = \quad \widetilde{F_1}(\alpha_2 + \alpha_3)$$

Similarly,

$p_{13} = \frac{\alpha_3}{\alpha_2+\alpha_3}[1 - \widetilde{F_1}(\alpha_2 + \alpha_3)]$ $\qquad p_{15}^{(2)} = [1 - \widetilde{F_1}(\alpha_3)] - \frac{\alpha_3}{\alpha_2+\alpha_3}[1 - \widetilde{F_1}(\alpha_2 + \alpha_3)]$

$p_{17}^{(2)} = \widetilde{F_1}(\alpha_3) - \widetilde{F_1}(\alpha_2 + \alpha_3)$ $\qquad p_{25} = 1 - \widetilde{F_1}\ (\alpha_3)$

$p_{27} = \widetilde{F_1}\ (\alpha_3)$ $\qquad p_{31} = \widetilde{F_3}(\alpha_2 + \beta)$

$p_{34} = \frac{\beta}{\alpha_2+\beta}[1 - \widetilde{F_3}(\alpha_2 + \beta)]$ $\qquad p_{35} = \frac{\alpha_2}{\alpha_2+\beta}[1 - \widetilde{F_3}(\alpha_2 + \beta)]$

$p_{41} = \widetilde{F_4}(\alpha_2)$ $\qquad p_{46} = [1 - \widetilde{F_4}(\alpha_2)]$

$p_{52} = \widetilde{F_3}\ (\beta)]$ $\qquad p_{56} = 1 - \widetilde{F_3}\ (\beta)$

$p_{70} = \widetilde{F_2}(\alpha_1 + \alpha_3)$ $\qquad p_{71} = \frac{\alpha_1}{\alpha_1+\alpha_3}[1 - \widetilde{F_2}(\alpha_1 + \alpha_3)]$

$p_{78} = \frac{\alpha_3}{\alpha_1+\alpha_3}[1 - \widetilde{F_2}(\alpha_1 + \alpha_3)]$ $\qquad p_{85} = \frac{\alpha_1}{\alpha_1+\beta}[1 - \widetilde{F_3}(\alpha_1 + \beta)]$

$p_{87} = \widetilde{F_3}(\alpha_1 + \beta)$ $\qquad p_{89} = \frac{\beta}{\alpha_1+\beta}[1 - \widetilde{F_3}(\alpha_1 + \beta)]$

$p_{96} = [1 - \widetilde{F}_4(\alpha_1)]$ $\quad$ $p_{97} = \widetilde{F}_4(\alpha_1)$

Thus, we observe the following relations:

$p_{10} + p_{13} + p_{15}^{(2)} + p_{17}^{(2)} = 1$ $\quad$ $p_{25} + p_{27} = 1$

$p_{52} + p_{56} = 1$ $\quad$ $p_{31} + p_{34} + p_{35} = 1$

$p_{41} + p_{46} = 1$ $\quad$ $p_{70} + p_{72} + p_{78} = 1$

$p_{85} + p_{87} + p_{89} = 1$ $\quad$ $p_{96} + p_{97} = 1$

$p_{01} = p_{62} = 1$

4.1. Mean Sojourn times

In reliability, Mean Sojourn time ψ_i, refers to the expected amount of time a system spends in a specific state before making a transition to another state. To find ψ_i for state S$_i$, we observe that there is no transition from S$_i$ to any otherstate as long as the system is in state S$_i$. Let T$_i$ denotes the sojourn time in state S$_i$ then mean sojourn time ψ_i in state S$_i$ is:

$$\psi_i = E[T_i] = \int P(T_i > t)dt$$

Hence, using the above formula following values for mean sojourn time are obtained:

$\psi_0 = \frac{1}{\alpha_1}$ $\quad$ $\psi_1 = \frac{1}{(\alpha_2+\alpha_3)}[1 - \widetilde{F}_1(\alpha_2 + \alpha_3)]$

$\psi_2 = \frac{1}{\alpha_3}[1 - \widetilde{F}_1(\alpha_3)]$ $\quad$ $\psi_3 = \frac{1}{(\alpha_2+\beta)}[1 - \widetilde{F}_2(\alpha_2 + \beta)]$

$\psi_4 = \frac{1}{\alpha_2}[1 - \widetilde{F}_4(\alpha_2)]$ $\quad$ $\psi_5 = \frac{1}{\beta}[1 - \widetilde{F}_3(\beta)]$

$\psi_6 = \int \bar{F}_4(t)dt$ $\quad$ $\psi_7 = \frac{1}{(\alpha_1+\alpha_3)}[1 - \widetilde{F}_2(\alpha_1 + \alpha_3)]$

$\psi_8 = \frac{1}{(\alpha_1+\beta)}[1 - \widetilde{F}_3(\alpha_1 + \beta)]$ $\quad$ $\psi_9 = \frac{1}{\alpha_1}[1 - \widetilde{F}_4(\alpha_1)]$

5. Analysis of Reliability and MTSF

Let T_i be any random variable which denotes the time to failiure of the system when system starts up from state S$_i \in E_i$, then the reliability of the system is given by

$$R_i(t) = P[T_i > t]$$

To obtain $R_i(t)$, we consider failed states as absorbing states. By referring to the state transition diagram, the recursive relations among $R_i(t)$ can be formulated on the basis of probabilistic arguments. Taking Laplace transform and solving the resultant set of equations for$R_0^*(s)$, we get

$$R_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{1}$$

where,

$$N_1(s) = Z_0^*[1 - q_{13}^*(q_{31}^* + q_{34}^*(q_{41}^* + Z_4^*))] + q_{01}^*[Z_1^* + q_{13}^*(Z_3^* + Z_4^* q_{34}^*)]$$

$$D_1(s) = 1 - q_{13}^*(q_{31}^* + q_{34}^* q_{41}^*) - q_{01}^* q_{10}^*$$

By taking inverse Laplace Transform of (1), we get reliability of the system.To get MTSF, we use the well known formula

$$E(T_0) = \int R_0(t)dt = \lim_{s\to 0} R_0^*(s) = \frac{N_1(0)}{D_1(0)} \tag{2}$$

where,

$$N_1(0) = \psi_0[1 - p_{13}(p_{31} + p_{34}(p_{41} + \psi_4))] + p_{01}[\psi_1 + p_{13}(\psi_3 + \psi_4 p_{34})]$$

and

$D_1(0) = 1 - p_{13}(p_{31} + p_{34}p_{41}) - p_{01}p_{10}$

Since we have $q^*_{ij}(0) = p_{ij}$ and $\lim_{s\to 0} Z^*_i(s) = \int Z_i(t)dt = \psi_i$ [1]

6. Availability Analysis

Availability refers to the probability that a system is available at time 't' given that initially it started from $S_i \in E_i$. Point wise availability refers to the availability of a system at a specific point in time. It is a measure of system performance and indicates whether a system is operational and providing its intended service at a particular moment. By using stochastic arguments, the recurrence relations among different point wise availabilities are obtained and taking the Laplace transforms and solving the resultant set of equations for$A^*_0(s)$, we have

$$A^*_0(s) = \frac{N_2(s)}{D_2(s)}$$

where,

$$\begin{aligned} N_2(s) =& Z^*_0[1 - q^*_{13}A(q^*_{31} + q^*_{34}q^*_{41})] + A(Z^*_1 + Z^*_3q^*_{13} + Z^*_4q^*_{13}q^*_{34}) + Z^*_5 \\ & [B(q^{(2)*}_{15} + q^*_{13}q^*_{35}) + q^{(2)*}_{17}C + q^*_{13}q^*_{34}D] + q^*_{27}(q^{(2)*}_{15} + q^{(2)*}_{17})[Z^*_7 \\ & + Z^*_8 + q^*_{78} + Z^*_9q^*_{78}q^*_{89}] + q^*_{13}q^*_{27}E[Z^*_7 + Z^*_8q^*_{78} + Z^*_9q^*_{78}q^*_{89}] \end{aligned}$$

Here,

A = $q^*_{27}(1 - q^*_{78})$

$B = 1 - q^*_{27}q^*_{72} - q^*_{78}q^*_{87} - q^*_{78}q^*_{89}q^*_{97} - q^*_{27}q^*_{78}q^*_{89}q^*_{96}$

$C = q^*_{25}q^*_{72} + q^*_{78}q^*_{87} - q^*_{78}q^*_{89}q^*_{97} - q^*_{27}q^*_{78}q^*_{89}q^*_{96}$

$D = q^*_{25}q^*_{46}{}_{q}{}^{*}_{46}q^*_{78}q^*_{87}(1 - q^*_{27}q^*_{97})$

$E = q^*_{34}q^*_{46} + q^*_{35}$

and,

$$\begin{aligned} D_2(s) =& q^*_{27}[1 - q^*_{72} - q^*_{78}q^*_{87} - q^*_{70}q^{(2)*}_{17} - q^*_{78}q^*_{89}q^*_{97} - q^*_{78}q^*_{85} + q^*_{70}q^{(2)*}_{15} - q^*_{13}q^*_{70} \\ & (q^*_{35} + q^*_{34}q^*_{46})] - F[q^*_{10} + q^*_{13}(q^*_{31} + q^*_{34}q^*_{41})] \end{aligned} \quad (3)$$

where,

$F = q^*_{27}[1 - q^*_{72} - q^*_{78}(q^*_{87} - q^*_{85} + q^*_{89}q^*_{97} - q^*_{89}q^*_{96})]$

The steady state availability is given by

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} sA^*_0(s) = \frac{N_2(0)}{D_2(0)}$$

[1]Limits of integeration whenever they are 0 to ∞ are not mentioned.

now we know that, $q_{ij}(t)$ is the pdf of the transition time from state Si to Sj and also $q_{ij}(t)$ is the probability of transition from state S_i to S_j during the interval (t, t + dt), thus

$$q^*_{ij}(s)|s = 0 = q^*_{ij}(0) = p_{ij}$$

We also know that

$$\lim_{s\to 0} Z^*_i(s) = \int Z_i(t)dt = \psi_i$$

Therefore,

$$\begin{aligned} N_2(0) =& \psi_0[1 - p_{13}A(p_{31} + p_{34}p_{41})] + A(\psi_1 + \psi_3 p_{13} + \psi_4 p_{13} p_{34}) + \psi_5[B(p^{(2)}_{15} + \\ & p_{13}p_{35}) + p^{(2)}_{17}C + p_{13}p_{34}D] + p_{27}(p^{(2)}_{15} + p^{(2)}_{17})[\psi_7 + \psi_8 + p_{78} + \psi_9 p_{78} p_{89}] \\ & + p_{13}p_{27}E[\psi_7 + Z\psi_8 p_{78} + Z\psi_9 q p_{78} p_{89}] \end{aligned}$$

Here,

$$A = p_{27}(1 - p_{78})$$

$$B = 1 - p_{27}p_{72} - p_{78}p_{87} - p_{78}p_{89}p_{97} - p_{27}p_{78}p_{89}p_{96}$$

$$C = p_{25}p_{72} + p_{78}p_{87} - p_{78}p_{89}p_{97} - p_{27}p_{78}p_{89}p_{96}$$

$$D = p_{25}p_{46} - p_{46}p_{78}p_{87}(1 - p_{27}p_{97})$$

$$E = p_{34}p_{46} + p_{35}$$

and

$$\begin{aligned} D_2(0) =& p_{27}[1 - p_{72} - p_{78}p_{87} - p_{70}p^{(2)}_{17} - p_{78}p_{89}p_{97} - p_{78}p_{85} + p_{70}p^{(2)}_{15} - p_{13}p_{70} \\ & (p_{35} + p_{34}p_{46})] - F[p_{10} + p_{13}(p_{31} + p_{34}p_{41})] \end{aligned}$$

where,

$$F = p_{27}[1 - p_{72} - p_{78}(p_{87} - p_{85} + p_{89}p_{97} - p_{89}p_{96})]$$

The steady state probability that the system will be up in long run is given by

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} sA^*_0(s)$$

$$\lim_{s\to 0} \frac{sN_2(s)}{D_2(s)} = \lim_{s\to 0} N_2(s) \lim_{s\to 0} \frac{s}{D_2(s)}$$

Since as s $\to$ 0, $D_2(s)$ becomes zero. Hence, on using L'Hospital's rule, A_0 becomes

$$A_0 = \frac{N_2(0)}{D'_2(0)} \tag{4}$$

where,

$$\begin{aligned} D'_2(0) =& \psi_0 A + p_{27}p_{70}(\psi_1 + \psi_3 p_{13}) + \psi_2 B + p_{27}(\psi_4 p_{13}p_{34}p_{78} + \psi_7 F) + \psi_5 C + \psi_6 D \\ & + p_{27}p_{78}(\psi_8 E + \psi_9 p_{89} F) \end{aligned} \tag{5}$$

here,

$$A = p_{10}[(p_{27}(1-p_{72}) - p_{78}p_{87} - p_{78}p_{89}p_{97}) - p_{78}(p_{25}p_{87} + p_{27}p_{85}) + p_{78}p_{89}(p_{25}p_{97} - p_{27}p_{96})] - p_{70}(p_{25}p_{17}^{(2)} - p_{27}p_{15}^{(2)}) + p_{13}p_{27}p_{70}(p_{35} + p_{34}p_{46})$$

$$B = 1 - p_{10}(1 - p_{78}p_{89}p_{97}) - p_{13}p_{34}p_{41}(p_{52} - p_{78}p_{89}p_{97}) - p_{13}p_{31}p_{56}(1 - p_{78}p_{87}) - p_{25}p_{70}p_{17}^{(2)} - p_{78}p_{87}(1 - p_{10}) - p_{78}p_{89}p_{97}(1 - p_{13}p_{31})$$

$$C = p_{25}[1 - E(p_{13}p_{31} + p_{10} + p_{13}p_{34}p_{41}) - p_{70}p_{17}^{(2)} - p_{78}(p_{87} + p_{89}p_{97})] + p_{27}[p_{78}p_{85} F(p_{70}(p_{15}^{(2)} + p_{13}p_{35}))]$$

$$D = p_{25}p_{56}[1 - E(p_{13}p_{31} + p_{10} + p_{13}p_{34}p_{41}) - p_{70}p_{17}^{(2)} - p_{78}(p_{87} + p_{89}p_{97})] + p_{27}p_{56}[p_{78}p_{85}F + p_{78}p_{89} + p_{70}(p_{15}^{(2)} + p_{13}p_{35})] + p_{13}p_{27}[p_{34}(p_{46}p_{70} - p_{41}p_{78}p_{89}p_{96}) - p_{13}p_{78}p_{89}p_{96}] - p_{10}p_{27}p_{78}p_{89}p_{96}$$

$$E = 1 - p_{78}p_{87} - p_{78}p_{89}p_{97}$$

$$F = 1 - p_{10} - p_{13}p_{31} - p_{13}p_{34}p_{41}$$

Using $N_2(0)$ and $D_2'(0)$ in equation [4], we get the expression for A_0. The expected up time of the system during (0,t] is given by

$$\mu_{up}(t) = \int_0^t A_0(u)du$$

So that,

$$\mu_{up}^* = \frac{A_0^*(s)}{s}$$

7. Busy Period Analysis

$B_i(t)$ refers to the probability that the system having started from regenerative state Si∈ E, at time t=0, is under repair at time t due to failure of the unit. To find these probabilities, we use simple probabilistic arguments and further on taking Laplace transform and solving the resultant set of equation for $B_0^*(s)$, we have

$$B_0^*(s) = \frac{N_3(s)}{D_2(s)}$$

where,

$$N_3(s) = Z_1q_{27}^*q_{70}^* + Z_2[q_{15}^{(2)*}A + q_{17}^{(2)*}B + Aq_{13}^*C]Z_7[q_{27}^*(q_{15}^{(2)*} + q_{17}^{(2)*}) + q_{13}^*q_{27}^*C]$$

here,

$$A = 1 - q_{78}^*q_{87}^* - q_{78}^*q_{89}^*q_{97}^*$$

$$B = q_{27}^* + q_{78}^*q_{85}^* + q_{78}^*q_{89}^*q_{96}^*$$

$C = q_{35}^{*} + q_{34}^{*} q_{46}^{*}$

and, D_2(s) is same as given by[3]. In the steady state, the probability that the repairman will be busy is given by

$$B_0 = \lim_{t\to\infty} B_0(t) = \lim_{s\to 0} B_0^*(s) = \frac{N_3(0)}{D_2^{'}(0)}$$

where,

$$N_3(0) = Z_1 p_{27} p_{70} + Z_2[p_{15}^{(2)} A + p_{17}^{(2)} B + A p_{13} C] Z_7[p_{27}(p_{15}^{(2)} + p_{17}^{(2)}) + p_{13} p_{27} C]$$

here,

$$A = 1 - p_{78} p_{87} - p_{78} p_{89} p_{97}$$

$$B = p_{27} + p_{78} p_{85} + p_{78} p_{89} p_{96}$$

$$C = p_{35} + p_{34} p_{46}$$

and $D_2^{'}(0)$ is same as obtained in [5]. The expected busy period of the repairman during (0,t] is given by

$$\mu_b(t) = \int_0^t B_0(u) du$$

So that,

$$\mu_b^*(s) = \frac{B_0^*(s)}{s}$$

8. Expected Number of Repairs

$V_i(t)$ is defined as the expeted number of repairs during the time interval (0,t] of the failed units, when the system initially starts from regenerative state S_i. Further, by the definition of $V_i(t)$ the recurrence relations are easily obtained and on taking their Laplace- Stieltjes transforms and solving the resultant set of equations for $\tilde{V}_0(s)$, we get

$$\tilde{V}_0(s) = \frac{N_4(s)}{D_3(s)}$$

where,

$$N_4(s) = \tilde{Q}_{27}[\tilde{Q}_{70}(\tilde{Q}_{10} - \tilde{Q}_{17}^{(2)}) + \tilde{Q}_{15}^{(2)} A + \tilde{Q}_{13} B(\tilde{Q}_{35} + \tilde{Q}_{34}\tilde{Q}_{46})]$$

here,

$$A = 1 + \tilde{Q}_{56}\tilde{Q}_{70} - \tilde{Q}_{78}\tilde{Q}_{87} - \tilde{Q}_{78}\tilde{Q}_{89}\tilde{Q}_{97}$$

$$B = 1 + \tilde{Q}_{70} - \tilde{Q}_{78}\tilde{Q}_{87} - \tilde{Q}_{78}\tilde{Q}_{89}\tilde{Q}_{97}$$

and $D_3(s)$ is written by replacing $\overset{*}{q}_{ij}$ and $q_{ij}^{(k)*}$ by $\tilde{Q}_{ij}$ and $\tilde{Q}_{ij}^{(k)}$ respectively in the equation (7). In the steady state, the expected number of repairs per unit time is given by

$$V_0 = \lim_{t\to\infty} V_0(t) = \lim_{s\to 0} s\tilde{V}_0(s) = \frac{N_4(0)}{D_2^{'}(0)}$$

where,

$$N_4(0) = p_{27}[p_{70}(p_{10} - p_{17}^{(2)}) + p_{15}^{(2)} A + p_{13} B(p_{35} + p_{34} p_{46})]$$

here,

$$A = 1 + p_{56} p_{70} - p_{78} p_{87} - p_{78} p_{89} p_{97}$$

$$B = 1 + p_{70} - p_{78} p_{87} - p_{78} p_{89} p_{97}$$

9. Profit Function Analysis

As the reliability charateristics are obtained, the profit function P(t) for the system can be obtained. Profit is defined as excess of revenue over the cost, therefore the expected total profits generated during(0,t] is given as :
P(t) = Expected total revenue in(0,t] - Expected total expenditure in(0,t]

$$= K_0 \mu_{up}(t) - K_1 \mu_b(t) - K_2 V_0(t)$$

where,
K_0 = Revenue per unit up time of the system.
K_1 = Cost per unit time for which repair man is busy in repairing the failed unit.
K_2 = Cost of repair per unit.
In steady state, the expected total profits per unit time, is given by:

$$P = \lim_{t \to \infty} \frac{P(t)}{t} = \lim_{s \to 0} s^2 \overset{*}{P}(s)$$

Therefore, we have

$$P = K_0 A_0 - K_1 B_0 - K_2 V_0 \tag{6}$$

10. Estimation of the Parameters, MTSF, And Profit Function

10.1. Classical Estimation

10.1.1 ML Estimation

Let

$$\underset{\sim}{X_1} = (x_{11}, x_{12}, ..., x_{1n_1}), \quad \underset{\sim}{X_2} = (x_{21}, x_{22}, ...x_{2n_2}), \quad \underset{\sim}{X_3} = (x_{31}, x_{32}, ..., x_{3n_3}), \underset{\sim}{X_4} = (x_{41}, x_{42}, ..., x_{4n_4}),$$

$$\underset{\sim}{X_5} = (x_{51}, x_{52}, ..., x_{5n_5}, \quad \underset{\sim}{X_6} = (x_{61}, x_{62}, ..., x_{6n_6}), \underset{\sim}{X_7} = (x_{71}, x_{72}, ..., x_{7n_7}, \quad \underset{\sim}{X_8} = (x_{81}, x_{82}, ..., x_{8n_8})$$

Therefore, Likelihood function of combined sample is :

$$L = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 | \alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)$$

Now pdf of exponential distribution is :$\lambda \exp(-\lambda x)$

$$L = {\alpha_1}^{n_1} {\alpha_2}^{n_2} {\alpha_3}^{n_3} {\lambda_1}^{n_4} {\lambda_2}^{n_5} {\lambda_3}^{n_6} {\gamma_1}^{n_7} {\gamma_2}^{n_8} \exp -(\alpha_1 W_1 + \alpha_2 W_2 + \alpha_3 W_3 + \lambda_1 W_4 + \lambda_2 W_5 + \lambda_3 W_6 + \lambda_4 W_7 + \beta W_8)$$

Here,

$$Wi = \sum_{n=1}^{n_i} x_{i_j}; 1 = 1, 2, 3, 4, 5, 6, 7, 8$$

Taking log b/s, we get

$$\begin{aligned} logL =& n_1 log\alpha_1 + n_2 log\alpha_2 + n_3 log\alpha_3 + n_4 log\lambda_1 + n_5 log\lambda_2 + n_6 log\lambda_3 + n_7 log\lambda_4 + n_8 log\beta_1 \\ & - (\alpha_1 W_1 + \alpha_2 W_2 + \alpha_3 W_3 + \lambda_1 W_4 + \lambda_2 W_5 + \lambda_3 W_5 + \lambda_4 W_6 + \beta W_8) \end{aligned} \tag{7}$$

Now we obtain, MLE $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3\ , \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4\ , \hat{\beta})$of the parameters $(\alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)$ as

$\hat{\alpha}_1 = \frac{n_1}{W_1}$, $\hat{\alpha}_2 = \frac{n_2}{W_2}$
$\hat{\alpha}_3 = \frac{n_3}{W_3}$, $\hat{\lambda}_1 = \frac{n_4}{W_4}$
$\hat{\lambda}_2 = \frac{n_5}{W_5}$, $\hat{\lambda}_3 = \frac{n_6}{W_6}$
$\hat{\lambda}_4 = \frac{n_7}{W_7}$, $\hat{\beta} = \frac{n_8}{W_8}$

The asymptotic distribution of$(\hat{\alpha_1} - \alpha_1, \hat{\alpha_2} - \alpha_2, \hat{\alpha_3} - \alpha_3, \hat{\lambda_1} - \lambda_1, \hat{\lambda_2} - \lambda_2, \hat{\lambda_3} - \lambda_3, \hat{\lambda_4} - \lambda_4, \hat{\beta} - \beta) \sim N_8(0, I^{-1})$, where I is the Fisher Information matrix with diagonal elements as

$I_{11} = \frac{n_1}{\alpha_1^2}$, $I_{22} = \frac{n_2}{\alpha_2^2}$, $I_{33} = \frac{n_3}{\alpha_3^2}$, $I_{44} = \frac{n_4}{\lambda_1^2}$, $I_{55} = \frac{n_5}{\lambda_2^2}$, $I_{66} = \frac{n_6}{\lambda_3^2}$, $I_{77} = \frac{n_7}{\lambda_4^2}$, $I_{88} = \frac{n_8}{\beta^2}$

and all non-diagonal elements are zero. The MLE $\hat{M}$ & $\hat{P}$ of MTSF and Profit function can be obtained using the invariance property of MLE. Also the asymptotic distribution of $(\hat{M} - M) is N(0, A' I^{-1} A)$ & that of $(\hat{P} - P) is N(0, B' I^{-1} B)$, where

$A' = (\frac{\delta M}{\delta \alpha_1}, \frac{\delta M}{\delta \alpha_2}, \frac{\delta M}{\delta \alpha_3}, \frac{\delta M}{\delta \alpha_2}, \frac{\delta M}{\delta \lambda_2}, \frac{\delta M}{\delta \lambda_3}, \frac{\delta M}{\delta \lambda_4}, \frac{\delta M}{\delta \beta})$

$B' = (\frac{\delta P}{\delta \alpha_1}, \frac{\delta P}{\delta \alpha_2}, \frac{\delta P}{\delta \alpha_3}, \frac{\delta P}{\delta \lambda_1}, \frac{\delta P}{\delta \lambda_2}, \frac{\delta P}{\delta \lambda_3}, \frac{\delta P}{\delta \lambda_4}, \frac{\delta P}{\delta \beta})$

10.2. Bayesian Estimation

Bayesian estimation is a statistical approach used to determine the impact of prior knowledge as well as the sample information.The Bayesian method of estimation is taken into consideration in this part for estimating the model parameters.The parameters involved in the model are random variables having independent Gamma prior distribution.Here, we estimate the unknown parameters while taking into account the gamma prior distribution and the corresponding PDFs.

$$\alpha_1 \sim Gamma(a_1, b_1) \qquad (\alpha_1, a_1, b_1) > 0, \tag{8}$$

$$\alpha_2 \sim Gamma(a_2, b_2) \qquad (\alpha_2, a_2, b_2) > 0, \tag{9}$$

$$\alpha_3 \sim Gamma(a_3, b_3) \qquad (\alpha_3, a_3, b_3) > 0, \tag{10}$$

$$\lambda_1 \sim Gamma(a_4, b_4) \qquad (\lambda_1, a_4, b_4) > 0, \tag{11}$$

$$\lambda_2 \sim Gamma(a_5, b_5) \qquad (\lambda_2, a_5, b_5) > 0, \tag{12}$$

$$\lambda_3 \sim Gamma(a_6, b_6) \qquad (\lambda_3, a_6, b_6) > 0, \tag{13}$$

$$\lambda_4 \sim Gamma(a_7, b_7) \qquad (\lambda_4, a_7, b_7) > 0, \tag{14}$$

$$\beta \sim Gamma(a_8, b_8) \qquad (\beta, a_8, b_8) > 0, \tag{15}$$

Here,$a_i\ and\ b_i$ (i = 1,2,3,4,5,6,7,8) denotes the shape and scale parameters. Now using likelihood function and taking prior distributions, the posterior distributions of these parameters is obtained as follows:

$$\alpha_1 | \underset{\sim}{X_1} \sim Gamma(n_1 + a_1, b_1 + W_1) \tag{16}$$

$$\alpha_2 | \underset{\sim}{X_2} \sim Gamma(n_2 + a_2, b_2 + W_2) \tag{17}$$

$$\alpha_3 | \underset{\sim}{X_3} \sim Gamma(n_3 + a_3, b_3 + W_3) \tag{18}$$

$$\lambda_1|\underset{\sim}{X_4} \sim Gamma(n_4 + a_4, b_4 + W_4) \tag{19}$$

$$\lambda_2|\underset{\sim}{X_5} \sim Gamma(n_5 + a_5, b_5 + W_5) \tag{20}$$

$$\lambda_3|\underset{\sim}{X_6} \sim Gamma(n_6 + a_6, b_6 + W_6) \tag{21}$$

$$\lambda_4|\underset{\sim}{X_7} \sim Gamma(n_7 + a_7, b_7 + W_7) \tag{22}$$

$$\beta|\underset{\sim}{X_8} \sim Gamma(n_8 + a_8, b_8 + W_8) \tag{23}$$

To obtain Bayes estimates and width of HPD intervals of the parameters,we generated observations from the above posterior distributions. For attaining Bayesian estimation of MTSF and profit function we substituted the above draws directly into the equations [2] & [6]. Assuming square error loss function, the sample means of respective draws are taken as Bayes estimates of parameters and reliability characteristics.

11. Simulation Study

A simulation research is carried out to investigate the behaviour of parameters estimates and reliability features. The Standard Error(SE)/Posterior Standard Error(PSE) and width of confidence/HPD intervals are computed and are given in Table 1-6. Samples of sizes $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8 = 100$ have been drawn from the six considered distributions by assuming various values of the parameters as given in Tables 1-6. The number of iterations taken are 10000. All calculations are performed in R software.

Table 1: *The values of MTSF for fixed $\lambda_1 = 0.15$ and varying α_1*

α_1	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	15.729	10.11	0.0042	0.068	10.05	0.00098	0.00076
0.2	9.114	5.03	0.0032	0.034	5.043	0.00075	0.00062
0.3	6.909	3.39	0.0029	0.024	3.369	0.00050	0.00049
0.4	5.807	2.55	0.0027	0.021	2.536	0.00044	0.00037
0.5	5.145	2.06	0.0025	0.019	2.032	0.00047	0.00028
0.6	4.704	1.70	0.0025	0.018	1.697	0.00042	0.00038
0.7	4.389	1.47	0.010	0.018	1.456	0.00044	0.00034
0.8	4.1536	1.28	0.0024	0.017	1.282	0.00049	0.00030
0.9	3.969	1.14	0.0024	0.017	1.138	0.00037	0.00025
1	3.822	1.02	0.0024	0.017	1.029	0.00047	0.00038

Table 2: *The values of MTSF for fixed λ_1=0.35 and varying α_1*

α_1	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	20.034	10.15	0.0068	0.069	10.109	0.0022	0.0012
0.2	11.267	5.10	0.0045	0.034	5.065	0.0011	0.00069
0.3	8.344	3.42	0.0037	0.025	3.387	0.00073	0.00055
0.4	6.883	2.55	0.0032	0.021	2.545	0.00061	0.00063
0.5	6.006	2.10	0.0031	0.019	2.038	0.00061	0.00056
0.6	5.422	1.70	0.0029	0.018	1.707	0.00060	0.00043
0.7	5.005	1.48	0.0028	0.018	1.464	0.00080	0.00049
0.8	4.691	1.32	0.0027	0.018	1.283	0.00061	0.00038
0.9	4.448	1.16	0.0026	0.017	1.145	0.00049	0.00044
1	4.253	1.08	0.0026	0.017	1.034	0.00059	0.00030

Table 3: *The values of MTSF for fixed λ_1=0.65 and varying α_1*

α_1	True MTSF	MlE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	26.493	10.18	0.011	0.070	10.1892	0.0024	0.0019
0.2	14.496	5.15	0.0066	0.035	5.114	0.0016	0.0014
0.3	10.497	3.47	0.0050	0.025	3.418	0.0010	0.00083
0.4	8.498	2.57	0.0043	0.021	2.563	0.0010	0.00088
0.5	7.298	2.10	0.0038	0.019	2.061	0.0010	0.00056
0.6	6.498	1.73	0.0036	0.018	1.719	0.0010	0.00063
0.7	5.927	1.48	0.0034	0.018	1.478	0.00085	0.00046
0.8	5.499	1.31	0.0032	0.018	1.297	0.00063	0.00060
0.9	5.165	1.18	0.0031	0.017	1.152	0.00057	0.00052
1	4.899	1.08	0.0030	0.017	1.040	0.00053	0.00052

Table 4: *The values of Profit for fixed λ_1=0.15 and varying α_1*

α_1	True profit	MlE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	565.66	504.78	498.53	575.20	479.28	397.45	291.21
0.2	402.36	383.26	397.36	810.27	350.27	323.22	159.32
0.3	311.70	317.39	635.72	636.85	265.39	63.56	46.57
0.4	254.26	275.92	526.51	951.11	204.74	45.93	33.66
0.5	214.62	247.42	877.84	464.26	159.05	743.24	544.48
0.6	185.61	226.48	376.34	961.33	123.31	25.80	18.91
0.7	163.44	210.38	631.29	451.36	94.53	24.65	18.06
0.8	145.92	197.58	662.10	739.46	70.84	20.19	14.79
0.9	131.71	187.11	718.50	516.67	50.96	81.40	59.64
1	119.95	178.39	643.89	545.35	34.03	15.60	10.70

Table 5: *The values of Profit for fixed λ_1=0.35 and varying α_1*

α_1	True profit	MlE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	821.06	543.02	602.82	626.98	629.57	319.08	233.79
0.2	620.12	419.43	689.35	691.46	547.35	275.89	161.56
0.3	496.74	345.93	678.57	543.09	483.00	141.41	103.61
0.4	414.10	298.73	653.09	802.17	430.65	195.75	143.42
0.5	355.20	266.05	632.30	583.35	387.01	89.59	65.64
0.6	311.25	242.08	652.66	1043.69	249.95	139.17	101.97
0.7	277.25	223.72	750.20	1033.47	318.03	28.31	20.74
0.8	250.25	209.17	527.56	876.81	290.21	156.78	114.87
0.9	228.19	3197.35	712.67	607.05	265.74	51.75	37.92
1	209.93	187.52	3884.21	993.43	244.01	89.38	65.49

Table 6: *The values of Profit for fixed λ_1=0.65 and varying α_1*

α_1	True profit	MlE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	1021.3	547.18	893.57	364.97	718.30	249.01	182.45
0.2	813.1	441.50	531.40	436.79	667.84	526.05	110.66
0.3	672.6	369.25	580.49	481.96	624.46	405.31	296.97
0.4	572.1	319.94	405.80	313.23	586.36	290.30	212.70
0.5	497.1	284.78	613.66	522.86	552.48	191.82	140.54
0.6	439.2	258.59	624.87	678.83	522.07	67.79	49.67
0.7	393.4	238.37	554.60	720.39	494.59	71.09	52.09
0.8	356.2	222.28	766.52	416.82	469.60	89.92	65.89
0.9	325.5	209.17	733.96	552.96	446.77	52.28	72.69
1	299.8	198.27	788.98	788.43	425.81	30.95	22.68

12. Graphical Study

A graphical analysis of the system model provides a more insightful and vivid representation of system behaviour, allowing for a clear visualization of failure transitions, repair processes, and overall system performance. So for more concrete study, we plot MTSF and Profit function wrt α_1 failure rate of unit A for different values of λ_1 repair rate of unit A as 0.15, 0.35 and 0.65. Here all other parameters are fixed α_2= 0.007, α_3= 0.009,λ_2=0.09,λ_3=0.06,$\lambda_4 = 0.07$ β=0.7,K_0=1500,K_1=300 and K_2=250.

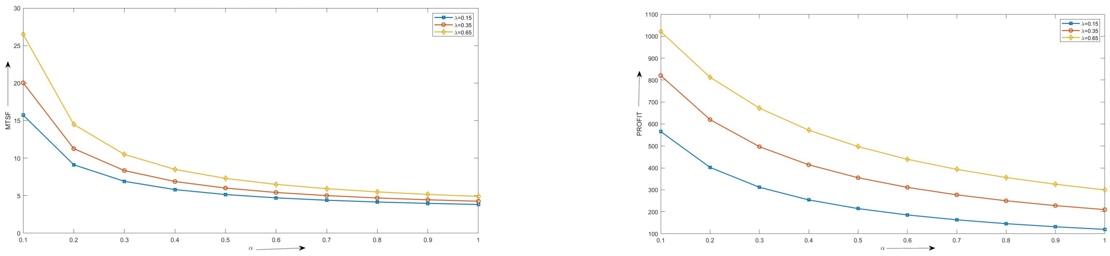

Figure 2: *(a)Behaviour of MTSF wrt to α_1 for different values of λ_1 and (b)Behaviour of Profit wrt to α_1 for different values of λ_1*

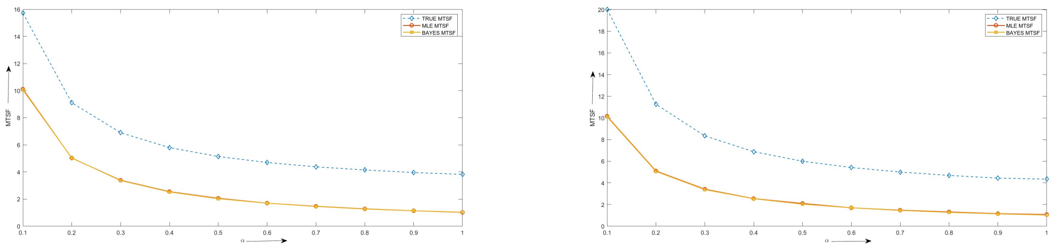

Figure 3: *(a)Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_1 for λ_1= 0.15 and (b) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_1 for λ_1= 0.35*

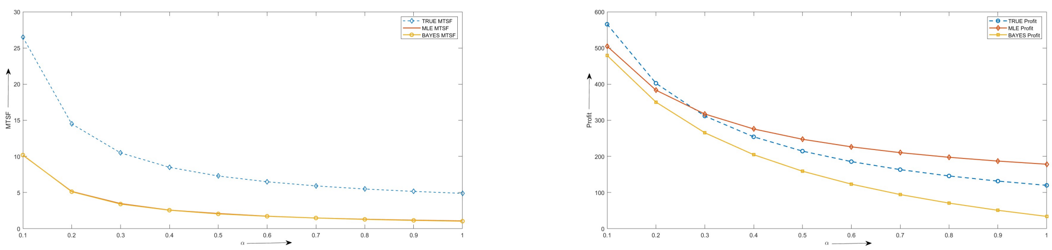

Figure 4: *(a) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_1 for λ_1= 0.65 and (b) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_1 for λ_1= 0.15*

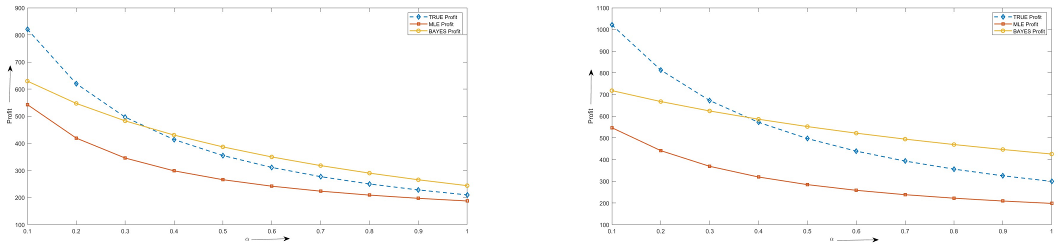

Figure 5: *(a)Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_1 for λ_1= 0.35 and (b) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_1 for λ_1= 0.65*

13. Discussion and Conclusion

From tables and figures we may conclude that MTSF decreases as the failure rate α_1 increases whereas it increases as the repair rate λ_1 increases. Same trends follows for profit function. Additionally, Tables 1-6 show that for fixed and variable parameters, Bayes estimates of the MTSF and profit function are better than MLEs because they have lower PSE, and the width of HPD intervals is smaller than the width of confidence intervals. Therefore based on this analysis we conclude that the Bayes approach is superior to the Classical approach for estimating the MTSF and profit function for the examined model.

14. Acknowledgement

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