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CONSTRUCTING A SEPARATE RESERVE IN A CHAIN OF LOW-RELIABILITY ELEMENTS

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Abstract

In this paper, we construct upper and lower estimates of the minimum separate reserve of low-reliability chain elements of length m, at which the probability of operability of the entire chain becomes close to unity. The appeal to a separate reserve is caused by the properties of this reserve, established by Barlow and Proschan. It is shown that the upper and lower estimates of the minimum separate reserve are quite close. If the probability of operability of an individual element of the chain is $1/\ln m$*, then the minimum reserve volume is of the order* $\ln^2 m$*.*

Keywords: a chain of low-reliability elements, a minimum amount of separate reserve, shortest paths in a two-pole.

1. Introduction

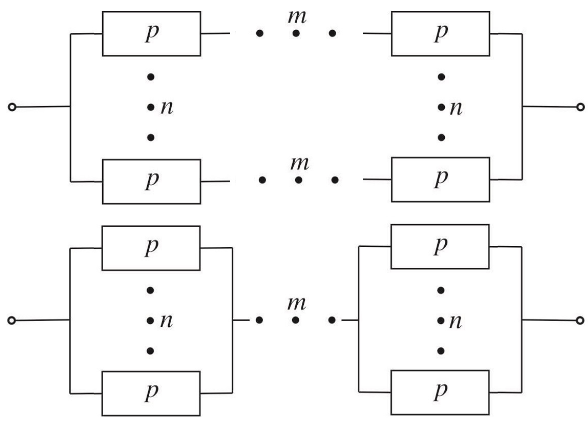
In the article of Erdos and Renyi [1], it was found that the probability of connectivity $P(m)$ for a complete graph with m vertices (in which there is an edge between any pair of vertices) and independently functioning edges with the probability $p(m) = \dfrac{c \ln m}{m}$, $m \to \infty$, satisfies the limit relations

$$\lim_{m\to\infty} P(m) = 1,\ c > 1,\ \lim_{m\to\infty} P(m) = 0,\ c < 1. \tag{1}$$

This property of a complete random graph resembles a phase transition in physical systems. The development of these results using modern graph-theoretic methods is described in the article of A.M. Raigorodsky [2] and in numerous articles on dynamic processes in random networks. A feature of the first limiting relation in (1) is the fact that a random graph with a high probability of connectivity is formed from low-reliability edges. Such random graphs are used to model the Internet and many other networks [3] –[6].

If, instead of a complete graph, we consider a chain of m sequentially connected edges, then with a high probability of operability of the edges, $p(m) = exp(-m^{-a}) \to 1,\ m \to \infty$ it is not difficult to establish that the probability of the chain ends being connected is $P(m) \to 0,\ a < 1$ and, conversely, $P(m) = \exp(-m^{1-a}) \to 1,\ a > 1$. In other words, for a chain of highly reliable edges, a phase transition is also observed with a variation of the parameter a in the vicinity of point 1.

However, chains with low-reliability edges do not have high reliability. Therefore, the problem arises of reserving parts of the chains, among which the most significant are the models of block and separate redundancy. Here, block redundancy is understood as the redundancy of two-poles in their entirety, i.e. their connection by the initial and final nodes. And separate redundancy means reserving each edge separately. In the classic monograph on the theory of reliability by Barlow and Proschan on [7], it was found that the probability of connectivity of the initial and final vertices of a bipolar in the case of a separate reserve is always higher than in the case of a block reserve.



A multiple n block reserve (top), a separate n reserve (bottom) of a chain with a length m.

In particular, when analysing two-poles represented by chains of length m, it was found that the minimum volume (the number of two-poles connected in parallel with the ends) of the block reserve exceeds a certain geometric progression growing by m. And the minimum volume of the split reserve (the number of individual edges connected in parallel by the ends) is limited from above by the logarithmic m function [8]. This result is also applicable to bipolar structures with m edges. With such an advantage of a separate reserve over a block one, it is natural to consider the task of constructing a highly reliable chain from low-reliability edges with separate redundancy. Moreover, the properties of the edge chain established for a separate reserve can be extended to two-poles.

2. Methods

Consider a chain (serial connection) m independently functioning edges with a probability of p, assuming $q = 1 - p,\ 0 < q < 1$. Because at $p \to 0$ the probability of the chain functioning If p^m also tends to zero, then convergence of this probability to unity is possible only if this chain is reserved.

Let each edge of the chain be n-fold reserved by independently functioning edges. Then the probability of a workable connection in such a chain between the initial and final vertices satisfies the equality $P_n(m) = (1 - q^n)^m$. Let us focus on determining the minimum volume of the separate reserve chain $n_*(m, \delta) = \min(n :\ P_n(m) \geq 1 - \delta)$ for some given $\delta,\ 0 < \delta < 1$. Following the tradition established in probability theory and mathematical statistics, it is natural to assume the value δ small ($\delta \ll 1$), for example, $\delta = 0.1,\ 0.05,\ 0.01$.

Estimation of the value of $n_*(m, \delta)$ We will carry out calculations using the following ratio. For any $a,\ 0 < a < 1$,, the obvious inequality holds

$$1 - ma \leq (1 - a)^m \leq 1 - ma + \frac{(ma)^2}{2}. \tag{2}$$

Then

$$\min\left(n :\ mq^n - \frac{(mq^n)^2}{2} \leq \delta\right) \leq n_*(m, \delta) \leq \min(n :\ mq^n \leq \delta) \tag{3}$$

To define $n_*(m, \delta)$ denote $x = mq^n$ and calculate the roots of the two equations $x = \delta,\ x^2/2 - x + \delta = 0$. The root of the first equation is $x_1 = \delta$, and the root of the second equation is $x_2 = 1 \pm \sqrt{1 - 2\delta}$. Hence, the inequality follows from the first equation

$$n_*(m, \delta) \leq \left[\frac{\ln m - \ln \delta}{-\ln q}\right] + 1, \tag{4}$$

where $[z]$ is the integer part of the real number z. In turn, the second equation implies the inequality

$$n_*(m,\delta) \geq \frac{\ln m - \ln(1-\sqrt{1-2\delta})}{-\ln q}. \tag{5}$$

Remark 1. For $\delta \ll 1$, we may approximately write $1-\sqrt{1-2\delta} \approx \delta$. This means that for large m, the upper and lower estimates of the value of $n_*(m,\delta)$ are almost the same.

3. Results

Inequalities (4), (5) are valid for any $q = 1-p,\ 0 < q < 1$, in particular for $q = q(m) \to 1,\ p = p(m) \to 0,\ m \to \infty$. By analogy with the technique of asymptotic estimation of the probability of connectivity of a complete graph [1], [2], we assume $p(m) \to 0,\ m \to \infty$. Then from formulas (4), (5) we obtain the following estimates for the required reserve volume

$$\frac{\ln m - \ln(1-\sqrt{1-2\delta}))}{\ln(1-p(m))} \leq n_*(m,\delta) \leq \left[\frac{\ln m - \ln\delta}{-\ln(1-p(m))}\right] + 1. \tag{6}$$

Let's now move on to the low-reliability edge model when $p(m) \to 0,\ m \to \infty$. Then, for example, for $p(m) = 1/\ln m \to 0,\ m \to \infty$, we get

$$\ln m(\ln m - \ln(1-\sqrt{1-2\delta})) \leq n_*(m,\delta) \leq \left[\frac{\ln m(\ln m - \ln\delta)}{1-1/\ln m}\right] + 1. \tag{7}$$

It follows from the formula (7) that the required volume of the separate reserve of the edges of the chain is proportional to $(\ln m)^2$ for $m \gg 1$.

If $p(m) = 1/\ln^b m$, then the minimum volume of the separate reserve becomes of the order $\ln^{b+1} m,\ 0 < b$. There are other ways to determine $p(m),\ n_*(m,\delta)$, for example, when $p(m) = 1/\ln\ln m$, the minimum reserve volume becomes on the order $\ln\ln m \cdot \ln m$.

4. Discussion

When switching from chains to bipolar circuits with low-reliability edges, the task arises of enumerating the shortest paths along which the minimum volumes of a separate reserve can be built. To do this, we can use a modification of the Dijkstra algorithm [9]. However, along with this, it will be necessary to consider the possible intersections of the shortest paths from the initial to the final vertex of the bipolar.

5. Conclusion

The upper bound in the formula (6) can be extended to a bipolar with m edges using the calculations of [8]. For $\delta \ll 1$, the upper bound in the formula (7) asymptotically coincides with the lower bound for $m \to \infty$. It should also be noted that by allocating the shortest path between the initial and final vertices in a bipolar using Dijkstra's algorithm (or some other algorithm, see, for example, [9]), it is possible to construct a separate reserve of all its edges using the algorithm of this paper.

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