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COMMON FIXED POINT THEOREMS IN COMPLEX VALUED B-METRIC SPACES AND USING WEAK COMPATIBLE MAPPINGS

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Abstract

Complex-valued metric spaces, where the distance between points is represented as a complex number, serve as a generalization of traditional metric spaces and present a variety of intriguing applications, particularly in advanced mathematics and theoretical physics, such as in quantum mechanics, quantum computing, functional analysis, operator theory, complex dynamics, fractal geometry, complex networks, and signal processing, as well as theoretical physics, relativity, geometry, topology, mathematical finance, and economics. Despite the increasing interest in complex-valued metric spaces, the current literature on fixed point theorems within these contexts is constrained in several ways. The available literature has notable shortcomings; consequently, this article investigates the results and implications of contractive mappings in complex-valued b-metric spaces by utilizing weak compatible mappings, and it also employs control functions while solving a system of Urysohn integral equations based on our primary result. These findings contribute to this area of study.

Keywords: fixed points, b-metric space, complex valued b-metric space, weakly compatible mappings.

1. Introduction

The classical Banach contraction principle [1] serves as a potent tool in nonlinear analysis and initiated a significant development in fixed point theory, owing to its broad applicability in diverse areas of mathematics, including numerical analysis and both differential and integral equations. A multitude of researchers has investigated various fixed point theorems, proposing new contractive conditions in metric spaces or their broader counterparts (also see [2], [3], [4], [5], [6] [7], [8], etc.).

In 2011, complex valued metric spaces were introduced for the first time by [9], followed by [10] in 2012, who improved the contraction theorem by applying it to control functions and establishing common fixed point theorems. Fixed point theorems in complex valued metric spaces were proven and applications were provided by [11]. Furthermore, in 2013, proposed a method for Urysohn integral equations employing common fixed points in metric spaces that include complex values by [12]. A common fixed point theorem for weakly compatible mappings in complex-valued metric spaces was established, avoiding continuity concepts as presented by [13]. Research was conducted by [14] on theorems concerning common fixed points, leveraging property (EA) in metric spaces that incorporate complex values. Additionally, illustrated various results related to common fixed points for pairs of mappings that satisfy broader contraction conditions expressed through rational functions, involving point-dependent control functions as coefficients within complex-valued metric space by [15]. The introduction of complex valued b-metric spaces occurred in 2013 by [16]. Common fixed point theorems were demonstrated in generalized complex valued b-metric spaces by [17]. Weak compatible mappings were initially introduced by [18]. This paper introduces new contractive mappings within the framework of completeness and the uniqueness of fixed point theorems in complex valued b-metric spaces, building upon previous research.

2. Preliminaries

Assume that $\omega_1, \omega_2 \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers that define a partial order $\leq$ on $\mathbb{C}$ such that $\omega_1 \leq \omega_2$ if and only if $Re(\omega_1) \leq Re(\omega_2)$ and $Im(\omega_1) \leq Im(\omega_2)$ that is $\omega_1 \leq \omega_2$. If any of the following is true [9],

1. $Re(\omega_1) = Re(\omega_2)$ and $Im(\omega_1) = Im(\omega_2)$;
2. $Re(\omega_1) < Re(\omega_2)$ and $Im(\omega_1) = Im(\omega_2)$;
3. $Re(\omega_1) = Re(\omega_2)$ and $Im(\omega_1) < Im(\omega_2)$;
4. $Re(\omega_1) < Re(\omega_2)$ and $Im(\omega_1) < Im(\omega_2)$;

we will write $\omega_1 \lneq \omega_2$ if $\omega_1 \neq \omega_2$ and one of (2),(3), and (4) satisfiy. Take note of that

$$0 \leq \omega_1 \lneq \omega_2 \Rightarrow |\omega_1| < |\omega_2|$$
$$\omega_1 \leq \omega_2, \omega_2 < \omega_3 \Rightarrow \omega_1 < \omega_3.$$

Let Δ be a set that is not empty. A mapping $\Omega : \Delta \times \Delta \to \mathbb{C}$ is considered to be congruent with the following conditions [9],

1. $0 \leq \Omega(\mu, \nu)$ for all $\mu, \nu \in \Delta$ and $\Omega(\mu, \nu) = 0$ if and only if $\mu = \nu$;
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$, for all $\mu, \nu \in \Delta$;
3. $\Omega(\mu, \nu) \leq \Omega(\mu, \omega) + \Omega(\omega, \nu)$ for all $\mu, \nu, \omega \in \Delta$.

In this case, the metric Ω is a complex valued metric on the set Δ, and the space (Δ, Ω) is a complex-valued metric space.
Let Δ be a set that is not empty and $s \geq 1$ a given real number. A mapping $\Omega : \Delta \times \Delta \to \mathbb{C}$ is considered to be congruent with the following conditions [16],

1. $0 \leq \Omega(\mu, \nu)$ for all $\mu, \nu \in \Delta$ and $\Omega(\mu, \nu) = 0$ if and only if $\mu = \nu$;
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$, for all $\mu, \nu \in \Delta$;
3. $\Omega(\mu, \nu) \leq s\left[\Omega(\mu, \omega) + \Omega(\omega, \nu)\right]$ for all $\mu, \nu, \omega \in \Delta$.

Then Ω is called a complex valued b-metric on Δ and (Δ, Ω) is called a complex valued b-metric space.
Let $\Delta = [0, 1]$. Define the mapping $\Omega : \Delta \times \Delta \to \mathbb{C}$ by [17],

$$\Omega(\mu, \nu) = |\mu - \nu|^2 + i\,|\mu - \nu|^2 \,\forall \mu, \nu \in \Delta$$

Then (Δ, Ω) is a complex valued b-metric space with $s = 2$.
Suppose that (Δ, Ω) is a complex-valued b-metric space if [16],

1. A point $\mu \in \Delta$ is referred to as an interior point of a set $A \subseteq \Delta$ if there exists a $r \in \mathbb{C}$ such that $B(\mu, r) = \{\nu \in \Delta : \Omega(\mu, \nu) < r\} \subseteq A$.
2. A point $\mu \in \Delta$ is called a limit point of a set $A \subseteq \Delta$ if there exists a $r \in \mathbb{C}$ such that $B(\mu, r) \cap (\Delta \setminus A) \neq \varnothing$ for all $0 < r \in \mathbb{C}$.

3. A subset $B \subseteq \Delta$ is referred to as open if each limit point of B is an interior point of B.
4. A subset $B \subseteq \Delta$ is considered closed if each limit point of B is included in B.
5. A subbase for a topology Δ is the family$F = \{B(\mu, r) : \mu \in \Delta\}$ and $0 < r$. This intricate topology is represented by the symbol τ_c. In fact, the topology τ_c is Hausdorff.

Assume that (Δ, Ω) is a space with complex values and b-metric, and let $\{\mu_n\}$ be a sequence in Δ, with $\mu \in \Delta$ [16].

1. μ_n is said to converge to μ and μ is a limit point of μ_n if for every$c \in \mathbb{C}$ with $0 < c$, there exists $n_0 \in N$ such that $\Omega(\mu_n, \mu) < c$ for all $n > n_0$. We denote this as $\mu_n \to \mu$ as $n \longrightarrow \infty$ or $\lim_{n\to\infty} \mu_n = \mu$.
2. $\{\mu_n\}$ is a Cauchy sequence if for every $c \in \mathbb{C}$ with $0 < c$, there exists $n_0 \in N$ such that for all $n > n_0$, $\Omega(\mu_n, \mu_{n+m}) < c$ where $m \in \mathbb{N}$.
3. If all Cauchy sequences are convergent in (Δ, Ω), then (Δ, Ω) is referred to as a complete complex valued b-metric space.

Lemma 1. (Δ, Ω) is a space that is a complex-valued b-metric space, and let $\{\mu_n\}$ be a sequence that is contained within Δ. If and only if the function $|\Omega(\mu_n, \mu)| \to 0$ converges to zero as $n \to \infty$ [16].

Lemma 2. (Δ, Ω) is a space that is a complex valued b-metric space, and let $\{\mu_n\}$ be a sequence that is contained within Δ. The series $\{\mu_n\}$ is considered to be a Cauchy sequence if and only if $|\Omega(\mu_n, \mu_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$ [16].

Given that L and M are self-mappings on a set Δ, if $w = L\mu = M\mu$ for some in Δ, then Δ is referred to as a coincidence point of L and M, and w is referred to as a point of coincidence of L and M [17].
If the self-mappings $L, M : \Delta \to \Delta$ commute at their respective coincidence point, then we refer to this pair of self-mappings as weakly compatible [18].

3. Main result

Theorem 1. Let (Δ, Ω) be a complete complex valued b-metric space and $L, M : \Delta \to \Delta$. If there exists mappings $\lambda, \delta : \Delta \to [0, 1)$ such that for all $\mu, \nu \in \Delta$,

1. $\lambda(L\mu) \leq \lambda(\mu)$, $\delta(L\mu) \leq \delta(\mu)$
2. $\lambda(M\mu) \leq \lambda(\mu)$ and $\delta(M\mu) \leq \delta(\mu)$
3. $(\lambda + \delta)\mu < 1$

Then the following holds

$$\Omega(L\mu, M\nu) \leq \lambda(\mu) \max \left\{ (\Omega(\mu, \nu), \frac{\Omega(\mu, L\mu)\Omega(\nu, M\nu)}{\Omega(\mu, M\nu) + \Omega(\nu, L\mu) + \Omega(\mu, \nu)} \right\}$$
$$+\delta(\mu) \{\Omega(\mu, L\mu) + \Omega(\nu, M\nu)\} \quad (1)$$

for all $\mu, \nu \in \Delta$ such that $\mu \neq \nu$, $\Omega(\mu, M\nu) + \Omega(\nu, L\mu) + \Omega(\mu, \nu) \neq 0$ and $s \geq 1$ then L and M have a unique common fixed point.

Proof. Let $\mu_0 \in \Delta$ be arbitrary. Since $L(\mu) \subseteq \Delta$ and $M(\mu) \subseteq \Delta$, we can construct the sequence $\{\mu_n\}$ in Δ such that $\mu_n = L\mu_{n-1}$ and $\mu_{n+1} = M\mu_n$ for all $n \geq 0$. Then from (1)

$$\begin{aligned}
\Omega(\mu_n, \mu_{n+1}) &= \Omega(L\mu_{n-1}, M\mu_n) \\
&\leq \lambda(\mu_n) \max\left\{\Omega(\mu_{n-1}, \mu_n), \frac{\Omega(\mu_{n-1}, L\mu_{n-1})\Omega(\mu_n, M\mu_n)}{\Omega(\mu_{n-1}, M\mu_n) + \Omega(\mu_n, L\mu_{n-1}) + \Omega(\mu_{n-1}, \mu_n)}\right\} \\
&+ \delta(\mu_n)\left\{\Omega(\mu_{n-1}, M\mu_n) + \Omega(\mu_n, L\mu_{n-1})\right\} \\
&\leq \lambda(\mu_n) \max\left\{\Omega(\mu_{n-1}, \mu_n), \frac{\Omega(\mu_{n-1}, \mu_n)\Omega(\mu_n, \mu_{n+1})}{\Omega(\mu_{n-1}, \mu_{n+1}) + \Omega(\mu_n, \mu_n) + \Omega(\mu_{n-1}, \mu_n)}\right\} \\
&+ \delta(\mu_n)\left\{\Omega(\mu_{n-1}, \mu_{n+1}) + \Omega(\mu_n, \mu_n)\right\} \\
&\leq \lambda(\mu_n) \max\left\{\Omega(\mu_{n-1}, \mu_n), \frac{\Omega(\mu_{n-1}, \mu_n)\Omega(\mu_n, \mu_{n+1})}{\Omega(\mu_{n-1}, \mu_{n+1}) + d(\mu_{n-1}, \mu_n)}\right\} \\
&+ \delta(\mu_n)\Omega(\mu_{n-1}, \mu_{n+1}) \\
&\leq \lambda(\mu_n)\Omega(\mu_{n-1}, \mu_n) + \delta(\mu_n)\Omega(\mu_{n-1}, \mu_{n+1}) \\
&\leq \lambda(\mu_n)\Omega(\mu_{n-1}, \mu_n) + s\delta(\mu_n)\left\{\Omega(\mu_{n-1}, \mu_n) + \Omega(\mu_n, \mu_{n+1})\right\} \\
&\Rightarrow \Omega(\mu_n, \mu_{n+1}) \leq \left\{\frac{\lambda(\mu_n) + \delta(\mu_n)s}{1 - \delta(\mu_n)s}\right\}\Omega(\mu_{n-1}, \mu_n) \\
&\Rightarrow |\Omega(\mu_n, \mu_{n+1})| \leq k\,|\Omega(\mu_{n-1}, \mu_n)|
\end{aligned}$$

Where $k = \left\{\frac{\lambda(\mu_n)+\delta(\mu_n)s}{1-\delta(\mu_n)s}\right\} \leqslant 1$

$$\Rightarrow |\Omega(\mu_n, \mu_{n+1})| \leq k^2\,|\Omega(\mu_{n-2}, \mu_{n-1})|$$

Continuing the above process, we get

$$|\Omega(\mu_n, \mu_{n+1})| \leq k^n\,|\Omega(\mu_0, \mu_1)|$$

on taking the limit as $n \longrightarrow \infty$ we obtain that $|\Omega(\mu_n, \mu_{n+1})| \longrightarrow 0$. Now, we show that $\{\mu_n\}$ is a Cauchy sequence in Δ, let $m, n \in N$, $m > n$ then

$$\begin{aligned}
\Omega(\mu_n, \mu_m) &\leq s\left\{\Omega(\mu_n, \mu_{n+1}) + \Omega(\mu_{n+1}, \mu_m)\right\} \\
&\leq s\left\{\Omega(\mu_n, \mu_{n+1})\right\} + s^2\left\{\Omega(\mu_{n+1}, \mu_{n+2}) + \Omega(\mu_{n+2}, \mu_m)\right\} \\
&\leq s\left\{\Omega(\mu_n, \mu_{n+1})\right\} + s^2\left\{\Omega(\mu_{n+1}, \mu_{n+2})\right\} \\
&\quad + s^3\left\{\Omega(\mu_{n+2}, \mu_{n+3}) + \Omega(\mu_{n+3}, \mu_m)\right\} \\
&\leq sk^n\left\{1 + sk + (sk)^2 + (sk)^3 + \dots\right\}\Omega(\mu_0, \mu_1) \\
&\leq sk^n\Omega(\mu_0, \mu_1)\left\{1 + sk + (sk)^2 + (sk)^3 + \dots\right\} \\
\Rightarrow \Omega(\mu_n, \mu_m) &\leq \left(\frac{sk^n}{1 - sk}\right)\Omega(\mu_0, \mu_1) \\
\Rightarrow |\Omega(\mu_n, \mu_m)| &\leq \left(\frac{sk^n}{1 - sk}\right)|\Omega(\mu_0, \mu_1)|
\end{aligned}$$

On taking the limit, we have $\lim_{n\to\infty} \Omega(\mu_n, \mu_m) = 0$ as $n, m \longrightarrow \infty$, since $\left(\frac{sk^n}{1-sk}\right) < 1$ hence sequence $\{\mu_n\}$ is Cauchy sequence in Δ. Since Δ is complete, then $\exists$ u in Δ such that

$\lim_{n\to\infty} \mu_n = u$. Now we show that u is the fixed point of T.

$$\begin{aligned}
\Omega(u, Lu) &\leq s\{\Omega(u, \mu_{n+1}) + \Omega(\mu_{n+1}, Lu)\} \\
&\leq s\{\Omega(u, \mu_{n+1}) + \Omega(M_n, Lu)\} \\
&\leq s\Omega(u, \mu_{n+1}) + s\lambda(\mu_n) \max\left\{\Omega(\mu_n, u), \frac{\Omega(\mu_n, M\mu_n)\Omega(u, Lu)}{\Omega(\mu_n, Lu) + \Omega(u, M\mu_n) + \Omega(\mu_n, u)}\right\} \\
&+ s\delta(\mu_n)\{\Omega(\mu_n, Lu) + \Omega(u, M\mu_n)\} \\
&\leq s\Omega(u, \mu_{n+1}) + s\lambda(\mu_n) \max\left\{\Omega(\mu_n, u), \frac{\Omega(\mu_n, \mu_{n+1})\Omega(u, Lu)}{\Omega(\mu_n, Lu) + \Omega(u, \mu_{n+1}) + \Omega(\mu_n, u)}\right\} \\
&+ s^2\delta(\mu_n)\{\Omega(\mu_n, u) + \Omega(u, Lu)\} + s\delta(\mu_n)\Omega(u, \mu_{n+1})
\end{aligned}$$

$$\Rightarrow \left\{1 - s^2\delta(\mu_n)\right\}\Omega(u, Lu) \leq s\{1 + \delta(\mu_n)\}\Omega(u, \mu_{n+1}) + s\lambda(\mu_n)A + s^2\delta(\mu_n)\Omega(\mu_n, u)$$

Where,

$$A = \max\left\{\Omega(\mu_n, u), \frac{\Omega(\mu_n, \mu_{n+1})\Omega(u, Lu)}{\Omega(\mu_n, Lu) + \Omega(u, \mu_{n+1}) + \Omega(\mu_n, u)}\right\}$$

$$\Rightarrow A \leqslant \max\{\Omega(\mu_n, u), \Omega(\mu_n, \mu_{n+1})\Omega(u, Lu)\}$$

Case-(i)

If $A = \Omega(\mu_n, u)$ then we get

$$\left\{1 - s^2\delta(\mu_n)\right\}\Omega(u, Lu) \leq s\{1 + \delta(\mu_n)\}\Omega(u, \mu_{n+1}) + s\lambda(\mu_n)\Omega(\mu_n, u) + s^2\delta(\mu_n)\Omega(\mu_n, u)$$

$$\left\{1 - s^2\delta(\mu_n)\right\}\Omega(u, Lu) \leq s\{1 + \delta(\mu_n)\}\Omega(u, \mu_{n+1}) + \left\{s\lambda(\mu_n) + s^2\delta(\mu_n)\right\}\Omega(\mu_n, u)$$

$$\Rightarrow \Omega(u, Lu) \leq \frac{s\{1 + \delta(\mu_n)\}}{1 - s^2\delta(\mu_n)}\Omega(u, \mu_{n+1}) + \frac{\left\{s\lambda(\mu_n) + s^2\delta(\mu_n)\right\}}{1 - s^2\delta(\mu_n)}\Omega(\mu_n, u)$$

Taking the limit as $n \longrightarrow \infty$, we get
$\lim_{n\to\infty} \Omega(u, Lu) = 0, \left[\frac{s\{1+\delta(\mu_n)\}}{1-s^2\delta(\mu_n)}, \frac{\left\{s\lambda(\mu_n)+s^2\delta(\mu_n)\right\}}{1-s^2\delta(\mu_n)} < 1\right]$
Hence u is the fixed point of L.

Case-(ii)

If $A = \Omega(\mu_n, \mu_{n+1})\Omega(u, Lu)$, then we obtain that
$A \leqslant k^n\Omega(\mu_0, \mu_1)\Omega(u, Lu)$, since $k < 1$.
On taking the limit as $n \longrightarrow \infty$, we have $A \longrightarrow 0$. Then as

$$\left\{1 - s^2\delta(\mu_n)\right\}\Omega(u, Lu) \leq s\{1 + \delta(\mu_n)\}\Omega(u, \mu_{n+1}) + s^2\delta(\mu_n)\Omega(\mu_n, u)$$

$$\Rightarrow \Omega(u, Lu) \leq \frac{s\{1 + \delta(\mu_n)\}}{1 - s^2\delta(\mu_n)}\Omega(u, \mu_{n+1}) + \frac{s^2\delta(\mu_n)}{1 - s^2\delta(\mu_n)}\Omega(\mu_n, u)$$

On taking the limit as $n \longrightarrow \infty$, we get that u is a Fixed point of L. Similarly, one can show that u=Mu. Therefore, u is a common fixed point of L and M.
The uniqueness of the fixed point.
Let there exists another common fixed point of L and M is u_1, that is, $u_1 = Lu_1 = Mu_1$.

$$\begin{aligned}
\Omega(u, u_1) &= \Omega(Lu, Mu_1) \\
&\leq \lambda(u) \max\left\{\Omega(u, u_1), \frac{\Omega(u, Lu)\Omega(u_1, Mu_1)}{\Omega(u, Mu_1) + \Omega(u_1, Lu) + \Omega(u, u_1)}\right\} \\
&\quad + \delta(u)\{\Omega(u, Lu) + \Omega(u_1, Mu_1)\} \\
&\leq \lambda(u)\Omega(u, u_1))
\end{aligned}$$

Since $\lambda(u) \in [0,1)$, we have $\Omega(u,u_1) = 0$. Therefore we have $u = u_1$ and thus u is a unique common fixed point of L and M. ∎

Corollary 1. Let (Δ, Ω) be a complete complex valued b-metric space and $L, M : \Delta \to \Delta$. If there exist two nonnegative real no's λ and δ such that for all $\mu, \nu \in \Delta$. If L and M satisfy the following

$$\Omega(L\mu, M\nu) \leq \lambda \max\left\{(\Omega(\mu,\nu), \frac{\Omega(\mu, L\mu)\Omega(\nu, M\nu)}{\Omega(\mu, M\nu) + \Omega(\nu, L\mu) + \Omega(\mu,\nu)}\right\} + \delta\left\{\Omega(\mu, L\mu) + \Omega(\nu, M\nu)\right\}$$

for all $\mu, \nu \in \Delta$, and $\lambda + \delta < 1$. Then L and M have a common fixed point.

Theorem 2. Let L, M, S and T be four self mappings of complete valued b-metric spaces if there exists mapping $\lambda, \delta : \Delta \longrightarrow [0,1)$ such that for all $\mu, \nu \in \Delta$ satisfying

1. $L(\Delta) \subseteq T(\Delta)$ and $M(\Delta) \subseteq S(\Delta)$
2. $$\Omega(S\mu, T\nu) \leq \lambda(\mu) \max\left\{\Omega(S\mu, T\nu), \frac{\Omega(S\mu, L\mu)\Omega(T\nu, M\nu)}{\Omega(S\mu, M\nu) + \Omega(T\nu, L\mu) + \Omega(S\mu, M\nu)}\right\} + \delta(\mu)\left\{\Omega(S\mu, L\mu) + \Omega(T\nu, M\nu)\right\}$$
3. Pairs (L,S) and (M,T) are weakly compatible and $M(\Delta)$ is closed subspace of Δ. Where $\Omega(S\mu, M\nu) + \Omega(T\nu, L\mu) + \Omega(S\mu, T\nu) \neq 0$ and $s\left\{\lambda(\mu) + \delta(\mu)\right\} < 1$

Then L, M, S, and T have a unique common Fixed point.

Proof. Consider a sequences $\{\nu_n\}$ and $\{\mu_n\}$ in Δ such that
$\nu_n = L\mu_n = T\mu_{n+1}$ and $\nu_{n+1} = M\mu_{n+1} = S\mu_{n+2}$. To prove $\{\nu_n\}$ is a Cauchy sequence of Δ.

$$\begin{aligned}
\Omega(\nu_n, \nu_{n+1}) &= \Omega(L\mu_n, M\mu_{n+1}) \\
&\leq \lambda(\nu_n) \max\left\{\Omega(S\mu_n, T\mu_{n+1}), \frac{\Omega(S\mu_n, L\mu_n)\Omega(T\mu_{n+1}, M\mu_{n+1})}{\Omega(S\mu_n, M\mu_{n+1}) + \Omega(T\mu_{n+1}, L\mu_n) + \Omega(S\mu_n, T\mu_{n+1})}\right\} \\
&+ \delta(\nu_n)\left\{\Omega(S\mu_n, L\mu_n) + \Omega(T\mu_{n+1}, M\mu_{n+1})\right\} \\
&\leq \lambda(\nu_n) \max\left\{\Omega(\nu_{n-1}, \nu_n), \frac{\Omega(\nu_{n-1}, \nu_n)\Omega(\nu_n, \nu_{n+1})}{\Omega(\nu_{n-1}, \nu_{n+1}) + \Omega(\nu_n, \nu_n) + \Omega(\nu_{n-1}, \nu_n)}\right\} \\
&+ \delta(\nu_n)\left\{\Omega(\nu_{n-1}, \nu_n) + \Omega(\nu_n, \nu_{n+1})\right\} \\
&\leq \lambda(\nu_n) \max\left\{\Omega(\nu_{n-1}, \nu_n), \frac{\Omega(\nu_{n-1}, \nu_n)\Omega(\nu_n, \nu_{n+1})}{\Omega(\nu_{n-1}, \nu_{n+1}) + \Omega(\nu_{n-1}, \nu_n)}\right\} \\
&+ \delta(\nu_n)\left\{\Omega(\nu_{n-1}, \nu_{n+1})\right\} \\
&\leq \lambda(\nu_n)\Omega(\nu_{n-1}, \nu_n) + s\delta(\nu_n)\left\{\Omega(\nu_{n-1}, \nu_n) + \Omega(\nu_n, \nu_{n+1})\right\} \\
&\Rightarrow \Omega(\nu_n, \nu_{n+1}) \leq \left\{\lambda(\nu_n) + s\delta(\nu_n)\right\}\Omega(\nu_{n-1}, \nu_n) \\
&\Rightarrow |\Omega(\nu_n, \nu_{n+1})| \leq \left\{\lambda(\nu_n) + s\delta(\nu_n)\right\}|\Omega(\nu_{n-1}, \nu_n)| \\
&\Rightarrow |\Omega(\nu_n, \nu_{n+1})| \leq \ell\,|\Omega(\nu_{n-1}, \nu_n)|
\end{aligned}$$

Where $\ell = \{\lambda(\nu_n) + s\delta(\nu_n)\}$

$$\Rightarrow |\Omega(\nu_n, \nu_{n+1})| \leq \ell^2\,|\Omega(\nu_{n-2}, \nu_{n-1})|$$

Continuing the above process, we get

$$|\Omega(\nu_n, \nu_{n+1})| \leq \ell^n\,|\Omega(\nu_0, \nu_1)|$$

on taking the limit as $n \longrightarrow \infty$ we obtain that $|\Omega(\nu_n, \nu_{n+1})| \longrightarrow 0$
Now, we show that $\{\nu_n\}$ is a Cauchy sequence in Δ, let $m, n \in N$, $m > n$ then

$$\begin{aligned}
\Omega(\nu_n, \nu_m) &\leq s\{\Omega(\nu_n, \nu_{n+1}) + \Omega(\nu_{n+1}, \nu_m)\} \\
&\leq s\{\Omega(\nu_n, \nu_{n+1})\} + s^2\{\Omega(\nu_{n+1}, \nu_{n+2}) + \Omega(\nu_{n+2}, \nu_m)\} \\
&\leq s\{\Omega(\nu_n, \nu_{n+1})\} + s^2\{\Omega(\nu_{n+1}, \nu_{n+2})\} \\
&\quad + s^3\{\Omega(\nu_{n+2}, \nu_{n+3}) + d(\nu_{n+3}, \nu_m)\} \\
&\leq s\ell^n\left\{1 + s\ell + (\ell k)^2 + (s\ell)^3 + \ldots\right\}\Omega(\nu_0, \nu_1) \\
&\leq s\ell^n\Omega(\nu_0, \nu_1)\left\{1 + s\ell + (s\ell)^2 + (s\ell)^3 + \ldots\right\} \\
\Rightarrow \Omega(\nu_n, \nu_m) &\leq \left(\frac{s\ell^n}{1 - s\ell}\right)\Omega(\nu_0, \nu_1) \\
\Rightarrow |\Omega(\nu_n, \nu_m)| &\leq \left(\frac{s\ell^n}{1 - s\ell}\right)|\Omega(\nu_0, \nu_1)|
\end{aligned}$$

On taking the limit, we have $\lim_{n\to\infty}\Omega(\nu_n, \nu_m) = 0$ as $n, m \longrightarrow \infty$, since $\left(\frac{s\ell^n}{1-s\ell}\right) < 1$ hence sequence $\{\nu_n\}$ is Cauchy sequence in Δ. Since Δ is complete then $\exists\ \omega$ in Δ such that $\lim_{n\to\infty} L\mu_n = \lim_{n\to\infty} T\mu_{n+1} = \lim_{n\to\infty} M\mu_{n+1} = \lim_{n\to\infty} S\mu_{n+2} = \omega$.
Now since $M(\Delta)$ is a closed subspace of Δ and so $\omega \in M(\Delta)$.
Since $M(\Delta) \subseteq S(\Delta)$, then there exist a point $v \in \Delta$, such that $\omega = Sv$. Now we show that $Lv = Sv = \omega$, we have

$$\begin{aligned}
&d(Lv, \omega) \leq s\{\Omega(Lv, \mu_n) + \Omega(M\mu_n, \omega)\} \\
&\qquad \frac{1}{s}\Omega(Lv, \omega) \leq \lambda(\mu_n)\max\left\{\Omega(Sv, T\mu_n), \frac{\Omega(Sv, Lv)\Omega(T\mu_n, \mu_n)}{\Omega(Sv, M\mu_n) + \Omega(T\mu_n, Lv)\Omega(Sv, T\mu_n)}\right\} \\
&\qquad + \delta(\mu_n)\{\Omega(Sv, Lv) + \Omega(T\mu_n, M\mu_n)\} + \Omega(M\mu_n, \omega) + \Omega(M\mu_n, \omega)
\end{aligned}$$

letting $n \longrightarrow \infty$

$$\begin{aligned}
&\Rightarrow \frac{1}{s}\Omega(Lv, \omega) \leq \lambda(\mu_n)\max\left\{\Omega(Sv, \omega), \frac{\Omega(Sv, Lv)\Omega(\omega, \omega)}{\Omega(Sv, \omega) + \Omega(\omega, Lv) + \Omega(Sv, \omega)}\right\} \\
&\qquad + \delta(\mu_n)\{\Omega(Sv, Lv) + \Omega(\omega, \omega)\} + \Omega(\omega, \omega) \\
&\qquad \Rightarrow \frac{1}{s}\Omega(Lv, \omega) \leq \lambda(\mu_n)\Omega(Sv, \omega) + \delta(\mu_n)\Omega(Sv, Lv) \\
&\qquad \Rightarrow \frac{1}{s}\Omega(Lv, \omega) \leq \lambda(\mu_n)\Omega(Sv, \omega) \\
&\qquad \Rightarrow \frac{1}{s}\Omega(Lv, \omega) = 0 \\
&\qquad \Rightarrow |\Omega(Lv, \omega)| = 0
\end{aligned}$$

Thus $Lv = Mv = \omega$ hence v is coincidence point of (L, S) Since $L(\Delta) \subseteq T(\Delta)$ and now $\omega \in L(\Delta)$, then there exist a point $w \in \Delta$ such that $\omega = Tw$, now we show that $Mw = \omega$, we also have $Lv = Sv = Tw = \omega$

$$\begin{aligned}
&\Omega(\omega, Mw) \leq s\{\Omega(\omega, L\mu_n) + \Omega(L\mu_n, Mw)\} \\
&\qquad \Rightarrow \frac{1}{s}\Omega(\omega, Mw) \\
&\qquad \leq \Omega(\omega, L\mu_n) + \lambda(\mu_n)\max\left\{\Omega(S\mu_n, Tw), \frac{\Omega(S\mu_n, L\mu_n)\Omega(Tw, Mw)}{\Omega(S\mu_n, Mw) + \Omega(Tw, L\mu_n) + \Omega(S\mu_n, Tw)}\right\}
\end{aligned}$$

letting $n \longrightarrow \infty$

$$
\begin{aligned}
\Rightarrow \frac{1}{s}d(\omega, Mw) &\leq \Omega(\omega,\omega) + \lambda(\mu_n)\max\left\{\Omega(\omega, Tw), \frac{\Omega(\omega,\omega)d(Tw,Mw)}{\Omega(\omega,Mw)+\Omega(Tw,\omega)+\Omega(\omega,Tw)}\right\} \\
&+ \delta(\mu_n)\left\{\Omega(\omega,\omega)+\Omega(Tw,Mw)\right\} \\
&\Rightarrow |\Omega(\omega, Mw)| = 0 \\
&\Rightarrow Mw = \omega
\end{aligned}
$$

Thus $Lv = Mv = \omega$ hence v is coincidence point of (M, T) Now we have $Lv = Sv = Tw = Mw = \omega$ since L and S are weakly compatible mapping then $LSv = SLv = L\omega = S\omega$, now we show that ω is fixed point of L on contrary if $L\omega \neq \omega$ then we have

$$
\begin{aligned}
\Omega(L\omega,\omega) &\leq s\left\{\Omega(L\omega, M\mu_{n+1}) + \Omega(M\mu_{n+1},\omega)\right\} \\
&\Rightarrow \frac{1}{s}\Omega(L\omega,\omega) \\
&\leq s\lambda(\mu_{n+1})\max\left\{\Omega(S\omega, T\mu_{n+1}), \frac{\Omega(S\omega,L\omega)\Omega(T\mu_{n+1},M\mu_{n+1})}{\Omega(S\omega,M\mu_{n+1})+\Omega(T\mu_{n+1},L\omega)+\Omega(S\omega,T\mu_{n+1})}\right\} \\
&+ s\delta(\mu_{n+1})\left\{\Omega(S\omega,L\omega)+\Omega(T\mu_{n+1},M\mu_{n+1})\right\} + s\Omega(M\mu_{n+1},\omega)
\end{aligned}
$$

letting $n \longrightarrow \infty$

$$
\begin{aligned}
\frac{1}{s}\Omega(L\omega,\omega) &\leq s\lambda(\mu_{n+1})\max\left\{\Omega(S\omega,\omega), \frac{\Omega(S\omega,L\omega)\Omega(\omega,\omega)}{\Omega(S\omega,\omega)+\Omega(\omega,L\omega)+\Omega(S\omega,\omega)}\right\} \\
&+ s\delta(\mu_{n+1})\left\{\Omega(S\omega,L\omega)+\Omega(\omega,\omega)\right\} + s\Omega(\omega,\omega) \\
&\Rightarrow \frac{1}{s}\Omega(L\omega,\omega) \leq s\lambda(\mu_{n+1})\Omega(S\omega,\omega) \\
&\Rightarrow |\Omega(L\omega,\omega)| \leq s\lambda(\mu_{n+1})\,|\Omega(L\omega,\omega))| \\
&\Rightarrow |\Omega(L\omega,\omega)| = 0
\end{aligned}
$$

Hence $L\omega = S\omega = \omega$. Similarly, we show that ω is a fixed point of M and $M\omega = T\omega = \omega$. Therefore, ω is a common fixed point of L, M, S, and T.
Uniqueness of the fixed point: Let ω_1 be another Fixed point of L, M, S, and T then $L\omega_1 = M\omega_1 = S\omega_1 = T\omega_1 = \omega_1$,

$$
\begin{aligned}
\Omega(\omega,\omega_1) &= \Omega(L\omega, M\omega_1) \\
&\leq \lambda(\omega)\max\left\{\Omega(S\omega, T\omega_1), \frac{\Omega(S\omega,L\omega)\Omega(T\omega_1,M\omega_1)}{\Omega(S\omega,M\omega_1)+\Omega(T\omega_1,L\omega)+\Omega(S\omega,T\omega_1)}\right\} \\
&+ \delta(\omega)\left\{\Omega(S\omega,L\omega)+\Omega(T\omega_1,M\omega_1)\right\} \\
&\Rightarrow |\Omega(\omega,\omega_1)| \leq \lambda(\omega)\,|\Omega(\omega,\omega_1)|
\end{aligned}
$$

Since $\lambda(\omega) < 1$

$$
\begin{aligned}
&\Rightarrow |\Omega(\omega,\omega_1)| = 0 \\
&\Rightarrow \omega = \omega_1
\end{aligned}
$$

Hence ω is the unique common fixed point of L, M, S, and T. ■

4. Applications

In this section, we apply Theorem 1 to the existence of a common solution of the system of Urysohn integral equations.
Let $\Delta = C([a,b],\mathbb{R}^n)$, where $[a,b] \subseteq \mathbb{R}^+$. and $d : \Delta\times\Delta \to \mathbb{C}$ is define by

$$
d(\mu,\nu) = \max_{t\in[a,b]} ||\mu(t)-\nu(t)||_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}
$$

Consider the Urysohn integral equations

$$\mu(t) = \int_a^b \kappa_1(t,s,\mu(s))\,ds + g(t) \tag{2}$$

$$\mu(t) = \int_a^b \kappa_2(t,s,\mu(s))\,ds + h(t) \tag{3}$$

where $t \in [a,b] \subset \mathbb{R}$ and $\mu, g, h \in \Delta$
Suppose that $\kappa_1, \kappa_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to R^n$ are such that $F_\mu, G_\mu \in \Delta$ for each $\mu \in \Delta$, where

$$F_\mu(t) = \int_a^b \kappa_1(t,s,\mu(s))\,ds + g(t),$$
$$G_\mu(t) = \int_a^b \kappa_2(t,s,\mu(s))\,ds + h(t)$$

for all $t \in [a,b]$. If there exists two mapping $\lambda, \delta : \Delta \to [0,1)$ such that for all $\mu, \nu \in \Delta$,

1. $\lambda(F_\mu + g) \leq \lambda(\mu)$ and $\delta(F_\mu + g) \leq \delta(\mu)$
2. $\lambda(G_\mu + h) \leq \lambda(\mu)$ and $\delta(G_\mu + h) \leq \delta(\mu)$
3. $(\lambda + \delta)\mu < 1$
4. $||F_\mu(t) - G_\nu(t) + g(t) - h(t)||_\infty \sqrt{1+a^2}e^{i\tan^{-1}a} \leq \lambda(\mu)A(\mu,\nu)(t) + \delta(\mu)B(\mu,\nu)(t)$

$$A(\mu,\nu)(t) =$$
$$\max_{t\in[a,b]} \left\{ ||\mu(t)-\nu(t)||_\infty, \frac{\left(||F_\mu(t)+g(t)-\mu(t)||_\infty ||G_\nu(t)+h(t)-\nu(t)||_\infty\right)\sqrt{1+a^2}e^{i\tan^{-1}a}}{||G_\nu(t)+h(t)-\mu(t)||_\infty + ||F_\mu(t)+g(t)-\nu(t)||_\infty + ||\mu(t)-\nu(t)||_\infty} \right\}$$

$$B(\mu,\nu)(t) = \left\{ ||F_\mu(t)+g(t)-\mu(t)||_\infty ||G_\nu(t)+h(t)-\nu(t)||_\infty \right\} \sqrt{1+a^2}e^{i\tan^{-1}a}$$

then the system of integral equations (2) and (3) have a unique common solution.
Proof. Let (Δ, Ω) be a complete complex valued b-metric space and $L, M : \Delta \times \Delta \to \Delta$ by $L\mu = F_\mu + g$ and $M\Delta = G_\mu + h$. Then

$$\Omega(L\mu, M\nu) = \max_{t\in[a,b]} ||F_\mu(t) - G_\nu(t) + g(t) - h(t)||_\infty \sqrt{1+a^2}e^{i\tan^{-1}a},$$
$$\Omega(\mu, L\mu) = \max_{t\in[a,b]} ||F_\mu(t) + g(t) - \mu(t)||_\infty \sqrt{1+a^2}e^{i\tan^{-1}a},$$
$$\Omega(\nu, M\nu) = \max_{t\in[a,b]} ||G_\nu(t) + h(t) - \nu(t)||_\infty \sqrt{1+a^2}e^{i\tan^{-1}a},$$
$$\Omega(\mu, M\nu) = \max_{t\in[a,b]} ||G_\nu(t) + h(t) - \mu(t)||_\infty$$

and

$$\Omega(\nu, L\mu) = \max_{t\in[a,b]} ||F_\mu(t) + g(t) - \nu(t)||_\infty$$

It is easily seen that for all $\mu, \nu \in \Delta$, we have

1. $\lambda(L\mu) \leq \lambda(\mu)$, $\delta(L\mu) \leq \delta(\mu)$

2. $\lambda(M\mu) \leq \lambda(\mu)$ and $\delta(M\mu) \leq \delta(\mu)$

3.

$$\Omega(L\mu, M\nu) \leq \lambda(\mu) \max \left\{ \Omega(\mu,\nu), \frac{\Omega(\mu, L\mu)\Omega(\nu, M\nu)}{\Omega(\mu, M\nu) + \Omega(\nu, L\mu) + \Omega(\mu,\nu)} \right\} + \delta(\mu) \left\{ \Omega(\mu, L\mu) + d(\nu, M\nu) \right\}$$

By theorem 1, we get L and M have a common fixed point. Thus, there is a unique point $\mu \in \Delta$ such as $\mu = L\mu = M\mu$. Now, we have
$\mu = L\mu = F_\mu + g$ and $\mu = M\mu = G_\mu + h$ that is

$$\mu(t) = \int_a^b \kappa_1(t, s, \mu(s))\, ds + g(t)$$

and

$$\mu(t) = \int_a^b \kappa_2(t, s, \mu(s))\, ds + h(t)$$

Therefore, the Urysohn integral (2) and (3) have a unique common fixed point. ■

5. Conclusion

Complex-valued metric spaces, which are a broadening of conventional metric spaces, hold significant importance in various fields, particularly in advanced mathematics and theoretical physics. Their applications extend to areas such as relativity, geometry, topology, mathematical finance, economics, quantum mechanics, quantum computing, functional analysis, operator theory, complex dynamics, fractal geometry, complex networks, and signal processing. Although interest in complex-valued metric spaces is increasing, the existing body of literature on fixed point theorems in these spaces is still quite limited and exhibits several shortcomings. In 2011, Azam and his colleagues developed a metric space that accommodates complex values and presented a prominent fixed point theorem that satisfies specific contractive conditions [9]. Subsequently, in 2013, another study proposed an approach for Urysohn integral equations that incorporates common fixed points in metric spaces with complex values [12]. Following this, Rao and his team expanded on these findings plying them to complex valued b-metric spaces and establishing a fixed point theorem [16]. This article reviews these advancements and presents a generalization of a new contraction mapping while employing weakly compatible mappings to investigate the completeness and uniqueness of fixed point theorems in complex valued b-metric spaces. In addition, we tackled a system of Urysohn integral equations by utilizing our primary results, leading to some fascinating discoveries and the formulation of several theories.

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