ON THE FAILURE RATE FUNCTION OF A SYSTEM UNDER THE δ-SHOCK MODEL

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Abstract

An interesting type of shock model in reliability theory is the delta-shock model, which is defined based on the length of time between successive shocks. Whenever this length is less than or equal to a pre-fixed threshold called delta, the system will fail. In this paper, the failure rate function of a system subjected to random shocks under the delta-shock model is discussed. To be more precise, a simple approximate formula for the failure rate function is established. Numerical studies are performed under the assumption that the arrival of shocks follows a Poisson process. Based on the numerical results, the accuracy of the approximation is evaluated, and it is also shown that reducing the mean of inter-arrival times between successive shocks leads to an increase in the failure rate of the system. Conclusions are presented at the end.

Keywords: δ-shock model, failure rate function, approximation

1. Introduction

Shock models are useful tools for analyzing systems that are subjected to random shocks from external sources. It is important to know how a system will behave when it gets shocks. Therefore, shock models have attracted much attention in applied probability, reliability theory, and engineering. Some basic types of shock models are cumulative shock models, extreme shock models, and run shock models. In a cumulative shock model, a system breaks down due to a cumulative effect; in an extreme shock model, a system breaks down due to a single large shock; in a run shock model, the range of a certain number of consecutive shocks is considered a failure criterion. For more on these, see, e.g., [1], [7], [13], and [15]. There are other shock models that have been developed in recent years. The so-called δ-shock model is one of them that has attracted more attention. According to the classical δ-shock model, if the inter-arrival time between two consecutive shocks falls below a critical threshold δ, the system fails. The δ-shock model was first introduced by Li et al. [8] in 1999, after which it was widely studied by many researchers; see, e.g., [9], [10], [11], [12], [18], and [19]. Many extensions and generalizations have been proposed for the δ-shock model. We briefly review some of them below. A generalized δ-shock model is studied in [2], in which the system fails when k consecutive inter-arrival times are less than a threshold δ. A discrete-time version of the δ-shock model is studied in [3], where the shocks occur according to a binomial process, i.e., the inter-arrival times between successive shocks follow a geometric distribution. In [4], the lifetime behavior of the δ-shock model and the censored δ-shock model is studied under the assumption that shocks arrive according to a renewal process with uniformly distributed inter-arrival times. In [16], the survival function and the mean lifetime of the system failure under the δ-shock model is investigated by considering the proportional hazard rate model. In [14], the reliability analysis of an extension of the discrete time version of the δ-shock model is studied by considering two different critical thresholds and a probable failure region. A general δ-shock model is introduced in [4], in which the recovery time depends on both the inter-arrival time and the magnitude of the shocks. The work by [5] revisited the classical δ-shock model and generalized it to the case of renewal processes of external shocks with arbitrary inter-arrival times and arbitrary distribution of the threshold δ.

In this paper, we will study the failure rate function of a system subjected to random shocks under the δ-shock model. In fact, it is not possible to provide an exact formula for this failure rate function, but here we are trying to establish a simple approximation. To the best of our knowledge, the failure rate function in δ-shock modeling has not been discussed in the literature. Therefore, the results of this paper constitute a significant advancement in this area.

The remainder of the paper is organized as follows. The description of the δ-shock model is derived in Section 2. The failure rate function for a system subjected to random shocks under the δ-shock model is discussed with a simple approximation in Section 3. Numerical results are presented in Section 4. Section 5 concludes the paper.

2. The δ-shock model

Assume that a system is subject to a sequence of external random shocks. Let X_i denote the ith intershock time, i.e., the time between ith and $(i+1)$th shocks for $i = 1, 2, \ldots$. Let also intershock times $X_1, X_2, \ldots$ are independent and identical (i.i.d.) with the cumulative distribution function (cdf) $F(x) = \Pr(X \leq x)$ (with $F(0) = 0$). Under the δ-shock model, the performance of the system is such that if $X_n \leq \delta$ for a given threshold δ, the system fails. Otherwise, the system continues to work without any performance problems under the influence of shocks. Thus, the lifetime of the system is defined as

$$T_\delta = \sum_{i=1}^{N} X_i,$$

where the stopping random variable N is

$$\{N = n\} \Leftrightarrow \{X_1 > \delta, X_2 > \delta, \ldots, X_{n-1} > \delta, X_n \leq \delta\},$$

for $n = 1, 2, \ldots$ and $\delta > 0$.

Obviously, the probability mass function of N is

$$\Pr(N = n) = F(\delta)(1 - F(\delta))^{n-1}, \quad for\ n = 1, 2, \ldots,$$

that is, N follows a geometric distribution with the mean $\frac{1}{F(\delta)}$.

Moreover, the reliability probability function (or survival function) of the system's lifetime T_δ is given by the following formula (see [4]):

$$\Pr(T_\delta > t) = \sum_{n=1}^{\infty} (1 - F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t - x\right) dF(x), \tag{1}$$

where S^*_n is the nth arrival time of a renewal process, whose inter-arrival times have the cdf,

$$F^*(x) = \frac{F(x) - F(\delta)}{1 - F(\delta)}, \quad for\ x > \delta.$$

Furthermore, the mean lifetime of the system, that is, the system's mean time to failure (MTTF), is calculated as $E(T_\delta) = \dfrac{E(X)}{F(\delta)}$.

3. Approximation of the failure rate function

Suppose the density function, cumulative distribution function, and survival function of the lifetime T_δ are denoted by $g(t)$, $G(t)$, and $\bar{G}(t)$, respectively. The survival function $\bar{G}(t)$ is the same in Eq. (1). The failure rate (or hazard rate) function is defined as

$$h(t) = \frac{g(t)}{\bar{G}(t)}. \tag{2}$$

Since it is difficult to obtain $g(t)$ from formula (1), it is not possible to obtain an exact formula for the failure rate function of a system subject to random shocks under the δ-shock model. Therefore, approximate formulas can be useful in this case. In the following theorem, we present an approximate formula for $h(t)$ by using Eq. (1).

Theorem 1. We have

$$h(t) \approx \frac{\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(t-x < S^*_{n-1} \leq t+\epsilon-x\right) dF(x)}{\epsilon \left(\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t-x\right) dF(x)\right)},$$

where $\epsilon > 0$ is a small number and S^*_n is the nth arrival time of a renewal process, whose inter-arrival times have the cdf, $F^*(x) = \dfrac{F(x)-F(\delta)}{1-F(\delta)}$ for $x > \delta$.

Proof. We have $g(t) = \frac{d}{dt}G(t)$. By using the definition of the derivative of a function, we can write

$$g(t) = \frac{d}{dt}G(t) = \lim_{\epsilon \to 0} \frac{G(t+\epsilon)-G(t)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\bar{G}(t)-\bar{G}(t+\epsilon)}{\epsilon}$$

This implies that when ϵ is small (or $t \approx t+\epsilon$), then

$$g(t) \approx \frac{\bar{G}(t)-\bar{G}(t+\epsilon)}{\epsilon}. \tag{3}$$

Using Eq. (3) in Eq. (2), we have

$$h(t) \approx \frac{\frac{\bar{G}(t)-\bar{G}(t+\epsilon)}{\epsilon}}{\bar{G}(t)} = \frac{\bar{G}(t)-\bar{G}(t+\epsilon)}{\epsilon \bar{G}(t)} \tag{4}$$

Now, by using Eq. (1) in Eq. (4), we obtain

$$\begin{aligned} h(t) &\approx \frac{\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \left(\Pr\left(S^*_{n-1} > t-x\right) - \Pr\left(S^*_{n-1} > t+\epsilon-x\right)\right) dF(x)}{\epsilon \left(\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t-x\right) dF(x)\right)} \\ &= \frac{\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(t-x < S^*_{n-1} \leq t+\epsilon-x\right) dF(x)}{\epsilon \left(\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t-x\right) dF(x)\right)}, \end{aligned}$$

where S^*_n is the nth arrival time of a renewal process, whose inter-arrival times have the cdf,

$$F^*(x) = \frac{F(x)-F(\delta)}{1-F(\delta)}, \quad for\ x > \delta.$$

The proof is complete. ■

Remark 1. The approximate formula proven in Theorem 1 can also be written in the following simpler form:

$$h(t) \approx \frac{1}{\epsilon}\left(1 - \frac{\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t+\epsilon-x\right) dF(x)}{\sum_{n=1}^{\infty} (1-F(\delta))^{n-1} \int_0^\delta \Pr\left(S^*_{n-1} > t-x\right) dF(x)}\right),$$

where ϵ and S^*_n hold under the assumptions of Theorem 1.

4. Numerical results

Assume that the inter-arrival times $X_1, X_2, \ldots$ have exponential distribution with mean $\frac{1}{\lambda}$ for $\lambda > 0$. Such a situation occurs when the arrival of shocks follows a Poisson process. Using Theorem 1, we obtain numerical results on the approximation of the failure rate function $h(t)$. The calculations have been performed considering different values of the parameter λ and the critical threshold δ at some different time points t. For the approximation accuracy trend in terms of ϵ, values 0.01, 0.001, and 0.0001 have been considered for ϵ. All numerical results are summarized in Table 1. It can be seen from Table 1 that as the parameter λ becomes larger and ϵ becomes smaller, the changes in the approximate values of the failure rate function $h(t)$ gradually decrease until the changes stop at least to four decimal places for $\lambda = 10$ and $\epsilon = 0.0001$. Regardless of the structural features of the approximation formula used, the numerical values show that the values of the failure rate function are increasing as the parameter λ increases. This seems reasonable because an increase in the parameter λ means a decrease in the mean of inter-arrival times between successive shocks, and this leads to the system being stressed by shocks and increasing the probability of system failure.

Table 1: Approximate values for the failure rate function $h(t)$.

λ	δ	t	ϵ	$h(t)$	ϵ	$h(t)$	ϵ	$h(t)$
5	3	4	0.01	4.7580	0.001	4.9973	0.0001	4.8517
		7		4.8587		4.9800		4.9166
		9		4.7457		4.9956		4.8908
		11		4.8784		5.0015		5.3862
		13		4.8778		4.9998		5.0845
	6	4	0.01	4.8768	0.001	4.9973	0.0001	5.3369
		7		4.8770		4.9959		5.0752
		9		4.8768		4.9956		5.2401
		11		4.8711		5.0019		4.6171
		13		4.8762		4.9830		4.9152
10	3	4	0.01	9.5167	0.001	9.9569	0.0001	9.8863
		7		9.5160		9.9612		10.0618
		9		9.5167		9.9585		10.0073
		11		9.5032		9.9473		10.0657
		13		9.5171		10.0830		10.0542
	6	4	0.01	9.5167	0.001	9.9569	0.0001	9.8863
		7		9.5184		9.9612		10.0618
		9		9.5167		9.9585		10.0073
		11		9.5209		9.9473		10.0657
		13		9.5171		9.9681		10.0542

5. Conclusions

In this paper, we establish an approximate formula to calculate the failure rate function of a system subjected to random shocks under the δ-shock model. The accuracy of the approximation was evaluated by numerical studies for the case where the arrival of the shocks follows a Poisson process. Future studies could focus on establishing more approximate formulas and comparing the accuracy of the formulas in approximating the system failure rate function.

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