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ON THE PROTECTION AGAINST NOISE FOR MEASUREMENT-BASED QUANTUM COMPUTATION

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Abstract

In this paper we establish conditions for some pairs of quantum logic gates which operate on one qubit to be protected against crosstalk. We represent these logic gates by self-adjoint operators acting on an unitary plane, consider the resembling bipartite quantum systems, and extend the operators naturally to a commuting pair. Thus the corresponding two measurements can be considered as simultaneous and we can define them as events in a classical probability space. In particular, we are able to study their dependence, the information noise thus produced, and to find necessary and sufficient conditions for being informationally independent events.

Keywords: Quantum computation, Logical gates, Noise, Informational independence.

1. Introduction, Notation

1.1. Notation

The following notation will be frequently used in this paper:
$\delta_{k,\ell}$: Kronecker's delta;
$\mathbb{Z}_2 = \{0, 1\}$: the additive group of two elements;
$\mathcal{H}$: 2-dimensional unitary space with inner product $\langle u|v\rangle$ of vectors $|u\rangle$ and $|v\rangle$;
$\{|0\rangle, |1\rangle\}$: orthonormal frame for $\mathcal{H}$, sometimes called the computational basis;
$\mathbb{I}_{\mathcal{H}}$: the identity linear operator on $\mathcal{H}$;
$\mathrm{Spec}(A)$: the spectre of a linear operator A on $\mathcal{H}$;

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}:$$

Pauli matrices and Hadamard matrix, respectively;
$\mathcal{E}$: the 3-dimensional real linear space of all self-adjoined operators on $\mathcal{H}$ with trace 0, which is furnished with orthonormal coordinates x, y, z;
$\mathbb{R}_+(A) = \{[(cx, cy, cz)|c \in \mathbb{R}, c \geq 0\}$: direction of the operator $A \in \mathcal{E}$ with coordinates (x, y, z);
$\mathcal{H}^{\otimes 2}$: the tensor square of $\mathcal{H}$ provided with the standard structure of a unitary space, the tensor product of vectors $|u\rangle$ and $|v\rangle$ being denoted $|uv\rangle$;
$\mathcal{U}^{(2)}$: the unit sphere of $\mathcal{H}^{\otimes 2}$;
$\psi_{s,t} = \frac{1}{\sqrt{2}}(|0t\rangle + (-1)^s|1(t+1)\rangle)$, $\psi_{s,t} \in \mathcal{U}^{(2)}$, $s, t \in \mathbb{Z}_2$: the four Bell's states;
We denote: $\mathrm{tr}_0\, \alpha = \cos\alpha$, $\mathrm{tr}_1\, \alpha = \sin\alpha$, $\alpha \in \mathbb{R}$.
We denote: $P(\alpha, \beta) = \mathrm{tr}_0\, \alpha\, \mathrm{tr}_0\, \beta\, \mathrm{tr}_1\, \alpha\, \mathrm{tr}_1\, \beta$, $\alpha, \beta \in \mathbb{R}$.

1.2. Introduction

Some of the logic gates in a quantum computational network can be represented by self-adjoined operators A, B, ... with spectre $\{1,-1\}$ on the unitary plane $\mathcal{H}$. Their set coincides with unit sphere of the linear space $\mathcal{E}$. After appropriate tensoring with the identity operator, any pair A, B of such logic operations generates a pair $\mathcal{A}$ and $\mathcal{B}$ of self-adjoined commuting operators with the same spectre, which are defined on the bipartite quantum system $\mathcal{H}^{\otimes 2}$. In particular, the measurements of the observables corresponding to these operators can be considered as simultaneous. In other words, we consider as an event "the outcome of measuring $\mathcal{A}$ is λ_k *and* the outcome of measuring $\mathcal{B}$ is λ_ℓ ", where $\lambda_k, \lambda_\ell \in \{1,-1\}$. We can suppose without loss of generality that $k, \ell \in \mathbb{Z}_2$, and that $\lambda_0 = 1$, $\lambda_1 = -1$. Born rule allows us to interpret the above conjunction as an intersection of two events in a sample space and this is done in Section 2. Namely, the tensor product of orthonormal frames of the unitary plane, consisting of eigenvectors of A and B constitutes an orthonormal frame of $\mathcal{H}^{\otimes 2}$ whose members are common eigenvectors of both $\mathcal{A}$ and $\mathcal{B}$. We chose this frame as the set of outcomes of a sample space $S(\psi; \mathcal{A}, \mathcal{B})$ with probability assignment $p_{k,\ell}$ created by quantum theory via fixing a state $\psi \in \mathcal{H}^{\otimes 2}$ and via Born rule. It turns out that $p_{k,\ell}$ is the probability of the intersection of the events $\mathcal{A} = \lambda_k$ and $\mathcal{B} = \lambda_\ell$ in this sample space. In order to find the probability $p_{k,\ell}$, we identify the self-adjoined operators with their matrices with respect to the computational basis for $\mathcal{H}$ and express the matrices A and B by using polar coordinates, $A = A_{\mu,\eta}$ and $B = A_{\nu,\zeta}$, see Theorem 2. Here μ, ν are polar angles and η, ζ are azimuthal angles.

Under the condition $\psi = \psi_{s,t}$ for some $s, t \in \mathbb{Z}_2$, (one of the four Bell's states) Theorem 3 yields that the probabilities of all events $\mathcal{A} = \lambda_k$ and $\mathcal{B} = \lambda_\ell$, $k, \ell \in \mathbb{Z}_2$, are equal and that the quantity $p_{k,\ell}$ is invariant with respect to the natural action of the group $\mathbb{Z}_2$ onto the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, the probability assignment of the sample space $S(\psi_{s,t}; \mathcal{A}, \mathcal{B})$ can be written in the form $(p_{k,k}, p_{k,k+1}, p_{k,k+1}, p_{k,k})$ for $k \in \mathbb{Z}_2$.

In Section 3 we consider the two binary trials $\mathfrak{A} = (\mathcal{A} = 1) \cup (\mathcal{A} = -1)$, $\mathfrak{B} = (\mathcal{B} = 1) \cup (\mathcal{B} = -1)$ and make use of the average quantity of information of one of the experiments $\mathfrak{A}$ and $\mathfrak{B}$ relative to the other (the *information flow*, or, *noise*, or, *crosstalk* between them), see [1, §1], defined in this particular case by the Shannon's formula [2, 5.3, (6)].

In accord with [2], the joint experiment of the above binary trials gives rise to the same probability distribution $(p_{k,k}, p_{k,k+1}, p_{k,k+1}, p_{k,k})$, $k \in \mathbb{Z}_2$. Via modification of its entropy, we bring out a function which measures the degree of dependence of the events $\mathcal{A} = \lambda_k$ and $\mathcal{B} = \lambda_\ell$. The above events are independent (the entropy is maximal) if and only if the crosstalk between $\mathfrak{A}$ and $\mathfrak{B}$ is zero. In this case we also say that the measurements of the observables corresponding to $\mathcal{A}$ and $\mathcal{B}$ are *informationally independent*. Thus, following [2, 5.2], we conclude that the above events are informationally independent exactly when the equation $p_{k,k} = \frac{1}{4}$ is satisfied.

Section 4 is devoted to three particular cases when the expression for $p_{k,k}$ from Theorem 3, (ii), (iii), has very simple form and the above equation can be solved explicitly in terms of sums or differences of polar or azimuthal angles. Those cases are defined by the property that both operators A, B have simultaneously directions $\mathbb{R}_+(A)$, $\mathbb{R}_+(B)$ laying in one of the three coordinate planes $x = 0$, $y = 0$, $z = 0$ of $\mathcal{E}$. In Subsection 4.1 we consider the set of operators with direction in coordinate plane $x = 0$ (a circle on the unit sphere of $\mathcal{E}$). Pauli matrices σ_2 and σ_3 belong to this set. The results of measurements performed on the observables $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$ are informationally independent if and only if $\mu + \nu = \frac{\pi}{2}$ or $\mu + \nu = \frac{3\pi}{2}$ in case $\psi = \frac{1}{\sqrt{2}}(|0t\rangle + |1(t+1)\rangle)$ for some $t \in \mathbb{Z}_2$, and if and only if $|\mu - \nu| = \frac{\pi}{2}$ in case $\psi = \frac{1}{\sqrt{2}}(|0t\rangle - |1(t+1)\rangle)$ for some $t \in \mathbb{Z}_2$.

We study the set of operators with direction in coordinate plane $y = 0$ in Subsection 4.2. Pauli matrices σ_1 and σ_3, as well as Hadamard matrix H, are members of this set. It turns out that the outcomes of measurement of observables $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$ are informationally independent exactly in case $\mu + \nu = \frac{\pi}{2}$ or $\mu + \nu = \frac{3\pi}{2}$, when $\psi = \frac{1}{\sqrt{2}}(|0(s+1)\rangle + (-1)^s|1s\rangle)$ for some $s \in \mathbb{Z}_2$, and in case $|\mu - \nu| = \frac{\pi}{2}$, when $\psi = \frac{1}{\sqrt{2}}(|0s\rangle + (-1)^s|1(s+1)\rangle)$ for some $s \in \mathbb{Z}_2$.

Finally, in Subsection 4.3 we examine the set of operators with direction in coordinate plane

$z = 0$. Pauli operators σ_1 and σ_2 belong there. In this case the measurements performed by the observables $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$ are informationally independent exactly in case $\eta + \zeta \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\}$, when $\psi = \frac{1}{\sqrt{2}}(|0,0\rangle + (-1)^s|1,1\rangle)$ for some $s \in \mathbb{Z}_2$, and in case $|\eta - \zeta| = \frac{\pi}{2}$ or $|\eta - \zeta| = \frac{3\pi}{2}$, when $\psi = \frac{1}{\sqrt{2}}(|0,1\rangle + (-1)^s|1,0\rangle)$ for some $s \in \mathbb{Z}_2$.

When all was said and done, we hope that the above conditions for absence of crosstalk between two interacting logic gates can be checked experimentally.

2. Two Commuting Operators on $\mathcal{H}^{\otimes 2}$

The self-adjoined operators on $\mathcal{H}$ with spectre $\{1, -1\}$ have, in general, the form

$$A_{\mu,\eta} = \begin{pmatrix} \cos\mu & e^{-i\eta}\sin\mu \\ e^{i\eta}\sin\mu & -\cos\mu \end{pmatrix},$$

where $\mu \in [0, \pi]$ is the polar angle and $\eta \in [0, 2\pi)$ is the azimuthal angle. The polar coordinates $x = \sin\mu\cos\eta$, $y = \sin\mu\sin\eta$, $z = \cos\mu$, establish an isomorphism of the set of above operators and the unit sphere in $\mathcal{E}$.

We denote $B_{\nu,\zeta} = A_{\nu,\zeta}$, $\lambda_0 = 1$, $\lambda_1 = -1$. Corresponding (normalized) eigenvectors are

$$u^{(k)}_{\mu,\eta} = (-1)^k e^{-i\eta}\,\mathrm{tr}_k\,\frac{\mu}{2}|0\rangle + \mathrm{tr}_{k+1}\,\frac{\mu}{2}|1\rangle, k \in \mathbb{Z}_2.$$

Moreover, $H^{(k)}_{\mu,\eta} = \mathbb{C}u^{(k)}_{\mu,\eta}$ is its λ_k-eigenspace. Note that

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \text{ and } u_{\mu,\eta}u_{\nu,\zeta} = \{|u^{(0)}_{\mu,\eta}u^{(0)}_{\nu,\zeta}\rangle, |u^{(0)}_{\mu,\eta}u^{(1)}_{\nu,\zeta}\rangle, |u^{(1)}_{\mu,\eta}u^{(0)}_{\nu,\zeta}\rangle, |u^{(1)}_{\mu,\eta}u^{(1)}_{\nu,\zeta}\rangle\}$$

are orthonormal frames for $\mathcal{H}^{\otimes 2}$.

Let us set $\mathcal{A}_{\mu,\eta} = A_{\mu,\eta} \otimes \mathbb{I}_{\mathcal{H}}$, $\mathcal{B}_{\nu,\zeta} = \mathbb{I}_{\mathcal{H}} \otimes B_{\nu,\zeta}$. It is a straightforward check that the last two linear operators on $\mathcal{H}^{\otimes 2}$ are also self-adjoined with spectre $\{1, -1\}$. Moreover, the λ_k-eigenspace $\mathcal{H}^{(k)}_{\mathcal{A}_{\mu,\eta}} = H^{(k)}_{\mu,\eta} \otimes \mathcal{H}$ of the operator $\mathcal{A}_{\mu,\eta}$ has orthonormal frame $\{|u^{(k)}_{\mu,\eta}u^{(0)}_{\nu,\zeta}\rangle, |u^{(k)}_{\mu,\eta}u^{(1)}_{\nu,\zeta}\rangle\}$, and the λ_ℓ-eigenspace $\mathcal{H}^{(\ell)}_{\mathcal{B}_{\nu,\zeta}} = \mathcal{H} \otimes H^{(\ell)}_{\nu,\zeta}$ of the operator $\mathcal{B}_{\nu,\zeta}$ has orthonormal frame $\{|u^{(0)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle, |u^{(1)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle\}$, $k, \ell \in \mathbb{Z}_2$.

Since $u_{\mu,\eta}u_{\nu,\zeta}$ is an orthonornal frame of $\mathcal{H}^{\otimes 2}$ consisting of eigenvectors of both $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$, then the last two operators commute. In other words, the outcomes of measurements of these observables can be thought as simultaneous (the order is irrelevant) and the quantum theory predicts the probabilities of both outcomes by employing the frame $u_{\mu,\eta}u_{\nu,\zeta}$. Formally, this is done below.

2.1. A Sample Space and Two Random Variables

Let $\psi \in \mathcal{U}^{(2)}$ and let $S(\psi; \mathcal{A}, \mathcal{B})$ be the sample space with set of outcomes $u_{\mu,\eta}u_{\nu,\zeta}$ and probability assignment $p_{k,\ell} = |\langle|u^{(k)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle|\psi\rangle|^2$, $k, \ell \in \mathbb{Z}_2$.

With an abuse of the language, we consider the observable $\mathcal{A}_{\mu,\eta}$ as a random variable

$$\mathcal{A}_{\mu,\eta} \colon u_{\mu,\eta}u_{\nu,\zeta} \to \mathbb{R}, \mathcal{A}_{\mu,\eta}(|u^{(0)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle) = 1, \mathcal{A}_{\mu,\eta}(|u^{(1)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle) = -1, \ell \in \mathbb{Z}_2,$$

on the sample space $S(\psi; \mathcal{A}, \mathcal{B})$ with probability distribution

$$p_{\mathcal{A}_{\mu,\eta}}(\lambda_k) = p_{k,0} + p_{k,1}, k \in \mathbb{Z}_2, p_{\mathcal{A}_{\mu,\eta}}(\lambda) = 0, \lambda \notin \mathrm{Spec}(\mathcal{A}_{\mu,\eta}).$$

Identifying the event $\{|u^{(k)}_{\mu,\eta}u^{(0)}_{\nu,\zeta}\rangle, |u^{(k)}_{\mu,\eta}u^{(1)}_{\nu,\zeta}\rangle\}$ with the "event" $\mathcal{A}_{\mu,\eta} = \lambda_k$ (the result of the measurement), we have $\mathrm{pr}(\mathcal{A}_{\mu,\eta} = \lambda_k) = p_{k,0} + p_{k,1}$, $k \in \mathbb{Z}_2$.

We also consider the observable $\mathcal{B}_{\nu,\zeta}$ as a random variable

$$\mathcal{B}_{\nu,\zeta}\colon u_{\mu,\eta}u_{\nu,\zeta}\to\mathbb{R}, \mathcal{B}_{\nu,\zeta}(|u^{(k)}_{\mu,\eta}u^{(0)}_{\nu,\zeta}\rangle)=1, \mathcal{B}_{\nu,\zeta}(|u^{(k)}_{\mu,\eta}u^{(1)}_{\nu,\zeta}\rangle)=-1, k\in\mathbb{Z}_2,$$

on the above sample space with probability distribution

$$p_{\mathcal{B}_{\nu,\zeta}}(\lambda_\ell)=p_{0,\ell}+p_{1,\ell}, \ell\in\mathbb{Z}_2, p_{\mathcal{B}_{\nu,\zeta}}(\lambda)=0, \lambda\notin\operatorname{Spec}(\mathcal{B}_{\nu,\zeta}).$$

Identifying the event $\{|u^{(0)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle, |u^{(1)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle\}$ with the "event" $\mathcal{B}_{\nu,\zeta}=\lambda_\ell$, we have $\operatorname{pr}(\mathcal{B}_{\nu,\zeta}=\lambda_\ell)=p_{0,\ell}+p_{1,\ell}$, $\ell\in\mathbb{Z}_2$. Moreover, the equality $(\mathcal{A}_{\mu,\eta}=\lambda_k)\cap(\mathcal{B}_{\nu,\zeta}=\lambda_\ell)=\{|u^{(k)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle\}$ yields $\operatorname{pr}((\mathcal{A}_{\mu,\eta}=\lambda_k)\cap(\mathcal{B}_{\nu,\zeta}=\lambda_\ell))=p_{k,\ell}$, $k,\ell\in\mathbb{Z}_2$.

We obtain immediately

Proposition 1. The following two statements are equivalent:
(i) One has $p_{0,0}=p_{1,1}$ and $p_{0,1}=p_{1,0}$.
(ii) For all $k,\ell\in\mathbb{Z}_2$ one has $\operatorname{pr}(\mathcal{A}_{\mu,\eta}=\lambda_k)=\operatorname{pr}(\mathcal{B}_{\nu,\zeta}=\lambda_\ell)$.

Theorem 2. Let $\psi=\psi_{s,t}$ where $s,t\in\mathbb{Z}_2$. Then for all $k,\ell\in\mathbb{Z}_2$ one has

$$p_{k,\ell}=\frac{1}{2}|(-1)^{k+\ell}e^{-\mathrm{i}(\eta+\zeta)}\operatorname{tr}_k\frac{\mu}{2}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t}+(-1)^k e^{-\mathrm{i}\eta}\operatorname{tr}_k\frac{\mu}{2}\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t}+$$
$$(-1)^{s+\ell}e^{-\mathrm{i}\zeta}\operatorname{tr}_{k+1}\frac{\mu}{2}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t+1}+(-1)^s\operatorname{tr}_{k+1}\frac{\mu}{2}\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t+1}|^2.$$

Proof. We have

$$p_{k,\ell}=|\langle u^{(k)}_{\mu,\eta}u^{(\ell)}_{\nu,\zeta}\rangle|\psi_{s,t}\rangle|^2=\frac{1}{2}|\langle u^{(k)}_{\mu,\eta}|0\rangle\langle u^{(\ell)}_{\nu,\zeta}|t\rangle+(-1)^s\langle u^{(k)}_{\mu,\eta}|1\rangle\langle u^{(\ell)}_{\nu,\zeta}|t+1\rangle|^2=$$
$$\frac{1}{2}|((-1)^k e^{-\mathrm{i}\eta}\operatorname{tr}_k\frac{\mu}{2}\delta_{0,0}+\operatorname{tr}_{k+1}\frac{\mu}{2}\delta_{1,0})((-1)^\ell e^{-\mathrm{i}\zeta}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t}+\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t})+$$
$$(-1)^s((-1)^k e^{-\mathrm{i}\eta}\operatorname{tr}_k\frac{\mu}{2}\delta_{0,1}+\operatorname{tr}_{k+1}\frac{\mu}{2}\delta_{1,1})((-1)^\ell e^{-\mathrm{i}\zeta}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t+1}+\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t+1})|^2=$$
$$\frac{1}{2}|(-1)^k e^{-\mathrm{i}\eta}\operatorname{tr}_k\frac{\mu}{2}((-1)^\ell e^{-\mathrm{i}\zeta}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t}+\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t})+$$
$$(-1)^s\operatorname{tr}_{k+1}\frac{\mu}{2}((-1)^\ell e^{-\mathrm{i}\zeta}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t+1}+\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t+1})|^2=$$
$$\frac{1}{2}|(-1)^{k+\ell}e^{-\mathrm{i}(\eta+\zeta)}\operatorname{tr}_k\frac{\mu}{2}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t}+(-1)^k e^{-\mathrm{i}\eta}\operatorname{tr}_k\frac{\mu}{2}\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t}+$$
$$(-1)^{s+\ell}e^{-\mathrm{i}\zeta}\operatorname{tr}_{k+1}\frac{\mu}{2}\operatorname{tr}_\ell\frac{\nu}{2}\delta_{0,t+1}+(-1)^s\operatorname{tr}_{k+1}\frac{\mu}{2}\operatorname{tr}_{\ell+1}\frac{\nu}{2}\delta_{1,t+1}|^2.$$

■

2.2. The Probability Assignment of the Sample Space $S(\psi;\mathcal{A},\mathcal{B})$

Below we express the probabilities $p_{k,\ell}$, $k,\ell\in\mathbb{Z}_2$, as functions in sums $\mu+(-1)^s\nu$ and $\eta+(-1)^t\zeta$, $s,t\in\mathbb{Z}_2$, by using Theorem 2.

Theorem 3. Let $\psi=\psi_{s,t}$ for some $s,t\in\mathbb{Z}_2$.
(i) One has $\operatorname{pr}(\mathcal{A}_{\mu,\eta}=\lambda_k)=\operatorname{pr}(\mathcal{B}_{\nu,\zeta}=\lambda_\ell)=\frac{1}{2}$ for any $k,\ell\in\mathbb{Z}_2$.
(ii) One has

$$p_{0,0}=p_{1,1}=$$
$$\frac{1}{2}\operatorname{tr}^2_t\frac{\mu+(-1)^s\nu}{2}+2(-1)^{s+t}\operatorname{tr}^2_t\frac{\eta+(-1)^t\zeta}{2}P\left(\frac{\mu}{2},\frac{\nu}{2}\right)=$$
$$\frac{1}{2}\operatorname{tr}^2_t\frac{\mu+(-1)^{s+1}\nu}{2}+2(-1)^{s+t+1}\operatorname{tr}^2_{t+1}\frac{\eta+(-1)^t\zeta}{2}P\left(\frac{\mu}{2},\frac{\nu}{2}\right)$$

for any $s,t \in \mathbb{Z}_2$.

(iii) One has

$$p_{0,1} = p_{1,0} =$$
$$\frac{1}{2}\operatorname{tr}^2_{t+1}\frac{\mu+(-1)^s\nu}{2} + 2(-1)^{s+t+1}\operatorname{tr}^2_t\frac{\eta+(-1)^t\zeta}{2}P\left(\frac{\mu}{2},\frac{\nu}{2}\right) =$$
$$\frac{1}{2}\operatorname{tr}^2_{t+1}\frac{\mu+(-1)^{s+1}\nu}{2} + 2(-1)^{s+t}\operatorname{tr}^2_{t+1}\frac{\eta+(-1)^t\zeta}{2}P\left(\frac{\mu}{2},\frac{\nu}{2}\right)$$

for any $s,t \in \mathbb{Z}_2$.

Proof. (i) In case $t=0$ Theorem 2 yields

$$p_{0,0} = \frac{1}{2}|e^{-\mathrm{i}(\eta+\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2} + (-1)^s\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2}|^2,$$

$$p_{1,1} = \frac{1}{2}|e^{\mathrm{i}(\eta+\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2} + (-1)^s\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2}|^2,$$

and hence $p_{0,0} = p_{1,1}$. Similarly, we obtain

$$p_{0,1} = \frac{1}{2}|e^{-\mathrm{i}(\eta+\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} + (-1)^{s+1}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}|^2,$$

$$p_{1,0} = \frac{1}{2}|e^{\mathrm{i}(\eta+\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} + (-1)^{s+1}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}|^2,$$

and therefore $p_{0,1} = p_{1,0}$.

In case $t=1$ Theorem 2 implies

$$p_{0,0} = \frac{1}{2}|e^{-\mathrm{i}(\eta-\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} + (-1)^s\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}|^2,$$

$$p_{1,1} = \frac{1}{2}|e^{\mathrm{i}(\eta-\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} + (-1)^s\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}|^2,$$

and therefore $p_{0,0} = p_{1,1}$. Similarly, we have

$$p_{0,1} = \frac{1}{2}|e^{-\mathrm{i}(\eta-\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2} + (-1)^{s+1}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2}|^2,$$

$$p_{1,0} = \frac{1}{2}|e^{\mathrm{i}(\eta-\zeta)}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2} + (-1)^{s+1}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2}|^2,$$

so $p_{0,1} = p_{1,0}$.

In accord with Proposition 1, for any $k,\ell \in \mathbb{Z}_2$ we have $\operatorname{pr}(\mathcal{A}_{\mu,\eta} = \lambda_k) = \operatorname{pr}(\mathcal{B}_{\nu,\zeta} = \lambda_\ell)$ and part (i) is proved.

(ii) In case $t=0$ we have

$$p_{0,0} = \frac{1}{2}|(\operatorname{tr}_0(\eta+\zeta) - \mathrm{i}\operatorname{tr}_1(\eta+\zeta))\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2} + (-1)^s\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2}|^2 =$$
$$\frac{1}{2}\operatorname{tr}^2_0(\eta+\zeta)\operatorname{tr}^2_0\frac{\mu}{2}\operatorname{tr}^2_0\frac{\nu}{2} + (-1)^s\operatorname{tr}_0(\eta+\zeta)\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} +$$
$$\frac{1}{2}\operatorname{tr}^2_1\frac{\mu}{2}\operatorname{tr}^2_1\frac{\nu}{2} + \frac{1}{2}\operatorname{tr}^2_1(\eta+\zeta)\operatorname{tr}^2_0\frac{\mu}{2}\operatorname{tr}^2_0\frac{\nu}{2} =$$
$$\frac{1}{2}(\operatorname{tr}^2_0\frac{\mu}{2}\operatorname{tr}^2_0\frac{\nu}{2} + 2(-1)^{s+1}\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} + \operatorname{tr}^2_1\frac{\mu}{2}\operatorname{tr}^2_1\frac{\nu}{2}) +$$
$$(-1)^s(1+\operatorname{tr}_0(\eta+\zeta))\operatorname{tr}_0\frac{\mu}{2}\operatorname{tr}_0\frac{\nu}{2}\operatorname{tr}_1\frac{\mu}{2}\operatorname{tr}_1\frac{\nu}{2} =$$
$$\frac{1}{2}\operatorname{tr}^2_0\frac{\mu+(-1)^s\nu}{2} + 2(-1)^s\operatorname{tr}^2_0\frac{\eta+\zeta}{2}P\left(\frac{\mu}{2},\frac{\nu}{2}\right).$$

In case $t = 1$ we obtain

$$p_{0,0} = \frac{1}{2}|(\mathrm{tr}_0\,(\eta-\zeta) - \mathrm{i}\,\mathrm{tr}_1\,(\eta-\zeta))\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2} + (-1)^s\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2}|^2 =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,(\eta-\zeta)\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} + (-1)^s\,\mathrm{tr}_0\,(\eta-\zeta)\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} +$$

$$\frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} + \frac{1}{2}\,\mathrm{tr}_1^2\,(\eta-\zeta)\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} + (-1)^s\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} + \frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} +$$

$$(-1)^{s+1}(1 - \mathrm{tr}_0\,(\eta-\zeta))\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} =$$

$$\frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu + (-1)^s\nu}{2} + 2(-1)^{s+1}\,\mathrm{tr}_1^2\,\frac{\eta-\zeta}{2}\,P\left(\frac{\mu}{2}, \frac{\nu}{2}\right).$$

(iii) When $t = 0$ we have

$$p_{0,1} = \frac{1}{2}|(\mathrm{tr}_0\,(\eta+\zeta) - \mathrm{i}\,\mathrm{tr}_1\,(\eta+\zeta))\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2} + (-1)^{s+1}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2}|^2 =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,(\eta+\zeta)\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} + (-1)^{s+1}\,\mathrm{tr}_0\,(\eta+\zeta)\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} +$$

$$\frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} + \frac{1}{2}\,\mathrm{tr}_1^2\,(\eta+\zeta)\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} + (-1)^s\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} + \frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} +$$

$$(-1)^{s+1}(1 + \mathrm{tr}_0\,(\eta+\zeta))\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} =$$

$$\frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu + (-1)^s\nu}{2} + 2(-1)^{s+1}\,\mathrm{tr}_0^2\,\frac{\eta+\zeta}{2}\,P\left(\frac{\mu}{2}, \frac{\nu}{2}\right).$$

When $t = 1$ we obtain

$$p_{0,1} = \frac{1}{2}|(\mathrm{tr}_0\,(\eta-\zeta) - \mathrm{i}\,\mathrm{tr}_1\,(\eta-\zeta))\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2} + (-1)^{s+1}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2}|^2 =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,(\eta-\zeta)\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} + (-1)^{s+1}\,\mathrm{tr}_0\,(\eta-\zeta)\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2} +$$

$$\frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} + \frac{1}{2}\,\mathrm{tr}_1^2\,(\eta-\zeta)\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,\frac{\mu}{2}\,\mathrm{tr}_0^2\,\frac{\nu}{2} + (-1)^{s+1}\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2} +$$

$$\frac{1}{2}\,\mathrm{tr}_1^2\,\frac{\mu}{2}\,\mathrm{tr}_1^2\,\frac{\nu}{2} + (-1)^s(1 - \mathrm{tr}_0\,(\eta-\zeta))\,\mathrm{tr}_0\,\frac{\mu}{2}\,\mathrm{tr}_0\,\frac{\nu}{2}\,\mathrm{tr}_1\,\frac{\mu}{2}\,\mathrm{tr}_1\,\frac{\nu}{2} =$$

$$\frac{1}{2}\,\mathrm{tr}_0^2\,\frac{\mu + (-1)^s\nu}{2} + 2(-1)^s\,\mathrm{tr}_1^2\,\frac{\eta-\zeta}{2}\,P\left(\frac{\mu}{2}, \frac{\nu}{2}\right).$$

Thus, we proved the third equalities from parts (ii) and (iii). The fourth equalities can be obtained by the trigonometric identity

$$P\left(\frac{\mu}{2}, \frac{\nu}{2}\right) = \frac{1}{4}(-1)^{s+t+1}(\mathrm{tr}_t^2\,\frac{\mu + (-1)^s\nu}{2} - \mathrm{tr}_t^2\,\frac{\mu + (-1)^{s+1}\nu}{2}).$$

■

3. The Noise

Here we follow [2, Section 5] and [3, Section 3] with $\alpha = \beta = \frac{1}{2}$, $A = (\mathcal{A}_{\mu,\eta} = 1)$, $B = (\mathcal{B}_{\nu,\zeta} = 1)$.

The joint experiment of the binary trials $\mathfrak{A}_{\mu,\eta} = (\mathcal{A}_{\mu,\eta} = 1) \cup (\mathcal{A}_{\mu,\eta} = -1)$ and $\mathfrak{B}_{\nu,\zeta} = (\mathcal{B}_{\nu,\zeta} = 1) \cup (\mathcal{B}_{\nu,\zeta} = -1)$ (see [4, Part I, Section 6]) produces the probability distribution $\xi_1 = p_{k,k}, \xi_2 = p_{k,k+1}, \xi_3 = p_{k,k+1}, \xi_4 = p_{k,k}, k \in \mathbb{Z}_2$, (that is, the probability assignment of the sample space $S(\psi_{s,t}; \mathcal{A}, \mathcal{B})$), see Theorem 3). In turn, we obtain Boltzmann-Shannon entropy function $E(\theta) = -2\theta \ln \theta - 2(\frac{1}{2} - \theta) \ln(\frac{1}{2} - \theta)$, where $\theta = \xi_1 = p_{k,k}$, and the corresponding degree of dependence function $e(\theta)$. In accord with Section 3 of [3], the two events $\mathcal{A}_{\mu,\eta} = \lambda_k$ and $\mathcal{B}_{\nu,\zeta} = \lambda_\ell$ are independent (that is, $e(\theta) = 0$), exactly when the corresponding binary trials $\mathfrak{A}_{\mu,\eta}$ and $\mathfrak{B}_{\nu,\zeta}$ are informationally independent. In this case we also say that the above events are *informationally independent*. Since their probabilities are both $\frac{1}{2}$, we accomplish the equality $\theta = \frac{1}{4}$ as an equivalent condition for independence. In accord with Theorem 3, if $\psi = \psi_{s,t}$ for some $s, t \in \mathbb{Z}_2$, then the equation

$$\frac{1}{2}\operatorname{tr}_t^2 \frac{\mu + (-1)^s \nu}{2} + 2(-1)^{s+t} \operatorname{tr}_t^2 \frac{\eta + (-1)^t \zeta}{2} P\left(\frac{\mu}{2}, \frac{\nu}{2}\right) = \frac{1}{4},$$

or, equivalently, the equation

$$\frac{1}{2}\operatorname{tr}_t^2 \frac{\mu + (-1)^{s+1} \nu}{2} + 2(-1)^{s+t+1} \operatorname{tr}_{t+1}^2 \frac{\eta + (-1)^t \zeta}{2} P\left(\frac{\mu}{2}, \frac{\nu}{2}\right) = \frac{1}{4},$$

is a necessary and sufficient condition for the events $\mathcal{A}_{\mu,\eta} = \lambda_k$ and $\mathcal{B}_{\nu,\zeta} = \lambda_\ell$ to be informationally idependent.

4. Special Types of Self-Adjoined Operators with Spectre $\{1, -1\}$

In this Section we apply conditions of informational independence from Section 3 in some particular cases where they have simple form and explicit solutions.

4.1. Operators with Direction in Coordinate Plane $x = 0$

Here we consider self-adjoined operators of the form

$$A_{\mu,\frac{\pi}{2}} = \begin{pmatrix} \cos\mu & -\mathrm{i}\sin\mu \\ \mathrm{i}\sin\mu & -\cos\mu \end{pmatrix},$$

where $\mu \in [0, \pi]$. In particular, $A_{0,\frac{\pi}{2}} = \sigma_3$ and $A_{\frac{\pi}{2},\frac{\pi}{2}} = \sigma_2$. The two events $\mathcal{A}_{\mu,\eta} = \lambda_k$ and $\mathcal{B}_{\nu,\zeta} = \lambda_\ell$ are informationally independent if and only if $\frac{1}{2}\operatorname{tr}_t^2 \frac{\mu+(-1)^s\nu}{2} = \frac{1}{4}$. Equivalently, $|\mu + (-1)^s\nu| = \frac{\pi}{2}$ or $|\mu + (-1)^s\nu| = \frac{3\pi}{2}$. Hence the outcomes of measurement of observables $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$ are informationally independent precisely in case $\mu + \nu = \frac{\pi}{2}$ or $\mu + \nu = \frac{3\pi}{2}$ when $\psi = \psi_{0,t}$, $t \in \mathbb{Z}_2$, and precisely in case $|\mu - \nu| = \frac{\pi}{2}$ when $\psi = \psi_{1,t}$, $t \in \mathbb{Z}_2$.

4.2. Operators with Direction in Coordinate Plane $y = 0$

Now, we consider the self-adjoined operators of the form

$$A_{\mu,0} = \begin{pmatrix} \cos\mu & \sin\mu \\ \sin\mu & -\cos\mu \end{pmatrix}$$

where $\mu \in [0, \pi]$. We have $A_{0,0} = \sigma_3$, $A_{\frac{\pi}{2},0} = \sigma_1$, and $A_{\frac{\pi}{4},0} = H$.

The two events $\mathcal{A}_{\mu,\eta} = \lambda_k$ and $\mathcal{B}_{\nu,\zeta} = \lambda_\ell$ are informationally independent if and only if $\frac{1}{2}\operatorname{tr}_t^2 \frac{\mu+(-1)^{s+t+1}\nu}{2} = \frac{1}{4}$. Equivalently, $|\mu + (-1)^{s+t+1}\nu| = \frac{\pi}{2}$. Therefore the results of measurement of observables $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$ are informationally independent exactly in case $\mu + \nu = \frac{\pi}{2}$ or $\mu + \nu = \frac{3\pi}{2}$ when $\psi = \psi_{s,s+1}$, and exactly in case $|\mu - \nu| = \frac{\pi}{2}$ when $\psi = \psi_{s,s}$, $s \in \mathbb{Z}_2$.

Remark 1. The case $t = s = 1$ is discussed also in [3].

4.3. Operators with Direction in Coordinate Plane $z = 0$

Finally, we consider self-adjoined operators of the form

$$A_{\frac{\pi}{2},\eta} = \begin{pmatrix} 0 & e^{-\mathrm{i}\eta} \\ e^{\mathrm{i}\eta} & 0 \end{pmatrix},$$

where $\eta \in [0, 2\pi]$. We have $A_{\frac{\pi}{2},0} = \sigma_1$ and $A_{\frac{\pi}{2},\frac{\pi}{2}} = \sigma_2$.

The two events $\mathcal{A}_{\mu,\eta} = \lambda_k$ and $\mathcal{B}_{\nu,\zeta} = \lambda_\ell$ are informationally independent if and only if $\frac{1}{2}\operatorname{tr}_s^2 \frac{\eta+(-1)^t\zeta}{2} = \frac{1}{4}$. Equivalently, $|\eta + (-1)^t\zeta| = \frac{\pi}{2}$. Therefore, the measurements performed by the observables $\mathcal{A}_{\mu,\eta}$ and $\mathcal{B}_{\nu,\zeta}$ are informationally independent precisely in case $\eta + \zeta \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\}$, when $\psi = \psi_{s,0}$, and precisely in case $|\eta - \zeta| = \frac{\pi}{2}$ or $|\eta - \zeta| = \frac{3\pi}{2}$, when $\psi = \psi_{s,1}$, $s \in \mathbb{Z}_2$.

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