BAYES ESTIMATES FOR THE PARAMETERS OF POISSON TYPE ONE PARAMETER RAYLEIGH CLASS SRGM USING GAMMA AND INVERTED GAMMA PRIORS

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Abstract

Research on the reliability of both software and hardware commonly utilizes the Rayleigh distribution as a model. During software testing, the failure occurrence behavior can be used to evaluate the software's reliability. In this paper, the Poisson pattern of failure occurrence is assumed to study the parameters of the Rayleigh Class software reliability growth model. It has been studied how the Rayleigh Class Model's scale parameter and the quantity of software's intrinsic failures behave. Additionally, using the Gamma and Inverted Gamma Priors, respectively, Bayes estimates for the number of intrinsic failures and the scale parameter are suggested. The comparative analysis of the proposed Bayes estimators and accompanying MLEs is done using the relative efficiency that are acquired via the use of Monte Carlo Simulation.

Keywords: Software Reliability Growth Model, Rayleigh Class, Poisson Type, Gamma and Inverted Gamma Priors, Maximum Likelihood Estimates (MLEs), Relative Efficiency (RE)

I. Introduction

In applied statistics and probability, the Rayleigh distribution is important for modelling. A variable lifespan distribution called the Rayleigh function is used in a lot of degradation process failure models. It has been noted that for a variety of hardware development procedures, the manpower curve can be roughly estimated using the Rayleigh distribution [10]. The Weibull distribution family includes the Rayleigh model. For many years, the Weibull distribution has been utilized for reliability studies in a variety of engineering domains, including the overflow incidence of rivers, electron tube failures, and the fatigue life of deep-groove ball bearings. It is recognized as one of the three distributions with extreme values, see [21]. The scale parameter of the Rayleigh distribution is estimated by [8] using the modified maximum likelihood. This study establishes the Rayleigh model, a formal model of software reliability. Given that it is predicated on a particular statistical distribution, this model is parametric. The amount of test-effort spent on software testing has been included into software reliability growth models suggested by [11] and [25] on the assumption that the test-effort during the testing can be described by the Rayleigh curve. [11] has created a constraint model that illustrates how the distribution of effort for software projects is like a set of Rayleigh curves. Software project error records were analysed month-by-month by [22], who discovered that the projects' composite error pattern matched a Rayleigh-like curve. In addition to work on software size and resource estimation, [12] suggested using the Rayleigh model to estimate the number of software flaws. It is claimed that by utilizing an appropriate model—of which there are numerous, some more appropriate than others—one may monitor and control the reliability growth of any software (or even hardware) development. The Rayleigh model is arguably the best suited for software development. According to [24], who looked at the relationship between test coverage and fault detection rate, there is a variation of the Rayleigh distribution in the relationship between structural coverage and fault coverage. The Rayleigh curve rises to a peak and then falls off at a slower pace. Software development projects have also been shown empirically to follow a lifespan pattern that is characterized by the Rayleigh curve [3] and [7]. The Rayleigh curve frequently provides accurate estimates of software development costs and timelines.

This study uses the Bayesian paradigm to estimate the parameters of the Rayleigh distribution, which is considered as a software reliability growth model. Comparable research has been done by [13] to [20] taking into account other models.

II. Model Formulation and Evaluation

The use of Rayleigh distribution in Software reliability has been shown by many authors such as [5], [10], [11], [22]-[25], etc. Here, in this chapter one parameter Rayleigh distribution has been established as a SRGM and some more details about it can be had from [2] and [4]. Let t be the positive random variable having Rayleigh distribution then its probability density function is

$$f(t) = \begin{cases} t\eta_1^{-2} e^{-\frac{1}{2}\left[\frac{t}{\eta_1}\right]^2} & , \; t > 0, \eta_1 > 0 \\ 0 & , \; Otherwise \end{cases} \tag{1}$$

The failure intensity function $\lambda(t) = \eta_0 f(t)$ is

$$\lambda(t) = \eta_0 \eta_1^{-2} t e^{-\frac{1}{2}\left[\frac{t}{\eta_1}\right]^2} \quad , \; t > 0, \eta_1 > 0, \eta_0 > 0 \tag{2}$$

and the expected number of failures at time t_e $\mu(t_e) = \int_0^{t_e} \lambda(t)\, dt$ which comes out to be,

$$\mu(t_e) = \eta_0 \left[1 - e^{-\frac{1}{2}\left(\frac{t_e}{\eta_1}\right)^2}\right] \tag{3}$$

It is seen that $\lambda(t)$ is very sensitive for values of η_1. Also, $\lambda(t)$ is high for the smaller values of η_1, becoming unimodal positively skewed distribution for the large values of η_1. The values of $\mu(t_e)$ are large for smaller failure rate η_1 and decreases as η_1 increases. Also, $\lambda(t)$ is less sensitive for increasing values of η_0 and for fixed η_1. The slope of $\lambda(t)$ and $\mu(t_e)$ remains same for increasing η_0.

III. MLEs of the Parameters

Let t_i, $i = 1,2,\dots,m_e$ be the failures of software up to the execution time $'t_e'$ then the likelihood function of η_0 and η_1 is given by $L(\eta_0, \eta_1, \underline{t}) = \left[\prod_{i=1}^{m_e} \lambda(t_i)\right] \exp[-\mu(t_e)]$. In present case, the likelihood function is obtained by considering failure intensity given by (2) and expected number of failures in (3) according to [9].

$$L(\eta_0, \eta_1) = \eta_0^{m_e} \eta_1^{-2m_e} \left[\prod_{i=1}^{m_e} t_i\right] e^{-\frac{1}{2}T\eta_1^{-2}}\, e^{-\eta_0} \exp\left\{\eta_0\, e^{-\frac{1}{2}\left(\frac{t_e}{\eta_1}\right)^2}\right\} \tag{4}$$

By applying the standard procedure of obtaining the maximum likelihood estimator for parameters η_0 and η_1, the estimators are

$$\hat{\eta}_{m0} = m_e \left[1 - e^{-\frac{1}{2}\left(\frac{t_e}{\hat{\eta}_{m1}}\right)^2}\right]^{-1} \tag{5}$$

and

$$2\hat{\eta}_{m1} + t_e^2 e^{-\frac{1}{2}\left(\frac{t_e}{\hat{\eta}_{m1}}\right)^2}\left[1 - e^{-\frac{1}{2}\left(\frac{t_e}{\hat{\eta}_{m1}}\right)^2}\right]^{-1} = \frac{T}{m_e} \tag{6}$$

The MLEs $\hat{\eta}_{m0}$ and $\hat{\eta}_{m1}$ can be obtained by getting the solution of simultaneous equations (5) and (6) using any available standard numerical method.

IV. Priors for Bayesian Estimation

Among the informative priors, the most preferred prior is gamma prior. The main reasons for its generous use are its conjugacy, general acceptability and mathematical tractability. [1] and other many researchers have noticed that the gamma distribution is sufficiently flexible for practical hardware reliability applications in life testing.

Here, it is supposed that, the experimenter is in a position to get prior information about the parameters η_0 and η_1 and find that the gamma distribution will be suitable for it. Hence, the gamma prior distributions can be proposed for the parameter η_0 as follows,

$$g(\eta_0) = \begin{cases} {\eta_0}^{\vartheta-1} e^{-\tau\eta_0} & ,\vartheta > 0, \tau > 0, 0 < \eta_0 < \infty \\ 0 & , \quad Otherwise \end{cases} \tag{7}$$

The inverse gamma distribution is a two-parameter family of continuous probability distributions on the positive real line, obtained as reciprocal of a gamma variate. The inverted gamma distributions may be useful in many situations e.g. if the mean number of events per time period has a Gamma distribution, then the mean time between events has an Inverse Gamma distribution. [6] has studied properties of characterization for the inverted gamma distributions and established some theorems based on conditional s-expectation and mean residual life. In the presents study, the inverted gamma prior distribution is considered for parameter η_1 as

$$g(\eta_1) = \begin{cases} {\eta_1}^{-\alpha-1} e^{-\beta/\eta_1} & ,\alpha > 0, \beta > 0, 0 < \eta_1 < \infty \\ 0 & , \quad Otherwise \end{cases} \tag{8}$$

where ϑ, τ, α, and β are parameters of considered priors for η_0 and η_1 respectively.

V. Posterior and Marginal Distributions

Since it is considered that prior information is available about both the parameters η_0 and $\eta_1$in the form of gamma and inverted gamma priors respectively and considering the total execution time is t_e and during this time m_e failures are experienced at times t_i $, i = 1,2, \dots, m_e$ then, combining likelihood function (4) with priors (5) and (6), the joint posterior of η_0 and η_1 given $\underline{t}$ is

$$\pi(\eta_0, \eta_1 | \underline{t}) \propto \eta_0^{m_e+\vartheta-1} \eta_1^{-2m_e-\alpha-1} e^{-\frac{1}{2}T\eta_1^{-2}} e^{-\eta_0} e^{-\tau\eta_0} e^{-\beta/\eta_1} \exp\left\{\eta_0\, e^{-\frac{1}{2}\left(\frac{t_e}{\eta_1}\right)^2}\right\}$$

$$m_e < \eta_0 < \infty, 0 < \eta_1 < \infty \tag{9}$$

where $\sum_{i=1}^{m_e} t_i^2 = T$.

The normalizing constant of above equation is

$$D = \int_0^\infty \int_{m_e}^\infty \eta_0^{m_e+\vartheta-1} \eta_1^{-2m_e-\alpha-1} e^{-\frac{1}{2}T\eta_1^{-2}} e^{-\eta_0} e^{-\tau\eta_0} e^{-\beta/\eta_1} \exp\left\{\eta_0\, e^{-\frac{1}{2}\left(\frac{t_e}{\eta_1}\right)^2}\right\} d\eta_0\, d\eta_1$$

$$m_e < \eta_0 < \infty, 0 < \eta_1 < \infty \tag{10}$$

Applying some mathematical simplification procedure, the value of D is

$$D = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j! {\tau_2}^{(m_e+\vartheta+j)}} \sum_{k=0}^{\infty} \frac{(-\beta)^k \Gamma(m_e+\vartheta+j, m_e\tau_2)\Gamma(2m_e+\alpha+k)/2}{k! {\tau_1}^{(2m_e+\alpha+k)/2}} \tag{11}$$

where, $\tau_1 = (T + jt_e^2)$ and $\tau_2 = (\tau + 1)$.

The marginal posterior of η_1, say $\pi(\eta_1|\underline{t}) = \int_{m_e}^{\infty} \pi(\eta_0, \eta_1|\underline{t})\, d\eta_0$ is solved and reduces to

$$\pi(\eta_1|\underline{t}) \propto \sum_{j=0}^{\infty}\left[\frac{\Gamma(m_e+\vartheta+j, m_e\tau_2)}{\tau_2^{(m_e+\vartheta+j)}j!}\right]\left[\eta_1^{-2m_e-\alpha-1}e^{-\frac{1}{2}\tau_1\eta_1^{-2}}e^{-\beta/\eta_1}\right] \qquad 0<\eta_1<\infty \qquad (12)$$

Similarly, the marginal posterior of η_0, say $\pi(\eta_0|\underline{t}) = \int_0^{\infty}\pi(\eta_0,\eta_1|\underline{t})\,d\eta_1$ can be obtained as

$$\pi(\eta_0|\underline{t}) \propto \sum_{j=1}^{\infty}\frac{1}{j!}\sum_{k=0}^{\infty}\frac{(-\beta)^k\Gamma(2m_e+\alpha+k)/2}{k!(\tau_1)^{(2m_e+\alpha+k)/2}}\,\eta_0^{m_e+\vartheta+j-1}e^{-\eta_0\tau_2} \qquad m_e<\eta_0<\infty \qquad (13)$$

VI. Bayes Estimates

Using the marginal posteriors (12) and (13), the Bayesian point estimates for the parameters η_0 and η_1 are obtained here. The Bayes estimator for η_0 say $\hat{\eta}_{B0}$ under the squared error loss function is the posterior mean i.e. $\hat{\eta}_{B0} = \int_{m_e}^{\infty}\eta_0\,\pi(\eta_0|\underline{t})d\eta_0$. On simplifying,

$$\hat{\eta}_{B0} = \frac{1}{D}\sum_{j=1}^{\infty}\frac{1}{j!\tau_2^{(m_e+\vartheta+j+1)}}\sum_{k=0}^{\infty}\frac{(-\beta)^k\Gamma(m_e+\vartheta+j+1, m_e\tau_2)\Gamma(2m_e+\alpha+k)/2}{k!\tau_1^{(2m_e+\alpha+k)/2}} \qquad (14)$$

Similarly, the Bayes estimator for η_1 say $\hat{\eta}_{B1}$ is the posterior mean i.e. $\hat{\eta}_{B1} = \int_0^{\infty}\eta_1\,\pi(\eta_1|\underline{t})d\eta_1$

$$\hat{\eta}_{B1} = \frac{1}{D}\sum_{j=1}^{\infty}\frac{1}{j!\tau_2^{(m_e+\vartheta+j)}}\sum_{k=0}^{\infty}\frac{(-\beta)^k\Gamma(m_e+\vartheta+j, m_e\tau_2)\Gamma(2m_e+\alpha+k-1)/2}{k!\tau_1^{(2m_e+\alpha+k-1)/2}} \qquad (15)$$

The performance of Bayes estimators of η_0 and η_1 can be checked by studying the risk efficiencies. The risk efficiency of $\hat{\eta}_{B0}$ over its MLE $\hat{\eta}_{m0}$ is defined by $RE_0 = \frac{E[\hat{\eta}_{m0}-\eta_0]^2}{E[\hat{\eta}_{B0}-\eta_0]^2}$ and that of $\hat{\eta}_{B1}$ over its MLE $\hat{\eta}_{m1}$ is defined by $RE_1 = \frac{E[\hat{\eta}_{m1}-\eta_1]^2}{E[\hat{\eta}_{B1}-\eta_1]^2}$.

VII. Discussion about Estimates

The Bayes estimators of total number of failures i.e. η_0 and failure rate i.e. η_1 are proposed and are compared with the corresponding maximum likelihood estimators. The performance of proposed Bayes estimators $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$ over maximum likelihood estimators $\hat{\eta}_{m0}$ and $\hat{\eta}_{m1}$ have been compared based on criterion of corresponding risks efficiencies. To study the performance, a sample of size m_e was generated sufficient number of times from the Rayleigh distribution considering the different values of η_0 and η_1. Then, using Monte Carlo simulation technique risks efficiencies RE_0 and RE_1 have been evaluated. The calculated values of risks efficiencies RE_0 and RE_1 have been obtained using various values of t_e and different values of η_0 and η_1 which are presented in the Figure 1 to Figure 6.

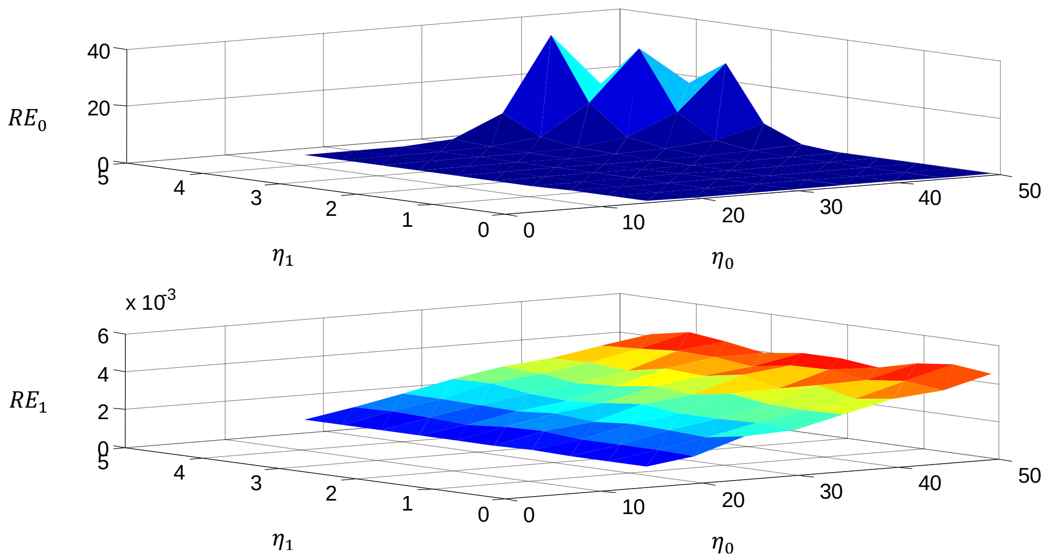

Figure 1: *Risk Efficiencies of* $\hat{\eta}_{B0}$ *and* $\hat{\eta}_{B1}$, $\vartheta = 0.01$, $\tau = 0.01$, $\alpha = 0.01$, $\beta = 0.01$, $t_e = 100$

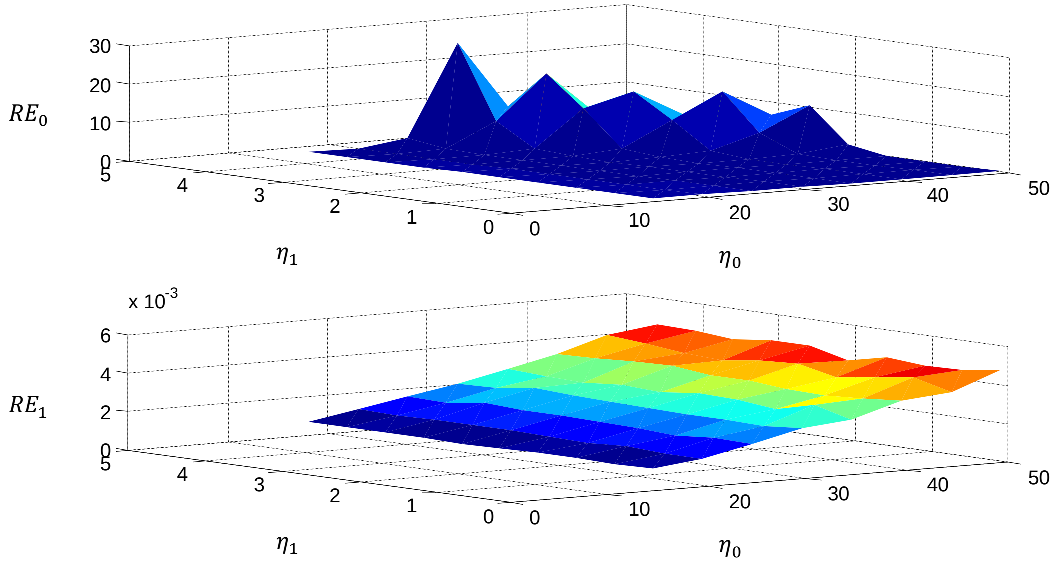

Figure 2: *Risk Efficiencies of* $\hat{\eta}_{B0}$ *and* $\hat{\eta}_{B1}$, $\vartheta = 1$, $\tau = 1$, $\alpha = 1$, $\beta = 1$, $t_e = 100$

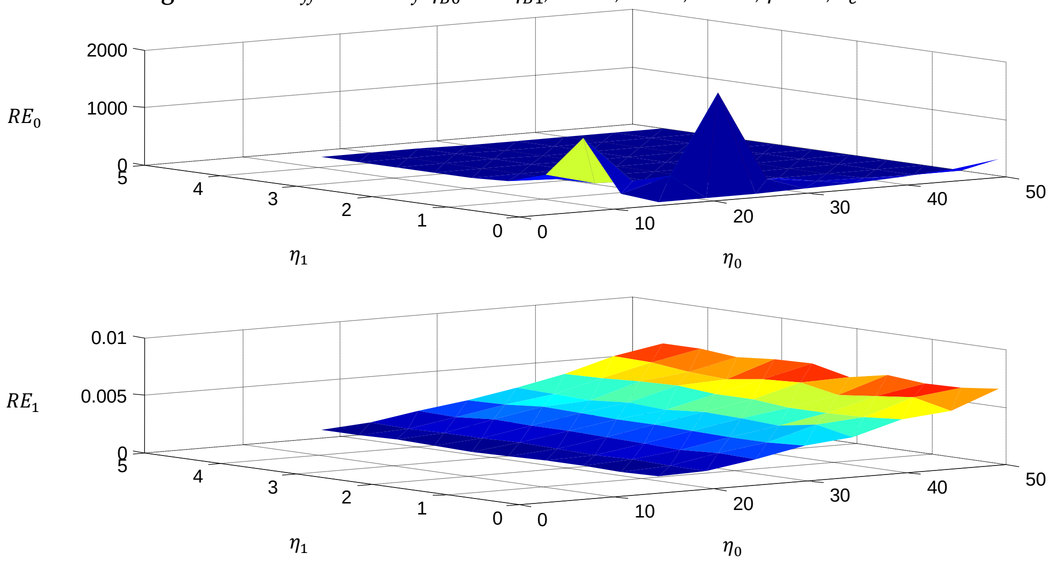

Figure 3: *Risk Efficiencies of* $\hat{\eta}_{B0}$ *and* $\hat{\eta}_{B1}$, $\vartheta = 10$, $\tau = 10$, $\alpha = 10$, $\beta = 10$, $t_e = 100$

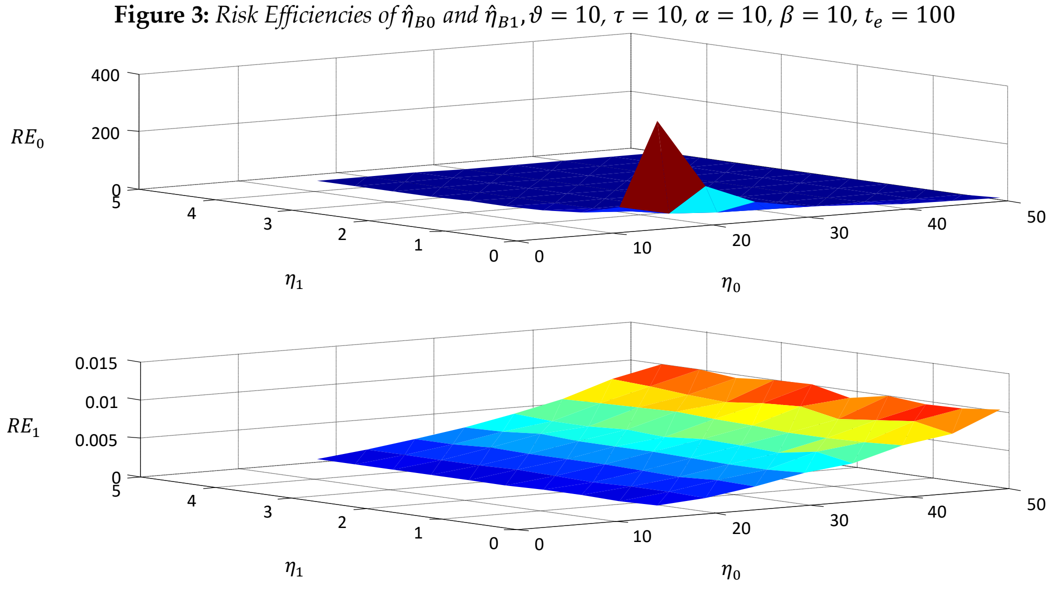

Figure 4: *Risk Efficiencies of* $\hat{\eta}_{B0}$ *and* $\hat{\eta}_{B1}$, $\vartheta = 0.01$, $\tau = 100$, $\alpha = 0.01$, $\beta = 100$, $t_e = 100$

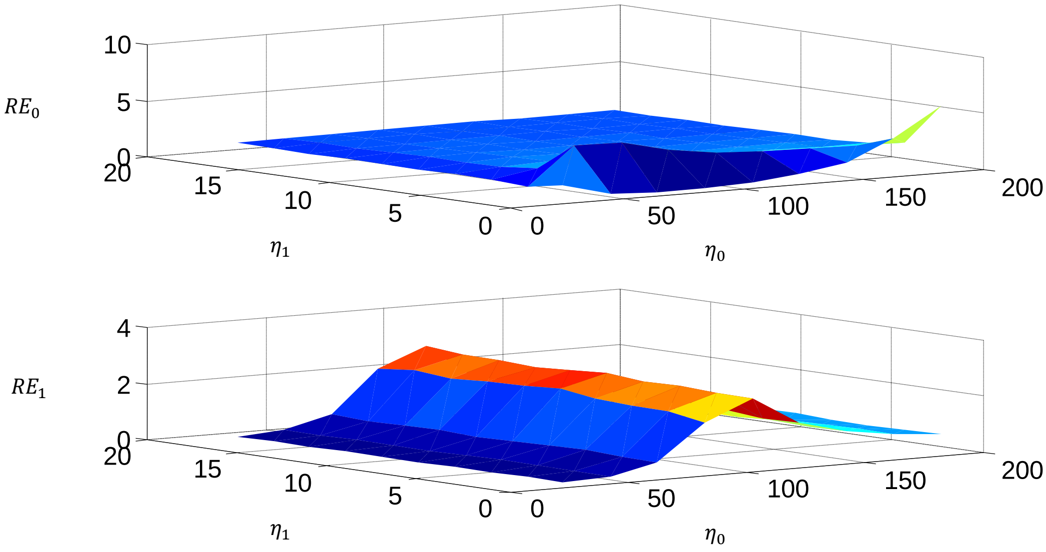

Figure 5: *Risk Efficiencies of* $\hat{\eta}_{B0}$ *and* $\hat{\eta}_{B1}$, $\vartheta = 100, \tau = 10; \alpha = 100, \beta = 10, t_e = 100$

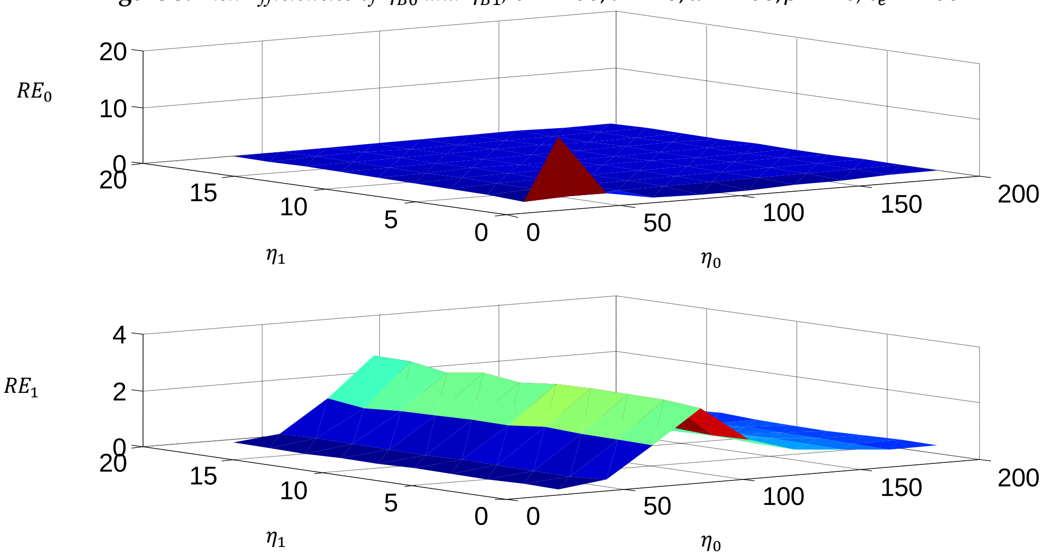

Figure 6: *Risk Efficiencies of* $\hat{\eta}_{B0}$ *and* $\hat{\eta}_{B1}$, $\vartheta = 100, \tau = 50; \alpha = 100, \beta = 50, t_e = 100$

The risk efficiencies of proposed Bayes estimator of η_0 over MLE decrease as the value of η_0 as well as η_1 increase (see Figure 1 to Figure 6) i.e. for large values of η_1 as well as $\eta_0$the risk efficiencies RE_0 become greater than one. Further, as the value of t_e increases the risk efficiencies RE_0 increase in most of the situations whereas the risk efficiencies RE_1 are almost unchanged (decrease very slowly) due to variation of t_e. The Bayes estimator $\hat{\eta}_{B1}$ is better than MLE for a propoer choice of prior constants ϑ, τ, α and β as well as η_0, η_1 and t_e. The risk efficiency of proposed Bayes estimator of η_1 increases as the value of η_0 as well as η_1 increase but this increase is very slow (almost constant) due to increase of η_0.

It is noticed that the prior constant τ and β are not much affecting the risk efficiencies RE_0 and RE_1 whereas the prior constants α and ϑ are significantly affecting the risk efficiencies. Due to increase in ϑ and α risk efficiencies of $\eta_0$first increase, attains a maximum and then decrease for certain value of t_e and prior constant. Whereas, the risk efficiencies RE_1 is increasing due to increase in ϑ as well as α. All the computations and their corresponding graphs are not being presented here due to limitation of the size of this project. Some selected graphs are presented only.

VIII. Conclusions

The proposed Bayes estimator of intrinsic failures η_0 and scale parameter η_1 i.e. $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$ can be preferred over MLE for a proper choice of execution time t_e, little large values of shape parameters of both the priors i.e. ϑ, α, and moderate values of scale parameters of both the priors τ, β as well as suitable expected value of intrinsic failures η_0 and scale parameter η_1 of Poisson Type Rayleigh Class Software Reliability Growth Model. The proposed Bayes estimator $\hat{\eta}_{B0}$ can be preferred if scale parameter η_1 is little high and execution time t_e is also large. Also, Bayes estimator $\hat{\eta}_{B0}$ is seen to be better than its corresponding MLE for moderately small value of prior constants.

The Bayes estimator $\hat{\eta}_{B1}$ is better than MLE for a proper choice of intrinsic failures η_0 and scale parameter η_1 and t_e. The Bayes estimator $\hat{\eta}_{B1}$ performs better than its corresponding MLE for moderately large values of prior constants ϑ, τ, α and β. It is noticed that the prior constant τ and β are not much affecting the risk efficiencies of both the proposed bayes estimators whereas the prior constants α and ϑ are significantly affecting the risk efficiencies. In particular, the risk efficiencies of $\hat{\eta}_{B1}$ are affected significantly by the change in values of prior constants. The Bayes estimators $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$ are better than MLEs if the above informative priors are selected in spite of non-informative priors.

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