ESTIMATION AND APPLICATION OF A NEW GENERALIZATION OF EXPONENTIAL DISTRIBUTION

D. Kumar*, P. K. chaurasia**, P. Kumar*** and A. Sahoo*

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* Department of Statistics, Banaras Hindu University, Varanasi-221005, India
**School of Business, UPES, Dehradun-248007, India
***Department of Statistics, Central University of Odisha, Koraput-763004, India
chaurasiaprashantvns@gmail.com

Abstract

In statistical literature, several lifetime distributions exist for real phenomena. And one of the methods to find new lifetime distribution by existing baseline distribution such a method is known as the transformation method. In this article, we proposed a generalization of the existing transformation by introducing the additional shape parameter. Here, we considered baseline distributions as an exponential distribution. Various statistical properties of the new lifetime distribution, such as survival function, hazard rate function, cumulative hazard rate function, moments, quantile function, and order statistics, have been discussed. Demonstrate the applicability and suitability of the proposed distribution. Here, we focus only on the estimation of the parameters likes MLE, LSE and also to check long-run behaviour of the estimators.

Keywords: Exponential distribution, Moments, Maximum likelihood estimator, Simulation Study

1. Introduction

In statistical literature, we have supposed that every real phenomenon is governed by some lifetime models. If we know the model, we can completely specify to our problem or phenomenon as various lifetime models have been developed for this purpose. The exponential distribution has been widely used in analyzing lifetime data due to its lack of memory property and its very simple form. However, the exponential distribution with only a constant hazard rate shape and it is not able to fit data sets with different hazard shapes like increasing, decreasing, bathtub, or unimodal (upside down bathtub) shaped failure rates, often encountered in medical, engineering and reliability, among others. In this way, the generalization of exponential distribution is Weibull distribution, it has increasing and decreasing hazard rate functions depends on the values of shape parameter. Gamma distribution has also increasing and decreasing hazard rate functions depends on its shape parameter. Generalized exponential distribution has similar characteristics according to Weibull and gamma distributions but it is more flexible as compared to Weibull and gamma distributions.

Recently, several authors have developed several generalizations of the exponential distribution to increase its flexibility in terms of their shapes of hazard rate and pdf. For example, the Marshall-Olkin exponential [1], exponentiated exponential by Gupta and Kundu [2], Harris extended exponential by Pinho et al. [3], DUS exponential distribution proposed and SS exponential distribution by Kumar et. al.[4], [5] , Kumaraswamy transmuted exponential by Afify et al. [6], a new transformation method based on trignometric function developed by Chaurasia et. al. [7], modified exponential by Rasekhi et al. [8], Minimum Guarantee exponential distribution by

Kumar et. al. [9], odd exponentiated half-logistic exponential by Afify et al. [10], Marshall-Olkin logistic-exponential by Mansoor et al. [11], odd log-logistic Lindley exponential by Alizadeh et al. [12], Kumar et. al. [13] investigate the statistical properties and application of SS-Lindley distribution and Marshall-Olkin alpha power exponential by Nassar et al. [14], among others. Application of exponential distribution has been discussed by Kumar et al. [15] in Bayesian setup.

Recently, Kumar et al. [9] introduced minimum guarantee (MG) transformation in order to generalize any available lifetime distribution called baseline distribution. The idea is very simple, as discussed below; If d(z) be the pdf of $Z > \sigma > 0$ and if $Y = \frac{\sigma}{Z}$, then its pdf k(y) is given by k(y)= d(z)$\left|\frac{\partial z}{\partial y}\right| = d\left(\frac{\sigma}{y}\right)\left|\frac{-\sigma}{y^2}\right| = \frac{\sigma}{y^2}.\ d\left(\frac{\sigma}{y}\right)$;$0 < y < 1$. Now, if F(x) and f(x) be the cdf and pdf of some available baseline distribution respectively, then the pdf g(x) of new lifetime distribution is given by,

$$g(x) = k(F(x))f(x); \quad x > 0 \tag{1}$$

The host authors Kumar et al. [9] named it as MG transformation. Now, we consider the following pdf,

$$\partial(z) = \alpha\beta^{\alpha}e^{-\alpha z}; \qquad z > \ln\beta, \alpha > 0, \beta > 1$$

Making the transformation by the substitution

$$Y = \frac{\ln\beta}{Z}$$
$$Z = \frac{\ln\beta}{Y}$$
$$J = \left|\frac{\partial z}{\partial y}\right| = \frac{-1}{y^2}\ln\beta$$
$$|J| = \frac{1}{y^2}\ln\beta$$

and as z ranges $\ln\beta$ to ∞, so that y ranges 0 to 1. If k(y) be the pdf of of Y, then

$$\begin{aligned} k(y) &= d(z)\left|\frac{\partial z}{\partial y}\right| \\ &= \alpha\beta^{\alpha}\frac{1}{y^2}\ln\beta e^{\ln\beta^{-\frac{\alpha}{y}}} \\ &= \frac{\alpha\ln\beta}{y^2}\beta^{\alpha-\frac{\alpha}{y}} \\ &= \frac{\alpha\ln\beta}{y^2}\beta^{\alpha(1-\frac{1}{y})}; \qquad 0 < y < 1, \alpha > 0,\ \beta > 1 \end{aligned} \tag{2}$$

Now, if F(x) and f(x) be the cdf and pdf, respectively of some available baseline distribution, then the pdf g(x) of new distribution corresponding to cdf F(x) and pdf f(x) can be obtained as follows,

$$g(x) = \frac{\alpha\ln\beta}{(F(x))^2}\beta^{\alpha\left(1-\frac{1}{F(x)}\right)}f(x); \qquad \forall\ x > 0,\ \alpha > 0,\ \beta > 1 \tag{3}$$

and the cdf G(x) corresponding to the pdf g(x) will be,

$$G(x) = \beta^{\alpha\left(1-\frac{1}{F(x)}\right)}; \quad \forall\ x > 0,\ \alpha > 0,\ \beta > 1 \tag{4}$$

We will call transformation (4) as generalized minimum guarantee (GMG) transformation.
In particular for $\alpha = 1$ and $\beta = e$, the proposed transformation (4) reduces to,

$$G(x) = e^{1-\frac{1}{F(x)}} \quad \forall\ x > 0 \tag{5}$$

Which is MG transformation as suggested by Kumar et al.[9], that is why we called (4) as GMG transformation. Now, to demonstrate the real-life application of GMG transformation, we consider the exponential distribution with pdf $f(x) = \theta e^{-\theta x};\ \ x > 0,\ \theta > 0$ and cdf $F(x) = 1 - e^{-\theta x};\ \ x > 0,\ \theta > 0$ as an available baseline distribution. Thus the cdf G(x) and pdf g(x) corresponding to the considered baseline distribution, using GMG transformation, are obtained as follows,

$$G(x) = \beta^{\alpha\left(1-\frac{1}{1-e^{-\theta x}}\right)} \quad \forall\ x > 0,\ \theta > 0,\ \alpha > 0,\ \beta > 1 \tag{6}$$

$$g(x) = \frac{\alpha \ln \beta}{(1-e^{-\theta x})^2} \beta^{\alpha\left(1-\frac{1}{1-e^{-\theta x}}\right)} \theta e^{-\theta x} \quad \forall\ x > 0,\ \theta > 0,\ \alpha > 0,\ \beta > 1 \tag{7}$$

We shall use the abbreviation $\mathrm{GMG}_E(\alpha, \beta, \theta)$- distribution for generalized minimum gurantee transformation as exponential distribution with parameter θ. The plots of cdf G(x) and pdf g(x) of $\mathrm{GMG}_E(\alpha, \beta, \theta)$- distribution for various combination of values of α, β and θ are shown in Figure 1 and 2 respectively.

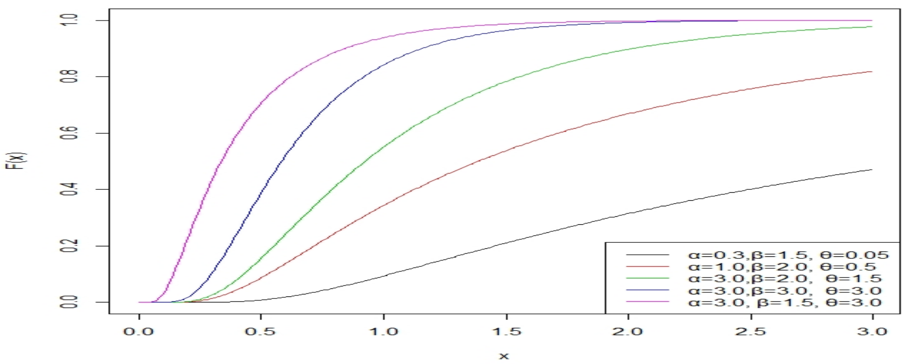

Figure 1: *Plots of cumulative distribution function G(x) for various combination of values of α, β and θ.*

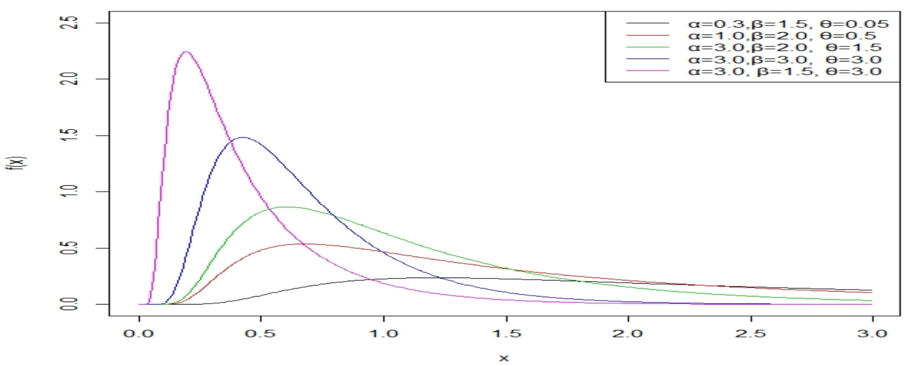

Figure 2: *Plots of probability density function g(x) for various combination of values of α, β and θ.*

2. Reliability Analysis

In this section, we have derived various reliability characteristics such as survival function, hazard rate function and cumulative hazard rate function of $\mathrm{GMG}_E(\alpha, \beta, \theta)$- distribution with cdf (6).

2.1. Survival Function

The Survival function S(x) of GMG$_E(\alpha, \beta, \theta)$- distribution at a mission time x is obtained as follows,

$$
\begin{aligned}
S(x) &= 1 - G(x) \\
&= 1 - \left[\beta^{\alpha - \frac{\alpha}{1-e^{-\theta x}}}\right]; \qquad x > 0, \theta > 0, \alpha > 0\ \beta > 1
\end{aligned} \tag{8}
$$

Figure 3 shows the behaviour of survival function S(x) of GMG$_E(\alpha, \beta, \theta)$- distribution for different combination of values of the parameters α, β and θ.

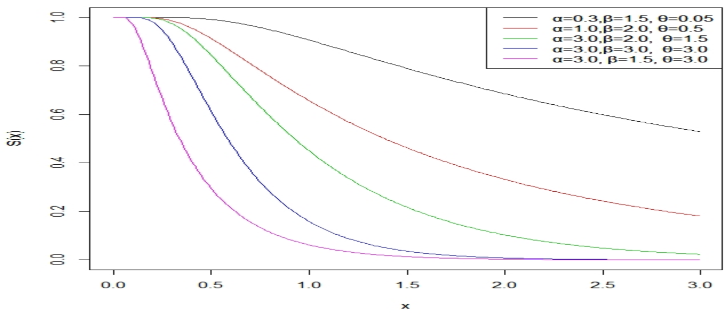

Figure 3: *Plots of survival function S(x) for different combination of values of α, β and θ.*

2.2. Hazard Rate Function

The hazard rate function h(x) of GMG$_E(\alpha, \beta, \theta)$- distribution is obtained as follows,

$$
\begin{aligned}
h(x) &= \frac{g(x)}{S(x)} \\
&= \frac{\alpha \ln \beta \left(\beta^{\alpha - \frac{\alpha}{1-e^{-\theta x}}}\right) \theta e^{-\theta x}}{\left(1 - e^{-\theta x}\right)^2 \left[1 - \beta^{\alpha - \frac{\alpha}{1-e^{-\theta x}}}\right]}; \quad x > 0, \theta > 0, \alpha > 0\ \beta > 1
\end{aligned} \tag{9}
$$

The shapes of hazard rate function h(x) for different combination of values of parameters α, β and θ are shown in Figure 4.

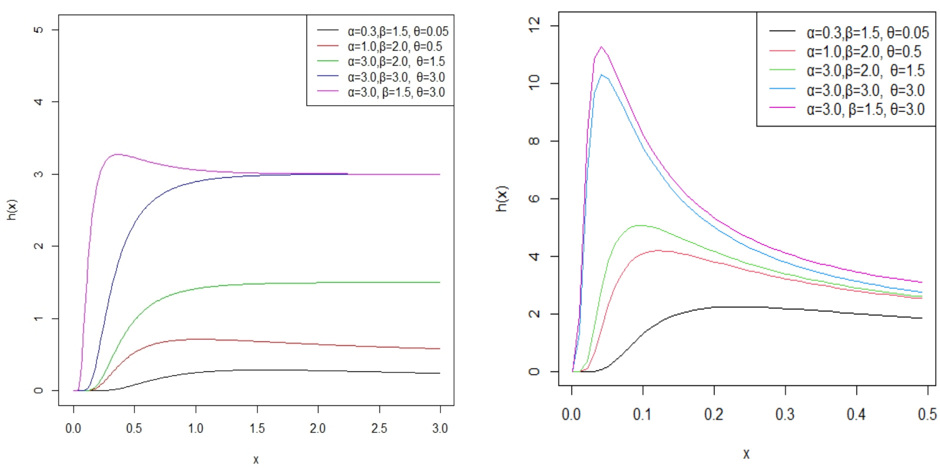

Figure 4: *Plots of hazard rate function h(x) for different combination of values of α, β and θ.*

2.3. Cumulative Hazard Rate Function

The cumulative hazard rate function H(x) of $\text{GMG}_E(\alpha, \beta, \theta)$- distribution is obtained as follows,

$$\begin{aligned} H(x) &= -\ln\{1 - G(x)\} \\ &= -\ln\left\{1 - \left(\beta^{\alpha - \frac{\alpha}{1-e^{-\theta x}}}\right)\right\}; \quad x > 0, \theta > 0, \alpha > 0\ \beta > 1 \end{aligned} \tag{10}$$

Figure 5 shows the behaviour of cumulative hazard rate function H(x) of $\text{GMG}_E(\alpha, \beta, \theta)$-distribution for different combination of values of the parameters α, β and θ.

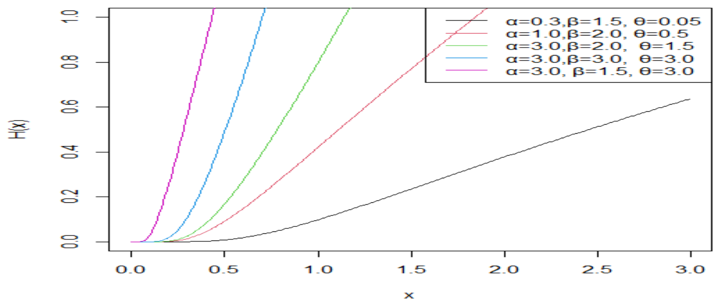

Figure 5: *Plots of cumulative hazard rate function H(x) for different combination of values of α, β and θ.*

3. Statistical Properties of $\text{GMG}_E(\alpha, \beta, \theta)$- distribution

3.1. Moments

The r^{th} order non-central moment, i.e, r^{th} moment about origin of $\text{GMG}_E(\alpha, \beta, \theta)$- distribution having pdf (7) is obtained as follows,

$$\begin{aligned} \mu_r' &= \int_0^\infty x^r f(x) dx \\ &= \theta\alpha \ln\beta \int_0^\infty \frac{x^r}{(1-e^{-\theta x})^2} \beta^{\alpha - \frac{\alpha}{1-e^{-\theta x}}} e^{-\theta x} dx \\ &= \theta\alpha \ln\beta \beta^\alpha \int_0^\infty \frac{x^r}{(1-e^{-\theta x})^2} \beta^{-\frac{\alpha}{1-e^{-\theta x}}} e^{-\theta x} dx \\ &= \theta\alpha \ln\beta \beta^\alpha \sum_{k=0}^\infty (k+1) \int_0^\infty x^r e^{-k\theta x} \beta^{-\frac{\alpha}{1-e^{-\theta x}}} e^{-\theta x} dx \\ &= \theta\alpha \sum_{k=0}^\infty (k+1) \ln\beta \beta^\alpha \int_0^\infty x^r e^{-k\theta x} \beta^{-\frac{\alpha}{1-e^{-\theta x}}} e^{-\theta x} dx \\ &= \theta\beta^\alpha \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{l+m}(k+1)\alpha^{l+1}(\ln\beta)^{l+1}}{l!} \int_0^\infty x^r e^{-k\theta x}(1-e^{-\theta x})^l e^{-\theta x} dx \\ &= \theta\beta^\alpha \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \frac{\binom{-l}{m}(-1)^{l+m}(k+1)\alpha^{l+1}(\ln\beta)^{l+1}}{l!} \int_0^\infty x^r e^{-\theta(k+m+1)x} dx \\ &= \theta\beta^\alpha \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \frac{\binom{-l}{m}(-1)^{l+m}(k+1)\alpha^{l+1}(\ln\beta)^{l+1}}{l!} \frac{\Gamma(r+1)}{(\theta(k+m+1))^{r+1}} \end{aligned} \tag{11}$$

3.2. Sample Generation

Let U follows continuous uniform distribution over [0,1], then

$$
\begin{aligned}
& G(x) = u \\
& \implies \beta^{\alpha\left(1-\frac{1}{F(x)}\right)} = u \\
& \implies \beta^{\alpha\left(1-\frac{1}{1-e^{-\theta x}}\right)} = u \\
& \implies \alpha\left(1-\frac{1}{1-e^{-\theta x}}\right)\ln\beta = \ln u \\
& \implies 1-\frac{1}{1-e^{-\theta x}} = \frac{\ln u}{\alpha\ln\beta} \\
& \implies \frac{1}{1-e^{-\theta x}} = 1-\frac{\ln u}{\alpha\ln\beta} \\
& \implies 1-e^{-\theta x} = \frac{1}{1-\frac{\ln u}{\alpha\ln\beta}} \\
& \implies e^{-\theta x} = 1-\frac{1}{1-\frac{\ln u}{\alpha\ln\beta}} \\
& \implies -\theta x = \ln\left(1-\frac{1}{1-\frac{\ln u}{\alpha\ln\beta}}\right) \\
& \implies x = -\frac{1}{\theta}\ln\left(1-\frac{1}{1-\frac{\ln u}{\alpha\ln\beta}}\right) \qquad (12)
\end{aligned}
$$

and consequently if $\underline{x}$= $(x_1, x_2, ..., x_n)$ be a random sample of size n from GMG$_E(\alpha, \beta, \theta)$-distribution, then it can be obtained as follows;

$$x_i = -\frac{1}{\theta}\ln\left(1-\frac{1}{1-\frac{\ln u_i}{\alpha\ln\beta}}\right) \quad \forall\, 1, 2, ...n. \qquad (13)$$

where $u_i \sim u[0,1]$ $\forall$ i= 1(1) n.

3.3. Quantile Function

For any p $\in$ (0,1), the p^{th} quantile function Q(p) of GMG$_E(\alpha, \beta, \theta)$- distribution is the solution of the following equation,

$$
\begin{aligned}
& G(Q(p)) = p \\
& \implies \beta^{\alpha\left(1-\frac{1}{1-e^{-\theta Q(p)}}\right)} = p \\
& \implies \alpha\left(1-\frac{1}{1-e^{-\theta Q(p)}}\right)\ln\beta = \ln p \\
& \implies 1-\frac{1}{1-e^{-\theta Q(p)}} = \frac{\ln p}{\alpha\ln\beta} \\
& \implies \frac{1}{1-e^{-\theta Q(p)}} = 1-\frac{\ln p}{\alpha\ln\beta} \\
& \implies -\theta Q(p) = \ln\left(1-\frac{1}{1-\frac{\ln p}{\alpha\ln\beta}}\right) \\
& \implies Q(p) = -\frac{1}{\theta}\ln\left(1-\frac{1}{1-\frac{\ln p}{\alpha\ln\beta}}\right)
\end{aligned}
$$

In particular for p = $\frac{1}{2}$, we can get, median $\tilde{\mu}$ of $\text{GMG}_E(\alpha,\beta,\theta)$- distribution as follows,

$$\begin{aligned}
\tilde{\mu} &= Q\left(\frac{1}{2}\right) \\
&= -\frac{1}{\theta}\ln\left(1-\frac{1}{1-\frac{\ln\left(\frac{1}{2}\right)}{\alpha\ln\beta}}\right) \\
&= -\frac{1}{\theta}\ln\left(1-\frac{1}{1-\frac{\ln 1-\ln 2}{\alpha\ln\beta}}\right) \\
&= -\frac{1}{\theta}\ln\left(1-\frac{1}{1+\frac{\ln 2}{\alpha\ln\beta}}\right) \qquad (14)
\end{aligned}$$

4. Estimation of the parameters α, β and θ of $\text{GMG}_E(\alpha,\beta,\theta)$- distribution:-

In this section, we will derive maximum likelihood estimators and least square estimators of the parameters α, β and θ of $\text{GMG}_E(\alpha,\beta,\theta)$- distribution with pdf (7) on the basis of complete sample $X_1, X_2, ...X_n$ of fixed size n from it.

4.1. Maximum Likelihood Estimators

The likelihood function for $X_1, X_2, ...X_n$ a random sample of fixed size n from $\text{GMG}_E(\alpha,\beta,\theta)$- distribution having pdf (7) is given by,

$$\begin{aligned}
L &= \prod_{i=1}^{n}(g(x_i)) \\
&= \prod_{i=1}^{n}\left(\frac{\alpha\ln\beta}{(1-e^{-\theta x})^2}\beta^{\alpha-\frac{\alpha}{1-e^{-\theta x}}}\theta e^{-\theta x}\right)
\end{aligned}$$

Taking natural logarithm of both sides of the above equation, we get,

$$\begin{aligned}
l &= \ln L \\
&= n\ln\alpha + n\ln(\ln\beta) + n\ln\theta - \theta\sum_{i=1}^{n}x_i + \\
&\sum_{i=1}^{n}\left(\alpha-\frac{\alpha}{1-e^{-\theta x_i}}\right)\ln\beta - 2\sum_{i=1}^{n}\ln(1-e^{-\theta x_i}) \qquad (15)
\end{aligned}$$

and hence the log-likelihood equations function estimating α, β and θ are given by

$$\begin{aligned}
&\frac{\partial l}{\partial\alpha} = 0 \\
\implies &\frac{n}{\alpha} + n\ln\beta - \ln\beta\sum_{i=1}^{n}\frac{1}{1-e^{-\theta x_i}} = 0 \qquad (16)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial l}{\partial\beta} = 0 \\
\implies &\frac{n}{\beta\ln\beta} + \frac{1}{\beta}\sum_{i=1}^{n}\left(\alpha-\frac{\alpha}{1-e^{-\theta x_i}}\right) = 0 \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial l}{\partial\theta} = 0 \\
\implies &\frac{n}{\theta} - \sum_{i=1}^{n}x_i + \ln\beta\sum_{i=1}^{n}\frac{\alpha x_i e^{-\theta x_i}}{\left(1-e^{-\theta x_i}\right)^2} - 2\sum_{i=1}^{n}\frac{x_i e^{-\theta x_i}}{1-e^{-\theta x_i}} = 0 \qquad (18)
\end{aligned}$$

It is clear that the above system of log-likelihood equations can not be solved analytically for α, β and θ. Therefore, some approximation technique will be used for approximate solution of them for α, β and θ. Here, we have again used nlm command in R software to maximize l and hence $\ln L$ for variation in α, β and θ.

4.2. Least Squares Estimators

The least square estimators of α, β and θ on the basis of complete sample $\underline{X} = (X_1, X_2, \ldots, X_n)$ of fixed size n from $\text{GMG}_E(\alpha, \beta, \theta)$- distribution can be obtained by minimizing the following function,

$$\begin{aligned} P(\alpha, \beta, \theta) &= \sum_{i=1}^{n} \left(G(x_i) - E(G_n(x_i))\right)^2 \\ &= \sum_{i=1}^{n} \left(G(x_i) - \frac{i}{n+1}\right)^2 \end{aligned}$$

where $E(G_n(x_i)) = \frac{i}{n+1}\ \forall i = 1(1)n$ is the empirical cdf of $\underline{X} = (X_1, X_2, \ldots, X_n)$.
Now, the normal equations for estimating α, β and θ are obtained as follows,

$$\frac{\partial P(\alpha, \beta, \theta)}{\partial \alpha} = 0 \Rightarrow \sum_{j=1}^{n} G'_\theta(x_j; \alpha, \beta, \theta) \left((G(x_j; \alpha, \beta, \theta)) - \frac{j}{n+1} \right) = 0$$

$$\frac{\partial P(\alpha, \beta, \theta)}{\partial \beta} = 0 \Rightarrow \sum_{j=1}^{n} G'_\theta(x_j; \alpha, \beta, \theta) \left((G(x_j; \alpha, \beta, \theta)) - \frac{j}{n+1} \right) = 0$$

$$\frac{\partial P(\alpha, \beta, \theta)}{\partial \theta} = 0 \Rightarrow \sum_{j=1}^{n} G'_\theta(x_j; \alpha, \beta, \theta) \left((G(x_j; \alpha, \beta, \theta)) - \frac{j}{n+1} \right) = 0$$

we have again used nlm command for the simultaneous solution of above normal equations for α, β and θ, as the system of above normal equations is not easy to solve analytically for α, β and θ. Let the least square estimators (LSEs) of α, β and θ, thus obtained are denoted by $\hat{\alpha}_L$, $\hat{\beta}_L$ and $\hat{\theta}_L$.

5. Simulation Study

In this section, the simulation study is carried out to know the performance of MLEs and LSEs of α, β and θ in terms of their mean squared error (MSEs) with varying sample sizes. The value of α, β and θ are arbitrarily chosen as α=0.5, 1, 2, β= 2 and θ=0.5, 1, 2 and the different considered values of n are 20, 50, 80, 100, 150 and 200. The process is repeated 5000 times in order to achieve convergence and MLEs $\hat{\alpha}_M$, $\hat{\beta}_M$ and $\hat{\theta}_M$ and LSEs $\hat{\alpha}_L$, $\hat{\beta}_L$ of α, β and θ have been calculated for each generated samples and consequently MSEs of these estimators have been calculated and reported in Tables 1,2 and 3. We have used the notation m$(\hat{\theta})$ to represent the mean squared error of the estimator.

Table 1: *MSEs of MLEs and LSEs of α, β and θ for fixed α=0.5, β=2 and θ=0.5 variation in the values of n.*

n	m$(\hat{\theta}_M)$	m$(\hat{\theta}_L)$	m$(\hat{\alpha}_M)$	m$(\hat{\alpha}_L)$	m$(\hat{\beta}_M)$	m$(\hat{\beta}_L)$
20	**2.3015**	3.1142	**0.2033**	0.5134	0.5113	**0.5090**
50	**0.3213**	0.7546	**0.0617**	0.3371	0.2866	**0.2172**
80	**0.1616**	0.4023	**0.0230**	0.2784	0.2462	**0.1932**
100	**0.1119**	0.2840	**0.0216**	0.1570	**0.1216**	0.1328
150	**0.0676**	0.1844	**0.0126**	0.0984	**0.0323**	0.1134
200	**0.0415**	0.1651	**0.0114**	0.0148	**0.0121**	0.1033

Table 2: *MSEs of MLEs and LSEs of α, β and θ for fixed α=1, β=2 and θ=1 variation in the values of n.*

n	m($\hat{\theta}_M$)	m($\hat{\theta}_L$)	m($\hat{\alpha}_M$)	m($\hat{\alpha}_L$)	m($\hat{\beta}_M$)	m($\hat{\beta}_L$)
20	**2.3015**	3.1142	**0.2033**	0.5134	0.5113	**0.5090**
50	**0.4947**	1.2120	**0.1898**	0.4298	0.2943	**0.2646**
80	**0.2407**	0.5236	**0.1742**	0.3615	0.2462	**0.1947**
100	**0.1850**	0.4319	**0.1591**	0.3450	**0.1957**	0.3879
150	**0.1121**	0.2800	**0.1110**	0.2553	**0.0471**	0.1623
200	**0.0819**	0.2582	**0.0928**	0.2234	**0.0335**	0.0898

Table 3: *MSEs of MLEs and LSEs of α, β and θ for fixed α=2, β=2 and θ=2 with variation in the values of n.*

n	m($\hat{\theta}_M$)	m($\hat{\theta}_L$)	m($\hat{\alpha}_M$)	m($\hat{\alpha}_L$)	m($\hat{\beta}_M$)	m($\hat{\beta}_L$)
20	**2.8192**	3.8851	**0.5871**	1.1527	**0.6192**	0.6768
50	**0.6540**	1.5958	**0.5612**	1.1485	**0.2997**	0.4288
80	**0.3716**	0.7409	**0.5391**	0.9697	**0.1803**	0.4053
100	**0.2846**	0.5625	**0.5297**	0.9310	**0.1612**	0.2594
150	**0.1725**	0.3679	**0.5029**	0.8069	**0.0896**	0.1655
200	**0.1248**	0.2809	**0.4903**	0.5781	**0.0797**	0.0842

From Tables 1 and 2, we observed that MLEs $\hat{\theta}_M$ and $\hat{\alpha}_M$ performs better as compared to their respective rivals LSEs $\hat{\theta}_L$ and $\hat{\alpha}_L$ in terms of having smallest values of MSEs, for all the considered values of n, where the values of (α, β and θ) have been as (0.5, 2, 0.5) and (1, 2, 1) respectively. But, in the case of estimators of β, the LSE $\hat{\beta}_L$ performs better as compared to MLE $\hat{\beta}_M$ for the considered small values of n and for larger values of n, reverse trend is noted.

Finally, from Table 3, we observed that MLEs $\hat{\theta}_M$, $\hat{\alpha}_M$ and $\hat{\beta}_M$ performs better as compared to their respective rivals LSEs $\hat{\theta}_L$, $\hat{\alpha}_L$ and $\hat{\beta}_L$ in terms of the well known having smallest values of MSEs, for all the considered values of n, where we have fixed α=2, β=2 and θ=2.

6. Data Analysis

Data set

In this section, we have considered two different real data sets, one from medical field and other from engineering field. In order to summarize the data sets, exploratory data analysis (EDA) has been carried out. To detect the shape of hazard rate function, scaled total-time on-test (TTT) plots has also been plotted. Further, the performance of $\mathrm{GMG}_E(\alpha, \beta, \theta)$- distributions has been checked out over some existing distribution in terms of fitting criterions AIC, BIC, and K-S distance.

Data Set-I:- A data of remission times (in months) of 128 bladder cancer patients (see, Kumar et al. [9]). The data is shown below:

X = (0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 1.46, 18.10, 11.79, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25 ,4.50, 6.25, 8.37, 12.02, 2.02, 13.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 12.07, 6.76, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69).

Data Set-II: :- A data of times between successive failures (in thousands of hours) in events of secondary reactor pumps see, suprawhardana et al.[16] and it contains 23 observations, which are

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

6.1. Exploratory data analysis (EDA)

For data sets I and II, we have applied basic exploratory data analysis (EDA), which is capable to exact the following informations from the data see,tukey et al.[17] .

• Displaying graphically the data in the form of bar charts, histogram and density curve that shows overall pattern and unusual observations.

• Computing descriptive statistics such as centre, spread, skewness and kurtosis that summarizes specific aspects of the data.

As the result of EDA, the descriptive statistics for data sets I and II are summarized in Tables 4 and 5 respectively and also the box plots, histogram and density curve for these data sets are shown in Figures 6 and 7 respectively.

Table 4: *Descriptive statistics for data set-I.*

Minimum	Q_1	Median	Mean	Q_3	Maximum	Variance	Skewness	Kurtosis
0.080	3.348	6.395	9.366	11.84	79.05	110.42	3.3257	19.153

Table 5: *Descriptive statistics for data set -II.*

Minimum	Q_1	Median	Mean	Q_3	Maximum	Variance	Skewness	Kurtosis
0.0620	0.310	0.614	1.578	2.041	6.560	3.7275	1.3643	3.5445

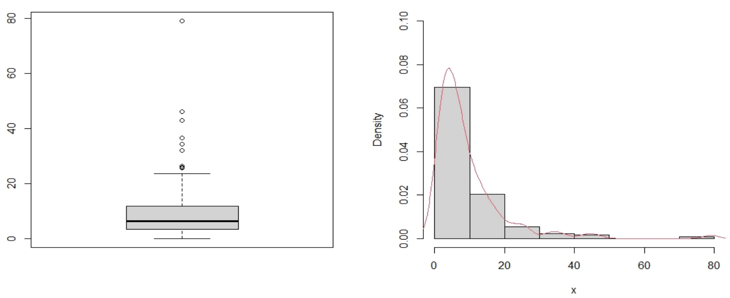

Figure 6: *The box plot, histogram and density plot for the data set I.*

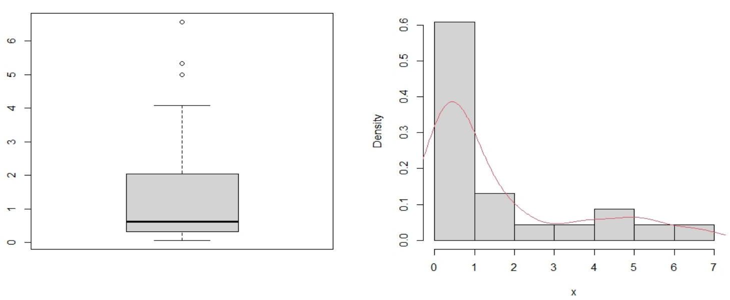

Figure 7: *The boxplot, histogram and density plot for the data set II.*

6.2. Scaled total-time-on-test (TTT) plot

In order to get an idea about the shape of the hazard functions of data sets I and II, we have drawn scaled TTT plots for these data sets, and the plots are shown in Figures 8 and 9 shows scaled TTT plots for data set I and II respectively. These plots shows that Figures 8, shows that the data sets I possesses reverse bathtub and Figures 9, shows that the data sets II possesses bathtub shapes of their hazard rate functions. Therefore, we can suggest to use our proposed distribution $\text{GMG}_E(\alpha, \beta, \theta)$- distribution for these data sets.

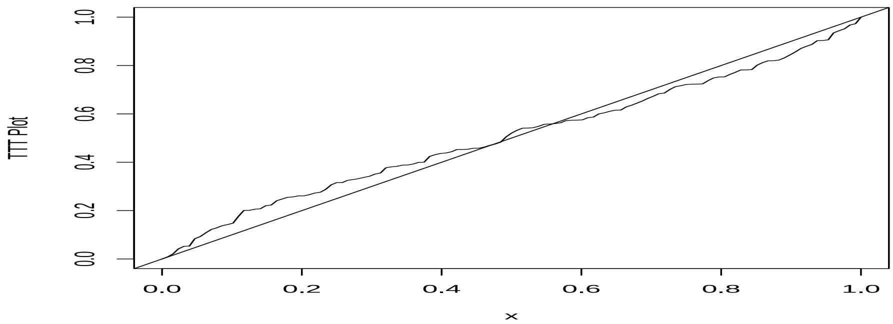

Figure 8: *Scaled total TTT plot for data set-I.*

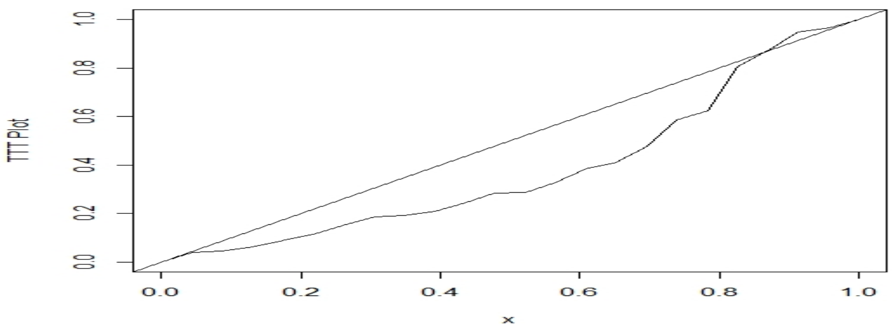

Figure 9: *Scaled total TTT for data set-II.*

6.3. Model Comparison

In this sub-section, we have compared the performance of our proposed distribution $\text{GMG}_E(\alpha, \beta, \theta)$-distribution over some distributions available in literature on the considered data sets I and II. The model selection criterions used are AIC, BIC and K-S distance.

Now, we have computed the maximum likelihood estimates of the parameters α, β and θ for the considered data sets I and II and these are shown in Table 6.

Table 6: *Maximum Likelihood Estimates.*

Parameter	α	β	θ
Data Set-I	0.24796	1.46763	0.03664
Data Set- II	0.84050	1.05509	0.14440

The values of AIC, BIC and K-S distance for the data set I and II for our proposed distribution have been calculated and are shown in Tables 7 and 8 respectively. Also, the values of AIC, BIC and K-S distance for the data set I for Exponentiated Pareto distributions ($E_P(\theta, \lambda)$) and Beta Pareto distribution (β_P(a,b)) have been extracted from Shawky and Abu Zinadh [18] and are shown in comparative model in Table 7. The values of these criterions for data set II for Reduced Additive Weibull distribution ($RA_W(\alpha, \beta, \lambda)$) and Flexible Weibull Extension ($FW_W(a, b)$) have been extracted from Mustafa et al.[19] are shown in Table 8.

Table 7: *Values of AIC, BIC, and K-S distance for $GMG_E(\alpha, \beta, \theta)$- distribution, $E_P(\theta, \lambda)$- distribution and β_P(a,b)- distribution for data set-I.*

Distributions	AIC	BIC	K-S distance
$\mathbf{GMG}_E(\alpha, \beta, \theta)$	**921.05**	**929.60**	**0.234**
$E_P(\theta, \lambda)$	992.20	997.90	0.251
β_P(a,b)	970.70	979.20	0.223

Table 8: *Values of AIC, BIC, and K-S distance for $GMG_E(\alpha, \beta, \theta)$- distribution, $RA_W(\alpha, \beta, \lambda)$ and $FW_W(a, b)$ for data set-II.*

Distributions	AIC	BIC	K-S distance
$\mathbf{GMG}_E(\alpha, \beta, \theta)$	**74.10**	**77.50**	**0.132**
$RA_W(\alpha, \beta, \lambda)$	178.14	181.55	0.162
$FW_W(a, b)$	170.68	172.95	0.134

From comparative Tables 7 and Table 8, we observed that the values of AIC, BIC and K-S distance for our proposed dsitribution is least as compared to those values for $E_P(\theta, \lambda)$-distribution and β_P(a,b)- distribution (Table 7) and $RA_W(\alpha, \beta, \lambda)$ -distribution and $FW_W(a, b)$-distribution (Table 8). Therefore, we can say that for the considered data sets I and II, our proposed distribution fits better as compared to other considered distributions in the sense of having smallest values of fitting criterions AIC, BIC and K-S distance.

7. Conclusions

In the present article, we have introduced a new transformation technique which is a generalization of minimum guarantee (MG) transformation proposed by Kumar et al. [9]. Thus the new transformation is called generalized minimum guarantee (GMG) transformation. To demonstrate its real life application, we have considered exponential distribution with the parameter θ as a baseline distribution. The distribution, thus obtained is abbreviated as $GMG_E(\alpha, \beta, \theta)$- distribution. Various statistical properties of the new distribution such as survival function, hazard rate function, cumulative hazard rate function, moments, quantile function and order statistics have been discussed. We have also obtained MLEs and LSEs of the parameters has also been discussed for estimating the parameters α, β and θ. For complete sample from $GMG_E(\alpha, \beta, \theta)$- distribution. Simulation study is carried out to know long-run behaviour of the estimatons and it was found that for almost all the considered values of α, β and θ, the MSEs of MLEs of α, β and θ is smallest as compared to those of MSEs of LSEs of α, β and θ respectively and as n increases, the MSEs are decreases. Finally, to demonstrate its real data application, we have considered two different real data sets and its found that our proposed distribution performs betters as compared to some available distributions in terms of AIC, BIC and K-S distance as fits model selection criterions.

References

[1] Marshall, A. W., and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika, 84(3), 641-652.

[2] Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. Biometrical Journal. Journal of Mathematical Methods in Biosciences, 43(1), 117-130.

[3] Pinho, L. G. B., Cordeiro, G. M. and Nobre, J. S. (2015). The Harris extended exponential distribution. Communications in Statistics-Theory and methods, 44(16), 3486-3502.

[4] Kumar, D., Singh, U., and Singh, S. K. (2015). A method of proposing new distribution and its application to Bladder cancer patients data. J. Stat. Appl. Pro. Lett, 2(3), 235-245.

[5] Kumar, D., Singh, U., and Singh, S. K. (2015). A new distribution using sine function-its application to bladder cancer patients data. Journal of Statistics Applications and Probability, 4(3), 417.

[6] Nofal, Z. M., Afify, A. Z., Yousof, H. M., Granzotto, D. C. and Louzada, F. (2016). Kumaraswamy transmuted exponentiated additive Weibull distribution. International Journal of Statistics and Probability, 5(2), 78-99.

[7] Kumar, D., Chaurasia, P. K., Kumar, P. and Chaurasia, A. (2022). A Novel Transformation: Based on Inverse Trigonometric Lindley Distribution. Reliability: Theory and Applications, 17(1 (67)), 303-315.

[8] Elbatal, I., Zayed, M., Rasekhi, M. and Butt, N. S. (2017). The exponential Pareto power series distribution: theory and applications. Pakistan Journal of Statistics and Operation Research, 603-615.

[9] Kumar, D., Singh, U. and Singh, U. (2017). Life time distributions: Derived from some minimum guarantee distribution. Sohag Journal of Mathematics, 4(1), 7-11.

[10] Alizadeh, M., Altun, E., Afify, A. Z. and Gamze, O. Z. E. L. (2018). The extended odd Weibull-G family: properties and applications. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1), 161-186.

[11] Mansoor, M., Tahir, M. H., Cordeiro, G. M., Provost, S. B. and Alzaatreh, A. (2019). The Marshall-Olkin logistic-exponential distribution. Communications in Statistics-Theory and Methods, 48(2), 220-234.

[12] Alizadeh, M., Altun, E., Ozel, G., Afshari, M., and Eftekharian, A. (2019). A new odd log-logistic lindley distribution with properties and applications. Sankhya A, 81, 323-346.

[13] Kumar, D., Singh, U., Singh, S. K. and Chaurasia, P. K. (2018). Statistical properties and application of a Lifetime model using sine function. Int. J. Creat. Res. Thoughts, 6, 993-1002.

[14] Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M., and Afify, A. Z. (2019). The Marshall–Olkin alpha power family of distributions with applications. Journal of Computational and Applied Mathematics, 351, 41-53.

[15] Kumar, D., Kumar, P., Singh, S. K., and Singh, U. (2019). A new asymmetric loss function: estimation of parameter of exponential distribution. J Stat Appl Probab Lett, 6(1), 37-50.

[16] M. S. Suprawhardana; S. Prayoto; Total time on test plot analysis for mechanical components of the RSGGAS reactor, Atom Indones, 25(2), 81-90 (1999).

[17] Tukey, J. W. (1977). Exploratory data analysis (Vol. 2, pp. 131-160). Reading, MA: Addison-wesley.

[18] Shawky, A. I., and Abu-Zinadah, H. H. (2009). Exponentiated Pareto distribution: different method of estimations. International Journal of Contemporary Mathematical Sciences, 4(14), 677-693.

[19] Mustafa, A. B. D. E. L. F. A. T. T. A. H., El-Desouky, B. S., and Al-Garash, S. H. A. M. S. A. N. (2016). The exponentiated generalized flexible Weibull extension distribution. Fundamental Journal of Mathematics and Mathematical Sciences, 6(2), 75-98.