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SOME ADDITIONAL PROPERTIES OF KUMARASWAMY LOG-LOGISTIC DISTRIBUTION BASED ON ORDER RANDOM VARIATES

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Abstract

The Exponentiated Kumaraswamy-Dagum distribution represents a significant advancement in the family of probability distributions. This family encompasses several well-established sub-models, including a variant of the Kumaraswamy-Fisk or Kumaraswamy log-logistic distribution, referred to as the Exponentiated Kumaraswamy-Dagum family distribution. In this paper, we characterize the Kumaraswamy log-logistic (KSLL) distribution by employing the moment features of generalized order statistics. Explicit expressions and recurrence relations are derived for both the single and product moments of generalized order statistics associated with the KSLL distribution. These formulations specifically address record values and moments of order statistics. Additionally, the distribution is further analyzed through conditional moments, and a recurrence relation for single moments of generalized order statistics is established. This rigorous investigation seeks to deepen the understanding of the KSLL distribution and its statistical properties as obtained by generalized order statistics.

Keywords: Generalized Order Statistics, Record Values, Order Statistics, Moments, Reliability, Characterization.

1. Introduction

As per historical review Dagum distrbution was introduced in 1977 by Camila Dagum [9]. This distribution if mainly fitted for the income and wealth data. Its was contained two formed as Dagum Type-I and Dagum Type-II specification distribution. This distrbution was a special case of generalized Beta distribution of second kind (GB2). Details McDonald [19], McDonald and Xu [20], Kleiber and Kotz [23]. Kleiber [22] introduced the rational properties of Dagum distribution. Further estimation properties for censored data dealt by Domma *et al.* [12]. After that there are several author find the generalized or mixed distribution with the Dagum distribution as different named as Beta-Dagum Distribution, Kumaraswamy Weibull distribution, Mc-Dagum distrbution, exponentiated Kumaraswamy-Dagum family distribution by the severals authors Domma and Condino [11], Cordiero *et al.* [6], Oluyede and Rajasooriya [24] and Huang and Oluyede [13], Alam *et al.* [1], [3].

Exponentiated Kumaraswamy-Dagum distribution was introduced by Huang and Oluyede [13]. It should be noted that the three parameter Kumaraswamy-Fisk or Kumaraswamy-log-

logistic distribution is a specific instance of the exponentiated version of the distribution. Due to its numerous uses in a variety of industries, including meteorology, acceptance sampling plans, areas of reliability, forests, modelling survival data or lifespan data, etc., it has attracted particular interest in recent decades.

A random variable X is said to have KSLL distribution, if its pdf is of the form

$$f(x) = \phi\delta\lambda x^{-\delta-1}(1+\lambda x^{-\delta})^{-2}[1-(1+\lambda x^{-\delta})^{-1}]^{\phi-1}, \quad x>0,\ \ \phi>0,\ \ \delta>0\ \ and\ \ \lambda>0 \tag{1}$$

Let $\lambda = 1$ without loss of generality,the corresponding pdf is

$$f(x) = \phi\delta x^{-\delta-1}(1+x^{-\delta})^{-2}[1-(1+x^{-\delta})^{-1}]^{\phi-1}, \quad x>0,\ \ \phi>0,\ and\ \ \delta>0 \tag{2}$$

and the corresponding distribution function (df)

$$F(x) = [1-(1+x^{-\delta})^{-1}]^{\phi},\ x>0,\ \ \phi>0,\ and\ \ \delta>0 \tag{3}$$

Now in view of (2) and (3), we have

$$\phi\delta\bar{F}(x) = (x+x^{-\delta+1})f(x) \tag{4}$$

The relation in (4) is the characterizing differential equation for distribution given in (3), where ϕ and δ are shape parameters.
Fisk or log-logistic distribution is a special case of Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution when $\phi = 1$. for more details about this distribution one may refers to Huang and Oluyede [13].

2. Statistical Properties

We provide various significant statistical and mathematical measurements for the KSLL distribution in this section, including quantiles, moments, mgf and hazard rate.
In general statistics, quantile functions are often employed and frequently have representations in the form of lookup tables for important percentiles. The p-th quantile function $F^{-1}(p)$ of the KSLL distribution is given by the following, which follows from (3).

$$Q(p) = [1-(1-p)^{1/\phi}-1]^{-1/\delta}, \qquad p \in (0,1) \tag{5}$$

In particular, the first three quantiles Q_1, Q_2 and Q_3, can be generated by setting p in (5) to, respectively, 0.25, 0.5, and 0.75. Be aware that KSLL distribution random variates can be produced using (5).

The necessity and significance of the moments in any statistical analysis, particularly in applied work, rarely needs to be emphasized. Moments can be used to study some of a distribution's most significant traits and attributes, such as central tendency, dispersion, skewness, and kurtosis. The k -th instant of the random variable X can be calculated as follows if it has the KSLL distribution.

Let $Z \sim KSLL(\phi, \delta)$, which is given in (2). Then the k-th moment is given by

$$\mu^k = \int_0^\infty z^k f(z)dz = \phi\delta\int_0^\infty z^{k-\delta-1}(1+z^{-\delta})^{-2}(1-(1+z^{-\delta})^{-1})^{\phi-1}dz. \tag{6}$$

Solving the (6), We get the k-th ordinary moment of KSLL distribution.

$$\mu^k = \phi\sum_{i=0}^{\infty}(-1)^i\binom{\phi-1}{i}B(1-\frac{k}{\delta}, i+1+\frac{k}{\delta}), \tag{7}$$

where $k \leq \delta$ and $Beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is a beta function.

If we consider $\phi > 0$ is integer. Then the k-th ordinary moment of KSLL distribution.

$$\mu^k = \phi \sum_{i=0}^{\phi-1} (-1)^i \binom{\phi-1}{i} B(1-\frac{k}{\delta}, i+1+\frac{k}{\delta}). \tag{8}$$

Let $\phi = 1$ in (8), we get the k-th moment of Log-logistic distribution as

$$\mu_{ll}^k = B(1-\frac{k}{\delta}, 1+\frac{k}{\delta}).$$

The above relationships allow for the computation of variance, skewness, and kurtosis measures. Through the moment generating function and moments, numerous intriguing characteristics and features of a distribution become accessible and can be derived.

Let $Z \sim KSLL(\phi, \delta)$, which is given in (2). Then the moment generating function is defined as

$$M_z(t) = E(e^{tz}) = \int_0^\infty e^{tz} f(z)dz = \phi\delta \int_0^\infty e^{tz} z^{k-\delta-1}(1+z^{-\delta})^{-2}(1-(1+z^{-\delta})^{-1})^{\phi-1}dz.$$

Expanding the exponential form

$$M_z(t) = \phi \sum_{i,j=0}^{\infty} (-1)^i \binom{\phi-1}{i} \frac{t^j}{j!} B(1-\frac{j+k}{\delta}, i+1+\frac{j+k}{\delta}).$$

Table 1: *Using* (6) *the Ordinary Moments of KSLL distribution with the parameter* $\delta = 5$

ϕ	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	$\mu_5^{'}$
0.5	1.5497	2.0000			
1.5	0.929818	0.952464			
2.5	0.805842	0.698473	0.64645		
3.5	0.741375	0.586718	0.491302	0.432905	
4.5	0.69901	0.519664	0.407079	0.333955	0.285714

Table 2: *Using Using* (7)*, the Ordinary Moments of KSLL distribution with the parameter* $\delta = 5$

ϕ	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	$\mu_5^{'}$
0.5	1.5497				
1.5	0.929818	0.952464			
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3.5	0.741375	0.586718	0.491302	0.432905	
4.5	0.69901	0.519664	0.407079	0.333955	0.285714

Table: 1 and 2 shows the verification of the moment expressions given in (7) with the general integration formula for the moment.

The hazard rate (HR) is one of the fundamental methods for researching the system's ageing and dependability characteristics. HR handles the remaining lifetime of the system. The HR provides the system's failure rate immediately following time x. As a result, the following gives the hazard rate function of the KSLL distribution.

$$h(x;\phi,\delta) = \frac{f(x;\phi,\delta)}{1-F(x;\phi,\delta)} = \frac{\phi * \delta}{x(1+x^{-\delta})}$$

3. Generalized Order Statistics

The introduction of generalized order statistics (GOS) by Kamps [15] encompasses various order models of random variables, including order statistics, upper record values, progressive Type II censoring order statistics, sequential order statistics, and Pfeifer's records.
Consider a sequence of independent and identically distributed (iid) random variables $X_1, X_2, ...$ (df) $F(x)$ and probability density function (pdf) $f(x)$. Assuming that $k > 0$, $n \in N$, $m \in R$ and $\gamma_r = k + (n-r)(m+1) > 0$. If the random variables $X(r,n,m,k)$, $r = 1,2,...,n$ possess a joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n), \tag{9}$$

on the cone $F^{-1}(0) < x_1 \leq \ldots . \leq x_n < F^{-1}(1)$.

In such a scenario, these random variables are referred to as generalized order statistics derived from a sample obtained from a distribution with a distribution function $F(x)$. It is noteworthy that when $m = 0, k = 1$, this model simplifies to the joint probability density function of regular order statistics. Similarly, when $m = -1$ the model corresponds to the joint probability density function of the k-th upper record values. We shall also take $X(0,n,m,k) = 0$. The pdf of the $r-$ th gos is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left(F(x)\right). \tag{10}$$

The joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k), 1 \leq r < s \leq n$ is

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ &\times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s - 1} f(y), \quad x < y. \end{aligned} \tag{11}$$

where $\bar{F}(x) = 1 - F(x)$, $\quad C_{r-1} = \prod_{i=1}^{r}\gamma_i, \ \ r = 1,2,...,n,$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \quad m \neq -1 \\ -\ln(1-x) & , \quad m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0,1).$$

Many authors utilized the concept of gos in their work. References may be made to Kamps and Gather[16], Keseling [17], Cramer and Kamps [7], Ahsanullah (2000), Habibullah and Ahsanullah (2000), Raqab (2001), Kamps and Cramer[18], Ahmad and Fawzy (2003), Beiniek and Syznal (2003), Cramer *et al.* [8], Khan and Alzaid (2004), Jaheen (2005), Khan *et al.* (2006), Khan *et al.* (2010), Khan and Zia (2014), Khan and Khan (2016), Singh *et al.*[28], Singh *et al.*[29], Alam *et al.*[2] and Singh *et al.* [27] among others.

4. Relation for Single Moments

We shall first establish the exact expression for $E(X^j(r,n,m,k))$. Using (10), we have when $m \neq -1$

$$E[X(r,n,m,k)]^j = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left(F(x)\right)dx. \tag{12}$$

On expanding $g_m^{r-1}(F(x)) = \left(\frac{1}{m+1}1 - (\bar{F}(x))^{m+1}\right)^{r-1}$ binomially in (12), we get

$$E[X(r,n,m,k)]^j = A \int_0^\infty x^j \left[\bar{F}(x)\right]^{\gamma_{r-u}-1} f(x)dx, \tag{13}$$

where,

$$A = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}$$

On using (3), we obtain

$$E[X(r,n,m,k)]^j = A \int_0^\infty (-1)^{j/\delta} [1 - (1 - \bar{F}(x)^{1/\phi})^{-1}]^{-j/\delta} [\bar{F}(x)]^{\gamma_{r-u}-1} f(x)dx \tag{14}$$

On using Maclaurine series expansion

$$(1-z)^{-t} = \sum_{k=0}^\infty \frac{(t)_k z^k}{k!} = \sum_{k=0}^\infty \frac{\Gamma(t+k) z^k}{\Gamma t k!} \tag{15}$$

$$(t)_k \;=\; \begin{cases} t(t+1)...(t+k-1) & , \quad k = 1,2 \\ \\ 1 & , \quad k = 0 \end{cases}$$

Using (15) in (14), we have

$$E[X(r,n,m,k)]^j = A \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(j/\delta)_p (p)_q}{p!q!} \int_0^\infty (-1)^{j/\delta} [\bar{F}(x)]^{\gamma_{r-u}+q/\phi-1} f(x)dx$$

$$E[X(r,n,m,k)]^j = A \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(j/\delta)_p (p)_q}{p!q!} (-1)^{j/\delta} \int_0^1 t^{\gamma_{r-u}+q/\phi-1} dt.$$

and simplifying the resulting expression, we find that

$$E[X(r,n,m,k)]^j = \frac{C_{r-1}(-1)^{j/\delta}}{(r-1)!(m+1)^{r-1}} \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(j/\delta)_p (p)_q}{p!q!} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \times \frac{1}{[k + (n-r+u)(m+1) + q/\phi]} \tag{16}$$

When $m = -1$, we have

$$E[X(r,n,m,k)]^j = \frac{0}{0} \quad \text{as} \quad \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} = 0.$$

Since (16)is of the form $\frac{0}{0}$ at $m = -1$, therefore, we have

$$E[X(r,n,m,k)]^j = B \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{[k + (n-r+u)(m+1) + q/\phi]^{-1}}{(m+1)^{r-1}}, \tag{17}$$

where

$$B = \frac{C_{r-1}}{(r-1)!} (-1)^{j/\delta} \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(j/\delta)_p (p)_q}{p!q!}$$

Differentiating numerator and denominator of (17) $(r-1)$ times with respect to m, we get

$$E[X(r,n,m,k)]^j = B \sum_{u=0}^{r-1} (-1)^{u+(r-1)} \binom{r-1}{u} \frac{[1.2.3...(r-1)](n-r+u)^{r-1}}{(r-1)![k + (n-r+u)(m+1) + q/\phi]^r}.$$

On applying L' Hospital rule, we have

$$\lim_{m\to -1} E[X(r,n,m,k)]^j = B\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}(r-n-u)^{r-1}\frac{[1.2.3...(r-1)]}{(r-1)!(k+q/\phi)^r}. \tag{18}$$

But for all integers $n \geq 0$ and for all real numbers x , we have Ruiz [25]

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^n = n!. \tag{19}$$

Now substituting (19) in (18) and simplifying, we find that

$$E[X(r,n,-1,k)]^j = E[(Y_r^k)^j] = k^r(-1)^{j/\delta}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\delta)_p(p)_q}{p!q!}\frac{1}{[k+q/\phi]^r} \tag{20}$$

where $(Y_r^k)^j$denotes the $k-$th upper record values.

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i) Putting $m=0$ and $k=1$ in (16), the exact expression for the single moments of order statistics from Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution can be obtained as

$$E(X_{r:n})^j = C_{r:n}(-1)^{j/\beta}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\sum_{p=0}^{r-1}\sum_{q=0}^{\infty}\frac{(j/\delta)_p(p)_q}{p!q!}\frac{1}{[n-r+u+1+q/\phi]^r} \tag{21}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

ii) Putting $k=1$ in (20), we deduce the explicit formula for the single moments of upper records for Kumaraswamy -Fisk or Kumaraswamy- log- logistic distribution in the form

$$E[(Y_r^k)^j] = E(X^j_{U(r)}) = (-1)^{-j/\delta}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\delta)_p(p)_q}{p!q!}\frac{1}{[1+q/\phi]^r} \tag{22}$$

Expressions (21) and (22)can be used to obtain the moments of order statistics and upper record values for arbitrary chosen values of ϕ, δ and various sample size $n = 1, 2, ..., 10$.

Remark 1. i) Putting $\phi = 1$ in (16), we get the explicit expression for the single moments of gos from Fisk or log- logistic distribution can be obtained.

$$E[X(r,n,m,k)]^j = \frac{C_{r-1}(-1)^{j/\delta}}{(r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\delta)_p(p)_q}{p!q!}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}$$
$$\times\frac{1}{[k+(n-r+u)(m+1)+q]}.$$

ii) Putting $\phi = 1$ in (22), we deduce the explicit formula for the single moments of upper records for Fisk or log- logistic distribution.

$$E[X(r,n,-1,k)]^j = E[(Y_r^k)^j] = k^r(-1)^{j/\delta}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\delta)_p(p)_q}{p!q!}\frac{1}{[k+q]^r}$$

Table 3: *First Moments of Order Statistics with the different parameter ϕ and $\delta = 1$*

n	r	$\phi = 0.5$	$\phi = 1.5$	$\phi = 2.5$	$\phi = 3.5$	$\phi = 4.5$	$\phi = 5.5$
1	1	1.5497	0.92982	0.80585	0.741375	0.69901	0.66795
2	1	1.0689	0.76966	0.68242	0.633289	0.59982	0.57475
	2	2.0304	1.08998	0.92926	0.84947	0.798206	0.76113
3	1	0.929821	0.69901	0.62385	0.58044	0.55053	0.52799
	2	1.34724	0.91093	0.799572	0.73898	0.69837	0.66828
	3	2.37203	1.17951	0.994105	0.9047	0.84812	0.807555
4	1	0.855167	0.65513	0.586486	0.546365	0.518563	0.497545
	2	1.15377	0.830662	0.735941	0.682682	0.646446	0.61934
	3	1.54072	0.99120	0.86320	0.795278	0.75031	0.717217
	4	2.64913	1.24228	1.03774	0.941174	0.888721	0.837668
5	1	0.805842	0.62385	0.559485	0.521595	0.495252	0.475299
	2	1.05247	0.78024	0.694489	0.645443	0.611804	0.586526
	3	1.30572	0.90630	0.798117	0.738541	0.698409	0.668569
	4	1.69738	1.0478	0.906592	0.833103	0.784914	0.749649
	5	2.88707	1.2909	1.07053	0.968192	0.904673	0.859673
6	1	0.769651	0.59982	0.538559	0.502329	0.477086	0.457942
	2	0.986799	0.744022	0.664116	0.617923	0.586083	0.562088
	3	1.18381	0.85266	0.755236	0.700483	0.663246	0.635403
	4	1.42763	0.959944	0.840999	0.776599	0.733572	0.701735
	5	1.83226	1.09173	0.939389	0.861354	0.810585	0.773606
	6	3.09803	1.33074	1.096775	0.98956	0.923491	0.876887
7	1	0.741375	0.580444	0.521595	0.486673	0.462303	0.443805
	2	0.939307	0.716045	0.640346	0.596267	0.56578	0.542762
	3	1.10553	0.813964	0.723543	0.672063	0.636837	0.610403
	4	1.28818	0.904256	0.797493	0.738377	0.698458	0.668737
	5	1.53221	1.00171	0.873629	0.805266	0.759907	0.726483
	6	1.95228	1.12774	0.965693	0.88379	0.830856	0.792455
	7	3.28899	1.36457	1.1186	1.00719	0.93893	0.890959
8	1	0.718341	0.564306	0.507403	0.473552	0.449903	0.431939
	2	0.902613	0.693406	0.620936	0.578516	0.549106	0.526865
	3	1.04939	0.783962	0.698576	0.64952	0.61581	0.59045
	4	1.1991	0.863966	0.765155	0.709634	0.671882	0.643658
	5	1.37727	0.944545	0.82983	0.76712	0.725035	0.693816
	6	1.62517	1.03601	0.89999	0.828154	0.78083	0.746083
	7	2.06132	1.15831	0.98762	0.902335	0.847532	0.807912
	8	3.46437	1.39404	1.13731	1.02217	0.951986	0.902822
9	1	0.69901	0.550533	0.495252	0.462303	0.439264	0.421754
	2	0.872983	0.674488	0.604608	0.563541	0.53501	0.513421
	3	1.00632	0.759617	0.67808	0.630927	0.59842	0.57392
	4	1.13553	0.832654	0.73956	0.686706	0.65058	0.62351
	5	1.27855	0.903107	0.797148	0.738294	0.698498	0.668844
	6	1.45625	0.97769	0.855976	0.79018	0.746266	0.713793
	7	1.70963	1.06517	0.921876	0.84714	0.798112	0.762228
	8	2.1618	1.18492	1.0064	0.918105	0.8616520	0.820964
	9	3.6272	1.42018	1.15367	1.03518	0.963278	0.913055

Table 4: *Continued...*

n	r	$\phi = 0.5$	$\phi = 1.5$	$\phi = 2.5$	$\phi = 3.5$	$\phi = 4.5$	$\phi = 5.5$
10	1	0.682424	0.538559	0.484662	0.452489	0.429976	0.412859
	2	0.848292	0.658304	0.590567	0.550635	0.522858	0.501811
	3	0.971747	0.739227	0.660773	0.615164	0.583647	0.55986
	4	1.08699	0.807192	0.718473	0.667708	0.63289	0.606728
	5	1.20834	0.870846	0.771193	0.715204	0.677138	0.648683
	6	1.34877	0.935369	0.823103	0.761384	0.719857	0.689005
	7	1.5279	1.00591	0.877891	0.809377	0.76387	0.730319
	8	1.78751	1.09056	0.940726	0.863325	0.812787	0.775904
	9	2.25537	1.20851	1.02282	0.9318	0.873868	0.832222
	10	3.77962	1.44369	1.16821	1.04666	0.973213	0.922035

Table 5: *First Moment Based on Upper Record Values*

n	$\phi = 1.5$	$\phi = 2.5$	$\phi = 3.5$	$\phi = 5.5$	$\phi = 6.5$	$\phi = 7.5$
1	0.929818	0.805842	0.741375	0.667943	0.643654	0.62385
2	1.21669	1.01358	0.918689	0.817397	0.785154	0.759243
3	1.46957	1.17176	1.04499	0.917449	0.8783	0.847274
4	1.73184	1.31691	1.15418	0.999098	0.953103	0.917135
5	2.01973	1.46037	1.25649	1.0716	1.01852	0.977535
6	2.34366	1.60757	1.35652	1.13902	1.07847	1.03229
7	2.71253	1.76185	1.45686	1.20356	1.13508	1.08344
8	3.13513	1.9256	1.55916	1.26655	1.18959	1.13222
9	3.62085	2.10081	1.66462	1.32887	1.24285	1.17941
10	4.1801	2.28925	1.77416	1.39115	1.29545	1.22558

Now we obtain the recurrence relations for single moments of Kumaraswamy-Fisk or Kumaraswamy-log- logistic distribution in the following theorem.

Theorem 1. For the distribution as given in (3) and $n \in N$, $m \in R$, $1 \le r \le n$, $j = 0, 1, 2, ...$

$$E[X(r,n,m,k)]^j[1 - \frac{j}{\phi\delta\gamma_r}] = E[X(r-1,n,m,k)]^j + \frac{j}{\phi\delta\gamma_r}E[X(r,n,m,k)]^{j-\delta} \tag{23}$$

Proof:. From (3) and (4), we have

$$E[X(r,n,m,k)]^j - E[X(r-1,n,m,k)]^j = \frac{j}{\phi\delta\gamma_r}[\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}(F(x))f(x)dx]$$

$$+\frac{j}{\phi\delta\gamma_r}[\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-\delta}[\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)dx]$$

Now (23) can be proved by noting that in view of Athar and Islam [4]

$$E[X(r,n,m,k)]^j - E[X(r-1,n,m,k)]^j = \frac{jC_{r-2}}{(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx.$$

Remark 2. i) Substituting $m = 0$, $k = 1$ in (23), we deduce the recurrence relation for single moments of order statistics for Kumaraswamy Fisk or Kumaraswamy log logistic distribution in the form

$$E(X_{r:n}^j)[1 - \frac{j}{\phi\delta(n-r+1)}] = E(X_{r-1:n}^j) + \frac{\lambda j}{\phi\delta(n-r+1)}E(X_{r:n}^{j-\delta})$$

ii) Putting $m = -1$ and $k = 1$ in (23), the result for single moments of upper record values from for Kumaraswamy Fisk or Kumaraswamy log logistic distribution is deduced as

$$E(X^j_{U(r)})[1 - \frac{j}{\phi\delta}] = E(X^j_{U(r-1)}) + \frac{j}{\phi\delta}E(X^{j-\delta}_{U(r)}).$$

(iii) Putting $\phi = 1$ in (23), we get the recurrence relation for single moments of *lgos* from the for Fisk or log logistic distribution.

$$E[X(r,n,m,k)]^j[1 - \frac{j}{\delta\gamma_r}] = E[X(r-1,n,m,k)]^j + \frac{j}{\delta\gamma_r}E[X(r,n,m,k)]^{j-\delta}$$

5. Characterization

Let $X(r,n,m,k),\ r = 1,2,...,n$ be *gos*, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x,\ 1 \le r < s \le n$, is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(x|y) = \frac{C_{s-1}}{(s-r-1)!\,C_{r-1}}[\bar{F}(x)]^{m-\gamma_r+1} \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y),\ \ x < y. \tag{24}$$

Theorem 2. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0) = 0$ and $0 \le F(x) \le 1$ for all $x \ge 0$, then

$$E[\xi\{X(s,n,m,k)\}|X(r,n,m,k) = x] = [1-(1+x^{-\delta})^{-1}]^{\phi}\prod_{j=1}^{s-l}\frac{\gamma_{l+j}}{\gamma_{l+j}+1}\ \ l = r,\ r+1, \tag{25}$$

if and only if

$$\bar{F}(x) = [1-(1+x^{-\delta})^{-1}]^{\phi},\ \ x > 0,\ \ \phi > 0,\ \ \delta > 0. \tag{26}$$

where,

$$\xi(y) = [1-(1+y^{-\delta})^{-1}]^{\phi},\ \ x > 0,\ \ \phi > 0,\ \ \delta > 0.$$

Proof. From (22) for $s > r+1$, we have

$$E[\xi\{X(s,n,m,k)\}|X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_x^{\infty}[1-(1+y^{-\delta})^{-1}]^{\phi}\Big[1-\Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{m+1}\Big]^{s-r-1}\Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{27}$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{[1-(1+y^{-\delta})^{-1}]^{\phi}}{[1-(1+x^{-\delta})^{-1}]^{\phi}}$, from (3) in (27), we find that

$$E[\xi\{X(s,n,m,k)\}|X(r,n,m,k) = x] = \frac{C_{s-1}[1-(1+x^{-\delta})^{-1}]^{\phi}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_0^1 u^{\gamma_s}\Big(1-u^{m+1}\Big)^{s-r-1}du \tag{28}$$

Again by setting $t = u^{m+1}$ in (28), we get

$$E[\xi\{X(s,n,m,k)\}|X(r,n,m,k) = x] = \frac{C_{s-1}[1-(1+x^{-\delta})^{-1}]^{\phi}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_0^1 t^{\frac{\gamma_s+1}{m+1}-1}(1-t)^{s-r-1}dt \tag{29}$$

and hence the necessary part.

To prove sufficient part, we have from (22) and (23)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty [1-(1+y^{-\delta})^{-1}]^\phi \left[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}\right]^{s-r-1}$$

$$\times [\bar{F}(y)]^{\gamma_s -1} f(y)dy = g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}}, \tag{30}$$

where

$$g_{s|r}(x) = [1-(1+x^{-\delta})^{-1}]^\phi \prod_{j=1}^{s-r} \frac{\gamma_{r+j}}{\gamma_{r+j}+1}.$$

Differentiating (30) both sides with respect to x, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_x^\infty [1-(1+y^{-\delta})^{-1}]^\phi \left[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}\right]^{s-r-2}$$

$$[\bar{F}(y)]^{\gamma_s-1} f(y)dy = g'_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1}\, g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x)$$

or

$$-\gamma_{r+1} g_{s|r+1}(x)[\bar{F}(x)]^{\gamma_{r+2}+m} f(x)$$

$$= g'_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x),$$

where,

$$g'_{s|r}(x) = -\phi\delta x^{-\delta-1}(1+x^{-\delta})^{-2}(1-(1+x^{-\delta})^{-1})^{\phi-1} \prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\Big),$$

$$g_{s|r+1}(x) = (1-(1+x^{-\delta})^{-1})^\phi \Big(\frac{\gamma_{r+1}+1}{\gamma_{r+1}}\Big)\Big(\prod_{j=1}^{s-r}\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\Big).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]}$$

$$= \frac{\phi\delta}{x + x^{-\delta+1}}. \tag{31}$$

Integrating (31) on both sides with respect to x between $(0, y)$, the sufficient part is proved.

Remark 3. i) As $m = -1$ in (25), we get the characterization results from the Kumaraswamy- Fisk or Kumaraswamy log-logistic distribution based on$k$- th upper record values.
ii) Setting $m = 0$, $k = 1$ in (25), we obtain the characterization results of the Kumaraswamy- Fisk or Kumaraswamy log-logistic distribution based on order statistics.

Following theorem contains characterization of this distribution by a recurrence relation for the single moments of *gos*.

Theorem 3. For a positive integer k and j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with pdf given by (2) is that

$$E[X(r,n,m,k)]^j[1-\frac{j}{\phi\delta\gamma_r}] = E[X(r-1,n,m,k)]^j + \frac{j}{\phi\delta\gamma_r}E[X(r,n,m,k)]^{j-\delta} \tag{32}$$

Proof. The necessary part follows from equation (23). On the other hand if the relation in (32)is satisfied, then on using (10), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\big[\bar{F}(x)\big]^{\gamma_r-1}f(x)g_m^{r-1}\big(F(x)\big)dx = \frac{C_{r-2}}{(r-2)!}\int_0^\infty x^j\big[\bar{F}(x)\big]^{\gamma_{r-1}-1}f(x)g_m^{r-2}\big(F(x)\big)dx$$
$$+\frac{jC_{r-1}}{\phi\delta\gamma_r(r-1)!}\int_0^\infty x^j\big[\bar{F}(x)\big]^{\gamma_r-1}f(x)g_m^{r-1}\big(F(x)\big)dx$$
$$+\frac{jC_{r-1}}{\phi\delta\gamma_r(r-1)!}\int_0^\infty x^{j-\delta}\big[\bar{F}(x)\big]^{\gamma_r-1}f(x)g_m^{r-1}\big(F(x)\big)dx$$

Let

$$h(x) = -\frac{1}{\gamma_r}\big[\bar{F}(x)\big]^{\gamma_r}g_m^{r-1}\big(F(x)\big) \tag{33}$$

Differentiating both the sides of (33), we get

$$h'(x) = -\frac{1}{\gamma_r}\big[\bar{F}(x)\big]^{\gamma_r-1}f(x)g_m^{r-2}\big(F(x)\big)\{\gamma_r g_m\big(F(x)\big)-(r-1)\big[\bar{F}(x)\big]^{m+1}\}$$

Thus,

$$\frac{C_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^j h'(x)dx = \frac{jC_{r-1}}{\phi\delta\gamma_r(r-1)!}\int_0^\infty x^j\big[\bar{F}(x)\big]^{\gamma_r-1}f(x)g_m^{r-1}\big(F(x)\big)dx$$
$$+\frac{jC_{r-1}}{\phi\delta\gamma_r(r-1)!}\int_0^\infty x^{j-\delta}\big[\bar{F}(x)\big]^{\gamma_r-1}f(x)g_m^{r-1}\big(F(x)\big)dx \tag{34}$$

Integrating right hand side in (34) by parts and using the value of $h(x)$ from (33), we find that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\big[\bar{F}(x)\big]^{\gamma_r-1}g_m^{r-1}\big(F(x)\big)\{\bar{F}(x)-\frac{x}{\phi\delta}f(x)-\frac{1}{\phi\delta}x^{-\delta+1}f(x)\}dx = 0 \tag{35}$$

Applying the extension of Mntz-Szsz Theorem, (see for example Hwang and Lin, [14]), to (28), we get

$$\phi\delta\bar{F}(x) = (x+x^{-\delta+1})f(x).$$

Which proves that $f(x)$ has the form as in (4).

6. Conclusion

The work by Huang and Oluyede [13] and Santana *et al.* [26] examines the KSLL distribution using features obtained from order random variables. Applications for order random variables can be found in many different domains, including distribution parameter estimation, production process optimisation, auto racing, insurance policy modelling, and auctions. The purpose of this essay is to go into great detail about the idea and importance of order random variables. Establishing the KSLL distribution's descriptive and distributional features, calculating its ordinary moments, and displaying the findings graphically are the main objectives of the first portion. Order statistics, record values, and their recurrence relations for single and product moments obtained from the KSLL distribution are among the related properties of order random variables that are examined in the second part. Theorems of characterisation for these qualities are also discussed. Two theorems describing the KSLL distribution based on conditional moments for single moments and recurrence relations are presented and proven in the work. This approach also analyses a number of deductions and specific circumstances.

Declarations

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Consent for Publication: I consent to the publication of the data and materials.

Code Availability We are using the software an R and MATHEMATICA to calculate the numerical values and graphics.

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