LEAST SQUARES, MAXIMUM LIKELIHOOD, AND ANDERSON-DARLING TYPES ESTIMATORS FOR ENTROPY TRANSFORMED EXPONENTIAL DISTRIBUTION

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Abstract

Aim: This research aim is to investigate the performance stability and determine the best estimator among Least Squares (LS), Maximum Likelihood (ML), and Anderson Darling (AD) type (right tail AD, left tail AD, second order right tail AD, and second order left tail AD), estimators for estimating the parameter of the Entropy Transformed Exponential distribution. Methods: A Monte Carlo simulation was conducted to evaluate the finite sample performance stability and efficiency at sample sizes of 20, 100, 250, 700, 1000, 1500, 2500, *and* 3000. *The evaluation measures used for assessment are the Absolute Bias (AB) and Root Mean Square Error (RMSE) as performance metrics. Real life data analysis was performed using a field intensity data measured in microvolts recorded at 5-second intervals. The data is a positive right skewed data and highly peaked. Results: The simulation results demonstrated that the LS method consistently outperformed the ML and AD type estimators. This is based on the lowest values of AB and RMSE obtained across all the eight sample sizes. Additionally, the findings were validated through real world data, where the LS estimator exhibited the lowest standard error and the narrowest confidence interval length. Based on the results, the LS is ranked 1st, right tail AD ranked 2nd, second order right tail AD ranked 3rd, MLE ranked 4th, left tail AD ranked 5th, and second order left tail AD ranked 6th. However, results from t-test in determining the significance of the parameter estimate from each estimator, shows that MLE and second order left tail AD estimates are not significant at* 5%. *The graphical representation of the field intensity data by each estimators fit clearly depict the LS estimator best describe the data distribution. Conclusion: Based on these findings, the study concludes that the LS estimator is the most effective method for estimating the parameter of the Entropy Transformed Exponential distribution followed by the right tail AD, second order right tail AD, MLE, left tail AD, and second order left tail AD being the least effective estimator.*

Keywords: Entropy Transformed Exponential distribution (EnTrExD), Least Squares (LS), Maximum Likelihood (ML), Anderson-Darling (AD), Estimators

I. INTRODUCTION

In the discussion of parameter estimation, the Maximum Likelihood (ML) estimator remains one of the most widely utilized techniques due to its simplicity and unbiased properties. While it may exhibit limitations when applied to small sample sizes, its overall effectiveness and versatility have solidified its prominence in various applications. Beyond the ML approach, numerous alternative estimation methods have been proposed in the literature, each with unique advantages and limitations. Comparative studies have also been conducted to evaluate the performance of these techniques across different models. Results from these studies indicate that the effectiveness of a given estimation method can vary depending on the specific characteristics of the model, with certain techniques demonstrating superior performance under particular conditions.

According to Almathkour *et al*. [2], the choice of an estimation method depends on the application area and performance standards. They argue that users may opt for a minimum variance unbiased estimator (MVUE) despite the absence of a closed-form expression, as it provides minimum variance among unbiased estimators. Similarly, Sanku Dey and Park [10] recommended using maximum likelihood (ML) or Bayes estimators for parameter estimation in the weighted exponential distribution, based on their comparative analysis of various estimation methods. Additionally, David *et al*. [3] proposed the estimated generalized least square technique with residual maximum likelihood estimation (EGLS-REML) for variance components in BertalanffyRichards split-plot design models (SPDM). Their study demonstrated that EGLS-REML outperformed ordinary least square (OLS) and EGLS-ML estimators for parameter estimation in SPDMs.

Mazucheli [9] conducted a comparative analysis of the L-moments and maximum likelihood estimation (MLE) techniques for the complementary Beta distribution using simulations and annual temperature extremes. Their simulation results indicated that the ML estimator exhibited lower bias and mean square error (MSE) compared to L-moments. However, they observed that ML estimates tended to be negatively biased, whereas L-moments estimates were positively biased. Despite these differences, both estimators were found to be asymptotically unbiased and consistent. For small sample sizes, they concluded that the ML estimator is more appropriate due to its superior performance. Almathkour *et al*. [2] compared eight estimation techniques—maximum likelihood estimation (MLE), ordinary least squares (OLS), weighted least squares (WLS), method of moments (MoM), maximum product of spacing (MPS), percentiles estimator (PCE), *L*2 distance estimator (L2DE), and Kullback-Leibler divergence of survival function (KLDSF)—using Monte Carlo simulations. Their findings revealed that MLE and L2DE produced the smallest mean square error (MSE) across all sample sizes, while PCE and KLDSF yielded the largest MSE. MPS and PCE provided the smallest absolute bias, with MoM, MPS, and PCE exhibiting negative bias, while the other estimators showed positive bias. Despite these differences, all estimators were asymptotically unbiased. However, based on the application of real-life data, the WLS estimator outperformed the other techniques. David *et al*. [4] found that the estimated generalized least squares with residual maximum likelihood (EGLS-REML) estimator outperformed OLS and EGLS-MLE in parameter estimation for a three parameter Weibull split-plot design model (SPDM). This conclusion was based on Median Adequacy Measures (MAM), Akaike Information Criterion (AIC), Corrected AIC, Bayesian Information Criterion (BIC), and Standard Error of Estimates (SEE). Similarly, David *et al*. [5, 6] demonstrated the superiority of EGLS-REML over OLS and EGLS-MLE for Johnson-Schumacher and Chapman-Richards SPDMs, based on the same performance criteria.

In this study, three estimation methods are compared: MLE, least squares (LS), and Anderson Darling (AD)-based estimators, which include right-tail AD (ADR), right-tail second-order AD (AD2R), left-tail AD (ADL), and left-tail second-order AD (AD2L). The comparison is conducted through Monte Carlo simulation and applied to real-life data on field intensity to identify the best-performing method for estimating the one-parameter Entropy Transformed Exponential (ETE) distribution introduced by [8].

II. ENTROPY TRANSFORMED EXPONENTIAL DISTRIBUTION

A continuous variable C is said to follow ETE distribution with scale parameter, α and shape parameter, φ. It is denoted as ETE ~ (C, φ) if its probability density function (PDF), and cumulative density function (CDF) are given by [8] as

$$f(c) = \varphi^2 c e^{-\varphi c} \tag{1}$$

and

$$F(c) = e^{-\varphi c}\left(e^{-\varphi c} - 1 - \varphi c\right) \tag{2}$$

The quantile function (q_c) of the ETE distribution was derived as

$$q_c = \frac{\log\left(\frac{W(k)e^{-1}+1}{k}\right)}{2\varphi} \tag{3}$$

The CDF and PDF plot of the model are presented in Figure 1. The plots indicated that the probability model is a right tailed distribution, thus will perform better on data that is right-skewed.

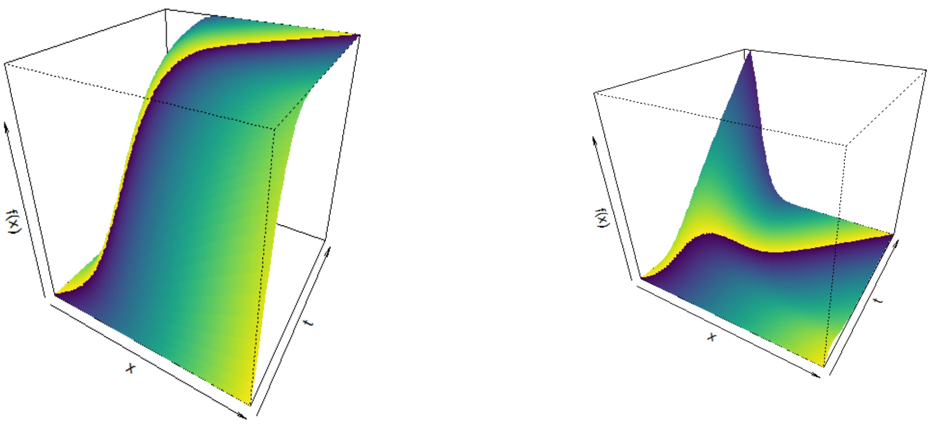

Figure 1: *Left to Right is the CDF and PDF of ETE Distribution*

III. METHODS OF ESTIMATION

In this section of the research, three estimation techniques are considered and presented for estimating the unknown parameter φ, for the ETE distribution. The techniques as mentioned earlier are LS, ML, and AD type's Right-tail AD, Left-tail AD, Right-tail AD of second degree and Left-tail AD of second degree of second degree (AD2) (ADR, ADL, AD2R, and AD2L) estimators.

I. The Least Squares Method (LS_m)

The LS_m estimator of the unknown parameter φ is obtained by minimizing the following equations with respect to φ:

$$LS_m = \sum_{i=1}^{n}\left[F(c_i) - \frac{i}{(n+1)}\right]^2 \tag{4}$$

$$LS_m = \sum_{i=1}^{n}\left((e^{-\varphi c_i}(e^{-\varphi c_i} - 1 - \varphi c_i)) - \frac{i}{(n+1)}\right)^2 . \tag{5}$$

Partially differentiating Equation (5) w.r.t φ and equate to zero gives

$$\frac{\partial}{\partial\varphi} = \sum_{i=1}^{n}\left((e^{-\varphi c_i}(e^{-\varphi c_i} - 1 - \varphi c_i)) - \frac{i}{(n+1)}\right)^2$$

$$2\sum_{i=1}^{n}\left((e^{-\varphi c_i}(e^{-\varphi c_i} - 1 - \varphi c_i)) - \frac{i}{(n+1)}\right)\left(c_i e^{-2\varphi c_i}(c_i \varphi e^{\varphi c_i} - 2)\right) = 0$$

$$\sum_{i=1}^{n}\left(e^{-2\varphi c_i} - e^{-\varphi c_i} - \varphi c_i e^{-\varphi c_i} - \frac{i}{(n+1)}\right)\left(c_i^2 \varphi e^{-\varphi c_i} - 2c_i e^{-2\varphi c_i}\right) = 0 \tag{6}$$

As shown in Equation (6), the LS_m estimator for the parameter of ETE distribution, does not exist in a closed form. Numerical estimation method using R software is applied in obtaining φ.

II. Maximum Likelihood Estimation (MLE) Method

To estimate the parameter of the ETE distribution, using the maximum likelihood method, let $c_1, c_2, ..., c_n$ be independent identically distributed of size (n) with probability function ETE distribution, then the likelihood function $L(\phi)$ for ETE distribution is given as

$$L(\phi) = \prod_{i=1}^{n} f(c) = \prod_{i=1}^{n} \varphi^2 c_i e^{-\varphi c_i} = \varphi^{2n} e^{-\varphi \sum_{i=1}^{n} c_i} \sum_{i=1}^{n} c_i \tag{7}$$

$$\log\left(L(\phi)\right) = \log\left(\varphi^{2n} e^{-\varphi \sum_{i=1}^{n} c} \sum_{i=1}^{n} c\right) = 2n\log(\varphi) - \varphi\sum_{i=1}^{n} c + \sum_{i=1}^{n}\log(c) \tag{8}$$

The likelihood in Equation (8) is partially differentiated w.r.t φ and equating to zero gives

$$\frac{\partial(\log(L(\phi)))}{\partial\varphi} = \frac{\partial(2n\log(\varphi))}{\partial\varphi} - \frac{\partial\left(\varphi\sum_{i=1}^{n} c\right)}{\partial\varphi} + \frac{\partial\left(\sum_{i=1}^{n}\log(c)\right)}{\partial\varphi} = 0 \tag{9}$$

$$\frac{(2n)}{\varphi} - \sum_{i=1}^{n} c = 0; \Rightarrow \varphi = 2n\left(\sum_{i=1}^{n} c\right)^{-1} \tag{10}$$

Therefore, the MLE for the parameter of ETE is given as;

$$\varphi = 2n\left(\sum_{i=1}^{n} c\right)^{-1} \tag{11}$$

III. The Minimum Distance Estimator (Anderson-Darling Method)

This subsection focuses on the Anderson-Darling (AD) type estimators, a class of minimum distance estimators used to measure the discrepancy between the empirical distribution and the true (theoretical) distribution [1]. The AD test is a modification of the Kolmogorov-Smirnov test, and it assigns greater weight to the tails of the distribution, making it particularly useful for assessing the fit of a distribution to data with significant deviations at the extremes. The AD type estimators come in different variants, namely ADR (Right-tail AD), ADL (Left-tail AD), AD2R (Second-order Right-tail AD), and AD2L (Second order Left-tail AD).

- The ADR estimator gives more weight to the right tail of the distribution.

- The ADL estimator gives greater emphasis to the left tail.
- The AD2R estimator assigns even larger weight to the right tail.
- The AD2L estimator places more emphasis on the left tail.

Derivation Notations: In the following, the set up framework for estimating the parameter φ using AD methods. Therefore, the following notations are introduced:

- $F_j = F(c_j)$: The CDF of the ETE distribution evaluated at the point c_j.
- $\hat{F}_j = 1 - F(c_j)$: The complement of the CDF representing the probability that the random variable is greater than c_j or the survival function evaluated at c_j.

For ADR:

$$ADR = \int_0^\infty \frac{\left[F_n(c) - F(c)\right]^2}{1 - F(c)}$$
$$= \frac{n}{2} - 2\sum_{j=1}^{n} F_j - \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left(\bar{F}_{n+1-j}\right)$$
$$= \frac{n}{2} - 2\sum_{j=1}^{n}\left[e^{-\varphi c_j}\left(e^{-\varphi c_j} - 1 - \varphi c_j\right)\right]$$
$$-\frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left(1 - e^{-\varphi c_{n+1-j}}\left(e^{-\varphi c_{n+1-j}} - 1 - \varphi c_{n+1-j}\right)\right) \tag{12}$$

Minimizing the function (12) with respect to φ and equating to zero gives the ADR estimator $\hat{\varphi}$;

$$\frac{\partial}{\partial \varphi} = \left(4\sum_{j=1}^{n} c_j e^{-2\varphi c_j} - 2\sum_{j=1}^{n} c_j \varphi e^{-2\varphi c_j}\right) + \frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)c_{n+1-j}(c_{n+1-j}\varphi e^{\varphi c_{n+1-j}} - 2)}{e^{2\varphi c_{n+1-j}} + (\varphi c_{n+1-j} + 1)e^{\varphi c_{n+1-j}} - 1} = 0 \tag{13}$$

For ADL:

$$ADL = \int_0^\infty \frac{\left[F_n(c) - F(c)\right]^2}{F(c)} dc = -\frac{3n}{2} - 2\sum_{j=1}^{n} F_j - \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log(F_j)$$
$$= -\frac{3n}{2} - 2\sum_{j=1}^{n}\left[e^{-\varphi c_j}\left(e^{-\varphi c_j} - 1 - \varphi c_j\right)\right] - \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[e^{-\varphi c_j}\left(e^{-\varphi c_j} - 1 - \varphi c_j\right)\right] \tag{14}$$

Minimizing the function (14) with respect to φ and equating to zero gives the ADL estimator $\hat{\varphi}$;

$$\frac{\partial}{\partial \varphi} = \left(4\sum_{j=1}^{n} c_j e^{-2\varphi c_j} - 2\sum_{j=1}^{n} c_j^2 \varphi e^{-\varphi c_j}\right) + \frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)c_j(c_j\varphi e^{\varphi c_j} - 2)}{(\varphi c_j + 1)e^{\varphi c_j} - 1} = 0 \tag{15}$$

For AD2R:

$$AD2R = \int_0^\infty \frac{[F_n(c) - F(c)]^2}{[1 - F(c)]^2} dc = 2\sum_{j=1}^{n}\log[\bar{F}_j] + \frac{1}{n}\sum_{i=1}^{n}\frac{(2j-1)}{\bar{F}_{n+1-j}}$$

$$= 2\sum_{j=1}^{n}\log\left(1 - e^{-\varphi c_j}\left(e^{-\varphi c_j} - 1 - \varphi c_j\right)\right) + \frac{1}{n}\sum_{i=1}^{n}\left|\frac{(2j-1)}{1 - e^{-\varphi c_{n+1-j}}\left(e^{-\varphi c_{n+1-j}} - 1 - \varphi c_{n+1-j}\right)}\right| \tag{16}$$

Minimizing the function (16) with respect to φ and equating to zero gives the AD2R estimator $\hat{\varphi}$;

$$\frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)c_{n+1-j}(c_{n+1-j}\varphi e^{\varphi c_{n+1-j}} - 2)e^{2\varphi c_{n+1-j}}}{\left(e^{2\varphi c_{n+1-j}} + (\varphi c_{n+1-j} + 1)e^{\varphi c_{n+1-j}} - 1\right)^2} - 2\sum_{j=1}^{n}\frac{c_j(c_j\varphi e^{\varphi c_j} - 2)}{e^{2\varphi c_j} + (\varphi c_j + 1)e^{\varphi c_j} - 1} = 0 \tag{17}$$

For AD2L:

$$AD2L = \int_0^\infty \frac{\left[F_n(c) - F(c)\right]^2}{\left[F(c)\right]^2} dc = 2\sum_{j=1}^{n}\log\left(F_j\right) + \frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)}{F_j}$$

$$= 2\sum_{j=1}^{n}\log\left(e^{-\varphi c_j}\left(e^{-\varphi c_j}-1-\varphi c_j\right)\right)+\frac{1}{n}\sum_{j=1}^{n}\left[\frac{(2j-1)}{e^{-\varphi c_j}\left(e^{-\varphi c_j}-1-\varphi c_j\right)}\right] \quad (18)$$

Minimizing the function (18) with respect to φ and equating to zero gives the AD2L estimator $\hat{\varphi}$;

$$\frac{1}{n}\sum_{j=1}^{n}\left[\frac{(2j-1)c_j(\varphi c_j e^{\varphi c_j}-2)e^{2\varphi c_j}}{[(c_j+1)e^{\varphi c_j}-1]^2}\right]-2\sum_{j=1}^{n}\frac{c_j(\varphi c_j e^{\varphi c_j}-2)}{(c_j+1)e^{\varphi c_j}-1}=0\,. \quad (19)$$

The parameter φ for the respective variants of the AD estimators is then obtained numerically by solving the homogeneous equations in (13), (15), (17), and (19) to obtain $\hat{\varphi}$.

To estimate distribution parameters using a minimum distance estimator in R software, the *mge* (method of generalized estimation) method must be specified in the fitdist function call. Additionally, the appropriate goodness-of-fit (GOF) distance argument should be included [7].

IV. ANALYSIS

The simulation and real data fitting for the various estimation methods are presented in subsections I and IV, respectively. Furthermore, the results from both the simulation and real-world data fittings are compared with plots illustrating the real data alongside the fitted data for the different estimation methods presented.

I. Simulation Study (Monte Carlo)

In this section, an extensive Monte Carlo simulation study was performed to compare the performances of the ML, LS, ADR, ADL, AD2R, and AD2L estimators of the parameter φ of the ETE distribution. All simulations were conducted in R environment for 1000 Monte-Carlo runs. In this study, finite sample sizes of n = 20, 100, 250, 700, 1000, 1500, 2500, 3000 are considered. Throughout the simulations, the parameter φ is assumed to be known with values 0.01, 0.04 and 0.06. The performances of the different estimators are assessed and compared based on the Absolute Bias (AB_s) and Root Mean Squares Error (RMSE) criteria. The expressions for AB_s and RMSE are given in Equations (20) and (21), respectively;

$$AB_s=\frac{\sum_{j=1}^{1000}\left|\hat{\varphi}_j-\varphi\right|}{1000} \quad (20)$$

$$RMSE=\sqrt{\frac{\sum_{j=1}^{1000}\left(\hat{\varphi}_j-\varphi\right)^2}{1000}} \quad (21)$$

where, the j^{th} estimate of φ is the simulated estimate of the parameter of interest while φ is the true parameter value. Also, "Minimum" is defined as the smallest AB_s and RMSE for the estimated parameter φ.

Table 1: *Comparison of Anderson Darling estimates (ADR, AD2R, ADL and AD2L)*

		ADR		AD2		ADL		AD2L	
Sizes (N)	True P	ABs	RMSE	ABs	RMSE	ABs	RMSE	ABs	RMSE
20	0.01	0.033	0.033	**0.026**	**0.026**	0.038	0.038	0.039	0.039
	0.04	0.134	0.134	**0.105**	**0.105**	0.151	0.151	6.452	6.452
	0.06	0.200	0.200	**0.158**	**0.158**	0.226	0.226	17.978	17.978
100	0.01	0.028	0.028	**0.018**	**0.018**	0.030	0.030	0.028	0.028
	0.04	0.114	0.114	**0.071**	**0.071**	0.118	0.118	0.113	0.113
	0.06	0.171	0.171	**0.106**	**0.106**	0.178	0.178	10.563	10.563
250	0.01	0.030	0.030	**0.021**	**0.021**	0.030	0.030	0.028	0.028
	0.04	0.120	0.120	**0.086**	**0.086**	0.120	0.120	0.113	0.113
	0.06	0.180	0.180	**0.128**	**0.128**	0.180	0.180	0.169	0.169
700	0.01	0.028	0.028	**0.022**	**0.022**	0.030	0.030	0.029	0.029
	0.04	0.114	0.114	**0.088**	**0.088**	0.119	0.119	0.114	0.114
	0.06	0.171	0.171	**0.132**	**0.132**	0.179	0.179	0.171	0.171
1000	0.01	0.028	0.028	**0.022**	**0.022**	0.030	0.030	0.028	0.028
	0.04	0.113	0.113	**0.087**	**0.087**	0.119	0.119	0.113	0.113
	0.06	0.169	0.169	**0.131**	**0.131**	0.178	0.178	0.169	0.169
1500	0.01	0.028	0.028	**0.022**	**0.022**	0.030	0.030	0.028	0.028
	0.04	0.112	0.112	**0.089**	**0.089**	0.118	0.118	0.112	0.112
	0.06	0.168	0.168	**0.133**	**0.133**	0.177	0.177	0.167	0.167
2500	0.01	0.028	0.028	**0.023**	**0.023**	0.030	0.030	0.028	0.028
	0.04	0.112	0.112	**0.091**	**0.091**	0.118	0.118	0.113	0.113
	0.06	0.168	0.168	**0.137**	**0.137**	0.178	0.178	10.500	10.500
3000	0.01	0.028	0.028	**0.023**	**0.023**	0.030	0.030	0.028	0.028
	0.04	0.113	0.113	**0.093**	**0.093**	0.119	0.119	0.113	0.113
	0.06	0.169	0.169	**0.139**	**0.139**	0.179	0.179	0.170	0.170

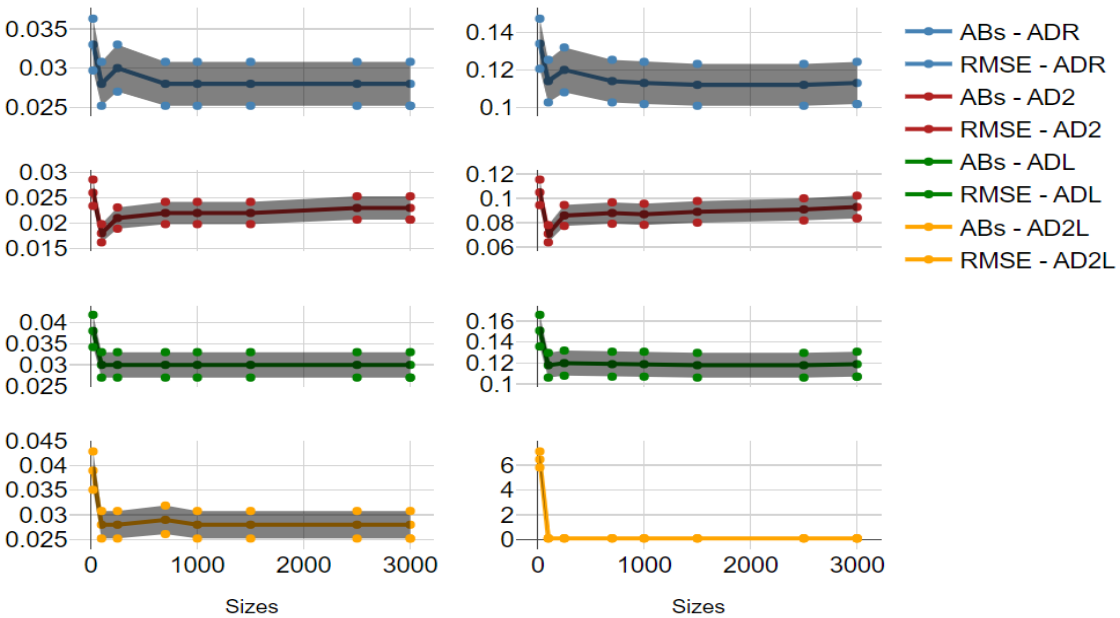

Figure 2a: *ABs and RMSE visualizations for ADR, AD2, ADL and AD2L methods*

Table 2: *Comparison of LS and ML Estimates*

		LS		MLE		Minimum	
Sizes (N)	**True P**	**ABs**	**RMSE**	**ABs**	**RMSE**	**ABs**	**RMSE**
20	0.01	**0.0099**	**0.0099**	0.0274	0.0285	0.0099	0.0099
	0.04	**0.0397**	**0.0397**	0.1096	0.1140	0.0397	0.0397
	0.06	**0.0595**	**0.0595**	0.1644	0.1709	0.0595	0.0595
100	0.01	**0.0087**	**0.0087**	0.0263	0.0265	0.0087	0.0087
	0.04	**0.0349**	**0.0349**	0.1054	0.1061	0.0349	0.0349
	0.06	**0.0525**	**0.0525**	0.1581	0.1592	0.0525	0.0525
250	0.01	**0.0080**	**0.0080**	0.0263	0.0263	0.0080	0.0080
	0.04	**0.0319**	**0.0319**	0.1050	0.1054	0.0319	0.0319
	0.06	**0.0479**	**0.0479**	0.1576	0.1580	0.0479	0.0479
700	0.01	**0.0074**	**0.0074**	0.0260	0.0261	0.0074	0.0074
	0.04	**0.0295**	**0.0295**	0.1041	0.1042	0.0295	0.0295
	0.06	**0.0443**	**0.0443**	0.1562	0.1563	0.0443	0.0443
1000	0.01	**0.0070**	**0.0070**	0.0260	0.0260	0.0070	0.0070
	0.04	**0.0282**	**0.0282**	0.1039	0.1039	0.0282	0.0282
	0.06	**0.0422**	**0.0422**	0.1558	0.1559	0.0422	0.0422
1500	0.01	**0.0067**	**0.0067**	0.0259	0.0260	0.0067	0.0067
	0.04	**0.0267**	**0.0267**	0.1038	0.1038	0.0267	0.0267
	0.06	**0.0400**	**0.0400**	0.1557	0.1558	0.0400	0.0400
2500	0.01	**0.0066**	**0.0066**	0.0260	0.0260	0.0066	0.0066
	0.04	**0.0265**	**0.0265**	0.1038	0.1038	0.0265	0.0265
	0.06	**0.0398**	**0.0398**	0.1557	0.1558	0.0398	0.0398
3000	0.01	**0.0066**	**0.0066**	0.0260	0.0260	0.0066	0.0066
	0.04	**0.0264**	**0.0264**	0.1039	0.1039	0.0264	0.0264
	0.06	**0.0396**	**0.0396**	0.1559	0.1559	0.0396	0.0396

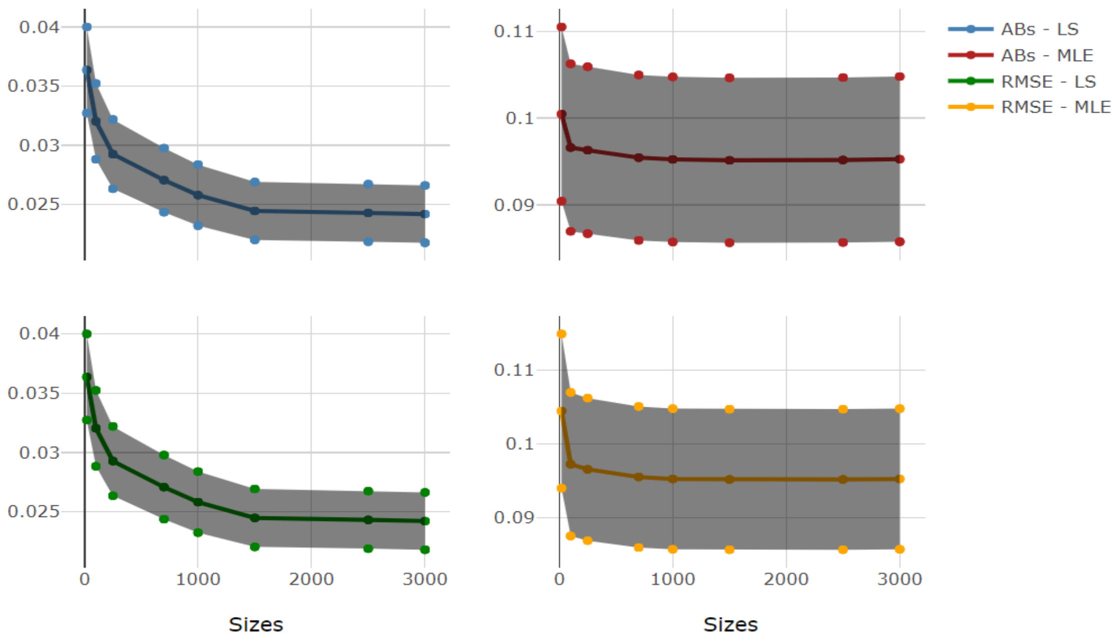

Figure 2b: *ABs and RMSE visualizations for LS and MLE methods*

II. Findings from AB_s and RMSE Values of the Estimation Methods

The results on the absolute biases of the parameter estimates obtained from the different estimation techniques are presented in Tables 1 and 2 and visualized in Figures 2*a* and 2*b*. This section considers the results and its implication with regards to the Anderson Darling methods (ADR, AD2R, ADL, and AD2L). Also, from the analysis results, for the methods of estimation, there was a general decrease in the magnitude of the AB_s and RMSE as the sample size increased from $20 \leq n \leq 3000$. The estimates obtained using the ADR, AD2R, ADL and AD2L revealed that the root AB_s and RMSE for the Anderson Darling methods at sample size $n = 20$ and assumed true parameter value of $\varphi = 0.01$ were as follows:

- ADR: $AB_s = 0.033$ & RMSE = 0.033.
- AD2R: $AB_s = 0.026$ & RMSE = 0.026.
- ADL: $AB_s = 0.038$ & RMSE = 0.038.
- AD2L: $AB_s = 0.039$ & RMSE = 0.039.

A further scrutiny of the results revealed that the Anderson Darling AD2R method of estimation consistently yielded the smallest AB_s and RMSE while AD2L yielded the largest values for the AB_s and RMSE. Thus, this implies that the for the Anderson Darling methods of estimation, the AD2R provides a better estimate for the ETE distribution.

Regarding the LS and ML estimation methods as shown in Table 2, it can be observed from the results presented that the ML and LS methods at sample size $n = 20$ and true $\varphi = 0.01$ were as follows:

- LS method: AB = 0.0099 & RMSE = 0.0099.
- ML Method: AB = 0.0274 & RMSE = 0.0285.

As further observed, the trajectory and behavior of the AB_s and RMSE as the sample size increases, showed a downward trend in the value of the AB_s and RMSE. Notably, the LS method consistently yielded the smallest values of the AB_s and RMSE, standing out as the best performing estimator. This trend reinforces the conclusion that the LS method provides the best estimates for the parameter of the ETE distribution.

III. Comparing the Minimum in Table 1 and 2

In this subsection, a comparison of the Anderson Darling methods, the ML and LS methods, are presented. Based on the simulation results, the LS estimator consistently demonstrated superior performance across all assumed true parameter values and sample sizes. These findings indicate that among the six estimators evaluated through the simulation study, the LS method emerges as the most effective estimation technique for the ETE distribution.

IV. Real World Application: Field Intensity Data

This section presents a numerical analysis using a real-world dataset to empirically evaluate the performance of different estimation methods. The objective is to determine which method provides the best fit to the data. The dataset consists of 80 observations sourced from [11], representing field intensity measured in microvolts and recorded at 5-second intervals. As illustrated in Figure 3, the data plot exhibits a right-skewed pattern. This skewness underscores the potential challenges for standard estimation techniques and emphasizes the need to select a method capable of providing robust and reliable modeling results. The data is presented as follows;

0.20, 0.71, 0.06, 0.05, 0.76, 0.32, 0.96, 0.63, 0.09, 0.18, 0.25, 0.45, 0.26, 0.10,0.95, 0.01, 0.50, 1.20, 1.99, 0.32, 0.51,0.01, 0.16, 0.56, 3.16, 1.27, 2.24, 1.00, 0.81,1.29, 0.28, 0.21, 0.35, 0.20, 0.39, 0.89,0.24, 0.08, 0.98,1.01, 0.49, 0.90,1.90, 1.42, 1.56,1.32,1.20,1.59, 2.24, 0.80, 0.56,0.18, 0.02, 0.28,0.81, 0.18,1.13, 0.64,1.95, 0.48, 0.55, 0.44, 0.28, 0.07, 0.71, 0.48, 0.06,0.79,1.01, 0.51, 0.70, 0.14, 0.16, 0.01,0.06, 0.03, 0.01.

Table 3: *Descriptive Statistics of Field Intensity*

Statistic	Values
Mean	0.670
Sd	0.630
Median	0.500
Mad	0.500
Min	0.010
Max	3.160
Range	3.150
Skew	1.440
Kurtosis	2.250
Se	0.070

Table 4 summarizes the estimates obtained from fitting the data to the distribution using the six different estimation methods. The results include the standard error (SE), the 95% confidence interval (CI), confidence interval length (CIL), *t*-value, *p*-value, and the rankings based on the CIL and SE.

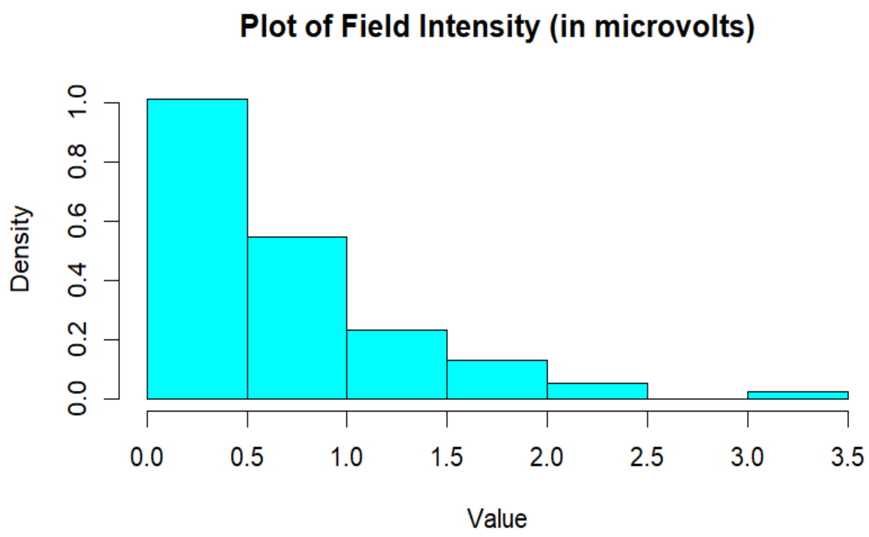

Figure 3: *Frequency plot of the Field Intensity Data*

Table 4: *Parameter estimates for the real-life data*

GOF	Estimate	SE	95% CI Lower	95% CI Upper	CIL	*t*-value	p-value	Rank
LS	1.1982	0.0246	1.1492	1.2472	0.0980	48.7167	0.0000	1
ADR	3.2878	0.4301	2.5714	4.2775	1.7061	7.6447	0.0000	2
AD2R	3.4377	0.4475	2.6775	4.4476	1.7701	7.6819	0.0000	3
MLE	1.0391	0.7718	-8.7679	10.8460	19.6139	1.3463	0.4067	4
ADL	72.4150	10.6967	55.1709	87.3595	32.1886	6.7698	0.0000	5
AD2L	92.1208	75.9367	9.3823	295.2279	285.8457	1.2131	0.2251	6

The rankings reveal that the least effective estimation method was AD2L, which ranked 6th, followed by ADL (rank 5), MLE (rank 4), AD2R (rank 3), and ADR (rank 2). The results indicate that the LS method of estimation performed best, producing the lowest standard error (0.0246) and the narrowest confidence interval length (0.0980). This conclusion is further supported by Figure 4 (a–f), which illustrates the fits generated by the various estimation methods in comparison to the plot of the real data.

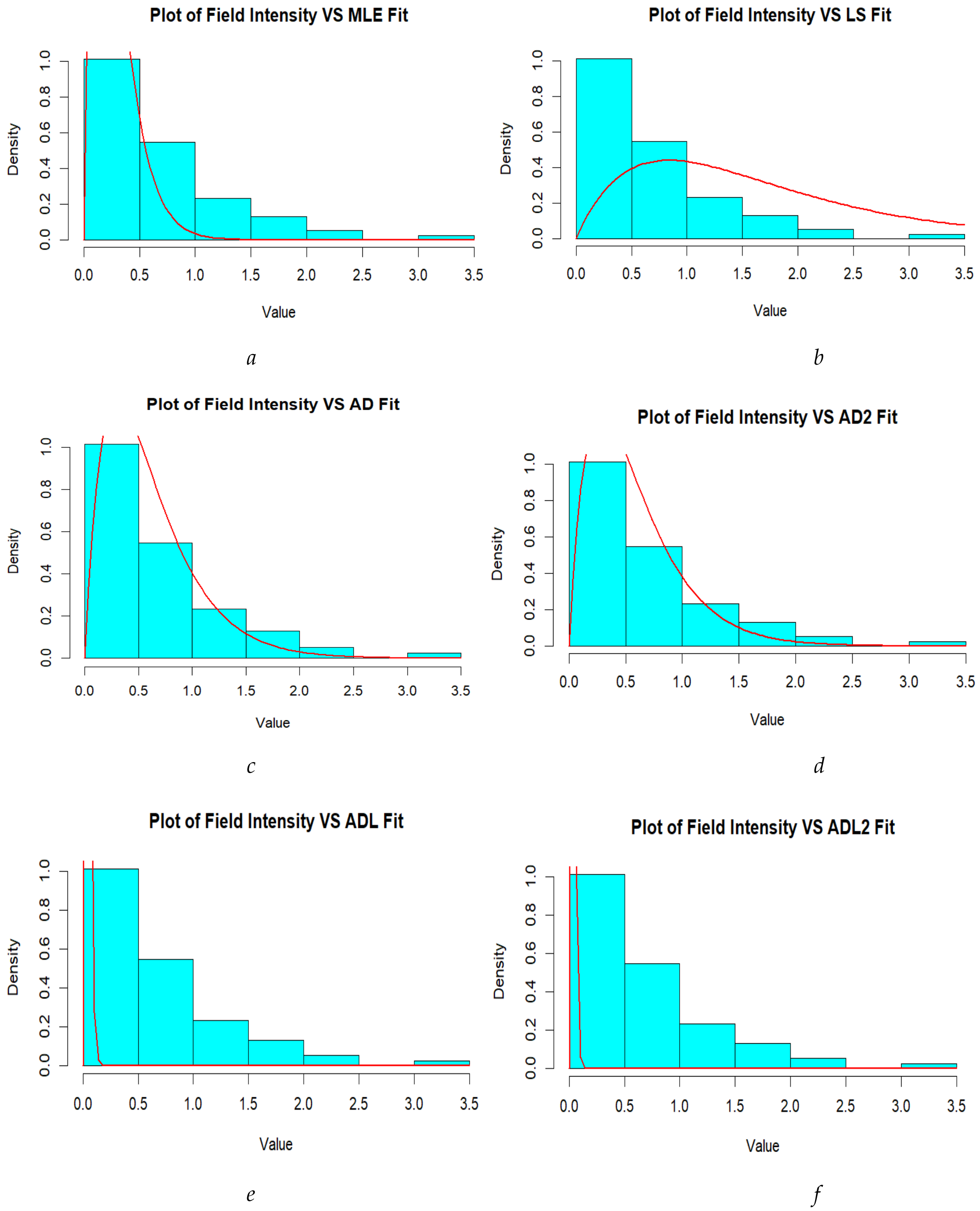

Figure 4: *Plots of the Various Fits Verse Real Data*

V. CONCLUSION

In this study, the performance of parameter estimates for the Entropy Transformed Exponential (ETE) distribution was evaluated using six estimation techniques: Anderson-Darling (AD) variants (ADR, AD2R, ADL, and AD2L), Maximum Likelihood (ML), and Least Squares (LS) estimation methods. A Monte Carlo simulation study was conducted to compare the average absolute biases (ABs), mean square errors (MSEs), and root mean square errors (RMSEs) for estimates obtained using each technique. The simulation results revealed that among the six methods studied, the LS estimation method consistently produced estimates with the most favorable performance metrics. Furthermore, real-world data analysis reinforced this finding, as the LS method demonstrated the narrowest confidence interval length and the smallest standard error of estimates. Based on these findings, this study concludes that the LS method is the most reliable and accurate estimation technique for the ETE distribution.

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