

TWO PARAMETRIC GENERALIZED FUZZY ENTROPY MEASURE WITH SOURCE CODING

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Abstract

In this manuscript, we introduce a new generalized fuzzy information measure for a fuzzy set. We establish its validity as a fuzzy entropy measure. Also, we define a new generalized fuzzy average code-word length for a fuzzy set, and explore its relationship with the fuzzy information measure. We prove coding theorems for discrete noiseless channels. The measures outlined in this communication are not only novel, but also encompass certain well-established measures from the existing literature on fuzzy information theory. We also show the proposed measure through a dataset and observe its monotonicity through tabular and graphical forms.

Keywords: Fuzzy set, Membership function, Shannon's entropy, Fuzzy entropy, Code-word length, Kraft inequality, Coding theorem, Holder's inequality.

1. INTRODUCTION

Fuzzy logic and probability both deal with uncertainty, but they do so in fundamentally different ways; Probability Theory deals with randomness and stochastic uncertainty. It describes how likely an event is to occur, assuming a well-defined sample space while Fuzzy Logic deals with vagueness and imprecision. Instead of assigning probabilities, it assigns degrees of truth to statements. Fuzziness and uncertainty are inherent aspects of human thinking and many real-world problems. Uncertainty is present in our decisions, language, and the way we process information. The primary goal of information is to reduce this uncertainty and fuzziness. In fact, the amount of information gained from an experiment is measured by the reduction of probabilistic uncertainty, and this reduction is often referred to as a measure of information. On the other hand, the measure of fuzziness quantifies the vagueness and ambiguity within uncertainties. The concept of entropy, which is central to understanding uncertainty, has been widely applied across various fields, including communication theory, statistical mechanics, finance, pattern recognition, and neural networks. Fuzzy set theory, developed by Lotfi A. Zadeh [20], has also found extensive applications in areas such as clustering, image processing, and decision-making due to its ability to model non-statistical imprecision and vague concepts. The significance of fuzzy sets lies in their ability to handle imprecise and uncertain information. Numerous fuzzy measures have been explored and developed by Kapur [11], Lowen [13], Nguyen and Walker [15], and Parkash [17], Pal and Bezdek [16], Zadeh [20], Hooda [8], Javid and Baig [2], Bhat and Baig [3], [4] etc. Application of fuzzy measures to engineering, fuzzy traffic control, fuzzy aircraft control, medicines, computer science and decision making etc, have already been established. The basic noiseless coding theorems see for instance; the papers, Aczel [1], Kapur [10], Renyi [18], Campbell [5] give the lower bound for the mean code-word length of a uniquely decipherable code in terms of Shannon's [19] entropy etc. Kapur has developed connections

between probability, entropy, and coding. However, there are instances where probabilistic entropy measures are not effective. To address these situations, rather than relying on probability, the concept of fuzziness can be considered.

Suppose there be a universe of discourse $X = \{x_1, x_2, x_3, \dots, x_n\}$, then a fuzzy subset of universe X is defined as:

$$A = \{(x_i, \mu_A(x_i)) : x_i \in X, \mu_A(x_i) \in [0, 1]\}$$

Where $\mu_A(x_i) \in [0, 1]$ is a membership function and represents the degree to which the element x_i belongs to the set A and is defined as follows:

$$\mu_A(x_i) = \begin{cases} 0 & \text{if } x_i \notin A \text{ and there is no ambiguity,} \\ 1 & \text{if } x_i \in A \text{ and there is no ambiguity,} \\ 0.5 & \text{if } x_i \in A \text{ or } x_i \notin A \text{ and there is maximum ambiguity.} \end{cases}$$

The membership function $\mu_A(x_i)$ associates each element $x_i \in X$ with a grade of membership in the set A . When $\mu_A(x_i)$ takes only the values 0 or 1, there is no uncertainty about the membership, and the set is considered crisp (i.e., non-fuzzy).

2. BASIC CONCEPTS

Let X be a discrete random variable that can take the values $x_1, x_2, x_3, \dots, x_n$ with corresponding probabilities $p_1, p_2, p_3, \dots, p_n$ where $p_i \geq 0 \quad \forall i = 1, 2, 3, \dots, n$ and $\sum_{i=1}^n p_i = 1$. Shannon [19] defines the entropy of this random variable as a measure of information which is given as:

$$H(A) = - \sum_{i=1}^n p_i \log_D p_i \quad (1)$$

The measure (1) acts as an appropriate way to quantify entropy. Let $p_1, p_2, p_3, \dots, p_n$ represent the probabilities of n code-words to be transmitted, and let their corresponding lengths $l_1, l_2, l_3, \dots, l_n$ adhere to the Kraft's [12] inequality,

$$\sum_{i=1}^n D^{-l_i} \leq 1 \quad (2)$$

Shannon [19] demonstrated that for uniquely decipherable codes, the mean code-word length,

$$L = \sum_{i=1}^n p_i l_i \quad (3)$$

has a lower bound for all codes that satisfy (2) and lies between $H(P)$ and $H(P)+1$, where D is the size of code alphabet.

Let $x_1, x_2, x_3, \dots, x_n$ be elements of the universe of discourse, with corresponding membership functions $\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)$. Each of these membership values lies between 0 and 1, but they are not probabilities because their sum does not equals unity. The $\mu_A(x_i)$ represents the degree to which the element x_i belongs to the set A . This function $\mu_A(x_i)$ assigns a grade of membership to each $x_i \in \mathbb{R}^n$ and is known as the membership function.

Define,

$$\text{Fuzzy Set} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \mu_A(x_1) & \mu_A(x_2) & \dots & \mu_A(x_n) \end{pmatrix}, \quad 0 \leq \mu_A(x_i) \leq 1, \forall x_i \in X \quad (4)$$

We refer to the scheme (4) as a finite fuzzy information scheme. Each finite scheme represents a state of uncertainty, which is analogous to Shannon's probabilistic entropy. For Havrda and Charvat's [7] information measure, Kapur [11] suggests the following fuzzy information measure:

$$H^\beta(A) = \frac{1}{1-\beta} \left[\frac{1}{n} \sum_{i=1}^n \left[\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right] - 1 \right], \quad \beta \neq 1, \beta \geq 0 \quad (5)$$

The measure (5) serves as a very suitable measure of fuzzy entropy for the finite fuzzy information scheme given in (4).

De-Luca and Termini [6] introduced a set of four properties and these properties are widely accepted as for defining new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness $H(A)$ in a fuzzy set A should have the following properties to be valid fuzzy entropy:

1. Sharpness: $H(A)$ is minimum if and only if A is a crisp set, i.e.,

$$\mu_A(x_i) = 0 \text{ or } 1, \quad \forall x_i, i = 1, 2, \dots, n$$

2. Maximality: $H(A)$ is maximum if and only if A is most fuzzy set, i.e.,

$$\mu_A(x_i) = \frac{1}{2}, \quad \forall x_i, i = 1, 2, \dots, n$$

3. Resolution: $H(A^*) \leq H(A)$, where A^* is sharpened version of A

4. Symmetry: $H(A) = H(A^c)$, where A^c is the complement of A , i.e.,

$$\mu_{A^c}(x_i) = 1 - \mu_A(x_i), \quad \forall x_i, i = 1, 2, \dots, n$$

2.1. Example of Fuzzy Sets

Example 1 Let $X = \{0, 2, 4, 6, 8, 10, 12\}$ represent the number of working hours in a day. Define the fuzzy set A as "desirable working hours" with the following membership grades:

- $\mu_A(0) = 0.1$ (not working at all is not desirable)
- $\mu_A(2) = 0.3$
- $\mu_A(4) = 0.7$
- $\mu_A(6) = 1.0$ (ideal working hours)
- $\mu_A(8) = 0.7$
- $\mu_A(10) = 0.3$
- $\mu_A(12) = 0.1$ (working too long is not desirable)

We have plotted the graphs using R software and from the below two graphs the ambiguity can easily be seen and it is maximum at $\mu_A(6) = 1.0$, which shows the ideal working hours. These graphs define a fuzzy set like "Moderate Working Hours" or "Optimal Working Time". This concept is widely used in decision-making, AI, control systems, and human behavior modeling. Here 6 hours is ideal (membership = 1) suggesting that it is the most optimal or representative number of working hours; 4 or 8 hours is still acceptable but less ideal (membership = 0.7) and 0 or 12 hours is not ideal (membership = 0.1) indicating these extremes are least preferred. Here the discrete membership function highlights how specific working hours "ranging from 0 to 12" align with varying degrees of membership, showing a peak at 6 hours where the membership reaches 1 and the continuous membership function graph provides a smooth, gradual transition between different working hours, rather than assigning membership values only to specific points. So, such a function can be used in decision-making processes where strict classification is impractical, such as determining an ideal workload for employees based on efficiency and fatigue considerations.



Figure 1: Membership Function on a Discrete Universe

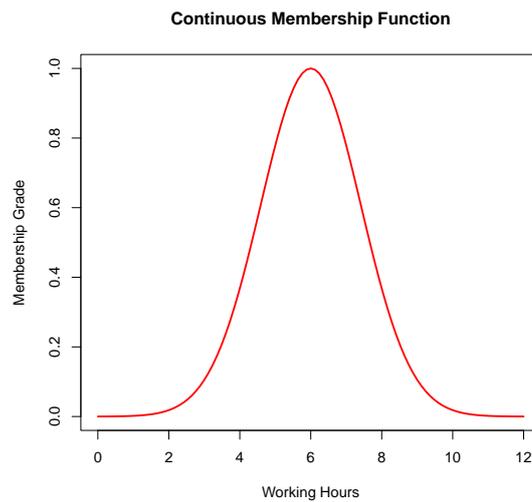


Figure 2: Membership Function on a Continuous Universe

3. NEW GENERALIZED FUZZY INFORMATION MEASURE AND ITS PROPERTIES

Suppose we have a fuzzy set A defined on a discrete universe of discourse consisting of the elements $x_1, x_2, x_3, \dots, x_n$, with corresponding membership values $\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)$. We introduce a new generalized fuzzy information measure of order α and type β as:

$$H_\alpha^\beta(A) = \frac{\alpha}{\alpha - \beta} \left[\frac{1}{n} \sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^{\frac{\beta}{\alpha}} \right) - 1 \right], \quad \alpha > \beta, \beta > 0, \alpha \neq \beta \quad (6)$$

To demonstrate that $H_\alpha^\beta(A)$ (as stated in equation (6)) is a valid fuzzy information measure, we will prove that it satisfies its key properties: sharpness, maximality, resolution, and symmetry (denoted as P1-P4).

- **P1. Sharpness:** $H_\alpha^\beta(A)$ is minimum if and only if A is a crisp set, i.e,

$$H_\alpha^\beta(A) = 0 \text{ if and only if } \mu_A(x_i) = 0 \text{ or } 1, \forall x_i, i = 1, 2, \dots, n$$

Proof: Let $H_\alpha^\beta(A)=0$

$$\frac{\alpha}{\alpha - \beta} \left[\frac{1}{n} \sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) - 1 \right] = 0$$

After some mathematical calculations, we have

$$\sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) = n$$

Conversely Suppose

$$\left[\frac{1}{n} \sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) \right] = 1$$

Or, Multiplying both sides of the above equation by $\frac{\alpha}{\alpha-\beta}$, we have

$$\frac{\alpha}{\alpha - \beta} \left[\frac{1}{n} \sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) - 1 \right] = 0$$

Which implies,

$$H_\alpha^\beta(P) = 0$$

Hence, $H_\alpha^\beta(A) = 0$ if and only if A is non fuzzy set or crisp set.

- **P2. Maximality:** $H_\alpha^\beta(A)$ is maximum if and only if A is most fuzzy set.

Proof: We have

$$\frac{\alpha}{\alpha - \beta} \left[\frac{1}{n} \sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) - 1 \right] \tag{7}$$

Differentiating equation (7) with respect to $\mu_A(x_i)$, we have

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} = \frac{\beta}{n(\alpha - \beta)} \left[\mu_A^{\beta-1}(x_i) + (1 - \mu_A(x_i))^{\beta-1} \right]$$

- When $0 \leq \mu_A(x_i) < 0.5$

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} > 0, \quad \forall 0 < \alpha < 1, 0 \leq \beta < 1.$$

Hence, $H_\alpha^\beta(A)$ is an increasing function of $\mu_A(x_i)$, whenever, $0 \leq \mu_A(x_i) < 0.5$

- Similarly, for $0.5 < \mu_A(x_i) < 1$, we have

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} < 0, \quad \forall 0 < \alpha < 1, 0 \leq \beta < 1.$$

Hence, $H_\alpha^\beta(A)$ is decreasing function of $\mu_A(x_i)$, whenever, $0 \leq \mu_A(x_i) < 0.5$

- And for $\mu_A(x_i) = 0.5$

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} = 0, \quad \forall 0 < \alpha < 1, 0 \leq \beta < 1.$$

Thus $H_\alpha^\beta(A)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$.

Hence $H_\alpha^\beta(A)$ is maximum if and only if A is the most fuzzy set, i.e. $\mu_A(x_i) = 0.5, i = 1, 2, 3, \dots, n$.

- **P3. Resolution:** $H_\alpha^\beta(A) \geq H_\alpha^\beta(A^*)$ where A^* is sharpened version of A.

Proof: Since $H_\alpha^\beta(A)$

is increasing function of $\mu_A(x_i)$ in the interval $[0,0.5)$ and is decreasing function of $\mu_A(x_i)$ in the interval $(0.5,1)$ therefore

$$\mu_{A^*}(x_i) \leq \mu_A(x_i)$$

Which implies,

$$H_\alpha^\beta(A^*) \leq H_\alpha^\beta(A), \quad \text{in } [0,0.5) \quad (8)$$

and,

$$\mu_{A^*}(x_i) \geq \mu_A(x_i)$$

Which implies,

$$H_\alpha^\beta(A^*) \geq H_\alpha^\beta(A), \quad \text{in } (0.5,1) \quad (9)$$

and

$$\mu_{A^*}(x_i) \geq \mu_A(x_i)$$

Then from above two equations, we conclude

$$H_\alpha^\beta(A) \geq H_\alpha^\beta(A^*)$$

- **P4. Symmetry:** $H_\alpha^\beta(A) = H_\alpha^\beta(A^c)$, where A^c is the complement of A.

Proof: It may be noted that from the definition of $H_\alpha^\beta(A)$ and $\mu_{A^c}(x_i) = 1 - \mu_A(x_i)$, we conclude that

$$H_\alpha^\beta(A) = H_\alpha^\beta(A^c)$$

Hence, It may be noted that from the definition of $H_\alpha^\beta(A)$ satisfies all the properties of fuzzy entropy, therefore, $H_\alpha^\beta(A)$ is a valid measure of fuzzy entropy.

4. NOISELESS CODING THEOREM OF TWO PARAMETRIC HARVDA AND CHARVAT'S GENERALIZED FUZZY INFORMATION MEASURE

Here we introduced a new corresponding code-word length according to equation (6) as:

$$L_\alpha^\beta(A) = \frac{\alpha}{\alpha - \beta} \left\{ \left[\left[\sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) \right] D^{-l_i \left(\frac{\beta-1}{\beta} \right)} \right]^\beta - 1 \right\} \quad (10)$$

$$; \alpha, \beta > 0, \alpha \neq \beta, \alpha > \beta$$

Now, we have determined the bounds of (10) in terms of equation (6) under the condition

$$\sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) D^{-l_i} \leq 1 \quad (11)$$

Or we can write

$$\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] D^{-l_i} \leq 1 \quad (12)$$

Where,

$$f(\mu_A(x_i), \mu_{A^c}(x_i)) = \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right)$$

Which is generalized fuzzy Kraft's inequality, where 'D' is the size of code alphabet.

Theorem 1. For all uniquely decipherable codes $L_\alpha^\beta(A) \geq H_\alpha^\beta(A)$ where,

$$L_\alpha^\beta(A) = \frac{\alpha}{\alpha - \beta} \left\{ \left[\left[\sum_{i=1}^n \left(\mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \right) \right] D^{-l_i \left(\frac{\beta-1}{\beta} \right)} \right]^\beta - 1 \right\}; \alpha, \beta > 0, \alpha \neq \beta, \alpha > \beta$$

The inequality holds if and only if

$$l_i = -\log_D \left[\frac{1}{\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))]} \right] \quad (13)$$

Proof: From Holder's Inequality, we have

$$\left(\sum_{i=1}^n a_i b_i \right) \geq \left(\sum_{i=1}^n a_i^\theta \right)^{\frac{1}{\theta}} \left(\sum_{i=1}^n b_i^\delta \right)^{\frac{1}{\delta}} \quad (14)$$

$$\forall, a_i, b_i > 0, i = 1, 2, 3, \dots, n \quad \text{and} \quad \frac{1}{\theta} + \frac{1}{\delta} = 1; \theta < 1 (\neq 0), \delta < 0 \text{ or } \delta < 1 (\neq 0), \theta < 0$$

The equality holds if and only if there exists a positive constant 'c' such that

$$a_i^\theta = c b_i^\delta$$

Make substitution

$$a_i = [f(\mu_A(x_i), \mu_{A^c}(x_i))]^{\frac{\beta}{\beta-1}} D^{-l_i} \quad b_i = [f(\mu_A(x_i), \mu_{A^c}(x_i))]^{\frac{1}{1-\beta}}$$

$$\theta_i = \frac{\beta-1}{\beta} \quad \delta_i = 1 - \beta$$

Now using the above values in (14), we have

$$\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i)) D^{-l_i}] \geq \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i)) D^{-l_i}]^{\frac{\beta-1}{\beta}} \right]^{\frac{\beta}{\beta-1}} \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right]^{\frac{1}{1-\beta}} \quad (15)$$

After some mathematical calculations, we have

$$\left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i)) D^{-l_i \left(\frac{\beta-1}{\beta} \right)} \right]^\beta - 1 \right\} \geq \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] - 1 \right\} \quad (16)$$

Multiplying $\frac{\alpha}{\alpha-\beta}$ both sides to (16) and the subtracting 1, we have

$$\frac{\alpha}{\alpha - \beta} \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i)) D^{-l_i \left(\frac{\beta-1}{\beta} \right)} \right]^\beta - 1 \right\} \geq \frac{\alpha}{\alpha - \beta} \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] - 1 \right\} \quad (17)$$

Which implies,

$$L_\alpha^\beta(A) \geq H_\alpha^\beta(A)$$

Now we have

$$l_i = -\log_D \left[\frac{1}{\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))]} \right]$$

Or,

$$D^{-l_i \left(\frac{\beta-1}{\beta} \right)} = \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right]^{\frac{1-\beta}{\beta}}$$

After some mathematical calculations, we have

$$\frac{\alpha}{\alpha - \beta} \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i)) D^{-l_i \left(\frac{1-\beta}{\beta}\right)}] \right]^\beta - 1 \right\} = \frac{\alpha}{\alpha - \beta} \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] - 1 \right\}$$

Or, we can write that

$$L_\alpha^\beta(A) = H_\alpha^\beta(A)$$

Theorem 2. For all uniquely decipherable codes, the following inequality holds:

$$L_\alpha^\beta(A) < H_\alpha^\beta(A) D^{(1-\beta)} + \frac{\alpha}{\alpha - \beta} (D^{(1-\beta)} - 1), \alpha > \beta, 0 \leq \beta < 1, 0 < \beta < 1 \quad (18)$$

Proof: From theorem (1)

$$L_\alpha^\beta(A) = H_\alpha^\beta(A) \quad (19)$$

Equality holds if and only if,

$$D^{-l_i} = \frac{1}{\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))]}$$

Or, we can write the above equation as

$$l_i = -\log_D \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right]$$

Code-word lengths l_i , ($i = 1, 2, 3, \dots, n$) are chosen in such a way that they satisfy the inequality

$$\log_D \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] \leq l_i < \log_D \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] + 1 \right\} \quad (20)$$

Consider the interval

$$\lambda_i = \left\{ \log_D \left[\frac{1}{n} \sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right], \left\{ \log_D \left[\frac{1}{n} \sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] + 1 \right\} \right\}$$

of length unity. In every λ_i , there lies exactly one positive integer l_i , such that

$$0 < \log_D \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] \leq l_i < \log_D \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] + 1 \right\} \quad (21)$$

We will first show the sequence l_1, l_2, \dots, l_n , thus defined satisfies the inequality given in (21) which is generalized fuzzy Kraft's Inequality of equation (21), we have

$$\log_D \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] \leq l_i$$

Again after some mathematical manipulation, we have

$$\left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] D^{-l_i \left(\frac{1-\beta}{\beta}\right)} < \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right]^{\frac{1}{\beta}} D^{\left(\frac{1-\beta}{\beta}\right)} \quad (22)$$

Again after simplifying, we have

$$\frac{\alpha}{\alpha - \beta} \left\{ \left[\left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] D^{-l_i \left(\frac{1-\beta}{\beta}\right)} \right]^\beta - 1 \right\} < \frac{\alpha}{\alpha - \beta} \left\{ \left[\sum_{i=1}^n [f(\mu_A(x_i), \mu_{A^c}(x_i))] \right] D^{(1-\beta)} - 1 \right\}$$

Or, we can write that,

$$L_\alpha^\beta(A) < H_\alpha^\beta(A) D^{(1-\beta)} + \frac{\alpha}{\alpha - \beta} (D^{(1-\beta)} - 1)$$

5. REAL WORLD APPLICATION

This section showcases the real-world applicability of the Proposed Fuzzy measure using an actual dataset. The dataset pertains to daily snowfall amounts recorded in inches of water, from 30 observations taken near Climax [14], and is also used by [9]. We evaluate the monotonic behavior by holding one parameter constant and changing the other, and similarly, by fixing the second parameter and adjusting the first.

Table 1: The behaviour of our proposed fuzzy measure When α is fixed and β varies in snowfall dataset.

	α	$H_\alpha^\beta(A)$		α	$H_\alpha^\beta(A)$
$\beta = 0.02$	0.10	0.6856531	$\beta = 0.02$	0.25	0.8532246
	0.11	0.7077139		0.26	0.8582822
	0.12	0.7269247		0.27	0.8630038
	0.13	0.7437958		0.28	0.8674217
	0.14	0.7587248		0.29	0.8715642
	0.15	0.7720250		0.30	0.8754563
	0.16	0.7839468		0.31	0.8791199
	0.17	0.7946922		0.32	0.8825745
	0.18	0.8044259		0.33	0.8858374
	0.19	0.8132833		0.34	0.8889243
	0.20	0.8213772		0.35	0.8918489
	0.21	0.8288016		0.36	0.8946236
	0.22	0.8356359		0.37	0.8972597
	0.23	0.8419475		0.38	0.8997674
	0.24	0.8477939		0.39	0.9021557

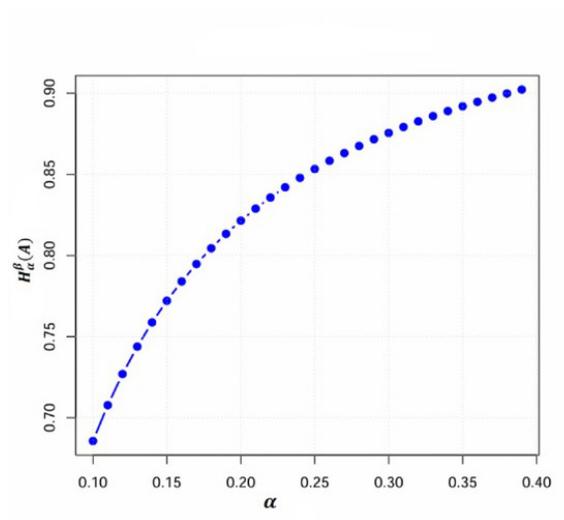


Figure 3: Graphical representation of $H_\alpha^\beta(A)$ when α is fixed and β varies

As observed in the table 1 above, when the value of α is held constant and β is varied, $H_\alpha^\beta(A)$ demonstrates an increasing trend. This behavior is illustrated graphically in Figure3. So, we can say that when α is held constant, there is an increase in snowfall level which could be attributed to changing climatic patterns or regional weather dynamics.

Table 2: The behaviour our proposed measure, when β is fixed and α varies in the snowfall data set.

$\beta = 1$	α	$H_\alpha^\beta(A)$	$\beta = 1$	α	$H_\alpha^\beta(A)$
	0.10	0.8213772		0.25	0.6301113
	0.11	0.8060979		0.26	0.6198356
	0.12	0.7912303		0.27	0.6098182
	0.13	0.7767615		0.28	0.6000514
	0.14	0.7626791		0.29	0.5905279
	0.15	0.7489710		0.30	0.5812403
	0.16	0.7356256		0.31	0.5721818
	0.17	0.7226318		0.32	0.5633456
	0.18	0.7099787		0.33	0.5547253
	0.19	0.6976558		0.34	0.5463145
	0.20	0.6856531		0.35	0.5381071
	0.21	0.6739607		0.36	0.5300972
	0.22	0.6625694		0.37	0.5222792
	0.23	0.6514698		0.38	0.5146476
	0.24	0.6406533		0.39	0.5071969

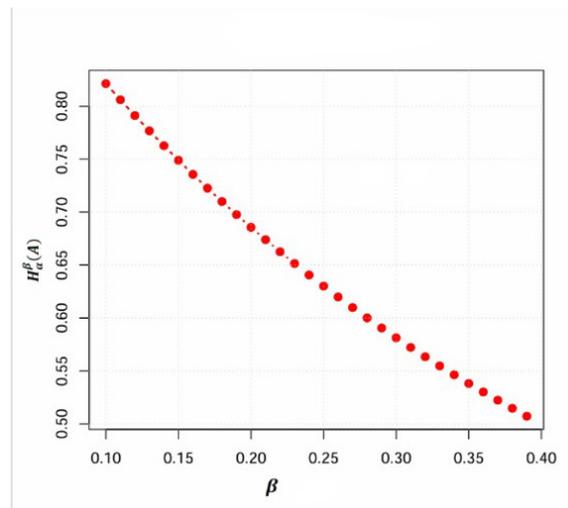


Figure 4: Graphical representation of $H_\alpha^\beta(A)$ when α is fixed and β varies

As observed in the table 2 above, when the value of β is held constant and α is varied, $H_\alpha^\beta(A)$ demonstrates an decreasing trend. This behavior is illustrated graphically in Figure4. So, we can say that when α is held constant, there is a decrease in snowfall level which could be attributed to changing climatic patterns or regional weather dynamics. This notable decrease may reflect broader climatic changes, with profound implications for ecosystems, water management, and local economies. By unraveling these factors, we can better anticipate future changes and develop adaptive strategies to mitigate potential impacts.

6. CONCLUSION

In this manuscript, we introduced a new generalized fuzzy entropy measure of order α and type β , and demonstrated that it is a valid measure of fuzzy entropy. This new measure generalizes several well-known fuzzy entropy measures found in existing literature. Additionally, we considered the

generalized fuzzy average code-word length and derived bounds in terms of the newly defined generalized fuzzy entropy measure of order α and type β . And finally, We have also shown the monotonicity of the proposed measure through a snowfall dataset and presented it in both tabular and graphical form.

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