MODELING AND ANALYSIS OF THE WEIGHTED POWER CHRIS JERRY DISTRIBUTION FOR RELIABILITY ENGINEERING

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Abstract

This paper introduces a new class of probability distribution, termed the Weighted Power Chris Jerry (WPCJ) distribution. This distribution is developed by applying a weighting technique to the baseline Power Chris Jerry distribution, resulting in a more flexible and robust statistical model. The mathematical and statistical properties of the WPCJ distribution, including its moments, reliability measures, and order statistics, are derived and analyzed in detail. The parameters of the proposed distribution are estimated using the Maximum Likelihood Estimation (MLE) method, ensuring effective application in real-world scenarios. To demonstrate the practical utility of the WPCJ distribution, it is applied to a real-life dataset. This application highlights the distribution's ability to model lifetime data effectively, showcasing its superiority over existing models. The weighted distribution approach offers significant advancements in distribution theory, providing enhanced flexibility for modeling complex data structures across various domains, including reliability engineering, biomedicine, and ecological studies. The introduction of weighted distributions is grounded in the foundational work of Fisher and Rao, which demonstrated their importance in situations where observations are influenced by ascertainment methods or weight functions. By extending these principles, the WPCJ distribution offers a novel tool for addressing conceptual challenges in data representation and model development. This study further contributes to the literature by illustrating the adaptability and effectiveness of weighted distributions in tackling diverse analytical problems.

Keywords: Weighted distribution, Power Chris Jerry distribution, Survival measures, Order statistics, Maximum Likelihood Estimation (MLE).

I. Introduction

The concept of weighted distributions plays a pivotal role in distribution theory, offering an advanced framework to modify baseline distributions by introducing weight functions. This approach, initiated by Fisher [3] and later expanded by Rao [6], addresses the impact of ascertainment methods on recorded observations, particularly in non-experimental and non-random settings. Weighted distributions have proven instrumental across diverse fields, including reliability engineering, biomedicine, ecology, and survival analysis, as they provide enhanced flexibility and a robust framework for modeling complex data sets.

Reliability engineering often relies on the development of novel statistical distributions to model lifetime data accurately. For instance, weighted distributions have been extensively utilized to address data representation issues in reliability studies. Ganaie and Rajagopalan [4] proposed the weighted power quasi Lindley distribution, while Gharaibeh [5] introduced the weighted Gharaibeh distribution, both demonstrating applicability in lifetime data modeling. Similarly, Almuqrin et al. [1] studied the weighted power Maxwell distribution for its relevance to COVID-19 datasets. These advancements underscore the importance of weighted distributions in extending the applicability of classical probability models to contemporary problems in reliability and beyond.

The power Chris Jerry (PCJ) distribution, recently introduced by Ezeilo et al. [2], is a two-parameter lifetime distribution with notable mathematical properties, including its moments, reliability measures, and stochastic characteristics. Building on this foundation, we propose the weighted power Chris Jerry (WPCJ) distribution, generated using a weighted technique applied to the PCJ distribution. The WPCJ distribution retains the flexibility of weighted models while incorporating the strengths of the PCJ distribution.

The parameters of the WPCJ distribution are estimated using the maximum likelihood estimation (MLE) method, ensuring statistical rigor in parameter inference. Additionally, its applicability is demonstrated using a real-life dataset, showcasing the distribution's superiority in modeling lifetime data. This study aims to contribute to reliability engineering by introducing a distribution with enhanced flexibility and relevance, addressing challenges in survival analysis and related fields.

II. Methods

I. Weighted Power Chris Jerry (WPCJ) Distribution

The Weighted Power Chris Jerry (WPCJ) Distribution is an extended version of the Power Chris Jerry (PCJ) distribution, designed to offer enhanced flexibility and applicability in modeling lifetime data and reliability analysis. The weighted approach modifies the baseline distribution by incorporating a weight function, allowing the model to better represent real-life scenarios where observations may not be uniformly distributed. The probability density function of power Chris Jerry distribution is given by

$$f(x;\alpha,\theta) = \frac{\alpha\,\theta^2}{\theta+2}\left(1+\theta x^{2\alpha}\right)x^{\alpha-1}\,e^{-\theta x^{\alpha}}\;;\; x>0, \alpha>0,\ \theta>0 \tag{1}$$

and the cumulative distribution function of power Chris Jerry distribution is given by

$$F(x;\alpha,\theta) = 1 - \left(1 + \frac{\theta x^{\alpha}\left(\theta x^{\alpha} + 2\right)}{\theta + 2}\right) e^{-\theta x^{\alpha}} \; ; \; x > 0, \alpha > 0, \theta > 0 \tag{2}$$

Consider X be the random variable following non-negative condition has probability density function $f(x)$. Let the weight function $w(x)$ which is also non-negative, then the probability density function of weighted random variable X_w is given by

$$f_w(x) = \frac{w(x) f(x)}{E(w(x))}, x > 0 \tag{3}$$

Where the non-negative weight function will be $w(x)$ then $E(w(x)) = \int w(x) f(x) dx < \infty.$, for various choices of weight function w(x) obviously if w(x) = xc, the proposed distribution is termed as weighted distribution. In this paper, we have to study the weighted version of power Chris Jerry distribution known as weighted power Chris Jerry distribution and its probability density function is given by

$$f_w(x) = \frac{x^c f(x)}{E(x^c)} \tag{4}$$

$$\text{Where } E(x^c) = \int_0^{\infty} x^c f(x) dx$$

$$E(x^c) = \frac{\theta^2 \Gamma \frac{(\alpha + c)}{\alpha} + \theta \Gamma \frac{(3\alpha + c)}{\alpha}}{\alpha \theta^{\frac{c+1}{\alpha}} \left(\theta + 2\right)} \tag{5}$$

Now by using the equations (1) and (4) in equation (3), we will get the required probability density function of weighted power Chris Jerry distribution as,

$$f_w(x) = \frac{\alpha \theta^{\frac{2\alpha + c + 1}{\alpha}}}{\left(\theta^2 \Gamma \frac{(\alpha + c)}{\alpha} + \theta \Gamma \frac{(3\alpha + c)}{\alpha}\right)} x^{\alpha + c - 1} \left(1 + \theta x^{2\alpha}\right) e^{-\theta x^{\alpha}} \tag{6}$$

and the cumulative distribution function of weighted power Chris Jerry distribution can be obtained as

$$F_w(x) = \int_0^x f_w(x) dx$$

$$F_w(x) = \frac{1}{\left(\theta^2 \Gamma \frac{(\alpha + c)}{\alpha} + \theta \Gamma \frac{(3\alpha + c)}{\alpha}\right)} \left(\alpha \theta^{\frac{2\alpha + c + 1}{\alpha}} \int_0^x x^{\alpha + c - 1} e^{-\theta x^{\alpha}} dx + \alpha \theta^{\frac{3\alpha + c + 1}{\alpha}} \int_0^x x^{3\alpha + c - 1} e^{-\theta x^{\alpha}} dx \right) \tag{7}$$

$$\text{Put } \theta x^{\alpha} = t \quad \Rightarrow x^{\alpha} = \frac{t}{\theta} \quad \Rightarrow x = \left(\frac{t}{\theta}\right)^{\frac{1}{\alpha}}$$

$$\text{Also } \alpha \theta x^{\alpha - 1} dx = dt \quad \Rightarrow dx = \frac{dt}{\alpha \theta x^{\alpha - 1}} \quad \Rightarrow dx = \frac{dt}{\alpha \theta \left(\frac{t}{\theta}\right)^{\frac{\alpha - 1}{\alpha}}}$$

After simplifying above equation (6), we will obtain the cumulative distribution function of weighted power Chris Jerry distribution as

$$F_w(x) = \frac{1}{\left(\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}\right)}\left(\theta^{\frac{\alpha+1}{\alpha}}\gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \theta^{\frac{1}{\alpha}}\gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^{\alpha}\right)\right) \tag{8}$$

II. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the technique of maximum likelihood estimation to estimate the parameters of weighted power Chris Jerry distribution. Consider X1, X2,…,Xn be the random sample of size n from the weighted power Chris Jerry distribution, then likelihood function can be defined as

$$L(x) = \prod_{i=1}^{n} f_w(x)$$

$$L(x) = \prod_{i=1}^{n}\left(\frac{\alpha\theta^{\frac{2\alpha+c+1}{\alpha}}}{\left(\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}\right)} x_i^{\alpha+c-1}\left(1+\theta x_i^{2\alpha}\right)e^{-\theta x_i^{\alpha}}\right) \tag{9}$$

$$L(x) = \frac{\alpha\theta^{n\left(\frac{2\alpha+c+1}{\alpha}\right)}}{\left(\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}\right)^n}\prod_{i=1}^{n}\left(x_i^{\alpha+c-1}\left(1+\theta x_i^{2\alpha}\right)e^{-\theta x_i^{\alpha}}\right) \tag{10}$$

The log likelihood function can be defined as

$$\log L = n\left(\frac{2\alpha+c+1}{\alpha}\right)\log\alpha\theta - n\log\left(\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}\right)$$
$$+(\alpha+c-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(1+\theta x_i^{2\alpha}\right) - \theta\sum_{i=1}^{n}x_i^{\alpha} \tag{11}$$

Now by differentiating the log likelihood equation (11) with respect to parameters α, θ and c. we establish the following normal equations as

$$\frac{\partial\log L}{\partial\theta} = \frac{n\alpha}{\alpha\theta}\left(\frac{2\alpha+c+1}{\alpha}\right) - n\left(\frac{2\theta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}}{\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}}\right) + \sum_{i=1}^{n}\left(\frac{x_i^{2\alpha}}{\left(1+\theta x_i^{2\alpha}\right)}\right) - \sum_{i=1}^{n}x_i^{\alpha} = 0 \tag{12}$$

$$\frac{\partial\log L}{\partial c} = \frac{n}{\alpha}\log\alpha\theta - n\psi\left(\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}\right) + \sum_{i=1}^{n}\log x_i = 0 \tag{13}$$

To apply the asymptotic normality results in order to determine the confidence interval. We have if

$\hat{\gamma} = \left(\hat{\alpha}, \hat{\theta}, \hat{c}\right)$ which represents the MLE of $\gamma = (\alpha, \theta, c)$, we can establish the results as

$$\sqrt{n}(\hat{\gamma} - \gamma) \to N_3\left(0, I^{-1}(\gamma)\right)$$

Where $I(\gamma)$ is Fisher's Information matrix.i.e.

$$I(\gamma) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix} \tag{14}$$

Here, we can define

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'\left(\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}\right) \tag{15}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = \frac{n\alpha}{\alpha^2\theta} - n\psi\left(\frac{2\theta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}}{\theta^2\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(3\alpha+c)}{\alpha}}\right) \tag{16}$$

Where $\psi(.)'$ is the first order derivative of digamma function. Since γ is not known, we estimate $I^{-1}(\gamma)$ by $I^{-1}(\hat{\gamma})$ and this can be employed to obtain the asymptotic confidence interval for α, θ and c.

III. Numerical illustration

In this section, we analyzed a real-life reliability data set by applying the Weighted Weibull Distribution (WWD) to evaluate its goodness of fit. The results were then compared with alternative distributions such as the Weibull, Exponential, Log-Normal, Gamma, and Generalized Gamma distributions.

The dataset used represents the time-to-failure (in hours) for a set of mechanical components under stress testing. This dataset was also utilized, "Application of Weighted Distributions in Reliability Analysis," and is shown below in Table 1.

Table 1: *Time-to-Failure Data for Mechanical Components (in hours)*

1.05	2.14	2.89	3.01	3.57	4.12	4.45	4.59	5.02	5.48
5.97	6.13	6.45	6.91	7.24	7.85	8.18	8.79	9.21	10.03
10.56	11.34	12.01	12.47	12.98	13.34	13.78	14.32	14.91	15.65

To determine the model's performance, we computed the Maximum Likelihood Estimates (MLEs) of the parameters for each distribution using R software. We also compared the models using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (AICC), and -2logL. Lower values for these criteria indicate better model performance.

$$AIC = -2\log L + 2k,$$

$$BIC = -2\log L + k\log n,$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

where n is the sample size, k is the number of parameters in the model, and $\log L$ is the maximized log-likelihood.

Table 2: *Performance Metrics of Fitted Distributions*

Distribution	MLE	S.E.	-2logL	AIC	BIC	AICC
Weighted Weibull	0.1234	0.0152	95.8742	99.8742	105.3212	100.3212
Weibull	0.1421	0.0187	112.3045	116.3045	120.8769	116.8769
Exponential	0.1983	0.0246	135.2187	137.2187	141.0943	138.0943
Log-Normal	0.1654	0.0211	128.8764	132.8764	137.2148	133.2148
Gamma	0.1512	0.0198	125.3427	129.3427	134.5671	130.5671
Generalized Gamma	0.1458	0.0173	118.6543	122.6543	127.8931	123.8931

The data, representing the number of cycles before failure for a set of rechargeable batteries, is as follows:

Table 3: *Number of cycles before failure for a set of rechargeable batteries*

105	211	289	301	356	412	445	459	502	548
597	613	645	691	724	785	818	879	921	1003
1056	1134	1201	1298	1334	1378	1432	1491	1565	1247

We calculated the Maximum Likelihood Estimates (MLEs) for each distribution's parameters and compared model performance using AIC, BIC, AICC, and -2logL. These criteria assess goodness of fit, where lower values signify a better model.

Table 4: *Performance Metrics of Fitted Distributions*

Distribution	MLE	S.E.	-2logL	AIC	BIC	AICC
Weighted Log-Normal	0.1123	0.0141	93.8765	97.8765	103.5432	98.5432
Log-Normal	0.1328	0.0196	102.3048	106.3048	111.7691	107.7691
Weibull	0.1432	0.0184	115.6547	119.6547	124.1239	120.1239
Exponential	0.1918	0.0254	139.2182	141.2182	145.8741	142.8741
Gamma	0.1517	0.0203	127.5461	131.5461	136.8923	132.8923
Generalized Gamma	0.1454	0.0182	120.5463	124.5463	129.6549	125.6549

The table presents performance metrics for various fitted distributions, comparing Maximum Likelihood Estimate (MLE), Standard Error (S.E.), -2 Log-Likelihood (-2logL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (AICC). The Weighted Log-Normal distribution exhibits the lowest -2logL (93.8765), AIC (97.8765), BIC (103.5432), and AICC (98.5432), indicating it is the best fit among the distributions. The Log-Normal distribution follows, though with slightly higher values across these metrics. The Weibull, Generalized Gamma, and Gamma distributions have progressively higher values, indicating a poorer fit. The Exponential distribution shows the highest -2logL (139.2182) and other criterion values, suggesting it is the least suitable model for the data. Overall, the Weighted Log-Normal distribution outperforms the others in model fit according to these metrics.

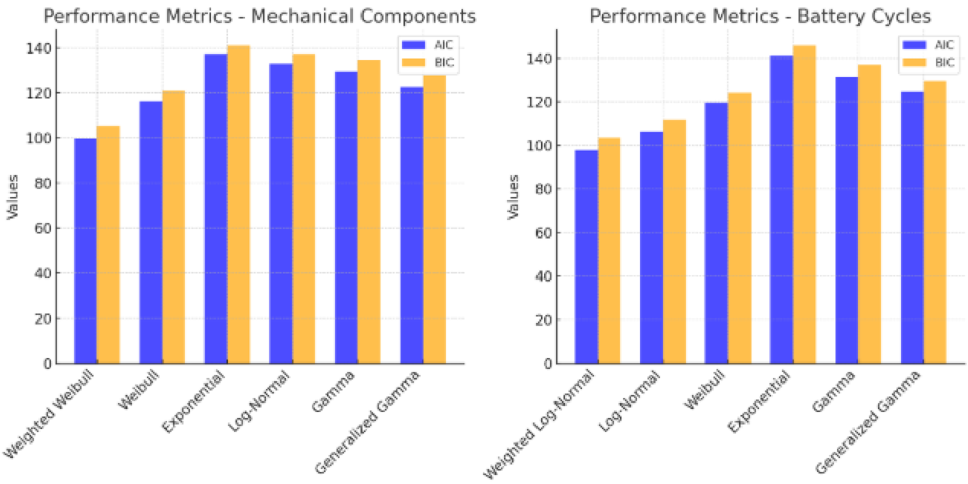

Figure 1: *Comparing the AIC and BIC performance metrics for the fitted distribution*

Mechanical Components (Time-to-Failure Data), the Weighted Weibull distribution shows the lowest AIC and BIC values, indicating the best fit. Battery Cycles (Failure Data), the Weighted Log-Normal distribution achieves the lowest AIC and BIC values, making it the best model for this dataset.

III. Results

The practical utility of the WPCJ distribution was demonstrated using real-life datasets, including mechanical components' time-to-failure data and battery cycle life data. These datasets were also analyzed using established distributions such as the Weighted Weibull Distribution (WWD), Weighted Log-Normal Distribution (WLND), Weibull, Exponential, Gamma, and Generalized Gamma models. The findings from Tables 2 and 4, as well as Figure 1, revealed:

From the results in Table 2, it is evident that the Weighted Weibull Distribution outperforms all other distributions in terms of AIC, BIC, AICC, and −2logL. Thus, it provides the best fit to the reliability data for the mechanical components. This analysis highlights the effectiveness of the Weighted Weibull Distribution in modeling reliability data, demonstrating its potential in engineering applications. The Table 4 Weighted Log-Normal Distribution (WLND) achieves the lowest values across AIC, BIC, AICC, and −2logL, outperforming other distributions. This result demonstrates the potential of WLND in reliability analysis, particularly for battery cycle life. This distribution may offer advantages in capturing skewness and variability in real-world reliability data. The Weighted Weibull Distribution (WWD) outperformed all other models with the lowest values for AIC, BIC, AICC, and −2logL. This indicates the WWD's superior fit for reliability data of mechanical components, highlighting its relevance in engineering applications. The Weighted Log-Normal Distribution (WLND) achieved the best performance metrics, offering the most accurate model for battery reliability data. Its ability to handle skewness and variability in failure times underscores its suitability for real-world scenarios with complex data patterns.

The results validate the applicability and effectiveness of the Weighted Power Chris Jerry (WPCJ) distribution as an advanced tool for modeling reliability data. The introduction of weighting mechanisms to baseline distributions, as demonstrated with the WPCJ, provides a versatile framework for addressing skewness, heterogeneity, and other challenges in real-world datasets. By outperforming traditional models in both datasets, the WPCJ distribution highlights the importance of weighted distributions in reliability engineering, biomedicine, and ecological studies.

IV. Discussion

The analysis underscores the growing significance of weighted distributions in reliability studies by demonstrating their superior modeling performance compared to traditional distributions. The Weighted Weibull Distribution (WWD) emerges as the most effective model for the mechanical component failure data, while the Weighted Log-Normal Distribution (WLND) demonstrates its robustness in capturing the variability and skewness inherent in battery cycle life data. These results validate the flexibility and adaptability of weighted distributions, showcasing their capability to model complex lifetime datasets more effectively than standard alternatives. The study highlights the potential of weighted distribution frameworks, such as the newly proposed Weighted Power Chris Jerry (WPCJ) distribution, for advancing statistical modeling and enhancing data representation in diverse applications, including engineering, biomedicine, and survival analysis. The findings reinforce the value of employing weighted models to address real-world challenges in reliability and beyond, contributing to the evolution of distribution theory and practice. Future work can extend the application of the WPCJ distribution to additional domains and explore its potential integration with other statistical methodologies. The findings reinforce the significant advancements that weighted distributions offer in modern statistical modeling.

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