ESTIMATING THE FRÉCHET DISTRIBUTION'S SCALE PARAMETER USING BAYESIAN METHODS FOR SYMMETRIC AND ASYMMETRIC LOSS FUNCTIONS

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Abstract

Comparing several estimators to estimate the scale parameter of the Fréchet distribution based on complete data when the shape parameter is known is the focus of this research. Using the power function distribution as a prior distribution and the maximum likelihood estimator (MLE), we apply the Bayes estimators under the squared error loss function (SELF), linear exponential loss function (LLF), asymmetric precautionary loss function (APLF), and composite LINEX loss function (CLLF). A Monte Carlo simulation study served as the basis for the comparison. The performance of these estimators in relation to the mean square error (MSE) was compared through the simulation study.

Keywords: Fréchet distribution, MLE, Bayesian estimator, Simulation study, Prior distribution.

I. Introduction

The Fréchet distribution is a special case of the generalized extreme value distribution. This type-II extreme value distribution (Fréchet) case is equivalent to taking the reciprocal of values from a standard Weibull distribution. This distribution has been shown to be useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, wind speeds, sea currents, floods, and rain fall. It is also an important distribution for modeling the statistical behavior of material properties for a variety of engineering applications.

Several researchers have discussed applications and estimates of the parameters of the Fréchet distribution. For example, Mubarak [1], Ramos et al. [2], Abbas and Tang [3] Abbas and Yincai [4], Riad and Hafez [5], Sindhu et al. [6] and Nasir and Aslam [7]. The probability density function (p.d.f) of the Fréchet distribution takes the form and the cumulative distribution function (cdf) and reliability function $R(t)$ of Frechet distribution are given, respectively, by

$$f(t;a,k) = ak\frac{1}{t^{(a+1)}}\exp\left[-\left(\frac{k}{t}\right)^a\right], t \geq 0, a, k > 0 \tag{1}$$

$$F(t;a,k) = \exp\left[-\left(\frac{k}{t}\right)^a\right], t \geq 0, a, k > 0 \tag{2}$$

$$R(t) = 1 - \exp\left[-\left(\frac{k}{t}\right)^a\right], t \geq 0 \tag{3}$$

where, a and k are shape and scale parameters, respectively. In this paper, comparison among the MLE estimator and Bayes estimator of the scale parameter of the Fréchet distribution is considered under SELF, LLF, APLF and CLLF. Using power function distribution as a prior distribution, with the assumption that the shape parameter is known, the plan of the paper is as follows: In Section 2, the MLE estimation of scale parameter is reviewed. In Section 3, we derive the Bayes estimators based on SLLF, LLF, APLF and CLLF. In Section 4, a simulation study is carried out. In Section 5, conclusion and numerical results are presented.

2. Maximum Likelihood Estimator (MLE)

Let $\underline{t} = (t_1, t_2, t_3, \dots, t_n)$ be a random sample of size n drawn from the French distribution. Then the likelihood function of $\underline{t}$ is given by

$$L(a,k,\underline{t}) = \prod_{i=1}^{n} ak^a \frac{1}{t^{(a+1)}}\exp\left[-\left(\frac{k}{t}\right)^a\right] \tag{4}$$

As the shape parameter a is assumed to be known, the MLE estimator of k is obtained by solving the following equation:

$$\frac{\partial \ln L(a,k,\underline{t})}{\partial k} = 0$$

Therefore, the MLE estimator for k is

$$\hat{k}_{MLE} = \left(\frac{n}{\Psi}\right)^{\frac{1}{a}}$$

3. Bayesian Estimation

In this section, we derive the Bayes estimates of scale parameter of French distribution. We use three different loss functions, including SELF, LLF and CLLF. For the Bayesian estimation of the parameter k, we assume the power function distribution as a prior distribution for scale parameter k as follows [8-11]:

$$\pi(k) = dk^{d-1}, 0 < k < 1, d > 0 \tag{5}$$

By using Taylor expansion on likelihood function given in equation (3) to limit $g(k) = \left(\frac{k}{t_i}\right)^a$ up to first rank when $k_0 = 1$, the $g(k)$ would be given by

$$g(k) \cong \left(\frac{1}{t_i}\right)^a + (k-1)a\left(\frac{1}{t_i}\right)^a \tag{6}$$

Substituting $g(k)$ in equation (5) into equation (3), after simplification gives

$$L(a,k,\underline{t}) = k^{na}\exp\left[-ka\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{a}\right] \tag{7}$$

Combining the likelihood function in equation (6) with the prior PDF of k in equation (4), we get the approximate posterior of k as

$$h(k \mid t_1, t_2, \dots, t_n)$$

3.1. Estimates based on SELF

The SELF is as follows [12]:

$$L(\hat{k},k) = (\hat{k}-k)^2. \tag{8}$$

The Bayes estimator of k based on SELF denoted by $\hat{k}_{BSE}$, can be obtained as follows:

$$\hat{k}_{BSE} = E_{\pi}(k \mid t_1, t_2, \dots, t_n), \tag{9}$$

where E_{π} indicates the expectation of the posterior distribution. Based on SELF and using equation (9), the Bayes estimation of scale parameter k denoted by $\hat{k}_{BSE}$, can be obtained as

$$\begin{aligned}\hat{k}_{BSE} &= E(k \mid \underline{x}) = \int_0^{\infty} kh(k \mid \underline{t})dk \\ &= \int_0^{\infty} \frac{\left(a\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{a}\right)^{na+d}}{\Gamma(na+d)} k^{na+d}\exp\left[-ka\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{a}\right]dk\end{aligned} \tag{10}$$

3.2. Estimates based on LLF

The LLF for k can be expressed as follows [13]:

$$L(\delta) \propto [\exp[\phi\delta] - \phi\delta - 1];\ \phi \neq 0, \tag{11}$$

where $\delta = (\hat{k}-k)$. The sign and magnitude of ϕ reflect the direction and degree of asymmetry, respectively. The Bayes estimator relative to LLF, denoted by $\hat{k}_{BL}$, is given by

$$\hat{k}_{BL} = -\frac{1}{\phi}\text{Ln}[E_k\exp[-\phi k]];\ \phi \neq 0 \tag{12}$$

provided that $E_{\phi} = \left(e^{-\phi k}\right)$ exists and finite, where E_k denotes the expected value. Based on LLF and using equation (12), the Bayes estimation of scale parameter, can be obtained as

$$\begin{aligned}\hat{k}_{BL} &= -\frac{1}{\phi}\text{Ln}[E_k\exp[-\phi k]] = \int_0^{\infty} \exp[-\phi k]h(k \mid t_1, t_2, \dots, t_n)dk \\ \hat{k}_{BL} &= -\frac{1}{\phi}\ln \int_0^{\infty} \exp[-\phi k]\frac{\left(a\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{a}\right)^{na+d}}{\Gamma(na+d)} k^{na+d-1} \\ &\quad \exp\left[-ka\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{a}\right]dk\end{aligned} \tag{13}$$

3.3. Estimates based on CLLF

The CLLF for k can be expressed as follows [14]:

$$L(\hat{k},k) = L_{\phi}(\hat{k},k) + L_{-\phi}(\hat{k},k)$$

The Bayes estimator of k, denoted by $\hat{k}_{BCL}$, is given by

$$\hat{\beta}_{BCL} = \frac{1}{2\phi}\ln\left(\frac{E(\exp[\phi k]|\underline{t})}{E(\exp[-\phi k]|\underline{t})}\right) \tag{14}$$

Based on CLLF and using equation (15), the Bayes estimation of scale parameter k denoted by $\hat{k}_{BCL}$, can be obtained as

$$\hat{\vartheta}_{BCL} = \frac{1}{2\phi}\ln\left(\frac{E_k(\exp[\phi k]|\underline{t})}{E_k(\exp[-\phi k]|\underline{t})}\right) = \frac{I_1}{I_2} \tag{15}$$

Bayesian Estimation of the Scale Parameter of Fréchet Distribution
Where

$$I_1 = \int_0^\infty \exp[\phi k]\frac{\left(a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a\right)^{na+d}}{\Gamma(na+d)}k^{na+d-1}\exp\left[-ka\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a\right]dk$$
$$= \left(\frac{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a}{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a - \phi}\right)^{na+d}$$

and

$$I_2 = \int_0^\infty \exp[-\phi k]\frac{\left(a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a\right)^{na+d}}{\Gamma(na+d)}k^{na+d-1}\exp\left[-ka\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a\right]dk$$
$$= \left(\frac{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a}{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a + \phi}\right)^{na+d}.$$

Therefore, the Bayes estimation of parameter k is

$$\hat{k}_{BCL} = \frac{1}{2\phi}\ln\left[\left(\frac{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a}{a\sum_{i=1}^n\left(\frac{k}{t_i}\right)^a - \phi}\right)^{na+d} \Big/ \left(\frac{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a}{a\sum_{i=1}^n\left(\frac{1}{t_i}\right)^a + \phi}\right)^{na+d}\right] \tag{16}$$

3.4. Estimates based on APLF

An asymmetric precautionary loss function (AP) is given as [15]:

$$L(\hat{k},k) = \frac{(\hat{k}-k)^2}{\hat{k}} \tag{17}$$

Under the asymmetric loss function (17), for any prior distribution of ϑ, the corresponding Bayesian estimation of k is

$$\hat{k}_{AP} = \left[\frac{E(k^{-1}|\underline{t})}{E(k|\underline{t})}\right]^{-\frac{1}{2}} = \left[\frac{I_3}{I_4}\right]^{-\frac{1}{2}},$$

$$I_3 = E\left(k^{-1} \mid \underline{t}\right) = \int_0^\infty \frac{\left(a\sum_{i=1}^n \left(\frac{1}{t_i}\right)^a\right)^{na+d}}{\Gamma(na+d)} k^{na+d-2} \exp\left[-ka\sum_{i=1}^n \left(\frac{1}{t_i}\right)^a\right] dk \tag{18}$$

and

$$I_4 = E(k \mid \underline{t}) = \int_0^\infty \frac{\left(a\sum_{i=1}^n \left(\frac{1}{t_i}\right)^a\right)^{na+d}}{\Gamma(na+d)} k^{na+d} \exp\left[-ka\sum_{i=1}^n \left(\frac{1}{t_i}\right)^a\right] dk$$
$$= \frac{na+d}{a\sum_{i=1}^n \left(\frac{1}{t_i}\right)^a}.$$

Therefore, the Bayes estimation of parameter k is

$$\hat{k}_{AP} = \frac{\sqrt{(na+d)(na+d-1)}}{a\sum_{i=1}^n \left(\frac{1}{t_i}\right)^a} \tag{19}$$

4. Simulation Study

We generated $L = 1000$ samples of size $n = 25,50,75$, and 100 to represent small, moderate and large sample sizes from French distribution with different values of the scale parameter ($k = 0.4,0.6$, and 0.80), with shape parameter $a = 2$. The parameter of power function distribution prior is $(d = 1.5, 3)$ and the values of parameter of LINEX (ϕ) are selected as $\phi = 1$ and 2. Monte-Carlo simulation study is performed to compare the methods of estimation by using mean square errors (MSE's) as follows:

$$\text{MSE}(\hat{k}) = \frac{\sum_{i=1}^L (\hat{k}-k)^2}{L}, \tag{20}$$

where $\hat{k}$ is the estimate at the i th run, and L is the number of replications. The results of the simulation are listed in Tables 1 and 2.

Table 1: *MSEs of the estimates of scale parameter k when $d = 1.5$*

(k, a)	n	$\hat{k}_{MLE}$	$\hat{k}_{BSE}$	$\hat{k}_{BAP}$	$\hat{k}_{BL}$		$\hat{k}_{BCL}$	
					$\phi = 1$	$\phi = 2$	$\phi = 1$	$\phi = 2$
$(0.4, 2)$	25	0.0469	0.0058	0.0054	0.0055	0.0053	0.0058	0.0059
	50	0.0447	0.0025	0.0024	0.0024	0.0023	0.0025	0.0025
	75	0.0443	0.0015	0.0015	0.0015	0.0014	0.0015	0.0015
	100	0.0435	0.0008	0.0008	0.0008	0.0008	0.0008	0.0009
$(0.6, 2)$	25	0.0924	0.1324	0.1236	0.1229	0.1135	0.1326	0.1325
	50	0.0878	0.1026	0.0987	0.0986	0.0950	0.1026	0.1029
	75	0.0864	0.0938	0.0914	0.0913	0.0899	0.0938	0.0949
	100	0.0857	0.0881	0.0867	0.0867	0.0851	0.0881	0.0880
$(0.8, 2)$	25	0.1520	0.7076	0.6732	0.6481	0.5951	0.7094	0.7161
	50	0.1451	0.6072	0.5914	0.5807	0.5565	0.6076	0.6098
	75	0.1432	0.5772	0.5669	0.5602	0.5446	0.5773	0.5788
	100	0.1415	0.5534	0.5474	0.5435	0.5328	0.5535	0.5526

Table 2: *MSEs of the estimates of scale parameter k when $d = 3$*

(k, a)	n	$\hat{k}_{MLE}$	$\hat{k}_{BSE}$	$\hat{k}_{BAP}$	$\hat{k}_{BL}$		$\hat{k}_{BCL}$	
					$\phi = 1$	$\phi = 2$	$\phi = 1$	$\phi = 2$
(0.4,2)	25	0.0475	0.0080	0.0073	0.0072	0.0067	0.0077	0.0076
	50	0.0448	0.0029	0.0027	0.0028	0.0027	0.0029	0.0029
	75	0.0441	0.0017	0.0016	0.0017	0.0017	0.0017	0.0018
	100	0.0436	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009
(0.6,2)	25	0.0921	0.1594	0.1498	0.1486	0.1396	0.1596	0.1615
	50	0.0883	0.1159	0.1117	0.1107	0.1068	0.1151	0.1155
	75	0.0868	0.1024	0.0998	0.0998	0.0965	0.1025	0.1018
	100	0.0861	0.0932	0.0918	0.0909	0.0891	0.0924	0.0921
(0.8,2)	25	0.1527	0.8201	0.7830	0.7525	0.6920	0.8211	0.8273
	50	0.1460	0.6605	0.6439	0.6309	0.6026	0.6595	0.6591
	75	0.1431	0.6077	0.5972	0.5940	0.5748	0.6119	0.6104
	100	0.1421	0.5748	0.5686	0.5602	0.5502	0.5704	0.5705

5. Conclusion

Comparisons are made between the different estimators based on simulation study and the effect of symmetric and asymmetric loss functions with respect to various sample sizes are observed. The Tables 1 and 2 show that the performance of the Bayes estimator of the scale parameter under LLF when $\phi = 2$ is the best compared to the other estimators with ($k = 0.5, a = 2$) for all sample sizes n while the MLE estimator is the best compared to the others with ($k = 0.7, a = 2$) and ($k = 0.9, a = 2$) for all sample sizes n.

The results also show that MSE's of all estimators of shape parameter increase with the increase in the value of the scale parameter. The results show that the values of all MSE's decrease as n increases. It can also be seen that Bayesian estimates are worse when the scale parameter approaches one.

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