PERFORMANCE MEASURES AND PARAMETRIC ESTIMATION OF A PARALLEL-SERIES SYSTEM MODEL WITH WEIBULL FAILURE AND REPPAIR TIME DISTRIBUTIONS

Shallu Sharma[1], Pawan Kumar[*2], Poonam Sharma[2]

•

[1,*2,2]Department of Statistics, University of Jammu, J&K.
shallusharma28udh@gmail.com[1], pkk_skumar@yahoo.co.in[*2], sharmapoonam1038@gmail.com[2]

Abstract

The present paper aims at the study of three non-identical unit parallel-series system model with Weibull failure and repair time distributions. The main focus is on the analysis of reliability characteristics and also estimation of parameters in Classical and Bayesian paradigms. The units in the system are named as A, B and C. In order that system works successfully, unit A and anyone of the units B or C should work.Upon failure a unit will immediately be taken up to repair facility for necessary remedial actions. A single repair facility is available to deal with any kind of failure/fault detected with any unit. First come first served service discipline is followed. The failure and repair time distributions for all the units are taken Weibull with varying parameters. Regenerative point technique is used to study various measure of effectiveness. MLE and Bayes estimates of various failure and repair parameters involved in the study have been obtained. A simulation study has also been undertaken to exhibit the behaviour of obtained characteristics in Classical and Bayesian setup and a comparison is made thereupon. Various conclusions have been drawn from the tables and graphs plotted for numerous performance measures for varying values of repair and failure parameters. Having obtained all the reliability characteristics, the profit incurred by the system has been obtained and studied graphically too.

Keywords: Reliability, Availability, Mean Time to System Failure, Regenerative Point Technique, Bayesian Estimation, Maximum Likelihood Estimation.

1. Introduction

In today's context, the configuration and design of industrial systems—such as transportation system, aircraft, telecommunication, automobiles, textile manufacturing systems, computer systems, and satellite systems—are becoming increasingly complex. The reliability, availability, and efficiency of any system largely depends on its design. As a result, system designers are constantly striving to create high-reliability systems through various techniques. Reliability denotes the capacity of a system, process, product, or service to consistently fulfill its intended purpose without failure over time or under particular conditions. It embodies dependability and trust, guaranteeing that results are predictable and align with expectations. In fields like engineering, manufacturing, and service industries, reliability is crucial for ensuring safety, customer satisfaction, and overall effectiveness. In the recent past, extensive research has been carried out by several authors on reliability characteristics and optimization of cost-benefit analysis for varieties of stochastic models under different sets of assumptions and different failure and repair time distributions. Sharma and Kumar[1] analyzed reliability measures of two identical unit system with one switching device and imperfect coverage. Sharma and Gupta[9] and Kumar and Saini[6] respectively undertaken analytical study on cost-benefit analysis of three-unit system model with correlated repair and failure time distributions and a single unit system model having Weibull repair and failure laws along with preventive maintenance and obtained various performance measures. Malik[7] also worked on cost benefit evaluation of a repairable system with alternative repair and Weibull distribution. Guilani et al.[3] discussed a simulation-based optimization approach for the redundancy allocation in a system with increasing failure rates of components based on Weibull distribution. Classical and Bayesian analysis of reliability characteristics was done by Kishan and Jain[4] for two-unit parallel system with Weibull failure and repair laws for each unit with common shape parameters but different scale parameters and derived important measures of system effectiveness using Regenerative point techniques. Devi et al.[2] and Saini at el. [8] demonstrated the performance of a non-identical unit system with priority and Weibull repair and failure laws by obtaining multiple measures of system effectiveness using regenerative point technique and semi-markov process. They also worked on stochastic modeling and profit evaluation of a redundant system with priority subject to Weibull densities for failure and repair. Casteren[10] made use of Weibull Markov model for the assessment of interruption costs in electric power systems. Kumar at el.[5] analysed the performance of computer systems with Weibull distribution subject to software upgrade and load recovery. Based on the literature reviewed above, it is evident that the research to a great extent has been conducted so far and still in progress on cost benefit analysis for stochastic system models under different configuration and different sets of assumptions. In view of above, an attempt has been made to analyze the performance of three unit redundant system model with Weibull repair and failure time distributions. Keeping in view the above ideas, this paper deals with the estimation of parameters and deriving important performance measures of three non-identical unit system model. Initially, all units are in operative state. Single repairman is always available to repair the failed units with FCFS service discipline. Repair and failure times for all the units are assumed to follow Weibull distributions. Regenerative point technique is used to analyze system model and derive various performance measures such as mean time to system failure, reliability, availability, and expected number of repairs. Furthermore, some characteristics of interest of the considered system model are examined and compared in both the Bayesian and Classical sets up using a simulation study and significant inferences are presented in the form of tables and graphs.

2. System Description and Assumptions

• The system consists of three non-identical units A,B and C. For successful operation of the system unit A and atleast one of the units B and C should function.
• The system failure occurs when either unit A or both the units B and C fail.
• The failure and repair times of units are assumed to have Weibull distributions with different scale parameters but common shape parameter.
• Only a single repair facility is available to repair failed units.
• Service discipline is FCFS.
• A repaired units work as good as new.

3. Notations and Symbols

$f_1(.)/f_2(.)/f_3(.)$: Failure rate function for Unit A/B/C.
$h_1(.)/h_2(.)/h_3(.)$: Repair rate function for unit A/B/C
$\alpha_i{}'s \& \beta_i{}'s, 1 \leq i \leq 3$: Scale parameter for failure and repair time distributions for Units A/B/C respectively.
γ: Shape parameter for failure and repair time distributions of units
$A_o/A_g/A_r/A_R/A_w$:Unit A is operative/ good/ under repair/ repair continued from earlier

state and waiting for repair respectively. Similar notations hold for units B and C.

3.1. SYMBOLS FOR THE STATES OF THE SYSTEM

UpStates	*DownStates*
$S_0 = [A_o, B_o, C_o]$	$S_3 = [A_r, B_g, C_g]$
$S_1 = [A_o, B_r, C_o]$	$S_4 = [A_g, B_r, C_w]$
$S_2 = [A_o, B_o, C_r]$	$S_5 = [A_g, B_w, C_r]$
$S_6 = [A_w, B_g, C_r]$	$S_7 = [A_w, B_r, C_g]$

The transition diagram of the model is shown in Figure 1.

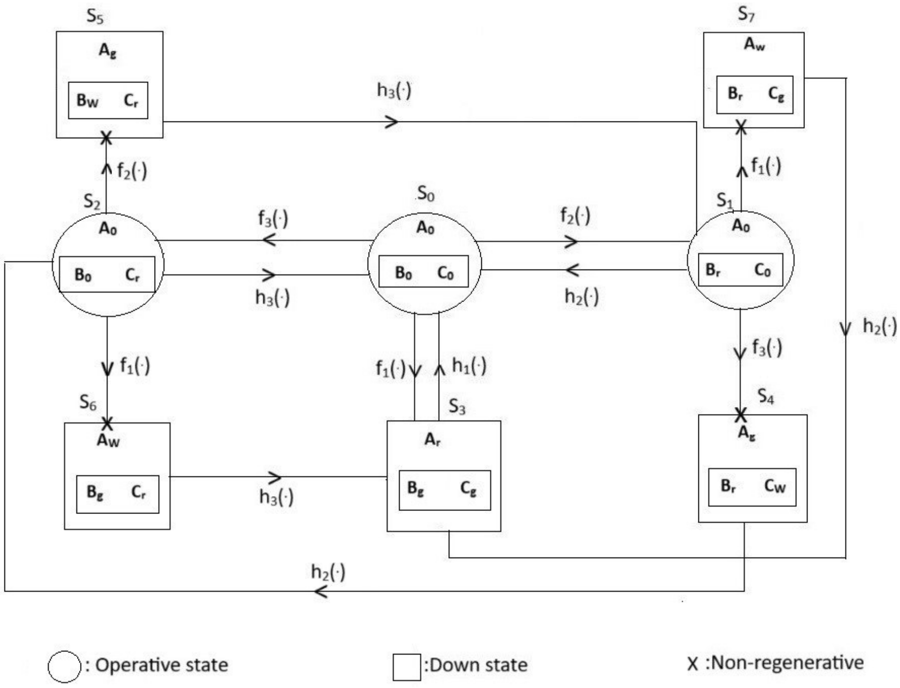


Figure 1: *Transition Diagram*

4. Transition Probabilities and Sojourn Times

The PDF of Weibull distribution with scale parameter α and shape parameter γ is given as:

$$f_T(t) = \begin{cases} \alpha\gamma t^{\gamma-1}\exp^{-}\alpha t^{\gamma}, & \text{if } \alpha,\gamma > 0 \;\; and \;\; t \geq 0 \\ 0 \qquad\qquad\qquad , & \text{if } otherwise \end{cases}$$

and CDF is:

$$F_T(t) = \begin{cases} 1 - \exp^{-}\alpha t^{\gamma}, & \text{if } \alpha,\gamma > 0 \;\; and \;\; t \geq 0 \\ 0 \qquad\qquad , & \text{if } otherwise \end{cases}$$

Also, the failure and repair rate functions respectively are:

$$f_i(t) = \alpha_i \gamma t^{\gamma-1}$$

and

$$h_i(t) = \beta_i \gamma t^{\gamma-1}; \;\; where, 1 \leq i \leq 3, \; \alpha,\gamma > 0 \;\; and \;\; t \geq 0$$

The reliability function is:

$$R_i(t) = \exp^{-} \alpha_i t^{\gamma}, 1 \leq i \leq 3,\ \alpha, \gamma > 0 \ \ and \ \ t \geq 0$$

The long-run or the steady state probabilities are obtained as under,
$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) = \int q_{ij}(t)dt,\quad p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t)$ and $p_{ij}^{(k,l)} = \lim_{t\to\infty} Q_{ij}^{k,l}(t)$.
In particular we have

$$p_{01} = \int \alpha_2 \gamma t^{\gamma-1} e^{-\alpha_2 t^{\gamma}} e^{-\alpha_3 t^{\gamma}} e^{-\alpha_1 t^{\gamma}} dt = \frac{\alpha_2}{(\alpha_2+\alpha_3+\alpha_1)}$$

Similarly,

$p_{02} = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}$ $\qquad p_{13}^{(7)} = \frac{\alpha_1}{\beta_2+\alpha_1+\alpha_3}$

$p_{03} = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}$ $\qquad p_{20} = \frac{\beta_3}{\beta_3+\alpha_1+\alpha_2}$

$p_{10} = \frac{\beta_2}{\beta_2+\alpha_1+\alpha_3}$ $\qquad p_{21}^{(5)} = \frac{\alpha_2}{\beta_3+\alpha_1+\alpha_2}$

$p_{12}^{(4)} = \frac{\alpha_3}{\beta_2+\alpha_1+\alpha_3}$ $\qquad p_{23}^{(6)} = \frac{\alpha_1}{\beta_3+\alpha_1+\alpha_2}$

Following relationships hold

$$p_{01} + p_{02} + p_{03} = 1$$
$$p_{10} + p_{12}^{(4)} + p_{13}^{(7)} = 1$$
$$p_{20} + p_{21}^{(5)} + p_{23}^{(6)} = 1$$
$$p_{30} = p_{42} = p_{51} = p_{63} = p_{73} = 1$$

4.1. Mean Sojourn times

The expected duration of time for which system remains in a particular state S_i before making transition to another state is known as the Mean Sojourn time ψ_i i.e. as long as the system is in state S_i, there is no transition from one state to another. With this information, we calculate ψ_i for state S_i. If T_i denotes sojourn time in state S_i, the mean sojourn time ψ_i can be calculated as follows.

$$\psi_i = E[T_i] = \int P(T_i > t)dt$$

Hence, using the above formula following values for mean sojourn time are obtained:

$\psi_0 = \frac{\sqrt{(\frac{1}{\gamma}+1)}}{(\alpha_1+\alpha_2+\alpha_3)^{\frac{1}{\gamma}}}$ $\qquad \psi_1 = \frac{\sqrt{(\frac{1}{\gamma}+1)}}{(\alpha_1+\alpha_3+\beta_2)^{\frac{1}{\gamma}}}$

$\psi_2 = \frac{\sqrt{(\frac{1}{\gamma}+1)}}{(\alpha_1+\alpha_2+\beta_3)^{\frac{1}{\gamma}}}$ $\qquad \psi_3 = \frac{\sqrt{(\frac{1}{\gamma}+1)}}{\beta_1^{\frac{1}{\gamma}}}$

$\psi_4 = \psi_7 = \frac{\sqrt{(\frac{1}{\gamma}+1)}}{\beta_2^{\frac{1}{\gamma}}}$ $\qquad \psi_5 = \psi_6 = \frac{\sqrt{(\frac{1}{\gamma}+1)}}{\beta_3^{\frac{1}{\gamma}}}$

5. Analysis of Reliability and MTSF

If the random variable T_i represents the system's life time when it starts from state $S_i \in E_i$, then the system's reliability is:

$$R_i(t) = P[T_i > t]$$

Here, the failed states are treated as absorbing states.
The state transition diagram and probabilistic arguments can be used to establish the recursive

relations among $R_i(t)$. After identifying the set of equations for $R_i(t)$ and applying the Laplace transform, we obtain

$$R_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{1}$$

[1] where,

$$N_1(s) = Z_0^* + Z_1^* q_{01}^* + Z_2^* q_{02}^*$$

and

$$D_1(s) = 1 - q_{01}^* q_{10}^* - q_{02}^* q_{20}^*$$

We obtain the system's reliability by taking the inverse Laplace transform of (1).
To obtain MTSF, we use well-known formula

$$E(T_0) = \int R_0(t)dt = \lim_{s \to 0} R_0^*(s) = \frac{N_1(0)}{D_1(0)} \tag{2}$$

where,

$$N_1(0) = \psi_0 + \psi_1 p_{01} + \psi_2 p_{02}$$

and

$$D_1(0) = 1 - p_{01}p_{10} - p_{02}p_{20}$$

Here we have used the results, $q_{ij}^*(0) = p_{ij}$ and $Z_i^*(0) = \psi_i$

6. Availability Analysis

Availability is the likelihood that a system, starting from $S_i \in E_i$, accomplishes its specified purpose at time 't'. The system's availability at a given time is referred to as point wise availability. It is a system performance metric that indicates if a system is potentially functional and capable of delivering the desired service at a specific moment. Establishing recursive relations among various pointwise availabilities and the applications of stochastic reasoning and Laplace transforms gives

$$A_0^*(s) = \frac{N_2(s)}{D_2(s)}$$

where,

$$N_2(s) = Z_0^*(1 - q_{12}^{*(4)} q_{21}^{*(5)}) + Z_1^*(q_{01}^* + q_{02}^* q_{21}^{*(5)}) + Z_2^*(q_{02}^* + q_{01}^* q_{12}^{*(4)})$$

and,

$$\begin{aligned} D_2(s) = &(1 - q_{03}^* q_{30}^*)(1 - q_{12}^{*(4)} q_{21}^{*(5)}) - q_{01}^*[q_{10}^* + q_{12}^{*(4)} q_{20}^* + q_{30}^*(q_{13}^{*(7)} + q_{12}^{*(4)} q_{23}^{*(6)})] \\ &- q_{02}^*[q_{20}^* + q_{10}^* q_{21}^{*(5)} + q_{30}^*(q_{23}^{*(6)} + q_{21}^{*(5)} q_{13}^{*(7)})] \end{aligned} \tag{3}$$

[1]Limits of integration whenever they are 0 to ∞ are not mentioned.

The steady state availability that the system will be up in long run is given as under

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0}\ sA_0^*(s) = \frac{N_2(0)}{D_2(0)}$$

Now using the results

$$\lim_{s\to 0} Z_i^*(s) = \int Z_i(t)dt = \psi_i \ \ and \ \ \lim_{s\to 0} q_{ij}^*(s) = p_{ij}$$

we get

$$N_2(0) = \psi_0(1 - \overset{(4)}{p_{12}}\overset{(5)}{p_{21}}) + \psi_1(p_{01} + p_{02}\overset{(5)}{p_{21}}) + \psi_2(p_{02} + p_{01}\overset{(4)}{p_{12}})$$

$$D_2(0) = (1 - p_{03})(1 - \overset{(4)}{p_{12}}\overset{(5)}{p_{21}}) - p_{01}[p_{10} + \overset{(4)}{p_{12}}p_{20} + \overset{(7)}{p_{13}} + \overset{(4)}{p_{12}}\overset{(6)}{p_{23}}] - p_{02}[p_{20} + p_{10}\overset{(5)}{p_{21}} + \overset{(6)}{p_{23}} + \overset{(5)}{p_{21}}\overset{(7)}{p_{13}}]$$

For a given system, the steady-state probability of its long-term operation is given by

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0}\ sA_0^*(s)$$

$$\lim_{s\to 0}\frac{sN_2(s)}{D_2(s)} = \lim_{s\to 0} N_2(s) \lim_{s\to 0}\frac{s}{D_2(s)}$$

Since as s $\to 0$, $D_2(s)$ becomes zero. Therefore, applying L'Hospital's rule, A_0 becomes

$$A_0 = \frac{N_2(0)}{D_2'(0)} \tag{4}$$

where,

$$D_2'(0) = (1 - \overset{(4)}{p_{12}}\overset{(5)}{p_{21}})(\psi_0 + \psi_3) + \psi_1(p_{01} + p_{02}\overset{(5)}{p_{21}}) + \psi_2(p_{02} + p_{01}\overset{(4)}{p_{12}}) \tag{5}$$

Using $N_2(0)$ and $D_2'(0)$ in equation[4], the expression for A_0 can be determined.
The system's expected uptime for (0,t] is provided by

$$\mu_{up}(t) = \int_0^t A_0(u)du$$

So that,

$$\mu_{up}^*(s) = \frac{A_0^*(s)}{s}$$

7. Busy Period Analysis

$B_i(t)$ is defined as the probability that the system, which initially starts from the regenerative state Si$\in$ E, is under repair at time t due to failure of the unit. By applying basic probabilistic logics and the Laplace transformations and solving the resulting set of equations for $B_0^*(s)$, we get

$$B^*{}_0(s) = \frac{N_3(s)}{D_2(s)}$$

$$N_3(s) = (q_{01}^* + q_{02}^* q^{*(5)}_{21})(Z_1^* + q_{14}^* Z_4^* + q_{17}^* Z_7^*) + (q_{02}^* + q_{01}^* q^{*(4)}_{12})(Z_2^* + q_{23}^* Z_3^* + q_{26}^* Z_6^*) + Z_3^*[q_{01}^*(q^{*(7)}_{13} + q^{*(4)}_{12} q^{*(6)}_{23}) + q_{02}^*(q^{*(6)}_{23} + q^{*(7)}_{13} q^{*(5)}_{21}) + q_{03}^*(1 - q^{*(4)}_{12} q^{*(5)}_{21})]$$

and, D_2(s) is same as given in equation [3].
The probability that the repairman will be busy in the long run is as follows:

$$B_0 = \lim_{t\to\infty} B_0(t) = \lim_{s\to 0} \ sB_0^*(s) = \frac{N_3(0)}{D_2^{'}(0)}$$

where,

$$N_3(0) =(p_{01} + p_{02}p_{21}^{(5)})(\psi_1 - p_{14}\psi_4 + p_{17}\psi_7) + (p_{02} + p_{01}p_{12}^{(4)})(\psi_2 + p_{25}\psi_5 + p_{26}\psi_6) + \psi_3[p_{01}(p_{13}^{(7)} + p_{12}^{(4)}p_{23}^{(6)}) + p_{02}(p_{23}^{(6)} + p_{13}^{(7)}p_{21}^{(5)}) + p_{03}(1 - p_{12}^{(4)}p_{21}^{(5)})$$

and $D_2^{'}(0)$ is same as obtained in [5].
During (0,t], the repairman's expected busy time is given by

$$\mu_b(t) = \int_0^t B_0(u)du$$

So that,

$$\mu_b^*(s) = \frac{B_0^*(s)}{s}$$

8. Expected Number of Repairs

Define $V_i(t)$ the expected number of repairs of the failed units during the time range (0,t] when the system starts from the regenerative state S_i. Furthermore, the recurrence relations may simply be framed using the definition of $V_i(t)$. Then, by making use of Laplace-Stieltjes transformations the solution of resulting set of equations for $\widetilde{V}_0(s)$, is given as
$\widetilde{V}_0(s) = N_4(s)/D_3(s)$
where, and $D_3(s)$ is written on replacing $\overset{*}{q}_{ij}$ and $q_{ij}^{(k)*}$ by $\widetilde{Q}_{ij}$ and $\widetilde{Q}_{ij}^{(k)}$ respectively in equation[3].

$$V_0 = \lim_{t\to\infty} V_0(t) = \lim_{s\to 0} s\ \widetilde{V}_0(s) = \frac{N_4(0)}{D_2^{'}(0)}$$

where,

$$N_4(s) =(\widetilde{Q}_{01} + \widetilde{Q}_{02} + \widetilde{Q}_{21}^{(5)})(\widetilde{Q}_{12}^{(4)} + \widetilde{Q}_{13}^{(7)}) + (\widetilde{Q}_{01}\widetilde{Q}_{12}^{(4)} + \widetilde{Q}_{02})(\widetilde{Q}_{20} + \widetilde{Q}_{21}^{(5)} + \widetilde{Q}_{23}^{(6)}) + \widetilde{Q}_{30}[\widetilde{Q}_{01}(\widetilde{Q}_{13}^{(7)} + \widetilde{Q}_{12}^{(4)}\widetilde{Q}_{23}^{(6)}) + \widetilde{Q}_{02}(\widetilde{Q}_{23}^{(6)} + \widetilde{Q}_{13}^{(7)}\widetilde{Q}_{21}^{(5)}) + \widetilde{Q}_{03}(1 - \widetilde{Q}_{12}^{(4)}\widetilde{Q}_{21}^{(5)})]$$

In steady state the expected number of repairs per unit time is given by

$$V_0 = \lim_{t\to\infty} V_0(t) = \lim_{s\to 0} s\ \widetilde{V}_0(s) = \frac{N_4(0)}{D_2^{'}(0)}$$

where,

$$N_4(0) =(p_{01} + p_{02} + p_{21}^{(5)})(p_{12}^{(4)} + p_{13}^{(7)}) + (p_{01}p_{12}^{(4)} + p_{02})(p_{20} + p_{21}^{(5)} + p_{23}^{(6)}) + p_{30}[p_{01}(p_{13}^{(7)} + p_{12}^{(4)}p_{23}^{(6)}) + p_{02}(p_{23}^{(6)} + p_{13}^{(7)}p_{21}^{(5)}) + p_{03}(1 - p_{12}^{(4)}p_{21}^{(5)})]$$

9. Profit Function Analysis

With the help of the characteristics obtained two profit functions $P_1(t)$ and $P_2(t)$ can be obtained for the system model under study. The expected total profit incurred during (0,t) are:
$P_1(t)$ = Expected total revenue in(0,t] - Expected total expenditure in(0,t]

$= K_0\mu_{up}(t) - K_1\mu_b(t)$

and

$P_2(t) = K_0\mu_{up}(t) - K_2V_0(t)$

where,
K_0 = revenue per unit up time of the system.
K_1 and K_2 = Amounts paid to the repairmen per unit of time and per unit repair while repairing the failed units.
The expected total gain per unit of time in steady state is provided by:

$$P_1 = \lim_{t\to\infty} \frac{P_1(t)}{t} = \lim_{s\to 0} s^2 \overset{*}{P}(s)$$

Therefore, we have

$$P_1 = K_0A_0 - K_1B_0 \tag{6}$$

Similarly,

$$P_2 = \lim_{t\to\infty} \frac{P_2(t)}{t} = K_0A_0 - K_1V_0 \tag{7}$$

10. Estimation of the Parameters, MTSF, And Profit Function

10.1. Classical Estimation

10.1.1 ML Estimation

Let us take
$\underset{\sim}{X_1} = (x_{11}, x_{12}, ..., x_{1n_1})$, $\underset{\sim}{X_2} = (x_{21}, x_{22}, ...x_{2n_2})$, $\underset{\sim}{X_3} = (x_{31}, x_{32}, ..., x_{3n_3})$, $\underset{\sim}{X_4} = (x_{4_1}, x_{42}, ..., x_{4n_4})$, $\underset{\sim}{X_5} = (x_{51}, x_{52}, ..., x_{5n_5})$, $\underset{\sim}{X_6} = and\ (x_{61}, x_{62}, ..., x_{6n_6})$
Therefore, Likelihood function of combined sample is :

$L = (X_1, X_2, X_3, X_4, X_5, X_6 | \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

The pdf of Weibull distribution is $f(x) = \alpha\gamma t^{\gamma-1} \text{exp}^{-} \alpha t^{\gamma}$ $,\alpha,\gamma\ \ and\ \ t \geq 0$

$$L = \alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\beta_1^{n_4}\beta_2^{n_5}\beta_3^{n_6}\gamma^{n_1+n_2+n_3+n_4+n_5+n_6} Z_1Z_2Z_3Z_4Z_5Z_6 \,\text{exp}^{-(\alpha_1W_1+\alpha_2W_2+\alpha_3W_3+\beta_1W_4+\beta_2W_5+\beta_3W_6)} \tag{8}$$

Here, $W_i = \sum_{j=1}^{n_i} x_{ij}^{\gamma}$ and $Z_i = \prod_{j=1}^{n_i} x_{ij}^{(\gamma-1)}$; i = 1,2,3,4,5,6
On solving, we get

$$logL = n_1log\alpha_1 + n_2log\alpha_2 + n_3log\alpha_3 + n_4log\beta_1 + n_5log\beta_2 + n_6log\beta_3 + logZ_1 + logZ_2 + logZ_3 + logZ_4 + logZ_5 + logZ_6 - (\alpha_1W_1 + \alpha_2W_2 + \alpha_3W_3 + \beta_1W_4 + \beta_2W_5 + \beta_3W_6) + (n_1 + n_2 + n_3 + n_4 + n_5 + n_6)log\gamma \tag{9}$$

The, MLE $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3\ , \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)$ of the parameters
$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ are as under

$\hat{\alpha}_1 = \frac{n_1}{W_1}$, $\hat{\alpha}_2 = \frac{n_2}{W_2}$
$\hat{\alpha}_3 = \frac{n_3}{W_3}$, $\hat{\beta}_1 = \frac{n_4}{W_4}$
$\hat{\beta}_2 = \frac{n_5}{W_5}$, $\hat{\beta}_3 = \frac{n_6}{W_6}$

The asymptotic distribution of$(\hat{\alpha_1} - \alpha_1, \hat{\alpha_2} - \alpha_2, \hat{\alpha_3} - \alpha_3, \hat{\beta}_1 - \beta_1, \hat{\beta}_2 - \beta_2, \hat{\beta}_3 - \beta_3,) \sim N_6(0, I^{-1})$, where I is the Fisher Information matrix with diagonal elements as

$I_{11} = \frac{n_1}{\alpha_1^2}, \quad I_{22} = \frac{n_2}{\alpha_2^2}, \quad I_{33} = \frac{n_3}{\alpha_3^2}, \quad I_{44} = \frac{n_4}{\beta_1^2}, \quad I_{55} = \frac{n_5}{\beta_2^2}, \quad I_{66} = \frac{n_6}{\beta_3^2}$

and all non-diagonal elements are zero. Using invariance property of MLE's, we can extract The MLE $\hat{M}$ & $\hat{P}$ of MTSF and Profit function. Also, asymptotic distribution of $(\hat{M} - M) is N(0, A'I^{-1}A)$ & that of $(\hat{P} - P) is N(0, B'I^{-1}B)$, where

$A' = (\frac{\delta M}{\delta\alpha_1}, \frac{\delta M}{\delta\alpha_2}, \frac{\delta M}{\delta\alpha_3}, \frac{\delta M}{\delta\beta_1}, \frac{\delta M}{\delta\beta_2}, \frac{\delta M}{\delta\beta_3})$

$B' = (\frac{\delta P}{\delta\alpha_1}, \frac{\delta P}{\delta\alpha_2}, \frac{\delta P}{\delta\alpha_3}, \frac{\delta P}{\delta\beta_1}, \frac{\delta P}{\delta\beta_2}, \frac{\delta P}{\delta\beta_3})$

10.2. Bayesian Estimation

A statistical method called Bayesian estimation is used to ascertain how sample information and past knowledge affect the prior distributions of the parameters being studied. It is assumed that the model's parameters are random variables with independent Gamma prior distributions. In this case, we use the gamma prior distribution and the associated PDFs to estimate the unknown parameters as

$$\alpha_1 \sim Gamma(a_1, b_1) \qquad (\alpha_1, a_1, b_1) > 0, \tag{10}$$

$$\alpha_2 \sim Gamma(a_2, b_2) \qquad (\alpha_2, a_2, b_2) > 0, \tag{11}$$

$$\alpha_3 \sim Gamma(a_3, b_3) \qquad (\alpha_3, a_3, b_3) > 0, \tag{12}$$

$$\beta_1 \sim Gamma(a_4, b_4) \qquad (\beta_1, a_4, b_4) > 0, \tag{13}$$

$$\beta_2 \sim Gamma(a_5, b_5) \qquad (\beta_2, a_5, b_5) > 0, \tag{14}$$

$$\beta_3 \sim Gamma(a_6, b_6) \qquad (\beta_3, a_6, b_6) > 0, \tag{15}$$

Here, a_i and b_i (i = 1,2,3,4,5,6) denotes the shape and scale parameters
The posterior distributions of these parameters are computed, taking into account the likelihood function and the prior distributions:

$$\alpha_1 | \underset{\sim}{X_1} \sim Gamma(n_1 + a_1, b_1 + W_1) \tag{16}$$

$$\alpha_2 | \underset{\sim}{X_2} \sim Gamma(n_2 + a_2, b_2 + W_2) \tag{17}$$

$$\alpha_3 | \underset{\sim}{X_3} \sim Gamma(n_3 + a_3, b_3 + W_3) \tag{18}$$

$$\beta_1 | \underset{\sim}{X_4} \sim Gamma(n_4 + a_4, b_4 + W_4) \tag{19}$$

$$\beta_2 | \underset{\sim}{X_5} \sim Gamma(n_5 + a_5, b_5 + W_5) \tag{20}$$

$$\beta_3 | \underset{\sim}{X_6} \sim Gamma(n_6 + a_6, b_6 + W_6) \tag{21}$$

Observations are created from the above-mentioned posterior distributions in order to obtain the width of HPD intervals and Bayes estimates for parameters. The aforementioned draws are directly entered into equations [2], [6], and [7] to obtain Bayesian estimates of the MTSF and profit function. From the sample averages of the relevant drawings, Bayesian estimates of parameters and reliability characteristics are obtained using a squared error loss function.

11. Simulation Study

A simulation study is conducted to investigate the behavior of parameters, estimates, and reliability features. Table 1-6 displays the values of the Standard Error (SE)/Posterior Standard Error (PSE) and the width of confidence/HPD intervals. Samples of sizes $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 100$ were taken from the six investigated distributions while presuming various parameter values as shown in Tables 1-6. R software is used to perform the computation with ten thousand iterations.

Table 1: *MTSF values for fixed $\beta_2 = 0.5$ and varying α_2*

α_2	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	3.59	3.55	0.00155	0.0059	4.79	0.0326	0.01173
0.2	2.87	2.71	0.00078	0.0036	4.71	0.0324	0.01160
0.3	2.41	2.22	0.00049	0.0027	4.53	0.0318	0.01131
0.4	2.10	1.86	0.00038	0.0022	4.50	0.0316	0.01129
0.5	1.86	1.65	0.00031	0.0020	4.38	0.0312	0.01124
0.6	1.68	1.44	0.00028	0.0018	4.21	0.0306	0.01113
0.7	1.54	1.32	0.00026	0.0016	4.20	0.0302	0.01111
0.8	1.42	1.19	0.00024	0.0015	4.19	0.0292	0.01086
0.9	1.32	1.11	0.00023	0.0014	4.06	0.0302	0.01082
1	1.24	1.01	0.00021	0.0013	3.99	0.0295	0.01077

Table 2: *MTSF values for fixed β_2=1 and varying α_2*

α_2	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	3.95	4.08	0.00244	0.0077	5.86	0.0330	0.01186
0.2	3.09	2.97	0.00113	0.0044	5.63	0.0332	0.01163
0.3	2.56	2.35	0.00068	0.0031	5.19	0.0318	0.02613
0.4	2.20	1.95	0.00049	0.0025	5.06	0.0316	0.02420
0.5	1.94	1.69	0.00039	0.0022	4.75	0.0318	0.02423
0.6	1.75	1.51	0.00034	0.0020	4.70	0.0309	0.02326
0.7	1.59	1.36	0.00030	0.0018	4.69	0.0308	0.01909
0.8	1.46	1.24	0.00028	0.0017	4.53	0.0305	0.02901
0.9	1.36	1.11	0.00027	0.0016	4.26	0.0301	0.02423
1	1.27	1.03	0.00025	0.0015	4.11	0.0297	0.02136

Table 3: *MTSF values for fixed β_2=1.5 and varying α_2*

α_2	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	4.25	4.47	0.00333	0.0093	5.90	0.0330	0.02450
0.2	3.27	3.15	0.00149	0.0050	5.83	0.0325	0.02118
0.3	2.68	2.49	0.00086	0.0035	5.67	0.0325	0.03697
0.4	2.29	2.08	0.00061	0.0027	5.59	0.0322	0.02498
0.5	2.01	1.76	0.00048	0.0023	5.23	0.0318	0.02648
0.6	1.79	1.55	0.00041	0.0021	5.01	0.0319	0.02250
0.7	1.63	1.37	0.00036	0.0019	4.99	0.0308	0.02110
0.8	1.50	1.26	0.00033	0.0018	4.91	0.0306	0.02223
0.9	1.39	1.15	0.00031	0.0017	4.70	0.0306	0.06198
1	1.29	1.05	0.00029	0.0016	4.69	0.0300	0.02095

Table 4: *Profit values for fixed β_2=0.5 and varying α_2*

α_2	True profit	MLE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	93.76	89.02	0.097	0.05770	180.56	0.160	0.2386
0.2	86.61	81.54	0.091	0.04630	180.52	0.163	0.2422
0.3	80.59	75.04	0.085	0.06095	180.33	0.167	0.2488
0.4	75.41	69.38	0.086	0.06123	180.23	0.170	0.2516
0.5	70.90	64.29	0.088	0.06591	180.00	0.172	0.2529
0.6	66.91	59.74	0.077	0.05611	179.98	0.171	0.2571
0.7	63.34	55.67	0.082	0.06343	179.56	0.172	0.2563
0.8	60.13	51.97	0.071	0.06374	179.21	0.173	0.2569
0.9	57.22	48.56	0.075	0.06991	179.01	0.175	0.2565
1	54.57	45.46	0.089	0.077320	178.88	0.173	0.2548

Table 5: *Profit values for fixed β_2=1 and varying α_2*

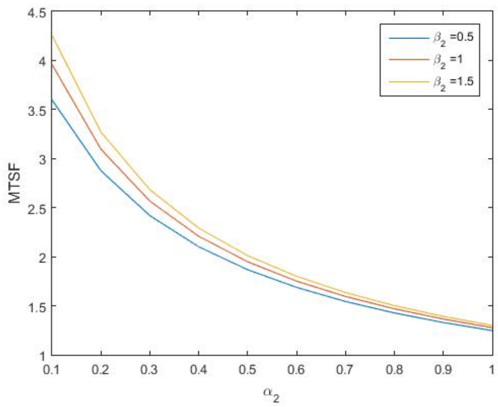
α_2	True profit	MLE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	101.49	107.01	0.084	0.06113	180.73	0.117	0.1734
0.2	94.31	99.72	0.086	0.07476	180.71	0.120	0.1779
0.3	88.19	93.34	0.094	0.07369	180.53	0.125	0.1832
0.4	82.88	87.67	0.093	0.06006	180.46	0.127	0.1877
0.5	78.20	82.53	0.092	0.06135	180.37	0.130	0.1903
0.6	74.03	77.87	0.094	0.06475	180.33	0.131	0.1929
0.7	70.30	73.61	0.085	0.06013	180.21	0.133	0.1954
0.8	66.91	69.70	0.089	0.07130	180.09	0.136	0.1956
0.9	66.83	66.07	0.080	0.07032	180.00	0.135	0.2023
1	61.00	62.70	0.077	0.06212	179.99	0.134	0.1995

Table 6: *Profit values for fixed $\beta_2 = 1.5$ and varying α_2*

α_2	True profit	MLE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	103.26	110.57	0.098	0.06532	181.82	0.110	0.1630
0.2	96.26	103.61	0.091	0.06521	181.62	0.114	0.1690
0.3	90.25	97.46	0.087	0.06222	181.53	0.118	0.1741
0.4	85.02	91.95	0.086	0.06509	181.36	0.120	0.1771
0.5	80.38	87.00	0.081	0.06011	181.09	0.122	0.1827
0.6	76.24	82.44	0.082	0.06529	181.00	0.124	0.1816
0.7	72.51	78.26	0.085	0.06246	180.76	0.124	0.1846
0.8	69.12	74.42	0.092	0.06479	180.53	0.125	0.1863
0.9	66.03	70.83	0.097	0.07351	180.19	0.128	0.1877
1	63.18	67.51	0.082	0.06274	180.00	0.128	0.1862

12. Graphical Study

A more detailed and vivid depiction of the behavior of the system can be obtained by performing a graphical analysis of the system model. For more extensive analysis, we plot MTSF and Profit function wrt α_2 failure rate of unit B for different values of β_2 repair rate of unit B as 0.5, 1 and 1.5 while keeping other parameters as fixed α_1= 0.9, α_3= 1.5, β_1=0.5, β_3=1.



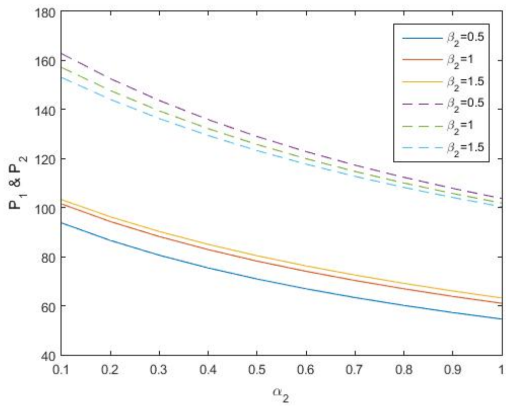


Figure 2: *(a) Behaviour of MTSF wrt to α_2 for different values of β_2 and (b) Behaviour of P_1&P_2 wrt to α_2 for different values of β_2*

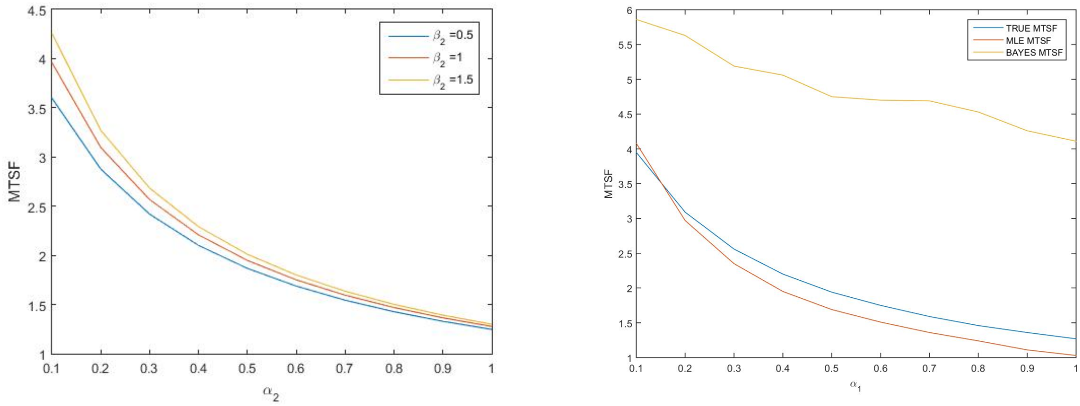


Figure 3: *(a) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_2 for β_2= 0.5 and (b) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_2 for β_2= 1*

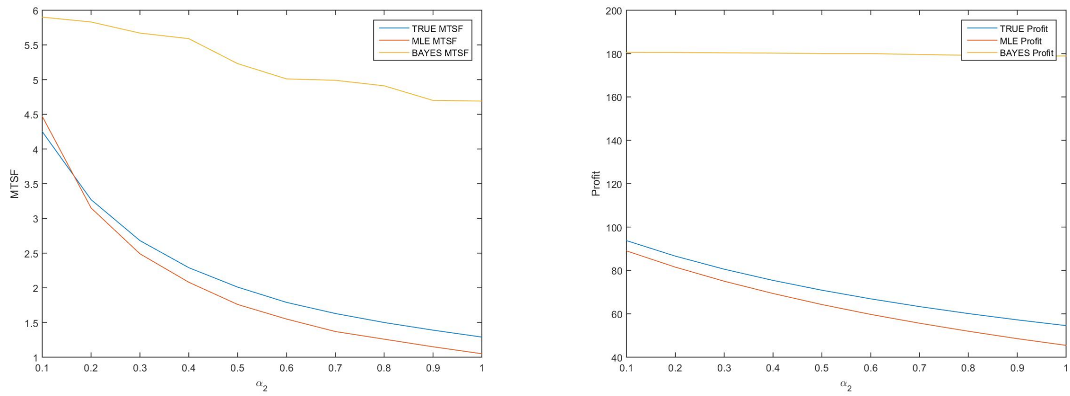


Figure 4: *(a) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_2 for β_2= 1.5 and (b) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_2 for β_2= 0.5*

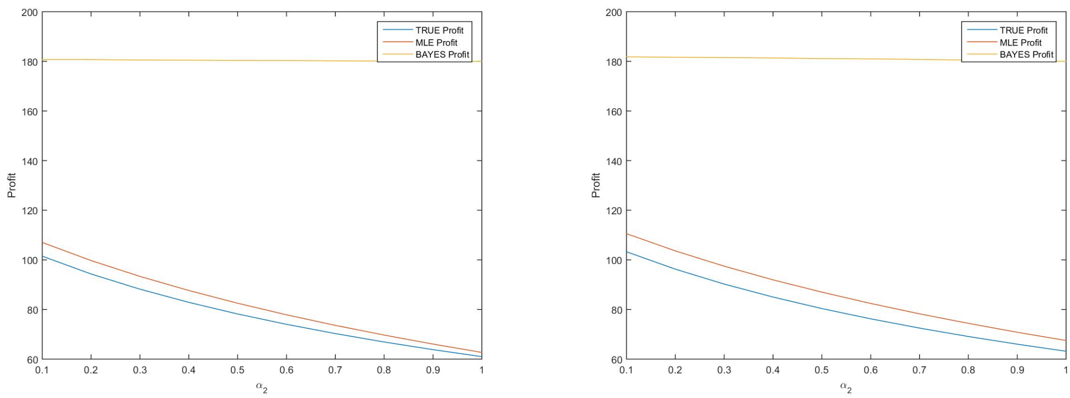


Figure 5: *(a) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_2 for β_2= 1 and (b) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_2 for β_2= 1.5*

13. Discussion and Conclusion

1. Tables and figures exhibit that MTSF decreases as the failure rate α_2 increases, but increases as the repair rate β_2 increases same trend is observed for profit function. Profit function decreases as the failure rate α_2 increases, and increases when value of repair rate β_2 increases.
2. Tables 1-6 indicate that for fixed and variable parameters, MLEs estimates of the profit function perform better than Bayes in terms of SE as well as in terms of the width of the confidence intervals as Bayes estimates have higher PSE and the width of HPD intervals as compared to the MLE's counterpart.
3. Based on the above discussions, we conclude that for estimating the MTSF and Profit function of the analyzed model, the Classical approach outperforms the Bayes approach.

References

[1] Sharma, A. and Kumar, P. (2019). Analysis of Reliability Measures of Two Identical Unit System with One Switching Device and Imperfect Coverage *Reliability: Theory & Applications*, 14(1):44–52.

[2] Kuntal D., Ashish K., and Monika S.(2017). Performance analysis of a non-identical unit system with priority and weibull repair and failure laws *International Journal of Computer Applications*, 975:8887.

[3] Pedram P. G., Parham A., Seyed T. A. N., and Seyed A. A. N. (2016). Redundancy allocation problem of a system with increasing failure rates of components based on weibull distribution: A simulation-based optimization approach *Reliability Engineering & System Safety*, 152:187–196.

[4] Kishan, R. and Jain, D. (2014). Classical and Bayesian analysis of reliability characteristics of a two-unit parallel system with Weibull failure and repair laws *International Journal of System Assurance Engineering and Management*, 5:252–261.

[5] Kumar, A., Garg, R. and Barak, MS (2023). Performance analysis of computer systems with Weibull distribution subject to software upgrade and load recovery *Life Cycle Reliability and Safety Engineering*, 12(1):51–63.

[6] Kumar, A. and Saini, M. (2014). Cost-benefit analysis of a single-unit system with preventive maintenance and Weibull distribution for failure and repair activities *Journal of Applied Mathematics, Statistics and Informatics*, 10(2):5–19.

[7] Malik, SC and others (2019). Cost-benefit evaluation of a repairable system with alternate repair and weibull distribution *International Journal of Agricultural & Statistical Sciences*, 15(2).

[8] Saini, M., Devi, K. and Kumar, A. (2021). Stochastic modeling and profit evaluation of a redundant system with priority subject to Weibull densities for failure and repair *Information and Communication Technology for Intelligent Systems: Proceedings of ICTIS 2020, Volume 1*, pages 11–20.

[9] Sharma, VK and Gupta, Surendra P. (2012). Cost Benefit Analysis of Three Unit Redundant System Model with Correlated Failures and Repairs *Journal of Pure & Applied Science & Technology*, 2(1).

[10] Van ,C., Jasper (2003). Assessment of interruption costs in electric power systems using the Weibull-Markov model *Department of Electric Power Engineering, Chalmers University of Technology*.

[11] Sharma, P. and Kumar, P. (2024) Analysis of two non-identical unit system having safe and unsafe failures with rebooting and parametric estimation in classical and bayesian paradigms *Reliability: Theory & Applications*, 19, 3(79):364–379.