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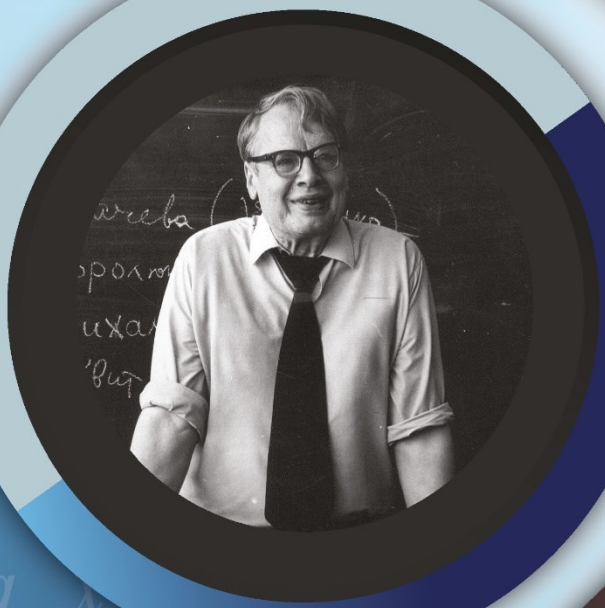

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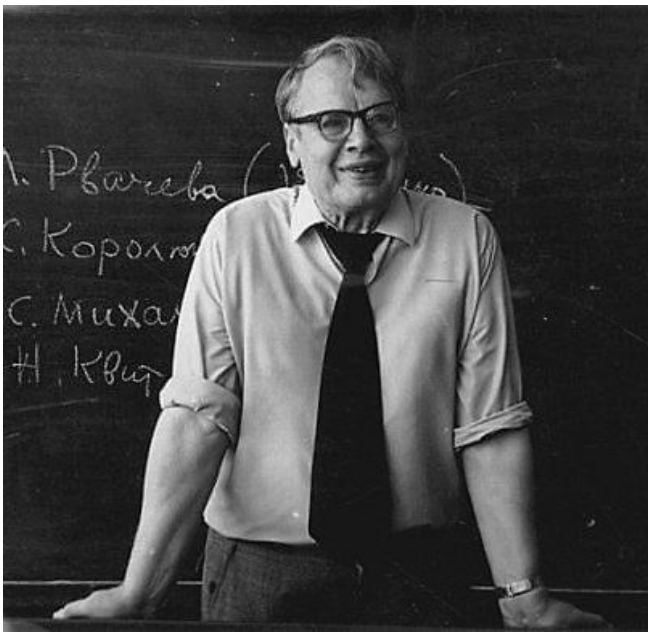

RELIABILITY:

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survey data of the COVID - 19 Booster shots acceptance and the non-acceptance values between [-1,0] and [0,1] converges precisely with the level of fixation [-0.5.0] and [0,0.5] alternatively by using the uncertain values in decision making process of the human behaviours in mathematical Analysis.

Effective management of production inventory for deteriorating items with dynamic demand patterns is crucial for businesses operating in today ' s competitive markets. This paper proposes a comprehensive model that addresses the complexities arising from the dual storage locations, item deterioration, and demand dependencies on both time and selling price. To optimize the decision variables associated with production and inventory control, we employ the Flower Pollination Optimization (FPO) algorithm, a nature-inspired meta-heuristic known for its ability to efficiently navigate complex search spaces. The two-storage production inventory model integrates the dynamics of item deterioration over time, capturing the real-world challenges faced by supply chain managers. The demand for items is modeled to be sensitive to both temporal variations and changes in selling prices, reflecting the intricate nature of market dynamics. Our approach leverages the FPO algorithm to explore and exploit the solution space, allowing for the identification of optimal or near-optimal strategies for production quantities, order quantities, and inventory levels. The FPO algorithm mimics the pollination process in nature, striking a balance between exploration and exploitation to efficiently search for solutions in a highly dynamic and nonlinear environment. The proposed model and optimization approach are validated through extensive simulations and sensitivity analyses. The results demonstrate the effectiveness of the FPO algorithm in finding robust solutions that enhance inventory management, mitigate deterioration-related losses, and adapt to varying demand scenarios. This research contributes to the field of supply chain optimization by offering a novel perspective on tackling the challenges associated with dual storage, item deterioration, and demand dependencies. The findings provide valuable insights for practitioners seeking advanced strategies for optimizing their production inventory systems in the face of evolving market conditions.

Modern machines and equipment's have a complex mechatronic structure consisting of various components, and their reliability depends on a large number of random factors that arise during design, production and operation, which are often impossible to predict. Each element of the modern machines is characterized by different performance criteria and corresponding failures. Various statistical models of failure distribution are widely used to quantify the reliability of machines and devices. The choice of a statistical model and its parameters is important for a proper assessment of reliability. The chosen statistical model should reflect the actual distribution of failures fairly correctly. In presented article is proposed a new failure distribution for reliability prediction of mechatronic components of modern machines and human-machine systems. A large number of sudden failures of modern complex technical facilities containing electronic and mechatronic structural elements seriously affect its λ-characteristic. Various studies have already shown that the failure behavior of complex systems cannot always be characterized by the "bathtub curve". This is especially true for modern complex machines, which, among other things, consist of numerous electronic components for which no wear and fatigue failures are assumed. For this reason, an alternative service life distribution for the description failure behavior of modern mechatronic components and human-machine systems is proposed. This is about the failure curves, which are initially characterized by a low or high failure rate and then tend to a constant failure rate. To determine the reliability indexes are provided analytical formulas. Methods for estimating the parameters of this distribution are presented based on failure statistic. To determine distribution parameters, statistical data on failures of the technical system are sufficient only in the first period of its operation. This is one of the main advantages of the presented distribution. On the example of practical cases, the hypothesis of compliance of the proposed theoretical distribution to the actual statistical data on failures of various mechatronic systems and human-machine system was tested.

The reliability of water supply systems plays a crucial role in ensuring sustainable water use, minimizing economic losses, and preventing failures in critical infrastructure. This paper proposes a mathematical approach to modeling the reliability of water systems based on probability theory and Markov processes. The main types of failures, their impact on operational characteristics, and economic consequences are examined. A simulation of the water supply network is conducted, considering the probabilistic characteristics of failures and recovery processes. The analysis of results demonstrates that the implementation of predictive monitoring methods and the optimization of maintenance strategies significantly enhance the resilience of water supply systems. The developed model can be applied in the planning of modernization and management of water supply infrastructure to improve its efficiency and economic feasibility.

This paper focuses on the Bayesian estimation of the shape parameter for the Inverse Topp-Leone (ITL) distribution. To achieve this, we employ both the extended Jeffrey's prior and the gamma prior, facilitating the derivation of posterior distributions for the shape parameter. The Bayesian estimators are calculated under various loss functions, including the squared error loss function (SELF), entropy loss function (ELF), precautionary loss function (PLF), and Linex loss function (LLF), each chosen to address different practical scenarios and estimator biases. In addition to the Bayesian approach, we also explore maximum likelihood estimation (MLE) to provide a comparative benchmark. The performance of these estimators is assessed and compared based on mean squared error (MSE) across multiple sample sizes, allowing for a detailed evaluation of estimator robustness and accuracy. A real-world dataset is then analyzed to further demonstrate the relative efficiency of each estimator under the different loss functions, providing practical insights into the applicability of each estimation approach for the ITL distribution. This analysis offers a comprehensive perspective on the versatility and precision of Bayesian and classical estimation methods for the ITL model.

This study presents a Bayesian spatial-temporal analysis for studying Dengue incidence in Tamil Nadu, aiming to provide insights into decision-making and risk assessment strategies. Statistical models that allow a more accurate depiction of true disease rates by borrowing information from neighboring regions will help mitigate the effects of sparsely populated regions and deliver better inference. Perhaps the most conspicuous manner of modeling spatial dependence is to introduce spatially associated random effects within a Bayesian hierarchical setting. The Bayesian modeling and inferential framework are flexible and extremely rich in its capabilities to accumulate various scientific hypotheses and assumptions. The spatial and spatial temporal epidemiology is concerned with the description and analysis of spatial and spatial temporal variations in disease risk with respect to risk factors. As the primary aim of this work is to quantify the spatial disease pattern of dengue incidences apart from the mapping of disease modelling the disease and finding spatial clusters/hotpots is one important aspect in epidemiology is to find the temporal trends in or outside of clusters. In this study, a spatial-temporal trends model is fitted using the Leroux CAR prior's set up for studying the spatial-temporal disease patterns with the estimation of the temporal trends with reference to dengue incidences in Tamil Nadu, India.

The special basis models have been taken from the exponential and Fréchet distributions. The parameters of the model are estimated using maximum likelihood techniques. There is a thorough analysis of percentage points. Three unique real data sets are used to demonstrate the significance of the new family. A comparison is drawn between the suggested distribution family and well-known two-, three-, and four-parameter components. To model actual data, it can be used as an alternative model to various lifetime distributions found in the statistical literature.

In many applications of multivariate statistical quality control, it is commonly observed that the number of quality characteristics exceeds the sample size. This poses significant challenges in monitoring high-dimensional data. In such conditions, it is challenging to detect sparse changes where an assignable cause leads to the deviation of only a few elements in the covariance matrix. On the other hand, the utilization of the multiple dependent state (MDS) sampling technique to enhance the sensitivity of control charts has recently attracted the attention of researchers. However, to the best of the authors' knowledge, no previous research has been conducted on equipping multivariate dispersion control charting methods with the MDS technique under high dimensionality. Therefore, this article integrates the adaptive thresholding Lasso statistic with the MDS and generalized MDS techniques to track all types of disturbances in the covariance matrix of high-dimensional processes, including diagonal, off-diagonal, and joint diagonal/off-diagonal deviations. The performance of the proposed control charts will be compared through a numerical example under seven out-of-control patterns in terms of three metrics: average, standard deviation, and median of run length. The results clearly indicate that the use of both sampling techniques significantly improves the run length properties of the adaptive thresholding Lasso chart.

This paper suggests a novel method for acceptance sampling that integrates neutrosophical fuzzy logic with the negative binomial distribution. The complexity and ambiguity that characterize real-world circumstances are typically overlooked by traditional acceptance sampling methodologies. The neutrosophic Fuzzy Acceptance Sampling Plan (NFASP) incorporates the negative binomial distribution, which is particularly well-suited for count data, to account for circumstances where defect occurrences are important. The efficacy of the methodology is demonstrated by theoretical study and simulations. This innovative method lifts acceptance sampling to a more accurate and sophisticated procedure by dealing with ambiguity and indeterminacy.

Preservation of a product is an important issue in the inventory control system. It prevents the deterioration effect of the products while these are stored in the warehouse/showroom. Considering deterioration effect of the product and preservation technology, an inventory model of non-instantaneous deteriorating items is developed with the demand dependent on the selling price of the product. Two different preservation rates are considered. Shortages are allowed partially with two different backlogging rates. Due to consideration of three-parameter Weibull distributed deterioration and preservation facility, the corresponding optimization problems are highly nonlinear. So, these problems cannot be solved analytically due to nonlinearity. To overcome this situation, different variants of quantum-behaved Artificial Gorilla Troops Optimizer (AGTO) are used. To illustrate and validate the proposed model, a numerical example is considered and solved for each case, and compared the results with the different variants of AGTO algorithms. Finally, a sensitivity analysis is performed to study the effect of changes of different parameters of the model on the optimal policy.

development of control systems through apparatus in the energy sector, taking into account modern requirements and technological possibilities. Control systems are widely used in various fields of technology, they are applied in the automation of production processes and calculations. Positive results are obtained when simulating the system using different parameter values for different types of interference signals. Management systems with the use of hardware can be successfully applied in the real working conditions of energy enterprises and can ensure optimal use of resources, reduction of operating costs and minimization of negative effects on the environment. This article discusses the characteristics of relay-contactor control systems. Relay contactor equipment controls electric drives powered by electric motors from a network with a constant voltage, which are widely used in all industries. Relay-contactor control systems are control systems built on a relay-contactor element base and designed to automate the operation of engines. With the help of such control systems, operations such as turning the engine on and off, choosing the direction and speed of rotation, starting and braking the engine, creating temporary pauses in movement, protective shutdown of the engine and stopping the mechanism are automated. These operations are necessary to perform the movement of the working body of the mechanism according to technological conditions. An electric drive, made on the basis of a relay-contactor control system, is a simple, unregulated electric drive of direct or alternating current, mainly for general industrial use, for example, electric drive of cranes, elevators, conveyors, fans, pumps, some transport devices, etc.

Tayeb Hamlat, Fatiha Mokhtari, Saadia Rahmani

The main goal of the present paper is to propose a systematic approach to model performance measurements within the context of continuous-time semi-Markov processes with a finite state space. Specifically, the mean performance is estimated using the kernel method. The uniform strong consistency and the asymptotic normality of the proposed estimator is investigated. Furthermore, a non-parametric kernel estimation of the expected cumulative operational time is addressed. The constructed estimator is proved to be consistent and to converge to a normal random variable as the time of observation becomes large. As an illustration example, a simulation study has been conducted in order to highlight the efficiency as well as the superiority of our method to the standard empirical method.

Haseeb Athar, Mohd. Amir

The Lindley distribution has been useful for fitting lifetime data. In recent times, several authors studied the extension of the original Lindley distribution. In this paper, we introduced the two general classes of distributions, which include all earlier versions of Lindley distributions. These general classes are characterized using conditional expectations of order statistics. Further, there results are applied to characterize several known distributions like Lindley, X-Lindley, power Lindley, Lindley-Pareto, Ailamujia, power Ailamujia, Lindley-Weibull, length-biased exponential, inverse Lindley, inverse power Lindley and inverted length biased exponential distributions.

Priyadharshini R, Shalini K

In a well-supervised production framework, non-conformities occur seldom, resulting in a more number of zeros in the count of non-conformities. The zero-inflated Poisson (ZIP) distribution is a suitable model for handling zero inflation. Double sampling plan (DSP) is a precise quality inspection method where a decision on the approval or rejection of a lot is made after reviewing two samples, providing stronger conclusions than single sampling plan (SSP). In practice, decision-making for submitted lots requires a consistent assessment of both within-lot and between-lot variations, which can be addressed using Bayesian methodology. A Bayesian approach integrates prior knowledge and provides more information for making decisions about the approval or rejection of a lot. This article focuses on the designing of Bayesian DSPs; employing a Gamma prior to the parameter in the Poisson component of ZIP distribution the operating characteristic (OC) function is derived. Examples are provided to assess Gamma-ZIP (GZIP) DSPs. The significance of GZIP DSPs over conventional ZIP DSPs is also presented.

Haripriya M, Radhika A, Jeslin J

This article explores into the Discrete Inverse Rayleigh Distribution, a novel discrete analogue of the continuous Inverse Rayleigh distribution, formulated by inverting a continuous Rayleigh random variable. The Discrete Inverse Rayleigh Distribution can effectively capture a range of hazard rate shapes, exhibiting either unimodal or monotonic decreasing behaviors depending on parameter values. To estimate the parameters of this distribution, we examine four distinct methods: a heuristic algorithm, a probability paper plotting technique designed for the Inverse Rayleigh, the method of moments, and the method of proportions. Each method offers unique strengths and presents different computational requirements and precision levels. Through rigorous simulation studies, we assess the accuracy and reliability of these methods, evaluating their performance across a variety of scenarios. Our results indicate that the methods of moments and proportions encounter significant difficulties when estimating parameters for right-skewed Discrete Inverse Rayleigh distributions. These challenges are primarily due to numerical instability and poor convergence properties under certain parameter configurations, which can limit their practical applicability in these cases. In contrast, both the probability paper plotting method and the heuristic algorithm demonstrate robustness and enhanced accuracy, especially in the context of right-skewed distributions. The probability paper plot is notably effective due to its reliance on graphical techniques that simplify parameter estimation in complex, non-monotonic datasets, whereas the heuristic algorithm provides a more computationally efficient solution without sacrificing precision. To validate the utility of the Discrete Inverse Rayleigh Distribution, we compare its performance with the Discrete Rayleigh Distribution by fitting both models to a real-world dataset. The comparative analysis leverages the Akaike Information Criterion (AIC) to quantitatively assess model fit. Our findings underscore the advantages of the Discrete Inverse Rayleigh Distribution, particularly in applications where discrete data exhibits non-monotonic hazard rates, highlighting its superior fit over the traditional Discrete Rayleigh in this context. This study contributes to the growing toolkit for discrete time-to-event data modeling, offering insights into effective parameter estimation strategies and demonstrating the value of the Discrete Inverse Rayleigh Distribution for specialized discrete hazard rate analysis.

Sangeetha S, Shalini K, Hemalatha R

Statistical Quality Control (SQC) involves the use of statistical techniques ... uality of a product. Acceptance sampling is a statistical technique that deter... batch of products using a sample instead of inspecting each one ind... ce and reduces the need for exhaustive inspection, balancin... es. A single sampling plan (SSP) involves inspe... the results whether the entire batch sho... , due to the evolution in technology ... acy of zeros in the count data of no... ero-inflated Binomial (ZIB) distrib... IB distribution addresses this issue ... counts for the excessive zeros, making ... onments where defectives are rare. This article ... nits follows ZIB distribution. By adopting the ZIB ... in terms of acceptance or rejection of product batches. The ... IB sampling plan is derived. Furthermore, the parameters of the SSP, s... ance number c, are derived for various sets of values, such as (p1, α, p2, β). Nu... ented to demonstrate the application of the ZIB SSPs. The risk efficiency of ZIB SSP is co... al SSP.

Dr. Komal Anadkat, Ayush Solanki, Dhruva Patel, Vraj Thakkar

Recognizing and extracting different emotions, and then validating those emotions have become important for enhancing human-computer interaction. Emotions play a crucial role in social interactions, facilitating rational decision-making and perception. Previously researched emotion recognition models have typically focused on a single input type like images, text, or audio, where each model can identify the emotion of a person through a single source like facial expressions, voice, social media posts, etc. However, these uni-model approaches are limited because they rely on just one type of data, which often misses the full range of emotional cues. To overcome these limitations, multi-model emotion recognition techniques are proposed which are useful for detecting emotions through a person's facial expressions, speech, social media status, and then EEG data. Model fusion techniques have been applied to detect the most accurate emotion for a particular person through fusion of all the models. A recognition rate-based weighting approach is proposed for model fusion, wherein models are assigned weights proportional to their individual recognition rates. This approach enhances overall performance by combining the outputs of various models with higher emphasis on those with better accuracy. The decision fusion-based multi- model emotion recognition model is proposed which achieved a maximum of 87%. accuracy using a bi-model approach and 92% accuracy with a tri-model approach. The weighted decision fusion approach assigns more weight to the model which is more accurate and achieved 93% accuracy. The proposed recognition rate-based weighting approach for fusion has provided significant results, achieving approximately 93% accuracy with 0.900 and 0.904 Cohen kappa and Mathew score respectively using facial expression, speech, and social media text modalities on combined dataset. The proposed model achieved 63% accuracy on a real-world collected dataset without considering EEG data and improved to 73% if EEG is also considered.

Nahid Mufidzade, Gulgaz Ismayilova

This article examines overvoltages at the inputs of high-voltage (HV) and low-voltage (LV) transformers rated at 110/6 kV and 110/10 kV, focusing on scenarios involving grounded and isolated neutrals during short circuits near the transformers. The study finds that with an isolated neutral, overvoltages resulting from a phase-to-ground short circuit reach the highest levels, as anticipated. However, the disconnection of all types of asymmetrical short circuits—whether with an isolated or grounded neutral—yields even greater, potentially excessive overvoltages. This occurs because the windings of undamaged transformer phases remain partially energized during disconnection, leading to significant currents being interrupted. The magnetic energy from these currents converts to electrical energy, resulting in substantial voltage increases, characterized as pulsed overvoltages lasting several microseconds. Implementing switches with shunt resistance can reduce these overvoltages considerably, though the remaining levels may still exceed acceptable thresholds. To mitigate the risk of such excessive overvoltages, installing surge arresters at the inputs of high-voltage transformers is recommended, ensuring that transformer input overvoltages remain within permissible limits.

P. Sivakumar1, V. Kaviyarasu, V. Devika

This article introduces an approach to optimize the design of Repetitive Group Sampling (RGS) plan in the context of quality control for modern manufacturing processes. The primary objective of this study is to enhance decision-making efficiency by applying Bayesian principles to develop optimal sampling plans. In modern manufacturing environment, the industries are using the advanced technologies and machineries to maintain the quality of their products. The existence of defects would consequently be highly rare in such production. In such situation, Zero Inflated Poisson (ZIP) distribution is a more appropriate probability distribution rather than the usual Poisson distribution. Further, manufacturing industries often use a variety of manpower and materials to produce their products in various

production streams. This may lead to have more quality variation in between lots and hence, the lot quality will vary over lots. The lots that arise from such a production process will be unstable, and quality variations among the units are often heterogeneous in nature. In such situation, the Bayesian sampling plans under Zero Inflated Poisson distribution would be more effective and alternative for traditional sampling plans. This paper studies the designing and selection of Bayesian Repetitive Group Sampling (BRGS) Plan under the conditions of Gamma-Zero Inflated Poisson distribution (G-ZIP). To investigate the effectiveness of this plan, a comparison between the proposed BRGS plan and various existing sample plans is made. Further, we provided the procedure and tables with the suitable numerical illustration to compute the optimal sampling plan.

Dr. V. P. Anuja

A traditional mathematical technique for analyzing line-waiting delays and overcrowding is queuing theory. It addresses the number of patrons in line as well as numerous other queue-related issues. Developing an Erlang service model in a fuzzy environment is our study ' s goal. This study aims to investigate the anticipated number of patients in the line as well as the queuing system ' s waiting time. To achieve this, we applied the L-R strategy under triangular fuzzy numbers and the alpha-cuts method. To measure various linguistic aspects in queuing systems, the fuzzy approach has been used. The findings showed that waiting times are determined using recommended techniques and that the fuzzy Erlang model is stable. Finally, we provide numerical examples to show the capabilities of the suggested method.

N.M. Piriyeva, N.S. Mammadov, S.V. Rzayeva

This article explores various methods and devices used for protecting overhead lines and substations from surges, particularly those induced by lightning strikes. Traditional surge protection methods such as lightning rods, arresters, and grounding systems are discussed, highlighting their limitations and challenges, especially in long-distribution networks. The study examines the development and implementation of novel surge protection devices, including nonlinear surge arresters, frequency-dependent devices (FDD), and multi-chamber arresters. Special attention is given to FDD, which utilizes ferromagnetic materials to create frequency-dependent resistance, effectively suppressing high-frequency overvoltages. Experimental results demonstrate the efficacy of FDD in reducing the amplitude of lightning-induced overvoltage pulses and enhancing the lightning resistance of overhead lines and substations. However, challenges such as insufficient information on device effectiveness, limited ohmic resistance at high frequencies, and size constraints hinder widespread adoption. The article concludes by emphasizing the need for further research to optimize FDD designs, increase active resistance, and assess operational effectiveness to facilitate broader deployment across different voltage classes.

Praseeja C B, Prasanth C B, C Subramanian, Unnikrishnan T

This research article attempts to establish and explore a case of two parameter Nwikpe distribution and termed it as Area Biased C2N distribution. As the characteristics of Hydrogen per Oxide(H2O2) is quite different from that of Water (H2O) even though both are the different combinations of the same elements Oxygen & Hydrogen, the characteristics of initial distribution is also entirely different from that of the area biased modified distribution. The implemented new distribution has distinct structural characteristics, and its parameters are estimating using maximum likelihood estimation. Utilizing biomedical data, the new distribution ' s application has been examining to ascertain its superiority and utility. One lifetime data set shows the mean reduction in blood glucose (mg/dL) after three days of the first usage of the Metformin medicine from a random sample of 130 patients from a hospital at Chennai, TamilNadu with type 2 diabetes mellitus by testing the FBS-Fasting Blood Glucose. The another set of lifetime data shows the mean reduction in blood glucose (mg/dL) after each dosage of the FIASP insulin-medicine in alternate days

analysis of various grounding device configurations is conducted, taking into account their resistance and the system's recovery time. The research results identify the optimal parameters of grounding devices that minimize the recovery time of power systems and enhance their overall stability. The findings can be utilized in the design and operation of power systems with improved reliability.

Many countries face problems in electricity generation. Boilers play an important role in a power plant. Sudden failures of a power plant boiler components cause loss of production and high maintenance cost. Due to unplanned and irregular maintenance, which can ultimately increase the production cost of electricity. This is a common challenge faced by power plant operators worldwide. The present study aims to examine and analyze the failure times of a boiler at a thermal power plant and identify its critical failure expectancy and system reliability. The data was collected over a long period and was analyzed using statistical methods. In this study, the hypothesis has been proposed to choose the best analysis. Furthermore, reliability, availability, and maintainability analysis were carried out under discrete analysis. The analysis included identifying the probability distribution of the failure times, identifying critical failure expectancy, and determining system reliability.

Using a pentagonal fuzzy framework, this research presents a probabilistic inventory model for deteriorating items under an uncertain demand. Degeneration of items puts a company's financial ability to meet its objectives at risk. Few models have synchronized optimization over this whole scenario with all components, according to a survey of the literature. It deals with the difficulties of inventory control in situations where demand is represented by fuzzy sets but is not precisely known. The model offers a clearer and more useful understanding of demand uncertainty by defuzzifying pentagonal fuzzy numbers using the Graded Mean Integration Representation (GMIR) approach. The goal of the study is to optimize inventory levels in order to minimize total costs, which include holding, degradation, shortage, and purchase. These components are included into a mathematical model, and numerical scenarios are shown to compare the both potential strategies. The sensitivity of the solution and decision variables with respect to different inventory characteristics is examined in both crisp and fuzzy settings. Fuzzy logic is integrated into the model to provide a strong framework for making decisions when dealing with ambiguous demand and the complications that come with deteriorating inventory. The paper includes numerical examples and sensitivity analyses to demonstrate the model™s effectiveness and practical relevance. These findings provide valuable guidance for inventory managers aiming to improve decision-making and operational efficiency in contexts with fuzzy demand and deteriorating products. At the optimal position, the total cost is relatively inelastic to an increase in base deterioration rate and more elastic to a decrease in it. Although the crisp example is marginally less efficient per unit cost, total costs are lower than in the fuzzy case, which is to be expected given the fuzzy case's potential for superior results.

The article explores the reliability and sensitivity of a power distribution substation. It includes an analysis based on real maintenance data collected from a 33/11kV electrical power distribution substation, which features a set of two 6 MVA power transformers supplying power through a total of six outgoing feeders (three feeders per transformer). The study documents faults observed in both transformers and all six outgoing feeders. The reliability of the substation is evaluated using various indices such as availability, repair durations, and expected repair frequencies for different failure types. The analysis employs Markov processes and regenerative point techniques. In addition to reliability, the study includes a profit analysis of the substation. It presents graphical representations of key parameters. Furthermore,

a sensitivity analysis is conducted to assess how variations in parameters impact the availability and profitability of the substation components. Substation economics is also established to assess the operational viability.

Several lifetime distributions have been developed in literature to handle different real-world scenario. Most of these distributions were developed to model a unimodal (symmetric or asymmetric) data. Only a hand-full of these distributions exhibits a bimodal property. This paper explores a new family of Lindley distributions featuring a bimodal property. We introduce five different sub-families of the T-Power Lindley{Y} family based on the quantile function of the uniform, exponential, Frechet, log-logistic and logistic distributions. Useful mathematical properties of the proposed T-Power Lindley{Y} family of distributions are derived and sub-models were the random variable T follows the one-parameter Topp- Leone, exponential, exponentiated exponential, Weibull and Gumbel distributions are introduced. From the graphical representation of the density function of these sub-models, we observe that the shape of the density function accommodates a decreasing (reversed-J), left-skewed, right-skewed, symmetric, as well as a bimodal shape. In order to illustrate the usefulness and performance of the proposed T-Power Lindley{Y} family of distributions, the Gumbel Power Lindley (GPL) distribution belonging to the proposed family of distribution was employed to fit a bimodal data set alongside with the beta-Normal distribution. Result obtained from the analysis revealed that the Gumbel Power Lindley (GPL) distribution compares favourably better than the beta-Normal distribution. The density fits of the distributions for the data set was also investigated to support the claim.

This paper presents a novel two-warehouse inventory model for degrading products, where the demand rate is governed over time by a generalized exponential function. Two real-world supply chain challenges that are taken into account in the model are the economic effects of inflation and partial backlog. By reducing the whole cost, which includes holding, shortage, and degradation charges, the Bacterial Foraging Optimization (BFO) method maximizes inventory management. The effectiveness of the model is validated through a comprehensive numerical example, and graphical representations demonstrate the impact of key factors on system performance. The results demonstrate how BFO may be used to complex inventory problems, giving supply chain managers crucial data as they try to balance cost-effectiveness and demand fluctuations in an inflationary environment. This approach highlights the need of advanced optimization techniques in improving decision-making processes for degrading products in a two-warehouse scenario.

The present paper deals with survival analysis of a stochastic model on cardiovascular system considering possibilities of damage, failure and recovery of heart. The analysis is based upon a stochastic model for the system considering different kinds of damage and failure of heart at different situations. The treatments and recovery of heart are taken in to account. On complete failure of heart, transplantation of the heart is also considered. The model has been analyzed by determining important measures of effectiveness using Markov process and regenerative point technique. Sensitivity analysis has also been done to select important parameters for enhancing the survivability of the system.

presents a GWO-SOMNN approach (Gray Wolf Optimization with Self-Organizing Map Neural Network) that combines Grey Wolf Optimization (GWO), Self-Organizing Maps (SOM) and Neural Networks (NN) for feature selection and classification on the UNSW-NB15 dataset. The proposed method leverages GWO to optimize feature selection, reducing the dataset ' s dimensionality and computational load, while SOM is employed for clustering and visualizing high-dimensional data. Neural Networks are then used for effective classification of network attacks. The GWO-SOMNN approach is evaluated on the UNSW-NB15 dataset, and its performance is measured in terms of 97.18% accuracy and 97.15% F1-score for binary classification and 82.41% accuracy and 78.92% F1-score for multiclass classification. The results demonstrate significant improvements over traditional methods, particularly in enhancing the classification of both binary and multi-class network attacks. These findings highlight the potential of this integrated approach in developing more efficient and accurate network intrusion detection systems.

In this paper, we analyze a single server markovian queueing model with encouraged arrivals that undergoes a single working vacation. Additionally, we consider the impatience and reneging behavior of customers in the queue during the working vacation period. Customers arrive at the system following a Poisson distribution. The server goes on vacation when the system is empty and stays on vacation for a random period that follows an exponential distribution. During the working vacation period, the server continues to provide service at a slower rate. After the vacation, the server returns to the regular service period and continues providing service at the regular busy period rate if there are one or more customers in the system, or it remains idle until a new customer arrives. During the working vacation, customers in the queue become impatient and renege from the system, with the reneging time assumed to follow an exponential distribution. The system is characterised as a quasi-birth-death process, and the stationary probabilities are derived using the probability generating function method. Some numerical analysis is also carried out to show the effect of encouraged arrivals on performance measures.

Errors in design and construction critically undermine the structural reliability of industrial buildings, putting property, the environment, and human safety at risk. In this regard, the present research work is intended to investigate how such mistakes influence the performance of the main structural components and the stability of steel industrial buildings. Detailed finite element analysis was performed using DIANA FEA for solid modeling and SAP2000 for beam modeling to assess global structural performance. This includes, among others, the insufficiency of local reinforcement in compressed members and eccentricity in column connections. It was performed to analyze the local and global buckling behaviors, deviations in symmetry, and inefficiency of the bracing systems. Consequently, it reveals a significant reduction in load-bearing capacity due to reinforcement deficiencies in the compressed elements and eccentricity, while a structural loss in integrity becomes highly significant at symmetry deviations, especially in horizontal loads. This study provides critical insights into mitigating design and construction errors to enhance the reliability of industrial steel buildings.

Modern engineering systems increasingly focus on multi-objective optimization. Nature-inspired optimization techniques have shown superior efficiency and effectiveness compared to many traditional methods across various parameters. This work demonstrates the reliability and cost optimization of a complex bridge system using the Multi-Objective Grey Wolf Optimization algorithm (MOGWO). The bridge system in question is a series-parallel system. A key performance highlight is the use of an archive for search agents to generate a Pareto optimal front (PoF) with a

minimal number of iterations. Among the various solutions in the PoF, the solution set that best meets the multi-objective criteria is preferred. Additionally, statistical analyses are conducted to further validate the competitiveness of the results.

In this manuscript, we propose a new extension of the Rayleigh distribution, named as Ratio Transformation Rayleigh Distribution (RTRD), which offers superior fits compared to the Rayleigh distribution and several of its known generalizations. We derive various properties of the proposed distribution, including moments, moment generating function, hazard rate, conditional moments, Bonferroni and Lorenz curves, mean residual life, mean waiting time, Renyi entropy and order statistics. The unknown parameters are estimated using the maximum likelihood estimation procedure. An extensive simulation study is conducted to illustrate the behavior of the maximum likelihood estimators (MLEs) based on Mean Square Errors. The flexibility of the new distribution is demonstrated by applying it to two real data sets. Comparative analysis with the Rayleigh distribution, Weighted Rayleigh distribution, Exponentiated Rayleigh distribution and Transmuted Rayleigh distribution reveals that RTRD outperforms these competing distributions based on Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion Corrected (AICC) and other goodness of fit measures.

In this paper, we introduce the Marshall–Olkin extended Shanker distribution, as an extension of the Shanker distribution, using the Marshall-Olkin approach. Several important properties of the new distribution, such as the hazard rate function, moments, incomplete moments, mean deviations, Lorenz and Bonferroni curves, and Rényi entropy are explored. The estimation of the parameters is discussed with the help of the maximum likelihood method. The performance of the estimators is evaluated using a simulation study. Two real data applications are developed in order to assess the flexibility and power of the new distribution. The goodness of fit criteria reveal that the new model may provide a better fit than the Shanker distribution and other competing models that belong to the Marshall-Olkin G family of distributions.

This article examines the reliability and risks associated with technical systems involved in the conversion of mechanical energy to electrical energy, focusing on the thermal dynamics of electric machines. It explores the processes of heat generation due to energy losses, primarily heat dissipation, and the effects of temperature increases on the longevity and performance of the machine. The cooling systems essential for managing heat transfer and minimizing overheating are analyzed, considering factors such as heat conduction, convection, and radiation, as well as the role of electrohydraulic and aerodynamic systems in optimizing heat exchange. Special attention is given to the impact of temperature fluctuations on the insulation materials of electric machines, with an emphasis on how overheating accelerates insulation degradation and reduces machine lifespan. The paper further discusses the intricate relationship between cooling efficiency, machine power, and the economic implications of designing effective thermal management systems. Moreover, the challenges of selecting and optimizing cooling strategies in electric machine design are highlighted, considering both technical and economic factors. Lastly, the study delves into ventilation calculations necessary to ensure efficient airflow and cooling, using practical equations and methods for determining pressure loss and fan performance, underscoring the complexity and importance of achieving optimal temperature conditions for long-term, reliable machine operation.

past residual life function are employed to analyze engineering and reliability aspects in informometric data. Truncation in probability distributions, particularly the doubly truncated distribution, is explored, revealing its broad applicability. The relationship between the Leimkuhler curve and truncated distributions will also be examined.

S.A. Khanahmedova

The paper analyzes the mechanical part of the design scheme of the electric drive of a hybrid electric machine, which is a key stage in the design and research of automatic control systems. The main elements of a mechanical system, a model of a real mechanical system connected to an electric drive, including all moving elements, transmission mechanisms, and actuators that convert electrical energy into mechanical work, are considered. The presented calculation scheme allows you to analyze dynamic processes, i.e. to study the system's behavior over time, to determine stability, fluctuations, and other characteristics. Calculations of various mass systems are performed using the capabilities of the MATLAB/Simulink software package for a three-mass and two-mass system. These models can be used for different systems with different parameters. To draw up a structural diagram, the elements of the mechanical part and the connections between the elements, the types of these connections (rigid, elastic) and the directions of motion transmission are determined. Structural diagrams are used to analyze the dynamic characteristics of the system, determine transients, stability, and vibrations.

B. Muniswamy, M. V. Lavanya

The number of children ever born is an important measure for understanding fertility patterns, which impact demographic structures and population growth. The problem relates to the modeling of count data that includes the truncation of zero values, specifically focusing on women who have experienced childbirth at least once. This study analyzes the factors that influence the number of children ever born (CEB) among women aged 15 to 50 in Andhra Pradesh, utilizing data from the National Family Health Survey (NFHS-5) conducted from 2019 to 2021. The study used Zero-Truncated Poisson (ZTP) and Zero-Truncated Generalized Poisson (ZTGP) models to identify major determinants, including religion, kind of cooking fuel used, place of delivery, wealth, age, and fertility choices. The ZTP regression model was found to be the best model and identifies significant determinants such as religion, wealth, age, and fertility preferences. The results show that rural residence, Muslim faith, and older age groups are associated with higher CEB, while wealthier women tend to have fewer children. The study shows the importance of implementing focused reproductive health activities, specifically in rural regions, to manage population growth and enhance the health outcomes of both mothers and children.

R. Keerthana

An M/G/1 retrial queue with working vacation, non persistent customers and a waiting server is taken into consideration in this study. Both retrial times and service times are assumed to follow general distribution and the waiting server follows an exponential distribution. Before switching over to a vacation the server waits for some arbitrary amount of time and so is called a waiting server. During the working vacation period customers are served at a lesser rate of service. We obtain the PGF for the number of customers and the mean number of customers in the invisible waiting area which is acquired by utilizing the supplementary variable technique. We compute the waiting time distribution. Out of interest a few special cases are conferred. Numerical outcomes are exhibited.

This article constructs a Repetitive Sampling Inspection Plan under Truncated life test (RSIPTL) when the lifetime follows the One Parameter Polynomial Exponential (OPPE) family of distributions. In RSIPTL, a lot can be accepted or rejected in the first, second, and so on, based on the number of defective items in each sample. The OPPE has infinite support. It has transformed into its unit form to utilize finite support, i.e., having the support (0, 1). The Lindley distribution, a particular choice of the OPPE, has been studied in detail. We obtained the minimum number of items required in a lot to satisfy the consumer risk. Extensive tables are prepared for easy understanding and use of the plan for industrial workers. The RSIPTL is compared with a single sampling plan (SSP) and a two-stage reliability acceptance sampling plan (TSRASP) for Lindley and Exponential distributions. Two data sets are discussed and comparative statements are made with respect to the proposed plan.

IN MEMORY OF YU. BELYAEV

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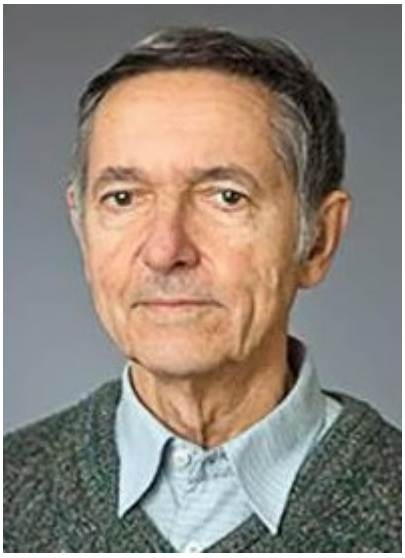
Dear Colleagues,

The Gnedenko Forum is deeply saddened to announce the death, at the age of 93 of one of the leading experts in probability theory, mathematical statistics and their applications, Doctor of Physical and Mathematical Sciences, Professor, Laureate of the State Prize of the USSR



YURI KONSTANTINOVICH BELYAEV
(31 August 1932, Moscow - 22 January 2025, Umeå, Sweden)

Yu.K. Belyaev On the memory of Prof. Yuri K. Belyaevwas born in Moscow. From 1951 to 1956 he was a student at the Department of Mathematics of the Faculty of Mechanics and Mathematics of the Lomonosov Moscow State University. From 1956 to 1959 he was a postgraduate student at the Mathematical Institute of the USSR Academy of Sciences (MIAN). During his postgraduate studies, on a topic suggested by his supervisor A.N. Kolmogorov, he studied the properties of the trajectories of Gaussian random processes. In 1960 he defended his PhD thesis on this subject at MIAN.

After this doctorate Yu.K. Belyaev started work at the Lomonosov Moscow State University. He began a long period (until the mid-1990s) of scientific collaboration with B.V. Gnedenko and A.D. Soloviev. They focused on the development of reliability theory, statistical methods of quality control, queueing theory, reliability models, on organization of conferences and seminars at MSU. When the "Reliability Cabinet" was established at the Polytechnic Museum in Moscow, Y.K. Belyaev lectured there on statistical methods for analyzing reliability test data.

The joint work with B.V. Gnedenko and A.D. Soloviev on the book "Mathematical Methods of Reliability Theory"; established the main directions in the study of statistical problems in reliability theory and statistical quality control.

In parallel, Y.K. Belyaev continued his studies on the properties of point processes. In this relation Yu.K. Belyaev selected, translated, and edited the famous monograph (written by D. Cox and W. Smith) on the renewal theory, which played a remarkable key role in the development, wide dissemination and application of this theory by Soviet specialists and those, familiar with the Russian language. He also supplemented this monograph with an excellent complete review of the modern (at that time) results of the renewal processes.

In 1970, for getting the degree of Doctor of Physical and Mathematical Sciences he defended a respective doctoral thesis at the Institute of Applied Mathematics (IAM) of the USSR Academy of Sciences. The thesis was on random point processes generated by random fields. Under his supervision, 24 doctoral students successfully defended their theses.

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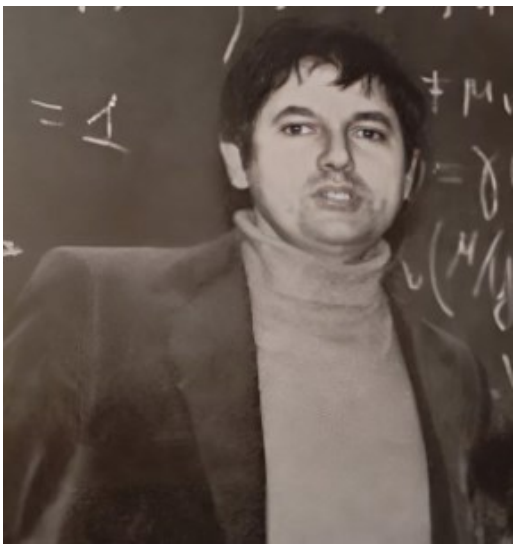

When I graduated from MIPT, V.V. Kalashnikov invited me to speak at a seminar on probability theory and mathematical statistics at Moscow State University. Kalashnikov invited me to speak at a seminar on probability theory and mathematical statistics at MSU. This seminar was held at MSU by Acad. A.N. Kolmogorov and at that time it was chaired by B.V. Gnedenko, A.D. Soloviev and Y.K. Belyaev. A special feature of this seminar was the close connection between probability theory and its applications in queueing theory, reliability theory and mathematical statistics. I was very lucky that the leaders of the seminar played an important role in my future scientific work in Vladivostok and actually gave me an admonition to work in science and to do what I was and still am interested in. This is how B.V. Gnedenko came to the Far Eastern Mathematical School with a number of very interesting papers. A.D. Solovyov acted as my opponent in my doctoral thesis. Y.K. Belyaev was remembered for his very interesting work on mathematical statistics. I remember how, when I was already working in Sweden, Y.K. Belyaev presented his original and very informative results on the application of mathematical statistics in ecology at an international conference on probability theory at Moscow State University. I remember some articles on the queueing theory and reliability published by Y.K. Belyaev in the journal "Technical Cybernetics". Y.K. Belyaev worked closely with I.A. Ushakov in solving some problems of reliability theory. As far as I can judge, Yu.K. Belyaev played a great role in the organisation of the journal 'Reliability: Theory and Applications'. Already while working in Vladivostok, I had the opportunity to see how some of Yu.K. Belyaev's results on asymptotic methods of probability theory echoed the work of V.M. Zolotarev on limit theorems of probability theory, which fascinated me. For me personally, Y.K. Belyaev played an important role in that group of remarkable specialists in probability theory who, in my opinion, formed the backbone of the probabilists on whom we, then young scientists working even far from Moscow, tried to orient ourselves and rely.

Gurami Tsitsiashvilli

Y.K. Belyaev founded and was the head of the Department of Reliability and Queueing Theory of the University Laboratory of Statistical Methods at MSU. Later this department was transformed into the Laboratory of Probability and statistics at the Department of Probability Theory of the Faculty of Mechanics and Mathematics of MSU. Y.K. Belyaev was a member of the editorial boards of the scientific journals "Technical Cybernetics"; and "Statistics". Y.K. Belyaev is one of the authors of the book "Questions of the Mathematical Theory of Reliability", published under the supervision of B.V. Gnedenko IN 1983.

The rapid development of computational techniques and the numeric methods reasonably created a need for new approaches in applications. For solving problems of estimating the accuracy of statistical inference, like study of distributions of point estimates, the biases, various classifiers, selection of the regression function in the presence of explanatory variables, etc. Y.K. Belyaev gave a theoretical justification of the validity of options for intensive use of computations in solving several such problems. Professor Belyaev has been invited to teach and conduct research at the Universities of Berlin and Magdeburg (Germany), Sofia (Bulgaria), Lund and Umeå (Sweden). Since 1993 he has been a professor at Umeå University where he supervised more than 30 doctoral students.

Y.K. Belyaev is a laureate of the USSR State Prize, a member of the International Statistical Institute (ISI) and the Institute of Mathematical Statistics (IMS). He is the author of more than 200 scientific publications, including 5 books. Since the establishment of the Gnedenko Forum, Yuri K. Belyaev has been its unchanging Honorary President. It is largely thanks to his support and attention that the Forum has taken a recognized role and has been active for more than 20 years.

The name of Yuri Konstantinovich Belyaev will remain in the history of the Probability, Statistics and its numerous applications.

He stays in the memory of all those who knew him and worked with him.

Honor to his blessed memory.

Gnedenko Forum Advisory Board

A NEW GENERALIZED EXPONENTIATED FAMILY OF CONTINUOUS DISTRIBUTIONS WITH APPLICATIONS TO ENVIRONMENTAL DATA SETS

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ABSTRACT

Different researchers in the field of distribution theory have derived new models for generalizing the classical ones to make them more flexible and to aid their application in various fields. This generalization and extension of the classical models is mostly done using families of distributions. This article presents a new family of distributions called the Exponentiated Pareto-G family of distributions with two positive shape parameters. Some statistical properties of the new family of distributions, such as explicit expressions for the quantile function, probability-weighted moments, moments, generating function, Reliability function, hazard function, and order statistics are discussed. A maximum likelihood estimation technique is employed to estimate the model parameters. Two submodels such as Weibull and Frechet distributions are employed to check the fit of the family of distributions with the aid of their pdf and hazard function plots. Also, a simulation study is presented to assess the performance of the maximum likelihood estimator. Furthermore, two real-life applications are carried out to assess the fit and flexibility of the new family using the Weibull model as the baseline. The results showed that the new distribution fits better in the two real data sets considered among the range of distributions considered.

Keywords: Exponentiated Pareto-G, maximum flood level, precipitation, consistent, flexibility

I. INTRODUCTION

Research in the field of statistical distribution theory has increased tremendously in the past few years and still growing rapidly. Different researchers in the field of distribution theory have

derived new models for generalizing the classical ones to make them more flexible and to aid their application in various fields. This generalization and extension of the classical models is mostly done using families of distributions. These families of distributions developed have aided the fit of many classical distributions with the addition of extra parameters to the baseline distributions.

Several considerations motivate the development of new generalized families of distributions. More adaptable, flexible, and robust models are required since current distributions frequently fall short of capturing the variability and patterns found in modern data. New distributions that can explain high-dimensional data and adjust to different contexts become crucial as data dimensionality and complexity rise. By offering a more accurate representation of the underlying processes, such distributions improve the resilience and accuracy of statistical analysis. Some of the well-known recently proposed modified families of distributions in the literature by different researchers to improve the standard theoretical distribution and also add flexibility to the classical distributions are: the new generalized family of distributions by [2], Topp-Leone Kumaraswamy-G family of distributions by [12], Topp-Leone Exponentiated-G family of distributions by[11], Rayleigh-exponentiated odd generalized-*X* family of distributions by [17], Type I half-logistic exponentiated-G family of distributions by [6], new generalized family of distributions by [15], Exponentiated type II generalized Topp-Leone-G family of distributions by [1].

II. THE EXPONENTIATED PARETO-G FAMILY (ETP-G) OF DISTRIBUTIONS

A new two-parameter distribution, called the exponentiated Pareto distribution introduced by [9] with cdf and pdf given as

$$F(x;\tau,\rho)=\left[1-\left(1+x\right)^{-\tau}\right]^{\rho} \tag{1}$$

$$f(x;\tau,\rho)=\tau\rho\left(1+x\right)^{-\tau-1}\left[1-\left(1+x\right)^{-\tau}\right]^{\rho-1} \tag{2}$$

According to [3], the cdf of the T-X family of distribution is given as

$$F\left(x\right)=\int_{a}^{W[G(x)]} r(t)dt=R\left[W\left[G\left(x\right)\right]\right] \tag{3}$$

Where $W\left[G(x)\right]$ satisfies the following conditions

(i) $W\left[G(x)\right]\in\left[a,b\right]$

(ii) $W\left[G(x)\right]$ is differentiable and monotonically non-decreasing, and

(4)

(iii) $W\left[G(x)\right]\to a$ as $x\to$ -∞ and $W\left[G(x)\right]\to b$ as $x\to\infty$

Let $r(t)$ be the pdf of a random variable $T\in\left[c,d\right]$ for $-\infty\le c<d<\infty$ and $W\left[G(x)\right]$ be a function of the cdf of a random variable X .

Then the pdf corresponding to equation (3) is given by;

$$f(x)=\left\{\frac{d}{dx}W\left[G(x)\right]\right\}r\{W\left[G(x)\right]\} \tag{4}$$

Proposition 1:

Let $G\left(x;\xi\right)$ be the cdf of any arbitrary random variable X . Also, let $T\in\left(c,d\right)$ be a random variable with a pdf, $r(t)$. Furthermore, let our proposed link function be given as $G(x)$, using the expoenentiated Pareto distribution as the generator, then the cdf of Exponentiated Pareto-G family of distributions is given as:

$$F(x;\tau,\rho,\zeta)=\left[1-\left(1+G(x;\zeta\right)^{-\tau}\right]^{\rho} \tag{5}$$

Proof:

$$F(x;\tau,\rho,\zeta)=\tau\rho\int_{0}^{G(x;\zeta)}\left(1+t\right)^{-\tau-1}\left[1-\left(1+t\right)^{-\tau}\right]^{\rho-1}\partial t$$

Let $y=1+t$, when $t=0,y=1$ and when $t=G(x;\zeta),y=1+G(x;\zeta)$

So, $\partial t=\partial y$

Now,

$$F(x;\tau,\rho,\zeta)=\tau\rho\int_{1}^{1+G(x;\zeta)}y^{-\tau-1}\left[1-y^{-\tau}\right]^{\rho-1}\partial y \tag{6}$$

From equation (6),

Let $k=1-y^{-\tau}$, when $y=1,k=0$ and when $y=1+G(x;\zeta),k=1-\left(1+G(x;\zeta)\right)^{-\tau}$

$\partial y=\frac{\partial k}{-\rho y^{-\rho-1}}$

So,

$$F(x;\tau,\rho,\zeta)=\tau\rho\int_{0}^{1-(1+G(x;\zeta))^{-\tau}}y^{-\tau-1}k^{\rho-1}\frac{\partial k}{\rho y^{-\rho-1}}$$

$$F(x;\tau,\rho,\zeta)=\rho\int_{0}^{1-(1+G(x;\zeta))^{-\tau}}k^{\rho-1}\partial k$$

$$F(x;\tau,\rho,\zeta)=\rho\int_{0}^{1-(1+G(x;\zeta))^{-\tau}}\frac{k^{\rho}}{\rho}\partial k$$

$$F(x;\tau,\rho,\zeta)=\rho\left[\frac{k^{\rho}}{\rho}\right]_{0}^{1-(1+G(x;\zeta))^{-\tau}}$$

$$F(x;\tau,\rho,\zeta)=\left[k^{\rho}\right]_{0}^{1-(1+G(x))^{-\tau}}$$

$$F(x;\tau,\rho,\zeta)=\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho},\ 0\le x\le\infty \tag{7}$$

Where $\tau,\rho>0$ are the shape parameters and $\zeta>0$ is a vector of parameters depending on the baseline distribution used.

The pdf to equation (7) is given as

$$f(x;\tau,\rho,\zeta)=\tau\rho g(x;\zeta)\left(1+G(x;\zeta)\right)^{-\tau-1}\left[1-\left(1+G(x;\zeta)\right)^{-\tau}\right]^{\rho-1} \tag{8}$$

III. EXPANSION OF DENSITY

This section presents the densities expansion which will be used to estimate some of the distributions properties.

$$\left(1+Z\right)^{-\omega}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\omega+i-1}{i}z^{i} \tag{9}$$

$$1-Z^{\ \omega}=\sum_{j=0}^{\infty}-1^{\ j}\binom{\omega}{j}Z^{j} \tag{10}$$

For $|z|<1$ and ω is a positive real non integer.

Applying equation (9) and equation (10) on the last term in equation (8), we have

$$\left[1-\left(1+G(x;\zeta)\right)^{-\tau}\right]^{\rho-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\rho-1}{i}\left[1+G(x;\zeta)\right]^{-\tau i}$$

$$\left[1+G(x;\zeta)\right]^{-\tau(i+1)-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\tau(i+1)+j-2}{j}\left[G(x;\zeta)\right]^{j}$$

$$f(x;\tau,\rho,\zeta)=\tau\rho g(x;\zeta)\sum_{i,j=0}^{\infty}\ (-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}\left[G(x;\zeta)\right]^{j} \tag{11}$$

In this vain, using equation (9) and equation (10) on equation (6), we have

$$\left[F(x;\tau,\rho,\zeta)\right]^{h}=\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho h}$$

$$\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho h}=\sum_{k=0}^{h}(-1)^{k}\binom{\rho h}{k}\left[1+G(x;\zeta)\right]^{-\tau k}$$

$$\left[1+G(x;\zeta)\right]^{-\tau k}=\sum_{d=0}^{\infty}(-1)^{d}\binom{\tau k+d-1}{d}\left[G(x;\zeta)\right]^{d}$$

$$\left[F(x;\tau,\rho,\zeta)\right]^h=\sum_{d=0}^{\infty}\sum_{k=0}^{h}(-1)^{d+k}\binom{\rho h}{k}\binom{\tau k+d-1}{d}\left[G(x;\zeta)\right]^d \tag{12}$$

IV. PROPERTIES OF ETP-G

I. PROBABILITY WEIGHTED MOMENTS (PWMS)

$$\xi_{r,s}=E\left[X^r F(X)^s\right]=\int_0^{\infty}x^r f(x)(F(x))^s dx \tag{13}$$

The PWMs of EtP-G is derive by substituting equation (11) and equation (12) into equation (13) by replacing *h* with *s*, we have

$$\xi_{r,s}=\int_0^{\infty}\tau\rho\sum_{i,j,d=0}^{\infty}\sum_{k=0}^{s}(-1)^{i+j+d+k}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}\binom{\rho s}{k}\binom{\tau k+d-1}{d}x^r g(x;\zeta)\left[G(x;\zeta)\right]^{d+j}dx \tag{14}$$

II. MOMENTS

$$E(X^r)=\int_0^{\infty}x^r f(x)dx \tag{15}$$

The r^{th} moments for EtP-G distribution is derive by substituting equation (11) into equation (15) to obtain

$$E(X^r)=\int_0^{\infty}x^r\tau\rho\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}g(x;\zeta)\left[G(x;\zeta)\right]^j dx \tag{16}$$

III. MOMENT GENERATING FUNCTION (MGF)

The Moment Generating Function of x is given as

$$M_x(t)=\int_0^{\infty}e^{tx}f(x)dx \tag{17}$$

The MGF for EtP-G distribution is derive by substituting equation (11) into equation (17) we obtain

$$M_x(t)=\int_0^{\infty}\tau\rho\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}e^{tx}g(x;\zeta)\left[G(x;\zeta)\right]^j dx \tag{18}$$

where the expansion of $e^{tx}=\sum_{z=0}^{\infty}\frac{t^z x^z}{z!}$ and following the process of moments above, we have the MGF for EtP-G distribution in equation (18) given as

$$M_x(t)=\int_0^{\infty}\tau\rho\sum_{i,j=0}^{\infty}\sum_{q=0}^{\infty}\frac{t^z}{z!}(-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}x^z g(x;\zeta)\left[G(x;\zeta)\right]^j dx \tag{19}$$

IV. RELIABILITY FUNCTION

$$R(x;\tau,\rho,\zeta)=1-\left[1-\left(1+G(x;\zeta\right)^{-\tau}\right]^{\rho} \tag{20}$$

V. HAZARD FUNCTION

$$T(x;\tau,\rho,\zeta)=\frac{\tau\rho g(x;\zeta)\left(1+G(x;\zeta)\right)^{-\tau-1}\left[1-\left(1+G(x;\zeta)\right)^{-\tau}\right]^{\rho-1}}{1-\left[1-\left(1+G(x;\zeta\right)^{-\tau}\right]^{\rho}} \tag{21}$$

VI. QUANTILE FUNCTION

$$\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho}=U$$

$$1-(1+G(x;\zeta))^{-\tau}=U^{\frac{1}{\rho}}$$

$$1-U^{\frac{1}{\rho}}=\left[1+G(x;\zeta)\right]^{-\tau}$$

$$\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}=1+G(x;\zeta)$$

$$G(x;\zeta)=\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1$$

$$x=Q(u)=G^{-1}\left[\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1\right] \tag{22}$$

VII. ORDER STATISTICS

$$f_{r:n}(x)=\frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}F(x)^{v+r-1} \tag{23}$$

The pdf of r^{th} order statistic for distribution is obtained also replacing h with v+r-1 in cdf expansion, we have

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}\sum_{i,j,d=0}^{\infty}\sum_{k=0}^{v+r-1}\sum_{v=0}^{n-r}(-1)^{v+i+j+d+k}\binom{\rho(v+r-1)}{k}\binom{\tau k+d-1}{d}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}\binom{n-r}{v}\left[G(x;\zeta)\right]^{j+d+v+r-1} \tag{24}$$

The pdf of the minimum order statistic of the EtP-G distribution is obtained by setting r=1 in equation (24)

$$f_{1:n}(x) = n \sum_{i,j,d=0}^{\infty} \sum_{k=0}^{v} \sum_{v=0}^{n-1} (-1)^{v+i+j+d+k} \binom{\rho v}{k} \binom{\tau k + d - 1}{d} \binom{\rho - 1}{i} \binom{\tau(i+1) + j - 2}{j} \binom{n-1}{v} [G(x;\zeta)]^{j+d+v} \tag{25}$$

Also, the pdf of the maximum order statistic of the distribution is obtained by setting r = n in equation (24)

$$f_{n:n}(x) = n \sum_{i,j,d=0}^{\infty} \sum_{k=0}^{v+n-1} (-1)^{v+i+j+d+k} \binom{\rho(v+n-1)}{k} \binom{\tau k + d - 1}{d} \binom{\rho - 1}{i} \binom{\tau(i+1) + j - 2}{j} [G(x;\zeta)]^{j+d+v+n-1} \tag{26}$$

VIII. MAXIMUM LIKELIHOOD ESTIMATION

This section explores the maximum likelihood estimation (mle) technique to estimate the unknown parameters of the EtP-G distribution. Let $x_1, x_2, ..., x_n$ be a random sample of size n from the EtP-G distribution. Then, the likelihood function based on observed sample for the vector of parameter $(\tau, \rho, \zeta)^T$ is given by

$$\log L = n \log \tau + n \log \rho + \sum_{i=1}^{n} \log g(x_i;\zeta) - \tau - 1 \sum_{i=1}^{n} \log 1 + G(x;\zeta) + \rho - 1 \sum_{i=1}^{n} \log \left[1 - 1 + G(x;\zeta)^{-\tau} \right] \tag{27}$$

The components of score vector $U = U_\tau, U_\rho, U_\zeta$ are given as

$$U_\tau = \frac{n}{\tau} - \sum_{i=1}^{n} \log 1 + G(x;\zeta) + \rho - 1 \sum_{i=1}^{n} \frac{1 + G(x;\zeta)^{-\tau} \log 1 + G(x;\zeta)}{\left[1 - 1 + G(x;\zeta)^{-\tau} \right]} = 0 \tag{28}$$

$$U_\rho = \frac{n}{\rho} + \sum_{i=1}^{n} \log \left[1 - 1 + G(x;\zeta)^{-\tau} \right] = 0 \tag{29}$$

$$U_\zeta = \sum_{i=1}^{n} \left[\frac{g(x_i;\zeta)^{\zeta}}{g(x_i;\zeta)} \right] - \tau - 1 \sum_{i=1}^{n} \left[\frac{G(x;\zeta)^{\zeta}}{1 + G(x;\zeta)} \right] + \rho - 1 \sum_{i=1}^{n} \left[\frac{\tau \, 1 + G(x;\zeta)^{-\tau-1} G(x;\zeta)^{\zeta}}{\left[1 - 1 + G(x;\zeta)^{-\tau} \right]} \right] \tag{30}$$

Equations (28), (29) and (30) cannot be solved analytically, so we have to resort to numerical method to estimate the unknown parameters.

V. SUB MODELS

I. EXPONENTIATED PARETO-WEIBULL (ETPW) DISTRIBUTION

The cdf and pdf of the Weibull distribution are given as

$$G(x;\theta,\beta)=1-e^{-(\theta x)^{\beta}} \tag{31}$$

$$g(x;\theta,\beta)=\theta\beta^{\theta}x^{\beta-1}e^{-(\theta x)^{\beta}} \tag{32}$$

Where $x\geq 0,\theta,\beta>0$.

The cdf for ETPW distribution is obtained by inserting equation (31) into equation (7) as

$$F(x;\tau,\rho,\theta,\beta)=\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho} \tag{33}$$

And the pdf for ETPW distribution is obtained by differentiating equation (33) with respect to x as

$$f(x;\tau,\rho,\theta,\beta)=\tau\rho\theta\beta^{\theta}x^{\beta-1}e^{-(\theta x)^{\beta}}\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau-1}\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho-1} \tag{34}$$

Where $x\geq 0,\tau,\rho,\theta,\beta>0$

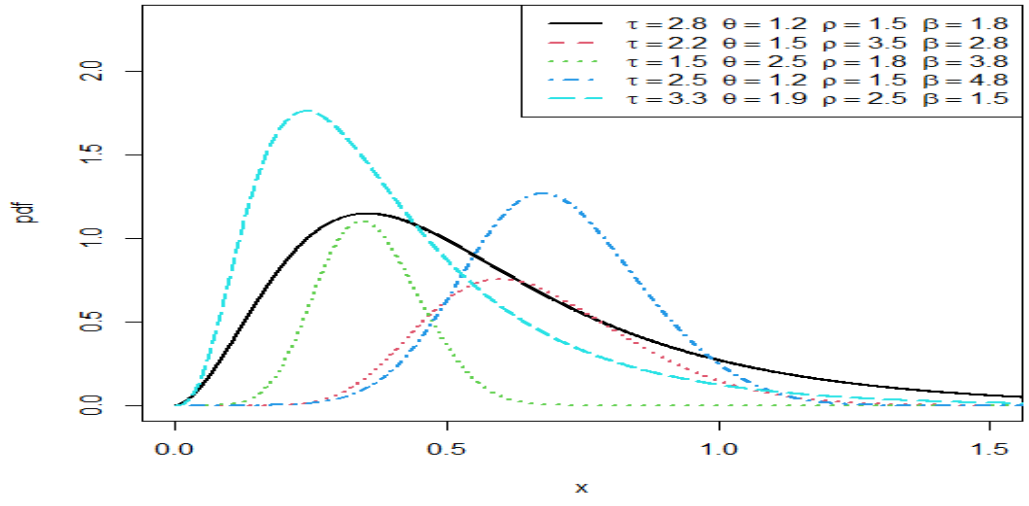

Figure 1: *Plots of pdf of ETPW distribution with different parameter values*

Reliability function for the ETPW distribution is given as

$$R(x;\tau,\rho,\theta,\beta)=1-\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho} \tag{35}$$

Hazard function for the ETPW distribution is given as

$$T(x;\tau,\rho,\theta,\beta)=\frac{\tau\rho\theta\beta^{\theta}x^{\beta-1}e^{-(\theta x)^{\beta}}\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau-1}\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho-1}}{1-\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho}} \tag{36}$$

Quantile function for the ETPW distribution is given as

$$x=Q(u)=\left[\frac{-1}{\theta}\log\left[1-\left[\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1\right]\right]\right]^{\frac{1}{\beta}} \tag{37}$$

II. EXPONENTIATED PARETO-FRECHET (ETPFr) DISTRIBUTION

The Frechet distribution's cdf and pdf are provided as

$$G(x;\theta,\delta)=e^{-(\frac{\theta}{x})^{\delta}}\quad,\quad x>0,\theta,\delta>0 \tag{38}$$

$$g(x;\theta,\delta)=\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\quad,\quad x>0,\theta,\delta>0 \tag{39}$$

The cdf for ETPFr distribution is given as

$$F(x;\tau,\rho,\theta,\delta)=\left[1-\left[1+e^{-(\frac{\theta}{x})^{\delta}}\right]^{-\tau}\right]^{\rho},x>0,\tau,\rho,\theta,\delta>0 \tag{40}$$

The pdf for ETPFr distribution is given as

$$f(x;\tau,\rho,\theta,\delta)=\tau\rho\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\left[1+e^{-(\frac{\theta}{x})^{\delta}}\right]^{-\tau-1}\left[1-\left[1+e^{-(\frac{\theta}{x})^{\delta}}\right]^{-\tau}\right]^{\rho-1},x>0,\tau,\rho,\theta,\delta>0 \tag{41}$$

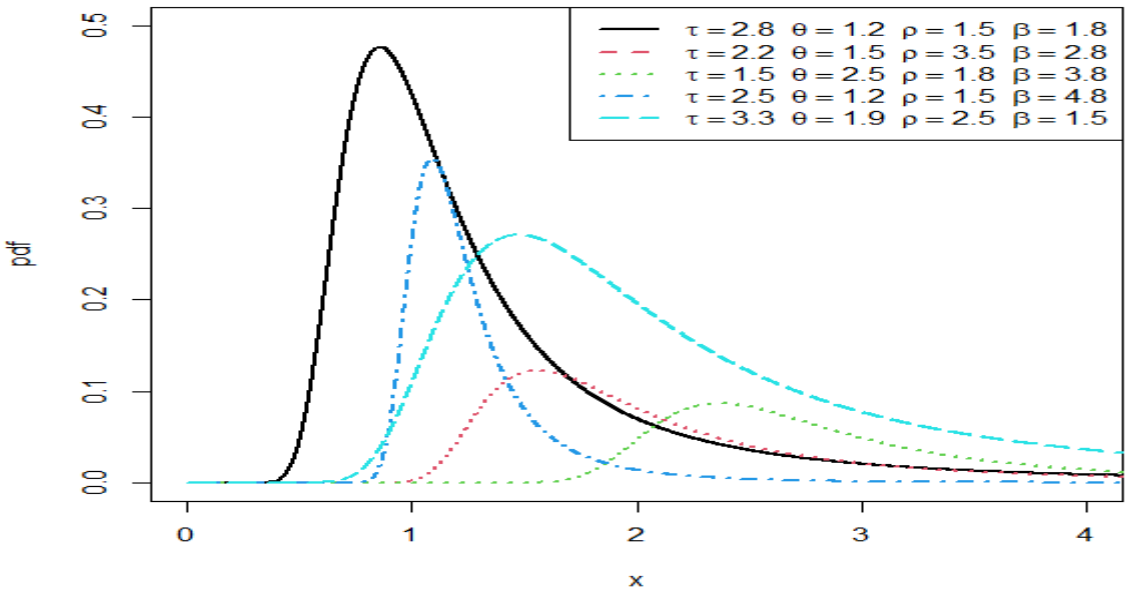

Figure 2: *Plots of pdf of ETPFr distribution with different parameter values*

Reliability function of the ETPFr distribution is given as

$$R(x;\tau,\rho,\theta,\beta)=1-\left[1-\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau}\right]^{\rho} \tag{42}$$

Hazard function of the ETPFr distribution is given as

$$T(x;\tau,\rho,\theta,\beta)=\frac{\tau\rho\beta\theta^{\beta}x^{-\beta-1}e^{-(\frac{\theta}{x})^{\beta}}\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau-1}\left[1-\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau}\right]^{\rho-1}}{1-\left[1-\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau}\right]^{\rho}} \tag{43}$$

Quantile function of the ETPFr distribution is given as

$$x=Q(u)=\frac{\theta}{\left[-\log\left[\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1\right]\right]^{\frac{1}{\beta}}} \tag{44}$$

VI. SIMULATION STUDY

This section addresses a numerical analysis to evaluate the performance of MLE for ETPW Distribution.

Table 1: *MLEs, biases and RMSE for some values of the parameters of ETPW distribution*

		(25,2,6,5)			(27,4,8,6)		
N	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
	τ	25.7129	0.7129	3.7808	27.7166	0.7166	4.1678
20	ρ	2.7524	0.7524	2.0073	5.5891	1.5891	4.0788
	θ	6.7789	0.7789	2.2489	9.1209	1.1209	2.8826
	β	5.0041	0.0041	0.4382	6.0493	0.0493	0.4292
	τ	25.5242	0.5242	3.1134	27.4919	0.4919	3.2601
50	ρ	2.2234	0.2234	0.9125	4.5111	0.5111	1.8794
	θ	6.4049	0.4049	1.5416	8.5580	0.5580	1.7238
	β	5.0226	0.0226	0.2663	6.0445	0.0445	0.2728
100	τ	25.3816	0.3816	2.3289	27.4562	0.4562	2.2990
	ρ	2.0852	0.0852	0.4054	4.2018	0.2018	0.9604
	θ	6.2106	0.2106	1.0451	8.2503	0.2503	1.1375
	β	5.0193	0.0193	0.1648	6.0314	0.0314	0.1609
250	τ	25.5657	0.5657	1.6176	27.3854	0.3854	1.4242
	ρ	2.0123	0.0123	0.2027	4.0360	0.0360	0.4619
	θ	5.9821	-0.0179	0.5937	8.0288	0.0288	0.6385
	β	5.0174	0.0174	0.0796	6.0192	0.0192	0.0843
500	τ	25.4954	0.4954	1.2777	27.2583	0.2583	0.8795
	ρ	1.9991	-0.0009	-0.1293	4.0074	0.0074	0.2968
	θ	5.9229	0.0771	0.4329	7.9697	-0.0303	0.4076
	β	5.0078	0.0078	0.0553	6.0076	0.0076	0.0495
1000	τ	25.4001	0.4001	0.9152	27.1578	0.1578	0.5850
	ρ	1.9956	-0.0044	-0.0896	4.0003	0.0003	0.1969
	θ	5.9244	0.0756	0.2979	7.9761	-0.0239	0.2774
	β	5.0045	0.0045	0.0323	6.0031	0.0031	0.0342

Table 1 displays the values of biases, estimated values and RMSEs It is noticed from the table that the RMESs approach zero and the estimates tend to the true parameter values as the sample increases. This is an indication that that the maximum likelihood estimates are efficient and consistent.

VII. APPLICATION

The fit of ETPW distribution is tested with applications to environmental data sets to assess its flexibility and robustness. The fit of the new model is compared with some existing distributions having Weibul distribution as the baseline. The comparators are: the Type I Half-Logistic Exponentiated Weibull (TIHLEtW) Distribution by [7], Type II Exponentiated Half Logistic Weibull (TIIEHLW) distribution by [4], Half-Logistic Generalized Weibull (HLGW) Distribution by [13], Exponentiated Weibull (EW) by [14]vand Weibull Distribution by [16].

The data set 1 consists of 20 observations with respect to maximum flood level data to see how the new model works in practice. The data has been obtained from [8] and is given as: 0.654,

0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

The data set 2 is obtained from [10] and also reported in [5]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 2: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{\tau}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\beta}$	ll	AIC	BIC
EtPW	38.9937	0.0354	206.5286	0.5620	16.3103	-24.6205	-20.6376
TIHLEtW	13.8158	2.3621	37.6306	0.5298	13.9359	-19.8717	-15.8888
TIIEtHLW	4.2059	0.6008	2.7424	6.0126	13.3035	-18.6071	-14.6242
HLGW	-	0.2951	6.5128	6888.7174	14.9716	-23.9432	-20.0560
EtW	-	1.4919	3.0333	2.3652	13.9497	-21.8993	-18.9121
W	-	3.5083	-	14.2303	13.2633	-22.5261	-20.5352

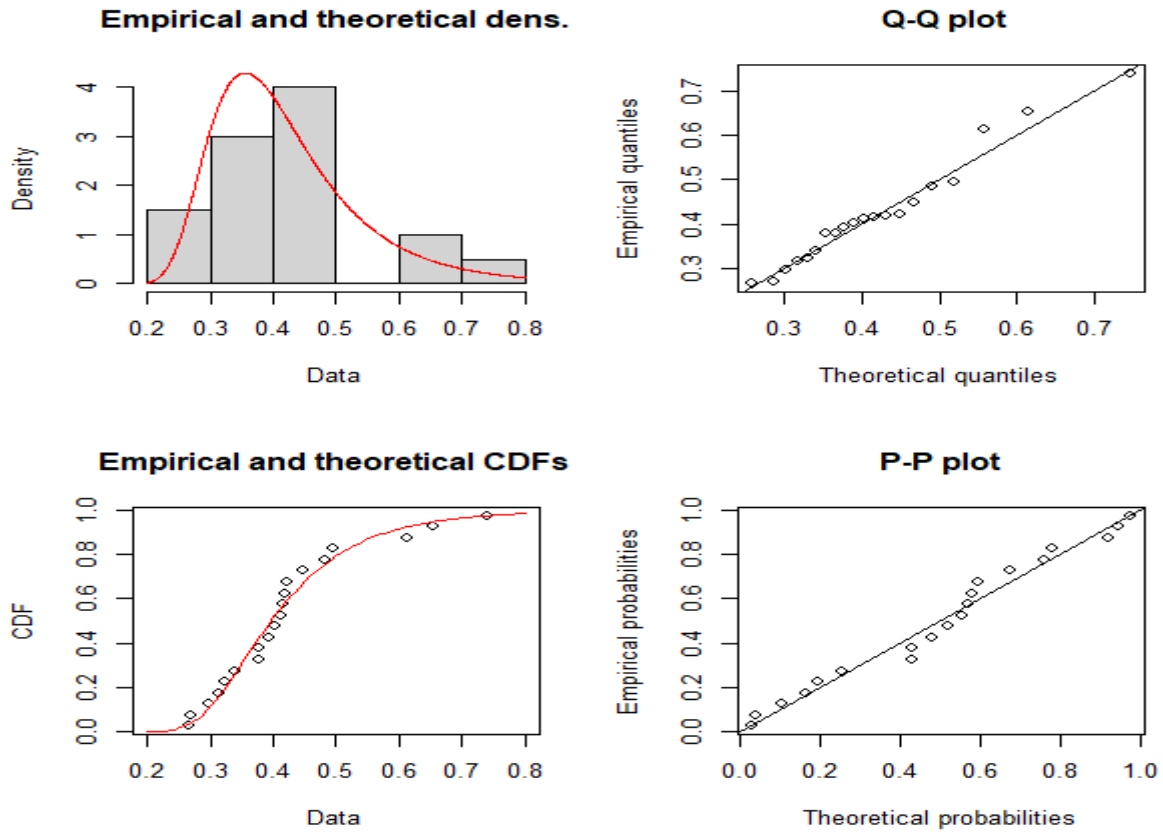

Figure 3: *Fitted cdf, pdf, Q-Q, and P-P plots for data set 1*

Table 3: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{\tau}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\beta}$	ll	AIC	BIC
EtPW	48.1901	0.0022	2.9823	1.0738	-38.0910	84.1820	89.7868
TIHLEtW	4.1437	0.6170	13.5135	0.5563	-38.4067	84.8135	90.4183

TIIEtHLW	0.7691	1.6782	0.4909	1.5904	-38.1060	84.2121	89.8168
HLGW	-	0.3416	2.7617	2.7351	-40.1181	86.2362	90.4398
EtW	-	2.4241	1.1680	0.8941	-39.8193	85.6386	89.8422
W	-	1.8088	-	0.3154	-41.6433	87.2866	90.0891

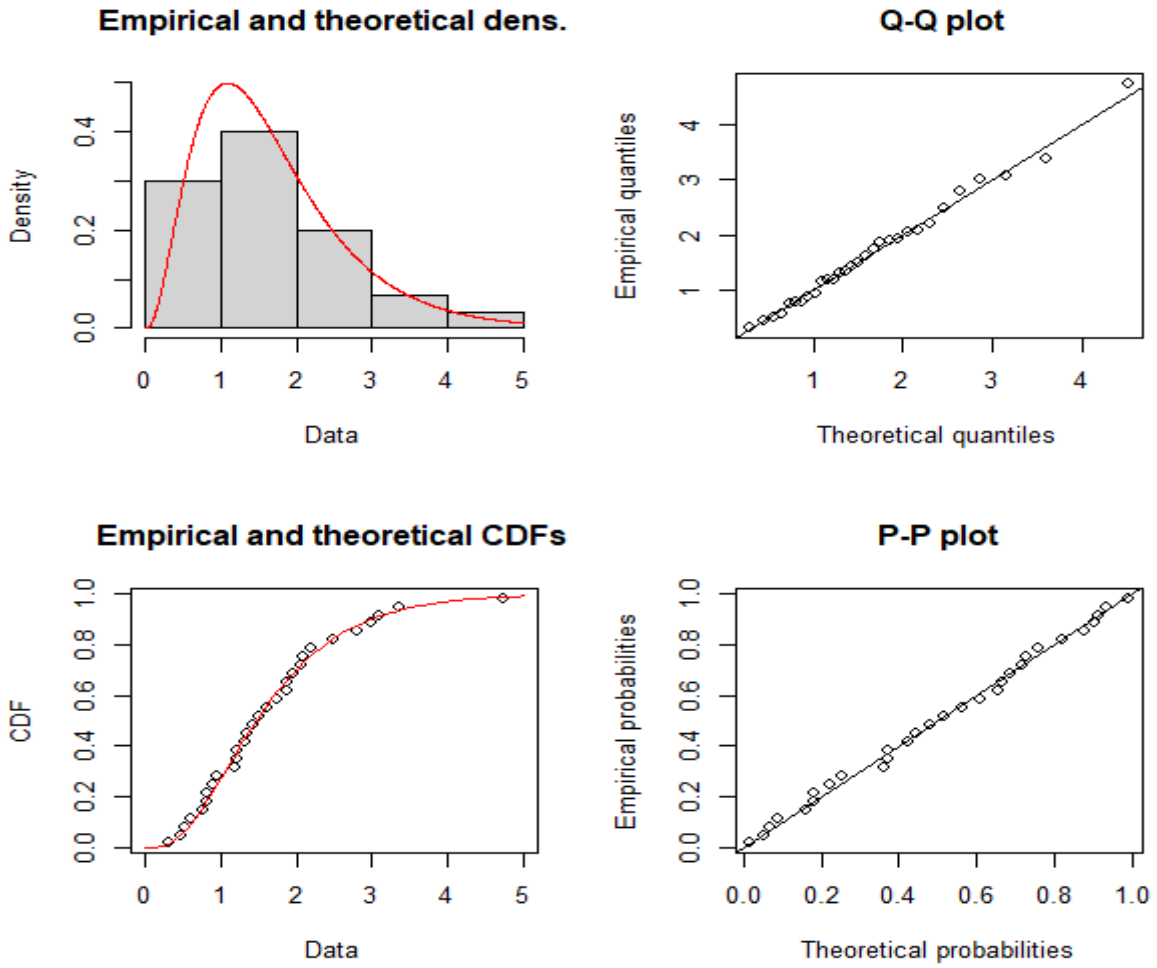

Figure 4: *Fitted cdf, pdf, Q-Q, and P-P plots for data set 2*

Tables 2 and 3 outline the results of the mle of the parameters of the EtPW distribution together with the comparator distributions. Based on the goodness of fit statistic AIC and BIC, the new probability model recorded the lowest AIC as well the lowest BIC value suggesting that the EtPW is best fits the two data sets. Figures 3 and 4 also buttress and reaffirm the fit of the EtPW distribution as it follows the pattern and shape of the data.

VIII. CONCLUSION

This research article proposed and studied a new family of distributions called the Exponentiated Pareto-G family of distributions. The family was derived from the exponentiated Pareto distribution using the T-X methodology proposed by [3]. The properties of the new family such as quantile function, probability-weighted moments, moments, generating function, reliability

function, hazard function, and order statistics were examined as statistical components of the newly proposed family of distributions. The parameters of the family are estimated using the method of maximum likelihood technique. Two submodels such as Weibull and Frechet are used to show the shape of the family as baseline distributions. A simulation results to evaluate the new distribution's performance is carried out using Weibull as the baseline distribution. This is to assess the efficiency of the estimation method used. Two real data sets are applied to ascertain the importance and flexibility of the new family of distributions. The results reveal that the new exponentiated Pareto Weibull distribution appears to be superior to the existing models considered. This implies that the new family has added flexibility to the baseline distribution and it can be used to model data in a variety of fields.

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A WEIGHTED THREE-PARAMETER XGAMMA DISTRIBUTION WITH PROPRTIES AND ITS APPLICATION TO REAL LIFE DATA

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Abstract

In this article, a weighted three-parameter Xgamma distribution has been proposed. It is an extension of two-parameter Xgamma distribution. The weighted three-parameter Xgamma distribution designed for modelling real-life data. The density function and cumulative distribution function, moments, hazard and survival function, moment-generating function and characteristic function, Bonferroni and Lorenz curve, renyi entropy of this distribution have all been derived. The parameter of this distribution is estimated by maximum likelihood estimation method. Finally, an application of the model to a real-life data set is presented and compared with some other existing distributions.

Keywords: weighted three-parameter Xgamma distribution, Reliability analysis, moments, maximum likelihood estimation, order statistics, renyi entropy.

I. Introduction

Choosing a sampling unit with equal probability for observational research on the population like fish, insects, plants and wild- animals is impossible. Due to this, the recorded observations are skewed, and the sampling frame is not well defined. These observations deviate from the original distribution, and as a result, their modeling gives rise to the weighted distribution theory. These studies pertain to survival analysis, family data analysis, and reliability. The weighted distribution theory is often used to handle model specification and data interpretation. Fisher developed the idea of weighted distribution first in 1934, and Rao formalized and improved it in 1965. Zelen introduced the weighted distribution of a length-biased sample in 1974, while Patil and Ord examined a size-biased sampling and related topics in 1976. In a series of articles with other co-authors, Patil has extensively pursued weighted distribution for the purpose of the encountering data analysis.

In this article, we introduce a new weighted distribution (see more reference [9][11][15][16]) known as weighted three-parameter Xgamma distribution, is the extension of the two parameter Xgamma distribution (see reference [14]). The alternative form of weighted three parameter Xgamma distribution is introduced in Section 3. The survival properties are studied in Section 4.

Moments, moment generating-function and characteristic function are described in Section 5. In Section 6, order statistics of the distribution are derived. The likelihood ratio test is discussed in Section 7. Bonferroni and Lorenz curve and Renyi entropy are discussed in Section 8 and 9, respectively. Methods of estimating parameters are discussed in section 10. In section11, two real data sets are analyzed to show of the application of weighted three-parameter Xgamma distribution; finally, Section 12 concludes.

II.Two-parameter Xgamma distribution

Two-parameter Xgamma distribution is the generalization of Xgamma distribution [14] by adding additional parameter α.
The probability density function (PDF) of a two-parameter Xgamma distribution is given by

$$F(x;\alpha,\theta)=\left(\frac{\theta^2}{\alpha+\theta}\right)\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)} \qquad \mathrm{x},\alpha,\theta>0 \tag{1}$$

The cumulative distribution function (CDF) is given by

$$F(x)=1-\frac{(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2)}{(\alpha+\theta)}e^{-\theta x} \qquad \mathrm{x},\alpha,\theta>0 \tag{2}$$

III. Weighted three-parameter Xgamma distribution

The concept of a weighted distribution, we have a definition, see Patil et al. (1988) as

$$f_w(X)=\frac{w(x)f(x)}{E\big(w(x)\big)} \qquad ;\ \mathrm{x}>0 \tag{3}$$

Where W(x) is the non-negative weight function and f_w(X) is pdf. We have different choices of weight function based on the different weight model (see the references [1], [2], [5], [6], [7], [13]). In the paper, use w(x) = x^c as a weight function. Then the probability function of weighted distribution is

$$fw(\mathrm{X})=\frac{(\mathrm{x^c})\mathrm{f(x)}}{\mathrm{E(x^c)}} \qquad ;\ \mathrm{c}>0 \tag{4}$$

Where

$$E(x^c)=\int_0^\infty (x^c)f(x;\alpha,\theta)dx$$

After simplification, we get

$$E(x^c)=\frac{c!}{2\theta^C(\alpha+\theta)})[2\theta+\alpha(1+c)(2+c)] \tag{5}$$

Substituting (1) and (5) in (4), we get

$$f\,w(x;\alpha,\theta,c)=\frac{x^{c}\left(\frac{\theta^{2}}{\alpha+\theta}\right)\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}{\left(\frac{c!}{2\theta^{c}(\alpha+\theta)}\right)[2\theta+\alpha(1+c)(2+c)]}$$

Definition 1: A non-negative continuous random variable X, is said to follow weighted three - parameter Xgamma distribution with parameters α,θ *and* c if its pdf is given by

$$fw(x,\alpha,\theta,c)=\frac{x^{c}\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)} \tag{6}$$

It is denoted by X~WTPXG (α,θ ,c).

Note:

(1). When we put c = 0 in (6), we obtain the two-parameter Xgamma distribution with parameter α *and* θ [14].

(2). Weighted three-parameter Xgamma distribution is positive skewed distribution.

Figure 1 show the density function for weighted three-parameter Xgamma distribution for different value of α,θ *and* c.

Then the cumulative distribution function of weighted three-parameter Xgamma distribution is given by

$$Fw(x,\alpha,\theta,c)=\frac{\theta\gamma(c+1,\theta x)+\frac{\alpha}{2}\gamma(c+3,\theta x)}{\frac{c!}{2}[2\theta+\alpha(c+1)(c+2)]} \tag{7}$$

Where, $\gamma(c+1,\theta x)$ and $\gamma(c+4,\theta x)$ is a lower incomplete gamma function [4].

Figure 2 show the distribution function for weighted three-parameter Xgamma distribution for different value of α,θ *and* c.

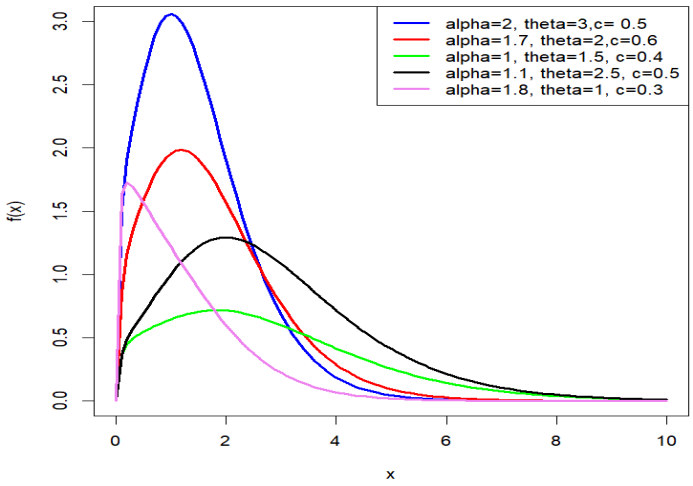

Figure 1: *probability density function of Weighted three-parameter Xgamma distribution*

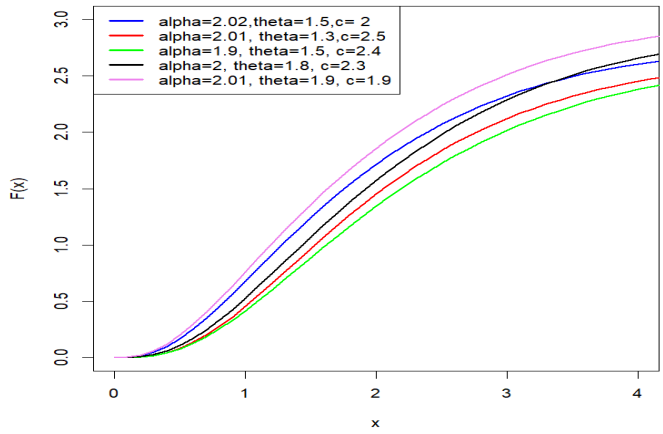

Figure 2: *cumulative distribution function of Weighted three-parameter Xgamma distribution*

IV. Reliability analysis

In this section, we study survival function, hazard function, reverse hazard function, mills ratio, odds rate function and cumulative hazard function for the weighted three-parameter Xgamma distribution with parameter α, θ and c given in (6).

I.Survival function:

The survival function is also known as the reliability function. The survival function of the weighted three-parameter Xgamma distribution is given by

$$s(x,\alpha,\theta) = 1 - F(x,\alpha,\theta,c) \tag{8}$$

Substituting equation (7) in (8)

$$s(x,\alpha,\theta,c) = 1 - \frac{\theta\gamma(c+1,\theta x) + \frac{\alpha}{2}\ \gamma(c+3,\theta x)\ ]}{\frac{c!}{2}\ \left(2\theta + \alpha(1+c)(2+c)\right)} \tag{9}$$

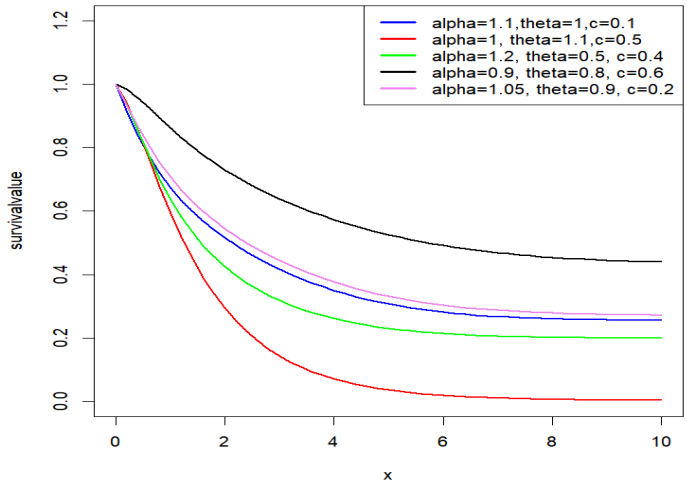

Figure 3: *survival function of Weighted three-parameter Xgamma distribution*

II. Hazard function

The hazard function of a weighted three-parameter Xgamma distribution is given

$$h\,(x,\alpha,\theta,c)\;=\frac{fw(x,\alpha,\theta,c)}{s(x,\alpha,\theta,c)}$$

$$\mathrm{h(x,\alpha,\theta,c)}=\frac{\dfrac{\mathrm{x^{c}\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}}{\frac{\mathrm{c!}}{2}\ \mathrm{(2\theta+\alpha(1+c)(2+c))}}}{1-\dfrac{\mathrm{\theta\,\gamma(c+1,\theta x)+\frac{\alpha}{2}\ \gamma(c+3,\theta x)\ ]}}{\frac{\mathrm{c!}}{2}\ \mathrm{(2\theta+\alpha(1+c)(2+c))}}}$$

After simplification, we get

$$h(x,\alpha,\theta,c)=\frac{x^{c}\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}{\frac{c!}{2}\ (2\theta+\alpha(1+c)(2+c))-\left[\theta\,\gamma\,(c+1,\theta x)+\frac{\alpha}{2}\ \gamma(c+3,\theta x)\right]}$$

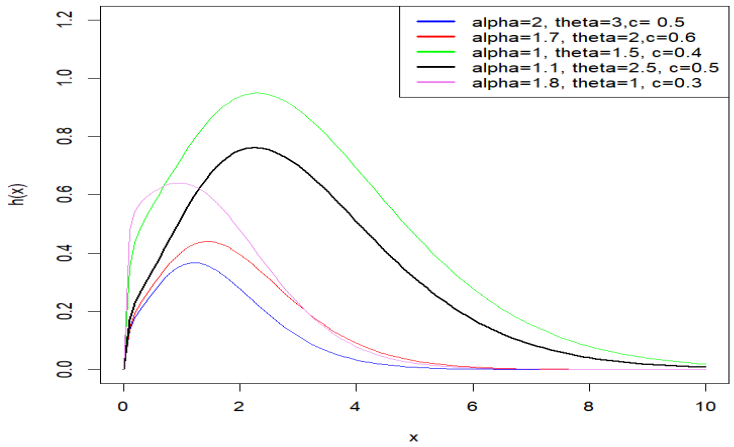

***Figure 4**: hazard plot of Weighted three-parameter Xgamma distribution*

III. Reverse Hazard function

The reverse hazard function of a weighted three-parameter Xgamma distribution is given

$$hr(x)=\frac{fw(x;\alpha,\theta,c)}{Fw(x;\alpha,\theta,c)}$$

$$hr(x)=\frac{\dfrac{x^{c}\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}{\frac{c!}{2}\ (2\theta+\alpha(1+c)(2+c))}}{\dfrac{[\theta\,\gamma(c+1,\theta x)+\frac{\alpha}{2}\,\gamma(c+3,\theta x)}{\frac{c!}{2}[2\theta+\alpha(c+1)(c+2)]}}$$

After simplification ,we get

$$hr(x) = \frac{x^c \theta^{c+2} \left(1 + \left(\frac{\alpha\theta}{2}\right) x^2\right) e^{-(\theta x)}}{\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\gamma(c+3, \theta x)}$$

IV. Mills Ratio

The mills ratio of a weighted three-parameter Xgamma distribution is given by

$$\mathrm{m(x)} = \frac{1}{h(x, \alpha, \theta, c)}$$

$$m(x) = \frac{1}{\frac{x^c \theta^{c+2} \left(1 + \left(\frac{\alpha\theta}{2}\right) x^2\right) e^{-(\theta x)}}{\frac{c!}{2}\ (2\theta + \alpha(1+c)(2+c)) \ -\ [\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)]}}$$

V. Odds rate function

The odds rate function of a weighted three-parameter Xgamma distribution is given

$$O(x) = \frac{Fw(x; \alpha, \theta, c)}{1 - Fw(x; \alpha, \theta, c)}$$

$$O(x) = \frac{\frac{[\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)]}{\frac{c!}{2}\ \left(2\theta + \alpha(1+c)(2+c)\right)}}{1 - \frac{[\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)]}{\frac{c!}{2}\ \left(2\theta + \alpha(1+c)(2+c)\right)}}$$

After simplification, we get

$$O(x) = \frac{\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)]}{\frac{c!}{2}\ \left(2\theta + \alpha(1+c)(2+c)\right) - [\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)]}$$

VI. Cumulative hazard function

The cumulative hazard function of a weighted three-parameter Xgamma distribution is given

$$H(x) = -\ln\left(1 - \frac{[\theta\ \gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)]}{\frac{c!}{2}\ \left(2\theta + \alpha(1+c)(2+c)\right)}\right)$$

V. Statistical Properties

In this section, discuss the statistical properties of the weighted three-parameter Xgamma distribution, that is moment, moment-generating function, and characteristic function.

I. Moments

If X is a random variable that follows a weighted three-parameter Xgamma distribution with pdf (6), then the rth order moment, that is μ_r' can be obtained as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_w(x) d(x)$$

$$\mu_r' = E(X^r) = \int_0^\infty x^r \frac{x^c \theta^{c+2} \left(1 + \left(\frac{\alpha\theta}{2}\right) x^2\right) e^{-(\theta x)}}{\frac{C!}{2} \left(2\theta + \alpha(1+c)(2+c)\right)} dx$$

After simplification we get rth order raw moment is

$$\mu_r' = \frac{\theta^{c+2}[2\Gamma(c+r+1) + \alpha\Gamma(c+1)]}{c!\left(2\theta + \alpha(1+c)(2+c)\right)\theta^{r+c+2}} \tag{10}$$

Putting r=1, 2 ,3, 4 in (10) we get 1st, 2nd, 3rd , 4th row moments of weighted three-parameter Xgamma distribution

$$\mu_1' = \frac{2(c+1) + \alpha(c+2)(c+1)}{2\theta + \alpha(c+1)(c+2)} \tag{11}$$

$$\mu_2' = \frac{2(c+1)\,(c+2) + \alpha(c+3)(c+2)(c+1)}{[2\theta + \alpha(c+1)(c+2)]\theta} \tag{12}$$

$$\mu_3' = \frac{2(c+1)\,(c+2)(c+3) + \alpha(c+3)(c+2)(c+1)(c+4)}{[2\theta + \alpha(c+1)(c+2)]\theta^{\,2}} \tag{13}$$

$$\mu_4' = \frac{2(c+1)\,(c+2)(c+3)((c+4) + \alpha(c+3)(c+2)(c+1)(c+4)(c+5)}{[2\theta + \alpha(c+1)(c+2)]\theta^{\,3}} \tag{14}$$

Note:

1.mean of the weighted three-parameter Xgamma distribution is given by

$$\text{Mean} = \frac{2(c+1) + \alpha(c+2)(c+1)}{2\theta + \alpha(c+1)(c+2)}$$

2. variance of the weighted three-parameter Xgamma distribution is given by

$$Variance = \frac{2(c+1)\,(c+2) + \alpha(c+3)(c+2)(c+1)}{[2\theta + \alpha(c+1)(c+2)]\theta} - \left(\frac{2(c+1) + \alpha(c+2)(c+1)}{2\theta + \alpha(c+1)(c+2)}\right)^2$$

$$= \frac{2(c+1) + \alpha(c+1)(c+2)}{\theta\left(2\theta + \alpha(c+1)(c+2)\right)^2}(\alpha(1-\theta)(c+1)(c+2) - 2\theta c)$$

3. Standard deviation

$$\sigma = \sqrt{\left(\frac{2(c+1) + \alpha(c+1)(c+2)}{\theta\left(2\theta + \alpha(c+1)(c+2)\right)^2}(\alpha(1-\theta)(c+1)(c+2) - 2\theta c)\right.}$$

4. Coefficient of variation (CV)

$$C\,V = \frac{\sqrt{2(c+1)+\alpha(c+2)(c+1)}}{\sqrt{\theta}\,(2(c+1)+\alpha(c+2)(c+1)}\,100$$

II. Moment generating function and Characteristic function

If a random variable X follows a weighted three-parameter Xgamma distribution, then the moment-generating function (MGF) is given as follows,

$$M_x(t) = E(e^{tx}) \tag{15}$$

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx$$

$$M_x(t) = \int_0^\infty e^{tx} \frac{x^c \theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}dx$$

$$M_x(t) = \frac{\theta^{c+2}}{\frac{c!}{2}\left(2\theta+\alpha(c+1)(c+2)\right)}\int_0^\infty e^{-x(\theta-t)}\,x^c\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)dx$$

After simplification, we get the moment-generating function is

$$M_x(t) = \frac{\theta^{c+2}(2(\theta-t)^2+\alpha\theta(c+1)(c+2)}{(\theta-t)^{c+3}(2\theta+\alpha(c+1)(c+2)} \tag{16}$$

Similarly, the characteristic function is obtained as

$$\phi_x(t) = \frac{\theta^{c+2}(2(\theta-it)^2+\alpha\theta(c+1)(c+2)}{(\theta-it)^{c+3}(2\theta+\alpha(c+1)(c+2)} \tag{17}$$

VI. Order Statistics

In this section, we derive the order statistics from weighted three-parameter Xgamma distribution. Let $x_{(1)}$, $x_{(2)}$, $x_{(3)}$,…. $x_{(n)}$ be the order statistics of the random variable x_1, x_2,…… x_n taken from weighted three parameter Xgamma distribution. Then the probability density function of the m^{th} order statistics $X_{(m)}$ is defined as

$$fw(m)(x) = \frac{n!}{(m-1)!\,(n-m)!}f(x)[F\,(x)]m-1\,[1-F(x)]n-m \tag{18}$$

Inserting equation (6) & (7) in (18)

$$\mathrm{fw(m)(x)} = \frac{\mathrm{n!}}{(\mathrm{m}-1)!\,(\mathrm{n}-\mathrm{m})!}\frac{\mathrm{x^c\theta^{c+2}}\left(1+\left(\frac{\alpha\theta}{2}\right)\mathrm{x}^2\right)\mathrm{e}^{-(\theta\mathrm{x})}}{\frac{\mathrm{c!}}{2}\ \left(2\theta+\alpha(1+\mathrm{c})(2+\mathrm{c})\right)}\left(\frac{\theta\gamma(\mathrm{c}+1,\theta\mathrm{x})+\frac{\alpha}{2}\,\gamma(\mathrm{c}+3,\theta\mathrm{x})}{\frac{\mathrm{c!}}{2}[2\theta+\alpha(\mathrm{c}+1)(\mathrm{c}+2)]}\right)^{\mathrm{m}-1}$$

$$\left(\frac{\theta\gamma(c+1,\theta x)+\frac{\alpha}{2}\gamma(c+3,\theta x)}{\frac{c!}{2}[2\theta+\alpha(c+1)(c+2)]}\right)^{n-m} \tag{19}$$

Put m=1 in equation (19), we get first order statistics $X_{(1)}$=mini ($x_1, x_2, \ldots\ldots x_n$) and m=n, we get n^{th} order statistics $X_{(n)}$=max ($x_1, x_2, \ldots\ldots x_n$)

VII. Likelihood Ratio Test

The likelihood-ratio test is a hypothesis test that compares two competing statistical models' goodness of fit. Assume that $x_1, x_2, \ldots\ldots\ldots x_n$ be n random sample taken from a weighted three-parameter Xgamma distribution to test the hypothesis.

$$H_0: f(x) = f(x,\alpha,\theta,c)$$
against
$$H_1: f(x) = f_w(x,\alpha,\theta,c)$$

$$\Delta = \frac{L_1}{L_0}\prod_{i=1}^{n}\frac{f_w\ (xi)}{f(xi)}$$

$$\Delta = \prod_{i=1}^{n}\frac{\frac{x^c{}_i\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^c{}_i\right)e^{-(\theta x_i)}}{\frac{C!}{2}\ (2\theta+\alpha(1+c)(2+c))}}{\left(\frac{\theta^2}{\alpha+\theta}\right)\left(1+\left(\frac{\alpha\theta}{2}\right)x^c{}_i\right)e^{-(\theta x_i)}}$$

$$\Delta = \prod_{i=1}^{n}\frac{x^c{}_i\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}$$

The null hypothesis is rejected if

$$\Delta = \frac{\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\prod_{i=1}^{n}x^c{}_i \ > \ K$$

$$\Delta = \prod_{i=1}^{n}x^c{}_i > \frac{k}{\left[\frac{\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right]^n}$$

$$\Delta^* = \prod_{i=1}^{n}x^c{}_i > K^*$$

Where

$$K^* = \frac{k}{\left[\frac{\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right]^n}$$

A Chi-square variate with one degree of freedom is distributed as -2log Δ for large sample size (n). The null hypothesis is rejected when the probability value is provided by

$$p(\Delta^*) > \alpha^*$$

Where $\alpha^* = \prod_{i=1}^{n} {x_i}^c$ is less than level of significance. And $\prod_{i=1}^{n} {x_i}^c$ is the observed value of the statistic Δ^*

VIII. Bonferroni and Lorenz Curves

In this section, we derive the Bonferroni and Lorenz curves from the weighted three-parameter Xgamma distribution [2]. The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures is given as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x\, f_a(x) dx$$

And

$$L(p) = \frac{1}{\mu_1'} \int_0^q x\, f_a(x) dx$$

Where

$\mu_1' = E(X) = \dfrac{2(c+1) + \alpha(c+2)(c+1)}{2\theta + \alpha(c+1)(c+2)}$ and $q = F^{-1}(p)$

$$B(p) = \frac{2\theta + \alpha(c+1)(c+2}{p\big(2(c+1) + \alpha(c+2)(c+1)\big)} \int_0^q x \frac{x^c \theta^{c+2} \left(1 + \left(\frac{\alpha\theta}{2}\right) x^2\right) e^{-(\theta x)}}{\frac{C!}{2}\ \big(2\theta + \alpha(1+c)(2+c)\big)}\, dx$$

$$B(p) = \frac{2\theta^{c+2}}{C!\, p\big(2(c+1) + \alpha(c+2)(c+1)\big)} \int_0^q x^c x\, e^{-\theta x} \left(1 + \left(\frac{\alpha\theta}{2}\right) x^2\right) dx$$

$$B(p) = \frac{2\theta^{c+2}}{C!\, p\big(2(c+1) + \alpha(c+2)(c+1)\big)} \{ \int_0^q x^{c+1} e^{-\theta x}\, dx + \int_0^q \frac{\alpha\theta}{2}\, x^{c+3} e^{-\theta x} dx \}$$

After simplification we get

$$B(p) = \frac{2\theta\, \gamma(c+2, \theta q) + \alpha\gamma(c+4, \theta q)}{C!\, p\theta\big(2(c+1) + \alpha(c+2)(c+1)\big)} \tag{20}$$

Where $L(p) = pB(p)$

$$L(p) = \frac{2\theta\, \gamma(c+2, \theta q) + \alpha\gamma(c+4, \theta q)}{C!\, \theta\big(2(c+1) + \alpha(c+2)(c+1)\big)} \tag{21}$$

IX. Renyi Entropy

The Renyi entropy is important in ecology and statistics as an index of diversity. For a given probability distribution (6), Renyi entropy is given by

$$R(\lambda) = \frac{1}{1-\lambda} \log \int_0^\infty ({f_w}^\lambda(x)) dx$$

Where $\lambda > 0\ and\ \lambda \neq 1$

$$R(\beta) = \frac{1}{1-\lambda} \log \int_0^\infty \left(\frac{x^c \theta^{c+2} \left(1 + \left(\frac{\alpha\theta}{2}\right) x^2\right) e^{-(\theta x)}}{\frac{C!}{2}\ \big(2\theta + \alpha(1+c)(2+c)\big)} \right)^\lambda dx$$

$$R(\beta)=\frac{1}{1-\lambda}\log\left(\frac{\theta^{c+2}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)^{\lambda}\int_0^{\infty}x^{c\lambda}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)^{\lambda}e^{-\theta\lambda x}dx$$

Using binomial expansion in above equation and can be obtain

$$R(\lambda)=\frac{1}{1-\lambda}log\left(\frac{\theta^{c+2}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\binom{\lambda}{i}\left(\frac{\alpha\theta}{2}\right)^{i}\int_0^{\infty}x^{c\lambda+2i}e^{-\theta\lambda x}dx$$

$$R(\lambda)=\frac{1}{1-\lambda}log\left(\frac{\theta^{c+2}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\binom{\lambda}{i}\left(\frac{\alpha\theta}{2}\right)^{i}\frac{(c\lambda+2i)!}{(\lambda\theta)^{c\lambda+2i+1}} \tag{22}$$

X. Estimations of Parameter

Let $x_1, x_2 \ldots\ldots x_n$ be the random sample of size n taken from weighted three-parameter Xgamma distribution, then the likelihood function is defined as below,

$$L(x,\alpha,\theta,c)=\prod_{i=1}^{n}f(x,\alpha,\theta,c)$$

$$L(x,\alpha,\theta,c)=\prod_{i=1}^{n}\frac{x^c\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}$$

$$L(x,\alpha,\theta,c)=\left(\frac{\theta^{c+2}}{\frac{c!}{2}\left(2\theta+\alpha(c+1)(c+2)\right)}\right)^{n}e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}x^c\left(1+\frac{\alpha\theta}{2}x_i^2\right)$$

Take logarithm on both sides we get

$$l\,n\,L(x,\alpha,\theta,c)=\ nln\left(\frac{\theta^{c+2}}{\frac{c!}{2}\left(2\theta+\alpha(c+1)(c+2)\right)}\right)-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}ln(\,(1+\frac{\alpha\theta}{2}x_i^2)x^C)$$

$$ln\ L(x,\alpha,\theta,c)=n(c+2)ln\theta-n\,lnc!-nln\theta-n\,ln\frac{c!}{2}-n\,ln\alpha-n\ln(c^2+3c+2)$$
$$-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}c\ lnx_i+\sum_{i=1}^{n}\ln\left(1+\frac{\alpha\theta}{2}x_i^2\right) \tag{23}$$

After differentiating equation (23) with respect to α,θ and c and equating to zero we get

$$\frac{\partial\ln L(x,\alpha,\theta,c)}{\partial\alpha}=\frac{\theta}{2}\sum_{i=1}^{n}\frac{x_i^2}{\left(1+\frac{\alpha\theta}{2}x_i^2\right)}-\frac{n}{\alpha}=0 \tag{24}$$

$$\frac{\partial\ln L(x,\alpha,\theta,c)}{\partial\theta}=\frac{\alpha}{2}\sum_{i=1}^{n}\frac{x_i^2}{\left(1+\frac{\alpha\theta}{2}x_i^2\right)}+\frac{n(c+1)}{\theta}-\sum_{i=1}^{n}x_i=0 \tag{25}$$

$$\frac{\partial \ln L(x,\alpha,\theta,c)}{\partial c} = n\ln\theta - \frac{2n}{c!}\psi(c+1)\Gamma(c+1 - \frac{4n}{c} + \sum_{i=1}^{n} ln\, x_i = 0 \qquad (26)$$

Where ψ is the Digamma function [15]. On solving equations (24), (25) and (26) we obtain the maximum likelihood estimator of parameters in weighted three-parameter Xgamma distribution. But we have the above equations in complex form; it can't be solved directly, so we estimate the parameter of the weighted three-parameter Xgamma distribution using R and Wolfram Mathematica**.**

XI. Applications

In this section, we have considered two real data sets for the purpose of showing that the distribution of weighted three-parameter Xgamma distribution shows a better fit over two parameter Xgamma distribution and Xgamma distribution.

The Akaike information criterion (AIC), Bayesian information criterion (BIC), Akaike information criterion corrected (AICC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC), and -2logl are using for model selection. It can be evaluated by using the formula as follows;

$$AIC = 2k - 2logL \qquad AICC = AIC + \frac{2k(k+1)}{(n-k-1)} \qquad BIC = klog(n) - 2logL$$

$$CAIC = \frac{2kn}{n-k-1} - 2logL \qquad HQIC = 2klog(\log(n)) - 2logL$$

Where n is the sample size, k is the number of parameters, and -2logL is the maximal value of the log likelihood function. After the calculation of AIC, AICC, HQIC, CAIC, and -2logL which the model with the minimum value is chosen as the best model to fit the data.

Data set 1: **blood cancer (leukemia)**

The following real lifetime data set consists of 37 patients suffering from blood cancer (leukemia) reported from one of the ministries of health hospitals in Saudi Arabia (see Abouammah et al. [3]). The ordered lifetimes (in years) are given below in Data Set 1 as:

1.145,1.208,1.263,1.414,2.025,2.036,2.162,2.211,2.37,2.532,2.693,2.805,2.91,2.912,3.192,3.263,3.348, 3.348,3.427,3.499,3.534,3.767,3.751,3.858,3.986,4.049,4.244,4.323,4.381,4.392,4.397,4.647,4.753,4.929, 4.973,5.074,5.381

Table 1: *The summary of blood cancer data set*

Min	1st Qu	Median	Mean	Variance	3rd Qu	Max
1.145	2.532	3.427	3.357	1.359	4.323	5.381

Data set 2: lung cancer

The following real lifetime data set consists of 39 lung cancer patients' survival periods (measured in months) as reported by Pena (2002). The ordered lifetimes (in months) are given below in Data Set 2 as:

2.99,3.06,3.15,3.45,3.71,3.75,3.81,4.11,4.27,4.34,4.40,4.63,4.73,4.93,5.03,5.16,5.17,5.49,5.68,5.72,5.98, 8.15,8.62,8.48,8.61,9.46,9.53,10.05,10.15,10.95,5.85,11.24,11.63,12.26,12.65,12.78,13.18,13.4

Table 2: *The summary of lung cancer data set*

Min	1st Qu	Median	Mean	Variance	3rd Qu	Max
2.99	4.355	5.700	7.120	11.377	9.920	13.400

Table 3: *MLEs AIC, BIC, AICC, CAIC, HQIC and -2log $\mathcal{L}$ of the fitted distribution for the given data set 1*

Distribution	WTPXG	TPXG	XG
MLE	$\hat{\theta}$=2.063455 (4.860449 e-05) $\hat{\alpha}$ = 2.171212 e+05 (1.677722 e+04) $\hat{c}$ =3.926777 (1.573179)	$\hat{\theta}$ = 8.935533e-01 (8.482223e-02) $\hat{\alpha}$ =2.478485 e+03 (1.677841e+04)	$\hat{\theta}$ =0.69602623 (0.07361946)
-2logL	119.3462	129.9725	151.4200
AIC	125.3203	133.9725	153.4202
BIC	130.1531	137.1944	155.0311
AICC	125.7648	134.4170	153.8646
HQIC	127.05	135.1084	153.9881
CAIC	126.0735	134.3255	153.5345

Table 4: *MLEs AIC, BIC, AICC, CAIC, HQIC and -2log $\mathcal{L}$ of the fitted distribution for the given data set 2*

Distribution	WTPXG	TPXG	XG
MLE	$\hat{\theta}$= 0.6597119 (0.1542389) $\hat{\alpha}$ = 614.8364779 (2836.4541338) $\hat{c}$ = 1.6986652 (1.0404110)	$\hat{\theta}$ = 4.211952e-01 (3.945576e-02) $\hat{\alpha}$ =1.113155e+03 (9.687653e+03)	$\hat{\theta}$=0.36242629 (0.03606323)
-2logL	192.5258	196.1067	208.8136
AIC	203.4356	203.4358	212.4512
BIC	198.5228	200.1067	210.8136
AICC	198.9552	200.5391	211.246
HQIC	200.2707	201.2719	211.3962
AIC	199.2287	200.4495	210.924

From tables 3 and 4 we can observe AIC, BIC, AICC, HQIC, CAIC, and -2logL values are lowest in the weighted three-parameter Xgamma distribution compared to the two-parameter Xgamma distribution and Xgamma distribution. Hence, it can be concluded that the weighted three-parameter Xgamma distribution model is the best model compared to the two-parameter Xgamma distribution and Xgamma distribution.

XII. Conclusion

This research deals with the selection of an appropriate model for fitting survival data. In this paper, the Two-parameter Xgamma distribution is extended to provide a new distribution, called

weighted three-parameter Xgamma distribution, which is the lifetime model for a real- life data set. In the section 3 to 10, discussing the statistical property of weighted three-parameter Xgamma distribution. The effectiveness of the suggested model is demonstrated by an examination of two real cancer data sets. The result indicates that the weighted three parameter Xgamma distribution is more flexible and practical than the two-parameter Xgamma distribution and Xgamma distribution.

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STRESS-STRENGTH MODELLING: RANKED SET AND SIMPLE RANDOM SAMPLING IN GENERALIZED INVERSE WEIBULL ANALYSIS

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Abstract

This study explores the stress-strength reliability model (P) for Generalized Inverse Weibull (GIW) distribution through transformation techniques. We compare two sampling methods: ranked set sampling (RSS) and simple random sampling (SRS), where stress and strength are two independent random variables from the GIW distribution respectively. RSS, is used for estimating stress-strength model, as this technique of sampling is more efficient alternative of SRS for obtaining the more informative sample. In this article, the maximum likelihood estimator (MLE) for stress-strength model is obtained through transforming technique. MLE estimates of stress-strength obtained through Ranked set sampling (RSS) methods are evaluated against corresponding estimates derived from simple random sampling (SRS) to understand their relative effectiveness and accuracy. The statistical estimators derived from Ranked Set Sampling (RSS) methodology exhibit superior efficiency relative to their Simple Random Sampling (SRS) counterparts. The empirical utility of RSS-based estimation procedures is subsequently validated through application to real datasets.

Keywords: Stress-strength reliability, simple random sampling, ranked set sampling, generalized inverse Weibull distribution, maximum likelihood estimation.

1. Introduction

The stress-strength model is a fundamental concept in reliability engineering and statistics. It is used to assess the probability of failure or success in a system subject to the random variations in stress and strength. This model is employed by many researchers in various fields, including engineering, materials science, quality control, and finance etc. The probability that a system's random stress Y is less than its random strength X is represented by $= P_r(Y < X)$ in the context of stress-strength model. In other words, it calculates the probability of failure in the stress-strength model. The system failure occurs when the stress exceeds the strength. Recently, the problem of stress-strength model is evaluated by an alternative approach of sampling proposed by Mclntyre [1] The pioneering investigations of Birnbaum [2] and Birnbaum and McCarty [3] represent the initial academic exploration of this fundamental problem. Church and Harris [4] were the first to use the phrase "stress-strength". Since then, a sizable amount of work has been completed from both a parametric and non-parametric perspective. For earlier bibliography one may refers to, Chaturvedi and Kumar [5], Kotz et al. [6], Kundu and Gupta [7][8], Raqab and Kundu [9], Kundu and Raqab [10], Krishnamoorthy et al. [11], Hassan [12], Wang et al. [13], Kayal et al. [14], Kumar and Chaturvedi [15]. In the above referred studies the estimation for the considered model is based on SRS.

The circumstances in which it is challenging to take the actual measurement for sample units (costly, destructive, time consuming), the RSS strategy can be used in under these circumstances which maintains the accuracy of our statistical judgements and reduces the sample size. Akgul and Senoglu [16] obtained the point estimators of stress-strength model when the stress and the strength both are independent Weibull random variables with common shape and different scale parameters based on RSS by using maximum likelihood (ML) and modified ML methodologies, Hassan et al. [17] used RSS for point and interval estimators of $P = P_r(Y < X)$ based on Gompertz distribution and MLES are compared by using MC simulation techniques. Hossein et al. [18] consider the RSS to estimate the parameters exponentiated pareto distribution and conclude that the estimator based on the ranked set sample have far better efficiency than the simple random sample at the same sample size. Akgul and Senoglu [19] constructed the asymptotic confidence interval for 'P' and obtained point and interval estimators for $P = P_r(Y < X)$ based on RSS, in addition the BCI for 'P' is constructed based on two distinct resampling methods.

In this paper we consider the point estimation of 'P' the stress-strength model, when the random stress and strength are two independent GIW random variables with different shape and scale parameters. A quick summary of the GIW distribution is given in section 2 and the point estimation of P using the maximum likelihood (ML) approach based on SRS is given in section 3. A brief explanation of RSS and its application in the point estimation is given in section 4. Monte Carlo simulation study is carried out in section 5 and a real life data study is performed for this model in section 6. Section 7 gives the concluding remarks for this study.

2. Preliminary

The GIW distribution is a continuous probability distribution which is proposed by de. Gosmao et al. [20]. It is an extended form of the Inverse Weibull distribution, introducing additional shape parameters to provide more flexibility in modelling. GIW has many applications in reliability, particularly in modelling the degradation of mechanical components such as pistons and crankshafts of diesel engines, as well as the breakdown of insulating fluid and in biological studies, where it is used to model a variety of failure characteristics such as infant mortality, useful life, and wear-out periods. Figure 1 and Figure 2 are showing the behaviour of probability density function and hazard rate function of GIW distribution respectively. The probability density function is positively skewed and the hazard rate function which is also known as failure rate function, during the initial phase, the hazard rate increases, indicating that the conditional probability of failure grows over time. This might represent a period where stress accumulation or wear-out effects dominate. However, after reaching a peak, the hazard rate begins to decrease, suggesting that units that have survived beyond a certain point have demonstrated their resilience and are less likely to fail immediately. This pattern can be observed in various real-world phenomena, such as certain mechanical systems or biological processes.

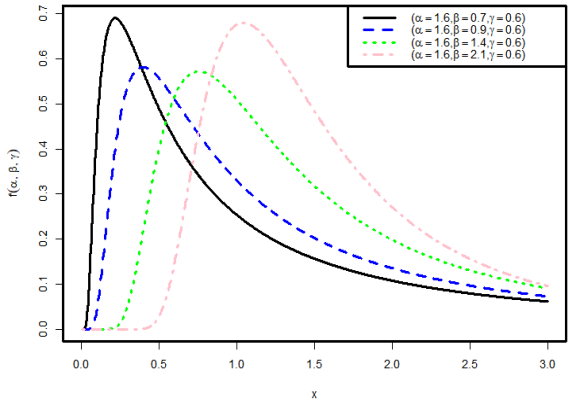
The probability density function (pdf) and cumulative distribution function (cdf) of GIW distribution are given respectively as

$$f(x, \alpha\, \beta, \gamma) = \gamma\beta\alpha^{\beta x^{-(\beta+1)}} \exp\left[-\gamma\left(\frac{\alpha}{x}\right)^{\beta}\right] \; ; \; x, \alpha, \beta, \gamma > 0 \tag{2.1}$$

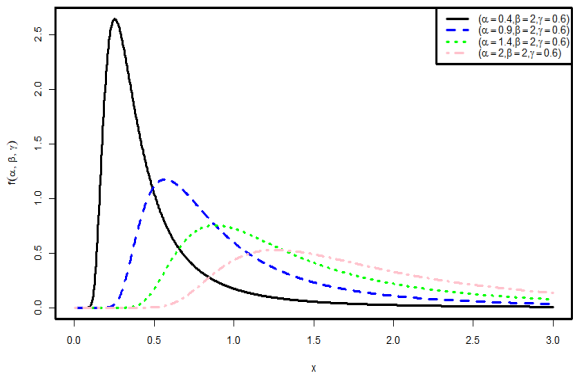
$$F(X) = \exp\left[-\gamma\left(\frac{\alpha}{x}\right)^{\beta}\right] \; ; \; x, \alpha, \beta, \gamma > 0 \tag{2.2}$$

Hazard rate equation of GIW distribution given as follows-

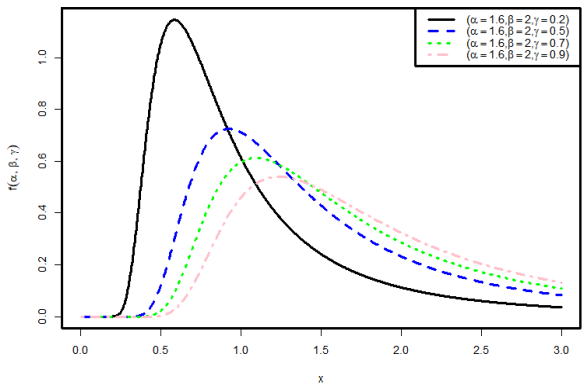
$$h(t) = \gamma\beta\alpha^{\beta} t^{-(\beta-1)} exp\left(-\gamma\left(\frac{\alpha}{t}\right)^{\beta}\right)\left[1 - exp\left(-\gamma\left(\frac{\alpha}{t}\right)^{\beta}\right)\right]^{-1} \; ; \; t, \alpha, \beta, \gamma > 0 \tag{2.3}$$



(a) For fixed α = 1.6 and γ = 0.6

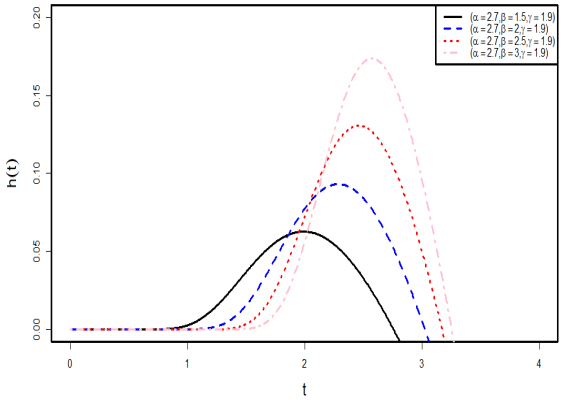


(b) For fixed β = 2 and γ = 0.6

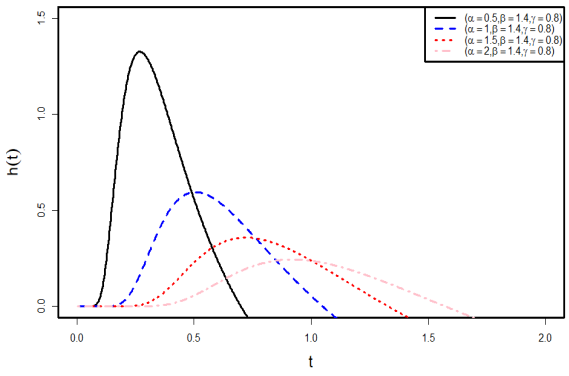


(c) For fixed α = 1.6 and β = 2

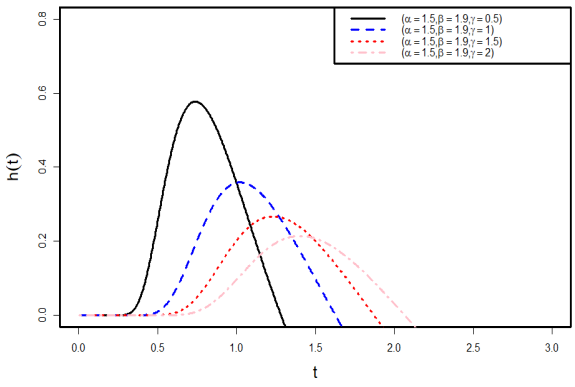
Figure 1: *Behaviour of pdf*



(a) For fixed α = 2.7 and γ = 1.9



(b) For fixed β = 1.4 and γ = 0.8



(c) For fixed α = 1.5 and β = 1.9

Figure 2: *Behaviour of Hazard rate*

3. Point Estimator for $P = P_r(Y < X)$ based on SRS

The pdf of GIW distribution is given by

$$f(x, \alpha, \beta, \gamma) = \gamma\beta\alpha^{\beta x^{-(\beta+1)}} \exp\left[-\gamma\left(\frac{\alpha}{x}\right)^{\beta}\right]; x, \alpha, \beta, \gamma > 0 \quad (3.1)$$

Let the rv's X and Y follow the GIW distribution given at (3.1) with the parameters (α, β, γ) and (θ, μ, χ).

Theorem 3.1: The MLE of $P = P_r(Y < X)$ is given by

$$P_{SRS}^{ML} = \frac{\bar{T}_Y}{(\bar{T}_X + \bar{T}_Y)}$$

$$where\ \bar{\Phi} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i{}^{-\beta} = \bar{T}_X\ (say), \qquad and\ \bar{\xi} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i{}^{-\mu} = \bar{T}_Y\ (\text{say})$$

Proof: Let us consider the transformation $x^{-\beta} = \Phi$ in (3.1), we get

$$f(\Phi|\lambda) = \lambda exp[-\lambda\Phi]\ ;\ \Phi, \lambda > 0 \tag{3.2}$$

which is exponential distribution with parameter λ, where $\lambda = \gamma\alpha^{\beta}$

Let us considered Φ and ξ be two independent rv's which follows exponential distribution with parameters λ_1 and λ_2 respectively, where $x^{-\beta} = \Phi$ and $y^{-\mu} = \xi$

Thus for

$$P = P_r(\xi < \Phi)$$

$$P = \int_0^{\infty}\int_{\Phi=0}^{\xi} f(\Phi|\lambda_1)d\Phi f(\xi|\lambda_2)d\xi$$

$$P = \frac{\lambda_1}{(\lambda_1+\lambda_2)} \tag{3.3}$$

If $\Phi_1, \Phi_2, \dots, \Phi_{n_1}\ and\ \xi_1, \xi_2, \dots, \xi_{n_2}$ are two independent random samples of size $n_1\ and\ n_2$ from the pdf's $f(\Phi|\lambda_1)$ and $f(\xi|\lambda_2)$ respectively then the joint pdf is given by

$$f(\Phi, \xi\,|\lambda_1, \lambda_2) = \lambda_1{}^{n_1}\lambda_2{}^{n_2} exp[-n_1\lambda_1\bar{\Phi} - n_2\lambda_2\bar{\xi}] \tag{3.4}$$

Taking likelihood function of (3.4) and derivatives w.r.to $\lambda_1\ and\ \lambda_2$ and equating to zero, we get MLES of $\lambda_1\ and\ \lambda_2$ respectively i.e.

$\lambda_1 = \frac{1}{\bar{\Phi}}$ and $\lambda_2 = \frac{1}{\bar{\xi}}$

The reliability function of P is

$$P_{SRS}^{ML} = \frac{\bar{\xi}}{(\bar{\Phi}+\bar{\xi})} \tag{3.5}$$

The equation (3.5) can be written as

$$P_{SRS}^{ML} = \frac{\bar{T}_Y}{(\bar{T}_X + \bar{T}_Y)}$$

4. Point Estimator for P = $P_r(Y < X)$ based on RSS

In this section, we derive the ML estimator of P based on RSS. We first discuss about RSS, RSS is a specialized statistical sampling method designed to improve the efficiency and accuracy of estimating population parameters, particularly when dealing with populations that are highly heterogeneous or contain outliers. This sampling technique was introduced as an alternative to traditional sampling methods, such as SRS, in order to tackle the challenges posed by extreme values or skewed distributions in the population. A significant increase in precision can occasionally be obtained by using RSS as an alternative to SRS. In a work by G. A. McIntyre, it was first suggested in relation to evaluating herbage productivity. RSS procedures are given below:

I. Consider random sample $x_1, x_2, \dots, x_m$ by SRS each of size m.

II. To obtain k observations from a population
III. Then, rank order them according to a pre-defined attribute.
IV. The unit that is judged the smallest is included in your ranked set sample.
V. This first unit is called the first judgement order statistics and denoted by $X_{[1]}$.
VI. Then we repeat the same process k time, there for the sample size is obtained as $n = km$.
VII. For better understanding this entire process, see the following table:

Cycle 1	$X_{[1]1}$	$X_{[2]1}$	$X_{[3]1}$	...	$X_{[k]1}$
Cycle 2	$X_{[1]2}$	$X_{[2]2}$	$X_{[3]2}$	...	$X_{[k]2}$
⋮	⋮	⋮	⋮	⋮	⋮
Cycle m	$X_{[1]m}$	$X_{[2]m}$	$X_{[3]m}$	...	$X_{[k]m}$

4.1 The maximum likelihood estimator of $P = P_r(Y < X)$

Let $x_{ij};\ i = 1,2,\dots,m_1$ and $j = 1,2,\dots,r_1$ denote the raked set sample of size $n_1 = r_1 m_1$ from GIW distribution with parameter (α, β, γ) where m_1 is the set size and r_1 is the number of cycles and $y_{kl}; k = 1,2,\dots,m_2$ and $l = 1,2,\dots,r_2$ denote the ranked set sample of size $n_2 = r_2 m_2$ from GIW distribution with parameter (θ, μ, χ) where m_2 is the set size and r_2 is the number of cycles. Then the pdf of x_{ij} and y_{kl}

$$f_i(x_{ij}) = \frac{m_1!}{(i-1)!\,(m_1-i)!}[F(x_{ij})]^{i-1}[1-F(x_{ij})]^{m_1-i} f(x_{ij})$$

$$f_k(y_{kl}) = \frac{m_2!}{(k-1)!\,(m_2-k)!}[F(y_{kl})]^{k-1}[1-F(y_{kl})]^{m_2-k} f(y_{kl})$$

Then the likelihood function based on RSS is given by

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k(y_{kl})$$

$$= \prod_{i=1}^{r_1}\left(\prod_{j=1}^{m_1}\frac{m_1!\ (\lambda_1 \exp(-\lambda_1\Phi_{ij}))}{(i-1)!\,(m_1-i)!}\exp(-\lambda_1\Phi_{ij})\right)^{i-1}\left(-\exp(-\lambda_1\Phi_{ij})\right)^{m_1-i+1}$$
$$\prod_{k=1}^{r_2}\left(\prod_{l=1}^{m_2}\frac{m_2!\ (\lambda_2 \exp(-\lambda_2\xi_{kl}))}{(k-1)!\,(m_2-k)!}\exp(-\lambda_2\xi_{kl})\right)^{k-1}[-\exp(-\lambda_2\xi_{kl})]^{m_2-k+1}$$

$$L = W\lambda^{n_1}\lambda^{n_2}\prod_{i=1}^{r_1}\left(\prod_{j=1}^{m_1}\exp(-\lambda_1\Phi_{ij})\right)^{i-1}\left[-\exp(-\lambda_1\Phi_{ij})\right]^{m_1-i+1}\exp(-\lambda_1\Phi_{ij})$$
$$\prod_{k=1}^{r_2}\left(\prod_{l=1}^{m_2}\exp(-\lambda_2\xi_{kl})\right)^{k-1}[1-\exp(-\lambda_2\xi_{kl})]^{m_2-k+1}\exp(-\lambda_2\xi_{kl}) \quad (4.1)$$

where

$$W = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\frac{m_1}{(i-1)!\,(m_1-i)!}\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\frac{m_2}{(k-1)!\,(m_2-k)!}$$

Taking log

$$log\,L = log\,W + n_1 log\lambda_1 + n_2 log\lambda_2 + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\,log\,[exp(-\lambda_1\Phi_{ij}) - \lambda_1\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\Phi_{ij} + \lambda_1\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i+1)\Phi_{ij} - \lambda_2\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\xi_{kl} + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\,log\,[exp(-\lambda_2\xi_{kl}) + \lambda_2\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2-k+1)\xi_{kl}$$

$$\frac{\partial logL}{\partial\lambda_1} = 0$$

Then,

$$\frac{n_1}{\lambda_1} - \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\frac{(i-1)\exp(-\lambda_1\Phi_{ij})\,\Phi_{ij}}{\exp(-\lambda_1\Phi_{ij})} - \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\Phi_{ij} + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i+1)\Phi_{ij} \tag{4.2}$$

$$\frac{\partial logL}{\partial\lambda_2} = 0$$

Then,

$$\frac{n_2}{\lambda_2} - \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\frac{(k-1)\exp(-\lambda_2\xi_{kl})\,\xi_{kl}}{\exp(-\lambda_2\xi_{kl})} - \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\Phi_{ij} + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_1-k+1)\xi_{kl} \tag{4.3}$$

Using a numerical method, we ascertain the values of the ML estimators for λ_1 and λ_2 based on RSS shown by ${\lambda_1}_{RSS}^{ML}$ and ${\lambda_2}_{RSS}^{ML}$ and using the invariance property of the ML estimator, we get the maximum of reliability parameter P based on RSS as

$$P_{RSS}^{ML} = \frac{{\lambda_1}_{RSS}^{ML}}{{\lambda_1}_{RSS}^{ML} + {\lambda_2}_{RSS}^{ML}} \tag{4.4}$$

where, $\lambda_1 = \frac{1}{\bar{\Phi}}$ and $\lambda_2 = \frac{1}{\bar{\xi}}$

$$P_{RSS}^{ML} = \frac{\bar{\xi}_{RSS}^{ML}}{\bar{\Phi}_{RSS}^{ML} + \bar{\xi}_{RSS}^{ML}}$$

where,

$$\bar{\Phi}_{RSS}^{ML} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^{-\beta} = \bar{T}_{x,RS\,S} \quad and \quad \bar{\xi}_{RSS}^{ML} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i^{-\mu} = \bar{T}_{y,RSS}$$

$$P_{SRS}^{ML} = \frac{\bar{T}_{y,RSS}}{(\bar{T}_{x,RSS} + \bar{T}_{y,RSS})}$$

5. Simulation Study

This section contains the simulation study that compares our suggested reliability estimator P based on RSS with the conventional reliability estimator of P based on SRS using the provided MSE and Bias values, $Bias(\hat{P}) = E(\hat{P} - P)$ and $MSE(P) = E(\hat{P} - P)^2$, respectively. The relative efficiency of the estimator of P is calculated as $= \frac{MSE(P_{MLE,SRS})}{MSE(P_{MLE,RSS})}$. If the value of relative efficiency is greater than one, it signifies that $\bar{P}_{SRS}$ is more efficient than the $\bar{P}_{RSS}$. Using the R programming language, all calculations were carried out. The following steps are used to explain the simulation study.

1. Generate 1000 simple random sample of $x_1, \dots, x_{n_1}$ and $y_1, \dots, y_{n_2}$ from Generalize Inverse Weibull distribution with the sample sizes (n_1, n_2).
2. Generate 1000 random sample $x_{11}, \dots, x_{m_1 r_1}$ and $y_{11}, \dots, y_{m_2 r_2}$ from Generalize inverse Weibull distribution with set sizes $m_1 = m_2 = 3, 4, 5$ in case of number of cycles $r_1 = r_2 = 5$ and when $r_1 = r_2 = 10$ then set size $m_1 = m_2 = 2, 3, 4$.
3. Initially the parameter for X ~ GIW (α, β, γ) distribution are taken as α = 2, β = 0.1, γ = 0.6 and Y ~ GIW (θ, μ, χ), θ = 3, μ= 0.2, χ = 0.5. After that we vary α = 2.5, 4 and θ = 3.5, 5 respectively and other parameters are fixed.
4. The MSEs relative efficiency and biased are calculated.

Table1: *Biases, MSES and RE of P under SRS and RSS when β = 0.1, γ = 0.6 and μ= 0.2, χ = 0.5 and r_1=r_2= 5, 10*

				SRS			RSS			
				$\boldsymbol{r_1}$= $\boldsymbol{r_2}$= 5						
	(n_1,n_2)	(m_1,m_2)	P_{True}	$\hat{P}_{SRS}$	Bias	MSE	$\hat{P}_{RSS}$	Bias	MSE	RE
α=2, θ=3	(15,15)	(3,3)	0.50797	0.51985	0.01187	0.000185	0.51981	0.01183	0.000165	1.12343
	(15,20)	(3,4)		0.51953	0.01156	0.000172	0.51989	0.01191	0.000161	1.07190
	(20,20)	(4,4)		0.51964	0.01167	0.000169	0.51966	0.01168	0.000152	1.11753
	(20,25)	(4,5)		0.51965	0.01167	0.000165	0.51987	0.01189	0.000153	1.08270
	(25,25)	(5,5)		0.51960	0.01163	0.000163	0.51996	0.01198	0.000154	1.06218
α=2.5, θ=3.5	(15,15)	(3,3)	0.50584	0.51947	0.01362	0.000218	0.51995	0.01410	0.000215	1.01260
	(15,20)	(3,4)		0.51946	0.01361	0.000212	0.51993	0.01408	0.000212	1.00436
	(20,20)	(4,4)		0.51956	0.01371	0.000212	0.51965	0.01380	0.000201	1.05510
	(20,25)	(4,5)		0.51956	0.01371	0.000209	0.51971	0.01386	0.000200	1.04142
	(25,25)	(5,5)		0.51952	0.01367	0.000207	0.51982	0.01397	0.000202	1.02271
α=4, θ=5	(15,15)	(3,3)	0.49976	0.51995	0.02018	0.000421	0.51995	0.02019	0.000416	1.01253
	(15,20)	(3,4)		0.51994	0.02017	0.000418	0.51993	0.02017	0.000413	1.01343
	(20,20)	(4,4)		0.51992	0.02015	0.000417	0.51998	0.02021	0.000414	1.00835
	(20,25)	(4,5)		0.51992	0.02015	0.000416	0.51991	0.02014	0.000410	1.01551
	(25,25)	(5,5)		0.51991	0.02014	0.000415	0.51998	0.02022	0.000412	1.00549
				$\boldsymbol{r_1}$= $\boldsymbol{r_2}$= 10						
	(n_1,n_2)	(m_1,m_2)	P_{True}	$\hat{P}_{SRS}$	Bias	MSE	$\hat{P}_{RSS}$	Bias	MSE	RE
α=2, θ=3	(20,20)	(2,2)	0.50797	0.51986	0.01189	0.000175	0.51953	0.01155	0.000154	1.13637
	(20,30)	(2,3)		0.51984	0.01186	0.000167	0.51984	0.01186	0.000156	1.07116
	(30,30)	(3,3)		0.51984	0.01186	0.000163	0.51975	0.01177	0.000150	1.08497
	(30,40)	(3,4)		0.51983	0.01186	0.000159	0.51980	0.01182	0.000149	1.06985
	(40,40)	(4,4)		0.51970	0.01172	0.000153	0.51992	0.01194	0.000150	1.02107
α=2.5, θ=3.5	(20,20)	(2,2)	0.50584	0.51974	0.01389	0.000217	0.51947	0.01362	0.00020	1.08420
	(20,30)	(2,3)		0.51972	0.01387	0.000211	0.51972	0.01387	0.000203	1.03920
	(30,30)	(3,3)		0.51972	0.01387	0.000208	0.51964	0.01379	0.000199	1.04873
	(30,40)	(3,4)		0.51971	0.01386	0.000205	0.51969	0.01384	0.000198	1.03801
	(40,40)	(4,4)		0.51960	0.01375	0.000201	0.51978	0.01394	0.000200	1.00415
α=4, θ=5	(20,20)	(2,2)	0.49976	0.51992	0.02015	0.000417	0.51997	0.02020	0.000416	1.00415
	(20,30)	(2,3)		0.51990	0.02013	0.000414	0.51989	0.02013	0.000410	1.00944
	(30,30)	(3,3)		0.51990	0.02013	0.000413	0.51984	0.02007	0.000407	1.01444
	(30,40)	(3,4)		0.51989	0.02013	0.000411	0.51987	0.02011	0.000407	1.00961
	(40,40)	(4,4)		0.51982	0.02005	0.000408	0.51994	0.02018	0.000392	1.04033

The data presented in Table 1 consistently demonstrates that the relative efficiency exceeds unity, indicating the superior performance of ranked set sampling over simple random sampling in stress-strength reliability estimation.

6. Real data application

In order to confirm the results from earlier portions of the paper, we looked at two actual datasets in this section. We use two real-life data sets proposed by Efron B. [21]. The dataset includes patients from two groups who have head and neck cancer diseases. The survival times of 58 patients with radiotherapy-treated head and neck cancer are shown in the first dataset, whereas the survival times of 45 patients receiving chemotherapy plus radiation treatment are shown in the second dataset. In the context of stress–strength reliability, Yadav et al. [22] analysed these datasets and they showed that the Inverse Weibull distribution could be used to model these datasets. These datasets can also be useful in the case of generalized Weibull distribution. The first dataset of 58 patient is used for the strength variable X and the second dataset of 45 patients is used for stress variable Y in the stress-strength model $P = P_r(Y < X)$. The datasets are as follows:

First data set of 58 patients

6.53	7	10.42	14.48	16.1	22.7	34	41.55
42	45.28	49.4	53.62	63	64	83	84
91	108	112	129	133	133	139	140
140	146	149	154	157	160	160	165
146	149	154	157	160	160	165	173
176	218	225	241	248	273	277	297
405	417	420	440	523	583	594	1101
1146	1417						

Second data set of 45 patients

12.2	23.56	23.74	25.87	31.98	37	41.35	47.38
55.46	58.36	63.47	68.46	78.26	74.47	81	43
84	92	94	110	112	119	127	130
133	140	146	155	159	173	179	194
195	209	249	281	319	339	432	469
519	633	725	817	1776			

The first dataset of 58 patient is used for the strength variable X ~ GIW (α, β, γ) and the second dataset of 45 patients is used for stress variable Y ~ GIW (θ, μ, χ) in the stress-strength model $P = P_r(Y < X)$. By using the iteration method in R software, the MLES of α, β, γ and θ, μ, χ is comes out as $\hat{\alpha} = 2.9057$, $\hat{\beta} = 0.7859$, $\hat{\gamma} = 12.3257$ and $\hat{\theta} = 6.8548$, $\hat{\mu} = 1.0248$, $\hat{\chi} = 11.5366$. Now if we take these MLEs values of the parameters as the true value for these datasets then the stress-strength model $P = P_r(Y < X)$ from the Eq. (3.3) is $P = 0.25574$

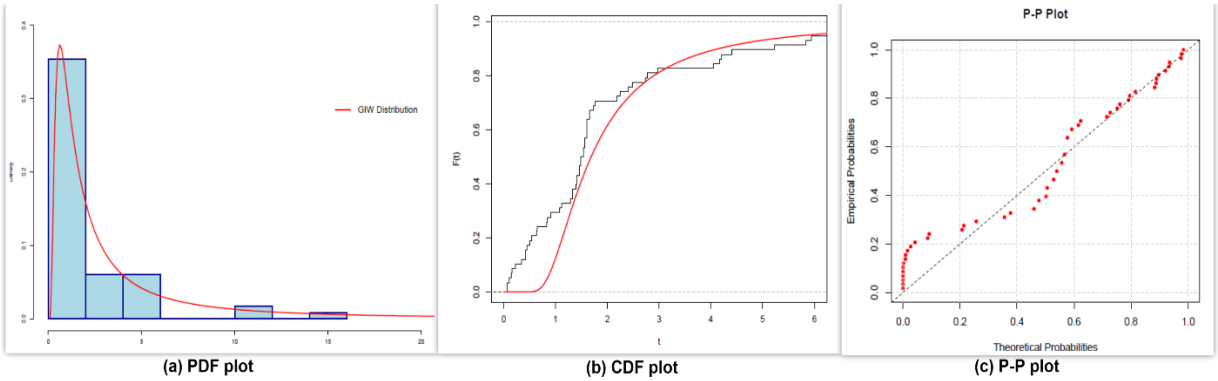


Figure 3: *The PDF, CDF and P-P Plots of the GIW distribution for First dataset*

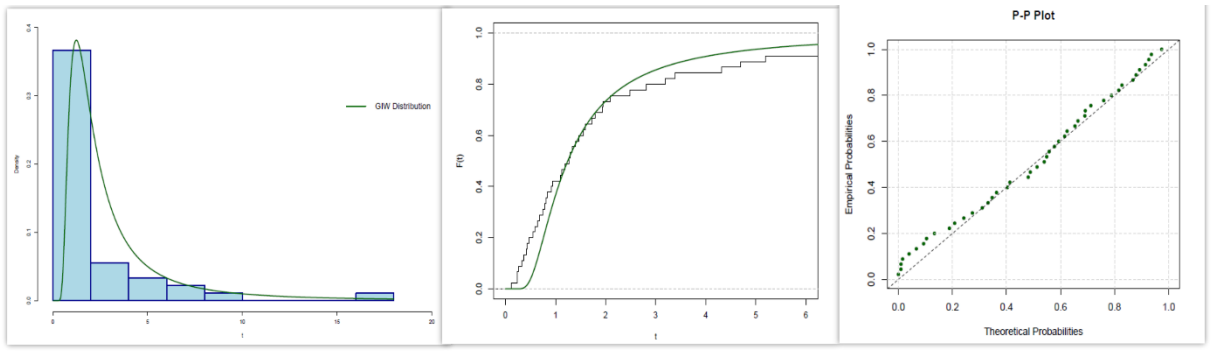


Figure 4: *The PDF, CDF and P-P Plots of the GIW distribution for First dataset*

Prior to delving into the core of our investigation, it is imperative to conduct a thorough examination of the salient characteristics of our dataset. To validate the robustness of our results, we employ a rigorous statistical methodology: the Kolmogorov-Smirnov (K-S) test, complemented by its associated P-value (P-V). This approach facilitates the quantification of the concordance between our empirical observations and theoretical expectations.

Our analysis yields promising results. For the initial dataset, we obtain a K-S distance of 0.31547 and a corresponding P-V of 0.42560. The secondary dataset exhibits comparable outcomes, with a K-S distance of 0.08889 and a P-V of 0.99520. These metrics provide substantial evidence supporting the close alignment of our model with the observed data.

To enhance comprehension and provide visual context, we have generated a series of graphical representations. These illustrations, presented in Figures 3 and 4, offer a comprehensive visualization of our statistical findings. They encompass probability-probability (PP) plots, as well as depictions of the estimated probability density function (PDF) and cumulative distribution function (CDF) for both datasets. These visual aids serve to corroborate and elucidate the numerical results of our analysis, thereby facilitating a more profound understanding of the data's underlying characteristics.

Now we draw 10 samples random sample of size 10 from each dataset and calculate the term $\bar{T}_x$ and $\bar{T}_y$ for each sample respectively. The simple random samples from each dataset are shown in Table 2 and Table 3, respectively.

Table 2: *Simple random samples from Data set 1*

											$\bar{T}_x$
Sample 1	157	176	63	129	7	53.62	140	149	218	225	0.04257
Sample 2	241	154	218	140	63	10.42	277	173	165	154	0.03312
Sample 3	157	149	49.4	440	165	154	140	34	273	22.7	0.03118
Sample 4	149	63	165	139	45.28	49.4	42	41.55	10.42	22.7	0.05443
Sample 5	157	112	14.48	173	440	218	160	139	417	140	0.02745
Sample 6	218	225	146	149	42	1417	176	7	297	53.62	0.04135
Sample 7	165	157	133	63	173	420	45.28	112	176	583	0.02213
Sample 8	225	140	165	277	149	14.48	91	129	108	157	0.03016
Sample 9	157	64	157	63	154	146	165	241	16.1	133	0.03187
Sample 10	140	22.7	417	176	139	149	146	41.55	583	241	0.02662

Table 3: *Simple random samples from Data set 2*

											$\bar{T}_y$
Sample 1	1776	119	74.47	725	81	37	469	43	63.47	58.36	0.01097
Sample 2	110	209	130	281	37	63.47	74.47	173	112	319	0.00889
Sample 3	173	55.46	37	68.46	63.47	281	319	25.87	31.98	195	0.01481
Sample 4	339	74.47	37	112	63.47	81	25.87	31.98	209	110	0.01492
Sample 5	249	281	173	47.38	81	633	37	23.56	74.47	140	0.01256
Sample 6	469	633	519	127	25.87	78.26	84	173	339	37	0.01019
Sample 7	1776	94	817	23.56	469	130	43	146	173	68.46	0.01044
Sample 8	127	110	78.26	249	209	41.35	94	43	432	155	0.00947
Sample 9	110	159	23.74	194	31.98	633	1776	155	112	179	0.01061
Sample 10	432	130	119	339	58.36	127	469	37	146	55.46	0.00902

In the next step, we draw 10 ranked set samples of size 10 from both the Data sets. To draw the ranked set sample of size n = 10, we take the set size m = 5 and run r = 2 cycles.

Table 4 : *Ranked set samples from Data set 1*

											$\bar{T}_x$
Sample1	7	112	157	165	405	10.42	108	139	160	225	0.05240
Sample 2	139	225	297	165	583	10.42	64	417	112	440	0.03091
Sample 3	7	165	165	160	173	112	63	49.4	146	160	0.04368
Sample 4	42	63	176	160	149	6.53	133	154	133	277	0.04495
Sample 5	22.7	7	165	405	1417	91	112	154	218	1417	0.04232
Sample 6	91	64	146	160	1101	112	108	157	154	1101	0.02010
Sample 7	84	139	405	149	1146	108	154	165	218	157	0.01795
Sample 8	6.53	140	64	165	594	140	133	84	420	241	0.04069
Sample 9	16.1	149	160	154	154	45.28	16.1	146	139	277	0.04040
Sample 10	10.42	45.28	160	225	594	6.53	149	149	277	417	0.05364

Table 5: *Ranked set samples from Data set 2*

											$\bar{T}_y$
Sample 1	43	55.46	195	319	249	23.56	41.35	155	130	173	0.01271
Sample 2	25.87	94	58.36	159	469	63.47	68.46	146	194	817	0.01070
Sample 3	12.2	63.47	173	146	469	41.35	155	112	155	633	0.01469
Sample 4	63.47	112	119	339	249	25.87	84	81	81	725	0.01053
Sample 5	47.38	55.46	195	249	817	41.35	92	155	281	1776	0.00855
Sample 6	12.2	37	119	155	817	23.74	37	194	432	519	0.01877
Sample 7	74.47	140	63.47	173	519	55.46	55.46	319	74.47	432	0.00887
Sample 8	23.56	119	78.26	195	725	23.56	130	133	281	519	0.01213
Sample 9	37	94	194	195	469	31.98	41.35	173	155	249	0.01100
Sample 10	37	112	68.46	179	319	81	58.36	339	94	817	0.00930

To get 10 samples of size 10, we totally run 20 cycles. Each two subsequent cycles constitute one sample of size 10. The 20 cycles we performed to get the 10 ranked samples from population X and displayed in Table 4. Similarly, the 20 cycles from population Y are drawn and the 10 ranked set samples are shown in Table 5.

Table 6: *Bias, MSE and Relative efficiency of stress-strength model P in case of SRS and RSS*

	SRS					RSS					
	$\bar{T}_x$	$\bar{T}_y$	$\hat{P}_{SRS}$	Bias	MSE	$\bar{T}_x$	$\bar{T}_y$	$\hat{P}_{RSS}$	Bias	MSE	RE (%)
Sample 1	0.04257	0.01097	0.25278	-0.00295	0.052	0.05240	0.01271	0.23154	-0.0242	0.00872	5.92
Sample 2	0.03312	0.00889				0.03091	0.01070				
Sample 3	0.03118	0.01481				0.04368	0.01469				
Sample 4	0.05443	0.01492				0.04495	0.01053				
Sample 5	0.02745	0.01256				0.04232	0.00855				
Sample 6	0.04135	0.01019				0.02010	0.01877				
Sample 7	0.02213	0.01044				0.01795	0.00887				
Sample 8	0.03016	0.00947				0.04069	0.01213				
Sample 9	0.03187	0.01061				0.04040	0.01100				
Sample10	0.02662	0.00902				0.05364	0.00930				

Based on Table 6, it can be inferred that the MSE of stress-strength model P under RSS conditions is lower than that of stress-strength model P under SRS circumstances. 5.92% of RSS is relative to SRS, or relative efficiency. Thus, in real-world scenarios, RSS techniques yield better outcomes than SRS strategies.

7. Conclusions

This work addresses the challenge of estimating the reliability function $P = P_r(Y < X)$, where X and Y are the independent random strength and stress variables from GIW distribution. The MLE of P is derived for SRS and RSS. Monte Carlo simulation study is performed to compare between point estimators of P in both cases SRS and RSS. The MLE of P based on RSS is more efficient results than the MLE of P based on the SRS. We further validate the advantages of RSS through an analysis of real-life data. Future research we shall explore the application of various type of ranked set sampling techniques in estimating the reliability models.

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AGRICULTURAL PRODUCTION PATTERNS IN TAMIL NADU: INSIGHTS FROM VECTOR AUTOREGRESSIVE ANALYSIS USING PYTHON PROGRAMMING

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Abstract

Understanding agricultural production patterns is crucial for enhancing productivity and ensuring food security. This study explores the dynamics of agricultural production in Tamil Nadu using the Vector Autoregressive (VAR) model to capture the interdependence among various crop yields and rainfall over time. Employing Python programming for data analysis and modeling, the study leverages historical time-series data to identify trends, forecast production, and analyze the impact of external shocks on agricultural outputs. The research incorporates preprocessing techniques to ensure stationarity, optimal lag selection using Akaike's Information Criterion(AIC) and Bayesian Information Criterion(BIC), and diagnostic checks for model accuracy and stability. The findings provide insights into the temporal relationships among various crops and rainfall. Additionally, Impulse Response Functions(IRF) and variance decomposition analyses offer a deeper understanding of how shocks to one variable propagate through the system. The study demonstrates the utility of Python-based VAR models in agricultural forecasting and decision-making, offering policymakers and stakeholders a robust tool to improve resource allocation and agricultural planning in Tamil Nadu. This work highlights the potential of data-driven approaches to address challenges in the agricultural sector effectively.

Keywords: Time Series Analysis, Stationary, Impulse Response Functions, Augmented Dickey-Fuller Test, Granger Causality Test, Vector Autoregressive.

I. Introduction

Agriculture plays a pivotal role in Tamil Nadu's economy, serving as a cornerstone for livelihood, food security, and economic growth. The state's diverse agro-climatic conditions enable the cultivation of various crops, making it a vital contributor to India's agricultural output. Farook and Kannan[8], investigate the influence of climate change on the yields of Kharif and Rabi rice crops, focusing on the impact of maximum temperature, minimum temperature, and rainfall using a VAR model that incorporates Granger causality, IRF, and variance decomposition. Their results reveal significant temperature effects on both types of rice, with rainfall negatively affecting Kharif yields and both maximum temperature and rainfall reducing Rabi yields.

This paper focuses on forecasting the production of major crops, including Paddy, Cumbu, Cholam, Ragi and Maize. These crops not only fulfill the dietary needs of Tamil Nadu's population but also play a pivotal role in the state's agricultural economy and resilience. Their significance lies in their ability to support food security, adapt to diverse climatic conditions, and provide livelihood opportunities to farmers across the state. Promoting sustainable cultivation practices and value addition for these crops can further enhance their role in Tamil Nadu's agricultural development. However, the agricultural sector faces several challenges, including erratic rainfall patterns, fluctuating market dynamics, and the increasing need for efficient resource allocation. Understanding the complex interdependencies among these factors is critical for informed decision-making and sustainable agricultural development.

Modern analytical methods provide powerful tools to examine the interdependencies and dynamics within agricultural systems. VAR analysis stands out as a versatile econometric model for understanding the relationships between multiple time-series variables. The VAR model allows researchers to identify causal relationships, forecast trends, and evaluate the impact of external shocks, making it a powerful tool for analyzing agricultural production patterns. This study applies the VAR model to explore the interdependencies of agricultural production variables in Tamil Nadu, such as crop yields and rainfall. Leveraging the computational power and versatility of Python programming, the research provides a comprehensive analysis of historical data to uncover trends and derive actionable insights. The objective of this research is to forecast agricultural production, understand the influence of key factors, and provide a data-driven foundation for policymaking and resource optimization. By integrating modern computational tools with advanced econometric techniques, this study contributes to enhancing agricultural planning and addressing the challenges of the sector in Tamil Nadu.

Granger[4], introduced testable definitions of causality and feedback using simple two-variable models. He developed a causality test to determine the directional relationships between variables, providing a systematic approach to identify causal links. Additionally, he addressed the critical issue of instantaneous causality, further enriching the understanding of temporal relationships in time series data.

II. Review of Literature

This section reviews and discusses several foundational and relevant papers that provide context and support for this article. Granger and Newbold[5], critically examines the pervasive issue of spurious regression in econometrics, particularly in time series analysis, and its implications for applied research. The authors highlight the frequent reporting of regression equations with high R^2 values but alarmingly low Durbin-Watson statistics, indicating significant autocorrelation in residuals. Despite warnings in econometric textbooks, such errors persist in respected literature, leading to inefficiencies, suboptimal forecasts, and invalid significance tests of coefficients. The findings of Dickey and Fuller[2] hold considerable importance for hypothesis testing and parameter estimation in autoregressive models, particularly in identifying whether a time series is stationary or nonstationary. By offering methods to test the unit root hypothesis ($H_0 : p = 1$), the paper contributes to the broader econometric literature on time series analysis. It underscores the importance of understanding the behavior of estimators in near-nonstationary environments, providing a foundation for more robust inference in such contexts.

Runkle[12], critically evaluates the utility of unrestricted VARs in understanding the interrelationships among key macroeconomic variables such as interest rates, money, prices, and output. The evidence highlights significant limitations in drawing strong conclusions using this approach. Granger[6] explored the intricate connection between causality, statistical methods, and their practical implications, emphasizing the importance of thoughtful evaluation in both

theoretical and applied settings. Johansen[9], explores the statistical framework for analyzing nonstationary VAR processes that are integrated of order 1, focusing on cointegration properties. The authors derive the maximum likelihood estimator for the cointegration space and propose a likelihood ratio test to evaluate the dimensionality of the cointegration vectors. Additionally, they develop tests for linear hypotheses about the cointegration vectors.

Barkley et al.,[1] provide an in-depth examination of the institutional framework of the Saudi economy using a VAR model. The study reveals that external variables, such as world inflation rates and Saudi oil policies, play a crucial role in shaping the country's economic outcomes. Waggoner and Zha[15], develops Bayesian methods to compute the exact finite-sample distribution of conditional forecasts in VAR models, addressing parameter uncertainty and expanding their applicability. The study highlights the practical utility of these methods in assessing monetary policy impacts and analyzing scenarios with specific economic conditions. This advancement enhances the reliability of VAR-based macroeconomic forecasts for policy and practical applications.

Stock and Watson[14], analyze the role of VARs in macro-econometrics, emphasizing their effectiveness in data description and forecasting. While VARs excel in capturing dynamic relationships among time series, the study highlights their limitations in structural inference and policy analysis due to the "identification problem," which requires economic theory or institutional insights to resolve. Zivot[16] offers a comprehensive overview of the VAR model, emphasizing its versatility in analyzing multivariate time series, especially in economic and financial contexts. Also primarily focused on VAR for stationary time series, while also previewing its extension to nonstationary series with cointegration. Sasikumar and Sheik[13], provided text highlights the critical role of financial market volatility in shaping investment decisions and regulatory policies. It effectively underscores the significance of time series modeling, particularly in the context of forecasting stock market dynamics. The discussion about exploring the interconnections between variables such as the dollar rate, crude oil, and fuel prices using a VAR model is particularly compelling.

Hamzah et al.,[7] present a thorough application of the VAR model to investigate the export dynamics of Indonesia's major agricultural commodities—coffee beans, cacao beans, and tobacco—over a ten-year period. The study effectively demonstrates the suitability of the VAR model for multivariate time series analysis, particularly for capturing the intricate dynamic relationships between endogenous and exogenous variables. By evaluating VAR models with varying lag structures (VAR(1) to VAR(5)), the researchers employ a rigorous model selection process based on well-established statistical criteria, including AIC, Corrected AIC, Schwarz Bayesian Criterion (SBC), and Hannan-Quinn Information Criterion (HQIC). The selection of the VAR(2) model as the best fit is robustly supported by these criteria, significantly enhancing the reliability and credibility of the findings.

III. Data Source and Basic Statistics

The data for this study were obtained from the official website of the Department of Economics and Statistics, Government of Tamil Nadu, India (https://www.tn.gov.in/crop/stat.html), covering the period from 1990-91 to 2022-23. This source provides reliable and comprehensive statistical information, ensuring the validity and accuracy of the analysis conducted in this research. Table 3.1 presents the basic statistical summary of the dataset used in this study. This summary provides an initial understanding of the data distribution and variability, which is essential for further analysis. Paddy and Ragi are more stable compared to Maize and Cholam, which exhibit higher fluctuations in yield. Cholam's skewness and kurtosis suggest potential outliers or irregular growth conditions.

Table 3.1: *Basic Statistics*

Basic Statistics	Rainfall (in mm)	Paddy (in Tonnes)	Cholam (in Tonnes)	Cumbu (in Tonnes)	Ragi (in Tonnes)	Maize (in Tonnes)
Maximum	1401.1	8141300	868940	296270	362343	2989945
Minimum	598.1	3222776	153856	56505	114429	43820
Mean	973.47	6253034.87	363440.03	155494.72	230889.75	1043979.06
Median	985.8	6610607	345820	146132	227476	759112
Skewness	0.1087	-0.75	1.25	0.48	0.1350	0.69
Kurtosis	-0.6067	-0.19	2.75	-0.65	-1.0332	-1.20
CV	0.19867	0.21404	0.40525	0.40354	0.29949	1.0408
SD	190.4448	1317966.677	145036.983	61790.6097	68093.8147	1070031.06

IV. Methodology

The VAR model is a statistical model used to capture the linear interdependencies among multiple time series. The VAR model is an extension of the univariate autoregressive (AR) model to multivariate time series data, allowing each variable to be a function not only of its past values but also of the past values of all other variables in the system. The following flowchart is representing the procedure of VAR model.

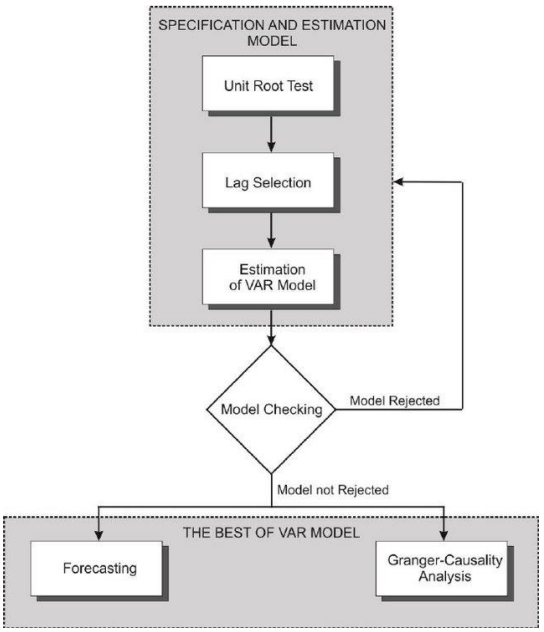

Figure 4.1: *Flow Chart of VAR Analysis Procedure*

4.1. Assumptions of VAR Model

The following are basic assumptions for performing VAR analysis

- *No Multicollinearity* - Variables should not be perfectly correlated.
- *Linearity* - The relationships between variables are assumed to be linear.
- *No Serial Correlation in Residuals* - Check for autocorrelation in residuals after fitting the model using tests like the Ljung-Boxtest.
- *Homoscedasticity* - The variance of residuals should be constant over time.

4.2. General Form of VAR(p) Model

For a system of *k* variables, the VAR(*P*) model is written as

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \in_t \tag{1}$$

where $Y_t = \begin{bmatrix} y_{1t} & y_{2t} & \cdots & y_{kt} \end{bmatrix}'$ vector of k variables at time *t*, A_i is $k \times k$ coefficient matrices for lag *i*, p is number of lags and $\in_t = \begin{bmatrix} \in_{1t} & \in_{2t} & \cdots & \in_{kt} \end{bmatrix}'$ Vector of error terms (innovations), assumed to be white noise, i.e., $\in_t \sim N(0, \Sigma)$.

4.3. VAR(1) Model with Two Variables

Consider a VAR(1) model(first order VAR) with two variables

$$Y_{1t} = \alpha_1 + \phi_{11} Y_{1(t-1)} + \phi_{12} Y_{2(t-1)} + \in_{1t} \tag{2}$$

$$Y_{2t} = \alpha_2 + \phi_{21} Y_{1(t-1)} + \phi_{22} Y_{2(t-1)} + \in_{2t} \tag{3}$$

where α_1 and α_2 are the intercept terms, $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$ are the coefficient for lagged values and $\in_{1t}, \in_{2t}$ are the error terms for each equation. The above equation (2) and (3) can be expressed in matrix form as

$$\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} Y_{1(t-1)} \\ Y_{2(t-1)} \end{bmatrix} + \begin{bmatrix} \in_{1t} \\ \in_{2t} \end{bmatrix} \tag{4}$$

4.4. Granger Causality Test

The Granger causality test evaluates whether past values of one time-series variable (X_t) provide statistically significant information about another variable (Y_t) beyond the information contained in its own past values. This is done by comparing two regression models: a restricted model (without X_t) and an unrestricted model (including X_t). Restricted model(No X_t) is defined as

$$Y_t = \alpha + \sum_{i=1}^{p} \beta_i Y_{t-i} + \in_t \tag{5}$$

where Y_t is current value of the dependent variable, β_i is coefficients of lagged Y_t, p is number of lags and ϵ_t is error term (white noise). Then the Unrestricted Model (Including X_t) is defined as

$$Y_t = \alpha + \sum_{i=1}^{p} \beta_i Y_{t-i} + \sum_{i=1}^{p} \gamma_i X_{t-i} + \in_t \tag{6}$$

where X_t is lagged values of the independent variable and γ_i coefficients for the lagged X_t. The Granger causality test uses an F-test to compare the restricted and unrestricted models is

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(n-k)} \tag{7}$$

here RSS_R is Residual Sum of Squares from the restricted model, RSS_U is Residual Sum of Squares from the unrestricted model, p is number of lags tested, n number of observations and k is Total number of parameters in the unrestricted model. Decision rule of the above F – Statistic, reject H_0 if the p-value is less than the significance level (0.05) and the rejection indicates that X_t Granger-causes Y_t.

4.5. Impulse Response Functions

Lütkepohl[11], offers an overview of IRFs in VAR models, emphasizing their role in analyzing dynamic relationships and responses to shocks. IRF is a fundamental method in time-series econometrics, widely applied within the framework of VAR models. This analysis is designed to evaluate the impact of an unexpected change, referred to as a shock or impulse, in one variable on the other variables within a system. By tracing the effects of such a shock over subsequent time periods, IRF reveals how interconnected variables respond and adapt dynamically.

This approach is particularly useful for understanding the magnitude, direction, and duration of these interactions, offering a comprehensive view of the causal relationships and feedback mechanisms present in complex systems. The IRF measures the effect of a one-time shock in one variable ($\in_{jt}$) on all variables in the system (Y_{it}) over time. The IRF formula is

$$\psi_h = \frac{\partial Y_{t+h}}{\partial \in_t} \tag{8}$$

4.6. Augmented Dickey-Fuller (ADF) Test

Dickey and Fuller[3], examines the statistical properties of a time series model of the form $Y_t = \alpha + \beta Y_{t-1} + \in_t$, where Y_t is fixed and $\in_t$ are independent and normally distributed random variables with mean 0 and variance σ^2. The focus is on the likelihood ratio test for the joint hypothesis (α, β) = (0, 1), which corresponds to a random walk without drift.

ADF test is also known as unit root test. It is a statistical procedure used in time-series analysis to determine whether a time series is stationary or non-stationary. A stationary time series has a constant mean and variance over time, while a non-stationary time series exhibits trends, seasonality, or varying variance. If a time series has a unit root, it means that it is non-stationary and requires differencing to achieve stationarity before applying models like VAR or Auto Regressive Integrated Moving Average (ARIMA). It is sensitive to the choice of lag length (p), which can be selected using criteria like AIC or BIC.

V. Results and Discussions

5.1. ADF Test

In this section, the results of the analysis are presented systematically, focusing on the key findings derived from the data. Before applying the VAR model, it is essential to verify whether the selected data is stationary. The ADF test evaluates this by testing the null hypothesis that the series has a unit root (non-stationary). A low p-value (< 0.05) indicates rejecting the null hypothesis, suggesting that the series is stationary. From the table 5.1, p-values of Rainfall, Paddy, and Ragi have p-values < 0.05, indicating that these data series are stationary. The remaining data series are non-stationary. Accordingly, the crops Paddy and Ragi are suitable for a VAR model, while for the other crops, a Vector Error Correction Model (VECM) or ARIMA model can be applied.

Table 5.1: *Augmented Dickey-Fuller Test*

Crops	ADF Test Statistic	p - value
Rainfall	-4.568784031659	0.0001474082433563
Paddy	-3.966250739089	0.0015979164166240
Cholam	-1.967101070968	0.3011817556291899
Cumbu	-2.607373774359	0.0914828150744177
Ragi	-3.585822200935	0.0060379338458998
Maize	-1.018083747077	0.7466116628126567

The assessment of linearity is conducted using a linear regression model. The calculated R^2 value is 0.08842 for the relationship between Rainfall and Paddy, and 0.000315 for Rainfall and Ragi. Based on these results, we can proceed with applying a VAR model only for Paddy and Ragi.

5.2. Lag Selection

Selecting the optimal lag length in a model is an important step because it determines how many past observations of the variables in the model are used to predict their future values. Incorrect lag selection can lead to misleading results.

Table 5.2: *Optimum Lag Selection*

Lag	AIC	BIC	HQIC
0	64.48	64.67	64.54
1	19.01	19.96	19.30
2	20.83	22.54	21.35
3	21.21	23.68	21.96
4	20.84	24.08	21.83
5	23.73	27.73	24.95

The table 5.2 presents the optimal lag order selection for a VAR model based on three selection criteria: AIC, BIC, and HQIC. The minimum AIC value is 19.01, the minimum BIC value is 19.96, and the minimum HQIC value is 19.30 at lag1. The optimal lag length is 1, as determined by all three criteria. This indicates that the relationships between the variables are best captured by considering only the most recent lag in the VAR model.

5.3. Impulse Response Function

In the IRF analysis of Rainfall to Paddy, a unit shock (unexpected change) in rainfall is used, and the graph illustrates its effect on paddy production over time. A one-unit shock in Rainfall results in a significant positive response in Paddy at the start, as indicated by the sharp rise in the impulse response curve. The shock in Rainfall has an immediate positive effect on Paddy production, indicating that rainfall is generally beneficial in the short term. The negative response observed in the subsequent periods may be due to the harmful effects of excessive rainfall, such as waterlogging, soil nutrient depletion, or flooding.

By Period 10, the effect of the shock in Rainfall on Ragi almost vanishes, indicating that the system stabilizes over time. In periods where the confidence intervals include zero, the impact of the shock may not be statistically significant. Initially, the impact is statistically significant (confidence intervals do not cross zero), but as the intervals narrow toward zero over time, the significance of the effect decreases.

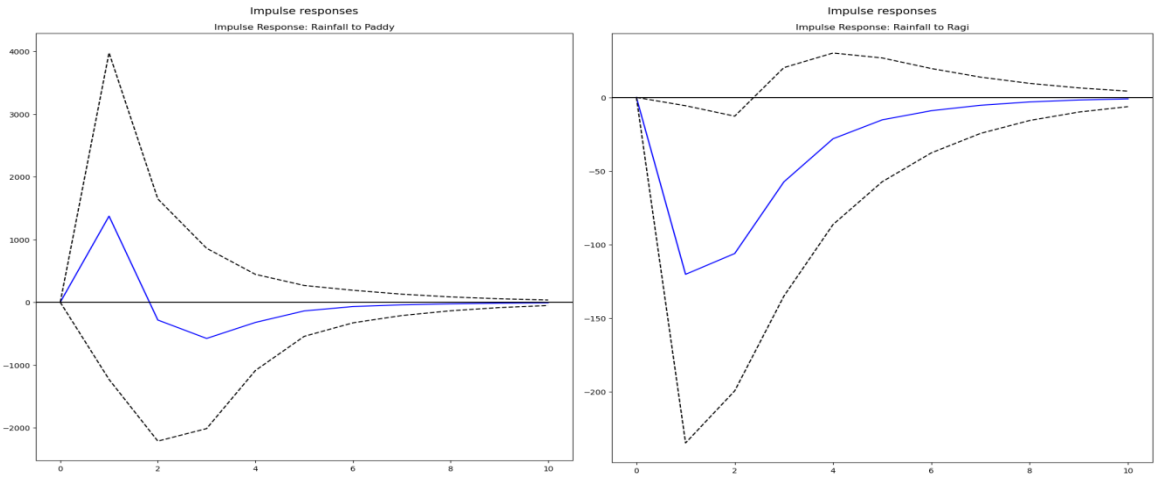

Figure 5.1: *Impulse Response Function Rainfall to Paddy and Ragi*

5.4. VAR Model

Lee[10], conducts an in-depth analysis of the causal relationships and dynamic interactions between asset returns, real activity, and inflation in the postwar United States using a multivariate VAR framework, providing valuable insights into the complex interplay between macroeconomic indicators and financial variables. The presented VAR model estimated using Ordinary Least Squares (OLS). It consists of two equations, each corresponding to one of the endogenous variables: Paddy and Ragi. A total of 33 observations were utilized for estimation, providing the basis for analyzing interdependencies among these variables. The log-likelihood value of -479.937 serves as a measure of the model's overall fit, with higher (less negative) values indicating better fit. To assess and compare model performance, information criteria such as the AIC, BIC, and HQIC are computed. These criteria penalize model complexity, and lower values indicate a better balance between fit and simplicity. Among them, BIC is often favoured due to its stricter penalty for model complexity, making it a preferred metric for model selection.

$$Paddy_t = \alpha_1 + \phi_{11} Year_{t-1} + \phi_{12} Rain\_fall_{t-1} + \phi_{13} Paddy_{t-1} + \in_{Paddy} \quad (9)$$

$$Ragi_t = \alpha_2 + \phi_{21} Year_{t-1} + \phi_{22} Rain\_fall_{t-1} + \phi_{23} Ragi_{t-1} + \in_{Ragi} \quad (10)$$

In equation (9), α_1 represent a Constant term or baseline value for Paddy that is independent of other variables, $\phi_{11} Year_{t-1}$ is the influence of year from the previous period on Paddy, $\phi_{12} Rain\_fall_{t-1}$ is the influence of the previous period's value of Rainfall on Paddy, $\phi_{13} Paddy_{t-1}$ captures the effect of Paddy's own previous value on its current state, reflecting consistency or persistence in its behavior over time. $\in_{Paddy}$ is error term, in other words random shocks or unexplained variation in Paddy. A similar interpretation applies to equation (10).

Table 5.3: *Actual and Forecasted Values*

	Actual Values			Forecasted Values		
Year	Rainfall (in mm)	Paddy (in Tonnes)	Ragi (in Tonnes)	Rainfall (in mm)	Paddy (in Tonnes)	Ragi (in Tonnes)
1990-91	714.6	5782440	316240	714.60	5782440.00	316240.00

1991-92	898.9	6596260	310610	899.76	6289285.80	301198.87
1992-93	862	6805720	291000	919.44	6536957.02	268547.41
1993-94	1171.9	6749810	330970	906.59	6389536.28	261099.76
1994-95	933.8	7558710	285020	982.96	7036997.82	245495.86
1995-96	750.6	5290030	221060	904.29	6492306.99	243275.31
1996-97	1121.2	5805300	190530	935.48	5804547.51	251970.63
1997-98	1133.8	6893730	217940	1008.06	6142549.85	185342.88
1998-99	1080.4	8141300	240610	979.00	6365657.99	189505.47
99-2000	896.8	7532100	245940	929.53	6479790.50	197842.16
2000-01	785.3	7366320	259490	908.25	6254337.95	229478.04
2001-02	795.2	6583630	235310	889.79	6187846.23	252689.53
2002-03	731	3577108	140169	919.15	6039911.11	245827.78
2003-04	1034.6	3222776	176381	1001.40	5301287.55	229395.41
2004-05	1078.9	5061622	154085	1085.72	5907828.88	215194.89
2005-06	1304.1	5209433	132172	1040.43	5910618.67	181648.21
2006-07	859.7	6610607	148148	1090.67	6093582.41	141017.99
2007-08	1164.8	5039954	175944	945.14	5658135.72	192788.63
2008-09	1023.1	5183385	169944	1068.17	6173918.51	184698.28
2009-10	937.8	5665258	160939	1032.94	5964952.35	198239.93
2010-11	1165.1	5792415	171096	1000.19	5825139.33	200128.79
2011-12	937.1	7458657	224862	1051.55	6196705.45	176887.79
2012-13	743.1	4050334	138011	948.11	6276321.94	220050.68
2013-14	790.6	7115195	362343	1012.75	5400115.14	227811.75
2014-15	987.9	7949437	349628	929.35	6886061.78	317211.53
2015-16	1118	7374681	282054	951.42	7110411.10	278908.07
2016-17	598.1	3554113	114429	1002.30	6876184.23	231659.58
2017-18	1017.2	6638450	321332	1003.97	5084997.01	239505.93
2018-19	698.9	6131550	255975	1006.66	6960801.78	273190.79
2019-20	985.8	7265161	274474	950.78	6149976.56	282109.08
2020-21	1232.8	6881725	288627	984.29	6684976.18	247023.14
2021-22	1401.1	7906373	227476	1056.34	7087419.11	228083.97
2022 -23	1170.6	7556567	206553	1065.69	6998552.23	165293.84
2023-24	-	-	-	1025.00	6567790.00	186125.00
2024-25	-	-	-	1025.00	6229389.00	202381.00
2025-26	-	-	-	1038.00	6319154.00	214837.00
2026-27	-	-	-	1040.00	6419746.00	219749.00
2027-28	-	-	-	1040.00	6463104.00	221745.00
2028-29	-	-	-	1041.00	6483788.00	223006.00
2029-30	-	-	-	1043.00	6500922.00	223927.00
2030-31	-	-	-	1045.00	6517247.00	224571.00
2031-32	-	-	-	1048.00	6532388.00	225030.00
2032-33	-	-	-	1050.00	6546546.00	225385.00

Table 5.3 provides an actual and forecasted of yearly values of Rainfall, Paddy and Ragi over the period from 1990-91 to 2032-33. The actual and estimated rainfall values over the time period are depicted in Figure 5.2(a). The estimated rainfall values are relatively more stable compared to the actual values, indicating that the forecasting model has smoothed out the extreme variations seen in the actual data. From the overall trend, it appears that both actual and forecasted rainfall have shown similar trajectories in recent years, with both stabilizing around the 1100 mm mark. The consistency of the forecasted values over time suggests that the forecasting method might be using a simplified approach that doesn't capture short-term weather fluctuations.

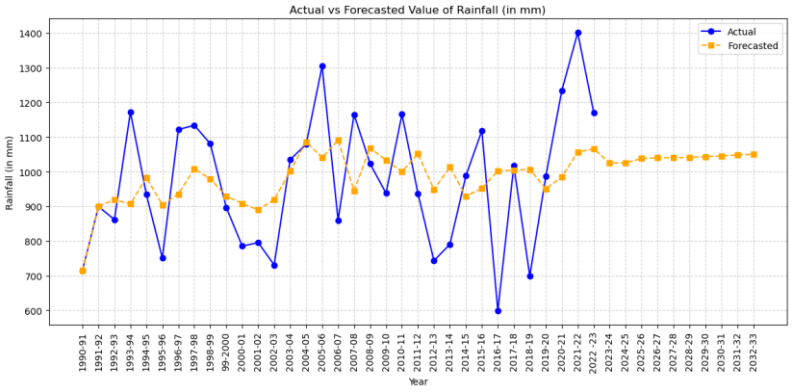

Figure 5.2(a): *Actual vs Forecasted Value of Rainfall (in mm) over the Years*

Figure 5.2(b) illustrates the historical variations in paddy production alongside the actual and forecasted values. From around 2020 onwards, the forecast remains steady, showing no major variation. This stability might suggest an assumption of consistent external conditions affecting paddy production. The closeness of the blue and red lines in many years suggests that the forecasting model performs well in capturing production trends.

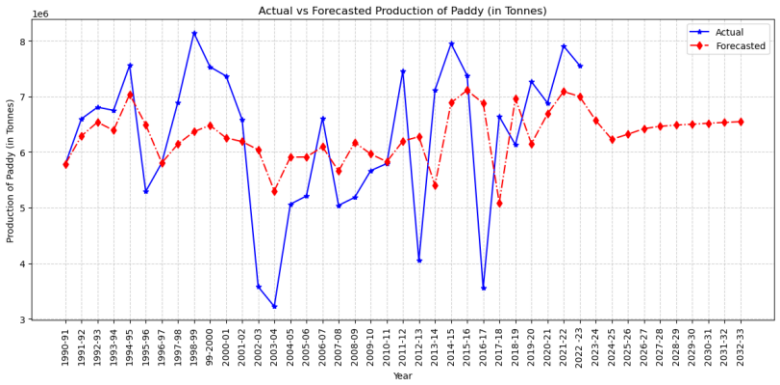

Figure 5.2(b): *Actual vs Forecasted Paddy Production (in Tonnes) over the Years*

Figure 5.3(c) depicts the historical trends in Ragi production along with the actual and forecasted values. The actual production of Ragi exhibits significant fluctuations over the years, with periods of both steep increases and decreases. The forecasted production generally smooths out the variability, following the overall trend of actual production but without sharp deviations. The forecasted production stabilizes in the later years, indicating an assumption of steadier production levels. This could be based on historical trends or constraints in the forecasting method.

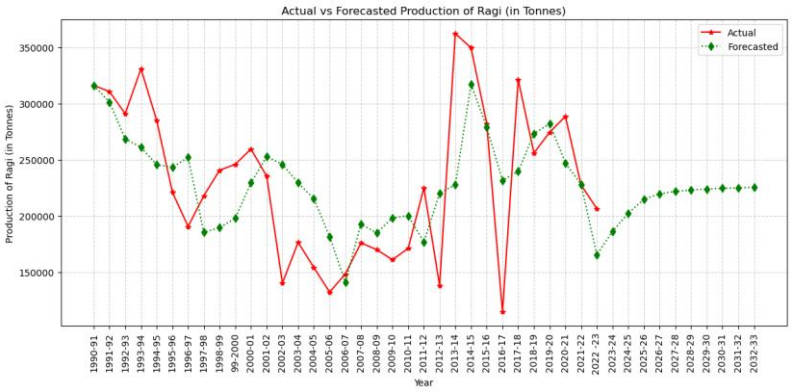

Figure 5.2(c): *Actual vs Forecasted Ragi Production (in Tonnes) over the Years*

VI. Conclusion

Rainfall shocks can have both positive and negative effects on Paddy production. Measures to mitigate short-term disruptions, such as drainage systems or adaptive farming techniques, may help stabilize yields. Over time, the negative impact subsides and stabilizes near zero, suggesting that the system recovers after the shock in Rainfall, and the long-term relationship between Rainfall and Ragi is relatively neutral. The forecast predicts a slow and steady improvement in Rainfall, which is expected to positively influence the production of both Paddy and Ragi. In Rainfall, the accuracy of the forecast can be assessed by comparing the actual data points with the forecasted ones. In some periods, the forecasted values align well with the actual values, while in others, there are noticeable discrepancies. The forecasting model provides a general sense of the production trend for Paddy but struggles with accurately predicting extreme changes. The forecast stabilizes future production, which might oversimplify the dynamics of agricultural production influenced by factors like climate, policy, and market conditions. Refining the model with additional variables or enhancing the methodology could improve accuracy, especially for capturing sharp variations in production. The forecasting model captures the general trend of Ragi production but struggles with periods of high volatility. Future adjustments to the model, such as incorporating additional variables or using more robust techniques, could improve accuracy, particularly during periods of extreme variability. Ragi demonstrates more resilience and consistent growth, whereas Paddy shows variability and slower recovery, signalling the need for specific attention to enhance its yield.

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APPLICATION OF FUZZY DYNAMIC GROUP MULTI-CRITERIA DECISION MAKING BASED ON Z-NUMBERS

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Abstract

Dynamic group multi-criteria decision making is essential for making informed, balanced, and adaptive decisions in complex and evolving environments. By integrating multiple methodologies and considering the dynamic nature of criteria and group interactions, dynamic group multi-criteria decision making provides a robust framework for decision-making across various fields and applications. Dynamic group fuzzy multi-criteria decision making under Z-information is a sophisticated approach that incorporates the dynamic aspects of decision making, the involvement of multiple stakeholders, and the use of fuzzy logic to handle uncertainties and imprecise information. Z-information refers to a type of uncertain information that combines fuzzy numbers and Z-numbers, where Z-numbers account for both the reliability of the information and its fuzziness. By integrating fuzzy logic and Z-numbers, it effectively handles dual uncertainties of fuzziness and reliability, while dynamically adapting to changes in criteria and stakeholder preferences. In this article, a dynamic multi-criteria decision-making model is proposed to solve strategic vendor selection problems that need to be evaluated in different time periods and involve uncertainty. Z-information is used to express uncertainty and in the proposed model, the decision-making group is asked to evaluate the alternatives in different time periods, and the evaluations made for these different periods are combined.

Keywords: Fuzzy logic, Z-numbers, multi-criteria decision making, vendor selection, dynamic group decision making

I. Introduction

Dynamic group multi-criteria decision making is a decision-making process that involves multiple criteria, stakeholders, and evolving scenarios over time. This approach integrates the complexities of group dynamics, changing environments, and various criteria that must be considered to reach an optimal decision. Multi-criteria decision making is indeed a critical branch in management science [1]. It involves evaluating and making decisions based on multiple conflicting criteria. This complexity is inherent in various real-world problems where decision-makers must consider several factors to arrive at the best possible solution [2]. A feature of several practical problems of multicriteria choice on a finite set of alternatives is not only a significant number of criteria and restrictions of various types, but also the presence of dependencies between the criteria, which appear when formalizing the preferences of the decision maker [3]. Additional difficulties arise when comparing qualitative criteria specified, for example, in linguistic scales [4]. Successful solution to the methodological and algorithmic problems that arise when forming a preference

function based on criteria of various types was implemented during the development of dynamic multi-criteria decision-making approach [5]. The information used in the decision-making process of many multi-criteria decision-making problems generally belongs to the same time period. When evaluating alternatives and criteria, the same time period, that is, the time period in which the evaluation is made, is taken into account. However, in some cases, it is necessary to evaluate the current performance of the alternatives as well as their performance in previous time periods. For example, the information required for decision making in investment decisions, medical diagnosis, personnel evaluation, and evaluation of the effectiveness of military systems must be collected in different time periods. For this purpose, Xu and Yager developed dynamic multi-criteria decision-making procedures based on fuzzy sets and dynamic fuzzy weighted environment operator for combining information collected in different time periods [6]. This authors also proposed a dynamic intuitionistic fuzzy multi-attribute decision making process in which all criteria are explained by intuitionistic fuzzy numbers collected at different periods. The authors defined intuitionistic fuzzy and imprecise intuitionistic fuzzy variables and used the intuitionistic fuzzy weighted environment operator when all decision information about attributes in different periods is represented by intuitionistic fuzzy numbers. Su et al. investigated the dynamic intuitionistic fuzzy multi-attribute group decision making problem. In this study, evaluations are made using fuzzy sets by a group of decision makers in different periods [7]. The authors present an interactive method for solving intuitionistic fuzzy multi-attribute group decision making problems. In this method, firstly, decision makers use intuitionistic fuzzy set-in different periods for evaluation alternatives. Dynamic multi-criteria decision making is particularly relevant in environments where information and circumstances continuously change, necessitating ongoing reassessment and adjustment of decisions [8]. Exploring dynamic multi-criteria decision making involves understanding how decision-making processes can adapt to changes over time, incorporating evolving criteria, preferences, and conditions [9]. Dynamic multi-criteria decision making is an advanced approach that integrates the temporal evolution of criteria and preferences, allowing for more flexible, adaptive, and robust decision-making processes. By leveraging methodologies such as dynamic AHP [10], dynamic TOPSIS [11], and fuzzy logic [12], and by continuously incorporating real-time data and feedback, dynamic MCDM can significantly enhance decision quality in complex, changing environments across various fields. The decision-making environment is usually uncertain due to some uncontrollable factors. Lots of applications in the real world are researched based on uncertainty, such as fault diagnosis [13] and reliability analysis [14]. To cope with such uncertainty, many tools have been put forward, including fuzzy set [15], evidence theory [16], linguistic information [17] and Z-numbers [18]. Dynamic decision-making with Z-numbers is an advanced approach that incorporates the uncertainty and reliability of information in evolving scenarios. Z-numbers, introduced by Lotfi A. Zadeh, are an extension of fuzzy numbers that consider both the fuzzy value of a piece of information and its reliability [19]. Dynamic decision-making with Z-numbers is an advanced approach to decision-making that incorporates both uncertainty and reliability in information. It provides a robust framework for handling uncertainty and reliability in evolving scenarios. This framework is particularly useful in dynamic decision-making scenarios, where conditions evolve over time and decision-makers must handle both vague information and varying degrees of trust in that information. By incorporating both the fuzzy value and reliability of information, this approach allows for more adaptive and informed decision-making. It is particularly valuable in complex and dynamic fields such as vendor selection, where continuous updates and adjustments are necessary to respond to changing conditions and new data.

The structure of this article is formed as follows. Section 2 introduces the basic definitions of the dynamic fuzzy multi-attribute decision making approach with Z numbers that is employed in this problem. Section 3 proposed statement and solution of the supplier selection problem. Section 4 represents the main results achieved in this article.

II. Preliminaries

Definition 1. Z-numbers extend the concept of fuzzy numbers to include both the uncertainty of a value and the reliability of that value. A Z-number $Z=\left(\tilde{A},\tilde{B}\right)$ consists of two components - $\tilde{A}$ is a fuzzy number representing an uncertain value, $\tilde{B}$ is fuzzy number representing the reliability of $\tilde{A}$ [19].

Definition 2. Basic operations on Z numbers represented as below [20]. Suppose are given two Z numbers, $z_1=\left(\left(a_1^l,a_1^m,a_1^u\right),\left(b_1^l,b_1^m,b_1^u\right)\right)$ and $z_2=\left(\left(a_2^l,a_2^m,a_2^u\right),\left(b_2^l,b_2^m,b_2^u\right)\right)$.

$$z_1+z_2=\left(\left(\alpha_1a_1^l+\alpha_2a_2^l,\alpha_1a_1^m+\alpha_2a_2^m,\alpha_1a_1^u+\alpha_2a_2^u\right),\right.$$
$$\left.\left(\max\left\{b_1^l/b_1,b_2^l/b_2\right\},\max\left\{b_1^m/b_1,b_2^m/b_2\right\},\max\left\{b_1^u/b_1,b_2^u/b_2\right\}\right)\right. \quad (1)$$

$$z_1*z_2=\left(\left(\alpha_1\alpha_2a_1^la_2^l,\alpha_1\alpha_2a_1^ma_2^m,\alpha_1\alpha_2a_1^ua_2^u\right),\right.$$
$$\left.\left(\min\left\{r_1^l/r_1,r_2^l/r_2\right\},\min\left\{r_1^m/r_1,r_2^m/r_2\right\},\min\left\{r_1^u/r_1,r_2^u/r_2\right\}\right)\right. \quad (2)$$

$$\lambda z_1=\left(\left(\lambda\alpha_1a_1^\lambda,\lambda\alpha_1a_1^m,\lambda\alpha_1a_1^u\right),\left(r_1^l/r_1,r_1^m/r_1,r_1^u/r_1\right)\right),\lambda>0 \quad (3)$$

$$z_1^\lambda=\left(\left(\left(\alpha_1a_1^l\right)^\lambda,\left(\alpha_1a_1^m\right)^\lambda,\left(\alpha_1a_1^u\right)^\lambda\right),\left(b_1^l/b_1,b_1^m/b_1,b_1^u/b_1\right)\right),\lambda>0$$

Definition 3. Z-number-valued pairwise comparison matrix (Z_{ij}) is a matrix of Z-numbers [21]:

$$(Z_{ij}=(A_{ij},B_{ij}))=\begin{pmatrix}Z_{11}=(A_{11},B_{11}) & \ldots & Z_{1n}=(A_{1n},B_{1n})\\ . & \ldots & .\\ Z_{n1}=(A_{n1},B_{n1}) & \ldots & Z_{nn}=(A_{nn},B_{nn})\end{pmatrix} \quad (4)$$

A Z-number $Z_{ij}=(A_{ij},B_{ij}),\ i,j=1,...,n$ describes partially reliable information on degree of preference for i-th criterion against j-th one. $\tilde{A}$ and $\tilde{B}$ can be linguistic terms selected from linguistic set $\tilde{A}_Z$ and $\tilde{B}_Z$, respectively. For example,

$\tilde{A}_Z=\{about\ a_1, nearly\ a_2, exactly\ a_3, over\ a_4\}$ *and* $\tilde{B}_Z=\{very\ low\ sure, low\ sure, shure, very\ sure\}$

where $a_i\ (i=1,\cdots,4)$ can be certain values of evaluation, such as percentages.

Let $D=\lfloor D_{ij}\rfloor$ be the decision-making matrix, where D_{ij} is the evaluation of any alternative with respect to each attribute. $D_{ij}=Z_{ij}\left(\tilde{A},\tilde{B}\right)$ and the Z-number $Z_{ij}\left(\tilde{A},\tilde{B}\right),\ i=1,2,\cdots,m,\ j=1,2,\cdots,n$ is the evaluation of the j - th criteria for i - th alternative, which contains the opinion of evaluators, $\tilde{A}$, and reliability of the opinion, $\tilde{B}$. Thus, the decision-making problem can be modelled as below.

$$D=\begin{bmatrix}Z_{11}\left(\tilde{A},\tilde{B}\right) & Z_{11}\left(\tilde{A},\tilde{B}\right) & \cdots & Z_{11}\left(\tilde{A},\tilde{B}\right)\\ Z_{11}\left(\tilde{A},\tilde{B}\right) & Z_{11}\left(\tilde{A},\tilde{B}\right) & \cdots & Z_{11}\left(\tilde{A},\tilde{B}\right)\\ \vdots & \vdots & \cdots & \vdots\\ Z_{11}\left(\tilde{A},\tilde{B}\right) & Z_{11}\left(\tilde{A},\tilde{B}\right) & \cdots & Z_{11}\left(\tilde{A},\tilde{B}\right)\end{bmatrix} \quad (5)$$

Definition 4. Assume $z_1 = \left(\left(a_1^l, a_1^m, a_1^u\right), \left(b_1^l, b_1^m, b_1^u\right)\right)$ and $z_2 = \left(\left(a_2^l, a_2^m, a_2^u\right), \left(b_2^l, b_2^m, b_2^u\right)\right)$ are Z -numbers, then normalize their second components as below [21].

$$z_1 = \left(\left(a_1^l, a_1^m, a_1^u\right), \left(b_1^l/b_1, b_1^m/b_1, b_1^u/b_1\right)\right) \tag{6}$$

$$z_2 = \left(\left(a_2^l, a_2^m, a_2^u\right), \left(b_2^l/b_2, b_2^m/b_2, b_2^u/b_2\right)\right) \tag{7}$$

where $b_1 = b_1^l + b_1^m + b_1^u$, $b_2 = b_2^l + b_2^m + b_2^u$

Definition 5. Let $\alpha(t_1), \alpha(t_2), \cdots, \alpha(t_p)$ be the arguments collected from P different periods. Weight vector $\omega(t)$ of the time series $t_k\,(k = 1, 2, \cdots, p)$, introduced a basic unit-interval monotonic function based approach to determining $\omega(t)$ [22].

$$\omega(t_k) = \frac{e^{\frac{\alpha k}{P}}\left(1 - e^{\frac{\alpha}{P}}\right)}{e^{\alpha} - 1} \tag{8}$$

P is size of periods, k is observation of period, and $0 < \alpha < 1$. $\omega(t_k)$ show the importance degrees of the arguments of the different periods and can reflect the change of the importance degrees of the periods.

Definition 6. A Z-number $Z = (A, B)$ is characterized by fuzzy number A, fuzzy number B and an underlying set of probability distributions G [23]. Distance between Z-numbers $D(Z_1, Z_2)$ determined as follows. Distance between A_1 and A_2 is computed as below.

$$D(A_1, A_2) = \sup_{\alpha \in (0,1]} D(A_1^{\alpha}, A_2^{\alpha}) \tag{9}$$

$$D(A_1^{\alpha}, A_2^{\alpha}) = \left| \frac{A_{11}^{\alpha} + A_{12}^{\alpha}}{2} - \frac{A_{21}^{\alpha} + A_{22}^{\alpha}}{2} \right| \tag{10}$$

A_1^{α} and A_2^{α} denote α -cuts of A_1 and A_2 respectively, $A_{11}^{\alpha}, A_{12}^{\alpha}$ denote lower and upper bounds of A_1^{α} ($A_{21}^{\alpha}, A_{22}^{\alpha}$ are those of A_2^{α}). Distance between B_1 and B_2 is computed analogously. A distance between sets G_1 and G_2 of probability distributions p_1 and p_2 can be expressed as

$$D(G_1, G_2) = \inf_{p_1 \in G_1,\, p_2 \in G_2} \left\{ \left(1 - \int_R (p_1 p_2)^{\frac{1}{2}} dx\right)^{\frac{1}{2}} \right\} \tag{11}$$

where the expression in figure brackets is the Hellinger distance between two probability distributions p_1 and p_2. The use of $\inf$ operator implies that among all the possible pairs of distributions, the pair of the closest $p_1 \in G_1$ and $p_2 \in G_2$ is found to define distance $D(G_1, G_2)$. Given $D(A_1, A_2)$, $D(B_1, B_2)$ and $D(G_1, G_2)$, the distance for Z-numbers is defined as below.

$$D(Z_1, Z_2) = \beta D(A_1, A_2) + (1 - \beta) D_{total}(B_1, B_2) \tag{12}$$

$D_{total}(B_1, B_2)$ is a distance for reliability restrictions which is computed as:

$$D_{total}(B_1, B_2) = wD(B_1, B_2) + (1-w)D(G_1, G_2) \tag{13}$$

$\beta, w \in [0,1]$ are DM's assigned importance degrees.

Definition 7. The basic concept of the simple additive weighted method is to find the weighted sum of performance ratings on each alternative on all attributes. This method needs the procedure of normalization of the decision matrix to a scale suitable to all alternative ratings [23]. The formula for normalization represented as follows:

$$R_{ij} = \begin{cases} \dfrac{X_{ij}}{Max\ X_{ij}} \\ \dfrac{Min\ X_{ij}}{X_{ij}} \end{cases} \tag{14}$$

R_{ij} is performance rating, $Max\ X_{ij}$ is maximum value of each row and column, $Min\ X_{ij}$ is minimum value of each row. The preference value for each alternative (V_i) is represented as below.

$$V_i = \sum_{j=1}^{n} W_j r_{ij} \tag{15}$$

V_i is final value of the alternative, W_j is specified weight, R_{ij} is normalization of the matrix. A larger value of V_i indicates that A_i alternatives are preferred.

Definition 8. An inconsistency index K for Z-number-valued pairwise comparisons method (Z_{ij}) is determined as follows [24]:

$$K\left(Z_{ij}\right) = \max_{i<j<k} \min \left\{ D\left(Z(1), \left(\frac{Z_{ik}}{Z_{ij} Z_{jk}} \right) \right) D\left(Z(1), \left(\frac{Z_{ij} Z_{ik}}{Z_{jk}} \right) \right) \right\} \tag{16}$$

where the components of Z-number $Z(1) = (A, B)$ are fuzzy singletons $A=1$, $B=1$.

Definition 9. For Z-numbers Z , Z' holds.

$$ZZ' \ if\ D\left(Z,\ (1,1)\right) \geq D(Z',\ (1,1)) \tag{17}$$

where D is distance defined above, (1,1) is a fuzzy singletons-based Z-number [24]. One can easily prove that ≤ is a partial order as it poses the following properties:
$Z \leq Z$ (reflexivity), If $Z \leq Z'$ and $Z' \leq Z$ then $Z = Z'$ (antisymmetric)
If $Z \leq Z'$ and $Z' \leq Z''$ then $Z \leq Z''$ (transitivity)

III. Case Study Example: Strategic vendor selection problem

The strategic vendor selection problem is a critical aspect of supply chain management and procurement, involving the evaluation and selection of suppliers that best meet an organization's strategic objectives. This problem is inherently multi-criteria and dynamic, as it must consider various factors such as cost, quality, delivery performance, technological capability, and more, which may change over time. Main steps in strategic vendor selection are problem definition, criteria identification, dynamic and multi-criteria approach utilization, data collection, applying

appropriate MCDM methodologies to handle the complexity and dynamics of the vendor selection process and incorporating Z-numbers to manage uncertainty and reliability in vendor evaluation. Problem definition includes defining the strategic goals and objectives of the vendor selection process, identifying the criteria that will be used to evaluate potential vendors. Suppose a company which wants to select strategic vendor. Decision maker uses dynamic fuzzy multi-attribute decision making approach by using Z-numbers for estimation alternatives and criteria [25]. Vendor selection is a multi-faceted decision-making process that involves evaluating potential suppliers based on various criteria to ensure they align with an organization's strategic goals and operational requirements [26].

Statement of the problem: Suppose, linguistic scale is employed to estimate the performance of the vendors A, B, C in the years 2021, 2022, and 2023 according to the criteria C_1, C_2, C_3, and C_4 with Z numbers. C_1 is quality, C_2 is flexibility, C_3 is sustainability, and C_4 is reliability. Quality criteria include such characteristics as compliance with specifications-the degree to which the vendor's products meet the required specifications and standards, defect rates- the frequency of defects or non-conformance in the supplied goods, certifications- quality certifications held by the vendor, which indicate adherence to recognized quality management systems. Flexibility is the vendor's ability to respond to changes in demand, market conditions, or buyer requirements, to offer customized solutions tailored to the buyer's specific needs, the capacity to scale operations up or down based on the buyer's needs. Sustainability is the vendor's practices related to environmental sustainability, such as waste management, carbon footprint, and energy use, also, vendor's commitment to social responsibilities, including fair labour practices, community engagement, ethical sourcing, and vendor's adherence to relevant laws and regulations concerning environmental and social standards. Reliability is historical performance data including reliability, consistency, and ability to meet commitments, the vendor's reputation in the market, including feedback from other customers and industry recognition, and the vendor's ability to identify, manage, and mitigate risks. Dynamic fuzzy multi-attribute decision making is used for selecting the best supplier. Dynamic fuzzy multi-attribute decision making refers to a decision-making process that deals with multiple criteria or attributes under conditions of uncertainty and vagueness, with the added complexity of decision elements changing over time. It is a powerful method for making decisions in environments where uncertainty and change are common. By combining fuzzy logic with dynamic weighting, it offers a flexible and robust approach to evaluating multiple alternatives over time. It allows decision-makers to handle uncertainty and vagueness in complex, real-world problems, adapts to changes in information or preferences over time, making it more flexible than static models, provides a more nuanced and comprehensive framework for evaluating alternatives compared to traditional methods. Linguistic scales are often used in fuzzy logic to handle qualitative and imprecise information. These scales allow for the representation of subjective judgments using linguistic terms, which are then converted into fuzzy numbers for analysis and decision-making. Graphical representation of linguistic terms for restriction (first) component of Z-number defined by triangular fuzzy numbers that represented in figure 1.

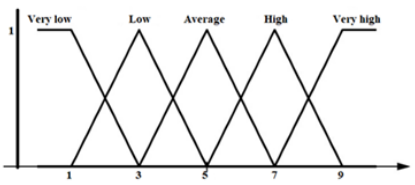

Figure 1: *Linguistic terms for restriction (first) component of Z-number*

Graphical representation of linguistic terms for reliability (second) component of Z-number defined by triangular fuzzy numbers is shown in figure 2.

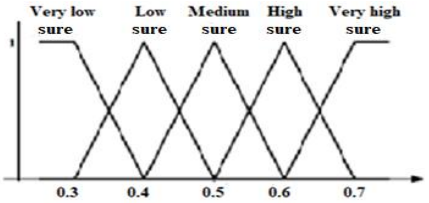

Figure 2: *Linguistic terms for reliability (second) component of Z-number*

Solution of the problem: In dynamic environments, supplier performance and market conditions can change over time. Traditional static decision models may not be effective in such scenarios because they don't account for the fluctuations and uncertainties that affect the performance of suppliers and the broader market. Dynamic approaches allow companies to be more responsive to changing supplier conditions and market dynamics, leading to more informed, by accounting for variability and uncertainty, dynamic models help reduce risks associated with poor supplier performance or unfavourable market conditions. The model incorporates time-dependent changes by updating Z-numbers periodically based on new information or performance reviews. Below is represented steps of how Z-numbers can be applied in this context.

Step 1: Construction decision matrices $D(t_1)$ for 2021, $D(t_2)$ for 2022, and $D(t_3)$ for 2023.

Construction of decision matrix $D(t_1)$ for 2021 represented in table 1.

Table 1: *Decision matrix* $D(t_1)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	5, 7, 9; 0.5,0.6,0.7	5, 7, 9; 0.5,0.6,0.7	3, 5, 7; 0.3,0.4,0.5	5, 7, 9; 0.5,0.6,0.7
B	3, 5, 7; 0.4,0.5,0.6	1, 3, 5; 0.5,0.6,0.7	3, 5, 7; 0.4,0.5,0.6	3, 5, 7; 0.4,0.5,0.6
C	5, 7, 9; 0.5,0.6,0.7	5, 7, 9; 0.5,0.6,0.7	3, 5, 7; 0.5, 0.6,0.7	5, 7, 9; 0.5,0.6, 0.7

Construction of decision matrix $D(t_2)$ for 2022 year represented in table 2.

Table 2: *Decision matrix* $D(t_2)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	3, 5, 7; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7	1, 3, 5; 0.5,0.6,0.7	5, 7, 9; 0.3,0.4,0.5
B	5, 7, 9; 0.3,0.4,0.5	3, 5, 7; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7
C	3, 5, 7; 0.5,0.6,0.7	5, 7, 9; 0.3,0.4,0.5	3, 5, 7; 0.4,0.5,0.6	5, 7, 9; 0.3,0.4,0.5

Construction of decision matrix $D(t_3)$ for 2023 year represented in table 3.

Table 3: *Decision matrix* $D(t_3)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	1, 3, 5; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7	1, 3, 5; 0.3,0.4,0.5	1, 3, 5; 0.5,0.6,0.7
B	3, 5, 7; 0.5,0.6,0.7	1, 3, 5; 0.5,0.6,0.7	3, 5, 7; 0.4,0.5,0.6	3, 5, 7; 0.4,0.5,0.6
C	5, 7, 9; 0.3,0.4,0.5	3, 5, 7; 0.5,0.6,0.7	5, 7, 9; 0.4,0.5,0.6	3, 5, 7; 0.4,0.5,0.6

Step 2. Construction comparative decision matrix of criteria for determining weight of each criterion as shown in table 4.

Table 4: *Comparative decision matrix of criteria*

Criteria C_i	Criteria C_i			
	C_1	C_2	C_3	C_4
C_1	1, 1, 1; 0.6, 0.7,0.8	0.2,0.25,0.33; 0.5, 0.6, 0.7	1, 2, 3; 0.4, 0.5, 0.6	3, 4, 5; 0.5, 0.6, 0.7
C_2	3, 4, 5; 0.5,0.6,0.7	1, 1, 1; 0.6, 0.7, 0.8	4.5, 5, 5.5; 0.6, 0.7, 0.8	3, 4, 5; 0.5, 0.6, 0.7
C_3	0.33, 0.5,1; 0.4, 0.5,0.6	0.18,0.2,0.23; 0.6, 0.7, 0.8	1, 1, 1; 0.6, 0.7, 0.8	0.2,0.25,0.3; 0.5, 0.6, 0.7
C_4	0.2,0.25,0; 0.5, 0.6,0.7	0.2,0.25,0.33; 0.5, 0.6, 0.7	3, 4, 5; 0.5, 0.6, 0.7	1, 1, 1; 0.6, 0.7, 0.8

When considering eigenvalues and eigenvectors in the context of Z-numbers, we deal with matrices whose entries are Z-numbers rather than real or complex numbers. The concept extends traditional linear algebra into the realm of uncertainty and fuzzy logic. Finding eigenvalues of Z-matrices involves solving the characteristic equation in the context of Z-numbers. This means extending the concept of determinants and characteristic polynomials to Z-numbers. The computational approach often involves the use of algorithms designed for fuzzy systems, where operations on Z-numbers are defined and used to compute the eigenvalues and eigenvectors. Once the eigenvalues are found, the corresponding eigenvectors are determined by solving the equation $Zv = \lambda v$ for each eigenvalue λ. In this equation, Z is $n \times n$ matrix, λ is a scalar (eigenvalue), and v is a vector (eigenvector).

Using Z-lab program we determine eigenvalues and weights of criteria [27]. The values of eigen vectors are represented below. Weights of criteria determined by defining eigenvalues.

For C_1 - [[0.0447, 0.3966, 0.3967] [0.3923, 0.483, 0.4998]],

For C_2 - [[0.1124, 0.9971, 0.9977] [0.3988, 0.411, 0.4241]],

For C_3 - [[0.0299, 0.2653, 0.2655] [0.442, 0.4709, 0.4877]],

For C_4 - [[0.0266, 0.2357, 0.2362] [0.4348, 0.498, 0.5011]]

Step 3. Calculation the time weight for each year. In dynamic fuzzy multi-attribute decision making, the decision-making process is influenced by the change of time. So, it is important to determine the weight of time. In dynamic fuzzy multi-attribute decision making, calculating the time weight for each year involves determining the importance or influence of each time period within the decision-making horizon. The time weight reflects how much each year's data impacts the final decision. Time weight is determined by using function (8) where P is size of periods, k is observation of period, and α is argument collected from different P periods $(0 < \alpha < 1)$. $\omega(t_k)$ show the importance

degrees of the arguments of the different periods and can reflect the change of the importance degrees of the periods. When size of periods (P) is 3 and $\alpha = 0.5$ time weights for different years will be calculated as below.

$$\omega(t_k) = \frac{e^{\frac{\alpha k}{P}}\left(1 - e^{\frac{\alpha}{P}}\right)}{e^{\alpha} - 1} = \frac{e^{\frac{0.5\times 1}{3}}\left(1 - e^{\frac{0.5}{3}}\right)}{e^{0.5} - 1} = 0.28,$$

$$\omega(t_2) = 0.33,\ \omega(t_3) = 0.39$$

Step 4. Determining weighted decision matrix. The weighted decision matrix is a powerful tool for making informed, transparent, and structured decisions. The weighted decision matrix is used to make more objective and transparent decisions, especially when multiple factors need to be considered, and it's difficult to rank options based on intuition alone. This matrix helps compare alternatives more objectively by considering both the performance of each alternative for each criterion and the relative importance of the criteria. By assigning weights and scoring alternatives, decision-makers can more easily evaluate multiple options based on the criteria that matter most. Each criterion is assigned a weight to reflect its relative importance, and each option is scored against these criteria. The option with the highest total score is typically the best choice. A weighted decision matrix is determined by multiplying matrix $D(t_1)$, $D(t_2)$, $D(t_3)$ into weights of criteria and time weights.

Weighted decision matrix of $D(t_1)$ represented in table 5.

Table 5: *Weighted decision matrix of* $D(t_1)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	0.06, 0.78, 0.9; 0.23, 0.32, 0.38	0.16,1.95, 2.5; 0.23, 0.28,0.32	0.03,0.37, 0.52; 0.17, 0.22, 0.28	0.04, 0.46, 0.6; 0.25, 0.33, 0.38
B	0.04,0.56,0.7; 0.19,0.27,0.33	0.03, 0.84, 1.4; 0.23, 0.28, 0.32	0.03,0.37, 0.52; 0.21, 0.27, 0.32	0.02,0.33, 0.46; 0.21, 0.28, 0.33
C	0.06,0.78,1; 0.23, 0.32, 0.38	0.16, 1.95, 2.51; 0.23, 0.28, 0.32	0.03,0.37, 0.52; 0.25, 0.31, 0.37	0.04, 0.46, 0.6; 0.25, 0.33, 0.38

Weighted decision matrix of $D(t_2)$ represented in table 6.

Table 6: *Weighted decision matrix of* $D(t_2)$

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.04,0.65,0.92; 0.23,0.32, 0.38	0.11,1.65, 2.30; 0.23, 0.28, 0.32	0.01, 0.26, 0.44; 0.25, 0.31, 0.37	0.04,0.54,0.7; 0.16,0.23,0.28
B	0.07,0.92,1.18; 0.15,0.22, 0.28	0.11, 1.65, 2.3; 0.23, 0.28, 0.32	0.03, 0.44, 0.61; 0.25, 0.31, 0.37	0.03,0.39,0.55; 0.25,0.33,0.38
C	0.04,0.65,0.92; 0.23,0.32, 0.38	0.19, 2.3, 2.4; 0.16, 0.2, 0.24	0.03, 0.44, 0.61; 0.21, 0.27, 0.32	0.04,0.54,0.7; 0.16,0.23, 0.28

Weighted decision matrix of $D(t_3)$ represented in table 7.

Table 7: *Weighted decision matrix of* $D(t_3)$

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.02,0.46,0.77; 0.23,0.32,0.38	0.13,1.94,2.72; 0.23,0.28, 0.32	0.01,0.31,0.52; 0.17, 0.22, 0.28	0.01,0.28, 0.46; 0.25, 0.33, 0.38
B	0.05,0.77,1.08; 0.23,0.32, 0.38	0.04, 1.17, 1.95; 0.23, 0.28, 0.32	0.03,0.52, 0.72; 0.21, 0.27, 0.32	0.03,0.46, 0.64; 0.21, 0.28, 0.33
C	0.09,1.08,1.39; 0.15,0.22, 0.28	0.13,1.94, 2.72; 0.23, 0.28, 0.32	0.06,0.72, 0.93; 0.21, 0.27, 0.32	0.03,0.46, 0.64; 0.21, 0.28, 0.33

Step 5. Aggregating decision matrices from different years is a way to combine decision-making data across multiple time periods to make more informed, long-term decisions. Aggregation of decision matrixes of different years represented in table 8.

Table 8: *Aggregated decision matrix*

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.12,1.89,2.69; 0.01, 0.05, 0.05	0.4, 5.54, 7.53; 0.01, 0.03,0.04	0.05,0.94,1.48; 0.01, 0.02, 0.09	0.09, 1.28, 1.76; 0.01, 0.03, 0.06
B	0.16,2.25, 3.04; 0.11, 0.15, 0.16	0.18,3.66,5.65; 0.01, 0.03,0.04	0.09,1.33,1.85; 0.02,0.05, 0.07	0.08, 1.18, 1.65; 0.06, 0.08, 0.1
C	0.19,2.51, 3.31; 0.01, 0.02, 0.05	0.48, 6.19, 7.63; 0.1, 0.2, 0.3	0.12, 1.53,2.06; 0.07, 0.09, 0.13	0.11, 1.46, 1.94; 0.01, 0.02, 0.05

Step 6. Normalizing aggregated decision matrix. Normalizing an aggregated decision matrix is an important step in multi-criteria decision-making processes. Normalization ensures that all criteria are comparable, especially when different criteria are measured on different scales (e.g., cost in dollars, quality in ratings, time in hours). This process converts the values of each criterion to a common scale, typically between 0 and 1, so that each criterion contributes proportionally to the final decision. It eliminates issues caused by different scales of measurement for different criteria, ensures each criterion contributes proportionally to the decision, preventing any single criterion from dominating because of its scale.

Normalized matrix represented as below.

Z_E_1_1- [[0.5696, 0.78, 0.9982], [0.01, 0.05, 0.07]]
Z_E_1_2- [[0.5422, 0.75, 0.9708], [0.01, 0.03, 0.04]]
Z_E_1_3- [[0.0, 0.0, 0.0, 0.0], [0.48, 0.48, 0.48, 0.49]]
Z_E_1_4- [[0.5643, 0.78, 0.9929], [0.01, 0.03, 0.06]]

Z_E_2_1- [[0.59, 0.92, 1.0], [0.45, 0.48, 0.49]]
Z_E_2_2- [[0.01, 0.21, 0.4274], [0.01, 0.03, 0.04]]
Z_E_2_3- [[0.01, 0.012, 0.013], [0.48, 0.48, 0.49]]
Z_E_2_4- [[0.59, 0.84, 1.0, 1.0], [0.45, 0.48, 0.48, 0.49]]

Z_E_3_1- [[0.5488, 0.92, 0.9774], [0.01, 0.02, 0.05]]
Z_E_3_2- [[0.532, 0.6392, 0.8534, 0.9606], [0.01, 0.02, 0.02, 0.03]]
Z_E_3_3- [[0.5372, 0.75, 0.9657], [0.07, 0.09, 0.13]]
Z_E_3_4- [[0.55, 0.76, 0.9827], [0.01, 0.02, 0.05]]

Table 9: *Normalizing aggregated decision matrix*

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.57,0.78,0.99; 0.01,0.05,0.07	0.54,0.5, 0.97; 0.01, 0.03, 0.04	0.01,0.012,0.013; 0.48, 0.481, 0.49	0.56, 0.78, 0.99; 0.01, 0.03, 0.06
B	0.59, 0.92, 1.0; 0.45, 0.48, 0.49	0.01,0.21,0.43; 0.01, 0.03, 0.04	0.01,0.012,0.013; 0.48, 0.481, 0.49	0.59, 0.92, 1.0; 0.45, 0.48, 0.49
C	0.55, 0.76, 0.98; 0.01, 0.02, 0.05	0.53,0.75,0.96; 0.01, 0.02, 0.03	0.54, 0.64, 0.96; 0.07, 0.09, 0.13	0.55, 0.76, 0.98; 0.01, 0.02, 0.05

Step 7. Using the SAW method, we get values for alternatives that are represented below.

A = [[1.67, 2.32, 2.96], [0.001, 0.01, 0.02]]

B = [[1.18, 2.05, 2.43], [0.001, 0.0011, 0.0012]]

C = [[2.17, 3.03, 3.89], [0.001, 0.0011, 0.0012]]

Step 8. Determining Hellinger distance for each alternative.

Hellinger distance between Alternative A and Z (1) is 2.1301350044600897

Hellinger distance between Alternative B and Z (1) is 2.015349684037576

Hellinger distance between Alternative C and Z (1) is 2.2936502538885915

$A = 2.13$, $B = 2.01$, $C = 2.29$

Comparison of alternatives represent that alternative B is best alternative.

IV. Conclusion

The strategic vendor selection problem refers to the process of choosing suppliers or vendors that align with a company's long-term strategic goals, rather than focusing solely on short-term needs like price or availability. Vendor selection is a critical decision that affects various aspects of an organization, including cost efficiency, product quality, innovation capacity, supply chain stability, and overall competitiveness. It requires a comprehensive and adaptive approach to handle the complexity and dynamics of modern supply chains. By integrating dynamic MCDM with Z-numbers, organizations can better manage uncertainty, incorporate real-time data, and continuously refine their vendor selection process. This approach ensures that the chosen vendors align with the organization's strategic goals and can adapt to changing conditions over time. In most of the methods developed to solve MCDM problems, the evaluation of alternatives is done over a certain period. However, in some MCDM problems, considering only the current performance of the alternatives may cause errors. For this reason, dynamic MCDM model with uncertainty factor was developed in the study, which allows the evaluation of alternatives in different time periods. Z-numbers theory was used to eliminate and express the uncertainty in the MCDM model and, it provides a closer evaluation to the human thinking structure. SAW methodology was used to select the most appropriate alternative. The proposed dynamic MCDM under model Z-information was applied as an example. The example showed that this model can be easily applied to MCDM problems and produce effective results. In today's competitive environment, selecting the right vendor is critical for the success of production systems. Dynamic fuzzy multi-attribute decision making utilization Z-numbers offers a sophisticated approach to handle the inherent uncertainties and complexities involved in this process. Dynamic fuzzy multi-attribute decision making approach is used to estimate the vendors A, B, C in the years 2021, 2022, and 2023 according to the criteria C_1, C_2, C_3 and C_4 with Z numbers. C_1 is technical capability, C_2 is quality, and C_3 is customer support and C_4 is reliability. Using Z-numbers in dynamic fuzzy multi-attribute decision making enhances the vendor selection process by effectively managing uncertainty and ensuring that

decisions are based on reliable information. Comparison of alternatives represent that alternative B is best alternative for vendor selection.

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A NEW CLASS OF COS-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

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Abstract

This paper introduces a novel family of probability distributions, termed the Cos-G family, which is derived from a trigonometric transformation approach. We present the general structural properties of this family and focus on one of its unique members. This newly proposed distribution, formulated from the inverse Weibull distribution, exhibits flexible hazard rate shapes, including reverse-J, increasing, and inverted bathtub forms. We investigate its fundamental statistical properties and employ the maximum likelihood estimation method to estimate its parameters. The performance of the estimation technique is assessed through a Monte Carlo simulation, revealing that biases and mean square errors decrease as sample size increases, ensuring reliable parameter estimation even for small samples. To illustrate its practical applicability, we fit the suggested model to three real-world datasets and compare its performance against existing models using various goodness-of-fit measures and model selection criteria. The results confirm the superiority of the proposed model in capturing complex data structures.

Keywords: Cos-G distribution, Inverse Weibull, Moment, Estimation, Goodness of fit.

1. Introduction

Real-world events are frequently studied using statistical distributions. Both novel developments for their application and the theory of statistical distributions are thoroughly researched. To explain a variety of real-world phenomena, several families of distributions have been developed. In fact, this fresh advancement in distribution theory is an ongoing practice. The majority of probability distributions suggested in the literature have a lot of parameters, which gives the model more adaptability. Some authors claim that it is challenging to acquire these estimates using numerical resources [1]. For modeling actual data, it is better to develop models with a limited number of parameters and a high level of flexibility. A team of scientists made the decision to use trigonometric functions to seek novel distributions in order to achieve this objective. Trigonometric models have gained popularity among scholars in recent years due to their adaptability and ability to be understood mathematically. Souza et al. [2] suggested a new class of trigonometric cosine distribution with a bathtub-shaped or increasing failure rate function called the Cos-G Class of distribution with base parameters ($\omega > 0$) among the various trigonometric G-family. The cumulative distribution function (CDF) for the Cos-G class of distribution are

$$F(x;\omega) = -\int_0^{\frac{\pi}{2}K(x;\omega)} sin(t)dt = 1 - \cos\left[\frac{\pi}{2}K(x;\omega)\right]; x \in \Re.$$

Souza et al. [3] utilized a similar methodology to propose the Sin-G family of distributions and include the Sin-Inverse Weibull distribution in the Sin-G class. Similarly, Souza et al. [4]

introduced a new Tan-G class with an increasing failure rate function or bathtub-shaped failure rate function, and focused on examining the Tan-BXII distribution as a member. A CDF exists for both the Sin-G and Tan-G classes of distributions.

$$F(x;\omega) = \int_0^{\frac{\pi}{2}K(x;\omega)} \cos(t)dt = \sin\left[\frac{\pi}{2}K(x;\omega)\right]; x \in \Re.$$

$$F(x;\omega) = \int_0^{\frac{\pi}{4}K(x;\omega)} sec^2(t)dt = \tan\left[\frac{\pi}{4}K(x;\omega)\right]; x \in \Re.$$

where $K(x;\omega)$ is the CDF of any parent distribution and ω is the vector of parameters of the parent distribution. The new sin-G family was created by [5], who also studied the sin-inverse Weibull model in specific. The CDF of the novel sin-G family of distribution are

$$F(x;\omega) = \int_0^{\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)} cos(t)dt = \sin\left[\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)\right]; x \in \Re.$$

Also, Chesneau and Jamal [6] have defined the sine Kumaraswamy-G family of distributions as having two extra parameters to this family. Muhammad et al. [7] have defined the exponentiated sine-G family and analyzed the particular distribution as an exponentiated sine-Weibull distribution. Another trigonometric function-related probability model introduced by [8] is called arctan generalized exponential distribution. Using the sine-G family of distribution [9] have developed a new two-parameter model called sine Burr XII distribution. A new family of distributions related to the Sine function was developed by [10] and used with medical data. As a result, we have observed that the simple functions have a trigonometric distribution and are tractable formally see [3]. Additionally, without the use of any extra parameters, the sine transformation can significantly increase $G(x)$ flexibility [6]. We are drawn to the cosine metamorphosis family because of these appealing qualities. In this research, we created a new family of trigonometric models using the cosine function, which we named the new class of cos-G family (NCC-G) of distributions.

This study is divided into several sections. In Section 2, we introduce the methodology of model development and key functions of the family of distributions. Section 3 presents some general properties of the NCC-G family, while Section 4 discusses methods of estimation. In Section 5, we introduce a specific member of the NCC-G family and present a detailed study, and in the application Section6, we provide the application of this model using three real datasets. Finally, Section 7 contains the conclusion.

2. The NCC-G family of Distribution (NCC-G FD)

In this study, a new family of distributions called NCC-G is suggested using the T-X approach as defined by [11]. Consider a baseline CDF, represented by $G(x;\xi)$, and a vector of associated parameters, denoted by $\xi > 0$. The ratio of $G(x;\xi)$ and $1 + G(x;\xi)$ can be treated as a function of the new family of distributions. For further information, refer to [12]. Mathematically it can be expressed as $\frac{G(x;\xi)}{1+G(x;\xi)} \to 0$ as $G(x;\xi) \to 0$; $\frac{G(x;\xi)}{1+G(x;\xi)} \to \frac{1}{2}$ as $G(x;\xi) \to 1$ The CDF $F(x;\xi)$ of the NCC-G family of distributions is defined as

$$F(x;\xi) = -\int_0^{\pi\frac{G(x;\xi)}{1+G(x;\xi)}} \sin(t) = 1 - \cos\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]; x \in \Re. \quad (1)$$

Differentiating the Equation (1), the PDF $f(x;\xi)$ of the family can be written as

$$f(x;\xi) = \pi \sin\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]\frac{g(x;\xi)}{(1+G(x;\xi))^2}; x \in \Re. \quad (2)$$

2.1. Survival Function

The survival function of NCC-G FD is presented as

$$R(x;\xi) = 1 - F(x;\xi) = \cos\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]; x \in \Re.$$

2.2. Hazard Function

The Hazard function of NCC-G FD can be expressed as

$$H(x;\xi) = \frac{f(x;\xi)}{R(x;\xi)} = \pi\sin\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]\frac{g(x;\xi)}{(1+G(x;\xi))^2}\left[\cos\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^{-1}; x \in \Re.$$

2.3. The Quantile Function

The quantile function is useful in statistical analysis and modeling, as it provides a way to estimate percentiles and other summary statistics of a probability distribution. Suppose $Q(p)$ is the smallest value of X for which the probability that $X \leqslant$ to that value is at least p. The quantile function $Q(p;\xi)$ of CDF $F(x;\xi)$ of NCC-G FD can be obtained as

$$Q(p;\xi) = G^{-1}\left[\frac{\cos^{-1}(1-p)}{\pi-\cos^{-1}(1-p)}\right]; \quad p \in (0,1). \tag{3}$$

Using equation (3) we can calculate the median, upper and lower quartile, quartile deviation (QD), coefficient of QD, skewness, and kurtosis, which are presented in Table 1.

Table 1: *Various measures based on quantiles of NCC-G FD*

Statistics	**Expressions**
Median	$G^{-1}\left[\frac{\cos^{-1}(0.5)}{\pi-\cos^{-1}(0.5)}\right]$
Lower Quartile	$G^{-1}\left[\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right]$
Upper Quartile	$G^{-1}\left[\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right]$
QD	$\frac{1}{2}\left[G^{-1}\left(\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right) - G^{-1}\left(\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right)\right]$
Coefficient of QD	$\frac{\left[G^{-1}\left(\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right)-G^{-1}\left(\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right)\right]}{\left[G^{-1}\left(\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right)+G^{-1}\left(\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right)\right]}$
Skewness ([13])	$\frac{Q\left(\frac{3}{4};\xi\right)-2Q\left(\frac{1}{2};\xi\right)+Q\left(\frac{1}{4};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$
Kurtosis ([14])	$\frac{Q\left(\frac{7}{8};\xi\right)-Q\left(\frac{5}{8};\xi\right)-Q\left(\frac{1}{8};\xi\right)+Q\left(\frac{3}{8};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$

3. Some Properties of NCC-G FD

3.1. Useful Expansion of NCC-G FD

Exponentiated distributions can be used to generate useful linear expansions. The CDF of the exponentiated-G (Exp-G) distribution for more information see [3, 15, 16], exponentiated distributions have well-known properties for a wide range of baseline CDF $G(x;\varphi)$ with parameter $z > 0$ is given by

$$G_z(x;\varphi) = [G(x;\varphi)]^z; x \in \Re, \text{ where } x \in \Re. \tag{4}$$

The PDF corresponding to (4) can be presented as

$$g_z(x;\varphi) = zg(x;\varphi)\left[G(x;\varphi)\right]^{(z-1)}, x \in \Re.$$

We can express the density function of the NCC-G FD in linear form using the series expansions shown below.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \frac{y^9}{9!} - \cdots; -\infty < x < \infty.$$

$$(1+y)^p = \sum_{n=0}^{\infty} \binom{p}{n} y^n = 1 + \frac{p}{1!}y + \frac{p(p-1)}{2!}y^2 + \frac{p(p-1)(p-2)}{3!}y^3 + \cdots; |y| < 1.$$

The PDF of NCC-G FD is

$$f(x,\xi) = g(x,\xi) \sum_{i=0}^{\infty} \frac{\pi^{2i+2}(-1)^i}{(2i+1)!} \left(1 + G(x,\xi)\right)^{2i-1} \left(G(x,\xi)\right)^{2i+1}. \tag{5}$$

Further expanding Equation (5) using generalized binomial series expansion. The expression for $f(x;\xi)$ becomes

$$f(x,\xi) = g(x,\xi) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} \left(G(x,\xi)\right)^{2i+j+1}, \tag{6}$$

where $\Delta_{ij} = \frac{\pi^{2i+2}(-1)^i}{(2i+1)!} \binom{2i-1}{j}$.

3.2. Moments

The r^{th} order moment ($\mu_r^{'}$) about the origin for the NCC-G FD is

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \tag{7}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} \int_{-\infty}^{\infty} x^r \left(G(x,\xi)\right)^{2i+j+1} g(x,\xi) dx. \tag{8}$$

Further moments can also be calculated using the quantile function for more detail see [17] as Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1$.

$$\mu_r^{'} = E(X^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} \int_0^1 p^{2i+j+1} Q_G^r(p) dp, 0 < p < 1.$$

where $G(x;\xi) = p$ and $Q_G(p)$ is the function of quantile.

3.3. Moment Generating Function (MGF)

The MGF ($M_X(t)$) for the NCC-G FD is

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_r^{'} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_{ij} \int_{-\infty}^{\infty} x^k g(x,\xi) \left(G(x,\xi)\right)^{2i+j+1} dx$$

Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1$.

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_{ij} \int_{-\infty}^{\infty} p^{2i+j+1} Q_G^l(p) dp, 0 < p < 1,$$

where $G(x;\xi) = p$ and $Q_G(p)$ is the quantile function of the baseline distribution.

3.4. Incomplete Moments

The Incomplete moments of the NCC-G FD can be defined as $M_r(y) = \int_0^y x^r f(x)dx$. Therefore incomplete moments for NCC-G FD are given by

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{-\infty}^{y} \Delta_{ij} x^r g(x;\xi)\left(G(x,\xi)\right)^{2i+j+1} dx. \tag{9}$$

Alternately, $M_r(y)$ may be defined in terms of quantile function as

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_0^{G(y)} p^{2i+j+1}Q_G^r(p)dp; 0 < p < 1.$$

3.5. Mean Residual Life

The mean residual life of the NCC-G FD can be defined as $\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \int_{-\infty}^{y} xf(x)dx\right] - y$. Therefore, the mean residual life for NCC-G FD is given by

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} xg(x;\xi)\left\{G(x;\xi)\right\}^{2i+j+1} dx\right] - y.$$

Alternatively, $\bar{M}(y)$ may be calculated in term of quantile function as

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_0^{G(y)} p^{2i+j+1}Q_G(p)dp\right] - y.$$

3.6. Inequality Measure

Lorenz and Bonferroni curves are utilized in various fields such as insurance, econometrics, and demography, among others, to analyze measures of inequality such as income and poverty.

i) **Lorenz Curve**

Lorenz curve is defined as $L_{F(y)} = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx$, where μ is the mean of x, hence Lorenz curve for NCC-G FD is given by

$$L_{F(y)} = \frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} xg(x;\xi)\left(G(x,\xi)\right)^{2i+j+1} dx. \tag{10}$$

Alternatively, in terms of quantile function as

$$L_{F(y)} = \frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{G(y)} p^{2i+j+1}Q_G(p)dp.$$

ii) **the Bonferroni Curve**

The Bonferroni curve is given by $B_{F(y)} = \frac{L_{F(y)}}{F(y)}$. From Equation (10), the Bonferroni curve for the NCC-G FD is obtained as

$$B_{F(y)} = \frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} xg(x;\xi)\left(G(x,\xi)\right)^{2i+j+1} dx.$$

3.7. Entropy

Entropy is a concept used to describe the degree of variation or uncertainty associated with a random variable. Its applicability is widespread and can be observed in various disciplines such as probability theory, medicine, insurance, engineering, life sciences, etc. in general.

i) **Renyi's Entropy**

Entropy, which serves as a measure of the amount of variation or uncertainty associated with a random variable, finds its applications across several disciplines, including engineering, econometrics, and financial mathematics. Renyi [18] proposed the concept of entropy as a metric for quantifying variability and uncertainty, and it can be computed as follows: $R_\rho(X) = \frac{1}{1-\rho}\log\int_{-\infty}^{\infty}\{f(x)\}^\rho\,dx$; $\rho > 0$ and $\rho \neq 1$. Applying Taylor™s series expansion $[f(x,\xi)]^\rho$ can be obtained in the form

$$[f(x;\xi)]^\rho = \pi^\rho\,(g(x;\xi))^\rho\left[\sin\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^\rho(1+G(x;\xi))^{-2\rho}.$$

By considering the function of Taylor series $\left[\sin\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^\rho$ at the point $s = 1/4$, we can write

$$[\sin(\pi s)]^\rho = \sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k\binom{k}{r}(-1)^{k-r}\left(\frac{1}{4}\right)^{k-r}s^r,$$

where $a_k = \frac{1}{k!}\left[\{\sin(\pi s)\}^\rho\right]^{(k)}\big|_{s=\frac{1}{4}}$. We have selected $s = \frac{1}{4}$ because $\{\sin(\pi s)\}^\rho$ is infinitely differentiable at this point and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$, which allows to have a more tractable expression for a_k at a given ρ

$$[f(x;\xi)]^\rho = \pi^\rho\,(g(x;\xi))^\rho\sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k\binom{k}{r}(-1)^{k-r}\left(\frac{1}{4}\right)^{k-r}(G(x;\xi))^r\,(1+G(x;\xi))^{-(2\rho+r)} \quad (11)$$

Further expanding Equation (11) using generalized binomial series expansion. The expression for $[f(x;\xi)]^\rho$ becomes

$$[f(x;\xi)]^\rho = \pi^\rho\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}(-1)^{m+k-r}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}(G(x;\xi))^{r+m}\,(g(x;\xi))^\rho \quad (12)$$

Substituting $[f(x,\xi)]^\rho$ into the expression for $R_\rho(X)$, the Renyi™s entropy for NCC-G family of distribution is given by

$$R_\rho(X) = \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{mkr}\int_{-\infty}^{\infty}(g(x;\xi))^\rho\,(G(x;\xi))^{r+m}\,dx\right],$$

where $\psi_{mkr} = (-1)^{m+k-r}\,\pi^\rho a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

ii) **q-Entropy**

The q-entropy is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\int_{-\infty}^{\infty}\{f(x)\}^\rho\,dx\right];\rho > 0 \text{ and } \rho \neq 1.$$

Substituting $[f(x,\xi)]^\rho$ from Equation (12) into the expression for $H(\rho)$, the q-Entropy for NCC-G FD is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{mkr}\int_{-\infty}^{\infty}(g(x;\xi))^\rho\,(G(x,\xi))^{r+m}\,dx\right];\rho > 0 \text{ and } \rho \neq 1.$$

iii) **Shannon's Entropy**
The Shannon's entropy for a random variable X with pdf $f(x)$ is a special case of the Renyi's entropy when $\rho \uparrow 1$. Shannon entropies are defined as $\eta_X = E(-\log f(x))$. For the NCC-G family of distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}g(x,\xi)\left(G(x,\xi)\right)^{2i+j+1)}\right\}\right].$$

4. Estimation Method

4.1. Maximum Likelihood Estimation (MLE)

The parameters of the NCC-G FD are estimated in this section using the method of maximum likelihood. Given random sample $x_1, ..., x_n$ of size n with parameters vector ξ from the NCC-G FD, let $u = \xi^T$ be $(p \times 1)$ parameter vectors, then the log density and total log-likelihood function respectively, are given by

$$l(x;\xi) = \log\pi + \log\left[\sin\left\{\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right\}\right] - 2\log\left(1+G(x;\xi)\right) + \log g(x;\xi)$$

and

$$l(\underline{x},\xi) = n\log\pi + \sum_{i=1}^{n}\log\left[\sin\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\right] - 2\sum_{i=1}^{n}\log\left(1+G(x_i;\xi)\right) + \sum_{i=1}^{n}\log g(x_i;\xi). \quad (13)$$

Differentiating Equation (13) gives the score function's components of $V(u) = \left(\frac{\partial l}{\partial\xi}\right)^T$ as follows,

$$\frac{\partial l}{\partial\xi} = \pi\sum_{i=1}^{n}\cot\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\frac{G_k^{'}(x_i;\xi)}{\left(1+G(x_i;\xi)\right)^2} - 2\sum_{i=1}^{n}\frac{G_k^{'}(x_i;\xi)}{\left(1+G(x_i;\xi)\right)} + \sum_{i=1}^{n}\frac{g_k^{'}(x_i;\xi)}{g(x_i;\xi)},$$

where $g_k^{'}(x_i;\xi) = \frac{dg(x_i;\xi)}{d\xi}$, $g_k^{''}(x_i;\xi) = \frac{d^2g(x_i;\xi)}{d^2\xi}$, $G_k^{'}(x_i;\xi) = \frac{dG(x_i;\xi)}{d\xi}$ and $G_k^{''}(x_i;\xi) = \frac{d^2G(x_i;\xi)}{d^2\xi}$.

5. Special member of NCC-G FD

Generalization of several distributions can be made using the NCC-G FD. The special distribution, a new class of cosine inverse Weibull distribution, is introduced in this section.

5.1. New class Cos inverse Weibull (NCC-IW) istribution

The CDF and PDF of the Inverse Weibull (IW) distribution are respectively given by

$$G(x) = 1 - exp(-\alpha x^{-\beta};\ x > 0, \alpha, \beta > 0$$

and

$$g(x) = \alpha\beta x^{-\beta-1}exp(-\alpha x^{-\beta}$$

Hence using the CDF and PDF of IW, the CDF and PDF of the NCC-IW distribution are given by

$$F(x;\alpha,\beta) = 1 - \cos\left[\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]; x > 0 \quad (14)$$

$$f(x;\alpha,\beta) = \pi\alpha\beta x^{-(\beta+1)}\sin\left[\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]\frac{\exp(-\alpha x^{-\beta})}{\left(1+\exp(-\alpha x^{-\beta})\right)^2}; x > 0 \quad (15)$$

The reliability and hazard functions, respectively, are given by

$$R(x;\alpha,\beta) = \cos\left[\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]; x > 0.$$

and

$$H(x;\alpha,\beta) = \pi\alpha\beta x^{-(\delta+1)}\frac{\exp(-\alpha x^{-\beta})}{\left(1+\exp(-\alpha x^{-\beta})\right)^2}\sin\left[\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]\left[\cos\left(\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right)\right]^{-1}.$$

The quantile function for the NCC-IW distribution is presented below

$$Q_X(p) = \left[-\frac{1}{\alpha}\log\left(\frac{\cos^{-1}(1-p)}{\pi-\cos^{-1}(1-p)}\right)\right]^{-\frac{1}{\beta}}, p \in (0,1)$$

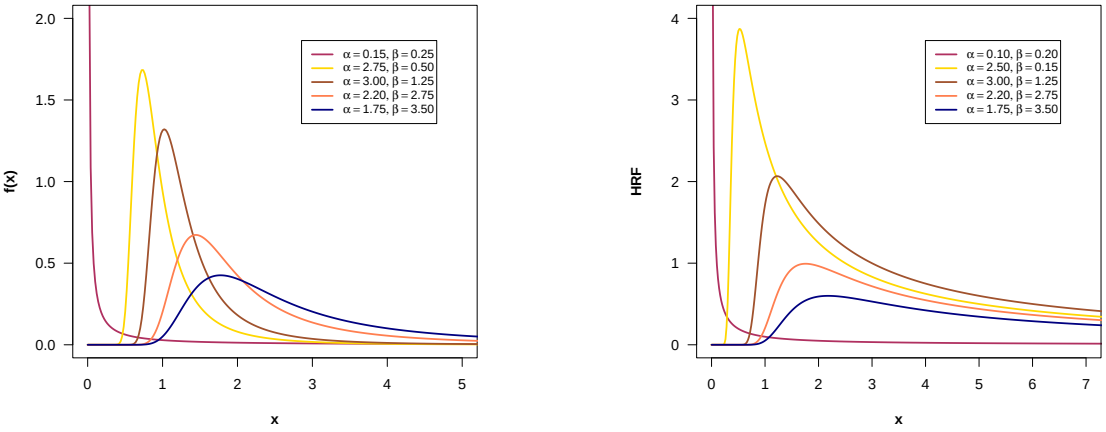

Figure 1: *Shapes of PDF and HRF of NCC-IW distribution*

5.2. Linear Expansion

Using Equation (6), Equation (15) can be expressed in linear form as

$$f(x;\xi) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}x^{-(\beta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}, \tag{16}$$

where $\Omega_{ij} = \frac{\pi^{2i+2}\alpha\beta(-1)^i}{(2i+1)!}\binom{2i-1}{j}$.

5.3. Moments

Using the PDF defined in Equation (16), the r^{th} order non-central moment (μ_r') for the NCC-IW distribution can be presented as

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-r}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-r}{\beta}}}; \quad \forall\beta > r, \tag{17}$$

where $\Omega_{ij}^{*} = \frac{\pi^{2i+2}\alpha(-1)^i}{(2i+1)!}\binom{2i-1}{j}$ and $\Gamma(.)$ is the gamma function.

5.4. Skewness and Kurtosis

Using the Equation (17) we can obtain the first four $(r = 1, 2, 3, 4)$ non-central moments as:

$$Mean = \mu_1^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\left[\alpha\{(2i+j)+2\}\right]^{-\frac{\beta-1}{\beta}}\Gamma\left(\frac{\beta-1}{\beta}\right); \quad \forall\ \beta > 1,$$

$$\mu_2^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-2}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-2}{\beta}}}; \quad \forall\ \beta > 2,$$

$$\mu_3^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-3}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-3}{\beta}}}; \quad \forall\ \beta > 3,$$

and

$$\mu_4^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-4}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-4}{\beta}}}; \quad \forall\ \beta > 4.$$

Similarly, we can calculate the central moments using the above non-central moments as

$$\mu_1 = \mu_1^{'},$$

$$\mu_2 = \mu_2^{'} - \mu_1^{'2},$$

$$\mu_3 = \mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2\mu_1^{'3},$$

and

$$\mu_4 = \mu_4^{'} - 4\mu_3^{'}\mu_2^{'} + 6\mu_2^{'}\mu_1^{'2} - 2\mu_1^{'4}$$

Therefore skewness and kurtosis for the NCC-IW distribution are $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$ respectively.

5.5. MGF

The MGF $(M_X(t))$ for the NCC-IW distribution is

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k\Omega_{ij}^{*}}{k!}\frac{\Gamma\left(\frac{\beta-r}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-r}{\beta}}}; \quad \forall\ \beta > r. \tag{18}$$

5.6. Incomplete moments

The incomplete moments for NCC-IW distribution are given by

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\int_0^y x^{r-(\delta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}dx$$

$$= \frac{1}{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\frac{\gamma\left(\frac{\beta-r}{\beta}, (2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-r}{\beta}}}.$$

where $\gamma(.)$ incomplete gamma function.

5.7. Mean Residual Life

The mean residual life for the NCC-IW distribution is given by

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\int_{0}^{y} x^{-\beta}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}\right] - y$$
$$= \frac{1}{F(y)}\left[\mu - \frac{1}{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\frac{\gamma\left(\frac{\beta-1}{\beta},(2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-1}{\beta}}}\right] - y.$$

5.8. Entropy

i) **Renyi™s Entropy** The Renyi™s entropy for NCC-IW distribution is given by

$$R_{\rho}(X) = \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{krm}\,(\alpha\beta)^{\rho}\int_{0}^{\infty} x^{-\rho(\beta+1)}\exp(-(r+m+\rho)\alpha x^{-\beta})dx\right]$$
$$= \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{krm}\frac{(\alpha\beta)^{\rho}}{\beta}\frac{\Gamma\left(\left\{\frac{(\rho-1)(\beta+1)}{\beta}+1\right\}\right)}{\{(r+m+\rho)\alpha\}^{\frac{(\rho-1)(\beta+1)}{\beta}+1}}\right]$$

where $\psi_{mkr} = (-1)^{m+k-r}\,\pi^{\rho}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

ii) **q-Entropy**

The q-Entropy for NCC-IW distribution is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1 - \psi_{krm}\,(\alpha\beta)^{\rho}\int_{0}^{\infty} x^{-\rho(\beta+1)}\exp(-(r+m+\rho)\alpha x^{-\beta})dx\right]$$
$$= \frac{1}{1-\rho}\log\left[1 - \psi_{krm}\frac{(\alpha\beta)^{\rho}}{\beta}\frac{\Gamma\left(\left\{\frac{(\rho-1)(\beta+1)}{\beta}+1\right\}\right)}{\{(r+m+\rho)\alpha\}^{\frac{(\rho-1)(\beta+1)}{\beta}+1}}\right];$$

where $\rho > 0$, $\rho \neq 1$ and $\psi_{krm} = (-1)^{m+k-r}\,\pi^{\rho}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

iii) **Shannon's Entropy**

The Shannon entropy for the NCC-IW distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}x^{-(\beta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}\right\}\right].$$

5.9. Inequality Measure

i) **Lorentz Curve:** The Lorenz curve for NCC-IW distribution is given by

$$L_{F(y)} = \frac{\alpha\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\int_{-\infty}^{y} x^{-\beta}\exp(-\alpha(2i+j+2)x^{-\beta})dx$$
$$= \frac{\alpha}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\frac{\gamma\left(\frac{\beta-1}{\beta},(2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-1}{\beta}}}.$$

ii) **Boneferroni Curve**
The Bonferroni curve for the NCC-IW distribution is given by

$$B_{F(y)} = \frac{1}{\mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} \int_{-\infty}^{y} x^{-\beta} \exp(-\alpha(2i+j+1)x^{-\beta})dx$$
$$= \frac{1}{\delta\mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij} \frac{\gamma\left(\frac{\beta-1}{\beta}, (2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-1}{\beta}}}.$$

5.10. Parameter estimation of NCC-IW distribution

Our current focus is on determining the parameters of the NCC-IW model through the MLE method. The objective is to compute the MLEs for the parameters α and β. To achieve this, we will examine the log-likelihood of a vector $X = (x_1, ..., x_n)^T$ of size n composed of independent random variables from the NCC-IW distribution.

$$l(x;\alpha,\beta) = n\log(\pi\alpha\beta) - (\beta+1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log sin\left[\pi\frac{\exp(-\alpha x_i^{-\beta})}{1+\exp(-\alpha x_i^{-\beta})}\right]$$
$$-2\sum_{i=1}^{n}\log\left(1+\exp(-\alpha x_i^{-\beta})\right) - \alpha\sum_{i=1}^{n}x_i^{-\beta} \tag{19}$$

Partially differentiating the Equation (19) with respect to β and α yields the components of the score function $V(u) = \left(\frac{\partial l}{\partial\beta}, \frac{\partial l}{\partial\alpha}\right)^T$ as follows

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log x_i + \pi\alpha\sum_{i=1}^{n}\frac{x_i^{-\beta}\log(x_i)\exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)^2}\cot\left[\pi\frac{\exp(-\alpha x_i^{-\beta})}{1+\exp(-\alpha x_i^{-\beta})}\right] +$$
$$2\alpha\sum_{i=1}^{n}\frac{x_i^{-\beta}\log(x_i)\exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)} + \alpha\sum_{i=1}^{n}x_i^{-\beta}\log(x_i).$$

and

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - \pi\sum_{i=1}^{n}\frac{x_i^{-\beta}exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)^2}\cot\left[\pi\frac{\exp(-\alpha x_i^{-\beta})}{1+\exp(-\alpha x_i^{-\beta})}\right] + 2\sum_{i=1}^{n}\frac{x_i^{-\beta}exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)} - \sum_{i=1}^{n}x_i^{-\beta}.$$

The MLEs of β and α are obtained by maximizing $l(x;\alpha,\beta)$ in β and α, which can be done by solving simultaneously the equation: $\frac{\partial l}{\partial\beta} = 0$ and $\frac{\partial l}{\partial\alpha} = 0$.

5.11. Simulation Study

We used the maxLik R package developed by [19] to create samples from the quantile function defined in Equation (14) for various parameter combinations of the NCC-IW distribution. The MLEs were calculated for each sample using the `maxLik()` function with the BFGS algorithm. This allowed us to test parameter estimation problems, such as the sharpness or flatness of the likelihood function and provided estimates for the size and direction (underestimate or overestimate) of the MLEs bias. We repeated the procedure 1000 times, with 25 samples of sizes ranging from 10 to 250. We then calculated the bias and mean square error (MSE) for each simulation. In addition, we provided lower confidence limit (LCL) and upper confidence limit (UCL) estimated values with a 5% level of significance. The results of the experiment are summarized in Tables 2 and 3, which show the bias and MSEs for each parameter, along with the LB and UB for the MLEs. As the table shows, the MLE method consistently estimates the

parameters of the proposed model. Moreover, as the sample size increases, the MLEs gradually approach the actual values of α and β. In Figures 2 and 3, we have displayed a clear picture of MSEs with 95% confidence bound (dark region) for α and β.

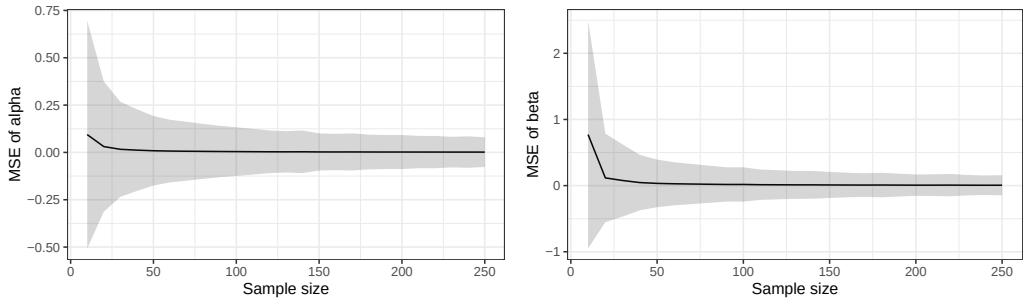

Figure 2: *MSE plots of α and β with 95% CI for initial values $\alpha = 0.5$ and $\beta = 1.25$.*

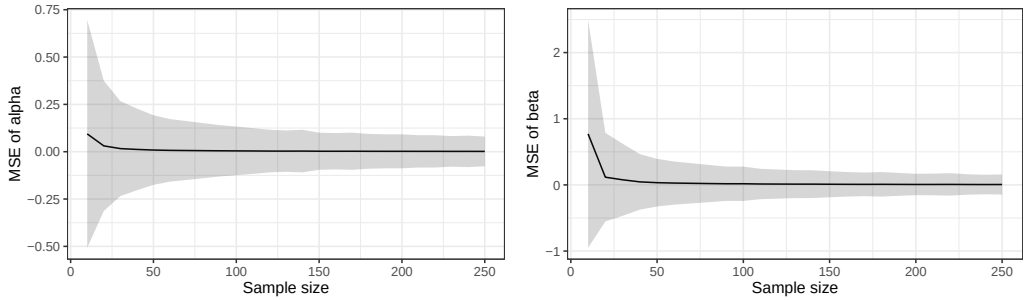

Figure 3: *MSE plots of α and β with 95% CI for initial values $\alpha = 0.75$ and $\beta = 1.5$.*

6. Application

Employing three real data sets, we demonstrate the applicability of the NCC-IW distribution in this section. The data sets employed for the application of the suggested distribution are given below:

6.1. Model Analysis

To analyze the data sets under study, we calculate several widely used goodness-of-fit statistics. The fitted models are then compared using various measures, including the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogorov-Smirnov (KS) with p-values, and Cramer-von Mises (CVM). For additional information see [1]. All the essential computations are carried out in R-software. For the comparison of fitting capability we have selected some models such as inverse Weibull (IW), arctan generalized exponential (ArcTGE) [8], arctan Lomax (ArcTLx) [20], arcsine exponential (ASE) [21], Tan Burr XII (TBXII) [4], New Cosine Weibull (NCW) [22], Exponentiated Cos Weibull (EcosW) [7], arcsine exponentiated Weibull (ASEW) [23], Cos Weibull (CosW) [2] and Sine inverse Weibull (Sin-IW) [3].

Data set I:

The dataset from [24] contains information on the relief times of 20 patients who were administered an analgesic. An analgesic is a type of medication that is commonly used to reduce pain, and the relief time refers to the duration for which the patients experience relief from their pain after taking the medication. The data are "1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, and 2.0".

Table 2: *Bias, MSEs, and LCL and UCL for MLEs with initial values* $\alpha = 0.5$ *and* $\beta = 1.25$.

n	$bias_\alpha$	$bias_\beta$	mse_α	mse_β	LCL_α	UCL_α	LCL_β	UCL_β
10	0.0904	0.1500	0.0444	0.4191	0.3381	1.0299	0.7585	2.5927
20	0.0445	0.0633	0.0143	0.0823	0.3761	0.7924	0.8920	1.9588
30	0.0248	0.0283	0.0079	0.0400	0.3893	0.7076	0.9577	1.7161
40	0.0194	0.0249	0.0056	0.0295	0.3918	0.6853	0.9747	1.6231
50	0.0152	0.0182	0.0040	0.0219	0.4117	0.6556	1.0243	1.5883
60	0.0136	0.0174	0.0034	0.0191	0.4155	0.6376	1.0242	1.5585
70	0.0126	0.0100	0.0027	0.0146	0.4284	0.6194	1.0481	1.512
80	0.0083	0.0138	0.0023	0.0140	0.4260	0.6108	1.0596	1.5091
90	0.0069	0.0141	0.0020	0.0119	0.4239	0.6032	1.0692	1.4905
100	0.0070	0.0074	0.0021	0.0105	0.4316	0.6074	1.0601	1.467
110	0.0072	0.0060	0.0017	0.0095	0.4308	0.5913	1.0866	1.4777
120	0.0037	0.0130	0.0014	0.0090	0.4359	0.5831	1.0944	1.4520
130	0.0060	0.0139	0.0014	0.0083	0.4393	0.5823	1.0997	1.4567
140	0.0062	0.0107	0.0013	0.0074	0.4405	0.5821	1.1054	1.4331
150	0.0058	0.006	0.0013	0.0068	0.4424	0.5814	1.1116	1.4269
160	0.0054	0.0102	0.0012	0.0062	0.4428	0.5776	1.1226	1.4222
170	0.0060	0.007	0.0011	0.0061	0.4505	0.5729	1.1146	1.4056
180	0.0032	0.0044	0.0010	0.0054	0.4455	0.5704	1.1236	1.4032
190	0.0041	0.0059	0.0010	0.0056	0.4487	0.5716	1.1212	1.4118
200	0.0031	0.0026	9.00E-04	0.0049	0.4472	0.5687	1.1261	1.3952
210	0.0044	0.0058	9.00E-04	0.0048	0.4493	0.5619	1.1206	1.3927
220	0.0033	0.0076	8.00E-04	0.0047	0.4526	0.5645	1.1295	1.4010
230	0.0046	-6.00E-04	8.00E-04	0.0047	0.4529	0.5606	1.1258	1.3934
240	0.0036	0.0042	7.00E-04	0.0041	0.4531	0.5606	1.1335	1.3816
250	0.0028	0.0014	7.00E-04	0.0040	0.4505	0.5561	1.1315	1.3801

Table 3: *Bias, MSEs and LCL and UCL for MLEs with initial values* $\alpha = 0.75$ *and* $\beta = 1.5$.

n	$bias_\alpha$	$bias_\beta$	mse_α	mse_β	LCL_α	UCL_α	LCL_β	UCL_β
10	0.1424	0.2518	0.0942	0.7713	0.5183	1.5550	0.9717	3.5167
20	0.0692	0.0882	0.0309	0.1166	0.5616	1.1901	1.0949	2.3295
30	0.0363	0.0789	0.0164	0.0774	0.581	1.0422	1.1612	2.1797
40	0.0294	0.0383	0.0121	0.0458	0.6083	1.0183	1.1821	2.0153
50	0.0227	0.0311	0.0088	0.0337	0.6022	0.9684	1.2209	1.9084
60	0.0193	0.0287	0.0071	0.0276	0.6224	0.9468	1.2437	1.8913
70	0.0162	0.025	0.0063	0.0240	0.6247	0.9314	1.2604	1.8680
80	0.0139	0.0238	0.0055	0.0206	0.6397	0.9192	1.2795	1.8384
90	0.0130	0.0131	0.0048	0.0175	0.6507	0.9126	1.2956	1.8022
100	0.0107	0.0176	0.0043	0.0175	0.6418	0.8926	1.2919	1.8124
110	0.0085	0.0112	0.0038	0.0137	0.6507	0.8873	1.3022	1.7490
120	0.0085	0.0098	0.0033	0.0126	0.652	0.8749	1.3022	1.7395
130	0.0071	0.0152	0.0031	0.0116	0.6577	0.8734	1.3151	1.7406
140	0.0091	0.0123	0.0033	0.0115	0.6556	0.8793	1.3232	1.7467
150	0.0065	0.0139	0.0025	0.0101	0.6637	0.8666	1.3344	1.7227
160	0.0075	0.0051	0.0024	0.0090	0.6693	0.8575	1.3315	1.7045
170	0.0074	0.0110	0.0025	0.0083	0.6709	0.8606	1.3356	1.7040
180	0.0080	0.0105	0.0022	0.0090	0.6712	0.8471	1.3412	1.7089
190	0.0053	0.0116	0.0021	0.0078	0.6685	0.8571	1.3481	1.6881
200	0.0052	0.0046	0.0021	0.0068	0.6694	0.8492	1.3471	1.6665
210	0.0040	0.0056	0.0019	0.0070	0.6731	0.8479	1.3615	1.6799
220	0.0055	0.0051	0.0019	0.0075	0.6758	0.8441	1.3438	1.6867
230	0.0055	0.0049	0.0017	0.0062	0.6799	0.8439	1.3583	1.6725
240	0.0049	0.0086	0.0018	0.0057	0.6770	0.8414	1.3603	1.6553
250	0.0055	0.0092	0.0016	0.0060	0.6822	0.8406	1.3621	1.6657

Table 4: *MLEs with SE (in parentheses) (dataset I)*

Model	Parameter(SE)		
NCC-IW(α, β)	3.2906(0.5941)	3.9558(1.0140)	–
IW(λ, θ)	4.0175(0.7060)	6.0224(2.0083)	–
ArcTGE(α, λ, θ)	0.0000(1.5645)	19.3864(6.0429)	1.8579(0.2245)
ArcTLx(α, β, θ)	147.2664(44.0127)	0.2871(0.2782)	12.3869(9.6739)
ASE(θ)	127.8946(4.8432)	–	–
ASEW($\lambda, \theta, \upsilon$)	1.0488(0.1284)	104.561(19.0921)	3.1656(0.1303)
NCW(λ, θ)	0.2505(0.0810)	2.2930(0.3402)	–
TBXII(λ, ν, θ)	1.3946(0.1597)	10.3624(5.0171)	0.3937(0.2735)
ECosW(β, λ, θ)	0.2386(0.0486)	0.2789(0.0712)	2.7222(0.1824)
CosW(β, δ)	2.2183(0.3323)	0.5655(0.0471)	–
SinIW(δ, θ)	5.3385(1.4594)	2.8386(0.4882)	–

In Tables 4, 6, and 8, we have presented the estimated values of the parameters and their corresponding standard error (SE in parentheses) of the models under study using the MLE method. Similarly, in Tables 5, 7, and 9, we have presented the model selection and goodness of fit statistics such as log-likelihood, AIC, HQIC, KS, AD, and CVM for all three data sets. It has been observed that the suggested model NCC-IW has the least statistics as compared to IW, ArcTGE, ArcTLx, ASE, ASEW, NCW, TBXII, ECosW, CosW, and Sin-IW. Hence NCC-G is more flexible (even four trigonometric distributions having three parameters) and provides a good fit. Also, we have displayed the graphical illustrations of the fitted models in Figures 5, 7, and 9. These figures also verified that the NCC-G model can perform well as compared to candidate models.

Table 5: *Some selection criteria and goodness-of-fit statistics (dataset-I)*

Model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCC-IW	31.1170	35.1170	35.5057	0.1148	0.9548	0.0306	0.9770	0.1772	0.9956
IW	30.8174	34.8174	35.2062	0.1020	0.9854	0.0266	0.9880	0.1545	0.9984
ArcGE	33.4131	39.4131	39.9962	0.1516	0.7473	0.0767	0.7169	0.4214	0.8256
ArcLmx	35.6262	41.6262	42.2094	0.1240	0.9182	0.0662	0.7806	0.5268	0.7175
ASE	154.7472	156.7472	156.9416	0.8863	0.0000	5.1247	0.0000	31.4397	0.0000
ASEW	31.1885	37.1885	37.7716	0.1170	0.9470	0.0363	0.9551	0.2096	0.9877
NCW	48.6870	52.6870	53.0757	0.1467	0.7829	0.1078	0.5521	0.7800	0.4940
Tan-BXII	31.0804	37.0804	37.6636	0.0919	0.9959	0.0231	0.9944	0.1377	0.9994
CosW	40.6035	46.6035	47.1867	0.1922	0.4508	0.1840	0.3022	1.0593	0.3267
NCosW	37.4854	41.4854	41.8742	0.1770	0.5576	0.1279	0.4681	0.7563	0.5118
Sin-IW	31.1572	35.1572	35.5460	0.1069	0.9763	0.0292	0.9813	0.1808	0.9949

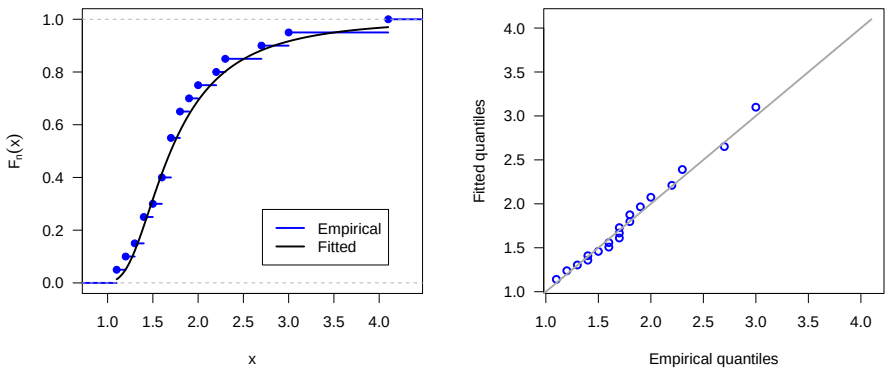

Figure 4: *KS and Q-Q plots (dataset-I).*

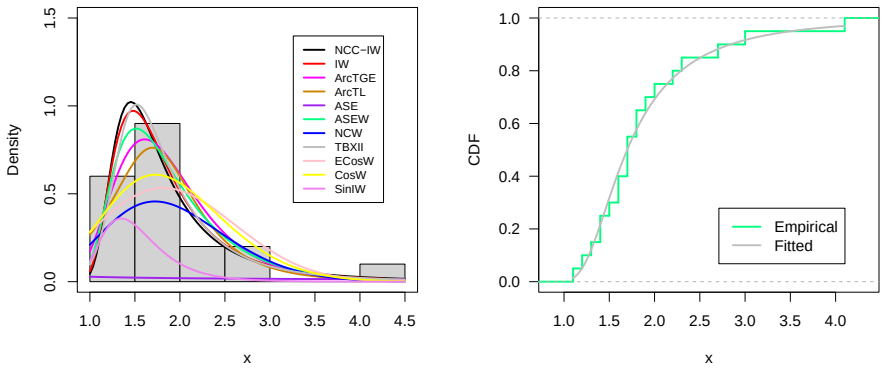

Figure 5: *Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-I).*

Data set-II

The following data set was obtained by [25] consisting of 128 observations on the time intervals, measured in seconds, between the arrivals of vehicles at a specific location on a road. 0.2, 0.5, 0.8, 0.8, 0.8, 1.0, 1.1, 1.2, 1.2, 1.2, 1.2, 1.2, 1.3, 1.4, 1.5, 1.5, 1.6, 1.6, 1.6, 1.7, 1.8, 1.8, 1.8, 1.8, 1.8, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 2.0, 2.1, 2.1, 2.2, 2.3, 2.3, 2.4, 2.4, 2.5, 2.5, 2.5, 2.6, 2.6, 2.7, 2.8, 2.8, 2.9, 3.0, 3.0, 3.1, 3.2, 3.4, 3.7, 3.9, 3.9, 3.9, 4.6, 4.7, 5.0, 5.1, 5.6, 5.7, 6.0, 6.0, 6.1, 6.6, 6.9, 6.9, 7.3, 7.6, 7.9, 8.0, 8.3, 8.8, 8.8, 9.3, 9.4, 9.5, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 24.7, 29.7, 30.6, 31.0, 33.7, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8, 125.3´

Table 6: *MLEs with SE (in parentheses) (dataset-II)*

model	Parameter(SE)		
NCC-IW(α,β)	0.6748(0.0476)	2.0442(0.1524)	–
IW(λ,θ)	0.8183(0.0540)	2.6651(0.2527)	–
ArcTGE(α,λ,θ)	1.00E-04(0.3004)	0.6685(0.0737)	0.0475(0.0063)
ArcTLx(α,β,θ)	1.00E-04(1.7292)	0.0833(0.0310)	1.6100(0.3969)
ASE(θ)	11.6716(1.5853)	–	–
ASEW$(\lambda,\theta,\upsilon)$	0.2832(0.0181)	36.6683(2.9617)	2.8476(0.1367)
NCW(λ,θ)	0.3294(0.0389)	0.5770(0.0359)	–
TBXII(λ,ν,θ)	1.5673(0.3041)	2.7668(0.7182)	0.2870(0.0962)
ECosW(β,λ,θ)	0.1949(0.0172)	0.4319(0.0000)	0.7179(1.00E-04)
CosW(β,δ)	0.5566(0.0350)	0.1655(0.0220)	–
SinIW(δ,θ)	3.0753(0.2472)	0.5944(0.0385)	–

Table 7: *Some selection criteria and goodness-of-fit statistics (dataset-II)*

model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCC-IW	924.4949	928.4949	930.8125	0.0616	0.717	0.1390	0.4253	1.0806	0.3175
IW	921.1562	925.1562	927.4738	0.0604	0.7381	0.1323	0.4488	0.974	0.3711
ArcGE	948.3638	954.3638	957.8402	0.1481	0.0073	0.8221	0.0064	4.4936	0.0050
ArcLmx	929.2980	935.2980	938.7744	0.0976	0.1745	0.2488	0.1899	1.7906	0.1202
ASE	956.6371	958.6371	959.7959	0.1501	0.0063	0.6261	0.0191	4.0879	0.0079
ASEW	912.5793	918.5793	922.0557	0.0886	0.2680	0.1858	0.2969	1.1082	0.3051
NCW	998.8923	1002.892	1005.210	0.1207	0.0480	0.3327	0.1096	2.8599	0.0323
Tan-BXII	917.7461	923.7461	927.2225	0.0787	0.4057	0.2086	0.2516	1.2355	0.2544
ECosW	937.7763	943.7763	947.2527	0.1170	0.0603	0.4531	0.0523	2.8386	0.0332
CosW	927.9831	931.9831	934.3006	0.1162	0.0632	0.3467	0.1003	2.1999	0.0716
Sin-IW	916.8895	920.8895	923.2071	0.0868	0.2903	0.1819	0.3056	1.1174	0.3010

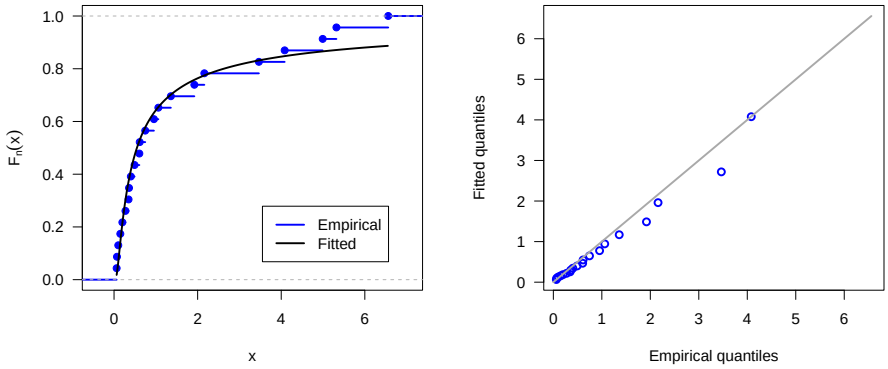

Figure 6: *KS and Q-Q plots (dataset-II).*

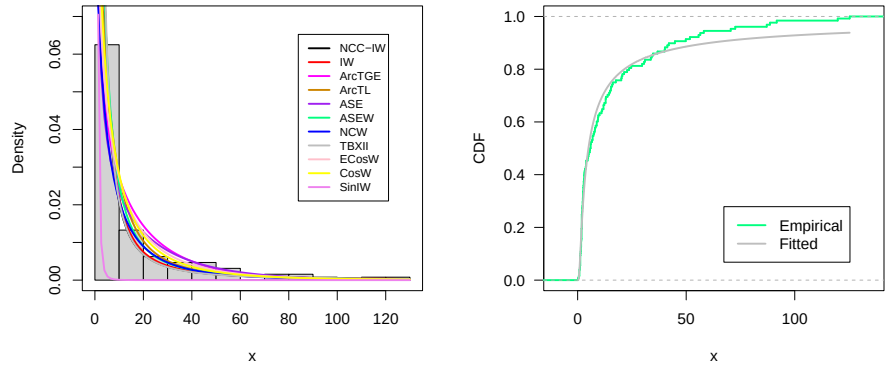

Figure 7: *Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-II).*

Data set-III

We have used the real data reported by [26] and it represents the failure time of 30 items "0.602, 0.603, 0.603, 0.615, 0.652, 0.663, 0.688, 0.705, 0.761, 0.770, 0.868, 0.884, 0.898, 0.901, 0.911, 0.918, 0.935, 0.953, 0.983, 1.009, 1.040, 1.097, 1.097, 1.148, 1.296, 1.343, 1.422, 1.540, 1.555, 1.653"

Table 8: *MLEs with SE (in parentheses) (dataset-III).*

Model	**Parameter(SE)**		
NCC-IW(α,β)	3.178(0.4814)	0.4482(0.1032)	–
IW(λ,θ)	3.8881(0.555)	0.4275(0.1108)	–
ArcTGE(α,λ,θ)	4.2E-07(1.3892)	31.5836(5.4017)	4.168(0.1055)
ArcTLx(α,β,θ)	143.7539(4.3514)	0.3264(0.2077)	18.9209(10.5266)
ASE(θ)	194.3346(5.9316)	–	–
ASEW$(\lambda,\theta,\upsilon)$	1.2673(0.1458)	41.0829(6.893)	5.4445(0.3188)
NCW(λ,θ)	1.172(0.1733)	2.662(0.3434)	–
TBXII(λ,ν,θ)	0.7768(0.1457)	7.3693(3.3169)	0.6354(0.5186)
ECosW(β,λ,θ)	0.1726(0.0315)	2.368(0.0046)	3.2934(5.00E-04)
CosW(β,δ)	2.5699(0.3352)	1.0907(0.0648)	–
SinIW(δ,θ)	0.8297(0.1375)	2.7635(0.3757)	–

Table 9: *Some selection criteria and goodness-of-fit statistics (dataset-III).*

Model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCC-IW	8.7135	12.7135	13.6100	0.1608	0.4197	0.0992	0.5920	0.6056	0.6413
IW	7.6494	11.6494	12.5459	0.1432	0.5700	0.0793	0.6995	0.5146	0.7306
ArcGE	7.2658	13.2658	14.6105	0.0901	0.9679	0.0479	0.8924	0.3760	0.8712
ArcLmx	12.3954	18.3954	19.7402	0.1262	0.7260	0.0542	0.8542	0.5154	0.7297
ASE	224.6097	226.6097	227.0579	0.9413	0.0000	8.6137	0.0000	62.4475	0.0000
ASEW	6.4640	12.4640	13.8087	0.1141	0.8293	0.0561	0.8419	0.4091	0.8385
NCW	26.5153	30.5153	31.4118	0.1328	0.6649	0.0797	0.6969	0.7164	0.5440
Tan-BXII	8.9087	14.9087	16.2535	0.1074	0.8792	0.0536	0.8575	0.4021	0.8456
ECosW	12.0969	18.0969	19.4416	0.1238	0.7477	0.1053	0.5622	0.7074	0.5514
CosW	9.8726	13.8726	14.7692	0.1004	0.9230	0.0741	0.7304	0.5449	0.7001
Sin-IW	7.1457	11.1457	12.0422	0.1111	0.8526	0.0567	0.8383	0.4210	0.8265

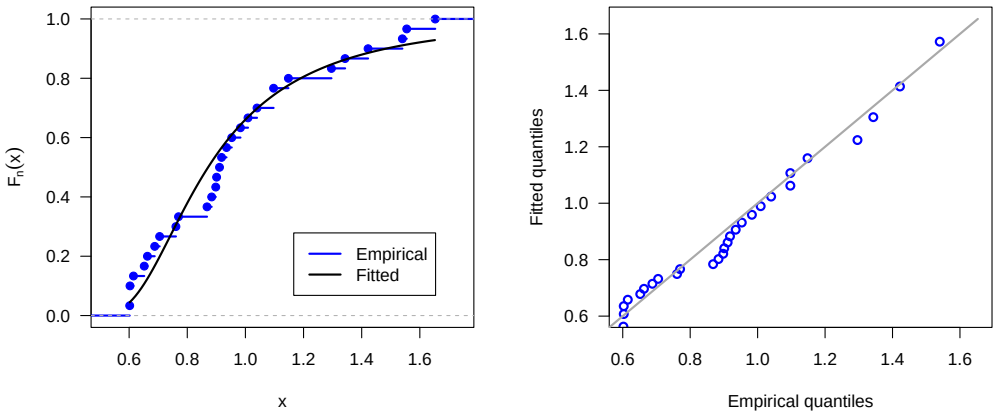

Figure 8: *KS and Q-Q plots (dataset-III).*

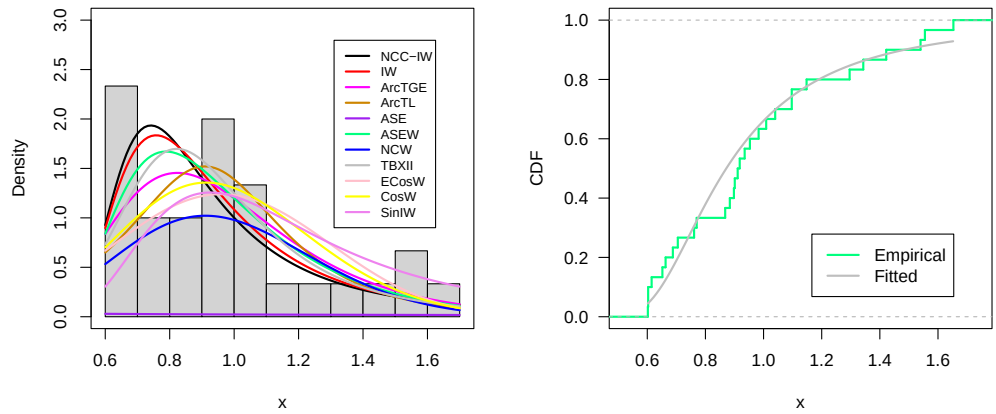

Figure 9: *Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-III).*

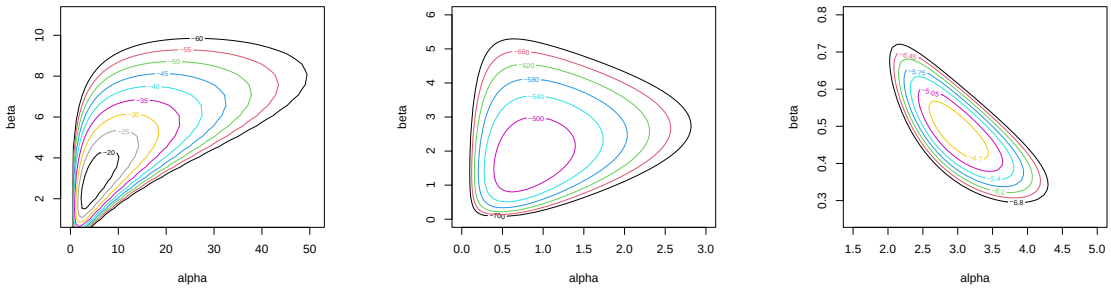

Figure 10: *Countour plots for three data sets under study respectively.*

7. Conclusion

A new family of distributions, known as the Cos-G family, has been developed by transforming the cosine function based on the ratio of CDF $G(x)$ and $1 + G(x)$ of a baseline distribution. The general properties of this distribution family have been described. To create a member of this family with a hazard function that is increasing, decreasing, or inverted bathtub-shaped, the Inverse Weibull distribution was used as a baseline distribution. The resulting distribution, called NCC-IW, was analyzed for its statistical properties, and its parameters were estimated using the MLE method. The estimation procedure was evaluated through a Monte Carlo simulation, which showed that the biases and mean square errors decrease as the sample size increases, even for small samples. The NCC-IW distribution was then applied to three real data sets, and using model selection criteria and goodness-of-fit test statistics, it was shown to outperform other existing models with more parameters. This suggests that the Cos-G family and its member distribution have wide applications in fields such as medical science, reliability engineering, and survival analysis, and can lead to the development of new models in the future.

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SOME APPLICATIONS OF EXPONENTIATED LOG-UNIFORM DISTRIBUTION

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Abstract

In this paper we introduced Exponentiated Log - Uniform distribution as a generalisation of the Log - Uniform distribution and its properties are studied. We provide graphical representations of its density function, cumulative distribution function, hazard rate function, and survival function. And derive various statistical properties such as moments, mean deviations, and quantile function of the new distribution. We also obtain the probability density functions of the order statistics of the Exponentiated Log-Uniform Distribution.To estimate the parameters of the distribution and the stress strength parameters, we use the maximum likelihood method, and validate the estimates of the model parameters through a simulation study. Our findings reveal that the Exponentiated Log-Uniform Distribution exhibits the least bias and that the values of the mean square error decrease as the sample size increases, indicating the effectiveness of this distribution in modeling real-world data. We applied the Exponentiated Log-Uniform distribution to a real data set and compared it with Exponentiated Quasi Akash Distribution and Exponentiated Weibull Distribution. It was found that the new distribution was a better fit than the other distributions based on the values of the AIC, CAIC, BIC, HQIC, the Kolmogorov-Smirnov (K-S) goodness-of-fit statistic and the p-values.

Keywords: Log - Uniform distribution, Exponentiated Log - Uniform distribution, Stress strength Reliability

1. Introduction

Statisticians are constantly pushing the boundaries of statistical analysis by exploring new families of distributions. Their goal is to increase the flexibility of existing distributions, which allows for the creation of compound distributions that can better analyze a wide range of data sets.The Log-Uniform distribution is a continuous probability distribution where the logarithm of a random variable is uniformly distributed [9].

One of the most attractive features of the Log-Uniform distribution is its scale-invariance property. This means that the distribution remains the same, regardless of the units of measurement used.The exponentiated family has become a fundamental concept in mathematical statistics and machine learning.

Ahmed [2] introduced a new version of the Exponentiated Burr X distribution, Exponentiated Transmuted Exponential distribution was proposed by Al-Kadim [3] which is more adaptable than other distributions, Alzaghal [4] derived Exponentiated TX family of distributions. The Beta Exponentiated Weibull distribution was derived by Cordeiro [5]. Ahmad Dar [6] introduced Exponentiated Quasi Akash distribution. Gupta and Kundu [8] proposed Exponentiated Exponential

family. Hassan [10] derived a new generalization of Ishita distribution and obtained properties of the distribution along with applications of the proposed model. The Exponential distribution and its applications was proposed by Jowett [11]. The Exponentiated Kumaraswamy distribution and its log-transform was studied by Lemonte [15]. Exponentiated weibull distribution is studied by Pal and Ali [16]. Ramires [17] introduced Exponentiated Uniform distribution: An interesting alternative to truncated models.

If X is a random variable with a Log Uniform distribution, then Log(X) is Uniformly distributed. A random variable X is said to have a Log-Uniform distribution if its probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{x\left[ln(b)-ln(a)\right]}; & \text{if }, a \leq x \leq b, 0 < a < b, a, b \in R \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where a and b are the parameters of the distribution and they are location parameters that define the minimum and maximum values of the distribution on the original scale and ln is the natural Log function (the logarithm to base e).

The Cumulative distribution function of the distribution is given by:

$$F(x) = \begin{cases} \frac{ln(x)-ln(a)}{ln(b)-ln(a)}; & \text{if }, a \leq x \leq b, 0 < a < b, a, b \in R \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

The exponentiated family of distributions is a set of distributions that can be derived by raising a positive real number to the cumulative distribution function (cdf) of a parent distribution.

The Cumulative distribution function of the family is given by:

$$F(x) = (H(x))^{\beta}; \quad \beta > 0 \tag{3}$$

where $H(x)$ is the distribution function of a random variable X.
The corresponding probability density function (pdf) is given by:

$$f(x) = \beta h(x)(H(x))^{\beta-1} \tag{4}$$

where $H(x)$ and $h(x)$ are cdf and pdf of the parent distribution and $\beta > 0$ is a shape parameter.

2. Exponentiated Log-uniform Distribution

On introducing Log-Uniform distribution to Exponentiated family of distributions we obtain a new distribution called the Exponentiated Log-Uniform distribution.

The distribution function has the form:

$$F(x) = (H(x))^{\beta}; \quad \beta > 0 \tag{5}$$

where $H(x)$ is the cdf of the base distribution. Therefore the cdf of Exponentiated Log-uniform Distribution is given by

$$F(x) = \begin{cases} \left(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right)^{\beta}, & \text{if } \beta > 0, a \leq x \leq b, 0 < a < b, a, b \in R \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where a and b are the parameters of the distribution and they are location parameters and β is a shape parameter.

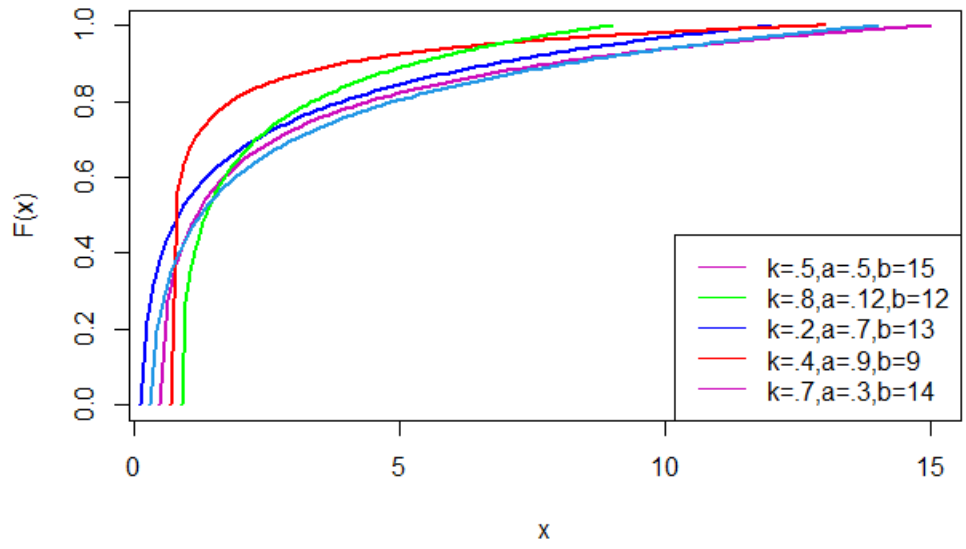

Figure 1: *Plot of cdf of Exponentiated Log-Uniform distribution*

The probability density function of the distribution is given by:

$$f(x) = \begin{cases} \beta \frac{1}{x ln(\frac{b}{a})} \left(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right)^{\beta-1}, & \text{if } \beta > 0, a \leq x \leq b, 0 < a < b, a, b \in R \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

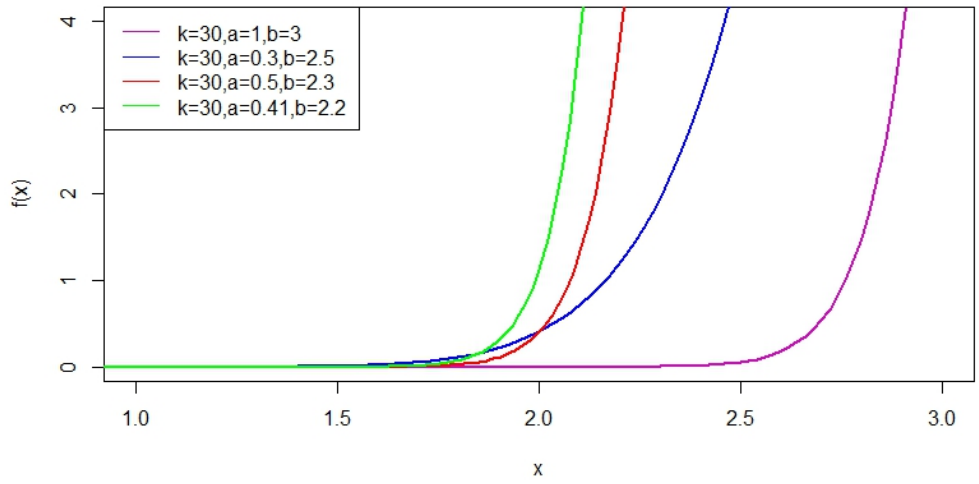

Figure 2: *Plot of pdf of Exponentiated Log-Uniform distribution*

The Survival function of Exponentiated Log-Uniform distribution is,

$$S(x) = \frac{\left(ln(\frac{b}{a})\right)^{\beta} - \left(ln(\frac{x}{a})\right)^{\beta}}{\left(ln(\frac{b}{a})\right)^{\beta}}; \beta > 0, a \leq x \leq b, 0 < a < b, b \in R \tag{8}$$

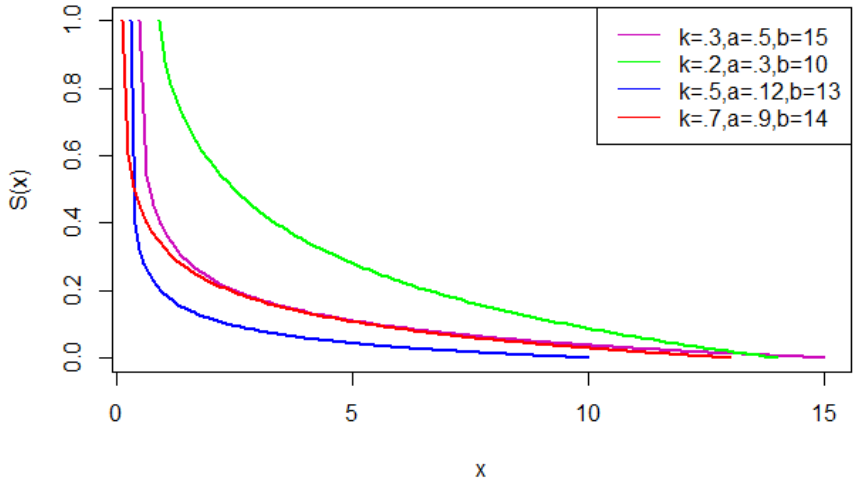

Figure 3: *Plot of survival function of Exponentiated Log-Uniform distribution*

The failure rate function of Exponentiated Log-Uniform distribution is,

$$h(x) = \frac{\beta(ln(\frac{x}{a}))^{\beta-1}}{x((ln\frac{b}{a})^{\beta} - (ln(\frac{x}{a}))^{\beta})}; \beta > 0, a \leq x \leq b, 0 < a < b, b \in R \tag{9}$$

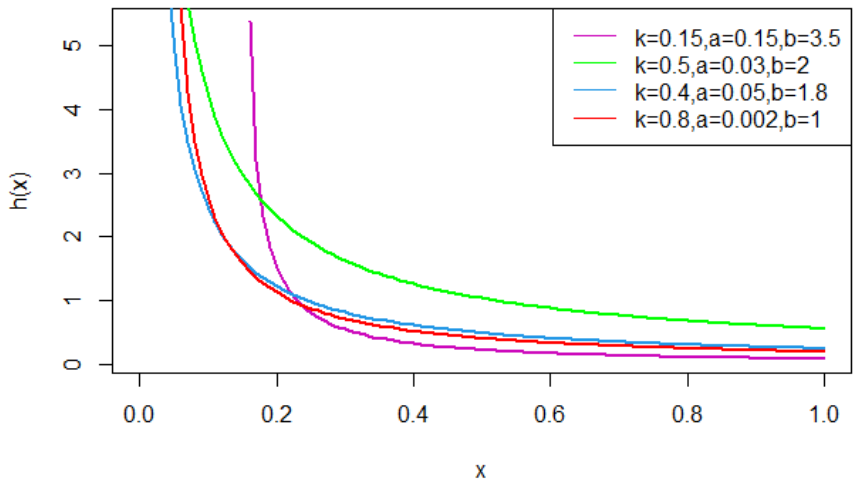

Figure 4: *Plot of hazard function of Exponentiated Log-Uniform distribution*

- Special Case:
  If we put $\beta = 1$, then Exponentiated Log-Uniform distribution reduces to Log-Uniform distribution.

2.1. Statistical Properties

Some structural properties of the Exponentiated Log-Uniform distribution have been evaluated in this section.The properties like Moments, Quantile function, Mean Deviation and Order Statistics are considered.

2.1.1 Moments

A distribution can have several moments, and shape of the distribution is determined by its moments. The first moment is the mean of the distribution. Suppose X is a random variable following Exponentiated Log-Uniform distribution with parameters a,b, β and then the r^{th} moment for a given probability distribution is given by:

$$\mu_r^{'} = E(X^r) = \int_a^b x^r f(x,a,b,\beta)dx \tag{10}$$

$$\mu_r^{'} = \int_a^b x^r \beta \left(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right)^{\beta-1} \left(\frac{1}{xln(\frac{b}{a})}\right) dx \tag{11}$$

$$= \frac{\beta}{(ln(\frac{b}{a}))^{\beta}} \int_a^b x^{r-1} \left(ln(\frac{x}{a})\right)^{\beta-1} dx$$

$$E(X^r) = \mu_r^{'} = a^r \left(\frac{\beta\Gamma\beta}{(-rln(\frac{b}{a}))^{\beta}} - \frac{\beta\Gamma(\beta, -rln(\frac{b}{a}))}{(-rln(\frac{b}{a}))^{\beta}}\right) \tag{12}$$

- On substituting r=1 on equation 12 results in mean of the distribution.

2.1.2 Quantile Function

The quantile function is important as it can be used to generate random numbers and to find quartiles, median, measures of skewness and kurtosis.
The Quantile function is given by,

$$x = Q(p) = F^{-1}(p), 0 < p < 1 \tag{13}$$

The Quantile function of Exponentiated Log-Uniform distribution is obtained by inverting distribution function.

$$p = \frac{\left(ln(\frac{x}{a})\right)^{\beta}}{\left(ln(\frac{b}{a})\right)^{\beta}}$$

Thus the quantile function is given by,

$$x = Q(p) = ae^{p^{\frac{1}{\beta}}\left(ln(\frac{b}{a})\right)} \tag{14}$$

- The second quartile (Median) of exponentiated log-uniform distribution is obtained by putting $p = \frac{1}{2}$ in 14

$$Q_2 = m = ae^{\left((\frac{1}{2})^{\frac{1}{\beta}} ln(\frac{b}{a})\right)} \tag{15}$$

- The r^{th} moment is defined as

$$\mu_r^{'} = E(X^r) = \int_a^b x^r f(x,a,b,\beta)dx$$

Which can be written in terms of the quantile function as

$$mu_r^{'} = \int_0^1 (Q(u))^r du$$

2.1.3 Mean Deviation

The mean deviation is a measure of amount of scatter in a random variable. Let X follows Exponentiated Log-Uniform distribution with mean μ and meadian M.

- Mean Deviation from the Mean is given by,

$$\delta_1(x) = \int_a^b |x-\mu| f(x)dx \ = 2\mu(F(\mu)-1) + 2T(\mu) \tag{16}$$

where μ is the mean of the distribution and

$$T(\mu) = \int_\mu^b xf(x)dx \tag{17}$$

$$T(\mu) = \int_\mu^b x\beta \frac{1}{xln(\frac{b}{a})} \Big(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\Big)^{\beta-1} dx$$

$$T(\mu) = \frac{a\beta}{(ln(\frac{b}{a}))^\beta} \left(\frac{(ln(\frac{\mu}{a}))^\beta \Gamma(\beta, (-ln(\frac{\mu}{a})))}{(-ln(\frac{\mu}{a}))^\beta} - \frac{(\ln(\frac{b}{a}))^\beta \Gamma(\beta, -ln(\frac{b}{a}))}{(-ln(\frac{b}{a}))^\beta} \right) \tag{18}$$

- Similarly,the Mean Deviation about Median is,

$$\delta_2(x) = \int_a^b |x-M| f(x)dx \ = 2T(M) - \mu \tag{19}$$

where M is the median of the distribution and μ is the mean of the distribution and

$$T(M) = \int_M^b xf(x)dx \tag{20}$$

$$T(M) = \frac{a\beta}{(ln(\frac{b}{a}))^\beta} \left(\frac{(ln(\frac{M}{a}))^\beta \Gamma(\beta, (-ln(\frac{M}{a})))}{(-ln(\frac{M}{a}))^\beta} - \frac{(\ln(\frac{b}{a}))^\beta \Gamma(\beta, -ln(\frac{b}{a}))}{(-ln(\frac{b}{a}))^\beta} \right) \tag{21}$$

The mean deviation about mean is obtained by substituting the mean, cdf and $T(\mu)$ in (16). The mean deviation about median is obtained by substituting the mean, cdf and $T(M)$ in (19).

2.1.4 Order Statistics

Let $X_{(1)}$,$X_{(2)}$,$X_{(3)}$,...,$X_{(n)}$ denote the order statistics of a random sample X_1,X_2,X_3,...,X_n drawn from the continuous distribution with pdf $f_X(x)$ and cdf $F_X(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ is given by,

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{(r-1)}[1-F(x)]^{(n-r)} \tag{22}$$

Using the equations (6) and (7) the probability density function of r^{th} order statistics $X_{(r)}$ of Exponentiated Log-Uniform distribution is given by,

$$f_{X_{(r)}}(x,\beta,a,b) = \frac{n!}{(r-1)!(n-r)!}\beta\left(\frac{1}{xln(\frac{b}{a})}\right)\left(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right)^{\beta-1}\left(\left(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right)^{\beta}\right)^{(r-1)}\left(1-\left(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\right)^{\beta}\right)^{(n-r)} \tag{23}$$

Then the probability density function of first order statistics $X_{(1)}$ of Exponentiated Log-Uniform distribution is given by,

$$f_{X_{(1)}}((x,\beta,a,b) = n\beta\left(\frac{1}{x\ln(\frac{b}{a})}\right)\left(\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})}\right)^{(\beta-1)}\left(1-\left(\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})}\right)^{\beta}\right)^{(n-1)} \tag{24}$$

and the probability density function of n^{th} order $X_{(n)}$ of Exponentiated Log-Uniform distribution is given as:

$$f_{x_{(n)}}((x,\beta,a,b) = n\beta\left(\frac{1}{x\ln(\frac{b}{a})}\right)\left(\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})}\right)^{(\beta-1)}\left(\left(\frac{\ln(\frac{x}{a})}{\ln(\frac{b}{a})}\right)^{\beta}\right)^{(n-1)} \tag{25}$$

2.2. Maximum Likelihood Estimation

In this section, we discuss the method of maximum likelihood(ML) for the estimation of the unknown parameters a, b, β of Exponentiated Log-Uniform distribution. Let X_1,X_2,X_3,...,X_n be the random sample of size n drawn from Exponentiated Log-Uniform distribution, the likelihood function is given by,

$$L(x_i;\beta,a,b) = \prod_{i=1}^{n}\beta\Big(\frac{1}{x_i ln(\frac{b}{a})}\Big)\Big(\frac{ln(\frac{x_i}{a})}{ln(\frac{b}{a})}\Big)^{\beta-1}$$

The log-likelihood function is given by,

$$lnL(x_i;\beta,a,b) = nln\beta - n\beta(ln(ln(\frac{b}{a}))) + (\beta-1)\sum_{i=1}^{n}(ln(ln(\frac{x_i}{a})) - \sum_{i=1}^{n}ln(x_i) \tag{26}$$

$$\frac{\partial lnL}{\partial a} = \frac{n\beta}{aln(\frac{b}{a})} - \frac{\beta-1}{a\sum_{i=1}^{n}ln(\frac{x_i}{a})} = 0 \tag{27}$$

$$\frac{\partial lnL}{\partial b} = \frac{-n\beta}{bln(\frac{b}{a})} = 0 \tag{28}$$

$$\frac{\partial lnL}{\partial \beta} = \frac{n}{\beta} - nln(ln(\frac{b}{a})) + \sum_{i=1}^{n}ln(ln(\frac{x_i}{a})) = 0 \tag{29}$$

Solving this system of equations, in a, b, β gives the MLEs of a, b, β as $\hat{a}$,$\hat{b}$,$\hat{\beta}$.

2.3. Stress Strength Reliability

In this section, the procedure of estimating reliability of $R = P(X_2 < X_1)$ model is considered. The expression $R = P(X_2 < X_1)$ measures the reliability of a component in terms of probability. The random variables X_1 representing the stress experienced by the component does not exceed X_2 which represents the strength of the component. If stress exceeds strength, the component would fail and vice-versa.
In order to estimate the stress-strength parameter, considering two random variables X and Y with Exponentiated Log-Uniform (β_1, a, b) and Exponentiated Log-Uniform (β_2, a, b) distributions

respectively. We assume that X and Y are independent random variables and the stress-strength parameter is obtained in the form,

$$R = P(Y < X) = \int_{X<Y} f(x,y)dxdy = \int_0^\infty f(x;\beta_1,a,b)F(x;\beta_2,a,b)dx \tag{30}$$

where $f(x,y)$ is the joint probability density function of random variables X and Y, having Exponentiated Log-Uniform distribution.so that

$$R = \int_a^b \beta_1 \frac{1}{xln(\frac{b}{a})}\Big(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\Big)^{\beta_1-1}\Big(\frac{ln(\frac{x}{a})}{ln(\frac{b}{a})}\Big)^{\beta_2}dx$$

On simplification we get,

$$R = \frac{\beta_1}{\beta_1+\beta_2} \tag{31}$$

To compute the maximum likelihood estimate of R, we need to compute the maximum likelihood estimate of β_1 and β_2. Suppose $X_1, X_2, ..., X_n$ is random sample of size n from the Exponentiated Log-Uniform (β_1, a, b) and $Y_1, Y_2, ..., Y_m$ is an independent random sample of size m from Exponentiated Log-Uniform (β_2, a, b). The likelihood function of the combined random sample can be obtained as follow:

$$L = \prod_{i=1}^n f(x_i;\beta_1,a,b)\prod_{i=1}^m f(y_i;\beta_2,a,b) \tag{32}$$

$$L = \prod_{i=1}^n \beta_1\Big(\frac{1}{x_i ln(\frac{b}{a})}\Big)\Big(\frac{ln(\frac{x_i}{a})}{ln(\frac{b}{a})}\Big)^{\beta_1-1}\prod_{i=1}^m \beta_2\Big(\frac{1}{y_i ln(\frac{b}{a})}\Big)\Big(\frac{ln(\frac{y_i}{a})}{ln(\frac{b}{a})}\Big)^{\beta_2-1} \tag{33}$$

The log-likelihood function is,

$$\begin{aligned} lnL = {} & nln\beta_1 - n\beta_1(ln(ln(\frac{b}{a}))) + (\beta_1-1)\sum_{i=1}^n (ln(ln(\frac{x_i}{a})) - \sum_{i=1}^n ln(x_i) \\ & + mln\beta_2 - m\beta_2(ln(ln(\frac{b}{a}))) + (\beta_2-1)\sum_{i=1}^m (ln(ln(\frac{y_i}{a})) - \sum_{i=1}^m ln(y_i) \end{aligned} \tag{34}$$

The maximum likelihood estimate (MLE) of β_1 and β_2 can be obtained as the solution of,

$$\frac{\partial lnL}{\partial \beta_1} = \frac{n}{\beta_1} - nln(ln(\frac{b}{a})) + \sum_{i=1}^n ln(ln(\frac{x_i}{a})) = 0 \tag{35}$$

$$\frac{\partial lnL}{\partial \beta_2} = \frac{m}{\beta_2} - mln(ln(\frac{b}{a})) + \sum_{i=1}^m ln(ln(\frac{y_i}{a})) = 0 \tag{36}$$

From the equation (36) and (37),we obtain,

$$\hat{\beta_1} = \frac{n}{nln(ln(\frac{b}{a})) - \sum_{i=1}^n ln(ln(\frac{x_i}{a}))} \tag{37}$$

$$\hat{\beta_2} = \frac{m}{mln(ln(\frac{b}{a})) - \sum_{i=1}^m ln(ln(\frac{y_i}{a}))} \tag{38}$$

The corresponding ML estimate of R is computed from (32) by replacing β_1 and β_2 by their ML estimates.

$$\hat{R} = \frac{\hat{\beta_1}}{\hat{\beta_1}+\hat{\beta_2}} \tag{39}$$

3. Simulation Study and Data Analysis

3.1. Simulation Study

Simulation studies are an important tool for statistical research. Here we take distinct combinations of parameters a, b, β with sample size n. Bias and the mean square error(MSE) of the parameter estimates are obtained using the following equations.
If $\hat{\theta} = T(X)$ is an estimator, then the bias of $\hat{\theta}$ is the difference between its expectation and the 'true' value:

$$Bias(\hat{\theta}) = E_\theta(\hat{\theta}) - \theta \tag{40}$$

An estimator T(X) is unbiased for θ if $E_\theta(T(X)) = \theta$ for all θ, otherwise it is biased.
The MSE of an estimator $\hat{\theta}$ is

$$MSE = E_\theta\left[(\hat{\theta} - \theta)^2\right] \tag{41}$$

The simulation is done by using different true parameter values. The chosen true parameter values are as follows:

- $a = 0.12$, $b = 12$, $\beta = 0.5$

  As the n increases, MSE decreases for the selected parameter values given in table 1. Moreover, the bias is close to zero as the sample size increases. Thus, as the sample size increases the estimates tend to be closer to the true parameter values.

Table 1: *Simulation study at* $a = 0.12$, $b = 12$,$\beta = 0.5$

n	Parameter	Estimate	Bias	MSE
30	a	0.1354	0.154	0.00238
	b	8.9662	3.0338	9.2039
	β	0.4904	0.5096	0.2596
75	a	0.12038	0.0824	0.00133
	b	12.009	0.962	0.0565
	β	0.4209	0.0790	0.0842
100	a	0.12005	0.04732	0.000276
	b	11.438	0.5520	0.0315
	β	0.3717	0.012827	0.0624
500	a	0.1200081	0.02019	0.0001108
	b	11.867	0.1328	0.01762
	β	0.4572	0.004270	0.0128
1000	a	0.120006	0.01650	0.00007403
	b	11.961	0.03891	0.001513
	β	0.49096	0.00904	0.00550

Comparing the performance of the estimators in Table 1, we can verify that the bias and MSE values decreases as the sample size increases.

3.2. Data Analysis

Here we consider the data set represents the strength of glass of the aircraft window reported by Fuller et al. (1994) and we fit these data to Exponentiated Log-Uniform distribution and compare the results with the Exponentiated Quasi Akash Distribution and Exponentiated Weibull Distribution.

Table 2: *Strength of glass of the aircraft window*

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.50
25.52	25.80	26.69	26.77	26.78	27.05	27.67	29.90
31.11	33.20	33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29	45.381	

Table 3: *AIC, CAIC, BIC,and HQIC statistics of the fitted model in data set*

Distribution	AIC	CAIC	BIC	HQIC
Exponentiated Log-Uniform Distribution	196.3994	197.2883	200.7014	197.8017
Exponentiated Quasi Akash Distribution	214.165	215.054	218.467	215.567
Exponentiated Weibull Distribution	214.0852	214.9741	218.3871	215.4875

From the table 3, it has been observed that the Exponentiated log-uniform Distribution possesses the lesser AIC, CAIC, BIC,and HQIC values as compared to Exponentiated Quasi Akash distribution and Exponentiated Weibull distribution. To check the model goodness of fit we had considered the Kolmogorov-Smirnov (K-S) test (goodness-of-fit) statistics for the strength of glass of the aircraft window data. Since the p-value of fitted model is highest than the other distributions we have considered. Therefore the results indicate, that the Exponentiated Log-Uniform distribution performed better than other distributions.

4. SUMMARY AND CONCLUSIONS

In this study, we introduce a new distribution, called the Exponentiated Log-Uniform Distribution, which is generated using an exponentiated technique based on the two-parameter Log-Uniform distribution as the base distribution. We provide graphical representations of its density function, cumulative distribution function, hazard rate function, and survival function. And derive various statistical properties such as moments, mean deviations, and quantile function of the new distribution. We also obtain the probability density functions of the order statistics of the Exponentiated Log-Uniform Distribution.

To estimate the parameters of the distribution and the stress strength parameters, we use the maximum likelihood method, and validate the estimates of the model parameters through a simulation study. Our findings reveal that the Exponentiated Log-Uniform Distribution exhibits the least bias and that the values of the mean square error decrease as the sample size increases, indicating the effectiveness of this distribution in modeling real-world data.

Furthermore, we apply the Exponentiated Log-Uniform Distribution to a real data set and compare it with Exponentiated Quasi Akash distribution and Exponentiated Weibull distribution. Our results indicate that the new distribution outperforms these models based on various criteria such as the Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan information criterion (HQIC), the Kolmogorov-Smirnov (K-S) goodness-of-fit statistic, and the p-values obtained for the models.

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APPLICATIONS OF SIMULATIONS AND QUEUING THEORY IN SUPERMARKET

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Abstract

This paper describes the role of queuing theory in supermarket or shopping complex. Generally, a supermarket is a place where people are gathered to purchase the daily requirement products and here, a queue represents the customers/items in ascending or descending order. An interesting aspect of queuing process resides in the measures of its system's performance especially in terms of average service rate and system's utilization. Simulation is a powerful and versatile tool for modeling facilities in supermarket. So, queuing process with simulation provide the average service rate and it helps in predicting queue lengths as well as waiting durations when multiple items are manufactured and distributed using first come first serve discipline. M/M/s model and poisson process are used to explore the supermarket with server arrival rate and service rate.

Keywords: Queues availability, service discipline, simulation and production analysis.

I. Introduction

In the queuing system like post office, bus stand, supermarket, bank etc. customers arrive at service facility to receive the service. The service facility has one or more servers to attend the customers. If all servers are already busy to attend the customers then arrival of new customer join the queue until a server is free. The first come first serve pattern is used to provide the services to the customers. If a customer is served then he leaves the system and can not rejoin the queue. It is viewed that queuing theory is the subfield of

- probability theory with mathematics
- operation research with engineer
- operation management with business
- performance analysis with computer science
- performance analysis in electrical engineering

Queuing theory is the part of operation research that examined the situations related with multiple items/products in queue. It can help organizations to manage the balance between waiting time and resource utilization. The first paper on queuing theory related with theory of probabilities and telephone conversations was published by Erlang [7]. He examined the problem of determining how many telephone circuits were necessary to provide phone service that would prevent customers from waiting too long for an available circuit. In developing a solution to this problem, he began to realize that the problem of minimizing waiting time was applicable to many

fields, and began developing the theory further. Baskett et al. [1] described the joint equilibrium distribution of queue sizes in a network of queues containing N service centers and R classes of customers. It is assumed that the equilibrium probabilities exist and are unique. Here, four types of service centers are considered to model central processors, data channels, terminals and routing delays. Reddy et al. [13] analyzed a bulk queue with N-policy multiple vacations and setup times.

Chakravarthy [4] discussed on disaster queue system with markovian arrivals and impatient customers. Arrivals occurring during the time the system undergoes repair are stored in a buffer of finite capacity. These customers can become impatient after waiting a random amount of time and leave the system. Jain and Jain [9] evaluate the repairable queuing system model with multiple breakdown. Igwe et al. [8] evaluate the performance of queue management in supermarkets subject to average service rate using first come first serve discipline. Zhang and Liu [15] analyzed the queue with server breakdown, working vacations and vacation interruption using the supplementary variable method.

Jhala and Bhathawala [10] analyzed applications of queuing theory in supermarket to provide the facility to the customers and conclude that single queue multi server is better in comparison to multi queue multi server. Here, The waiting time of customers waiting in the queue is reduced almost 3 times to the previous one. We also proved that expected total cost is less in case of single queue multi server as compared to multi queue multi server model. Saraswat [13] evaluated the effects of a single counter markovian queuing model with multiple inputs. Chakravarthy and Kulshrestha [3] described a queuing model with backup server in the absence of the main server to continue the process. Bura [2] examined the queuing system model with low quality of services. Generally, customers are impatient due to long waits in queue but here, it is considered that customers are not impatient due to long waits but they are impatient due to the poor quality of service. Daş et al. [6] examined the two stage stochastic model for an industrial symbiosis network under uncertain demand. It is analyzed that the profitability of these volunteer companies is critical as it affects the sustainability of these networks.

Divya and Indhira [7] explained the literature survey on queuing model with working vacation. Narmadha and Rajendran [12] examined the literature review on development of queuing networks. Here, development is a process of gradual change that takes place over many years, during which a theory slowly progress and attain a good state. Yadav et al. [14] described the applications of simulation and queuing theory in scooter industry.

One of the expected gains from studying queuing systems is to review the efficiency of the models in terms of utilization and waiting length. Hence, increasing the number of queues so customers will not have to wait longer when servers are too busy. In queuing theory, Little's theorem, states that the long term average number L of customers in a stationary system is equal to the long term average effective arrival rate λ multiplied by the average time T that a customer spends in the system. Mathematically, it is expressed as

$$L = \lambda * T.$$

This relationship has been shown to be valid for a wide class of queuing models. Consider the example of a supermarket where the customer's arrival rate (λ) doubles but the customers still spend the same amount of time (T) in the bill paying area. These facts double the number of customers (L). By the same logic, if the customer arrival rate (λ) remains the same but the customer service time doubles. These will also double the total number of customers in the system. This indicates that in order to control the three variables, managerial decisions are only required for any two of the three variables. Three fundamental relationships can be derived from Little's theorem as

- L increases if λ or T increases.
- λ increases if L increases or T decreases.
- T increases if L increases or λ decreases.

II. Assumptions

To describe the performance of the supermarket, there are following assumptions

- All similar items are grouped in a particular sequence.
- There should be multiple queues to provide the services to the customers.
- There should be some spare service channels to provide the services when any working channel failed.
- Daily items availability and selling records should be maintained.
- The average number of waiting customers and average number of available service facilities are analyzed by using simulation.
- Arrivals time follow a poisson probability distribution at an average rate of λ customers per unit of time.
- Service times are distributed exponentially with an average of p customers per unit time.
- Service rate is independent of line length.
- The average arrival rate is greater than average service rate.
- First come first serve discipline is utilized for services.

III. Notations

There are following notations

- n = number of costumers in the system
- P_n = probability of n customers in the system
- λ = Average customer arrival rate
- μ = Average number of customers served per unit time at the place of service
- $1 / \mu$ = Average service completion time
- $1 / \lambda$ = Average inter arrival time
- P_0 = Probability of no customers in the system
- s = Number of service channels
- N = Maximum number of customers allowed in the system
- L_s = Average number of customers in the system
- L_q = Average number of customers in the queue
- L = Average length of non-empty queue
- W_s = Average waiting time in the system
- W_q = Average waiting time in the queue
- P_w = Probability that an arriving customer has to wait
- For achieving a steady-state condition and analytical results to be valid, we must have $\lambda / \mu < 1$.

IV. Supermarket Analysis

A Supermarket has three salesmen at the sales counters if the service time for each customer is exponential with a mean of 5 minutes and if people arrive in the poisson fashion at the rate of 20 an hour. Now, using the M/M/s queuing model and determine the following

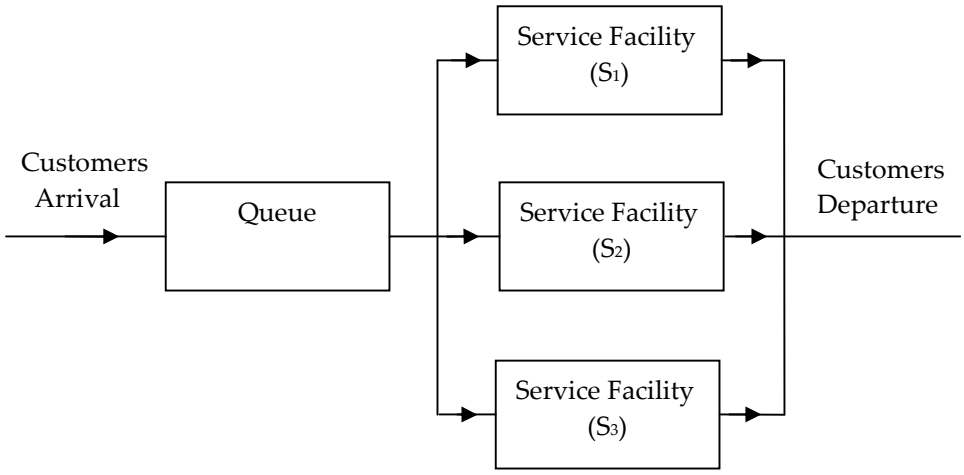

Figure 1: *Queuing System at Supermarket*

Solution: From the given data, get

- Mean service time (1/μ) = 5 minutes=1/12 hour
- Service Rate (μ)= 1/ mean service time = 1/(1/12) = 12 customers per hour
- Arrival Rate (λ) = 20 customers per hour
- Number of server (s) = 3

Now, calculate the following values

(a) Traffic Intensity (ρ)

$$\rho = \frac{\lambda}{s.\mu} = \frac{20}{3\times 12} = \frac{20}{36} = \frac{5}{9} \approx 0.55$$

(b) Probability of zero customer in the system (P_o)

$$P_o = \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \cdot \frac{1}{1-\rho} \right]^{-1} \quad \text{, where } \frac{\lambda}{\mu} = \frac{20}{12} \approx 1.66$$

Now calculate the sum:

For $n = 0$, $\frac{(1.66)^0}{0!} = 1$, For $n = 1$, $\frac{(1.66)^1}{1!} = 1.66$

For $n = 2$, $\frac{(1.66)^2}{2!} = \frac{2.77}{2} \approx 1.38$, For $n = 3$, $\frac{(1.66)^3}{3!} = \frac{4.62}{6} \approx 0.77$

Adding these values, get

$$\text{Sum} = 1 + 1.66 + 1.38 + +0.77 \approx 4.71$$

Now, calculate $\frac{(\lambda/\mu)^s}{s!} \cdot \frac{1}{1-\rho} = \frac{(1.66)^3}{3!} \cdot \frac{1}{1-0.55}$

$$= 0.77 \cdot \frac{1}{0.45} \approx 1.73$$

Now compute $P_o = (4.81 + 1.73)^{-1} = 0.15$

(c) Average number of customers in the system (L)

.

$$L = \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2}{s!\times(1-\rho)}$$

Here, $\frac{\lambda}{\mu} = \frac{20}{12} \approx 1.66$

and $\frac{(\lambda/\mu)^2}{s!\times(1-\rho)} = \frac{(1.66)^2}{3!\times(1-0.55)} = \frac{2.77}{6\times0.45} = \frac{2.77}{2.70} = 1.02$

Thus, $L = 1.66 + 1.04 = 2.70$

(d) Average time in the system (W)

$$W = \frac{L}{\lambda} = \frac{2.7}{20} \approx 0.1354 \text{ hours} \approx 8.12 \text{ minutes}$$

(e) Average number of customers in the queue (L_q)

$$L_q = L - \frac{L}{\lambda} = 2.7 - 1.66 \approx 1.04$$

(f) Average time in the queue (W_q)

$$W = \frac{L_q}{\lambda} = \frac{1.04}{20} \approx 0.05 \text{ hours} \approx 3.1 \text{ minutes}$$

V. Discussion

In the present model, first come first serve discipline is used to provide the services to the customers. If large number of customers arrived together then there are multiple counters to provide the services to customers to reduce the length of queue in one counter. Here, number of entry gate is one and number of exit gates is three to facilitate the customers. It is analyzed that the probability of queue existence depends on the utilization of products during the day. The utilization is directly proportional to the average number of customers. So, the average number of customers increases as the utilization increases.

Thus, from the above study it is clear that utilization of system is 0.55 and probability of all idle customers is 0.15. Average number of customers in the system is less than average number of customers in the queue and average time in the system is more than average time in the queue.

VI. Conclusion

From the above study, it is clear that servers are utilized about 55% of time and it means that there is some spare capacity available. The increase in the number of servers will reduce the time of customers have to wait in line before been served. So, it will increase the efficiency of the supermarket due to the utilization of their services to the customers as and when required.

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ALPHA POWER TRANSFORMED WEIBULL LOMAX DISTRIBUTION: PROPERTIES AND ITS APPLICATIONS

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Abstract

We proposed a new model called the Alpha Power Transformed Weibull-Lomax (APTWL) distribution which extends the Weibull Lomax distribution and have an increasing, decreasing and bathtub shapes for the hazard rate function. Various structural properties of the new distribution are derived including moments, probability weighted moments, generating and quantile function. The Renyi and q entropies are also obtained. Statistical inference is presented for the APTWL distribution using the method of maximum likelihood estimation to estimate the parameters of proposed distribution. The potentiality of the new model is illustrated by means of three real life datasets. The results of the analysis of the datasets show the superiority of APTWL distribution over some compared distributions.

Keywords: Alpha power transformation, Hazard function, Likelihood estimation, Lomax distribution, Weibull Lomax distribution

1. Introduction

The Lomax distribution also known as the Pareto Type II distribution is a continuous distribution that is used to model extreme events that follow a power law distribution. It is named after K.S Lomax [12] who first introduced. It is a heavy tailed distribution. That is, the extreme events are more likely to occur in other distributions like the normal distribution. It is often used to model phenomena like city sizes, earthquake magnitudes and financial returns. The Lomax distributions has applications in various fields including income and wealth inequality, acturial science, biological and medical sciences, engineering, longevity and reliability modeling, economics, insurance and geology. Overall, the Lomax distribution is a useful tool for modeling and analyzing extreme events.

Various authors extend the classical distributions to make them applicable in various fields. For example, Marshall-Olkin Extended Lomax (MOEL) was introduced by Ghitany *et al.* [7], Exponentiated Lomax (EL) was studied by Abdul-Moniem and Abdel-Hameed [1], Beta Lomax (BL) was examined by Lemonte and Cordeiro [11], Gamma Lomax (GL) was presented by Cordeiro *et al.* [4], Power Lomax (PL) was studied by Rady *et al.* [17] and Weibull Power Lomax(WPL) distribution was introduced by Hussain *et al.* [8]. The Weibull Lomax (WL) distribution was recently introduced and its mathematical and statistical features were examined by Tahir *et al.* [19].

Weibull-Lomax(WL) distribution which is introduced by Tahir *et al* [19] which extends the Lomax distribution has increasing and decreasing shapes for the hazard rate function. It has wider application in areas such as engineering, survival and lifetime data, hydrology and economics

(income inequality).
The cdf of WL distribution is given by,

$$F(x) = 1 - exp\left\{-\alpha\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha} - 1\right\}^{b}\right\}. \tag{1}$$

The pdf of WL distribution is given by,

$$f(x) = \frac{ab\alpha}{\beta}[1+\frac{x}{\beta}]^{b\alpha-1}\left\{1-[1+\frac{x}{\beta}]^{-\alpha}\right\}^{b-1}\exp\left\{-\alpha([1+\frac{x}{\beta}]^{-\alpha}-1)^{b}\right\}. \tag{2}$$

where $x > 0$; $a > 0$ and $b > 0$ are two parameters.

We are motivated to do this work for developing distribution which follow increasing, decreasing and constant failure rates. The main aim of this study is to provide another extension of the Weibull Lomax distribution introduced by Tahir *et al.*[19] using the Alpha power transformation defined by Mahdavi and Kundu [13].

Mahdavi and Kundu [13] proposed a new class of distributions called the Alpha Power Transformation (APT) family. It is introduced to analyse lifetime data obtained from systems that exhibit variety of monotonic and non-monotonic failure patterns.
The cdf of the APT family is defined by,

$$\begin{aligned} F_{APT}(x) &= \frac{\alpha^{G(x)}-1}{\alpha-1}, if\ \alpha > 0, \alpha \neq 1 \\ &= G(x), if\ \alpha = 1. \end{aligned} \tag{3}$$

The corresponding pdf of APT family is,

$$\begin{aligned} f_{APT}(x) &= \frac{log\,\alpha}{\alpha-1}\,\alpha^{G(x)}g(x), if\ \alpha > 0, \alpha \neq 1 \\ &= g(x), if\ \alpha = 1. \end{aligned} \tag{4}$$

2. Alpha Power Transformed Weibull Lomax Distribution

2.1. Probability Density Function (pdf) and Cumulative Distribution Function (cdf)

The Alpha Power Transformed Weibull Lomax (APTWL) distribution is obtained by using Weibull Lomax distribution as baseline distribution in alpha power transformation which is proposed by Mahdavi and Kundu [13]. Inserting equation (1) in (3), we get the five parameter APTWL cdf which is given by,

$$F(x) = \frac{\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}-1}{\alpha-1}. \tag{5}$$

And the corresponding pdf is obtained by inserting (2) in (4) and is given by,

$$f(x) = \frac{log\,\alpha}{\alpha-1}\,\frac{ab\theta}{\beta}\,\left(1+\frac{x}{\beta}\right)^{b\theta-1}\,\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1}exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\} \alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}. \tag{6}$$

where $x > 0$; $\alpha > 0$ is the additional shape parameter.

Figure 1 and Figure 2 provides plots of the pdf for some selected values of parameters. It is clear from the graph that the APTWL densities appears to be right Skewed and flexible heavy tailed distribution.

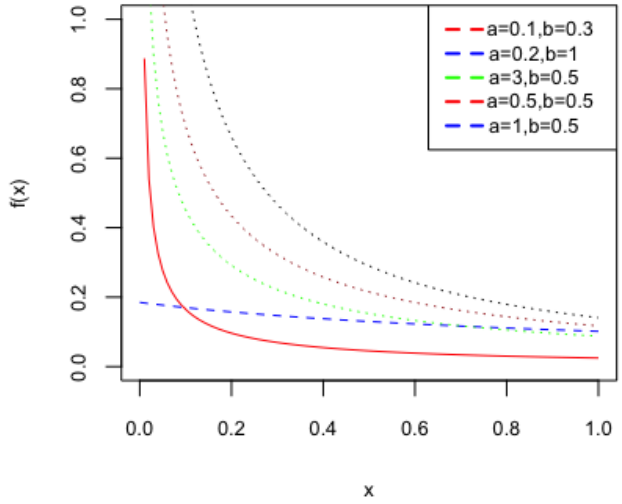

Figure 1: *pdf plot for* $\alpha = 0.5, \beta = 0.75, \theta = 0.5$

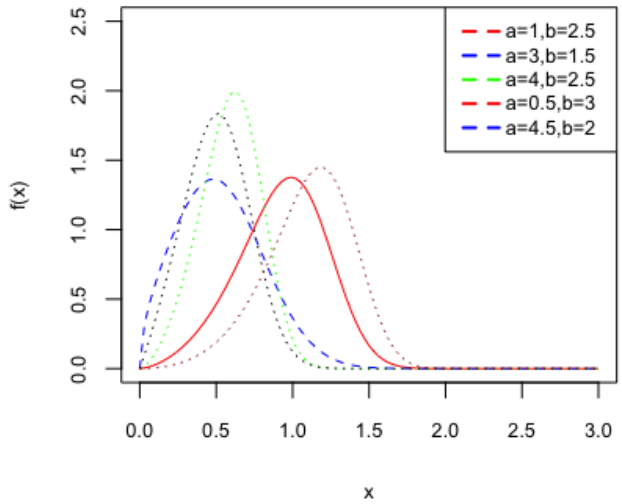

Figure 2: *pdf plot for* $\alpha = 3, \beta = 5, \theta = 4$

2.2. Quantile Function

Quantile functions are used in statistical analysis to summarize distributions and are essential in constructing box plots, quantile-quantile plots and in statistical methods (Mood *et.al.*) [16].

Quantile Function can be obtained by taking the inverse of the cumulative distribution function of a distribution. For $q \in (0,1)$,the quantile function of x is obtained by,

$$x_q = \beta \left\{ \left(\left[-\frac{1}{\alpha} log \left\{ 1 - \left\{ (log\,\alpha)^{-1} log\,[q(\alpha-1)+1] \right\} \right\} \right]^{\frac{1}{b}} + 1 \right)^{\frac{1}{\theta}} - 1 \right\}. \tag{7}$$

The root of equation (7) which is x_q, gives the unique solution for every value of $q \in (0,1)$ for a particular combination of parameter values of (α,β,θ,a,b). If q = 0.5, the median of APTWL distribution denoted by $x_{0.5}$ can be obtained from equation (7) with its expression given as

$$x_{0.5} = \beta \left\{ \left(\left[-\frac{1}{\alpha} log \left\{ 1 - \left\{ (log\,\alpha)^{-1} log\,[0.5\,(\alpha-1)+1] \right\} \right\} \right]^{\frac{1}{b}} + 1 \right)^{\frac{1}{\theta}} - 1 \right\}. \tag{8}$$

2.3. Reliability Function

The reliability function is a fundamental concept in reliability engineering used to model and analyze the longevity and performance of systems and components over time. It is crucial for determining maintenance schedules, warranty analysis, and risk assessment(Elsayed) [6]. Reliability function can be obtained from,

$$\begin{aligned} R(x) &= 1 - F(x) \\ &= (\alpha-1)^{-1} \left[\alpha - \alpha^{1-exp\left\{ -a\left\{ \left[1+\left(\frac{x}{\beta}\right)\right]^{\theta} - 1 \right\}^{b} \right\}} \right]. \end{aligned} \tag{9}$$

2.4. Hazard Function

The hazard function provides insights into the risk of failure at any given time point which helps to understand the dynamics of the failure process. It is a crucial concept in survival analysis and reliability engineering (Kalbfleisch and Prentice) [9]. The hazard function is the probability of failure in an infinitely small time period between x and x + dx given that the subject has survived up to time x. The hazard function is defined by,

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)} \\ &= \frac{\frac{log\,\alpha}{\alpha-1}\,\frac{ab\theta}{\beta}\,\left(1+\frac{x}{\beta}\right)^{b\theta-1}\,\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1}\,exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}{(\alpha-1)^{-1}} \\ &\qquad \left[\frac{\alpha^{1-exp\left\{ -a\left\{ \left[1+\left(\frac{x}{\beta}\right)\right]^{\theta} - 1 \right\}^{b} \right\}}}{\left[\alpha - \alpha^{1-exp\left\{ -a\left\{ \left[1+\left(\frac{x}{\beta}\right)\right]^{\theta} - 1 \right\}^{b} \right\}} \right]} \right]. \end{aligned} \tag{10}$$

Figure 3 and Figure 4 shows that hazard rate shapes can take different shapes such as constant, increasing and decreasing shape.

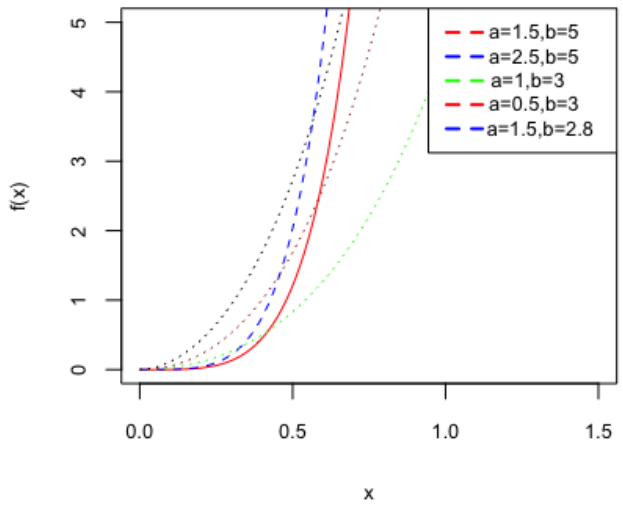

Figure 3: *hrf plot for* $\alpha = 1.5, \beta = 1.5, \theta = 1.5$

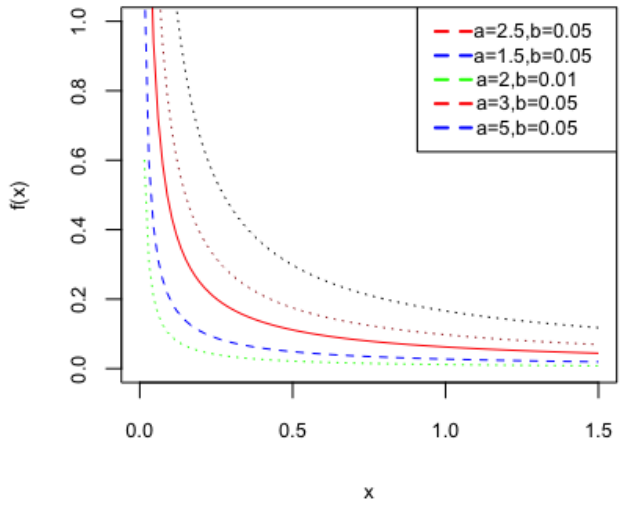

Figure 4: *hrf plot for* $\alpha = 3$, $\beta = 0.5$, $\theta = 1.5$

3. Properties

3.1. Moments

Moments are applied in various statistical methods including parameter estimation, hypothesis testing and in the method of moments for deriving estimators (DeGroot and Schervish) [5]. The rth moment of a random variable X is obtained as follows,

$$\mu_r^{'} = E[X^r]$$

$$\mu_r^{'} = \int_0^{\infty} x^r \frac{log\,\alpha}{\alpha - 1} \frac{ab\theta}{\beta} \left(1 + \frac{x}{\beta}\right)^{b\theta - 1} \left\{1 - \left(1 + \frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^{\theta} - 1\right\}^{b}\right\}$$

$$\alpha^{1 - exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^{\theta} - 1\right\}^{b}\right\}} dx. \quad (11)$$

Using the power series expansion $\alpha^k = \sum_{j=0}^{\infty} \frac{(log\alpha)^j}{j!} k^j$ in equation (11) and then applying binomial expansion $(1-z)^{\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i$ we get,

$$\mu_r^{'} = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+1}\, a^{k(ij+1)+2}\, b\theta\, (log\,\alpha)^{j+1}}{i!\, k!\, (j-1)!\, \beta(\alpha - 1)} \int_0^{\infty} x^r \left[\left(1 + \frac{x}{\beta}\right)^{-\theta}\right]^{-\frac{1}{\theta} - b - bk(ij+1)} \left[1 - \left(1 + \frac{x}{\beta}\right)^{-\theta}\right]^{(b-1) + bk(ij+1)}.$$

Put $y = \left(1 + \frac{x}{\beta}\right)^{-\theta}$ and then integrating we get,

$$\mu_r^{'} = \sum_{m=0}^{r} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+m+1}\, a^{k(ij+1)+2}\, \beta^r\, b (log\,\alpha)^{j+1} \binom{r}{m}}{i!\, k!\, (j-1)!\, (\alpha - 1)} B(\frac{1}{\theta}(m - r) - b - bk(ij+1), b + bk(ij+1)). \quad (12)$$

3.2. Moment Generating Function

The moment generating function(mgf) of a random variable X is defined by $M_x(t) = E\left[e^{tx}\right]$, for values of t in some neighborhood of zero for which this expectation exists (DeGroot and Schervish) [5]. It is an alternate method for analyzing results instead of working directly with the pdf and cdf of a random variable X. We obtain the moment generating function of a random variable X of the APTWL distribution as,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'}$$

$$= \sum_{m=0}^{r} \sum_{i,j,k,r=0}^{\infty} \frac{(-1)^{i+m+1}\, a^{k(ij+1)+2}\, \beta^{r-1}\, t^r b (log\,)^{j+1} \binom{r}{m}}{i!\, k!\, r!\, (j-1)!(\alpha - 1)} B(\frac{1}{\theta}(m - r) - b - bk(ij+1), b + bk(ij+1)). \quad (13)$$

3.3. Probability Weighted Moments

Probability Weighted Moments (PWMs) are a set of statistical measures used to summarize the probability distribution of a random variable. It is used to derive estimators of the parameters

and quantiles of generalized distributions. The (h,s)th PWM of X is given by,

$$\rho_{h,s} = E\left[X^h F(x)^s\right]$$
$$= \int_0^\infty x^h \frac{log\,\alpha}{\alpha-1}\frac{ab\theta}{\beta}\left(1+\frac{x}{\beta}\right)^{b\theta-1}\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^\theta-1\right\}^b\right\}$$
$$\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^\theta-1\right\}^b\right\}}\left[\frac{\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^\theta-1\right\}^b\right\}}-1}{\alpha-1}\right]^s. \quad (14)$$

Using the power series expansion and binomial expansion in equation (14), we get,

$$\rho_{h,s} = \sum_{i=0}^{s}\sum_{j,k=0}^{\infty}(-1)^{k+s-i}\binom{s}{i}\binom{j}{k}\frac{ab\theta(log\,\alpha)^{j+1}(i+1)^j}{j!\,\beta\,(\alpha-1)^{s+1}}\int_0^\infty x^h\left(1+\frac{x}{\beta}\right)^{b\theta-1}$$
$$\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{(k+1)\left\{-a\left(1+\frac{x}{\beta}\right)^\theta-1\right\}^b\right\}dx\cdot$$

Put $y = \left(1+\frac{x}{\beta}\right)^{-\theta}$ and then integrating, we get,

$$\rho_{h,s} = \sum_{i=0}^{s}\sum_{j,k,m,n=0}^{\infty}(-1)^{n+k+s-i}\binom{s}{i}\binom{j}{k}\binom{h}{m}\frac{a^{n+1}\beta^h\,b\,(log\,\alpha)^{j+1}\,(i+1)^j\,(k+1)^n}{(\alpha-1)^{(s+1)}\,j!\,n!}$$
$$B\left(\frac{1}{\theta}(m-h)-b(1+n)-1\,,b(1+n)\right)\cdot \quad (15)$$

3.4. Renyi Entropy and q Entropy

The Renyi and q entropies are two important measures in information theory for examining the unpredictability associated with random variables that follow a lifetime distribution. The entropy of a random variable X is a measure of the uncertain variation.
The Renyi entropy is defined by,

$$\delta_R(\omega) = (1-\omega)^{-1}\,log\left[\int_0^\infty (f(x))^\omega\,dx\right]\cdot$$

By applying the binomial expansion and exponential expansion in the pdf, we get,

$$\delta_R(\omega) = (1-\omega)^{-1}\,log\left[a^\omega\left(\frac{b\theta log\,\alpha}{\beta(\alpha-1)}\right)^\omega\sum_{i,j,k=0}^{\infty}(-1)^{i+k}\binom{j}{i}j^k\frac{(\omega log\,\alpha)^i}{j!\,k!}\int_0^\infty\left(1+\frac{x}{\beta}\right)^{\omega(b\theta-1)}\right.$$
$$\left.\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{\omega(b-1)}\left\{\left(1+\frac{x}{\beta}\right)-1\right\}^{bk}dx\cdot\right.$$

Put $y = \left(1+\frac{x}{\beta}\right)^{-\theta}$ and then integrating, we get,

$$\delta_R(\omega) = (1-\omega)^{-1}\,log\left[a^{\omega+k}\,\beta^{1-\omega}\,\theta^{\omega-1}\left(\frac{b\,log\,\alpha}{\alpha-1}\right)^\omega\sum_{i,j,k=0}^{\infty}(-1)^{i+k}\binom{j}{i}j^k\frac{(\omega\,log\,\alpha)^j}{j!\,k!}\right.$$
$$\left.B\left(\frac{1}{\theta}(\omega-1)-b(\omega+k)\,,1+bk+\omega(b-1)\right)\cdot \quad (16)\right.$$

The q entropy $H_q(f)$ is defined by,

$$H_q(f) = \frac{1}{q-1} log\left[1 - I_q(f)\right]$$

$$where\ I_q(f) = \int_R f^q(x)dx\cdot$$

$$H_q(f) = \frac{1}{q-1} log\left[1 - a^{q+k}\,\beta^{1-q}\,\theta^{q-1}\,\left(\frac{b\,log\,\alpha}{\alpha-1}\right)^q \sum_{i,j,k=0}^{\infty}(-1)^{i+k}\binom{j}{i}\,j^k\,\frac{(q\,log\,\alpha)^j}{j!\,k!}\right.$$
$$\left. B\left(\frac{1}{\theta}\,(q-1) - b(q+k)\,, 1 + bk + q(b-1)\right)\cdot\right. \quad (17)$$

3.5. Order Statistics

In a random sample of size n drawn from the APTWL distribution, we estimate the density of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$. We have (for i = 1,...,n),

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)}\,F^{i-1}(x)\,\left[1 - F(x)\right]^{n-i}\,f(x) \quad (18)$$

Now we can write,

$$F(x)^{i+j-1} = \frac{1}{(\alpha-1)^{i+j-1}}\left[\alpha^{\left\{1-exp\left\{-a\left[\left(1+\frac{x}{\beta}\right)^{\theta}-1\right]^b\right\}\right\}} - 1\right]^{i+j-1} \quad (19)$$

Using the binomial expansion $(1-z)^{\beta} = \sum_{i=0}^{\infty}(-1)^i\,\binom{\beta}{i}\,z^i$ in equation (18) and then applying power series expansion $\alpha^k = \sum_{j=0}^{\infty}\frac{(log\alpha)^j}{j!}k^j$ we get,

$$F(x)^{i+j-1} = \frac{1}{(\alpha-1)^{i+j-1}}\,(-1)^{i+j+k-1}\sum_{m,k=0}^{\infty}\binom{i+j-1}{k}\,\frac{(log\,\alpha)^m}{m!}\,k^m\,\sum_{t=0}^{\infty}(-1)^t\binom{m}{t}$$
$$exp\left\{-at\left\{\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^b\right\} \quad (20)$$

Substituting equation (20) in equation (18), we get,

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)}\,\frac{(log\,\alpha)^m\,k^m\,(at)^s}{m!\,\alpha^{i+j-1}}\sum_{j=0}^{n-i}\sum_{m,k,t,s=0}^{\infty}(-1)^{i+j+k+t+s-1}\binom{n-i}{j}\binom{i+j-1}{k}$$
$$\binom{m}{t}\left\{\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{bs} \quad (21)$$

4. Estimation

4.1. Maximum Likelihood Estimation

We consider the estimation of the unknown parameters of the APTWL distribution by the maximum likelihood method. The maximum likelihood approach is the most commonly used of the several parameter estimating techniques which have been validated in the literature. Let

$x_1, x_2, x_3, ..., x_n$ be a sample of size n from the APTWL distribution. The log likelihood (ll) function is given by

$$ll = n\,log\,(\frac{log\,\alpha}{\alpha-1}) \;+\; n\,log\,a \;+\; n\,log\,b \;+\; n\,log\,\theta \;-\; n\,log\,\beta \;+\; (b\theta-1)\sum_{i=1}^{n} log\,(1+\frac{x}{\beta}) + (b-1)\sum_{i=1}^{n} log\left\{1-(1+\frac{x}{\beta})^{-\theta}\right\} + \sum_{i=1}^{n}\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b} + log\,\alpha\sum_{i=1}^{n}\left[1-exp\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b}\right]. \quad (22)$$

Differentiating ll with respect to each parameter a,b,α,β and θ and setting the result equals to zero, we obtain maximum likelihood estimates (MLEs).

$$\frac{\partial ll}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\left[(1+\frac{x}{\beta})^{\theta}-1\right]^{b}\left\{1 + log\,\alpha\, e^{[-a(1+\frac{x}{\beta})-1]^{b}}\right\}. \quad (23)$$

$$\frac{\partial ll}{\partial b} = \frac{n}{b} \;+\; \theta\sum_{i=1}^{n} log(1+\frac{x}{\beta}) \;+\sum_{i=1}^{n} log\left\{1-(1+\frac{x}{\beta})^{-\theta}\right\} - a\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}\left[log\left\{(1+\frac{x}{\beta})^{\theta}-1\right\} - log\,\alpha\, e^{-a\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right]. \quad (24)$$

$$\frac{\partial ll}{\partial \alpha} = \frac{n}{\frac{log\,\alpha}{\alpha-1}}\;\frac{1}{\alpha} \;+\sum_{i=1}^{n}\left[1-log\,\alpha\, e^{-a\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right]\;\frac{1}{\alpha} = \frac{1}{\alpha}\left[\frac{n(\alpha-1)}{log\,\alpha} \;+\sum_{i=1}^{n}\left(1-log\,\alpha\, e^{-a\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right)\right]. \quad (25)$$

$$\frac{\partial ll}{\partial \beta} = -\frac{n}{\beta} - \frac{b\theta-1}{\beta^2}\sum_{i=1}^{n}\frac{x}{\left(1+\frac{x}{\beta}\right)^{3}} + \frac{\theta(b-1)}{\beta^2}\sum_{i=1}^{n}\frac{x(1+\frac{x}{\beta})^{-\theta-1}}{\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}} - \frac{ab\theta}{\beta^2}\sum_{i=1}^{n}\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b-1} x\left(1+\frac{x}{\beta}\right)^{\theta-1}\left\{1-log\sum_{i=1}^{n} exp\left(-a(1+\frac{x}{\beta})^{\theta}-1\right)^{b}\right\}. \quad (26)$$

$$\frac{\partial ll}{\partial \theta} = \frac{n}{\theta} \;+\; b\sum_{i=1}^{n} log(1+\frac{x}{\beta}) \;+b-1)\;\theta\sum_{i=1}^{n}(1+\frac{x}{\beta})^{\theta}\,log\left\{(1+\frac{x}{\beta})^{\theta}\right\} - a\,b\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b-1}\;(1+\frac{x}{\beta})^{\theta}\,log(1+\frac{x}{\beta})\;\left[1-log\,\alpha\, e^{-a\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right]. \quad (27)$$

Solving equation (23),(24),(25),(26) and (27) by equating to zero, we can find the maximum likelihood estimates of a,b,α, β and θ.

5. Simulation

In this section a small simulation study to illustrate the efficiency of the ML estimators of APTWL distribution is given. We generate data from the APTWL distribution using the

Table 1: *Simulation study at a=2 , b=2 ,*$\alpha = 2$, $\beta = 2$, $\theta = 2$

n	Parameters	Means	Bias	MSE
	a=2	1.10565	0.89434	0.91377
	b=2	2.39277	-0.39277	0.16138
100	l=2	1.02928	0.97071	0.94228
	p =2	1.01551	1.01550	1.00138
	t=2	1.29873	0.70126	0.49176
	a=2	1.75194	0.24805	0.06152
	b=2	2.38924	-0.38924	0.15151
500	l=2	1.21045	0.78954	0.62337
	p =2	1.27939	0.72060	0.51926
	t=2	1.38891	0.61109	0.47937
	a=2	2.17948	0.17948	0.03221
	b=2	2.03179	-0.03179	0.00101
2000	l=2	1.23165	0.76834	0.61244
	p =2	2.39977	-0.39977	0.15982
	t=2	2.35516	-0.35515	0.12613

Table 2: *Simulation study at a=0.5 , b=0.5 ,*$\alpha = 0.5$, $\beta = 0.5$, $\theta = 0.5$

n	Parameters	Means	Bias	MSE
	a=0.5	0.81558	-0.31558	0.11254
	b=0.5	0.60494	-0.10494	0.03801
100	l=0.5	2.87124	-2.37124	6.35831
	p =0.5	0.13112	0.36887	0.13789
	t=0.5	0.33607	0.16392	0.03643
	a=0.5	0.75076	-0.25076	0.07087
	b=0.5	0.55669	-0.05669	0.00539
500	l=0.5	2.49185	-1.99184	4.46373
	p=0.5	0.14258	0.35741	0.13791
	t=0.5	0.38565	0.11435	0.01722
	a=0.5	0.63824	-0.13824	0.04047
	b=0.5	0.48755	0.01244	0.00071
2000	l=0.5	2.42174	-1.92174	4.08359
	p =0.5	0.24247	0.25752	0.08332
	t=0.5	0.40056	0.09943	0.01081

quantile function given in Eq.(7).The behaviour of the parameters of the APTWL distribution was investigated by conducting simulation studies with the aid of R software. Data sets were generated from the APTWL distribution with a replication number m = 1000; random samples of sizes n = 100, 500 and 2000 were further selected. The simulation was conducted for two different cases using varying true parameter values. The selected true parameter values are $a = 2, b = 2, \alpha = 2, \beta = 2, \theta = 2$ and $a = 0.5, b = 0.5, \alpha = 0.5, \beta = 0.5, \theta = 0.5$ for the first and second cases respectively.

It can be understood from the table 1 and table 2 that mean square error (MSE) reduces for all the selected parameter values as the sample size increases. Also bias reduces as the sample size increases. Hence as sample size increases, the estimates tend towards the true parameter values.

6. Applications

In this study, three lifetime datasets are fitted to demonstrate the adaptability and usefulness of the APTWL distribution. We have fitted the APTWL distribution to the dataset using MLE and

compared the proposed APTWL distribution with Weibull Lomax distribution(WL), Exponentiated Lomax distribution(EL), Gamma Lomax distribution(GL), Power Lomax distribution(PL) and Weibull Power Lomax distribution(WPL). The Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are the goodness of fit statistics that were employed to compare the performances. AIC and BIC are computed as follows:

$$AIC = -2ll + 2k$$
$$BIC = -2ll + k \log(n),$$

where ll is the log likelihood function, k is the parameter number and n is the sample size. We use optim package in R to estimate parameters. Smaller values of the AIC and BIC statistics indicates better model fittings.

6.1. First data set: Strengths of 1.5 cm glass fibres

The glass fibres data set analyzed by Smith and Naylor [18] was used for this comparison. The data set originate from 63 observations of strengths of 1.5cm glass fibres primitively obtained by workers at the UK National Physical Laboratory as reported by Bourguignon *et al.*[2]. The data set is presented below:
0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

The parameter estimates values are shown in Table 3 and the performances of the APTWL distribution with the other competing distributions are shown in Table 4.

Table 3: *MLE for the strengths of 1.5 cm glass fibres*

Distribution	M.L.ESTIMATES				
	α	β	a	b	λ
APTWL	12.7181	2.7532	0.0147	2.1803	4.9414
WL	5.956	6.0375	0.17	3.5281	-
EL	0.0355	74.871	0.0355	-	-
GL	18304.4	20.9077	1312.05	-	-
PL	4.5151	16.196	-	-	77.3271
WPL	62.1361	0.8587	0.019	1.65	63.7374

Table 4: *AIC and BIC measures*

Distribution	-2loglik	AIC	BIC
APTWL	-26.0325	36.03849	46.75417
WL	-30.6798	38.6798	47.2524
EL	-31.95475	69.9095	76.33891
GL	-24.5296	55.05919	61.4886
PL	-32.775	38.6151	39.0218
WPL	-35.1032	36.48121	47.19688

Table 4 compare the APTWL model with the WL, EL, GL, PL and WPL models. We note that the APTWL model gives the lowest values for the AIC and BIC values among all fitted models. So, the APTWL model could be chosen as the best model.

Figure 5 shows the histogram of the data and the estimated pdfs for the fitted models. It is obvious that the APTWL distribution fits the histogram better than the other distributions suggesting that it might be the best model for the given set of data.

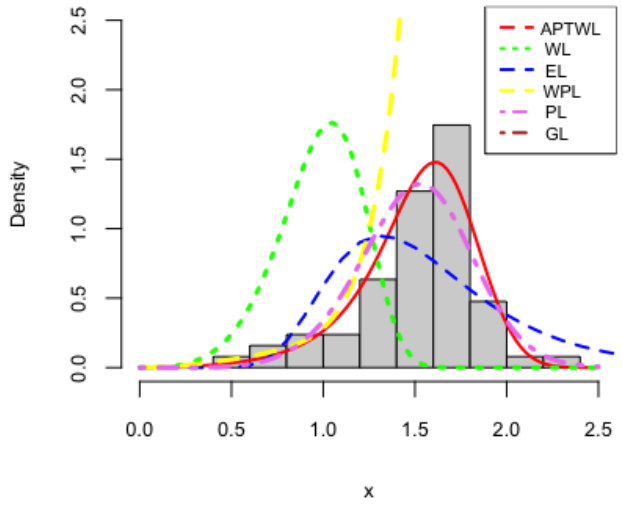

Figure 5: *Estimated pdfs for the first data set*

6.2. Second data set: Remission times of bladder cancer patients

We consider a dataset corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang [10]. The observations are as follows:
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 6.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

For data set 2, the parameter estimates values are shown in Table 5 and further discrepancy criteria for the competing distributions are shown in Table 6.

Table 5: *MLE for the remission times of 128 bladder cancer patients*

Distribution	M.L.ESTIMATES				
	α	β	a	b	λ
APTWL	8.35054	5.78478	87.7104	1.22317	0.04308
WL	0.25661	1.57945	2.42151	1.86389	-
EL	4.5857	24.7414	1.5862	-	-
GL	4.754	20.581	1.5858	-	-
PL	2.07012	1.4276	-	-	34.8626
WPL	62.1361	0.8587	0.0190	1.6500	63.7374

Table 6: *AIC and BIC measures*

Distribution	-2loglik	AIC	BIC
APTWL	-406.8466	817.6932	823.3972
WL	-410.811	829.622	841.03
EL	-407.5037	821.0074	829.5635
GL	-407.5165	821.0331	829.5891
PL	-409.74	825.48	834.036
WPL	-407.0033	824.0066	838.2667

The APTWL model is compared with the WL, EL, GL, PL and WPL models in Table 6. It is evident that out of all the fitted models, the APTWL model provides the lowest values for the AIC and BIC values. Thus the APTWL model may be selected as the optimal model.

The histogram of the data and the estimated pdfs for the fitted models are displayed in Figure 6. It is clear from Figure 6 that the APTWL distribution provides a better fit to the histogram and therefore could be chosen as the best model for the data set.

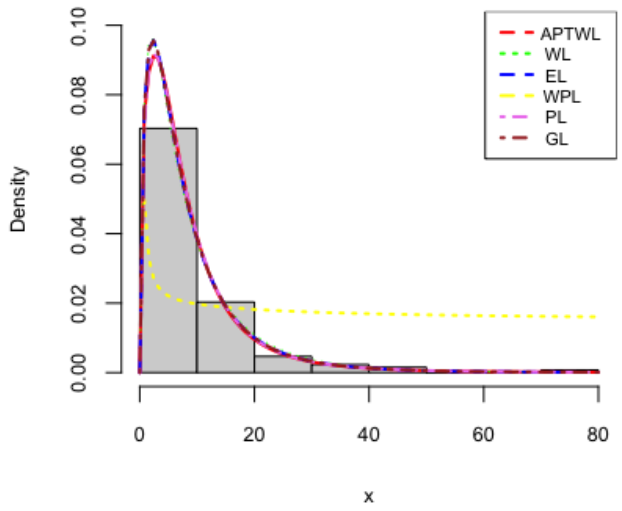

Figure 6: *Estimated pdfs for the second data set*

6.3. Third data set: Breaking stress of carbon fibers

For the third data set, we consider the uncensored data which consist of 100 observations on breaking stress of carbon fibers (in Gba) from Nichols *et al.* [15]. The data are:
3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65.

For data set 3, the parameter estimates values are shown in Table 7 and further discrepancy criteria for the competing distributions are shown in Table 8.

Table 7: *MLE for the breaking stress of carbon fibers (in Gba)*

Distribution	M.L.ESTIMATES				
	α	β	a	b	λ
APTWL	4.7392	0.1325	2.1899	5.4454	0.2104
WL	0.15361	2.57945	4.46252	0.56295	-
EL	8.9648	8.2838	14.2225	-	-
GL	1.6499	6.1510	6.9435	-	-
PL	1.6240	3.1692	-	-	29.4556
WPL	0.6134	1.4387	1.2636	2.2953	2.5218

Table 8: *AIC and BIC measures*

Distribution	-2loglik	AIC	BIC
APTWL	“141.1817	292.3634	305.3892
WL	-143.1656	294.7656	306.1296
EL	-146.652	300.7922	308.6077
GL	-143.6743	293.3486	301.1642
PL	-144.0012	296.914	304.7295
WPL	-141.3484	292.6968	305.7226

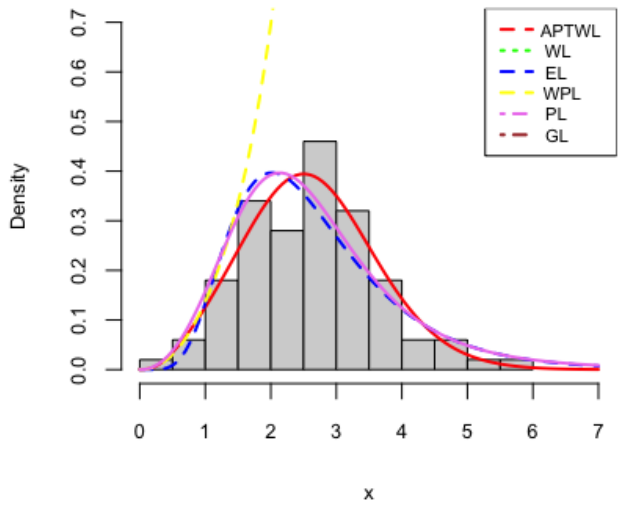

Figure 7: *Estimated pdfs for the third data set*

Table 8 compares the APTWL model with the WL, EL, GL, PL, and WPL models. It is clear that the APTWL model provides the lowest values for the AIC and BIC values among all the fitted models. As a result, the APTWL model might be chosen as the best one.

The histogram of the data and the estimated pdfs for the fitted models are displayed in Figure 7. Figures 7 affirm the results of the analysis that the APTWL distribution is more suitable for the data than the other competing distributions.

7. Conclusion

In this paper we introduced a new model called the Alpha Power Transformed Weibull-Lomax (APTWL) distribution which has bathtub, decreasing, and increasing shapes for the hazard rate function. We derived the structural characteristics of the new distribution including moments, probability weighted moments, generating function and quantile function. Additionally, the Renyi and q entropies are obtained. The APTWL distribution is statistically inferred with the parameters estimated by maximum likelihood estimation. We use three real life datasets for analysing the new model. The new model provides consistently a better fit than the other models namely Weibull Lomax (WL) distribution, Exponentiated Lomax (EL) distribution, Gamma Lomax (GL) distribution, Power Lomax (PL) distribution and Weibull Power Lomax (WPL) distribution. The proposed model will attract wider application in areas such as engineering, survival and lifetime data, economics (income inequality) and others.

8. Acknowledgements

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ALTERNATE QUADRA-SUBMERGING POLAR FUZZY GRAPH AND ITS DECISION MAKING ANALYSIS

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Abstract

In this article, the two extreme values [-1,1] is proposed with it's uncertain submerging values [-0.5,0.5] as the Alternate Quadra Submerging Polar (AQSP) Fuzzy Graph. The AQSP Fuzzy graph COVID-19 vaccines survey model has been analyzed to find the highest and the lowest membership and the non-membership value of the five influencing factors effectively. The notion of the AQSP fuzziness has been considered from the various points of view, in the specification of variables with the multiple input of single output rule. The self-reporting nature of the collected survey data of the COVID - 19 Booster shots acceptance and the non-acceptance values between [-1,0] and [0,1] converges precisely with the level of fixation [-0.5.0] and [0,0.5] alternatively by using the uncertain values in decision making process of the human behaviours in mathematical Analysis.

Keywords: Alternate Quadra - Submerging Polar Fuzzy Graph, AQSP Fuzzy Sub-merging Polar Relations, AQSP Fuzzy Submerging level of Fixation and AQSP Fuzzy Regular, Totally Regular, Strong and Complete graphs.

1. Introduction

The origin expression of fuzzification was first introduced by L.A. Zadeh [30] in his well-known concept 'Fuzzy sets'. The fuzzy arena is rising exponentially in the arena of fuzzy graph which is extended by A. Rosenfeld [22] who introduced the 'fuzzy graph' in 1975. He originated the fuzzy relation by considering fuzzy sets and functions as a structure of fuzzy graph. And in 1973 Kauffman [13] presented the first definition of fuzzy graph. Different types of fuzzy graph analogues and concepts were hosted by many fuzzy mathematicians. Fuzzy set theory and operations on fuzzy graph with several properties are introduced and defined by Klir[14] and Yuan[29] in and M.S.Sunitha.[26] Many important perceptions on fuzzy graphs were presented by Moderson and NagoorGani A,[17] Nair. Akram[1] has proposed fuzzy graphs concepts. J.N. Moderson[16] presented the fuzzy line graph.

In this analysis of AQSP fuzzy graph COVID 19 vaccine survey, we find the membership value of conflict feelings and frustration which is vague and uncertain is measured precisely. To highlight human beings, alternate conflict feelings, attitudinal behavior with an alternate equal association of membership and non - membership principles are polarized in alternate quadrant. The concept, level of fixation is defined as [-0.5,0] and [0, 0.5] will precise the many submerging level of uncertain human behavior with an Alternate Quadra values as the level of presumption along with the level of membership values in the given fuzzy graph which is denoted as{ [0,0],[0,1],[1,1],[1,0] } and { [0,0],[-1,0],[-1,-1],[0,-1] } of fuzzy sets. This mid-sub merging alternate quadra - values can be used to describe the increasing or decreasing level towards

destination of certain values, which is consistent with mid-submerging uncertain fuzzified values in AQSP Fuzzy graphs are{ [0,0],[0,0.5],[0.5,0.5],[0.5,0] } and { [0,0],[-0.5,0],[-0.5,-0.5],[0,-0.5] }.The submerging alternate quadrant sets with equal opposite polarized reaction which implement the destination of the mind to decide whether to accept or not accept the Booster shot in future. The fixed level of confidence in AQSP fuzzy graph, indicates all possible preferential membership and non-membership values of reaching certain level of precise value which is reliable.

2. Preliminaries

2.1. Fuzzy Graph [22]

Let ϑ is a non-empty set. A fuzzy graph $G : (\vartheta, \sigma, \mu)$ matching to the crisp graph $G^* : (\sigma^*, \mu^*)$ is a non-empty set ϑ together with pair of functions $\sigma : \vartheta \to [0,1]$ where σ is a fuzzy subset of ϑ and $\mu : \vartheta \times \vartheta \to [0,1]$, μ is a symmetric fuzzy relation on σ, for all values of $x, y \in \vartheta$, such that $\mu(x,y) \leq \min(\sigma(x), \sigma(y)) \forall x, y \in \vartheta$

2.2. Partial Fuzzy Sub Graph [23]

The fuzzy graph $\varphi = (\vartheta_1, \alpha, \beta)$ is called a partial fuzzy subgraph of $G : (\vartheta, \sigma, \mu)$ *if* $\alpha \leq \sigma$ *and* $\beta \leq \mu$. In precise mod, $\varphi = (\vartheta_1, \alpha, \beta)$ is called a fuzzy subgraph of $G : (\vartheta, \sigma, \mu)$ persuaded by by ϑ_1 if $\vartheta_1 \subseteq \vartheta$ and $\alpha(x) = \sigma(x)\ \forall\ x \in \vartheta_1$ with $\beta(x,y) = \mu(x, y)\ \forall\ x, y \in \vartheta_1$. The fuzzy graph $G : (\vartheta, \sigma, \mu)$ is trivial if $|\varphi| = 1$.

2.3. Alternate Quadra Sub - merging Polar(AQSP) Fuzzy Graph [5]

An Alternate Quadra - Submerging Polar (AQSP) Fuzzy Graph $G = (\sigma_{AQSP}, \mu_{AQSP})$ is a fuzzy graph with crisp graph $G^* = (\sigma^*_{AQSP}, \mu^*_{AQSP})$ is given as $V = (\sigma^P(x), \sigma^N(x), \rho^P(x), \rho^N(x))$ which is the membership value of vertices along with the uncertain membership value of edges is given as, $E = V \times V = (\mu^P(x,y), \mu^N(x,y), \gamma^P(x,y), \gamma^N(x,y))$.

Here the vertex set V is defined with the given condition in a unique method which is an alternate contrast submerging polarized uncertain transformation.Here $\sigma^P = V \to [0,1]$, $\sigma^N = V \to [-1,0]$, $\rho^P = d \mid 0.5, \sigma^P(x) \mid$ and $\rho^N = -d \mid -0.5, \sigma^N(x) \mid$. Here (-0.5, 0.5) is the fixation of uncertain alternate contrast polarized submerging transformation into certain consistent preferable position. And the edge set E satisfies the following sufficient conditions.

$(i)\quad \mu^P(x,y) \leq \min(\sigma^P(x), \sigma^P(y)), \quad (ii)\ \mu^N(x,y) \geq \max(\sigma^N(x), \sigma^N(y))$

$(iii)\ \gamma^P(x,y) \leq \min(\rho^P(x), \rho^P(y)) \quad (iv)\ \gamma^N(x,y) \geq \max(\rho^N(x), \rho^N(y)),$

$\forall (x,y) \in E$. By definition, $\mu^P = V \times V \to [0,1] \times [1,0]$, $\mu^N = V \times V \to [-1,0] \times [0,-1]$ and the submerging mappings, $\gamma^P = V \times V \to [0,0.5] \times [0.5,0]$,
$\gamma^N = V \times V \to [-0.5,0] \times [0,-0.5]$, which denotes the impact of the alternate quadrant polarized fuzzy mapping. The maximum of submerging presumption to be at the level of confidence $[0,0.5] \subseteq [0,1]$ and the minimum of submerging presumption level of confidence is $[-0.5,0] \subseteq [-1,0]$ extension of the graph with its membership and non - membership values portrait the unique level of submerging destination in an AQSP fuzzy graph.

Also it must satisfy the condition, $-1 \leq \sigma^P(x) + \sigma^N(x) \leq 1$ *and* $|\rho^P(x) + \rho^N(x)| \leq 1$ with constrains $0 \leq \sigma^P(x) + \sigma^N(x) + |\rho^P(x) + \rho^N(x)| \leq 2$ such that the uncertain status of submerging presumption, transform into its precise consistent level with fixation mid - value 0.5, which implies that level of confidence 0.5 in an AQSP as the valuable membership of its position which is real and valid in the fuzzification. The example of AQSP fuzzy graph is given in the Figure.1.

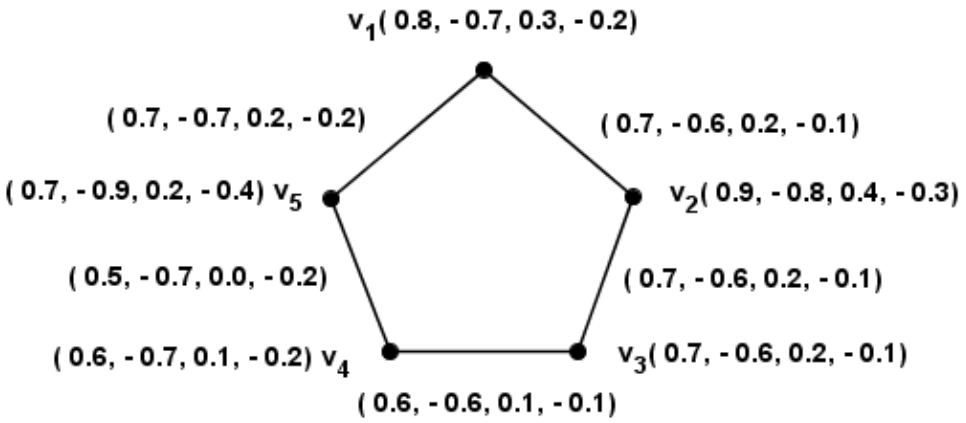

Figure 1: *AQSP Fuzzy Graph*

2.4. Complement of Fuzzy Graph [17]

Let $G = (\vartheta,\ \sigma, \mu)$ be a fuzzy graph corresponding to the crisp graph. The complement of uncertain graph G is defined as $\bar{G}:\ \bar{\sigma}, \bar{\mu}$ where, $\bar{\sigma} = \sigma,\ \ \bar{\mu}\ = \mu$ and the definition is given as $\bar{\mu}\ (x,y) = \sigma\,(x) \wedge \sigma\,(y) -\ \mu\,(x,y)\ \ \forall\ x,y \in \sigma$.

2.5. Complete Fuzzy Graph [17]

Let $G = (\sigma, \mu)$ is called as complete fuzzy graph with underlying crisp graph $G^* = (\sigma^*, \mu^*)$ such as $\mu\,(x,y) = \min\,(\sigma\,(x)\,, \sigma\,(y))\ \forall\ x,y\ \in V$ with given fuzzy node and edge sets.

2.6. Strong Fuzzy Graph [17]

Let fuzzy graph is a pair of functions $G : (\sigma, \mu)$ where σ is a uncertain fuzzy subset of X, μ is a symmetrical uncertain fuzzy relation on σ, where $\sigma : X \to [0,1]$ and $\mu : X \times X \to [0,1]$ such that, $\mu\,(x,y) = \min\,(\sigma\,(x)\,, \sigma\,(y)),\quad x,y\ \in E$ with satisfying membership values constrain in edge set is called as strong fuzzy graph.

2.7. Intersection of two fuzzy graphs [18]

Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with $V_1 \cap V_2 = \emptyset$ and then $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $G^* = G_1^* \cap G_2^* =\ (V_1 \cap V_2, E_1 \cap E_2\ )$ be the intersection of crisp graph of G_1^* and G_2^*. Then the intersection of $G_1 \cap\ G_2$ is defined as $(\sigma_1 \cap \sigma_2, \mu_1 \cap \mu_2\ )$ of the crisp graph G_1^* and G_2^* ,

$(i)\ (\sigma_1 \cap \sigma_2)\,(x) \quad = \sigma_1\,(x) \quad if\ x \in\ V_1 \cap \bar{V}_2$

$(ii)\ (\sigma_1 \cap \sigma_2)\,(x) \quad = \sigma_2\,(x) \quad if\ x \in\ V_2 \cap \bar{V}_1$

$(iii)\ \ (\mu_1 \cap \mu_2)\,(x,y)\ =\ \mu_1\,(x,y)\ if\ x,y \in\ E_1 \cap \bar{E}_2$

$(iv)\ \ (\mu_1 \cap\ \mu_2)\,(x,y)\ =\ \mu_2\,(x,y)\ if\ x,y \in\ E_2 \cap \bar{E}_1.$

3. Classifications of AQSP Fuzzy Graphs

3.1. Order of AQSP Fuzzy graph

Let $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ be an AQSP Fuzzy graph with vertex set V, then the order of G is defined as, $O\left(G_{AQSP}\right) = (\sum_{x\in V}\ \sigma^P(x), \sum_{x\in V}\sigma^N(x), \sum_{x\in V}\rho^P(x), \sum_{x\in V}\rho^P(x))$.

3.2. Size of AQSP Fuzzy graph

Let $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ be an AQSP Fuzzy graph with edge set E, then the size of G is defined as $S\left(G_{AQSP}\right) = (\sum_{(x,y)\in V}\mu^P(x,y), \sum_{(x,y)\in V}\mu^N(x,y), \sum_{(x,y)\in V}\gamma^P(x,y), \sum_{(x,y)\in V}\gamma^N(x,y))$.

3.3. Degree of AQSP Fuzzy graph

Let $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ be an AQSP Fuzzy graph ,then the degree of a vertex 'x' is defined with an example presented in Figure.2,
$deg_G(x) = (\sum_{x\neq y}\ \mu^P\ (x,y)), (\sum_{x\neq y}\ \mu^N\ (x,y)), (\sum_{x\neq y}\ \gamma^P\ (x,y)), (\sum_{x\neq y}\ \gamma^N\ (x,y))$.

3.4. AQSP Partial Fuzzy Sub graph

The fuzzy graph $S : (\rho, \vartheta)$ is called a partial AQSP fuzzy sub graph of the AQSP fuzzy graph, $G = \left(\sigma_{AQSP}\ ,\ \mu_{AQSP}\right)$ if $\rho \subseteq \sigma$ and $\gamma \subseteq \mu$ if vertex set $\rho\ (x) \leq\ \sigma\ (x)\ \forall\ x\ \in V$ and the edge set of membership value is, $\gamma^P\ (x,y)\ \leq\ \mu^P\ (x,y)$ then the non - membership value such as, $\gamma^N\ (x,y)\ \geq\ \mu^N\ (x,y)$ for all values of $(x,y)\ \in E \subseteq v \times v$.

In a unique way an AQSP membership values are defined and compared with the fixation (-0.5, 0.5) level of confidence. If the sufficient condition of fuzzy sub graph is defined with the membership and non - membership values such as,

$\rho\,(x) =\ \sigma\,(x)\ \forall\ x\ \in\ \rho^*$ and $\gamma\,(x,y) =\ \mu\ (x,y)\ \forall\ (x,y)\ \in\ \gamma^*$ satisfied then it is obvious $S : (\rho, \gamma)$is a fuzzy sub graph of $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ and it is denoted as $|(\sigma, \rho)| = 1$. Figure.2 represents The AQSP Partial fuzzy graph.

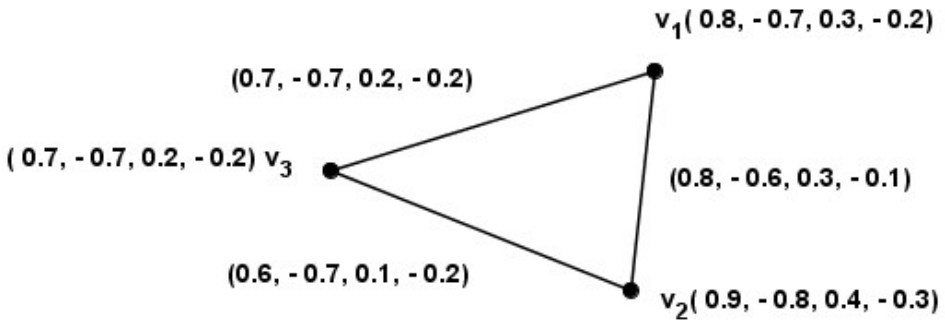

Figure 2: *AQSP Partial Fuzzy Sub graph* $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$

4. Regular and Totally Regular AQSP fuzzy graph

4.1. Regular AQSP Fuzzy graphs

Let $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ be the given AQSP fuzzy graph. If $d_G\,(v) = k\ \forall\ v\ \in V$ if each vertex has same degree of k- elements then G is called as the regular AQSP Fuzzy graph, is of degree k (or) a k - regular AQSP fuzzy graph. An AQSP fuzzy Regular graph is shown in Figure.3. Here $d_G\,(v_1) = \ d_G\,(v_2) = \ d_G\,(v_3) = \ \{\ 1.4,\ -1.2,\ 0.4,\ -0.2\}$.

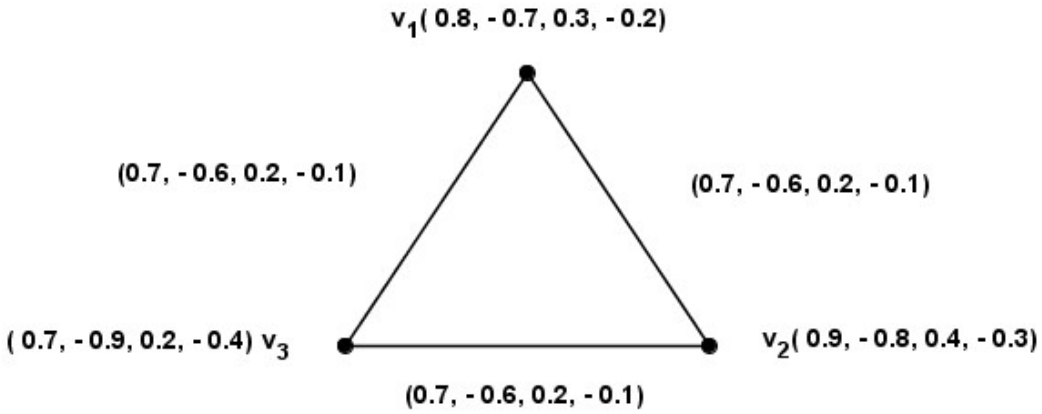

Figure 3: *Regular AQSP Fuzzy graph* $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$

4.2. Totally Regular AQSP Fuzzy graphs

Let $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$be an AQSP fuzzy graph. Then the AQSP fuzzy graph total degree of each vertex $x \in V$ and $T\,[d_G\,(x)]$ is denoted as with the constraints.

$(\sum_{(x,y)\in E}(\mu^p(x,y) + \sigma^P(x)) + \sum_{(x,y)\in E}(\mu^p(x,y) + \sigma^N(x))$

$+ \sum_{(x,y)\in E}(\rho^p(x,y) + \rho^P(x)) + \sum_{(x,y)\in E}(\rho^p(x,y) + \rho^N(x)))$.

Here each vertex of $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ has the same total degree 'k' then the AQSP is said to be a totally regular AQSP fuzzy graph with degree 'k'. The following illustration presents in Figure.4, implement that the AQSP is totally regular fuzzy graph of $G^* = \left(\sigma^*_{AQSP}\ ,\ \ \mu^*_{AQSP}\right)$.

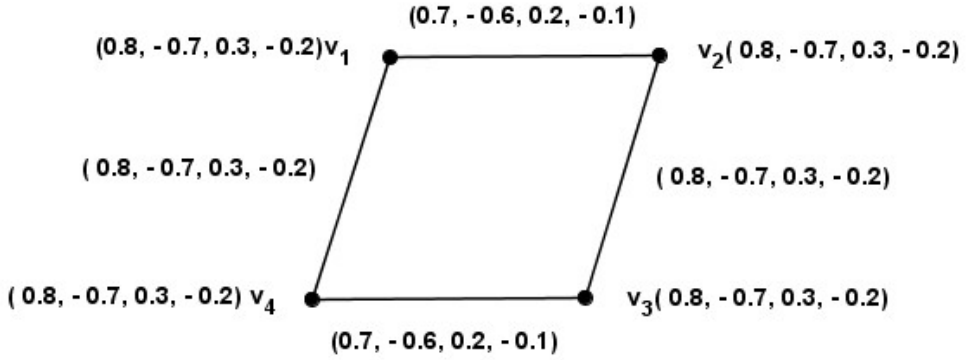

Figure 4: *Totally Regular AQSP Fuzzy graph* $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$

4.3. Example of Totally Regular AQSP Fuzzy graphs

Let $v_1 = v_2 = v_3 = v_4 = (0.8, -0.7, 0.3, -0.2)$

$v_1v_2 = v_3v_4 = (0.7, -0.6, 0.2, -0.1)$ and $v_2v_3 = v_1v_4 = (0.8, -0.7, 0.3, -0.2)$, is totally regular and regular AQSP. The AQSP fuzzy graph in Figure.3 with vertex and edge set, shows that it is regular and regular AQSP fuzzy graph .

$v_1 = v_2 = v_3 = v_4 = (0.8, -0.7, 0.3, -0.2)$

$v_1v_2 = v_3v_4 = (0.7, -0.6, 0.2, -0.1)$ and

$v_2v_3 = v_1v_4 = (0.8, -0.7, 0.3, -0.2)$,

But the example in Figure.4 shows that it's only regular AQSP fuzzy graph. And it is not totally regular AQSP fuzzy graph .

4.4. Theorem of Totally Regular AQSP Fuzzy graphs

Let $G = (\sigma_{AQSP}, \mu_{AQSP})$ be an AQSP fuzzy graph with underlying crisp graph $G^* = (\sigma^*_{AQSP}, \mu^*_{AQSP})$ and $V = (\sigma^P(a), \sigma^N(a), \rho^P(a), \rho^N(a))$ is a constant function if and only if the following conditions are equivalent.

(i) $G = (\sigma_{AQSP}, \mu_{AQSP})$ is a regular AQSP fuzzy graph.

(ii) $G = (\sigma_{AQSP}, \mu_{AQSP})$ is a totally regular AQSP fuzzy graph.

Proof.

Let $G = (\sigma_{AQSP}, \mu_{AQSP})$ be a regular AQSP fuzzy graph. Suppose V is a constant function.

Then $deg_G(a) = c \ \forall \ a \in V$ where c is a constant.

Here $\sigma^P(a) = c_1$, $\sigma^N(a) = c_2$, $\rho^P(a) = c_3$ and $\rho^N(a) = c_4 \ \forall \ a \in V$.

(i) $\Longrightarrow$ (ii)

Consider, $G = (\sigma_{AQSP}, \mu_{AQSP})$ be a regular AQSP Fuzzy graph, then ,

$deg_G\sigma^P(a) = k_1$, $deg_G\sigma^N(a) = k_2$,

$deg_G\rho^P(a) = k_3$ *and* $deg_G\rho^N(a) = k_4 \ \forall \ a \in V$.

Now,

$tdeg_G(\sigma^P(a)) = deg_G(\sigma^P(a) + \sigma^P(a))$, $tdeg_G(\sigma^N(a)) = deg_G(\sigma^N(a) + \sigma^N(a)$,

$tdeg_G(\rho^P(a)) = deg_G(\rho^P(a) + \rho^P(a))$, and

$tdeg_G(\rho^N(a)) = deg_G(\rho^N(a) + \rho^N(a) \forall \ a \in V$.

Hence ,

$tdeg_G(\sigma^P(a)) = k_1 + c_1$,

$tdeg_G(\sigma^N(a)) = k_2 + c_2$,

$tdeg_G(\rho^P(a)) = k_3 + c_3$ and

$tdeg_G(\rho^P(a)) = k_3 + c_3 \ \forall \ a \in V$.

Thus , $G = (\sigma_{AQSP}, \mu_{AQSP})$ is a totally regular AQSP fuzzy graph.

(ii) $\Longrightarrow$ (i)

Suppose that $G = (\sigma_{AQSP}, \mu_{AQSP})$ is a totally regular AQSP fuzzy graph,

$tdeg_G(\sigma^P(a)) = t_1$,

$tdeg_G(\sigma^N(a)) = t_2$,

$tdeg_G(\rho^P(a)) = t_3$, and

$tdeg_G(\rho^N(a)) = t_4 \ \forall \ a \in V$. Otherwise,

$deg_G(\sigma^P(a) + \sigma^P(a) = t_1$, $deg_G(\sigma^N(a) + \sigma^N(a) = t_2$,

$deg_G(\rho^P(a) + \rho^P(a) = t_3$ and $deg_G(\rho^N(a) + \rho^N(a) = t_4 \ \forall \ a \in V$.

The other notion of the result such as,

$deg_G(\sigma^P(a)) + c_1 = t_1$,

$deg_G(\sigma^N(a)) + c_2 = t_2$,

$deg_G(\rho^P(a)) + c_3 = t_3$ and

$deg_G(\rho^N(a)) + c_4 = t_4 \ \forall \ a \in V$ or,

$deg_G(\sigma^P(a) = t_1 - c_1$, $deg_G(\sigma^N(a) = t_2 - c_2$,

$deg_G(\rho^P(a) = t_3 - c_3$ and $deg_G(\rho^N(a) = t_4 - c_4 \ \forall \ a \in V$.

Theis implies that $G = (\sigma_{AQSP}, \mu_{AQSP})$ is a regular AQSP Fuzzy graph.

Hence it is obvious that (i) and (ii) are equivalent.

5. Strong AQSP fuzzy graph

A fuzzy graph $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ is said to be a strong AQSP fuzzy graph if it satisfies the following conditions.

$(i)\ \mu^P(x,y) = \min(\sigma^P(x)\ ,\ \sigma^P(y)\ )$,

$(ii)\mu^N(x,y) = \max(\sigma^N(x)\ ,\ \sigma^N(y)\ )$

$(iii)\gamma^P(x,y) = \min(\rho^P(x)\ ,\ \rho^P(y)\ )$,

$(iv)\gamma^N(x,y) = \max(\rho^N(x)\ ,\ \rho^N(y)\ )$ for $x, y \in \mu_{AQSP}$.

Figure.5. is an illustration of the Strong AQSP Fuzzy graph.

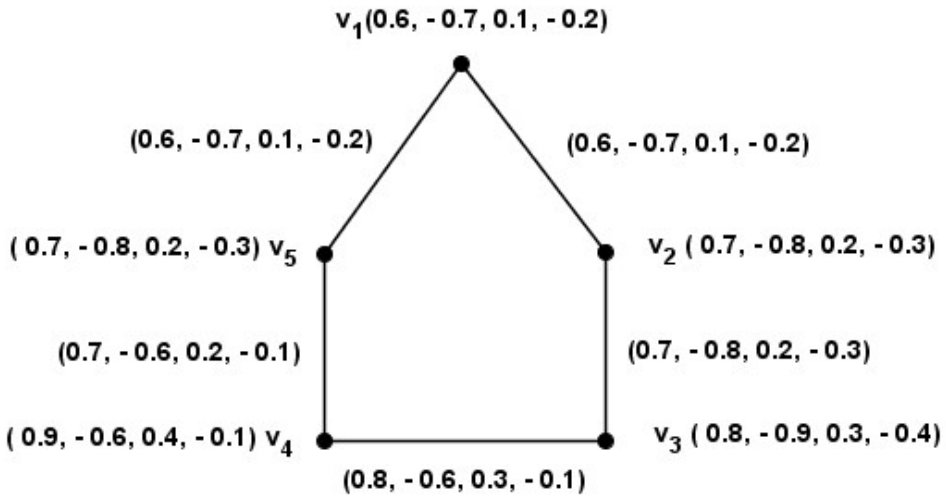

Figure 5: *Strong AQSP Fuzzy graph* $G = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$

5.1. Theorem of the Strong AQSP Fuzzy graph

If $G_1 \times G_2$ is strong AQSP fuzzy graphs then at least $G_1 = (\sigma_{AQSP}, \mu_{AQSP})$ or $G_2 = (\sigma_{AQSP}, \mu_{AQSP})$ must be strong.

Proof. Consider that $G_1 = (\sigma_{AQSP}, \mu_{AQSP})$ and $G_2 = (\sigma_{AQSP}, \mu_{AQSP})$ are not strong AQSP fuzzy graphs. Then there exists $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$ such that, for membership and non - membership values of AQSP fuzzy graph is given with the conditions,

$\mu_1^p(u_1,v_1) < \wedge((\sigma_1^P)(u_1),(\sigma_1^P)(v_1)),\ \mu_2^p(u_1,v_1) < \wedge((\sigma_2^P)(u_1),(\sigma_2^P)(v_1))$
$\mu_1^N(u_1,v_1) > \vee((\sigma_1^N)(u_1),(\sigma_1^N)(v_1)),\ \mu_2^N(u_1,v_1) > \vee((\sigma_2^N)(u_1),(\sigma_2^N)(v_1))$
$\forall\ (u_1,v_1) \in E_1\ ,\ \forall\ (u_2,v_2) \in E_2$. And for submerging membership and
non - membership values of AQSP fuzzy graph is,
$\gamma_1^p(u_1,v_1) < \wedge((\rho_1^P)(u_1),(\rho_1^P)(v_1)),\ \ \gamma_2^p(u_1,v_1) < \wedge((\rho_2^P)(u_1),(\rho_2^P)(v_1))$
$\gamma_1^N(u_1,v_1) > \vee((\rho_1^N)(u_1),(\rho_1^N)(v_1)),\ \gamma_2^N(u_1,v_1) > \vee((\rho_2^N)(u_1),(\rho_2^N)(v_1))$
Consider membership and non - membership values of AQSP fuzzy graph,

$$\mu_2^P(u_2,v_2) \leq (\mu_1^p(u_1,v_1)) < \wedge(\sigma_1^P(u_1),\sigma_1^P(v_1)) \leq \sigma_1^P(u_1) \tag{1}$$

$$\mu_2^N(u_2,v_2) \geq (\mu_1^N(u_1,v_1)) > \vee(\sigma_1^P(u_1),\sigma_1^N(v_1)) \geq \sigma_1^N(u_1) \tag{2}$$

Consider submerging membership values of AQSP fuzzy graph,

$$\gamma_2^P(u_2,v_2) \leq (\mu_1^p(u_1,v_1)) < \wedge(\rho_1^P(u_1),\rho_1^P(v_1)) \leq \rho_1^P(u_1) \tag{3}$$

$$\gamma_2^N(u_2,v_2) \geq (\mu_1^N(u_1,v_1)) > \vee(\rho_1^P(u_1),\rho_1^N(v_1)) \geq \rho_1^N(u_1) \quad (4)$$

Let us assume that,
$E = \{(u_1,v_1),(u_1,v_2)/u_1 \in V_1, u_2,v_2 \in E_2\} \cup \{(u_1,w),(v_1,w)/w \in V_2, u_1,v_1 \in E_1\}$
Consider $(u,u_2),(u,v_2) \in E$ we have,
$(\mu_1^P \times \mu_2^P)((u_1,u_2),(u_1,v_2)) = \wedge(\sigma_1^P(u_1),\mu_2^P(u_2,v_2))$
$(\mu_1^P \times \mu_2^P)((u_1,u_2),(u_1,v_2)) < \wedge(\sigma_1^P(u_1),\sigma_2^P(u_2),\sigma_2^P(v_2))$ and
$(\sigma_1^P \times \sigma_2^P)(u_1,u_2) = \wedge(\sigma_1^P(u_1),\sigma_2^P(u_2))$, $(\sigma_1^P \times \sigma_2^P)(u_1,v_2) = \wedge(\sigma_1^P(u_1),\sigma_2^P(v_2))$
Therefore,
$\wedge((\sigma_1^P \times \sigma_2^P)(u_1,u_2),(\sigma_1^P \times \sigma_2^P)(u_1,v_2)) = \wedge(\sigma_1^P(u_1),\sigma_2^P(u_2),\sigma_2^P(v_2))$ Hence, the membership and non - membership value of AQSP fuzzy graph is,
(i) $(\mu_1^P \times \mu_2^P)((u_1,u_2),(u_1,v_2)) < \wedge((\sigma_1^P \times \sigma_2^P)(u_1,u_2),(\sigma_1^P \times \sigma_2^P)(u_1,v_2))$
(ii) $(\mu_1^N \times \mu_2^N)((u_1,u_2),(u_1,v_2)) > \vee((\sigma_1^N \times \sigma_2^N)(u_1,u_2),(\sigma_1^N \times \sigma_2^N)(u_1,v_2))$ Similarly we get for the submerging membership / non - membership value of AQSP fuzzy graph is,
(iii) $(\gamma_1^P \times \gamma_2^P)((u_1,u_2),(u_1,v_2)) < \wedge((\rho_1^P \times \rho_2^P)(u_1,u_2),(\rho_1^P \times \rho_2^P)(u_1,v_2))$
(iv) $(\gamma_1^N \times \gamma_2^N)((u_1,u_2),(u_1,v_2)) > \vee((\rho_1^N \times \rho_2^N)(u_1,u_2),(\rho_1^N \times \rho_2^N)(u_1,v_2))$
Therefore, $G_1 \times G_2$ is not strong AQSP fuzzy graph, it is a contradiction. Hence $G_1 \times G_2$ is strong AQSP fuzzy graph, then at least G_1 or G_2 must be strong AQSP fuzzy graph.

6. Complement of a Strong AQSP fuzzy graph

Complement of a strong AQSP fuzzy graph $G = (\sigma_{AQSP}, \mu_{AQSP})$ of
$G^* = (\sigma_{AQSP}, \mu_{AQSP})$ is a strong AQSP fuzzy graph $\overline{G} = (\overline{\sigma}_{AQSP}, \overline{\mu}_{AQSP})$ on $\overline{G^*} = (\overline{\sigma^*}_{AQSP}, \overline{\mu^*}_{AQSP})$,
$\overline{\sigma}_{AQSP} = (\overline{\sigma}^P(x), \overline{\sigma}^N(x), \overline{\rho}^P(x), \overline{\rho}^N(x))$ and
$\overline{\mu}_{AQSP} = ((\overline{\mu}^P(x), (\overline{\mu}^N(x), (\overline{\gamma}^P(x), (\bar{\gamma}^N(x))$ are defined by with the conditions,

(i) $\overline{V} = V$
(ii) $\overline{\sigma}^P(x) = \sigma^P(x)$, $\overline{\sigma}^N(x) = \sigma^P(x)$,
$\overline{\rho}^P(x) = \rho^P(x)$, $\overline{\rho}^N(x) = \rho^N(x) \quad \forall\, x \in V$,
(iii)

$$\overline{\mu}^P(x,y) = \begin{cases} 0 & if \ \ \mu^P(x,y) > 0, \\ \wedge(\mu^P(x),\mu^P(y)) & if \ \ \mu^P(x,y) = 0, \end{cases}$$

$$\overline{\mu}^N(x,y) = \begin{cases} 0 & if \ \ \mu^N(x,y) > 0, \\ \vee(\mu^N(x),\mu^N(y)) & if \ \ \mu^N(x,y) = 0, \end{cases}$$

$$\overline{\gamma}^P(x,y) = \begin{cases} 0 & if \ \ \gamma^P(x,y) > 0, \\ \wedge(\gamma^P(x),\gamma^P(y)) & if \ \ \gamma^P(x,y) = 0, \end{cases}$$

$$\overline{\gamma}^N(x,y) = \begin{cases} 0 & if \ \ \gamma^N(x,y) > 0, \\ \vee(\gamma^N(x),\mu^N(y)) & if \ \ \gamma^N(x,y) = 0, \end{cases}$$

6.1. Theorem of the AQSP Fuzzy Bijective Map

Let $G_1 = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ and $G_2 = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ be AQSP fuzzy graphs. Then the $G_1 = (\sigma_{AQSP}\ ,\ \mu_{AQSP}) = G_2 = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ if and only if $G_1 = (\sigma_{AQSP}\ ,\ \mu_{AQSP}) \approx \overline{G_2} = (\overline{\sigma}_{AQSP}\ ,\ \overline{\mu}_{AQSP})$.

Proof. Consider that $G_1 = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ and $G_2 = (\sigma_{AQSP}\ ,\ \mu_{AQSP})$ isomorphic, There exists a bijective map $\phi : v_1 \to v_2$ satisfying the AQSP fuzzy graph with submerging membership and non - membership values. $\sigma_1^P(x) = \sigma_2^P(x)\ ,\ \sigma_1^N(x) = \sigma_2^N(x)$,

$\rho_1^P(x) = \rho_2^P(x)$, $\rho_1^N(x) = \rho_2^N(x)$ $\forall$ $x \in V$.
$\mu_1^P(x,y) = \mu_2^P(\phi(x),\phi(y))$, $\mu_1^N(x,y) = \mu_2^N(\phi(x),\phi(y))$,
$\gamma_1^P(x,y) = \gamma_2^P(\phi(x),\phi(y))$, $\gamma_1^N(x,y) = \gamma_2^N(\phi(x),\phi(y))$ $\forall$ $x,y \in E$.
By definition of complement of AQSP fuzzy graph we have,
$\overline{\mu_1}^P(x,y) = \wedge(\sigma_1^P(x),\sigma_1^P(y)) = \wedge(\sigma_1\phi(x),\sigma_1\phi(y)) = \overline{\mu_1}^P(\phi(x),\phi(y))$,
$\overline{\mu_1}^N(x,y) = \vee(\sigma_1^N(x),\sigma_1^N(y)) = \vee(\sigma_1\phi(x),\sigma_1\phi(y)) = \overline{\mu_1}^N(\phi(x),\phi(y))$,
$\overline{\gamma_1}^P(x,y) = \wedge(\rho_1^P(x),\rho_1^P(y)) = \wedge(\rho_1\phi(x),\rho_1\phi(y)) = \overline{\gamma_1}^P(\phi(x),\phi(y))$,
$\overline{\gamma_1}^N(x,y) = \vee(\rho_1^N(x),\rho_1^N(y)) = \vee(\rho_1\phi(x),\rho_1\phi(y)) = \overline{\gamma_1}^N(\phi(x),\phi(y))$
$\forall$ x , y $\in E_1$. Hence, $G_1 = (\sigma_{AQSP}$, $\mu_{AQSP}) \approx \overline{G_2} = (\overline{\sigma}_{AQSP}$, $\overline{\mu}_{AQSP})$.
The converse is true is shown in Figure. 11 as an example.

7. Complete AQSP fuzzy graph

An AQSP fuzzy graph $G = (\sigma_{AQSP}$, $\mu_{AQSP})$ is said to be a complete AQSP fuzzy graph if the necessary and sufficient conditions are satisfied. Figure.6. represents the complete AQSP fuzzy graph in the following.
(i) $\mu^P(x,y) = \min(\sigma^P(x)$, $\sigma^P(y))$, $(ii)\mu^N(x,y) = \max(\sigma^N(x)$, $\sigma^N(y))$
$(iii)\gamma^P(x,y) = \min(\rho^P(x)$, $\rho^P(y))$, $(iv)\gamma^N(x,y) = \max(\rho^N(x)$, $\rho^N(y))$.
$\forall$ $(x,y) \in \mu^*_{AQSP}$.

7.1. Theorem

Let $G_1 = (\sigma_{AQSP}, \mu_{AQSP})$ and $G_2 = (\sigma_{AQSP}, \mu_{AQSP})$ be complete AQSP Fuzzy graphs, then $G_1 \cap G_2$ is also a complete AQSP Fuzzy graph.

Proof. If $(x_1,y_1),(x_2,y_2) \in E$, since $G_1 = (\sigma_{AQSP}, \mu_{AQSP})$ and
$G_2 = (\sigma_{AQSP}, \mu_{AQSP})$ are complete AQSP Fuzzy graphs, then ,
for the membership value of AQSP fuzzy graph is,
$(\mu_1^P \cap \mu_2^P)((x_1,y_1),(x_2,y_2)) = \mu_1^P(x_1,x_2) \wedge \mu_2^P(y_1,y_2)$,
$= \sigma_1^P(x_1) \wedge \sigma_2^P(x_2) \wedge \sigma_2^P(y_1) \wedge \sigma_2^P(y_2)$,
$= \sigma_1^P(x_1) \wedge \sigma_2^P(y_1) \wedge \sigma_1^P(x_2) \wedge \sigma_2^P(y_2)$,
$= \sigma_1^P \cap \sigma_2^P(x_1,y_1) \wedge \sigma_1^P \cap \sigma_2^P(x_2,y_2)$.
Similarly for the non - membership values of AQSP fuzzy graph is ,
$(\mu_1^N \cap \mu_2^N)((x_1,y_1),(x_2,y_2)) = \mu_1^N(x_1,x_2) \wedge \mu_2^N(y_1,y_2)$,
$= \sigma_1^N(x_1) \vee \sigma_2^N(x_2) \vee \sigma_2^N(y_1) \vee \sigma_2^N(y_2)$,
$= \sigma_1^N(x_1) \vee \sigma_2^P(y_1) \vee \sigma_1^N(x_2) \vee \sigma_2^N(y_2)$,
$= \sigma_1^N \cap \sigma_2^N(x_1,y_1) \vee \sigma_1^N \cap \sigma_2^N(x_2,y_2)$.
For the submerging AQSP fuzzy graph membership value is,
$(\gamma_1^P \cap \gamma_2^P)((x_1,y_1),(x_2,y_2)) = \gamma_1^P(x_1,x_2) \wedge \gamma_2^P(y_1,y_2)$,
$= \rho_1^P(x_1) \wedge \rho_2^P(x_2) \wedge \rho_2^P(y_1) \wedge \rho_2^P(y_2)$,
$= \rho_1^P(x_1) \wedge \rho_2^P(y_1) \wedge \rho_1^P(x_2) \wedge \rho_2^P(y_2)$,
$= \rho_1^P \cap \rho_2^P(x_1,y_1) \wedge \rho_1^P \cap \rho_2^P(x_2,y_2)$.
For the submerging non - membership values of AQSP fuzzy graph is,
$(\gamma_1^N \cap \gamma_2^N)((x_1,y_1),(x_2,y_2)) = \gamma_1^N(x_1,x_2) \wedge \gamma_2^N(y_1,y_2)$,
$= \rho_1^N(x_1) \wedge \rho_2^N(x_2) \vee \rho_2^N(y_1) \vee \rho_2^P(y_2)$,
$= \rho_1^N(x_1) \vee \rho_2^N(y_1) \vee \rho_1^N(x_2) \vee \rho_2^N(y_2)$,
$= \rho_1^N \cap \rho_2^N(x_1,y_1) \vee \rho_1^N \cap \rho_2^N(x_2,y_2)$.
Hence, $G_1 \cap G_2$ is complete AQSP Fuzzy graph is proved.

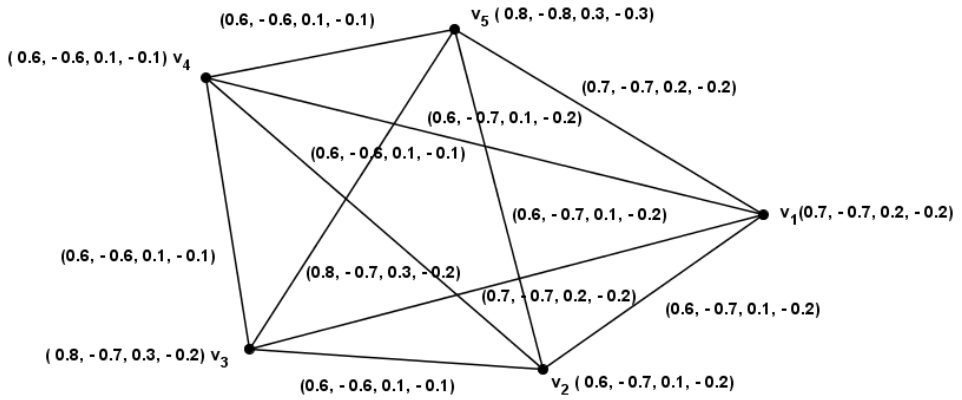

Figure 6: *Complete AQSP Fuzzy Graph*

8. AQSP Fuzzy Graph COVID - 19 Vaccines Survey Analysis from different college Teaching and Non - Teaching staff and Students

In this application of AQSP Fuzzy graph, the determination of accepting/non - accepting Booster shot in future is a Decision - making problem in the experience of COVID - 19 Vaccine, which is taken by willingness/compulsion of each person in pandemic time. If a person 'A' wants to decide in taking booster shot after taking one or two doses during pandemic time was influenced by five influencing factors (IF) which we considered for survey vertices such as, v_1 = Government, v_2 = Health care Authorities, v_3 = Family Members, v_4 = Personal Interest, v_5 = Social Media. Each influencing factor has some common alternate conflict fuzzification attitude, that we considered as $s_1(\sigma,\rho)$ =Gender, $s_2\ (\sigma,\rho)$ =Age, $s_3(\sigma,\rho)$ = Number of Doses, $s_{4(}\ \sigma,\rho)$ = Weightage of IF, $s_5\ (\sigma,\rho)$ = Effects of Vaccines. Here the Influencing factor cannot be measured specifically, since it depends on the decision maker with attitudinal submerging conflict feelings which we considered as edges, v_1v_2 = Positive and Negative effects, v_1v_3 = Reduction of Risk, v_1v_4 = Compulsion and Willingness, v_1v_5 = Immunity, v_2v_3 = Necessity of Booster, v_2v_4 = Behavioral feelings, v_2v_5 = Psychological proses, v_3v_4 = Conflict thoughts, v_3v_5 = Acceptance and non - acceptance of Booster Shotter and v_4v_5 = Essential age group to receive Booster Shotter.

The edge is drawn between the apexes if they have at minimum of communal and attitudinal conflict feelings of accepting / not accepting of third dose booster shot. Thus, each node has multiple attributes compressing the feelings of uncertainty.

Table 1: *Tabular representation of Impact on FIVE Influencing Factors with Vertices of AQSP Fuzzy Graph.*

Vertices	Influencing Factors (IF)	Recurrences	AQSP fuzzy values
v_1	Government	(36%)	(1.0,-0.8,0.5,-0.3)
v_2	Health care Authorities	(11.4%)	(0.9,-0.8,0.4,-0.3)
v_3	Family Members	(15.6%)	(1.0,-0.9,0.5,-0.4)
v_4	Personal Interest	(36.2%)	(1.0,-1.0,0.5,-0.5)
v_5	social media	(0.8%)	(0.5,-0.7,0.0,-0.2)

$(i)\,(\mu,\gamma)^P s_i \ \leq\ (\ (\sigma,\rho)\,(x)\ \wedge\ (\sigma,\rho)\,(y))\,,\forall\ (x,y)\ \epsilon\ \mu^P_{(x,y)}.$
$(ii)\,(\mu,\gamma)^N s_i \ \geq\ (\ (\sigma,\rho)\,(x)\ \vee\ (\sigma,\rho)\,(y))\,,\ \forall\ (x,y)\ \epsilon\ \mu^N_{(x,y)}.$
We get the score values and functions of the AQSP Fuzzy Graph using the constrain,

$\frac{1}{n}\ \ (\frac{1}{S^P_{AQSP}}\ \sum\theta^P_x\ -\ \frac{1}{S^N_{AQSP}}\ \sum\theta^N_x\ \ ).$

Table 2: *Tabular representation of AQSP Survey Analysis of Attitudinal Conflict Feelings .*

Vertices	Sequences	Recurrences	AQSP fuzzy values
Gender	Female	(85.7%)	$(0.9, -0.6, 0.4, -0.1)$
	Male	(14.3 %)	
Age	< 40	(89.1%)	$(1.0, -0.8, 0.5, -0.3)$
	> 40	(10.4 %)	
No. of doses	Two	(89.7%)	$(0.9, -0.5, 0.4, 0.0)$
	One	(4.8 %)	
	Booster	(8%)	
	None	(0.6 %)	
Weightage of (IF)	Agree	(60.5%)	$(0.9, -0.8, 0.4, -0.3)$
	Not Agree	(40 %)	
Effects of Vaccines	High	(69.9%)	$(0.7, -0.6, 0.2, -0.1)$
	Low	(30 %)	
Immunity level	High	(71.8 %)	$(1.0, -0.8, 0.5, -0.3)$
	Low	(35.8 %)	
Necessity of Booster	Strongly Agree	(52.7 %)	$(0.8, -0.5, 0.3, 0.0)$
	Strongly Disagree	(47.2 %)	
Behavioral feelings	Desire	(52.3 %)	$(0.8, -0.5, 0.3, 0.0)$
	Non desire	(47.6 %)	
Psychological Conflict feelings	Positive	(65.3%)	$(0.5, -0.7, 0.0, -0.2)$
	Negative	(34.6 %)	
Submerging thought process	Less Immunity	(48.7%)	$(1.0, -0.8, 0.5, -0.3)$
	More Immunity	(51.3 %)	
Acceptance of Booster	Agree	(37.8%)	$(0.5, -0.7, 0.0, -0.2)$
	Not Agree	(62.2 %)	
Merits/Demerits(Booster)	Merits	(72%)	$(1.0, -0.6, 0.5, -0.1)$
	Demarits	(28 %)	
Neccessity of Booster	Important	(53.3%)	$(0.8, -0.6, 0.3, -0.1)$
	Not Important	(46.7 %)	
Essential Age(Booster)	< 40	(56.9 %)	$(0.9, -0.7, 0.4, -0.2)$
	> 40	(43.1 %)	

Table 3: *Tabular representation of Combinataric Factors in Boost Shotter survey analysis in AQSP Fuzzy graphs.*

Variables	$(\sigma,\rho)s_i$	v_1	v_2	v_3	v_4	v_5
Gender	$s_1(\sigma,\rho)$	0.575	0.550	0.600	0.625	0.425
	$(0.9,-0.6,0.4,-0.1)$					
Age	$s_2(\sigma,\rho)$	0.600	0.575	0.675	0.700	0.500
	$(1.0,-0.8,0.5,-0.3)$					
No. of Doses	$s_3(\sigma,\rho)$	0.550	0.525	0.575	0.600	0.525
	$(0.9,-0.5,0.4,0.0)$					
Weightage of IF	$s_4(\sigma,\rho)$	0.625	0.600	0.650	0.675	0.475
	$(0.9,-0.8,0.4,-0.3)$					
Effects of Vaccines	$s_5(\sigma,\rho)$	0.525	0.500	0.550	0.575	0.375
	$(0.7,-0.6,0.2,-0.1)$					
Average Score	$s_i(\sigma,\rho)$	0.585	0.560	0.550	0.575	0.375

9. AQSP fuzzy graphs Multi input with single output Method

Influencing Factor represents the vertex set of AQSP fuzzy graph = (IF)
Attitudinal Feelings is the edge set of AQSP fuzzy graph = (AF)

(i)Total score weightage (IF) = $\sum(IF_i\ (\bar{A}_i)) + (IF + \frac{(\sigma,\rho)s_1}{10})$, $\alpha_{cut} = S_1$

(ii)Total score weightage (AF)= $\sum(AF_i\ (\bar{A}_i)) + (AF + \frac{(\mu,\gamma)s_i}{10})$, $\alpha_{cut} = IF_5$

$$A(\sigma,\rho) \to B(\mu,\gamma) = I_{AQSP}\ ((\sigma,\rho)\ ,\ (\mu,\gamma)\ ) = \begin{cases} 1 & (\sigma,\rho) \le (\mu,\gamma) \\ (\mu,\gamma) & (\sigma,\rho) \ge (\mu,\gamma) \end{cases}$$

$A(\sigma,\rho) = [0.9,\ 0.7,\ 0.8,\ 1.0,\ 0.6]$,

$B(\mu,\gamma) = [0.8,\ 0.9,\ 1.0,\ 1.0,\ 0.7,\ 0.8,\ 0.6,\ 1.0,\ 0.6,\ 0.6]$

$R_{AQSP}[A(\sigma,\rho)\ ,\ B(\mu,\gamma)] = (\sigma,\rho) \to (\mu,\gamma)$

$$R_{AQSP} = \begin{bmatrix} 0.9 \\ 0.7 \\ 0.8 \\ 1.0 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.8, & 0.9, & 1.0, & 1.0, & 0.7, & 0.8, & 0.6, & 1.0, & 0.6, & 0.6 \end{bmatrix}$$

$$R_{AQSP} = \begin{bmatrix} 0.8 & 1.0 & 1.0 & 1.0 & 0.7 & 0.8 & 0.6 & 1.0 & 0.6 & 0.6 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 0.6 & 1.0 & 0.6 & 0.6 \\ 0.8 & 1.0 & 1.0 & 1.0 & 0.7 & 1.0 & 0.6 & 1.0 & 0.6 & 0.6 \\ 0.8 & 1.0 & 1.0 & 1.0 & 0.7 & 0.8 & 0.6 & 1.0 & 0.6 & 0.6 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$A'(\sigma,\rho) = [0.7,\ 0.6,\ 0.8,\ 1.0,\ 0.5]$, $B'(\mu,\gamma) = A'\ T_M{}^{\circ} R_{AQSP}$

$$[0.7,\ 0.6,\ 0.8,\ 1.0,\ 0.5]\quad T_M{}^{\circ}\quad \begin{bmatrix} 0.8 & 1.0 & 1.0 & 1.0 & 0.7 & 0.8 & 0.6 & 1.0 & 0.6 & 0.6 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 0.6 & 1.0 & 0.6 & 0.6 \\ 0.8 & 1.0 & 1.0 & 1.0 & 0.7 & 1.0 & 0.6 & 1.0 & 0.6 & 0.6 \\ 0.8 & 1.0 & 1.0 & 1.0 & 0.7 & 0.8 & 0.6 & 1.0 & 0.6 & 0.6 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$B'(\mu,\gamma) = \vee(A'(x) \wedge R(A,B))$

$$B'(\mu,\gamma) = \begin{bmatrix} \vee(0.7,0.6,0.8,0.8,0.5) & \vee(0.7,0.6,0.8,1.0,0.5) \\ \vee(0.7,0.6,0.8,1.0,0.5) & \vee(0.7,0.6,0.8,1.0,0.5) \\ \vee(0.7,0.6,0.7,0.7,0.5) & \vee(0.7,0.6,0.8,0.8,0.5) \\ \vee(0.6,0.6,0.6,0.6,0.5) & \vee(0.7,0.6,0.8,1.0,0.5) \\ \vee(0.6,1.0,0.6,0.6,0.5) & \vee(0.6,1.0,0.6,0.6,0.5) \end{bmatrix}$$

$B'(\mu,\gamma) = [0.8,\ 1.0,\ 1.0,\ 1.0,\ 0.7,\ 0.8,\ 0.6,\ 1.0,\ 0.6,\ 0.6]$

$B'(\mu,\gamma) = [0.6\ \ 1.0]$, Using AQSP Fuzzy graph MISO method, we get the result of the following,

Highest Influencing Factor = v_4 (Personal Interest)

Lowest Influencing Factor = v_5 (social media)

Acceptance/ Non acceptance of Booster shot AQSP fuzzy membership and non - membership value = [0.6 , 1.0]

10. Conclusion

Fuzzy graph remains an essential mathematical tool to solve the complex uncertain problems. Its applications are renowned in the evolving fields of Engineering Mathematics, Control Engineering, Real Analysis, Topology, Operations Research, Optimization, Data Science and Computer Science. Specially, Fuzzy graph modules provides a more generalized notion to resolve the problems with vagueness. Based on the study and analysis of the various fuzzy graph modules, a new fuzzy graph Alternate Quadra Submerging Polar Fuzzy Graph (AQSP) has been proposed. This novel method enables to precisely identify the membership, the non-membership values that improves and drives the decision-making analysis in the uncertain situations. Specifically, the application of the AQSP fuzzy graph is based on the behavioral transformation of the human beings due to the various influencing factors. It has been proved that the potential association between the influencing factors and the attitudinal human feelings lead to take a concrete decision of either the acceptance or the non-acceptance of the booster shot. In future, the AQSP fuzzy graph method can be further explored for the possibilities of its applications in the interdisciplinary fields of Artificial intelligence, Approximate reasoning, and Machine learning process to be precise in the identification of the membership or non-membership values due to the uncertain thought process of the human behaviors. Furthermore, the AQSP Fuzzy soft graph and the Matrix representations of the AQSP fuzzy graphs can be further analyzed and explored for the identification and the evaluation of the reliable values in the uncertain situations.

Declarations

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Conflict of interest

The authors declared that they have no conflict of interest regarding the publication of the research article.

Contributions

The authors worked equally regarding the publication of the research article.

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A PRODUCTION INVENTORY MODEL FOR DETERIORATING ITEMS WITH TIME AND PRICE RELIANT DEMAND USING FLOWER POLLINATION OPTIMIZATION

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Abstract

Effective management of production inventory for deteriorating items with dynamic demand patterns is crucial for businesses operating in today's competitive markets. This paper proposes a comprehensive model that addresses the complexities arising from the dual storage locations, item deterioration, and demand dependencies on both time and selling price. To optimize the decision variables associated with production and inventory control, we employ the Flower Pollination Optimization (FPO) algorithm, a nature-inspired meta-heuristic known for its ability to efficiently navigate complex search spaces. The two-storage production inventory model integrates the dynamics of item deterioration over time, capturing the real-world challenges faced by supply chain managers. The demand for items is modeled to be sensitive to both temporal variations and changes in selling prices, reflecting the intricate nature of market dynamics. Our approach leverages the FPO algorithm to explore and exploit the solution space, allowing for the identification of optimal or near-optimal strategies for production quantities, order quantities, and inventory levels. The FPO algorithm mimics the pollination process in nature, striking a balance between exploration and exploitation to efficiently search for solutions in a highly dynamic and nonlinear environment. The proposed model and optimization approach are validated through extensive simulations and sensitivity analyses. The results demonstrate the effectiveness of the FPO algorithm in finding robust solutions that enhance inventory management, mitigate deterioration-related losses, and adapt to varying demand scenarios. This research contributes to the field of supply chain optimization by offering a novel perspective on tackling the challenges associated with dual storage, item deterioration, and demand dependencies. The findings provide valuable insights for practitioners seeking advanced strategies for optimizing their production inventory systems in the face of evolving market conditions.

Keywords: Production Inventory Model, Deteriorating Items, Two-Storage, Shortages, Time and Selling Price Dependent Demand, Flower Pollination Optimization.

1. Introduction and related work

In the realm of supply chain management, the effective control and optimization of inventory systems play a pivotal role in ensuring the success and sustainability of businesses. As markets become increasingly dynamic and customer demands evolve, the complexities associated with

managing production inventory for deteriorating items intensify. This is particularly true in scenarios where the demand for items is not only influenced by temporal variations but also by changes in selling prices. To address these challenges, we propose a two-storage production inventory model that accounts for the intricacies of dual storage locations, item deterioration, and demand dependencies on time and selling price. The management of deteriorating items presents a unique set of challenges due to the perishable nature of certain goods over time. Incorporating this deterioration factor into inventory models is crucial for avoiding unnecessary losses and ensuring that products reaching customers are of the highest quality. Moreover, the consideration of time-dependent demand recognizes the influence of various temporal factors, such as seasonality or market trends, on the overall demand pattern. Adding an additional layer of complexity, our model acknowledges the impact of selling prices on demand. Price elasticity is a well-established concept in economics, and understanding how changes in selling prices affect the demand for items is vital for making informed decisions in a competitive marketplace.To address the optimization problem inherent in this multifaceted inventory management model, we turn to the Flower Pollination Optimization (FPO) algorithm. FPO, inspired by the pollination process in flowers, offers a nature-inspired meta-heuristic that excels in navigating complex and dynamic search spaces. By mimicking the pollination behavior of flowers, the FPO algorithm strikes a balance between exploration and exploitation, making it well-suited for finding optimal or near-optimal solutions in intricate and non-linear environments.

This research aims to contribute to the field of supply chain optimization by proposing a novel approach to managing production inventory in the face of deteriorating items with time and selling price dependent demand. Through extensive simulations and sensitivity analyses, we evaluate the effectiveness of the FPO algorithm in optimizing production quantities, order quantities, and inventory levels. The outcomes of this study provide valuable insights for practitioners seeking advanced strategies to enhance their production inventory systems, adapt to changing market conditions, and minimize losses associated with item deterioration.

Supply chain management can be defined as: "Supply chain management is the coordination of production, storage, location and transport between players in the supply chain to achieve the best combination of responsiveness and efficiency for a given market. Many researchers in the inventory system have focused on a product that does not overcome spoilage. However, there are a number of things whose meaning doesn't stay the same over time. Yadav et al. [1-10] developed the deterioration of these substances plays an important role and cannot be stored for long. Yadav, et al. [11-20] studied deterioration of an object can be described as deterioration, evaporation, obsolescence and loss of use or restriction of an object, resulting in less inventory consumption than under natural conditions. When raw materials are put in stock as a stock to meet future needs, there may be a deterioration of the items in the arithmetic system which could occur for one or more reasons, etc. Storage conditions, weather or humidity. Yadav, a. al. [21-53] explore the inach generally states that management has a warehouse to store the purchased warehouse. However, for various reasons, management may buy or lend more than it can store in the warehouse and call it OW, with an extra number in a rented warehouse called RW near OW or just off it. Yadav and swami. [54-61] developed an inventory costs (including maintenance costs and depreciation costs) in RW are generally higher than OW costs due to additional costs of running, equipment maintenance, etc. Reducing inventory costs will cost-effectively utilize RW products as quickly as possible. Actual customer service is only provided by OW, and to reduce costs, RW stock is cleaned first. Such arithmetic examples are called two arithmetic examples in the shop. Yadav and Kumar [62] established the management of the supply of electronic storage devices and integration of environmental and nerve networks. Yadav, A.S. [63-65] analysize of seven supply chain management measures to improve inventory of electronic storage devices by submitting a financial burden using GA and PSO and supply chain management analysis to improve inventory and inventory of equipment using genetic computation and model design and chain inventory analysis from bi inventory and economic difficulty in transporting goods by genetic computation. Swami, et. al. [66-68] developed inventory policies for inventory and inventory needs and miscellaneous inventory costs based

on allowable payments and inventory delays An example of depreciation of various types of goods and services and costs by keeping a business loan and inventory model with pricing needs low sensitive, inventory costs versus inflationary business expense loans. Gupta, et. al. [69- 70] established the objectives of the Multiple Objective Genetic Algorithm and PSO, which include the improvement of supply and deficit, inflation and a calculation model based on a genetic calculation of the scarcity and low inflation of PSO. Singh, et. al. [71, 72] studied an example with two stock depreciation on assets and inventory costs when updating particles and an example with two inventories of property damage and inventory costs in inflation and soft computer techniques. Kumar, et. al. [73-75] delayed control of alcohol supply and particle refinement and green cement supply system and inflation by particle enhancement and electronic inventory system and distribution center by genetic computations. Chauhan and Yadav [76-77] depreciation example at two stores and warehouses based on inventory using one genetic stock and one vehicle stock for demand and inflation inventory with two distribution centers using genetic stock. Pandey, et. al. [78] analysize of marble Improvement of industrial reserves based on genetic technology and improvement of multiple particles. Ahlawat, et. al. [79] studied the white wine industry in supply chain management through nerve networks. Singh, et. al. [80] examines the best policy to import damaged goods immediately and pay for conditional delays under the supervision of two warehouses. The study by Yadav et al. [81] focuses on enhancing inventory management for degrading commodities within the framework of green technology investment, accounting for factors including selling price, carbon emissions, and time-sensitive demand. It emphasizes sustainable practices in inventory models by fusing conventional economic principles with environmental factors. Using an interval number technique, Yadav, Yadav, and Bansal [82] examine a two-warehouse inventory management model for degrading commodities in order to take demand and cost uncertainty into consideration. Their analytical optimization methods demonstrate how spending money on preservation technology can save waste and increase inventory efficiency. With an emphasis on a two-warehouse system to maximize inventory levels, Yadav, Yadav, and Bansal [83] offer an inventory model that tackles the deterioration of commodities during storage. Their strategy emphasizes how crucial it is to successfully control degradation costs in order to raise total inventory efficiency.

2. Assumptions and notations

2.1. Assumptions

The following assumptions were used in the formulation of the mathematical model.

1. The unit production cost is a function of the rate of production.
2. The rate of production is considered to be decision variable.
3. The demand rate is a function of time and selling price, which is $D(t,p) = (\alpha + \beta t - \gamma t^2)p^{-\lambda}$; $\alpha > 0$, $\beta \in [0,1)$, $\gamma \in [0,1)$.
4. The rate of deterioration is constant and different for both the warehouses.
5. The OW has limited capacity of W units and the RW has unlimited capacity.
6. Deterioration units can't be repaired or replaced.
7. The RW is located near the OW and thus the transportation cost between them is negligible.
8. The inventory cost (including carrying cost and deterioration cost) in RW is higher than that in OW.
9. Shortages are not allowed.
10. The holding cost is constant for both the warehouses.

2.2. Notations

Table 2 is provided a description of the notations utilised for the constructed mathematical model.

Table 1: *Notations*

Notation	Units	Description
α	Constant	Coefficient of demand function
β	Constant	Coefficient of demand function
γ	Constant	Coefficient of demand function.
θ_1	Constant	Deterioration rate in OW.
θ_2	Constant	Deterioration rate in RW.
P	–	Production rate.
p	\$/Units	Selling price of product.
U	capability constraint	The owned warehouse capacity
SUC_i	\$/unit	Set-up cost
C_D	\$/Units	Deterioration cost.
h	\$/Units	Holding cost.
TVC	\$/Units	The function for total inventory cost.

3. Mathematical Model Formulation

The mathematical model for a production inventory system handling decaying products with two-storage includes time-dependent and selling price-dependent demand. Assume that I(t) reflects the inventory level at time t and that there are two storage facilities: one for immediate sales and another for buffer stock. The demand function D(p(t),t) is influenced and altered by the selling price p(t). The degradation rate affects the inventory's usefulness, therefore manufacturing costs, storage costs, and potential revenue loss due to spoiling must all be balanced. The objective is to minimize the total cost, which includes holding costs, production expenses, and lost revenue due to deterioration, subject to constraints on inventory levels, demand, and production rates. (See Fig. 1).

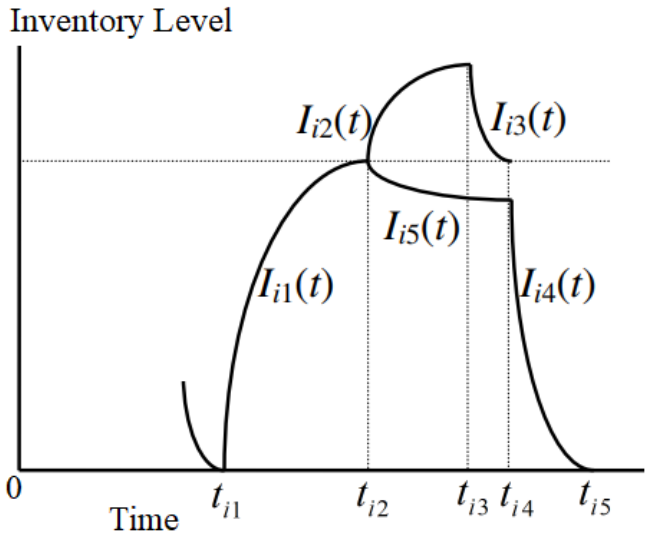

Figure 1: *A graphical depiction of the two-warehouse production inventory model.*

The inventory level is characterized by the following differential equations:

$$\frac{dI_{i1}(t)}{dt} + \theta_1 I_{i1}(t) = P - (\alpha + \beta t - \gamma t^2)p^{-\lambda}; \quad t \in [0, t_{i1}] \tag{1}$$

with the boundary condition (B.C.) $I_{i1}(0) = 0$.

$$\frac{dI_{i2}(t)}{dt} + \theta_2 I_{i2}(t) = P - (\alpha + \beta t - \gamma t^2)p^{-\lambda}; \quad t \in [t_{i1}, t_{i2}] \tag{2}$$

with the boundary conditions (B.C.) $I_{i2}(t_{i1}) = 0$.

$$\frac{dI_{i3}(t)}{dt} + \theta_2 I_{i3}(t) = -(\alpha + \beta t - \gamma t^2)p^{-\lambda}; \quad t \in [t_{i2}, t_{i3}] \tag{3}$$

with the boundary conditions (B.C.) $I_{i3}(t_{i3}) = 0$.

$$\frac{dI_{i4}(t)}{dt} + \theta_1 I_{i4}(t) = -(\alpha + \beta t - \gamma t^2)p^{-\lambda}; \quad t \in [t_{i3}, t_{i4}] \tag{4}$$

with the boundary conditions (B.C.) $I_{i4}(t_{i4}) = 0$.

$$\frac{dI_{i5}(t)}{dt} + \theta_1 I_{i5}(t) = 0; \quad t \in [t_{i1}, t_{i3}] \tag{5}$$

with the boundary conditions (B.C.) $I_{i5}(t_{i1}) = W$.

The solutions of the differential Eqs. (1) -(5) are (6) -(10), respectively:

$$I_{i1}(t) = \left\{ \frac{1}{\theta_1^3}\left(\frac{2\gamma}{P^\lambda} + P\theta_1^2 - \frac{2\alpha\theta_1^2}{P^\lambda} + \frac{\beta\theta_1}{P^\lambda}\right) - \frac{1}{\theta_1^3}\left[e^{-t\theta_1}\left(\frac{2\gamma}{P^\lambda} + P\theta_1^2 - \frac{2\alpha\theta_1^2}{P^\lambda} + \frac{\beta\theta_1}{P^\lambda}\right)\right] + \frac{\gamma t^2}{P^\lambda\theta_1} - \frac{t(2\gamma + \beta\theta_1)}{P^\lambda\theta_1^2}\right\} \tag{6}$$

$$I_{i2}(t) = \left\{ \frac{1}{\theta_2^3}\left(\frac{2\gamma}{P^\lambda} + P\theta_2^2 - \frac{2\alpha\theta_2^2}{P^\lambda} + \frac{\beta\theta_2}{P^\lambda}\right) - e^{-t\theta_2}e^{-t_{i1}\theta_2}\left[\frac{1}{\theta_2^3}\left(\frac{2\gamma}{P^\lambda} + P\theta_2^2 - \frac{2\alpha\theta_2^2}{P^\lambda} + \frac{\beta\theta_2}{P^\lambda} + \frac{\gamma t_{i1}^2}{P^\lambda\theta_2} - \frac{t_{i1}(2\gamma + \beta\theta_2)}{P^\lambda\theta_2^2}\right)\right] + \frac{\gamma t^2}{P^\lambda\theta_2} - \frac{t(2\gamma + \beta\theta_2)}{P^\lambda\theta_2^2}\right\} \tag{7}$$

$$I_{i3}(t) = \frac{(-\alpha\theta_2^2 + \beta\theta_2 + 2\gamma)}{P^\lambda\theta_2^3} + \frac{\gamma t^2}{P^\lambda\theta_2} - \frac{t(2\gamma + \beta\theta_2)}{P^\lambda\theta_2^2} - e^{\theta_2(t_{i3}-t)}\left[\frac{(-\alpha\theta_2^2 + \beta\theta_2 + 2\gamma)}{P^\lambda\theta_2^3} + \frac{\gamma t_{i3}^2}{P^\lambda\theta_2} - \frac{t_{i3}(2\gamma + \beta\theta_2)}{P^\lambda\theta_2^2}\right] \tag{8}$$

$$I_{i4}(t) = \frac{(-\alpha\theta_1^2 + \beta\theta_1 + 2\gamma)}{P^\lambda\theta_1^3} + \frac{\gamma t^2}{P^\lambda\theta_1} - \frac{t(2\gamma + \beta\theta_1)}{P^\lambda\theta_1^2} - e^{\theta_1(t_{i4}-t)}\left[\frac{(-\alpha\theta_1^2 + \beta\theta_1 + 2\gamma)}{P^\lambda\theta_1^3} + \frac{\gamma t_{i4}^2}{P^\lambda\theta_1} - \frac{t_{i4}(2\gamma + \beta\theta_1)}{P^\lambda\theta_1^2}\right] \tag{9}$$

$$I_{i5}(t) = We^{\theta_1(t_{i1}-t)} \tag{10}$$

Therefore, the relevant costs of the production inventory system are as follows.

1. **Set up costs for the cycle :**

$$SUC_i = C_{SU} \tag{11}$$

2. **Holding costs in RW for the cycle :**

$$HC_{RW} = h\left[\int_{t_{i1}}^{t_{i2}} I_{i2}(t)dt + \int_{t_{i2}}^{t_{i3}} I_{i3}(t)dt\right]$$

$$\begin{aligned}HC_{RW} = \Bigg\{ &\frac{he^{-\theta_2 t_{i2}}}{6\theta_2^4}\Big[12\gamma e^{\theta_2 t_{i1}} - 12\gamma e^{\theta_2 t_{i2}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i1}} + 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} + 6\beta\theta_2 e^{\theta_2 t_{i1}} \\ &- 6\beta\theta_2 e^{\theta_2 t_{i2}} + 3\beta\theta_2^3 t_{i1}^2 e^{\theta_2 t_{i1}} - 3\beta\theta_2^3 t_{i1}^2 e^{\theta_2 t_{i2}} + 6\gamma\theta_2^2 t_{i1}^2 e^{\theta_2 t_{i1}} - 6\gamma\theta_2^2 t_{i1}^2 e^{\theta_2 t_{i1}} \\ &- 2\gamma\theta_2^3 t_{i1}^3 e^{\theta_2 t_{i1}} + 2\gamma\theta_2^3 t_{i1}^3 e^{\theta_2 t_{i1}} - 12\gamma\theta_2 t_{i1}^3 e^{\theta_2 t_{i1}} + 12\gamma\theta_2 t_{i1}^3 e^{\theta_2 t_{i1}} + 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i1}} \\ &- 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i1}} + 6\alpha\theta_2^3 t_{i1} e^{\theta_2 t_{i2}} - 6\alpha\theta_2^3 t_{i1} e^{\theta_2 t_{i2}} - 6\beta\theta_2^2 t_{i1} e^{\theta_2 t_{i1}} + 6\beta\theta_2^2 t_{i1} e^{\theta_2 t_{i1}} \\ &- 6P\theta_2^3 P^\lambda t_{i1} e^{\theta_2 t_{i1}} + 6P\theta_2^3 P^\lambda t_{i1} e^{\theta_2 t_{i1}}\Big] + \frac{he^{-\theta_2 t_{i2}}}{6P^\lambda\theta_2^4}\Big[12\gamma e^{\theta_2 t_{i2}} - 12\gamma e^{\theta_2 t_{i3}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} \\ &+ 6\alpha\theta_2^2 e^{\theta_2 t_{i3}} + 6\beta\theta_2 e^{\theta_2 t_{i2}} - 6\beta\theta_2 e^{\theta_2 t_{i3}} + 3\beta\theta_2^3 t_{i2}^2 e^{\theta_2 t_{i2}} - 3\beta\theta_2^3 t_i^2 e^{\theta_2 t_{i3}} \\ &+ 6\gamma\theta_2^2 t_{i2}^2 e^{\theta_2 t_{i2}} - 6\gamma\theta_2^2 t_{i3}^2 e^{\theta_2 t_{i3}} - 2\gamma\theta_2^3 t_{i2}^3 e^{\theta_2 t_{i2}} + 2\gamma\theta_2^3 t_{i3}^3 e^{\theta_2 t_{i3}} - 12\gamma\theta_2 t_{i2}^3 e^{\theta_2 t_{i2}} \\ &+ 12\gamma\theta_2 t_{i3}^3 e^{\theta_2 t_{i3}} + 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i2}} - 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i3}} + 6\alpha\theta_2^3 t_{i2} e^{\theta_2 t_{i2}} - 6\alpha\theta_2^3 t_{i3} e^{\theta_2 t_{i3}} \\ &- 6\beta\theta_2^2 t_{i2} e^{\theta_2 t_{i2}} + 6\beta\theta_2^2 t_{i3} e^{\theta_2 t_{i3}} - 6P\theta_2^3 P^\lambda t_{i2} e^{\theta_2 t_{i2}} + 6P\theta_2^3 P^\lambda t_{i3} e^{\theta_2 t_{i3}}\Big]\Bigg\} \end{aligned} \quad (12)$$

3. **Holding costs in OW for the cycle :**

$$HC_{OW} = h\left[\int_{t_0}^{t_{i1}} I_{i1}(t)dt + \int_{t_{i1}}^{t_{i3}} I_{i5}(t)dt + \int_{t_{i3}}^{t_{i4}} I_{i4}(t)dt\right]$$

$$\begin{aligned}HC_{OW} = h\Bigg\{ &\frac{he^{-\theta_1 t_{i1}}}{6P^\lambda\theta_1^4}\Big[12\gamma e^{\theta_1 t_{i4}} - 12\gamma e^{\theta_1 t_{i4}} - 6\alpha\theta_1^2 e^{\theta_1 t_{i3}} + 6\alpha\theta_1^2 e^{\theta_2 t_{i4}} + 6\beta\theta_1 e^{\theta_2 t_{i3}} \\ &- 6\beta\theta_1 e^{\theta_2 t_{i4}} + 3\beta\theta_1^3 t_{i3}^2 e^{\theta_1 t_{i4}} - 3\beta\theta_1^3 t^{i3} e^{\theta_1 t_{i3}} + 6\gamma\theta_1^2 t_{i3}^2 e^{\theta_1 t_{i3}} - 6\gamma\theta_1^2 t_{i4}^2 e^{\theta_2 t_{i4}} \\ &- 2\gamma\theta_1^3 t_{i3}^3 e^{\theta_1 t_{i3}} + 2\gamma\theta_1^3 t_{i4}^3 e^{\theta_1 t_{i4}} - 12\gamma\theta_1 t_{i3}^3 e^{\theta_1 t_{i3}} + 12\gamma\theta_1 t_{i4}^3 e^{\theta_1 t_{i4}} + 6P\theta_1^2 P^\lambda e^{\theta_1 t_{i3}} \\ &- 6P\theta_1^2 P^\lambda e^{\theta_1 t_{i4}} + 6\alpha\theta_1^3 t_{i3} e^{\theta_1 t_{i3}} - 6\alpha\theta_1^3 t_{i4} e^{\theta_1 t_{i4}} - 6\beta\theta_1^2 t_{i3} e^{\theta_1 t_{i4}} + 6\beta\theta_1^2 t_{i3} e^{\theta_1 t_{i4}} \\ &- 6P\theta_1^3 P^\lambda t_{i3} e^{\theta_1 t_{i3}} + 6P\theta_1^3 P^\lambda t_{i4} e^{\theta_1 t_{i4}}\Big] + \frac{t_{i1}}{\theta_1^3}\left(\frac{2\gamma}{P^\lambda} + P\theta_1^2 - \frac{2\alpha\theta_1^2}{P^\lambda} + \frac{\beta\theta_1}{P^\lambda}\right) \\ &- W\left(\frac{e^{\theta_1(t_{i1}-t_{i3})} - 1}{\theta_1}\right) + P\left(\frac{e^{\theta_1(t_{i1})} - 1}{\theta\alpha_1^2}\right) - \frac{t_{i1}^2(2\gamma + \beta\theta_1)}{2p\lambda\theta_1^2} + \frac{\gamma t_{i1}^3}{3p\lambda\theta_1} \\ &- \frac{\alpha(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^2} + \frac{\beta(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^3} + \frac{2\gamma(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^4}\Bigg\} \end{aligned} \quad (13)$$

4. **Deterioration costs in RW for the cycle :**

$$DC_{RW} = C_D\theta_2\left[\int_{t_{i1}}^{t_{i2}} I_{i2}(t)dt + \int_{t_{i2}}^{t_{i3}} I_{i3}(t)dt\right]$$

$$DC_{RW} = C_D\theta_2 e^{-\theta_2 t_{i2}}\left\{\frac{1}{6\theta_2^4}\left[12\gamma e^{\theta_2 t_{i1}} - 12\gamma e^{\theta_2 t_{i2}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i1}} + 6\alpha\theta_2^2 e^{\theta_2 t_{i2}}\right.\right.$$
$$+ 6\beta\theta_2 e^{\theta_2 t_{i1}} - 6\beta\theta_2 e^{\theta_2 t_{i2}} + 2\gamma\theta_2^3 t_{i2}^3 e^{\theta_2 t_{i2}} - 12\gamma\theta_2 t_{i1}^3 e^{\theta_2 t_{i1}} + 3\beta\theta_2^3 t_{i2}^2 e^{\theta_2 t_{i2}}$$
$$- 3\beta\theta_2^3 t^{i2} e^{\theta_2 t_{i2}} + 6\gamma\theta_2^2 t_{i1}^2 e^{\theta_2 t_{i1}} - 6\gamma\theta_2^2 t_{i2}^2 e^{\theta_2 t_{i2}} - 2\gamma\theta_2^3 t_{i1}^3 e^{\theta_1 2 t_{i2}} + 12\gamma\theta_2 t_{i2}^3 e^{\theta_2 t_{i2}}$$
$$+ 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i1}} - 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i2}} + 6\alpha\theta_2^3 t_{i1} e^{\theta_2 t_{i1}} - 6\alpha\theta_2^3 t_{i2} e^{\theta_2 t_{i2}} - 6\beta\theta_2^2 t_{i1} e^{\theta_2 t_{i1}}$$
$$\left. + 6\beta\theta_2^2 t_{i2} e^{\theta_1 t_{i2}} - 6P\theta_2^3 P^\lambda t_{i1} e^{\theta_1 t_{i1}} + 6P\theta_2^3 P^\lambda t_{i2} e^{\theta_2 t_{i42}}\right] + \frac{1}{6P^\lambda\theta_2^4}\left[12\gamma e^{\theta_2 t_{i2}}\right.$$
$$- 12\gamma e^{\theta_2 t_{i3}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} + 6\alpha\theta_2^2 e^{\theta_2 t_{i3}} + 6\beta\theta_2 e^{\theta_2 t_{i2}} - 6\beta\theta_2 e^{\theta_2 t_{i3}}$$
$$- 2\gamma\theta_2^3 t_{i2}^3 e^{\theta_2 t_{i2}} + 2\gamma\theta_2^3 t_{i3}^3 e^{\theta_2 t_{i3}} - 12\gamma\theta_2 t_{i2}^3 e^{\theta_2 t_{i2}} + 12\gamma\theta_2 t_{i3}^3 e^{\theta_2 t_{i3}}$$
$$\left.\left. + 6\alpha\theta_2^3 t_{i2} e^{\theta_2 t_{i2}} - 6\alpha\theta_2^3 t_{i3} e^{\theta_2 t_{i3}} - 6\beta\theta_2^2 t_{i2} e^{\theta_2 t_{i2}} + 6\beta\theta_2^2 t_{i3} e^{\theta_2 t_{i3}}\right]\right\} \quad (14)$$

5. **Deterioration costs in OW for the cycle :**

$$DC_{OW} = C_D\theta_1\left[\int_{t_0}^{t_{i1}} I_{i1}(t)dt + \int_{t_{i1}}^{t_{i3}} I_{i5}(t)dt + \int_{t_{i3}}^{t_{i4}} I_{i4}(t)dt\right]$$

$$DC_{OW} = C_D\theta_1\left\{\frac{e^{-\theta_1 t_{i3}}}{6P^\lambda\theta_1^4}\left[12\gamma e^{\theta_1 t_{i3}} - 12\gamma e^{\theta_1 t_{i4}} - 6\alpha\theta_1^2 e^{\theta_1 t_{i3}} + 6\alpha\theta_1^2 e^{\theta_1 t_{i4}}\right.\right.$$
$$+ 6\beta\theta_1 e^{\theta_1 t_{i3}} - 6\beta\theta_1 e^{\theta_1 t_{i4}} + 3\beta\theta_1^3 t_{i3}^2 e^{\theta_1 t_{i3}} - 3\beta\theta_1^3 t^{i4} e^{\theta_1 t_{i4}} + 6\gamma\theta_1^2 t_{i3}^2 e^{\theta_1 t_{i3}}$$
$$- 6\gamma\theta_1^2 t_{i4}^2 e^{\theta_1 t_{i4}} - 2\gamma\theta_1^3 t_{i3}^3 e^{\theta_1 t_{i3}} + 2\gamma\theta_1^3 t_{i4}^3 e^{\theta_1 t_{i4}} - 12\gamma\theta_1 t_{i3}^3 e^{\theta_1 t_{i3}}$$
$$+ 12\gamma\theta_1 t_{i4}^3 e^{\theta_1 t_{i4}} + 6\alpha\theta_1^3 t_{i3} e^{\theta_1 t_{i3}} - 6\alpha\theta_1^3 t_{i4} e^{\theta_1 t_{i4}} - 6\beta\theta_1^2 t_{i3} e^{\theta_1 t_{i4}}$$
$$\left. + 6\beta\theta_1^2 t_{i3} e^{\theta_1 t_{i4}}\right] + \frac{t_{i1}}{\theta_1^3}\left(\frac{2\gamma}{P^\lambda} + P\theta_1^2 - \frac{2\alpha\theta_1^2}{P^\lambda} + \frac{\beta\theta_1}{P^\lambda}\right) - W\left(\frac{e^{\theta_1(t_{i1}-t_{i3})} - 1}{\theta_1}\right)$$
$$+ P\left(\frac{e^{\theta_1(t_{i1})} - 1}{\theta\alpha_1^2}\right) - \frac{t_{i1}^2(2\gamma + \beta\theta_1)}{2p\lambda\theta_1^2} + \frac{\gamma t_{i1}^3}{3p\lambda\theta_1}$$
$$\left. - \frac{\alpha(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^2} + \frac{\beta(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^3} + \frac{2\gamma(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^4}\right\} \quad (15)$$

6. **Production cost for the cycle :**

$$PC_i = n_o(P)\left[\int_0^{t_{i1}} Pdt + \int_{t_{i1}}^{t_{i2}} Pdt\right]$$

$$PC_i = P^2 n_o t_{i2} \quad (16)$$

Therefore, the present worth of total variable cost for the cycle

$$TVC = \frac{1}{T}[SUC_i + HC_{RW} + HC_{OW} + DC_{RW} + DC_{OW} + PC_i] \quad (17)$$

Note that for the detailed version of Equation (17), see Appendix A.

4. Flower Pollination Optimization Methodology

Flower Pollination Optimization (FPO) is a naturalistic approach to solving complex optimization issues by modeling flower pollination. FPO takes inspiration from the biological mechanisms of pollination, which involve the transport of pollen from flowers to pollinators and the subsequent reproduction of plants, to effectively explore and exploit the search space (see Fig. 2). In many

fields, including artificial intelligence, finance, and engineering, this method excels at determining the optimal solutions to continuous and discrete optimization problems. The algorithm details of the FPO technique which were brought al. multi-purpose optimization level (Darwin, [83]) after gaining the first literature (Yang, [82]) and Investigation of Artificial Intelligence Based Optimization Algorithms. Okula, et, al., [84] are as follows:

Algorithm:

Step 1: (Installation Phase): Randomly distribute N-flower particle (potential solution variables) in solution space. Assign algorithm values, specify the transition probability parameter (go). Perform the necessary arrangements for the problem to be solved.
Step 2: Calculate the objective function value (fitness) according al. position of the flowers - particles (potential solution variables). Find out what's best.
Step 3: Repeat the following steps throughout the iterative process (eg until you reach a certain number of iterations or until you reach a desired value in the objective function): (For each particle; for each purpose function size).
Step 3.1: (Global - Local Pollination Phase): Generate a random value. If the value produced is less than the value of equation and Levy Flights (step vector: L). If the value produced is equal to or greater than the value of go, uniform distribution in the range [0, 1]. Run the local pollination process in the context.14
Step 3.2: Calculate the purpose function value (fitness) according al. updated position of flowers - particles (potential solution variables).
Step 3.3: Update the global best value (and hence the variable position) if the best objective at that time is found to be better than the function value.
Step 4: Iteration - At the end of the cycle the value (s) obtained according al. global best position is considered to be the optimum value (s).

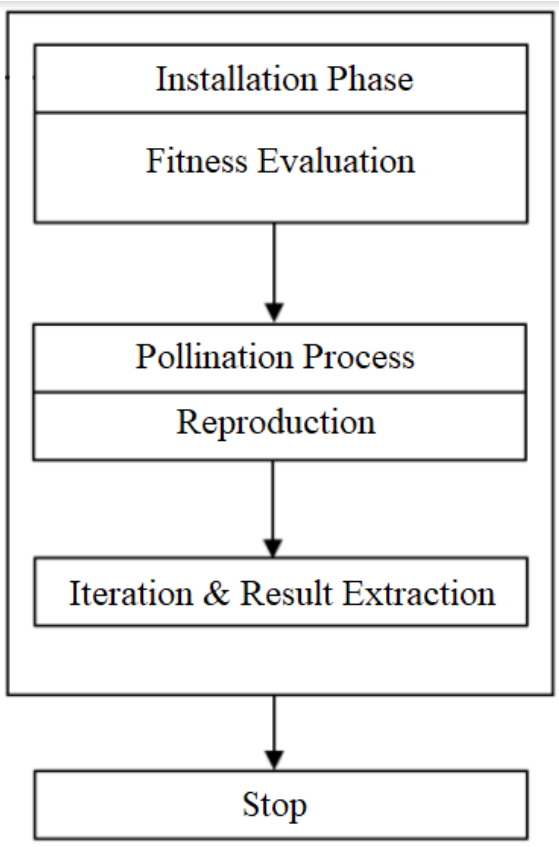

Figure 2: *Flowchart for the Flower Pollination Optimization.*

5. Graphical representation

The primary objective of the graphical representations of the suggested model is a flowchart listing the key components. To demonstrate how to manage inventory for things that are deteriorating, it begins with two storage systems. The diagram highlights the relationship between time, selling price, and demand, showing how these factors dynamically change inventory levels. The flowchart also demonstrates the Flower Pollination Algorithm approach, which shows how potential solutions are developed and improved to cut costs and boost the efficacy of inventory management. All things considered, the complex relationships revealed in the model and the optimization technique employed are well communicated by this visual aid.

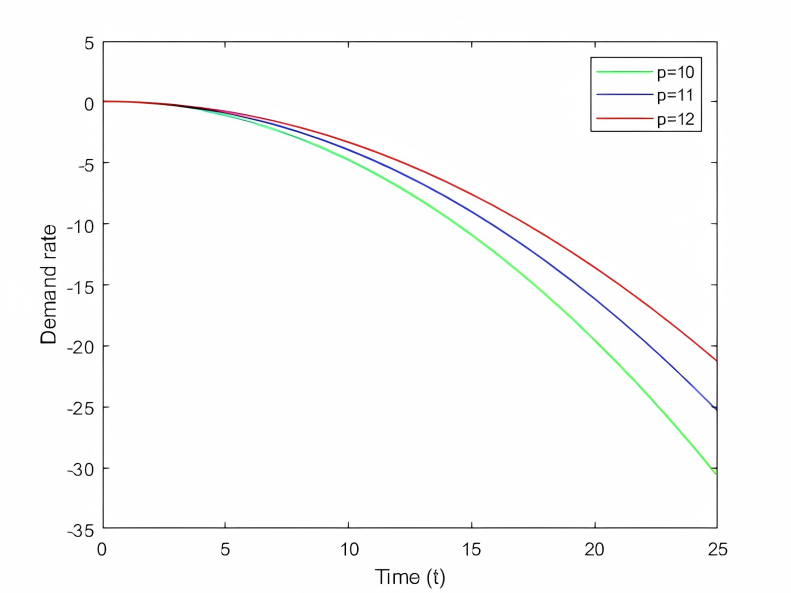

Figure 3: *Relation between Demand rate and time using different values of p.*

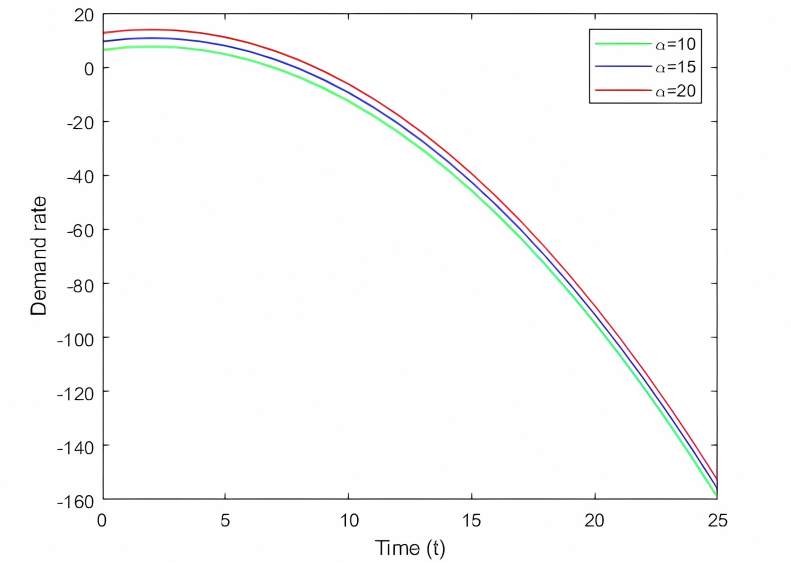

Figure 4: *Relation between Demand rate and time using different values of α.*

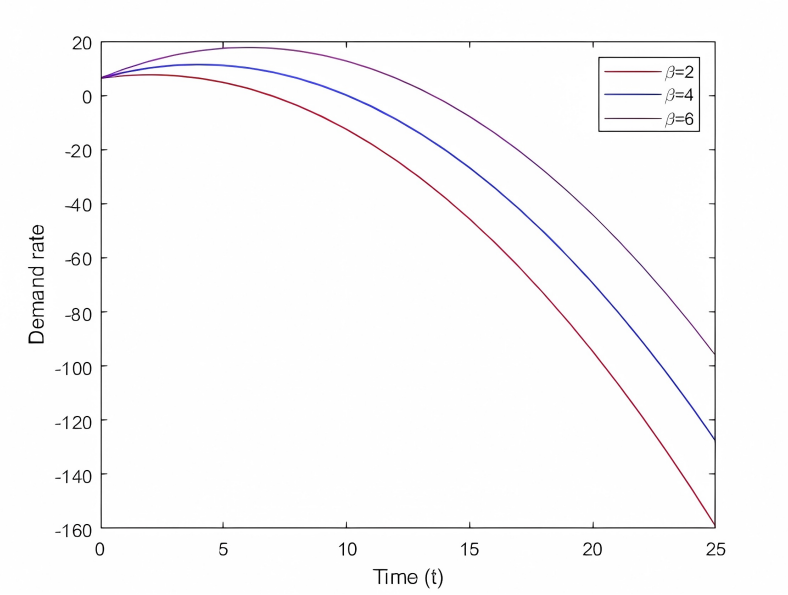

Figure 5: *Relation between Demand rate and time using different values of β.*

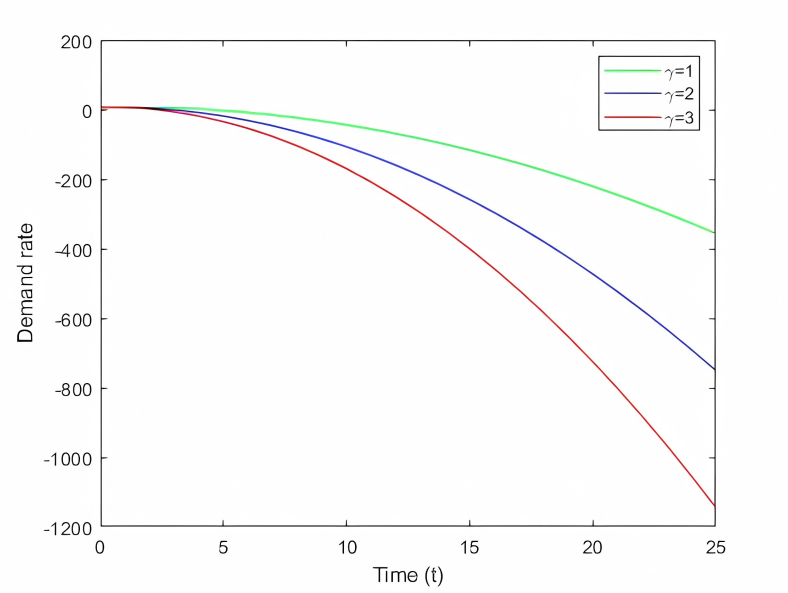

Figure 6: *Relation between Demand rate and time using different values of γ.*

Following are a few insights drawn from the graphical representation's observations.

1. If the selling price parameter p are increased, the total average inventory cost (TVC) rises due to the higher demand rate (see Fig. 3).
2. If the parameters α and β are increased, the total average inventory cost (TVC) rises quickly due to the extended running time of the warehouses. Simultaneously, both the cycle length and the product price decrease (see Fig. 4 and Fig. 5).
3. If the demand rate parameter (γ) increases, the total average inventory cost (TVC) rises rapidly due to the higher amount of product waste (see Fig. 6).

6. Conclusion

In this study, we presented a comprehensive approach to address the challenges associated with managing a two-storage production inventory model for deteriorating items with time and selling price dependent demand. The complexities of dual storage locations, item deterioration, and dynamic demand patterns were considered, reflecting the real-world intricacies faced by supply chain managers. To optimize decision variables and navigate the complex solution space, we employed the Flower Pollination Optimization (FPO) algorithm, a nature-inspired metaheuristic known for its efficacy in solving complex optimization problems.

The proposed model demonstrated its relevance by integrating the impact of item deterioration over time, allowing for a more accurate representation of inventory dynamics. The consideration of time-dependent demand and selling price dependencies further enriched the model, capturing the nuances of market fluctuations and consumer behavior. Our choice of the FPO algorithm proved effective in finding solutions that strike a balance between exploration and exploitation. By simulating the pollination process in flowers, the FPO algorithm efficiently explored the solution space, leading to robust strategies for production quantities, order quantities, and inventory levels. The adaptability of FPO to dynamic and non-linear environments was crucial in addressing the intricate nature of the proposed inventory model. Through extensive simulations and sensitivity analyses, we validated the effectiveness of our approach, showcasing its ability to enhance inventory management, mitigate deterioration-related losses, and adapt to varying demand scenarios. The findings contribute valuable insights for supply chain practitioners seeking advanced strategies to optimize their production inventory systems in the face of evolving market conditions. As we conclude, it is important to emphasize the practical implications of our research. The proposed model and optimization approach offer a forward-looking perspective on addressing the challenges in dual storage inventory systems. The integration of FPO provides a powerful tool for decision-makers to refine their inventory strategies, ultimately improving overall supply chain efficiency and resilience. While this study has provided significant contributions, there are opportunities for further research. Future investigations could explore the applicability of the proposed model in different industry contexts and evaluate the performance of other meta-heuristic algorithms for comparison. Additionally, incorporating more nuanced factors such as supply chain disruptions or sustainability considerations could enrich the model further.

In conclusion, this research contributes to advancing the field of supply chain optimization by proposing an innovative solution to a complex inventory management problem. The integration of a two-storage production inventory model with FPO optimization provides a robust framework for addressing real-world challenges and paves the way for more resilient and adaptive supply chain strategies.

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Appendix A

$$
\begin{aligned}
TVC = \frac{1}{T}\Bigg[& (C_{SU} + P^2 n_o t_{i2}) \\
& + \Bigg\{ \frac{he^{-\theta_2 t_{i2}}}{6\theta_2^4}\Bigg[12\gamma e^{\theta_2 t_{i1}} - 12\gamma e^{\theta_2 t_{i2}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i1}} + 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} + 6\beta\theta_2 e^{\theta_2 t_{i1}} \\
& - 6\beta\theta_2 e^{\theta_2 t_{i2}} + 3\beta\theta_2^3 t_{i1}^2 e^{\theta_2 t_{i1}} - 3\beta\theta_2^3 t_{i1}^2 e^{\theta_2 t_{i2}} + 6\gamma\theta_2^2 t_{i1}^2 e^{\theta_2 t_{i1}} - 6\gamma\theta_2^2 t_{i1}^2 e^{\theta_2 t_{i1}} \\
& - 2\gamma\theta_2^3 t_{i1}^3 e^{\theta_2 t_{i1}} + 2\gamma\theta_2^3 t_{i1}^3 e^{\theta_2 t_{i1}} - 12\gamma\theta_2 t_{i1}^3 e^{\theta_2 t_{i1}} + 12\gamma\theta_2 t_{i1}^3 e^{\theta_2 t_{i1}} + 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i1}} \\
& - 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i1}} + 6\alpha\theta_2^3 t_{i1} e^{\theta_2 t_{i2}} - 6\alpha\theta_2^3 t_{i1} e^{\theta_2 t_{i2}} - 6\beta\theta_2^2 t_{i1} e^{\theta_2 t_{i1}} + 6\beta\theta_2^2 t_{i1} e^{\theta_2 t_{i1}} \\
& - 6P\theta_2^3 P^\lambda t_{i1} e^{\theta_2 t_{i1}} + 6P\theta_2^3 P^\lambda t_{i1} e^{\theta_2 t_{i1}}\Bigg] + \frac{he^{-\theta_2 t_{i2}}}{6P^\lambda\theta_2^4}\Bigg[12\gamma e^{\theta_2 t_{i2}} - 12\gamma e^{\theta_2 t_{i3}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} \\
& + 6\alpha\theta_2^2 e^{\theta_2 t_{i3}} + 6\beta\theta_2 e^{\theta_2 t_{i2}} - 6\beta\theta_2 e^{\theta_2 t_{i3}} + 3\beta\theta_2^3 t_{i2}^2 e^{\theta_2 t_{i2}} - 3\beta\theta_2^3 t_i^2 e^{\theta_2 t_{i3}} \\
& + 6\gamma\theta_2^2 t_{i2}^2 e^{\theta_2 t_{i2}} - 6\gamma\theta_2^2 t_{i3}^2 e^{\theta_2 t_{i3}} - 2\gamma\theta_2^3 t_{i2}^3 e^{\theta_2 t_{i2}} + 2\gamma\theta_2^3 t_{i3}^3 e^{\theta_2 t_{i3}} - 12\gamma\theta_2 t_{i2}^3 e^{\theta_2 t_{i2}} \\
& + 12\gamma\theta_2 t_{i3}^3 e^{\theta_2 t_{i3}} + 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i2}} - 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i3}} + 6\alpha\theta_2^3 t_{i2} e^{\theta_2 t_{i2}} - 6\alpha\theta_2^3 t_{i3} e^{\theta_2 t_{i3}} \\
& - 6\beta\theta_2^2 t_{i2} e^{\theta_2 t_{i2}} + 6\beta\theta_2^2 t_{i3} e^{\theta_2 t_{i3}} - 6P\theta_2^3 P^\lambda t_{i2} e^{\theta_2 t_{i2}} + 6P\theta_2^3 P^\lambda t_{i3} e^{\theta_2 t_{i3}}\Bigg]\Bigg\} \\
& + h\Bigg\{ \frac{he^{-\theta_1 t_{i1}}}{6P^\lambda\theta_1^4}\Bigg[12\gamma e^{\theta_1 t_{i4}} - 12\gamma e^{\theta_1 t_{i4}} - 6\alpha\theta_1^2 e^{\theta_1 t_{i3}} + 6\alpha\theta_1^2 e^{\theta_2 t_{i4}} + 6\beta\theta_1 e^{\theta_2 t_{i3}} \\
& - 6\beta\theta_1 e^{\theta_2 t_{i4}} + 3\beta\theta_1^3 t_{i3}^2 e^{\theta_1 t_{i4}} - 3\beta\theta_1^3 t^{i3} e^{\theta_1 t_{i3}} + 6\gamma\theta_1^2 t_{i3}^2 e^{\theta_1 t_{i3}} - 6\gamma\theta_1^2 t_{i4}^2 e^{\theta_2 t_{i4}} \\
& - 2\gamma\theta_1^3 t_{i3}^3 e^{\theta_1 t_{i3}} + 2\gamma\theta_1^3 t_{i4}^3 e^{\theta_1 t_{i4}} - 12\gamma\theta_1 t_{i3}^3 e^{\theta_1 t_{i3}} + 12\gamma\theta_1 t_{i4}^3 e^{\theta_1 t_{i4}} + 6P\theta_1^2 P^\lambda e^{\theta_1 t_{i3}} \\
& - 6P\theta_1^2 P^\lambda e^{\theta_1 t_{i4}} + 6\alpha\theta_1^3 t_{i3} e^{\theta_1 t_{i3}} - 6\alpha\theta_1^3 t_{i4} e^{\theta_1 t_{i4}} - 6\beta\theta_1^2 t_{i3} e^{\theta_1 t_{i4}} + 6\beta\theta_1^2 t_{i3} e^{\theta_1 t_{i4}} \\
& - 6P\theta_1^3 P^\lambda t_{i3} e^{\theta_1 t_{i3}} + 6P\theta_1^3 P^\lambda t_{i4} e^{\theta_1 t_{i4}}\Bigg] + \frac{t_{i1}}{\theta_1^3}\left(\frac{2\gamma}{P^\lambda} + P\theta_1^2 - \frac{2\alpha\theta_1^2}{P^\lambda} + \frac{\beta\theta_1}{P^\lambda}\right) \\
& - W\left(\frac{e^{\theta_1(t_{i1}-t_{i3})} - 1}{\theta_1}\right) + P\left(\frac{e^{\theta_1(t_{i1})} - 1}{\theta\alpha_1^2}\right) - \frac{t_{i1}^2(2\gamma + \beta\theta_1)}{2p\lambda\theta_1^2} + \frac{\gamma t_{i1}^3}{3p\lambda\theta_1} \\
& - \frac{\alpha(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^2} + \frac{\beta(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^3} + \frac{2\gamma(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^4}\Bigg\} \\
& + C_D\theta_2 e^{-\theta_2 t_{i2}}\Bigg\{ \frac{1}{6\theta_2^4}\Bigg[12\gamma e^{\theta_2 t_{i1}} - 12\gamma e^{\theta_2 t_{i2}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i1}} + 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} \\
& + 6\beta\theta_2 e^{\theta_2 t_{i1}} - 6\beta\theta_2 e^{\theta_2 t_{i2}} + 2\gamma\theta_2^3 t_{i2}^3 e^{\theta_2 t_{i2}} - 12\gamma\theta_2 t_{i1}^3 e^{\theta_2 t_{i1}} + 3\beta\theta_2^3 t_{i2}^2 e^{\theta_2 t_{i2}} \\
& - 3\beta\theta_2^3 t^{i2} e^{\theta_2 t_{i2}} + 6\gamma\theta_2^2 t_{i1}^2 e^{\theta_2 t_{i1}} - 6\gamma\theta_2^2 t_{i2}^2 e^{\theta_2 t_{i2}} - 2\gamma\theta_2^3 t_{i1}^3 e^{\theta_1 2 t_{i2}} + 12\gamma\theta_2 t_{i2}^3 e^{\theta_2 t_{i2}} \\
& + 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i1}} - 6P\theta_2^2 P^\lambda e^{\theta_2 t_{i2}} + 6\alpha\theta_2^3 t_{i1} e^{\theta_2 t_{i1}} - 6\alpha\theta_2^3 t_{i2} e^{\theta_2 t_{i2}} - 6\beta\theta_2^2 t_{i1} e^{\theta_2 t_{i1}} \\
& + 6\beta\theta_2^2 t_{i2} e^{\theta_1 t_{i2}} - 6P\theta_2^3 P^\lambda t_{i1} e^{\theta_1 t_{i1}} + 6P\theta_2^3 P^\lambda t_{i2} e^{\theta_2 t_{i42}}\Bigg] + \frac{1}{6P^\lambda\theta_2^4}\Bigg[12\gamma e^{\theta_2 t_{i2}} \\
& - 12\gamma e^{\theta_2 t_{i3}} - 6\alpha\theta_2^2 e^{\theta_2 t_{i2}} + 6\alpha\theta_2^2 e^{\theta_2 t_{i3}} + 6\beta\theta_2 e^{\theta_2 t_{i2}} - 6\beta\theta_2 e^{\theta_2 t_{i3}} \\
& + 3\beta\theta_2^3 t_{i2}^2 e^{\theta_2 t_{i2}} - 3\beta\theta_2^3 t_{i3}^2 e^{\theta_2 t_{i3}} + 6\gamma\theta_2^2 t_{i2}^2 e^{\theta_2 t_{i2}} - 6\gamma\theta_2^2 t_{i3}^2 e^{\theta_2 t_{i3}} \\
& - 2\gamma\theta_2^3 t_{i2}^3 e^{\theta_2 t_{i2}} + 2\gamma\theta_2^3 t_{i3}^3 e^{\theta_2 t_{i3}} - 12\gamma\theta_2 t_{i2}^3 e^{\theta_2 t_{i2}} + 12\gamma\theta_2 t_{i3}^3 e^{\theta_2 t_{i3}} \\
& + P\left(\frac{e^{\theta_1(t_{i1})} - 1}{\theta\alpha_1^2}\right) - \frac{t_{i1}^2(2\gamma + \beta\theta_1)}{2p\lambda\theta_1^2} + \frac{\gamma t_{i1}^3}{3p\lambda\theta_1} - \frac{\alpha(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^2} \\
& + \frac{\beta(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^3} + \frac{2\gamma(e^{-\theta_1 t_{i1}} - 1)}{p\lambda\theta_1^4}\Bigg\} + P^2 n_o t_{i2}\Bigg] \qquad (18)
\end{aligned}
$$

A FAILURE DISTRIBUTION FOR RELIABILITY PREDICTION OF MECHATRONIC COMPONENTS AND HUMAN-MACHINE SYSTEM

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•
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Abstract

Modern machines and equipment's have a complex mechatronic structure consisting of various components, and their reliability depends on a large number of random factors that arise during design, production and operation, which are often impossible to predict. Each element of the modern machines is characterized by different performance criteria and corresponding failures. Various statistical models of failure distribution are widely used to quantify the reliability of machines and devices. The choice of a statistical model and its parameters is important for a proper assessment of reliability. The chosen statistical model should reflect the actual distribution of failures fairly correctly. In presented article is proposed a new failure distribution for reliability prediction of mechatronic components of modern machines and human-machine systems. A large number of sudden failures of modern complex technical facilities containing electronic and mechatronic structural elements seriously affect its λ-characteristic.

Various studies have already shown that the failure behavior of complex systems cannot always be characterized by the "bathtub curve". This is especially true for modern complex machines, which, among other things, consist of numerous electronic components for which no wear and fatigue failures are assumed. For this reason, an alternative service life distribution for the description failure behavior of modern mechatronic components and human-machine systems is proposed. This is about the failure curves, which are initially characterized by a low or high failure rate and then tend to a constant failure rate.

To determine the reliability indexes are provided analytical formulas. Methods for estimating the parameters of this distribution are presented based on failure statistic. To determine distribution parameters, statistical data on failures of the technical system are sufficient only in the first period of its operation. This is one of the main advantages of the presented distribution. On the example of practical cases, the hypothesis of compliance of the proposed theoretical distribution to the actual statistical data on failures of various mechatronic systems and human-machine system was tested.

Keywords: failure distribution, reliability indexes, failure rate, mechatronic components, human-machine system

I. Introduction

Modern machines and equipments have a complex mechatronic structure consisting of various components, and their reliability depends on a large number of random factors that arise during

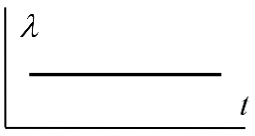
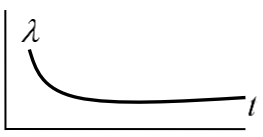
design, production and operation, which are often impossible to predict. Each element of the modern machines is characterized by different performance criteria and corresponding failures. Various statistical models of failure distribution are widely used to quantify the reliability of machines and devices. The choice of a statistical model and its parameters is important for a proper assessment of reliability. The chosen statistical model should reflect the actual distribution of failures fairly correctly. Currently, dozens of different statistical models are used to assess the reliability of machines and their structural elements [1-6]. Each of the existing distribution has certain advantages and disadvantages. When choosing a statistical failure model for any technical system, it is necessary to take into account the operating conditions, the type and nature of failures, the design features of the unit, and other factors. [1] it was proposed to use the Weibul distribution to assess the reliability of transmission mechanisms. However, the need for different price perception of the Weibul distribution parameters at different stages of operation creates certain difficulties in calculations.

The conducted research shows that the laws of dependence of failure rate on time in modern mechatronic systems and human-machine systems depend on many factors. A large number of sudden failures of modern complex technical facilities containing electronic and mechatronic structural elements seriously affect its λ-characteristic.

II. Problem statement

The investigations show that the failure behavior according to Table 1, which are characteristic of complex machines and mechatronic components, occur most frequently in today's mechanical engineering [7]. Therefore, the description of these failure behavior by statistical distributions is of interest. The failure behavior according to the pattern "B" can of course be well described with the exponential distribution. The other two failure behavior, can in principle be described with the Weibull distribution, but with different parameters for the two ranges [1].

Table 1: *Failure behavior for random failures*

	Failure behavior		Overall Characteristic
Random failures	A		complex machines with high-stress tests after commissioning
	B		Well-designed complex machines
	C		Electronic components and complex components after repair

For this reason, an alternative service life distribution for the description of the three above-mentioned failure behavior types of modern mechatronic components and human-machine systems is proposed in the present work. The failure rate is described in the following form:

$$\lambda(t) = \lambda\left[1 + (\alpha - 1)e^{-\beta t}\right], \quad (1)$$

where λ - the failure rate in the range of random failures; α - the shape parameter characterizes the value of the failure rate at the beginning of the first range:

$$\alpha = \frac{\lambda_0}{\lambda}. \quad (2)$$

The parameter β describes the duration of the characteristic early failure time t_f (Fig.1). It has been determined by calculations that by α=2 and λ=0,25 results $\beta \approx 4/t_f$. As follows from the figure, by α=1 results in exponential distribution that determines the failure behavior according to the pattern "B" (Tab. 1) describes. With the presented distribution, the failure behavior according to the patterns "A" (for $0<\alpha<1$) and "C" (for $\alpha>1$) can be described very well.

If, in the case of a failure behavior, the failures begin only from a point in time t_0, such a course can be described with a corresponding four-parameter distribution (Fig. 2). For the failure rate in this case, the following applies

$$\lambda(t) = \lambda\left[1 + (\alpha - 1)e^{-\beta(t-t_0)}\right]. \quad (3)$$

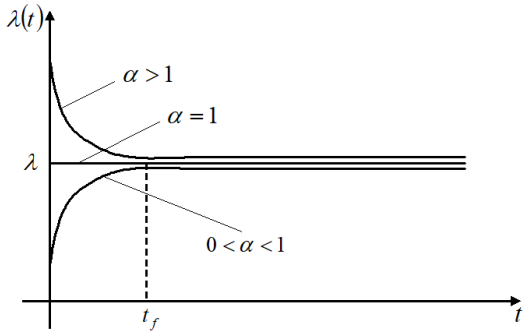


Figure 1: *Course of the failure rate in the three-parameter distribution*

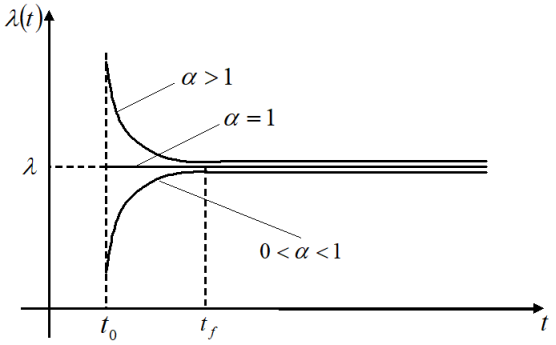


Figure 2: *Course of the failure rate in the four-parameter distribution*

III. Analytical determination of the remaining failure functions

By integration according to [1], the analytical formulas for the remaining failure functions of the three-parameter and four-parameter distribution (survival probability R (t), failure probability F (t) and density function f (t)) can be determined. Table 2 provides formulas for determining the failure functions of a three-parameter distribution.

Table 2: *Analytical formulas for the failure functions of the three-parameter distribution*

Failure functions	Failure rate	$\lambda(t) = \lambda\left[1 + (\alpha - 1)e^{-\beta t}\right]$
	Survival probability	$R(t) = \exp\left(-\int_0^t \lambda(t)dt\right) = \exp\left[\frac{\lambda}{\beta}(\alpha - 1)\left(e^{-\beta t} - 1\right) - \lambda t\right]$
	Failure probability	$F(t) = 1 - R(t) = 1 - \exp\left[\frac{\lambda}{\beta}(\alpha - 1)\left(e^{-\beta t} - 1\right) - \lambda t\right]$
	Failure density function	$f(t) = \lambda(t) \cdot R(t) = \lambda\left[1 + (\alpha - 1)e^{-\beta t}\right] \cdot \exp\left[\frac{\lambda}{\beta}(\alpha - 1)\left(e^{-\beta t} - 1\right) - \lambda t\right]$

In Figure 3, the graphical curves of the distribution failure functions (survival probability R(t) and failure density function f(t)) for different shape parameters α are shown. As can be seen from the figure, different density functions can be generated with the presented lifetime distribution in contrast to the exponential distribution. Depending on the shape parameter α, the density function changes significantly. For $\alpha \geq 1$, there is a decreasing density function, as with the exponential distribution. For $\alpha < 1$, the density function for decreasing α-values starts at lower failure densities, then reaches a steep maximum and finally falls flat.

Other statistical measures of the present distribution (mathematical expectation, variance, etc.) can only be calculated using numerical methods, since the resulting integral expressions cannot be solved analytically. A significant advantage of the illustrated distribution is that the parameters of the distribution can be determined by an incomplete test during the period of early failures. In this way, the testing effort of lifetime tests can be significantly reduced.

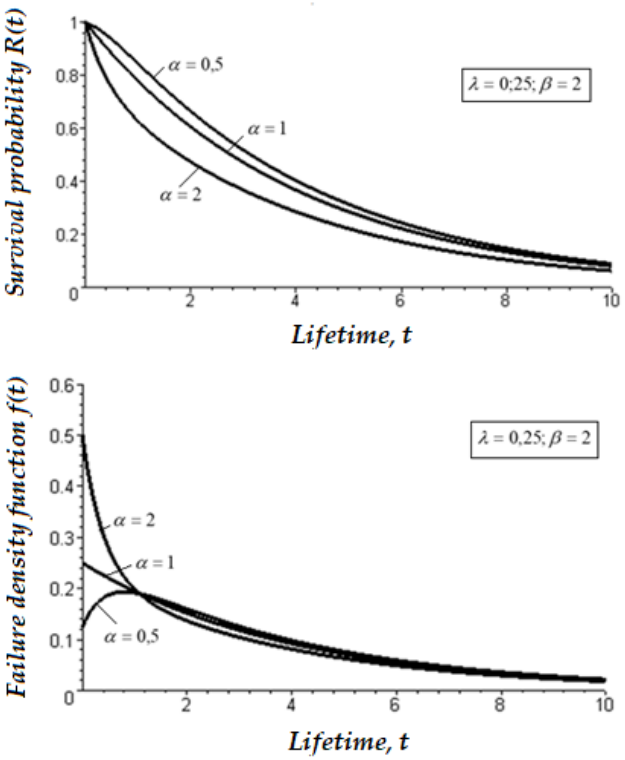


Figure 3: *Distribution failure functions*

IV. Determination of the distribution parameters

The parameters of the distribution can be determined by lifetime tests or failure statistics for an order statistics for m test specimens of the size n. For the test specimens i under consideration of the size n, let the failure times, ordered according to their size be $t_{1,i}$, $t_{2,i}$, $t_{3,i}$, ..., $t_{k,i}$.... A value of the failure rate corresponds to each failure times. According to the definition of the failure rate, these can be determined in the following form[1]:

$$\lambda_{1,i} = \frac{1}{n \cdot t_{1,i}}, \quad \lambda_{2,i} = \frac{1}{(n-1)(t_{2,i}-t_{1,i})}, \ldots, \lambda_{k,i} = \frac{1}{(n-k+1)(t_{k,i}-t_{k-1,i})}, \ldots \tag{4}$$

For the entire order statistics, the average values of the failure times and failure rates can be determined according to the respective rank to:

$$t_1 = \frac{1}{m}\sum_{i=1}^{m} t_{1,i}, \ t_2 = \frac{1}{m}\sum_{i=1}^{m} t_{2,i}, \ldots, t_k = \frac{1}{m}\sum_{i=1}^{m} t_{k,i}, \ldots \tag{5}$$

$$\lambda_1 = \frac{1}{m}\sum_{i=1}^{m} \lambda_{1,i}, \ \lambda_2 = \frac{1}{m}\sum_{i=1}^{m} \lambda_{2,i}, \ldots, \lambda_k = \frac{1}{m}\sum_{i=1}^{m} \lambda_{k,i}, \ldots \tag{6}$$

By entering the obtained pairs of values in the graph (Fig. 4), it is possible to assess in a first approximation whether the investigated failure behavior corresponds to the present distribution. The failure times t_f is called the characteristic early failure time when the difference $\lambda(t_f) - \lambda(t_{f+1})$ is very small. Thus, we can approximately assume that the parameter $\lambda \approx \lambda(t_{f+1})$, and the failure rate tends to this value. The shape parameter α can be determined approximately in the following form:

$$\alpha = \frac{\lambda_0}{\lambda} \approx \frac{\lambda(t_1)}{\lambda(t_{f+1})}. \tag{7}$$

In this case, it is assumed that the value of the failure rate at the beginning of the first range λ_0 is equal to $\lambda(t_1)$.

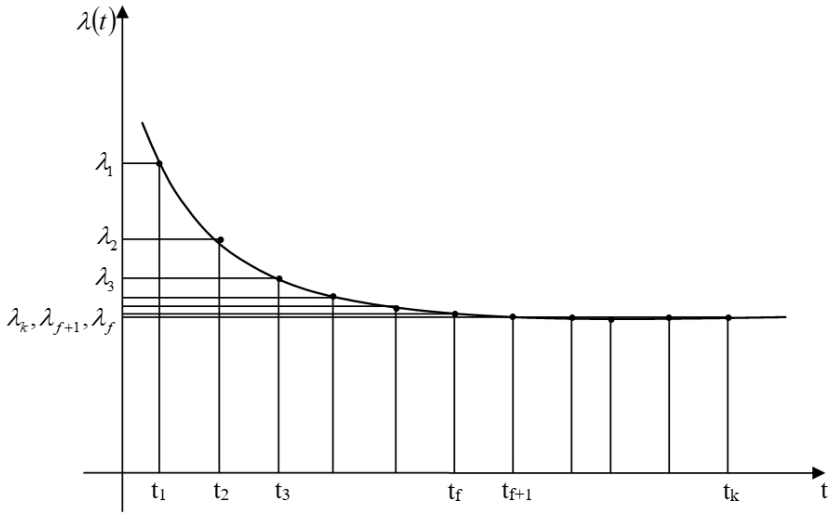


Figure 4: *The graph for determining the distribution parameters*

For the determination of the parameter β, formula (1) for the failure time $t = t_f$ is represented in the form:

$$\lambda(t_f) - \lambda = \lambda(\alpha - 1)e^{-\beta \cdot t_f}. \tag{8}$$

By logarithmizing, one obtains:

$$\ln[\lambda(t_f) - \lambda] = \ln[\lambda(\alpha - 1)] - \beta \cdot t_f. \quad (9)$$

It follows from this

$$\beta = \frac{1}{t_f} \ln \frac{\lambda(\alpha-1)}{\lambda(t_f)-\lambda}. \quad (10)$$

Suppose that for α >1, the condition $\lambda(t_f)-\lambda=e^{-4}\cdot\lambda\approx 0{,}018\ \lambda$ pays off at the time t_f. Then, based on the expression (10), we get:

$$\beta = \frac{1}{t_f}\big(4 + \ln(\alpha - 1)\big). \quad (11)$$

By analogy, one can write for the case α < 1:

$$\beta = \frac{1}{t_f}\big(4 + \ln(1 - \alpha)\big). \quad (12)$$

The parameters of the distribution can also be determined by three pairs of values (t_L,λ_L), (t_M,λ_M), (t_N,λ_N) (Figure 5). For this, the following system of equations must be solved:

$$\begin{cases} \lambda\big[1 + (\alpha - 1)e^{-\beta \cdot t_L}\big] = \lambda_L \\ \lambda\big[1 + (\alpha - 1)e^{-\beta \cdot t_M}\big] = \lambda_M \\ \lambda\big[1 + (\alpha - 1)e^{-\beta \cdot t_N}\big] = \lambda_N \end{cases} \quad (13)$$

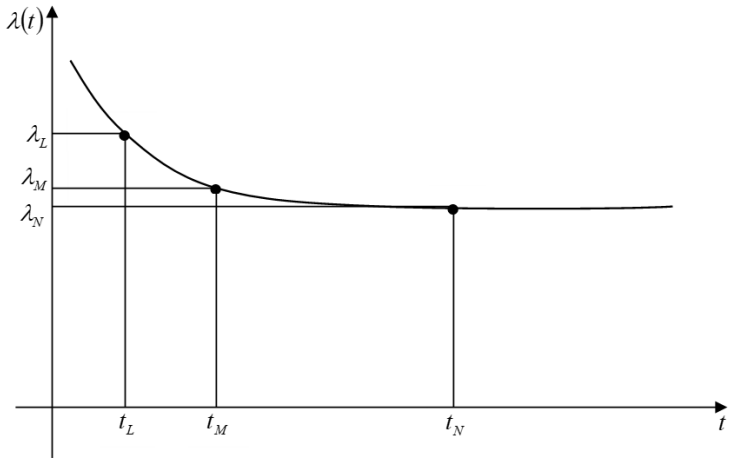


Figure 5: *Determination of distribution parameters based on three test results*

V. Application and evaluation

In order to evaluate the possibility of applying the proposed distribution, it is necessary to consider some practical cases.

In [8], the results of tests for the service lifetime of submersible electric motors of oil field equipment PED-32 and PED-45 were presented. The sample size was n=197 failures during 1548 days of testing. The class number was determined by the formula of Sturgess and after rounding it was taken K=8. Thus, the class length was taken h=193,5 days. Statistical data on failures were presented in Table 3. As can be seen from the table, the actual values of failure rates correspond to the case "A" of the proposed distribution (Tab. 1), since in the first period these values gradually increase, and then remain almost constant during normal operation. Only in the last period, the

failure rate increases sharply. This is due to the fact that by the end of the test period, the number of serviceable electric motors remains very small. This has little effect on the reliability of the electric motors in question during the first period and normal operation. Therefore, this difference can be ignored.

Based on the calculations of the distribution parameters using the formulas (7) and (12), the following results were obtained: λ=0,003 day^{-1}; α=0,3; β=0,012 day^{-1}.

Figure 5 shows a histogram of the failure rate of a submersible electric motor of oilfield equipment based on test results and a curve of changes in the values of this parameter according to the present distribution.

The test of the hypothesis about the conformity of the present theoretical distribution to the actual statistical data on failures was carried out according to the χ^2-Pearson criteria. According to [9], the measure of the discrepancy between the theoretical and empirical probability density functions can be determined by the expression below:

$$\chi^2 = \sum_{i=1}^{8} \frac{(n_i - np_i)^2}{np_i}. \quad (14)$$

Here p_i is the theoretical probability of a random variable falling into each of the intervales.

Table 3: *Statistical data on failures of submersible electric motors*

N-r of classes	Lifetime intervals, in days	Number of failures, n_i	Number of operable motors in the middle of the interval, N_i	Actual survival probability $R_i=N_i/N$	Actual failure rate $\lambda_i = \frac{n_i}{N_i \cdot \Delta t}$
1	0 – 193,5	58	168	0,85	0,0018
2	193,5- 387	62	108	0,55	0,003
3	387- 580,5	34	60	0,31	0,003
4	580,5- 774	17	34	0,17	0,0026
5	774- 967,5	12	20	0,1	0,0031
6	967,5- 1161	6	11	0,06	0,0028
7	1161- 1354,5	4	6	0,03	0,0034
8	1354,5- 1548	4	2	0,01	0,01
Total		197			

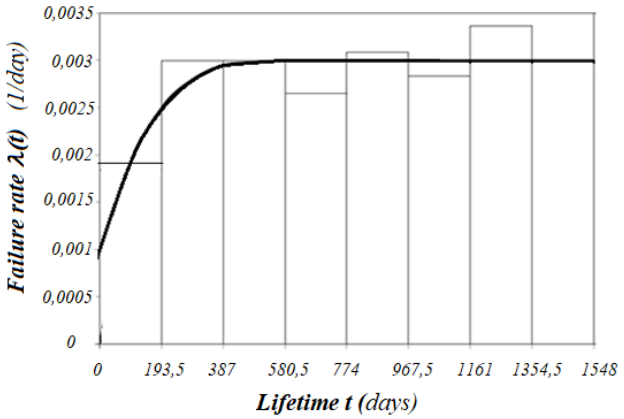


Figure 6: *Histogram of the failure rate of the submersible electric motor according to the test results and the curve of the values change according to the proposed distribution.*

The results of calculation based on the χ^2-Pearson test were presented in Table 4. The critical value χ_α^2 for the significance level α=0,80 and degrees of freedom m=K-1=8-1=7 is χ_α^2=3,82 [9]. Since the calculated χ^2=3,18 is less than χ_α^2 at a high level of significance, we can safety conclude that the proposed distribution is well suited for describing the actual distribution of failures of the tested submersible electric motors of oilfield equipment.

Table 4: *The results of calculation based on the χ^2 test for the summersible electric motors*

Nr.	Lifetime, t, days	Theoretical survival probability, R(t)	Theoretical failure frequency p_i=R(t_{i-1})-R(t_i)	Number of failures, n_i	$\frac{(n_i - n \cdot p_i)^2}{n \cdot p_i}$
1	0	1,0			
2	193,5	0,655	0,345	58	1,461
3	387	0,372	0,283	62	0,70
4	580,5	0,209	0,163	34	0,111
5	774	0,117	0,092	17	0,070
6	967,5	0,065	0,052	12	0,301
7	1161	0,037	0,028	6	0,042
8	1354,5	0,02	0,017	4	0,126
9	1548	0,005	0,015	4	0,369
Total				n=197	3,18

The following case from practice for the application of the present distribution realates to traffic-related mortality. Statistical data on traffic-related mortality in Germany in 2002 show the dependency of the human failure rate on age as shown in Figure 7 [10]. Since the right to drive a car is allowed from the age 18, here you can take t_0=18 year.

As can be seen from Figure 7, the traffic-related mortality at t=t_0=18 year, is equal to λ_0=0,00032 dead/(year·person). Over time, due to increased experience of drivers, the mortality decreases and takes a constant minimum value – λ=0,000052 dead/(year·person). The parameter α is determined based on the formula (7): α=6,15. For α >1, from the condition $\lambda(t_f)$-λ=e^{-4}·λ≈0,018 λ pays the time t_f. Since $\lambda(t_f)$≈1,018 λ=0,000053 dead/(year·person), according to the graph in Fig. 7, we get t_f≈33 year.

Using the formula (12), the following value for β was obtained: β=0,171 day^{-1}.

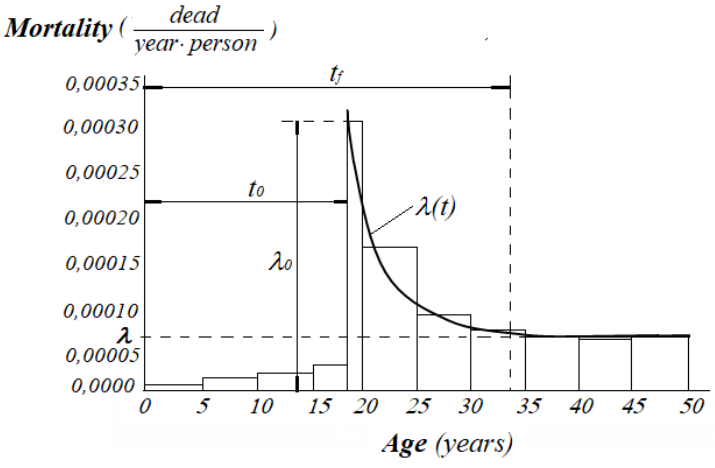


Figure 7: *Traffic-related mortality in Germany in 2002*

After determining all distribution parameters, it is possible to calculate other indexes of reliability of a man-machine system using present distribution. A graph of the human survival probability for this case is presented in Figure 8.

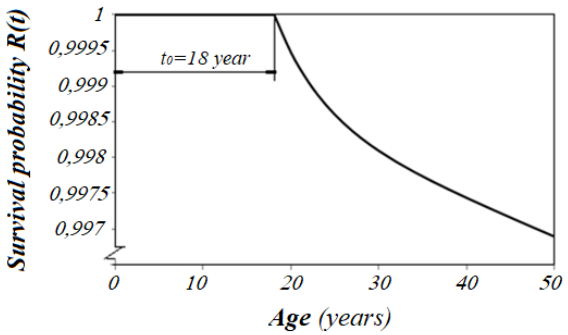


Figure 8: *Traffic-related human survival probability on age*

VI. Conclusions

The presented new theoretical distribution can be applied to describe the failures of various modern mechatronic components and human-machine systems. Using the presented distribution, reliability prediction in many cases can produce more effective results. This model can also be applied to assess the reliability of the human-machine system. To determine distribution parameters, statistical data on failures of the technical system are sufficient only in the first period of its operation. This is one of the main advantages of the presented distribution. Using the example of practical cases, a hypothesis was tested on the conformity of the present distribution to the actual statistical data on failures of various technical systems. Based on the obtained positive results, we can conclude that the proposed distribution can be successfully applied to describe the actual distribution of failures and assess the reliability of various modern mechatronic machines and human-machine system.

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MODERN APPROACHES TO MODELING RELIABLE AND EFFICIENT WATER SUPPLY SYSTEMS

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Abstract

The reliability of water supply systems plays a crucial role in ensuring sustainable water use, minimizing economic losses, and preventing failures in critical infrastructure. This paper proposes a mathematical approach to modeling the reliability of water systems based on probability theory and Markov processes. The main types of failures, their impact on operational characteristics, and economic consequences are examined. A simulation of the water supply network is conducted, considering the probabilistic characteristics of failures and recovery processes. The analysis of results demonstrates that the implementation of predictive monitoring methods and the optimization of maintenance strategies significantly enhance the resilience of water supply systems. The developed model can be applied in the planning of modernization and management of water supply infrastructure to improve its efficiency and economic feasibility.

Keywords: water supply reliability, mathematical modeling, probability theory, Markov processes, fault tolerance, economic efficiency.

I. Introduction

The reliability of water supply systems plays a crucial role in ensuring the sustainable development of cities and industrial facilities. Disruptions in water supply can lead to significant economic losses, a decline in the population's quality of life, and failures in critical infrastructure operations. Modern water systems are exposed to various risk factors, including the physical deterioration of pipelines, insufficient equipment modernization, and external environmental influences, making the task of enhancing their reliability particularly relevant.

Failures in water supply systems have a substantial economic impact, increasing emergency recovery costs, reducing industrial production efficiency, and imposing additional financial burdens on municipal budgets. From a social perspective, water supply disruptions can lead to deteriorating sanitary and hygienic conditions and a lower level of public comfort, especially in regions with limited access to alternative water sources [1].

The reliability assessment of water systems is traditionally conducted using mathematical modeling, probabilistic analysis, and simulation modeling. In recent years, researchers have increasingly focused on intelligent failure prediction methods based on big data analysis and machine learning applications. However, despite the development of numerous approaches, the integration of reliability models with economic parameters—allowing for the consideration of both technical and financial aspects of system operation—remains insufficiently explored.

This study aims to develop a reliability model for water supply systems that accounts not only for the technical characteristics of the infrastructure but also for the economic consequences of

failures. The research examines reliability forecasting principles, analyzes the main types of failures, and assesses their impact on operational costs [2]. The proposed model will enable the optimization of water resource management and enhance the resilience of water supply systems to potential disruptions.

Research in the field of water supply system reliability covers a wide range of approaches, including analytical, statistical, and simulation methods. Classical reliability assessment methods are based on failure analysis, probabilistic models, and reliability theory for engineering systems. One of the most common approaches is the calculation of the system's availability factor, which is defined as the ratio of the time of failure-free operation to the total duration of operation, including downtime periods. This method allows for a quantitative assessment of the impact of failures on the operational performance of the network.

Modern mathematical models of water supply reliability use probabilistic processes, including Markov chains and Monte Carlo network models. For example, the application of Markov processes allows for accounting for the transient states of the system when failures of varying criticality occur, enabling the prediction of the system's behavior in dynamic conditions. In studies dedicated to simulation modeling, the importance of considering the spatial distribution of consumers and the hydraulic characteristics of the network is emphasized, as different sections of the system have varying degrees of wear and load.

Economic aspects of water system reliability are also actively studied in the scientific literature. Cost analysis related to failures typically includes direct costs for repair and restoration, as well as indirect losses caused by reduced water supply quality and potential social consequences [3]. Some studies propose optimization models that link system reliability with the economic efficiency of investments in modernization. For example, a comparison of different maintenance strategies shows that proactive monitoring and predictive maintenance can reduce long-term operational costs despite higher initial investments. Table 1 presents comparative data on various reliability management methods in terms of their effectiveness and economic feasibility.

Table 1.: *Comparative data on different reliability management methods in terms of their effectiveness and economic feasibility*

Reliability Management Method	Mean Time Between Failures (hours)	Average Annual Operational Costs ($)	Availability Factor
Reactive Maintenance	8,500	1,200,000	0.89
Planned Maintenance	11,000	900,000	0.94
Predictive Maintenance	14,500	750,000	0.98

Among the technological solutions aimed at enhancing the reliability of water supply systems, intelligent monitoring systems based on real-time data analysis stand out. The implementation of sensor networks and Internet of Things (IoT) technologies significantly improves the accuracy of pipeline condition diagnostics, while machine learning helps identify potential failures before they actually occur. Several contemporary studies highlight that the integration of digital technologies not only improves system reliability but also optimizes maintenance costs. Thus, modern research demonstrates the importance of a comprehensive approach to ensuring water supply reliability, where mathematical modeling, economic analysis, and advanced technological solutions play a key role.

II. Formulation of the problem

The reliability of water supply systems is determined by several key parameters, including Mean Time Between Failures (MTBF), the probability of failure at a specific moment in time, and

the Availability factor [4]. These indicators allow for a quantitative assessment of the infrastructure's performance and the development of strategies for its optimal operation.

As an example, consider a municipal water supply system consisting of a pumping station, a network of main pipelines, and a distribution system. If the mean time between failures of the pumping station is 10,000 hours and the mean time to repair (MTTR) is 50 hours, the system's availability can be calculated as:

$$A = \frac{MTBF}{MTBF + MTTR} = \frac{10000}{10000 + 50} \approx 0.995$$

This means the system will be operational 99.5% of the time. However, if the number of failures increases and the repair time grows, the availability factor decreases, leading to significant water losses and higher operational costs.

Failures in water supply can be classified by their nature and consequences. Hydraulic failures are related to pressure losses, leaks, and pipe blockages; mechanical failures are associated with wear of pumps, valves, and connectors; while technological failures are caused by breakdowns in automated control systems, sensors, or software. For instance, a rupture in a 500 mm diameter main pipeline may result in a loss of 5-10 thousand cubic meters of water per hour, requiring urgent intervention and significant repair costs.

The economic consequences of failures are expressed in direct and indirect losses. Direct costs include repair expenses, equipment replacement, and labor wages, while indirect costs cover business losses, fines for violating environmental regulations, and damage to consumers. To assess these consequences, simulation modeling in Colab can be used, analyzing the statistics of emergency situations. For example, if the failure probability of the pumping station is 2% per year, and the average damage from a single failure is $50,000, the expected annual losses can be estimated as:

$$P \times C = 0.02 \times 50000 = 1000\ (dollars\ per\ year)$$

More complex models can account for the dynamic behavior of failures using Markov processes or Monte Carlo methods. To visualize the relationship between losses and equipment reliability, a graph can be constructed using Python (Figure 1). This graph demonstrates how the increasing probability of failures affects annual losses. Thus, mathematical modeling allows predicting risks and determining optimal maintenance strategies to minimize financial costs.

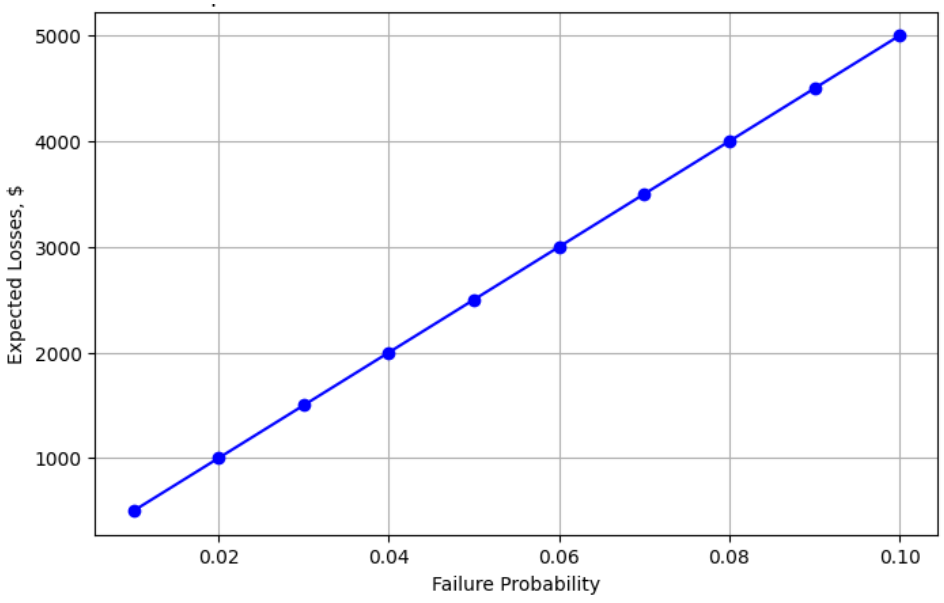

Figure 1: *Expected economic losses at different failure probabilities*

III. Problem solution

Development of a Water Supply System Reliability Model

When modeling the reliability of a water supply network, it is important to consider the probability of equipment failures, the impact of external factors, and the economic consequences. The mathematical foundation can be based on probability theory and Markov processes, which allow describing the system's behavior over time [5-7].

Mathematical Reliability Model

To analyze the reliability of a water supply system, we consider the water pipeline network as a collection of elements with probabilistic failure characteristics. Let the system consist of n elements, each of which can be in one of two states: operational (S_1) or failed (S_0).

The probability of the system being in an operational state can be expressed using the exponential distribution of time to failure:

$$P(t) = e^{-\lambda t}$$

where λ is the failure rate. If the elements operate independently, the overall probability of system failure will depend on the network structure. For example, for series-connected nodes:

$$P_{sys}(t) = \prod_{i=1}^{n} e^{-\lambda_i t}$$

For parallel connection:

$$P_{sys}(t) = 1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right)$$

Markov Reliability Model

Suppose that the water supply network can be in one of several states: fully operational, partially degraded, or failed. Let X(t) represent the number of operational nodes at time t. The dynamics of transitions between states are described by a Markov process with a transition probability matrix:

$$\begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \dots & 0 \\ \mu_1 & -(\lambda_2 + \mu_1) & \lambda_2 & \dots & 0 \\ 0 & \mu_2 & -(\lambda_3 + \mu_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mu_{n-1} & -\lambda_n \end{bmatrix}$$

Failure Simulation Modeling

For a more precise analysis, simulation modeling can be applied. In this case, the Monte Carlo method is used, where random failure events with exponential distribution are generated. During the simulation, the mean time to failure-free operation, the average number of failures over a period, and the recovery costs are evaluated.

An example of code for simulating the reliability of a water pipeline considering failures and repairs is provided below (figure 2).

The graph shows the distribution of failure and repair times. Analyzing this data allows for forecasting the probability of system failure and calculating optimal maintenance intervals.

Economic Component of the Model

Incorporating economic parameters into the model allows for considering repair costs and downtime losses. For example, if a failure of a component leads to financial losses of Closs per hour of downtime, and the repair requires an expenditure of Crepair, the average annual costs can be expressed as:

$$C_{total} = C_{repair} \cdot N_{repair} + C_{loss} \cdot T_{downtime}$$

where N_{repair} is the average number of repairs per year, and $T_{downtime}$ is the total downtime. Including these parameters in the model helps justify preventive maintenance strategies and the optimal allocation of budget resources.

```python
import numpy as np
import matplotlib.pyplot as plt

# System parameters
n_elements = 10  # Number of nodes in the system
lambda_fail = 0.002  # Failure rate (1/hour)
mu_repair = 0.005  # Repair rate (1/hour)
time_horizon = 1000  # Simulation time (hours)
n_simulations = 1000  # Number of simulations

def simulate_water_network():
    failures = np.random.exponential(1/lambda_fail, n_elements)
    repairs = np.random.exponential(1/mu_repair, n_elements)
    return min(failures), min(repairs)

failure_times = []
repair_times = []

for _ in range(n_simulations):
    f_time, r_time = simulate_water_network()
    failure_times.append(f_time)
    repair_times.append(r_time)

# Plotting the distribution of failures and repairs
plt.figure(figsize=(10, 5))
plt.hist(failure_times, bins=30, alpha=0.7, label="Failures", color='red')
plt.hist(repair_times, bins=30, alpha=0.7, label="Repairs", color='blue')
plt.xlabel("Time (hours)")
plt.ylabel("Frequency")
plt.title("Distribution of Failure and Repair Times in Water Network Nodes")
plt.legend()
plt.show()
```

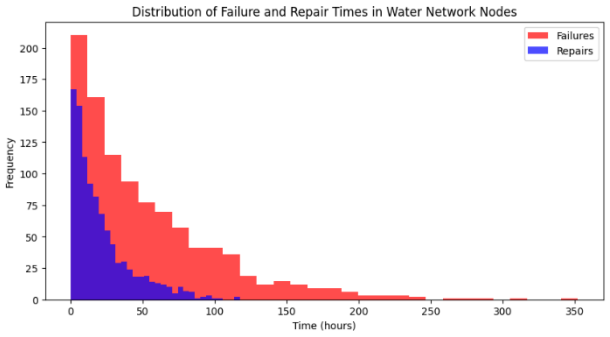

Figure 2: *Distribution of failure and repair times in water network nodes*

The developed model enables the evaluation of the reliability of water supply systems, considering probabilistic failures, state transitions, and financial consequences. Simulation modeling confirms that increasing the intensity of repairs reduces overall downtime but increases maintenance costs. Thus, the proposed approach allows finding a balance between system costs and reliability.

Analysis of Results

To verify the proposed reliability model, a real medium-sized water supply network was chosen, which provides drinking water to an urban area with a population of approximately 500,000 people. The analysis considered the main nodes of the system, including pumping stations, trunk and distribution pipelines, as well as reservoirs and shut-off valves. The focus was on failures related to pipeline wear, pump equipment breakdowns, and malfunctioning automated control systems.

The conducted modeling allowed for the determination of the mean time between failures for various system components and their failure probabilities [8-9]. For instance, the calculation showed that the trunk pipelines with a diameter of 600 mm have a failure probability of 0.0025 per year, while pumping stations demonstrate higher failure rates – up to 0.015 per year. The data analysis identified critical areas of the network where failure concentration exceeded acceptable standards, indicating the need for timely preventive measures and equipment upgrades.

To assess the effectiveness of the proposed reliability enhancement strategies, a comparative evaluation of different operational scenarios was conducted. Three options were considered: the baseline (current operation without changes), preventive (regular maintenance and planned equipment replacement), and intelligent (application of predictive monitoring and digital twins). The results are presented in Table 2.

Table 2. *Impact of operational strategies on reliability indicators*

Operational strategy	Mean time between failures (years)	Failure probability per year	Reduction in repair costs (%)
Baseline	8.2	0.012	0
Preventive	12.5	0.0065	22
Intelligent	16.8	0.0032	37

As seen from the calculations, the introduction of preventive measures increases the mean time between failures by 52%, while the use of an intelligent approach, based on predictive analytics and automated control, reduces the failure probability by more than three times compared to the current system state.

From an economic perspective, the implementation of the intelligent system led to a 37% reduction in annual costs for emergency repairs and downtime [10-12]. Data visualization (Figure 3) shows the relationship between operational costs and the system's reliability level under different management strategies.

The analysis demonstrates that improving the reliability of the water supply system leads to a reduction in economic losses associated with unscheduled repairs, water losses, and disruptions in consumer supply. This confirms the feasibility of applying predictive monitoring and implementing digital technologies for water resource management.

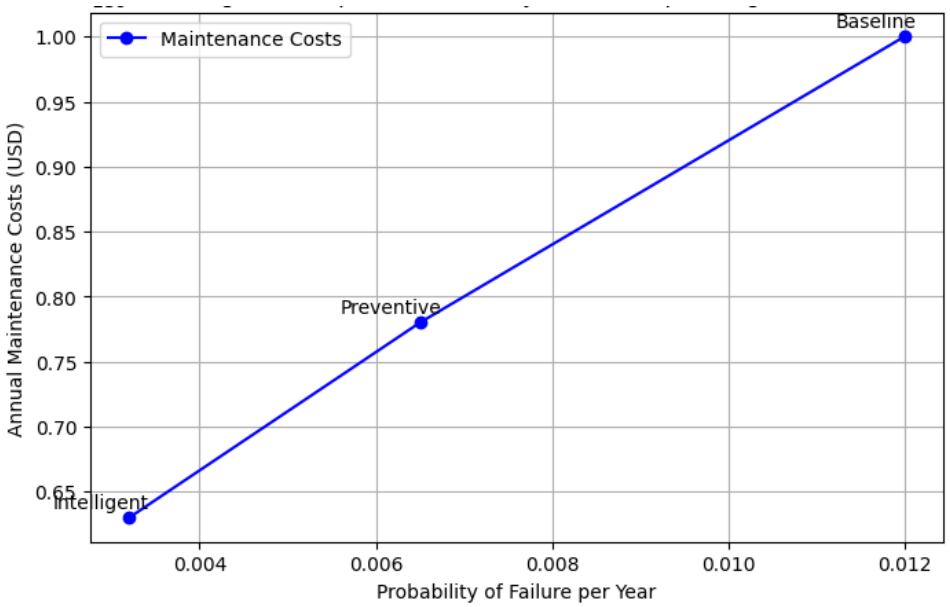

Figure 3. *Impact of reliability level on operating costs*

In the context of ensuring the reliability of water supply systems, a strategic approach to maintenance, predictive monitoring, and the implementation of intelligent technologies plays a key role [13]. The application of modern management methods allows for minimizing failure risks and improving operational efficiency.

One of the most effective solutions is the optimization of maintenance strategies. Traditional preventive methods are often based on scheduled maintenance plans, which do not always account for the actual condition of the equipment [14-15]. The introduction of risk-based and predictive maintenance based on data analysis helps reduce unplanned downtime and lower operational costs. The table 3 presents a comparative analysis of different maintenance strategies.

Table 3. *Comparative characteristics of different maintenance strategies*

Approach	Advantages	Limitations
Scheduled Preventive Maintenance	Ease of implementation, reduced failure probability	High operational costs, potential for unnecessary repairs
Reactive Maintenance	Minimal preventive costs	High risk of emergency failures and significant financial losses
Predictive Maintenance (Data-Based)	Minimization of unplanned shutdowns, cost reduction	Requires the implementation of sensors and data processing systems

Predictive monitoring using digital technologies is also becoming a crucial element in ensuring a reliable water supply. The deployment of pressure, flow, and vibration sensors enables real-time infrastructure condition tracking. Machine learning algorithms for data analysis provide the ability to detect potential failures in advance and reduce the likelihood of breakdowns.

Intelligent water supply management systems integrated with digital platforms optimize network performance through automatic flow redistribution, leakage prevention, and timely detection of critical equipment conditions. The implementation of such solutions enhances resilience, reduces water losses, and lowers operational costs. Collectively, these measures form a comprehensive approach to water system reliability management, ensuring their efficient and uninterrupted operation.

IV. Conclusions

The conducted study has identified key factors affecting the reliability of water supply systems and proposed a mathematical model that accounts for the probability of failures and their economic consequences. The simulation results demonstrate that considering fault tolerance in the planning and operation of water systems significantly reduces the risks of supply disruptions and associated financial losses.

The developed approach can be used to optimize maintenance strategies and infrastructure modernization. The implementation of digital monitoring technologies and failure prediction opens up opportunities to enhance the reliability of water systems by accurately identifying critical network elements and enabling timely decision-making.

Further research may focus on expanding the model by incorporating climatic factors, changes in water consumption, and the impact of external influences on infrastructure. The integration of machine learning methods and the Internet of Things (IoT) into reliability management systems also represents a promising direction for improving the efficiency of water supply systems.

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COMPARATIVE BAYESIAN ANALYSIS OF THE INVERSE TOPP-LEONE DISTRIBUTION

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Abstract

This paper focuses on the Bayesian estimation of the shape parameter for the Inverse Topp-Leone (ITL) distribution. To achieve this, we employ both the extended Jeffrey's prior and the gamma prior, facilitating the derivation of posterior distributions for the shape parameter. The Bayesian estimators are calculated under various loss functions, including the squared error loss function (SELF), entropy loss function (ELF), precautionary loss function (PLF), and Linex loss function (LLF), each chosen to address different practical scenarios and estimator biases. In addition to the Bayesian approach, we also explore maximum likelihood estimation (MLE) to provide a comparative benchmark. The performance of these estimators is assessed and compared based on mean squared error (MSE) across multiple sample sizes, allowing for a detailed evaluation of estimator robustness and accuracy. A real-world dataset is then analyzed to further demonstrate the relative efficiency of each estimator under the different loss functions, providing practical insights into the applicability of each estimation approach for the ITL distribution. This analysis offers a comprehensive perspective on the versatility and precision of Bayesian and classical estimation methods for the ITL model.

Keywords: Bayesian analysis, Jeffery's prior, Gamma prior, Maximum likelihood estimation, Loss functions

I. Introduction

Topp-Leone distribution belongs to the distribution family which has support $[0,1]$. It indicates the j-shape form of density function along with bathtub shape of its hazard function. This distribution is used for the analysis of failure data. The probability density function of Topp-Leone is given by

$$g(x,\theta) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1};\ \ 0 \le x \le 1, \theta > 0 \tag{1}$$

Since the Topp-Leone distribution is newly formulated distribution proposed by Topp and Leone [20]. This distribution has been studied by several authors such as Nadarajah [12], Ghitney et al [6, 7], Genc [8], Al-Zahrani [5], MirMostafaee [11], Vicari et al. [21]. Recently Hassan et al. [9] explored

the inverse of Topp-Leone distribution. Researchers have extensively explored and generalized a variety of probability distributions. For instance, Rather and Ozel [17, 18] investigated the weighted Power Lindley distribution, and also examined the length-biased Power Lindley distribution, providing insights into its properties and applications. In addition, Rather et al. [19] introduced the exponentiated Ailamujia distribution and discussed its real-life applications. Qayoom and Rather [13] conducted a comprehensive study of the Weighted Transmuted Mukherjee-Islam distribution, analysing its statistical properties. Qayoom and Rather [14] also explored a new generalization of the Transmuted Mukherjee-Islam distribution. More recently, Qayoom et al. [15] presented an extension of the Lindley distribution, examining its practical utility in real-world scenarios.

Let X follows the probability distribution function of Topp-Leone distribution, then the transformation $Y=\frac{1}{X}-1$ is said to follow inverse of Topp-Leone distribution having probability density function (p.d.f) as

$$f(y,\theta)=2\theta y\frac{(1+2y)^{\theta-1}}{(1+y)^{2\theta+1}};y>0,\theta>0 \tag{2}$$

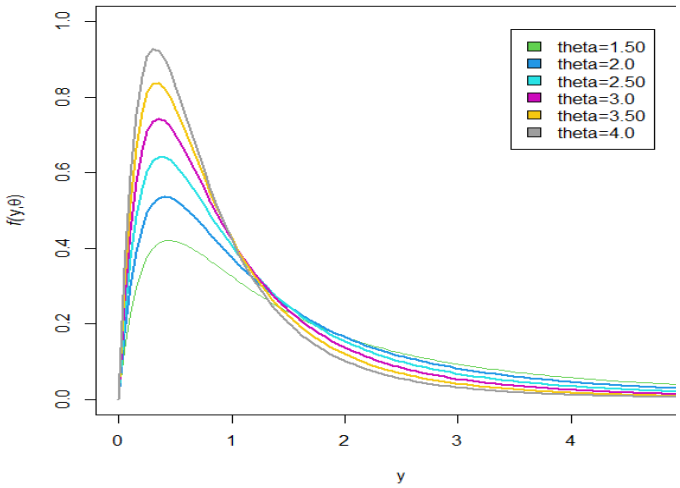

Figure 1: *p.d.f plot of ITLD under different values of parameters*

Figure 1, illustrates some possible shapes of p.d.f for varying parameters. The corresponding cumulative distribution function (c.d.f) of (2) is given by

$$F(y,\theta)=1-\left[\frac{(1+2y)^{\theta}}{(1+y)^{2\theta}}\right];y\geq 0,\theta>0 \tag{3}$$

II. Maximum Likelihood Estimation

Let $y_1,y_2,\ldots,y_n$ be random samples from inverse Topp-Leone distribution given by (2), then the likelihood function becomes

$$l=\prod_{i=1}^{n}f(y_i,\theta) \tag{4}$$

$$l=\prod_{i=1}^{n}(2\theta)y_i(1+y_i)^{-(2\theta+1)}(1+2y_i)^{\theta-1} \tag{5}$$

The log-likelihood function is

$$\log l = n\log 2 + n\log\theta + \sum_{i=1}^{n}\log y_i - (2\theta+1)\sum_{i=1}^{n}\log(y_i+1) + (\theta-1)\sum_{i=1}^{n}\log(2y_i+1) \tag{6}$$

Differentiate w.r.t θ we get

$$\frac{\partial \log l}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\log(y_i+1) + \sum_{i=1}^{n}\log(2y_i+1) \tag{7}$$

Solving $\frac{\partial \log l}{\partial\theta} = 0$, we get

$$\hat{\theta} = \frac{n}{2\sum_{i=1}^{n}\log(y_i+1) - \sum_{i=1}^{n}\log(2y_i+1)} \tag{8}$$

III. Bayesian Estimation of Inverse Topp-Leone Distribution

Bayesian estimation is a highly effective approach for estimating the parameters of distribution models. This method incorporates prior knowledge to determine the posterior distribution of a lifetime distribution's parameters. From a Bayesian perspective, selecting an appropriate prior is flexible, as no single prior can be universally preferred; the choice depends on the available information about the parameter. When little prior knowledge about the parameter is available, a non-informative prior is typically chosen to minimize bias. However, when sufficient prior information is available, using an informative prior enhances the accuracy of the estimation. The goal of the current study is to derive a Bayesian estimation of the parameter for the inverse Topp-Leone distribution, specifically employing an extended Jeffreys prior and a gamma prior. In recent years, Bayesian estimation methods have gained considerable attention. For instance, Ahmad et al. [2] investigated Bayesian parameter estimation for the two-parameter exponentiated gamma distribution, while Mudasir et al. [10] focused on the weighted Erlang distribution. Raqab and Madi [16] explored Bayesian estimation for the exponentiated Rayleigh distribution. Recently, Ahmad et al. [3] examined Bayesian parameter estimation for the inverse Ailamujia distribution using various loss functions. In this study, both extended Jeffreys and gamma priors are considered. The extended Jeffreys prior is a non-informative prior, useful when interpretive information about the parameters is limited, while the gamma prior provides a more informative approach when substantial parameter knowledge is available. This Bayesian framework aims to enhance parameter estimation accuracy for the inverse Topp-Leone distribution, with relevance across various applied fields.

IV. Posterior Distribution of Inverse Topp-Leone Distribution

Suppose $y = (y_1, y_2, .., y_n)$ denotes the n recorded values of (2). Then its likelihood function is given by

$$L(y|\theta) = \prod_{i=1}^{n} 2\theta \frac{y_i}{(y_{i+1})^2}\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]^{\theta-1} \tag{9}$$

$$L(y|\theta) = 2^n\theta^n \prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2} e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]} \tag{10}$$

We assume the prior distribution of θ to be extended Jeffery's prior proposed by Alkutubi [4], is given by

$$g(\theta) \propto [I(\theta)]^c, c \in R^+$$

Where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f(y,\theta)}{\partial^2 \theta}\right]$ is the Fisher information matrix, for the distribution (2),then

$$g(\theta) = K\left[\frac{n}{\theta^2}\right]^c \tag{11}$$

The posterior distribution of θ under the assumption of extended Jeffrey's prior i.e $g(\theta) \propto \frac{1}{\theta^{2c}}$ is given by

$$h(\theta|y) \propto L(y|\theta)g(\theta)$$

$$\Rightarrow h(\theta|y) \propto \prod_{i=1}^{n} 2^n \frac{y_i}{(y_i+1)^2} e^{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} \theta^{n-2c} e^{-\theta \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} \tag{12}$$

$$\Rightarrow h(\theta|y) \propto K\theta^{n-2c} e^{-\theta \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} \tag{13}$$

Where K is independent of θ

$$K^{-1} = \int_0^\infty \theta^{n-2c} e^{-\theta \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} d\theta$$

$$K^{-1} = \frac{\Gamma(n-2c+1)}{\left\{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]\right\}^{n-2c+1}}$$

Therefore

$$K = \frac{\left\{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]\right\}^{n-2c+1}}{\Gamma(n-2c+1)} \tag{14}$$

Hence the posterior distribution of θ is given by

$$h(\theta|y) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \theta^{n-2c} e^{-\theta T} \tag{15}$$

Where,

$$T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]$$

I. Estimation under square error loss function

The squared error loss function is defined as $l(\hat{\theta},\theta) = c_1(\hat{\theta}-\theta)^2$ for some constant c_1. The risk function is given by

$$R(\hat{\theta},\theta) = E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta) = \int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta) = \int_0^\infty c_1(\hat{\theta}-\theta)^2 \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \theta^{n-2c} e^{-\theta T} d\theta \tag{16}$$

$$R(\hat{\theta},\theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} c_1 \int_0^\infty (\hat{\theta}-\theta)^2 \theta^{n-2c} e^{-\theta T} d\theta \tag{17}$$

$$R(\hat{\theta},\theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} c_1 \left[\hat{\theta}^2 \int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta + \int_0^\infty \theta^{n-2c+2} e^{-\theta T} d\theta - 2\hat{\theta} \int_0^\infty \theta^{n-2c+1} e^{-\theta T} d\theta\right]$$

After solving the integral, we get

$$R\left(\hat{\theta},\theta\right)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}c_1\left[\begin{matrix}\hat{\theta}\dfrac{\Gamma(n-2c+1)}{T^{n-2c+1}}+\dfrac{(n-2c+1)(n-2c+1)\Gamma(n-2c+1)}{T^{n-2c+2}}\\ -2\hat{\theta}\dfrac{(n-2c+1)\Gamma(n-2c+1)}{T^{n-2c+1}}\end{matrix}\right]$$

$$R\left(\hat{\theta},\theta\right)=c_1\left[\hat{\theta}^2+\frac{(n-2c+1)(n-2c+2)}{T^2}-2\hat{\theta}\frac{(n-2c+1)}{T}\right]$$

Now solving $\frac{\partial R\left(\hat{\theta},\theta\right)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_s=\frac{(n-2c+1)}{T} \tag{18}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

II. Estimation under entropy loss function

The entropy loss function is defined as $l(\delta)=b[\delta-\log(\delta)-1];b>0$, $\delta=\frac{\hat{\theta}}{\theta}$. The risk function is given by

$$R\left(\hat{\theta},\theta\right)=E[l(\delta)]$$

$$R\left(\hat{\theta},\theta\right)=\int_0^\infty l(\delta)h(\theta|y)d\theta$$

$$R\left(\hat{\theta},\theta\right)=\int_0^\infty b[\delta-\log(\delta)-1]h(\theta|y)d\theta \tag{19}$$

$$R\left(\hat{\theta},\theta\right)=b\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\int_0^\infty\left[\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\theta^{n-2c}e^{-\theta T}d\theta \tag{20}$$

$$R\left(\hat{\theta},\theta\right)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}b\left[\hat{\theta}\int_0^\infty\theta^{n-2c-1}e^{-\theta T}d\theta-\log\left(\hat{\theta}\right)\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta+\int_0^\infty\log(\theta)\theta^{n-2c}e^{-\theta T}d\theta-\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$R\left(\hat{\theta},\theta\right)=b\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}\frac{\Gamma(n-2c)}{T^{n-2c}}-\log\left(\hat{\theta}\right)\frac{\Gamma(n-2c+1)}{T^{n-2c+1}}+\frac{\Gamma'(n-2c+1)}{T^{n-2c+1}}-\frac{\Gamma(n-2c+1)}{T^{n-2c+1}}\right]$$

$$R\left(\hat{\theta},\theta\right)=b\left[\hat{\theta}\frac{T}{n-2c}-\log\left(\hat{\theta}\right)+\psi'(n-2c+1)-1\right] \tag{21}$$

Where $\psi'(.)$ denotes the digamma function

Now solving $\frac{\partial R\left(\hat{\theta},\theta\right)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_e=\frac{(n-2c)}{T} \tag{22}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

III. Estimation under precautionary loss function

The entropy loss function is defined as $l(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$. The risk function is given by

$$R(\hat{\theta},\theta)=E\left[l(\hat{\theta},\theta)\right]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\int_0^\infty\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\theta^{n-2c}e^{-\theta T}d\theta \tag{23}$$

$$R(\hat{\theta},\theta)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta+\frac{1}{\hat{\theta}}\int_0^\infty\theta^{n-2c+2}e^{-\theta T}d\theta-2\int_0^\infty\theta^{n-2c+1}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=\left[\hat{\theta}+\frac{1}{\hat{\theta}}\frac{(n-2c+1)(n-2c+2)}{T^2}-\frac{2(n-2c+1)}{T}\right] \tag{24}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_p=\frac{[(n-2c+1)(n-2c+2)]^{\frac{1}{2}}}{T} \tag{25}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

IV. Estimation under linex loss function

The linex loss function is defined as $l(\hat{\theta},\theta)=\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1$.The risk function is given by

$$R(\hat{\theta},\theta)=E\left[l(\hat{\theta},\theta)\right]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\int_0^\infty\left[\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1\right]\theta^{n-2c}e^{-\theta T}d\theta \tag{26}$$

$$R(\hat{\theta},\theta)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[e^{b_1\hat{\theta}}\int_0^\infty\theta^{n-2c}e^{-\theta(b_1+T)}d\theta-b_1\hat{\theta}\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta+b_1\int_0^\infty\theta^{n-2c+1}e^{-\theta T}d\theta-\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=\left[e^{b_1\theta}\left(\frac{T}{b_1+T}\right)-b_1\hat{\theta}+\frac{b_1(n-2c+1)}{T}-1\right] \tag{27}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}}=0$, we get

$$\hat{\theta}_l=\frac{(n-2c+1)}{b_1}\log\left(\frac{b_1+T}{T}\right) \tag{28}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

V. Posterior Distribution of Inverse Topp-Leone Distribution under Gamma Prior

Suppose $y=(y_1,y_2,..,y_n)$ denotes the n recorded values of (2). Then its likelihood function is given by

$$L(y|\theta)=\prod_{i=1}^{n}2\theta\frac{y_i}{(y_{i+1})^2}\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]^{\theta-1} \tag{29}$$

$$L(y|\theta)=2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]} \tag{30}$$

We assume the prior distribution of θ to be gamma prior

The posterior distribution of θ under the assumption of gamma prior i.e., $g(\theta)\propto\frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta}$ is given by

$$h(\theta|y)\propto L(y|\theta)g(\theta)$$

$$\Rightarrow h(\theta|y)\propto 2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]}\frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta}$$

$$\Rightarrow h(\theta|y)\propto\prod_{i=1}^{n}2^n\frac{y_i}{(y_i+1)^2}e^{\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]}\frac{a^b}{\Gamma(b)}\theta^{n+b-1}e^{-\theta\left(a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}$$

$$\Rightarrow h(\theta|y)\propto K\theta^{n+b-1}e^{-\theta\left(a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}$$

Where K is independent of θ

$$K^{-1}=\int_0^{\infty}\theta^{n+b-1}e^{-\theta\left(a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}d\theta \tag{31}$$

$$K^{-1}=\frac{\Gamma(n+b)}{\left\{a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n+b}} \tag{32}$$

Therefore,

$$K=\frac{\left\{a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n+b}}{\Gamma(n+b)} \tag{33}$$

Hence the posterior distribution of θ is given by

$$h(\theta|y)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}e^{-(a+T)\theta} \tag{34}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

I. Estimation under square error loss function

The squared error loss function is defined as $l(\hat{\theta},\theta)=c_1(\hat{\theta}-\theta)^2$ for some constant c_1. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=c_1\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty(\hat{\theta}-\theta)^2\theta^{n+b-1}e^{-(a+T)\theta}d\theta \tag{35}$$

$$R(\hat{\theta},\theta)=c_1\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}^2\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta+\int_0^\infty\theta^{n+b+1}e^{-(a+T)\theta}d\theta-2\hat{\theta}\int_0^\infty\theta^{n+b}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=c_1\left\{\hat{\theta}^2+\frac{(n+b)(n+b+1)}{(a+T)^2}-2\hat{\theta}\frac{(n+b)}{a+T}\right\} \tag{36}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_s=\frac{n+b}{a+T} \tag{37}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

II. Estimation under entropy loss function

The entropy loss function is defined as $l(\delta)=b[\delta-\log(\delta)-1];b>0$, $\delta=\frac{\hat{\theta}}{\theta}$.The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\delta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\delta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\int_0^\infty b[\delta-\log(\delta)-1]h(\theta|y)d\theta \tag{38}$$

$$R(\hat{\theta},\theta)=b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\left[\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\theta^{n+b-1}e^{-(a+T)\theta}d\theta \tag{39}$$

After solving the integral, we get

$$R\left(\hat{\theta},\theta\right)=b\left\{\hat{\theta}\frac{(a+T)}{(n+b-1)}-\log\hat{\theta}+\psi'(n+b)-1\right\} \tag{40}$$

Where $\psi'(.)$ denotes the digamma function

Now solving $\frac{\partial R\left(\hat{\theta},\theta\right)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_e=\frac{n+b-1}{a+T} \tag{41}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

III. Estimation under precautionary loss function

The entropy loss function is defined as $l\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}$ .The risk function is given by

$$R\left(\hat{\theta},\theta\right)=E\left[l\left(\hat{\theta},\theta\right)\right]$$

$$R\left(\hat{\theta},\theta\right)=\int_0^\infty l\left(\hat{\theta},\theta\right)h(\theta|y)d\theta$$

$$R\left(\hat{\theta},\theta\right)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}\theta^{n+b-1}e^{-(a+T)\theta}d\theta \tag{42}$$

$$R\left(\hat{\theta},\theta\right)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta+\frac{1}{\hat{\theta}}\int_0^\infty\theta^{n+b+1}e^{-(a+T)\theta}d\theta-2\int_0^\infty\theta^{n+b}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$R\left(\hat{\theta},\theta\right)=\left\{\hat{\theta}+\frac{1}{\hat{\theta}}\frac{(n+b)(n+b+1)}{(a+T)^2}-2\frac{(n+b)}{(a+T)}\right\} \tag{43}$$

Now solving $\frac{\partial R\left(\hat{\theta},\theta\right)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_p=\frac{\left[(n+b)(n+b+1)\right]^{\frac{1}{2}}}{a+T} \tag{44}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

IV. Estimation under linex loss function

The linex loss function is defined as $l\left(\hat{\theta},\theta\right)=\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1$.The risk function is given by

$$R\left(\hat{\theta},\theta\right)=E\left[l\left(\hat{\theta},\theta\right)\right]$$

$$R\left(\hat{\theta},\theta\right)=\int_0^\infty l\left(\hat{\theta},\theta\right)h(\theta|y)d\theta$$

$$R\left(\hat{\theta},\theta\right)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^{\infty}\left[\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1\right]\theta^{n+b-1}e^{-(a+T)\theta}d\theta \quad (45)$$

After solving the integral, we get

$$\hat{\theta}_l=\frac{1}{b_1}\log\left(\frac{a+b_1+T}{a+T}\right)^{n+b} \quad (46)$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

VI. Simulation Analysis

This section is dedicated to the simulation analysis, we generate $N=1500$ random samples of size n= 50, 100 and 150 to represent a small, medium and large data set from inverse Topp-Leone distribution for specific values of $\theta=0.5$ and 1. The shape parameter is estimated with maximum likelihood estimation and Bayesian using extended Jeffery's prior and gamma prior. For extended Jeffrey's prior we chose $c=0.5$ and 1 and the value of loss function $b_1=0.06$ and 0.09. In case of gamma prior we chose $a=0.5,1.0$ and $b=0.5,1.0$ with loss function $b_1=0.06$ and 0.09. R software is used for simulation analysis in order to examine and compare the efficiency of the estimates for different sample sizes with different values of loss functions. The results are presented in table 1 and 2.

Table 1: *Mean Square Error for $\hat{\theta}$ Using Jeffery's Prior*

n	$\hat{\theta}$	C	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
						$b_1=0.06$	$b_1=0.09$
50	0.5	0.5	0.01019658	0.01019497	0.01019739	0.01019656	0.01019654
		1.0	0.01006666	0.01006506	0.01006746	0.01006662	0.0100666
	1	0.5	0.1605382	0.1605446	0.160535	0.1605383	0.1605383
		1.0	0.1591694	0.1591759	0.1591662	0.1591696	0.1591697
100	0.5	0.5	0.01003337	0.01003257	0.01003377	0.01003335	0.01003335
		1.0	0.01002572	0.01002492	0.01002612	0.0100257	0.0100257
	1	0.5	0.1603843	0.1603875	0.1603827	0.1603844	0.1603844
		1.0	0.1602739	0.1602771	0.1602723	0.160274	0.160274
150	0.5	0.5	0.01001125	0.01001071	0.01001151	0.01001123	0.01001123
		1.0	0.01004535	0.01004483	0.01004563	0.01004535	0.01004533
	1	0.5	0.1600195	0.1600216	0.1600184	0.1600195	0.1600196
		1.0	0.1600716	0.1600738	0.1600706	0.1600717	0.1600717

$\hat{\theta}_s=$ Square error loss function, $\hat{\theta}_e=$ Estimation under Entropy,

$\hat{\theta}_p=$ Estimation under Precautionary, $\hat{\theta}_l=$ Estimation under LINEX

In table 1, Bayes estimation with squared error loss function under extended Jeffery's prior the lesser values in most cases. Moreover, when sample size increase from 50 to 150, the mean square error decreases quite significantly.

Table 2: *Mean Square Error for $\hat{\theta}$ Using gamma Prior*

n	$\hat{\theta}$	a	b	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
							$b_1 = 0.06$	$b_1 = 0.09$
50	0.5	0.5	0.5	0.02337317	0.02337461	0.02337391	0.0233732	0.0233732
		0.5	1.0	0.02335441	0.02335582	0.02335512	0.02335444	0.02335444
	1	0.5	0.5	0.2494094	0.2494161	0.2494127	0.2494095	0.2494096
		0.5	1.0	0.4267061	0.4267121	0.4267091	0.4267061	0.4267062
100	0.5	0.5	0.5	6.735e-06	6.7374e-06	6.7365e-06	6.7356e-06	6.7356e-06
		0.5	1.0	7.1562e-06	7.1517e-06	7.1540e-06	7.1561e-06	7.1561e-06
	1	0.5	0.5	0.2500997	0.250103	0.2501014	0.2500998	0.2500998
		0.5	1.0	0.2490105	0.2490138	0.2490121	0.2490105	0.2490104
150	0.5	0.5	0.5	4.5992e-06	4.6011e-06	4.6002e-06	4.59938e-06	4.59938e-06
		0.5	1.0	4.6406e-06	4.6426e-06	4.6416e-06	4.64060e-06	4.64060e-06
	1	0.5	0.5	0.2502639	0.2502661	0.250265	0.2502639	0.2502639
		0.5	1.0	0.2497457	0.2497479	0.2497468	0.2497457	0.2497457

In table 2, Bayes estimation with squared error loss function under gamma prior the lesser values in most cases. Moreover, when sample size increase from 50 to 150, the mean square error decreases quite significantly.

VII. Application

In this section we provide a real life data sets through which the efficiency of the estimators and posterior risks of different loss function has been obtained.

The data of 40 patients suffering from blood cancer(leukaemia) from one ministry of health hospitals in Saudi Arabia (see Abouammah et al. [1]).

By using different loss functions, the Bayesian estimates and posterior risks of the posterior distribution through both priors are as follows where posterior risks are in parenthesis.

Table 3: *Bayes Estimation and Posterior Risks Using Jeffery's Prior*

$\hat{\theta}$	C	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
					$b_1 = 0.06$	$b_1 = 0.09$
1.0	0.5	0.5836 (0.0085)	0.5690 (3.622)	0.5908 (8.616)	0.5833 (0.0350)	0.5832 (0.0525)
	1.0	0.5690 (0.0083)	0.5544 (3.597)	0.5763 (8.403)	0.5688 (0.0341)	0.5686 (0.0512)
	1.5	0.5544 (0.0080)	0.5398 (3.571)	0.5617 (8.191)	0.5542 (0.0332)	0.5541 (0.0499)
2.0	0.5	0.5836 (0.0085)	0.5690 (3.622)	0.5908 (8.616)	0.5833 (0.0350)	0.5832 (0.0525)
	1.0	0.5690 (0.0083)	0.5544 (3.597)	0.5763 (8.403)	0.5688 (0.0341)	0.5686 (0.0512)
	1.5	0.5544 (0.0080)	0.5398 (3.571)	0.5617 (8.191)	0.5542 (0.0332)	0.5541 (0.0499)

$\hat{\theta}_s$ =Square error loss function, $\hat{\theta}_e$ =Estimation under Entropy, $\hat{\theta}_p$ =Estimation under Precautionary, and $\hat{\theta}_l$ = Estimation under LINEX

Table 4: *Bayes Estimation and Posterior Risks Using Gamma Prior*

$\hat{\theta}$	a	b	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
						$b_1 = 0.06$	$b_1 = 0.09$
1.0	0.5	0.5	0.5866 (0.0084)	0.5721 (4.247)	0.5793 (0.5699)	0.5864 (0.0351)	0.5862 (0.0527)
	0.5	1.0	0.5939 (0.0086)	0.5794 (4.247)	0.5866 (0.5771)	0.5936 (0.0356)	0.5935 (0.0534)
	1.0	0.5	0.5824 (0.0083)	0.5680 (4.254)	0.5752 (0.5637)	0.5821 (0.0349)	0.5820 (0.0524)
2.0	0.5	0.5	0.5866 (0.0084)	0.5721 (4.247)	0.5793 (0.5699)	0.5864 (0.0351)	0.5862 (0.0527)
	0.5	1.0	0.5939 (0.0086)	0.5794 (4.247)	0.5866 (0.5771)	0.5936 (0.0356)	0.5935 (0.0534)
	1.0	0.5	0.5824 (0.0083)	0.5680 (4.254)	0.5752 (0.5637)	0.5821 (0.0349)	0.5820 (0.0524)

Among other loss functions, it is evident from table 3 and table 4. That the square error loss function shows smaller Bayes posterior risk under the both assumptions (extended Jeffery's prior and gamma prior). According to decision rule of less Bayes posterior risk, we accomplish that square error loss function is more useful than others.

VIII. Conclusion

In this paper, we have initially obtained the Bayes posterior distribution and estimation of parameter of the inverse Topp-Leone distribution under both informative and non-informative priors. We have discussed different loss functions among them square error loss function provides less Bayes posterior risk. Eventually through simulation analysis and application, the performance of the estimators has been achieved.

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BAYESIAN SPATIAL TEMPORAL TREND ANALYSIS FOR DECISION MAKING AND RISK ASSESSMENT IN DENGUE INCIDENCE STUDIES: A CASE OF TAMILNADU

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Abstract

This study presents a Bayesian spatial-temporal analysis for studying Dengue incidence in Tamil Nadu, aiming to provide insights into decision-making and risk assessment strategies. Statistical models that allow a more accurate depiction of true disease rates by borrowing information from neighboring regions will help mitigate the effects of sparsely populated regions and deliver better inference. Perhaps the most conspicuous manner of modeling spatial dependence is to introduce spatially associated random effects within a Bayesian hierarchical setting. The Bayesian modeling and inferential framework are flexible and extremely rich in its capabilities to accumulate various scientific hypotheses and assumptions. The spatial and spatial temporal epidemiology is concerned with the description and analysis of spatial and spatial temporal variations in disease risk with respect to risk factors. As the primary aim of this work is to quantify the spatial disease pattern of dengue incidences apart from the mapping of disease modelling the disease and finding spatial clusters/hotpots is one important aspect in epidemiology is to find the temporal trends in or outside of clusters. In this study, a spatial-temporal trends model is fitted using the Leroux CAR prior's set up for studying the spatial-temporal disease patterns with the estimation of the temporal trends with reference to dengue incidences in Tamil Nadu, India.

Keywords: Spatial temporal, Bayesian modeling, Bayesian hierarchical modeling, Leroux CAR prior

I. Introduction

In spatial epidemiology, the main interest is to describe the spread of a disease or infection through models that attempt to summarize the spatial and temporal effects. After detecting disease clusters, further analysis about those clusters leads to the finding of the temporal trend of the cluster. The Bayesian methodology is highly useful to study this behaviour which may be better than the classical procedure, because of the fact that the procedure of Bayesian inference combines prior distribution of model parameters and the data likelihood, for deriving the posterior distribution of parameters which portray the behaviour of the parameter in a better manner. The Bayesian hierarchical model that involves time and regional effects yield more information to the problem of study based on the neighbourhood structures of the regions and adjacent times.

But a model developed so would be quite complex in nature. However, the computational procedures based on MCMC methodology are very much useful to approximate the posterior distribution in Bayesian hierarchical models. In this article, a Bayesian model is used to describe the spatial patterns with the estimation of the temporal trends with reference to dengue incidences in Tamil Nadu, India. The spatio-temporal model was proposed to study the spatial and temporal patterns which allows for spatial temporal discontinuities between areas [12] by using this model the study has been made to address the climatic variability of dengue cases in Makassar Indonesia [1]. The rate of cases in neighborhood i and time j, mosquito density data, fixed scaling factors, lagged time for specific variables and different weighting functions between neighborhood effects which consists of economic value of the neighborhood, population density and travel distance between neighborhoods are included. The nearest-neighborhood effects, (local) and all between-neighborhood effects (global) are compared in order to predict the association between mosquito density and human cases of dengue. Models that preferred were contains global between-neighborhood effects and the covariates mosquito density and human cases of dengue and their interaction. In this study we have used the district wise dengue incidences data collected from the Government of Tamil Nadu over the period of 2007 – 2018.

II. The Spatial and Temporal Models

The Bayesian methodology is highly useful to study this behaviour which may be better than the classical procedure, because of the fact that the procedure of Bayesian inference combines prior distribution of model parameters and the data likelihood, for deriving the posterior distribution of parameters which portray the behaviour of the parameter in a better manner. As already stated, a large number of models have been proposed for estimating the Spatio-temporal trends in disease risk and as our disease outcome variable is a count they have the general form,

$$Y_{it} \sim \text{Poisson}(E_{it}R_{it}) \qquad \text{for } i = 1, \dots, N, t = 1, \dots, T \tag{1}$$

$$\text{In}(R_{it}) = \beta_0 + \beta_1 \text{Rainfall}_{it} + \beta_2 \text{Temperature}_{it} + \psi_{it} \tag{2}$$

$$\beta \sim N(\mu_\beta, \Sigma_\beta) \tag{3}$$

where the number of observed disease count is denoted by $Y = (Y_1, \dots, Y_N)_{i \times T}$, where $Y_t = (Y_{1t}, \dots, Y_{it})$ denoted by $i \times 1$ column vector of observed disease count for all regions i for time t, E_{it} is the expected number of disease cases, R_{it} is the relative risk of dengue disease in area i and time t. The vector of covariate regression parameters is denoted by $\beta$and a multivariate Gaussian prior is assumed with mean μ_β and diagonal dispersion matrix Σ_β, ψ_{it} is the random effect for the study region i and time t. Taking $\psi_{it} = \rho_T \psi_{t-1} + \epsilon_t$. The temporal autocorrelation is thus induced through the mean $\rho_T \phi_{t-1}$, while spatial autocorrelation is induced by the variance $\tau^2 Q(W, \rho_s)^{-1}$. The precision matrix is given by,

$$Q(W, \rho_s) = \rho_s(\text{diag}[W1] - W) + (1 - \rho_s)I \tag{4}$$

where $(1, I)$ is a $N \times 1$ vector of ones and the $N \times N$ identity matrix respectively. Hence, the spatial autocorrelation is induced by the neighbourhood matrix W defined above, and if $\omega_{ij} = 1$ then the random errors $(\epsilon_{kt}, \epsilon_{jt})$ are modelled as spatially autocorrelated, while if $\omega_{ij} = 0$ then $(\epsilon_{it}, \epsilon_{jt})$ are assumed to be conditionally independent. Thus ρ_s, ρ_T respectively control the levels of spatial and temporal autocorrelation, with values of 0 corresponding to independence while a value 1

corresponds to string autocorrelation. The precision matrix $Q(W, \rho_s)$ corresponds to the conditional autoregressive (CAR) prior proposed by Leroux [15], given by,

$$\epsilon_{it}|\epsilon_{-it}, W \sim N\left(\frac{\rho_s \sum_{j=1}^{N} \omega_{ij}\epsilon_{jt}}{\rho_s \sum_{j=1}^{N} \omega_{ij}+1-\rho_s}, \frac{\tau^2}{\rho_s \sum_{j=1}^{N} \omega_{ij}+1-\rho_s}\right) \quad (5)$$

With the temporal informative priors $\tau^2 \sim$Inverse Gamma$(1, 0.01)$ $\rho_s, \rho_T \sim$Uniform$(0, 1)$ and $\epsilon_{-it} = (\epsilon_{1t}, \ldots, \epsilon_{i-1t}, \epsilon_{i+1t}, \ldots, \epsilon_{Nt})$. If $\rho_s = 1$ the model simplifies to the intrinsic CAR prior proposed Besag et al.,[3] and if $\rho_s = 0$ the errors ϵ_{kt} are independent and normally distributed with mean zero and a constant variance τ^2.

III. Results

The analysis was performed using the model assumed in the previous section. As the primary aim of this work is to quantify the spatial disease pattern of dengue incidences risk over time the spatially autoregressive model is used. The MCMC samples are generated from the three independent Markov chains and each chain was run for 20,000 samples. To check whether the Markov chains are converged the trace plot of the samples for each parameter are observed and since the samples show no trend in their means or variances, convergence is assured. These trace plots are presented in Figure:1

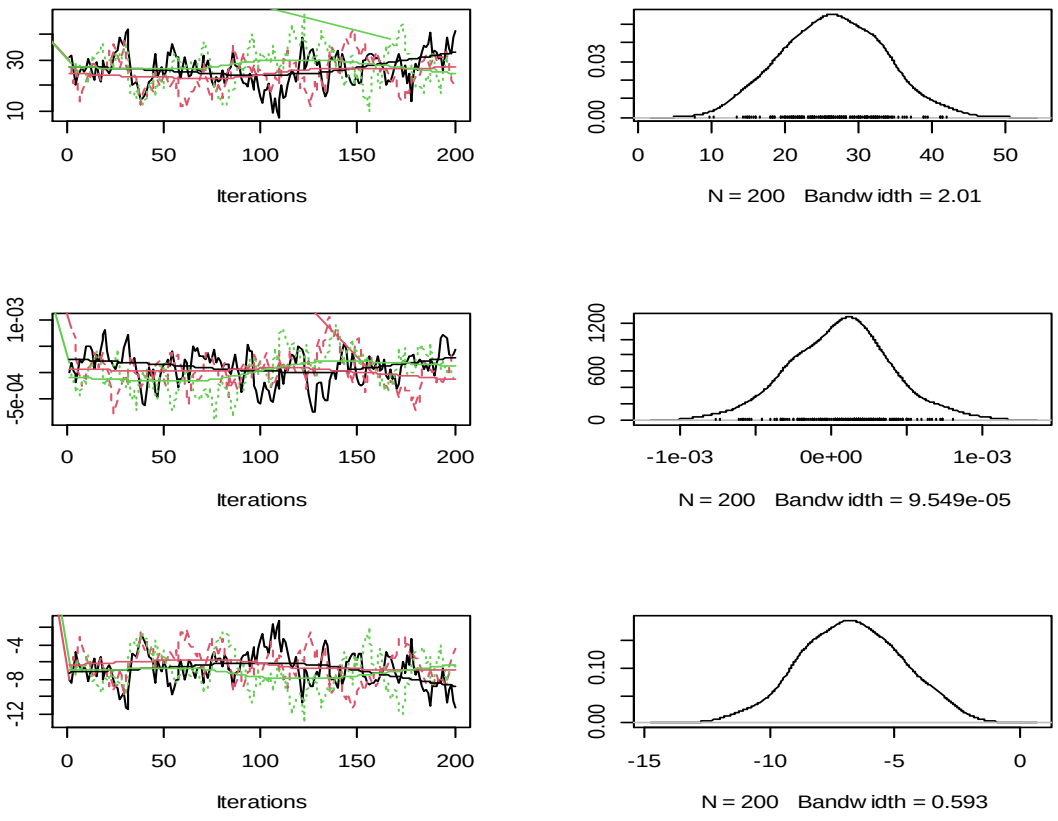

Figure 1: *Trace plots of the MCMC samples from each chain*

The Gelman and Rubin diagnostic [7] is used as an additional check for testing the between to within chain variation reduction in the MCMC samples. It is observed that the samples are well mixed both separately and then jointly as the values of the point estimates are less than 1.1.

Table1: *Gelman-Rubin statistic*

MCMC Chain	Point Estimates	Upper Credible Interval
Chain 1	1.04	1.13
Chain 2	1.07	1.25
Chain 3	1.04	1.14

Table 2: *Posterior Quantities for Selected Parameters and DIC of the Autoregressive CAR Model (Chain 1)*

Model paramters	Median	2.5%	97.5%	n.effective	Geweke.diag
Intercept	26.1415	14.3764	39.2920	22.9	-0.2
Rainfall	0.0001	-0.0006	0.0007	29.3	1.5
Temperature	-6.7187	-10.6279	-3.1630	22.3	0.1
tau2	2.2096	1.8642	2.6795	231.0	-1.5
rho.S	0.9083	0.8233	0.9542	200.0	-2.0
rho.T	0.6794	0.5806	0.7782	91.1	-2.0

DIC=2365.255

The above Table.2 provides the parameter summaries and posterior median point estimates with 95% credible intervals. The Deviance Information Criterion is given at the bottom. It is observed that the covariate temperature has a negative relationship with dengue incidences and all other covariates have a positive relationship. The spatial and temporal parameters ρ_s and ρ_T exhibit the presence of spatial and temporal autocorrelation after adjusting for the effects of covariates. It is observed that the condition for convergence according to Geweke diagnostic [8] is satisfied as the corresponding values lie between -2 to +2.

Table 3: *Posterior Median Relative Risk for Covariates*

Credible Interval	50%	2.5%	97.5%
Temperature	1.0	0.999	1.002
Rainfall	1.004	0.780	1.437

The estimated relative risk of the covariates obtained from MCMC samples for the regression parameters β_1 and β_2 are obtained and given in Table.3 It is seen that the covariate temperature is not significantly related to dengue incidence risk as 95% credible interval consist of the null risk of 1. A similar thing is observed for the variable rainfall.

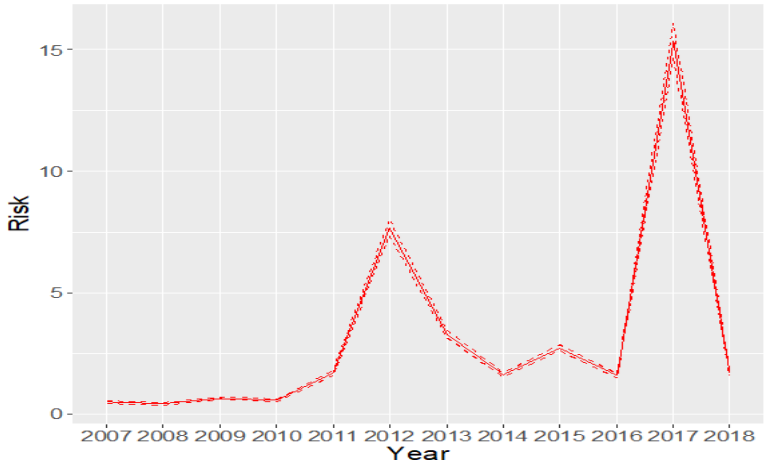

Figure 2: *Posterior Median and 95% Credible Interval for the Temporal Trend in Dengue*

With reference to Spatio-temporal trends in disease risk, the following graph (Figure: 2) has been plotted based on posterior risk distributions, in which the samples of fitted values are divided by the fixed expected number of disease cases.

Table 4: *Posterior Median and 95% credible interval for the Temporal Trend in Dengue Disease Risk.*

Year	Median	Lower Credible Interval	Upper Credible Interval
2007	0.0133	0.0119	0.0147
2008	0.0171	0.0152	0.0192
2009	0.0168	0.0154	0.0184
2010	0.0137	0.0124	0.0152
2011	0.0417	0.0391	0.0445
2012	0.2014	0.193	0.2101
2013	0.1064	0.1015	0.1115
2014	0.0437	0.0412	0.0463
2015	0.0155	0.0719	0.0794
2016	0.0475	0.0448	0.0504
2017	0.5473	0.5252	0.5712
2018	0.0505	0.0477	0.0535

To estimate the average temporal trend, the average risk across the study areas for each year is estimated which yields the posterior distribution of spatial averages for each year. The corresponding posterior median and 95% credible intervals are given in the following Table 4. The estimated temporal trend in disease risk is plotted in the following Figure 3. The figure clearly shows a downward trend in dengue incidences over 12 years study period. The peaks of risk are observed in 2012 and 2017.

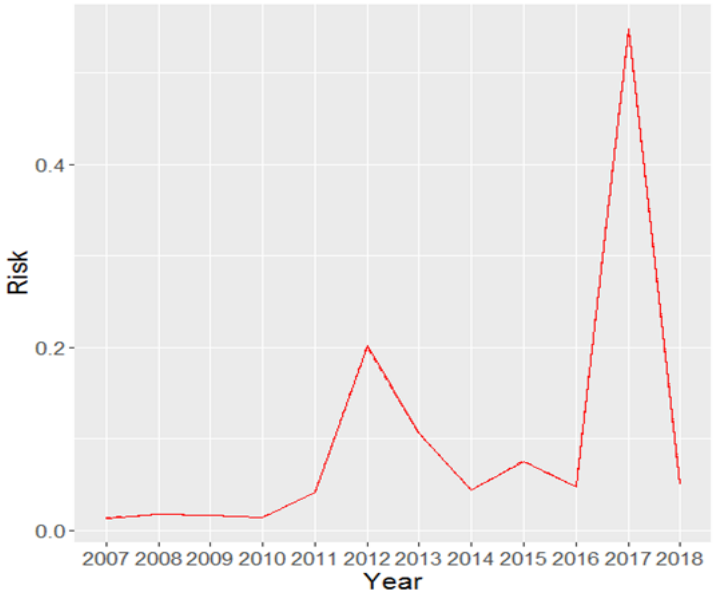

Figure 3: *Estimated temporal trend in Temporal Trend in Dengue incidence risk as measured by The Spatial Interquartile Range*

The measure of variations in disease risk over the study region called "Total Inequality" is measured by the interquartile range (IQR) variation for each year and is given in Table 4. and the corresponding plot is given in Figure 3 The figure clearly shows that the total inequality in Dengue incidences when using the interquartile range has increased over the years till 2017 and decreased in 2018 which suggest that the population is becoming uneven in terms of later years.

Table 4: Inter Quartile Ranges

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
IQR	0.008	0.013	0.009	0.005	0.015	0.142	0.096	0.036	0.078	0.038	0.475	0.038

The spatial pattern is computed in two ways, the one is with reference to the posterior median risk surface and the one is the posterior exceedance probability. The maps of posterior exceedance probability and the median risk have been generated and shown in figures: 4 and 5.

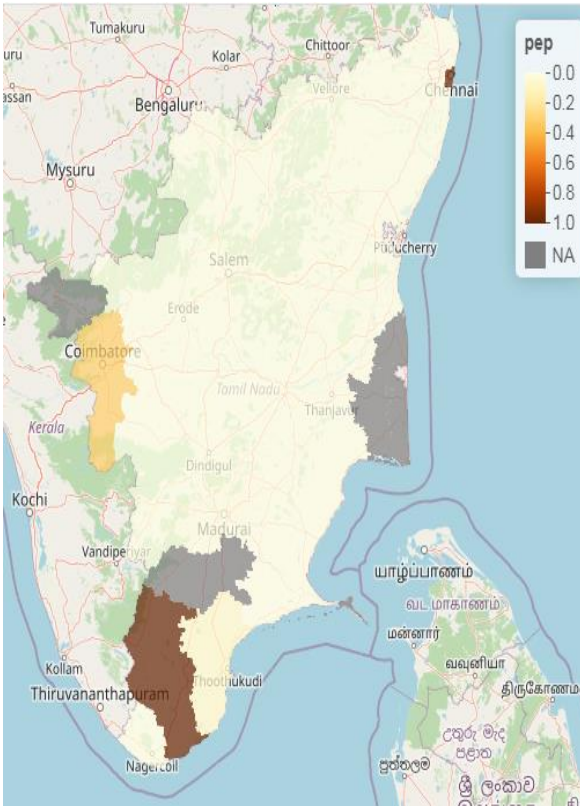

Figure 4: *Estimated (Posterior Median) Risk Surface for 2017*

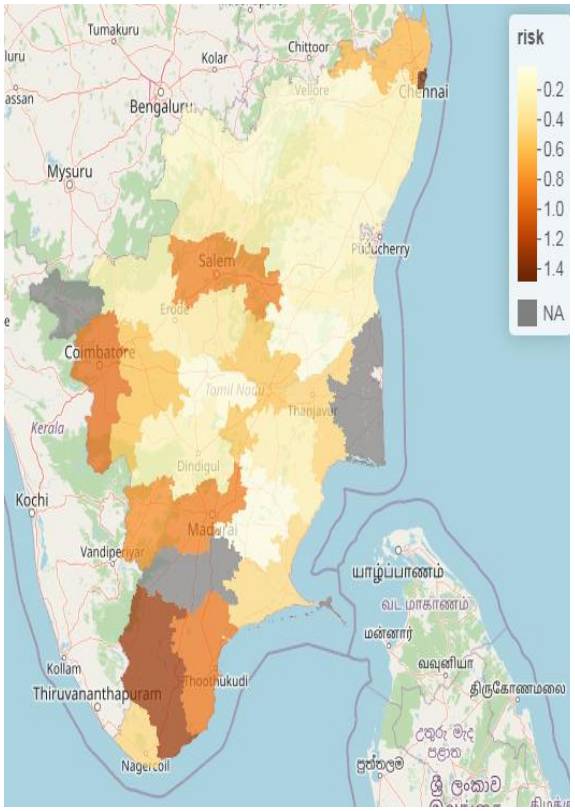

Figure 5: *The Posterior Exceedance Probabilities that the risk in 2017 is greater than 1*

From figure: 5, it is observed that the areas of highest risk are found in Chennai and the next level of higher risk is observed in Tirunelveli. Moderate levels of risk are found in Coimbatore, Thoothukudi, Madurai, Theni, Salem and Thiruvallur. Lower levels of risk are found in Tiruppur, Tiruchirapalli, Thanjavur, Ramanathapuram and Kanyakumari. The Posterior Exceedance Probabilities map shows that the majority of areas have zero probability of exceeding the risk of 1, except the areas Chennai and Tirunelveli.

IV. Discussion

Spatial-temporal disease mapping is a familiar approach for studying disease patterns in an effective manner. In this article, a spatial-temporal trends model is fitted using the Leroux CAR prior's set up for studying the spatial-temporal disease patterns of Dengue incidences are over the 12 years. From the analysis, a downward trend of dengue incidences is observed and the peaks of risk is observed in 2012 and 2017. Areas of highest risk are found in Chennai and the next level of higher risk is observed in Tirunelveli. Moderate levels of risk are found in Coimbatore, Thoothukudi, Madurai, Theni, Salem and Thiruvallur. These results are observed both spatially and temporally. If sufficient preventive measures are taken up by the health authorities on the areas quoted above the disease incidences may get lower and even null.

Conflict of Interest

The Authors declare that there is no conflict of Interest.

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OPTIMIZATION OF RESOURCE ALLOCATION USING INTEGER PROGRAMMING OF IMPROVED RATIO ESTIMATOR UNDER STRATIFIED RANDOM SAMPLING

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Abstract

This paper provides a case study that illustrates how integer programming may be used to optimize resource allocation. With the known population median of the study variable acting as auxiliary data, an exponential ratio estimator is shown for estimating the finite population mean under stratified random sampling. The objective is to minimize a cost function within specific bounds. Using integer programming techniques and the Lagrange multiplier approach, we transform the proposed problem into an optimization problem with a linear cost function. This allows us to propose an optimal way for minimizing total costs while maintaining desired accuracy levels. We found that the suggested estimator performed better than methods involving stratified random sampling. Additionally, a numerical example is given to verify the theoretical conclusions for real-world applications. We go over how the problem was formulated, how to use LINGO software to solve it, and the results. It is advised to choose the estimator with the lowest MSE in real-world stratified random sampling situations. The strategy shows significant cost savings and efficient use of resources. The effectiveness of the recommended approach is demonstrated by testing the methodology on both simulated and real-world datasets.

Keywords: linear cost function, integer programming, optimization, resource allocation, lingo software, cost minimization

1. Introduction

The problem of effectively estimating the mean of a study variable in the presence of auxiliary information using different sample procedures has been attempted several times in the literature on sampling theory. The problem of creating effective estimators has been thoroughly researched by a number of authors. Regression estimators, products, and ratios are common examples. Stratified random sampling is the suggested sample design for collecting data from a variety of populations due to its low cost and high efficiency. Allocating resources optimally is essential for increasing productivity and cutting expenses in operations research and management science. Because stratified random sampling can yield estimates that are more accurate than those obtained from plain random sampling, it is a widely used technique in statistical surveys. In order to

maximize estimate precision within budgetary limits, sample sizes must be distributed among different strata. Conventional methods, like Cochran [1] suggested, make use of continuous optimization techniques, which might not be useful when sample sizes have to be integers. In order to determine the best integer solutions for sample size allocation in stratified random sampling, this work investigates the application of Lagrange multipliers and integer programming. Numerous studies have been conducted on the use of simple random sampling [1, 2, 5].

In order to increase estimate precision, a number of scholars have concentrated on maximizing sample size allocation using auxiliary information [2, 5]. Cochran [4] has discussed a number of sampling strategies, including stratified sampling, systematic sampling, simple random sampling, and others. In the topic of survey sampling, Cochran's work is essential since it offers thorough instructions on various methods. In order to increase the efficiency of population parameter estimation, Bahl and Tuteja [6] presents ratio and product-type exponential estimators. Under some circumstances, the suggested techniques perform better in basic random sampling than conventional estimators. The application of optimization theory to large-scale systems is covered in [3], with a focus on computational and mathematical methods for complex system optimization. Neyman [7] contrasted two techniques: purposive selection, which is a non-probabilistic approach, and stratified sampling, which is a probabilistic approach. In order to guarantee representative samples, author suggested stratified sampling. The optimization problem has been expanded to include linear cost functions in more recent research [8, 10]. By adding integer restrictions to the optimization issue, this work expands on these foundations and offers a more useful solution for real-world scenarios. Shi et al. [9] examines methods based on optimization, fusing theoretical underpinnings with real-world applications. In order to determine the best integer solutions for sample size allocation in stratified random sampling, [10] investigates the application of Lagrange multipliers and integer programming. In stratified sampling, [11] suggest a technique for calculating the interquartile range under a nonlinear cost function. Their method guarantees accurate and economical estimations for all stratified populations. While the method for creating effective stratum borders in stratified sampling while taking survey expenses into consideration is developed in [12]. The technique lowers the overall cost of the survey while improving sampling efficiency. Recently In stratified sampling, the study [14] suggests the best method for determining the population mean under a linear cost function. Comparing the results to current estimators, they show increased cost-effectiveness and accuracy. In [15], a linear cost function is used to present an efficient and cost-effective estimator for the population mean in stratified sampling. Superior efficiency is demonstrated by the approach, which has been confirmed using real-world data. In order to minimize a cost function under predetermined limits, a resource allocation issue is studied using integer programming techniques. We employ LINGO software to determine the best option and show that this strategy works.

2. Material and Methods

The methodology and optimization strategies employed in this work to create and assess an enhanced median based ratio estimator in stratified random sampling under cost functions are described in this part. The integer programming technique and langrage's multiplier technique were used to solve the optimization issue. Furthermore, the suggested estimator's mathematical characteristics, such as its bias and mean squared error (MSE), are calculated and contrasted with those of other estimators.

I. Study Design

- The study variable (Y) and auxiliary variable (X) are used to split the population into four strata. In order to guarantee that the sample sizes are integer values optimized using integer programming and langrage's multiplier technique, a stratified random sampling design is utilized. The suggested optimization method is validated using the real-world dataset, which is derived from census data. Under the restriction of decreasing the overall survey cost while preserving precision, the ideal sample sizes for each stratum are determined.

Four strata are given in the population, one for each research variable (Y) and auxiliary variable (X).

II. Problem Formulation

- The optimization problem is formulated as follows:
- Minimize the objective function:

$$\text{Minimize } \sum_{I=1}^{4} \frac{c_i}{n_i} \tag{1}$$

Subject to the constraints:

$$c_1 = 2,\ c_2 = 3,\ c_3 = 4,\ c_4 = 5$$
$$c_0 = 500$$
$$2 \leq n_h \leq N_h$$
$$h = 1, 2, 3, 4.$$

The suggested approach was used to ascertain the ideal sample sizes using actual data from [https://censusindia.gov.in/census.website/data/census-tables]. The findings suggest that when compared to conventional techniques, the integer programming and langrage's methodology produces a more economical use of resources.

3. Solution Techniques

In this instance, a real population from the literature [13] is used to compare the effectiveness of the suggested median-based estimator by [13] with existing estimators. The number of households and the square kilometers of villages and cities, which provide information on study variables and auxiliary variables, respectively, are significant features.

The Neyman allocation is then used to divide the population into four non-crossover strata, and a numerical depiction is finished.

$$n_h = n \frac{N_h S_h}{\sum_{h=1}^{k} N_h S_h}$$

where $i = 1, 2, p$.

Table 1: *Data statistics (source: [13])*

Population $(N = 645; h = 4)$											
H	N_h	n_h	$\bar{Y}_h$	M_h	C^2_{yh}	C_{ymh}	C^2_{mh}	S_{yh}	S_{ymh}	λ_h	θ_h
1	237	4.13025	116.236	116.81	0.31485	0.20065	0.14554	65.2218	2724.33	0.2379	1.37869
2	164	5.78153	307.603	292.295	0.18397	0.14238	0.30406	131.936	12801	0.16687	0.46825
3	90	16.8718	547.444	548.77	1.64244	2.49501	3.84895	701.592	749552	0.04816	0.64823
4	154	68.2164	757.1	727.165	4.79469	6.20317	8.78042	1657.81	3415068	0.00817	0.70648

Table 2: *MSE values of different estimators*

Estimators	MSE
$\mu_{0(st)}$ Stratified	50064.21813
$\mu_{p(st)}$ Bahl and Tuteja 1991	298413.7926
$\mu_{pe(st)}$ Bahl and Tuteja 1991	156446.8056
$\mu_{1(st)}$ Kadilar and Cingi 2004	73914.17572
$\mu_{2(st)}$ Kadilar and Cingi 2004	73610.17851
$\mu_{3(st)}$ Kadilar and Cingi 2004	73581.24647
$\mu_{4(st)}$ Kadilar and Cingi 2004	73764.64679
$\mu_{5(st)}$ Kadilar and Cingi 2004	73585.78191
$\mu_{6(st)}$ Kadilar and Cingi 2004	73794.12042
$\mu_{7(st)}$ Kadilar and Cingi 2004	73599.63542
$\mu_{8(st)}$ Kadilar and Cingi 2004	73841.96289
$\mu_{9(st)}$ Kadilar and Cingi 2004	73572.84621
$\mu_{10(st)}$ Kadilar and Cingi 2004	73824.07952
$\mu_{11(st)}$ Kadilar and Cingi 2004	73291.58656
$\mu_{12(st)}$ Kadilar and Cingi 2004	46271.34602
$\mu_{subr(st)}$ Subramani 2016	17357.5585
$\mu_{CR(st)}$ Cochran estimator 1940	8660.837079
$\mu_{**(st)}$ Yadav 2019	6020.730985
$\mu_{prop(st)}$ Estimator	4267.075487

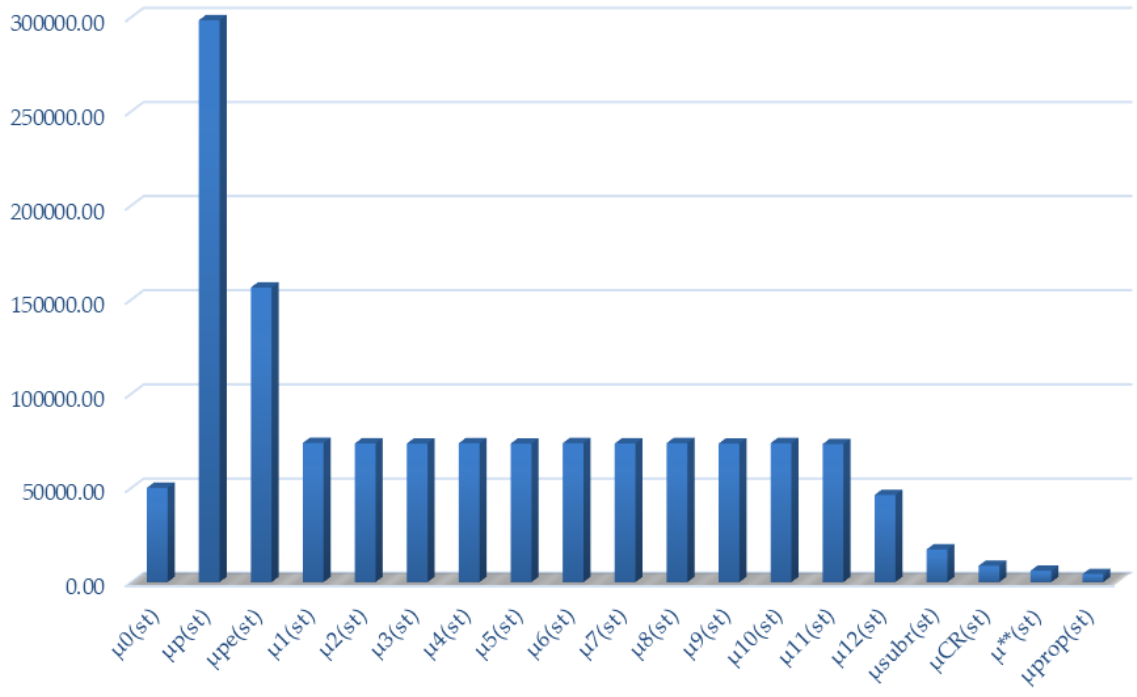

Figure 1: *Standard MSE*

Table 3: *PRE of different estimators*

Estimators	PRE
$\mu_{0(st)}$ Stratified	100
$\mu_{CR(st)}$ Cochran estimator 1940	578.0529
$\mu_{p(st)}$ Bahl and Tuteja 1991	16.77678
$\mu_{pe(st)}$ Bahl and Tuteja 1991	32.00079
$\mu_{1(st)}$ Kadilar and Cingi 2004	67.7329
$\mu_{2(st)}$ Kadilar and Cingi 2004	68.01263
$\mu_{3(st)}$ Kadilar and Cingi 2004	68.03937
$\mu_{4(st)}$ Kadilar and Cingi 2004	67.87021
$\mu_{5(st)}$ Kadilar and Cingi 2004	68.03518
$\mu_{6(st)}$ Kadilar and Cingi 2004	67.8431
$\mu_{7(st)}$ Kadilar and Cingi 2004	68.02237
$\mu_{8(st)}$ Kadilar and Cingi 2004	67.79914
$\mu_{9(st)}$ Kadilar and Cingi 2004	68.04714
$\mu_{10(st)}$ Kadilar and Cingi 2004	67.81557
$\mu_{11(st)}$ Kadilar and Cingi 2004	68.30827
$\mu_{12(st)}$ Kadilar and Cingi 2004	108.197
$\mu_{subr(st)}$ Subramani 2016	288.4289
$\mu_{**(st)}$ Yadav 2019	831.5306
$\mu_{prop(st)}$ Estimator	1173.268

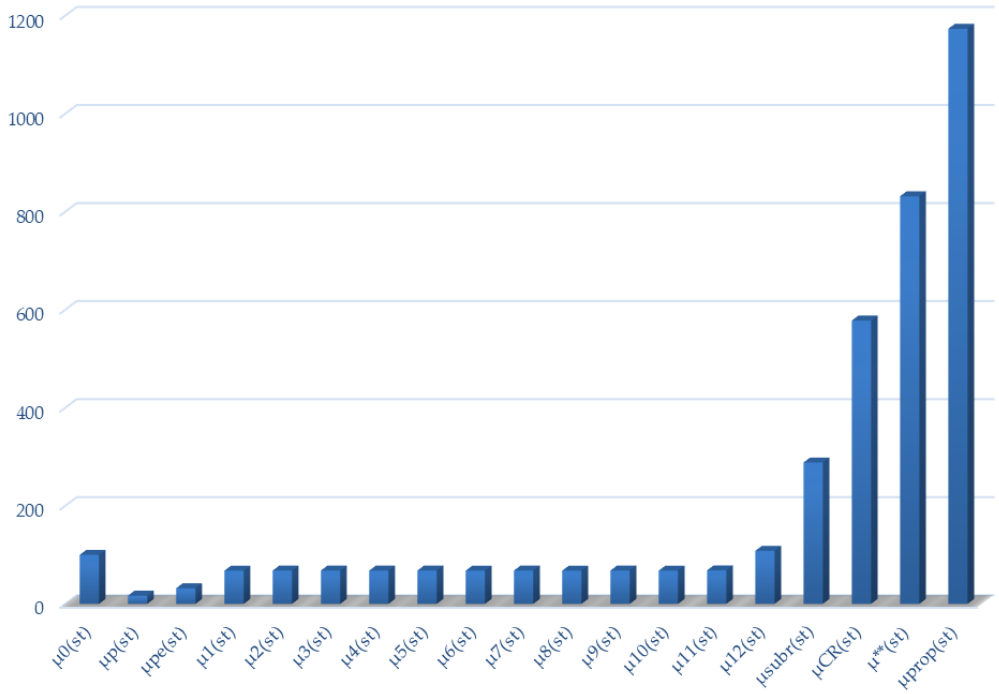

Figure 2: *Standard PRE*

The comparison of the proposed estimator with existing estimators utilizing stratified random sampling, Tables 2 and 3 unequivocally demonstrate that the proposed estimator has the greatest PRE and the lowest MSE value and their graphs were also given as Figure 1 and 2.

4. Cost Function

The main factor that influences of the number of samples across strata is survey expenditure. [8] introduced linear cost and fixed total cost C_0 of the survey as a linear function of n_h; $h = 1, 2, \ldots, L$.

$$C_0 = \sum_{h=1}^{L} c_h n_h \tag{2}$$

where c_h denotes the cost per unit of measuring each characteristic in the hth stratum.; $h = 1, 2, \ldots, L$. In this instance, our goal is to determine the fixed linear cost function's least mean square error. Thus, the optimization issue for the proposed estimator in [9] may be described as follows:

$$\begin{aligned} &\text{Minimize } MSE(t_{pr(st)}) \\ &\text{subject to } \sum_{h=1}^{L} c_h n_h \leq C_0 \\ &\qquad\qquad 2 \leq n_h \leq N_h \\ &\text{and} \qquad\quad n_h \text{ are integers; } \ h = 1, 2, \ldots, L. \end{aligned}$$

Using the cost function, the mean square error will now be

$$\widehat{\mu}_{P(st)_{\min}} = \sum_{h=1}^{L} \bar{Y}_h^2 \left(\frac{1 - f_h}{n_h} \right) [C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}]. \tag{3}$$

Integer Programing and Lagrange's Multiplier Technique

Integer Programing:

With a constant linear cost function and actual data, we get the least mean square error. This allows the optimization issue to be stated as follows:

$$\begin{aligned} &\text{Minimize } \frac{507.3364037}{n_1} + \frac{10707.94895}{n_2} + \frac{6113.182684}{n_3} + \frac{131645.0146}{n_4} \\ &\textit{Subject to} \\ &\qquad \sum_{h=1}^{L} c_h n_h \leq C_0 \\ &\qquad c_1 n_1 + c_2 n_2 + c_3 n_3 + c_4 n_4 \leq C_0 \\ &\qquad 2n_1 + 3n_2 + 4n_3 + 5n_4 \leq 500 \end{aligned}$$

Bounds on variables:

$$\begin{aligned} &2 \leq n_h \leq N_h \\ &\textit{and } n_h \textit{ are integers; } \ h = 1, 2, 3, 4 \\ &2 \leq n_1 \leq 237, \ \ 2 \leq n_2 \leq 164 \\ &2 \leq n_3 \leq 90, \ \ 2 \leq n_4 \leq 154 \end{aligned}$$

The Lagrange multiplier method produces an optimality criterion in some applications. Additionally, the conditions are suitable to set a minimum or maximum. Therefore, the most optimal n value may be found using the Lagrange multiplier method.

The Lagrange function is so defined as:

$$L(x,\lambda) = f(x) - \lambda g(x),$$

where L = Lagrangian, λ = Lagrange multiplier, $f(x)$ = Function, x = integer.
Now

$$L(n_h,\lambda) = MSE + \lambda\left(\sum_{h=1}^{L} C_h n_h - C_0\right)$$

$$L = \sum_{h=1}^{L} \bar{Y}_h^2 \left(\frac{1-f_h}{n_h}\right)[C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}] + \lambda\left(\sum_{h=1}^{L} C_h n_h - C_0\right). \tag{4}$$

Now let us partially differentiate the above equation (4) with respect to n_h, we get

$$\frac{dL}{dn_h} = 0$$

$$\frac{d\left(\sum_{h=1}^{L} \bar{Y}_h^2 \left(\frac{1-f_h}{n_h}\right)[C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}] + \lambda\left(\sum_{h=1}^{L} C_h n_h - C_0\right)\right)}{dn_h} = 0$$

Then

$$n_h = \sqrt{\frac{\bar{Y}_h^2(1-f_h)(C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}})}{\lambda C_h}}.$$

Again, differentiate the equation (4) with respect to λ, we get

$$\frac{dL}{d\lambda} = 0$$

$$\frac{d\left(\sum_{h=1}^{L} \bar{Y}_h^2 \left(\frac{1-f_h}{n_h}\right)[C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}] + \lambda\left(\sum_{h=1}^{L} C_h n_h - C_0\right)\right)}{d\lambda} = 0$$

Using the value of equation (4) after differentiating above equation, we get

$$\sqrt{\lambda} = \frac{\sqrt{\bar{Y}_h^2(1-f_h)(C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}})C_h}}{C_0}. \tag{5}$$

Now putting the value of equation (5) in equation (4) to find out the value of n_h, we get

$$n_h = \frac{C_0\sqrt{\bar{Y}_h^2(1-f_h)(C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}})}}{\sqrt{(\bar{Y}_h^2(1-f_h)(C_{y_h}^2 + {\theta_h}^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}))C_h^2}}$$

$$n_h = \frac{C_0}{C_h}.$$

5. Empirical Study with Cost Function

In this part, we prove the efficiency of the proposed estimator using the real data set. The actual population as reported by the Indian census conducted in Lucknow, Uttar Pradesh, is taken into account in the data set (`https://censusindia.gov.in/census.website/data/census-tables`). The data $N = 645$, $h = 4$, which were used to apply the recommended estimator, contain information on the number of households and the area in square kilometers of certain cities and villages, respectively. These details provide information on the auxiliary variable and the variable under investigation. The population is then split up into four distinct, non-overlapping strata. Integer programming and Lagrange multiplier approaches have been

used in numerical illustration. A reference to the data summary may be found in Table 1. When variables in an optimization problem have to handle integer values, the problem is known as integer programming. If all of the functions are linear, then an integer linear programming problem can be considered. Now, using real data and a fixed linear cost function, we can calculate the least mean square error. Next, the following is a description of the optimization scenario:

Problem Formulation of Proposed Estimator
Objective Function

$$\widehat{\mu}_{P(st)_{\min}} = \sum_{h=1}^{k} \bar{Y}_h^2 \delta_h [C_{y_h}^2 + {\theta_h}^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}]$$

Limited population factor will be ignored,

$$\widehat{\mu}_{P(st)_{\min}} = \sum_{h=1}^{k} \bar{Y}_h^2 \frac{1}{n_h} [C_{y_h}^2 + \theta_h^2 C_{mh}^2 - 2\theta_h C_{y_{mh}}].$$

The objective is to minimize the cost function defined as:

$$\text{Minimize } \frac{507.3364037}{n_1} + \frac{10707.94895}{n_2} + \frac{6113.182684}{n_3} + \frac{131645.0146}{n_4}$$

Subject to

$$\sum_{h=1}^{L} c_h n_h \leq C_0$$
$$c_1 n_1 + c_2 n_2 + c_3 n_3 + c_4 n_4 \leq C_0$$
$$2n_1 + 3n_2 + 4n_3 + 5n_4 \leq 500$$

Bounds on variables:

$$2 \leq n_h \leq N_h$$
$$\textit{and } n_h \textit{ are integers; } h = 1, 2, 3, 4$$
$$2 \leq n_1 \leq 237, \;\; 2 \leq n_2 \leq 164$$
$$2 \leq n_3 \leq 90, \;\; 2 \leq n_4 \leq 154$$

We apply integer programming techniques along with the Lagrange multiplier approach to solve this optimization issue. To determine the best integer values for the sample sizes, the LINGO program is used. Integer variables are used in the model formulation to represent resource allocations, together with an objective function to minimize costs and restrictions to guarantee workable solutions. The variables' ideal values were determined to be n_1, n_2, n_3, and n_4.

These numbers show effective resource allocation by minimizing the cost function while meeting all restrictions.

Table 4: *Optimized MSE and PRE of different estimators using integer programming*

Population $(N,h) = (645,4)$							
Estimators	n_1	n_2	n_3	n_4	n	*MSE*	*PRE*
$\mu_{0(st)}$ Stratified	5	9	37	63	114	37811.71301	100
$\mu_{p(st)}$ Bahl and Tuteja 1991	4	6	26	74	110	254647.4931	14.84864922
$\mu_{pe(st)}$ Bahl and Tuteja 1991	5	6	28	72	111	132124.2009	28.6183097
$\mu_{1(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62734.46491	60.27263173
$\mu_{2(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62470.09829	60.52769892
$\mu_{3(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62444.95167	60.55207345
$\mu_{4(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62606.61364	60.3957167
$\mu_{5(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62439.33502	60.55752034
$\mu_{6(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62632.11269	60.37112814
$\mu_{7(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62460.7517	60.53675624
$\mu_{8(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62675.90102	60.32895002
$\mu_{9(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62437.71155	60.55909492
$\mu_{11(st)}$ Kadilar and Cingi 2004	4	6	26	74	110	62233.04837	60.75825305
$\mu_{12(st)}$ Kadilar and Cingi 2004	4	5	28	73	110	38419.51093	98.41799671
$\mu_{subr(st)}$ Subramani 2016	4	16	36	60	116	12234.51456	309.0577304
$\mu_{CR(st)}$ Cochran estimator 1940	2	3	18	83	106	7361.991722	513.6071112
$\mu_{**(st)}$ Yadav19	5	20	15	74	114	4412.062967	857.0075561
$\mu_{prop(st)}$ Estimator	7	26	17	68	118	2779.875899	1360.194282

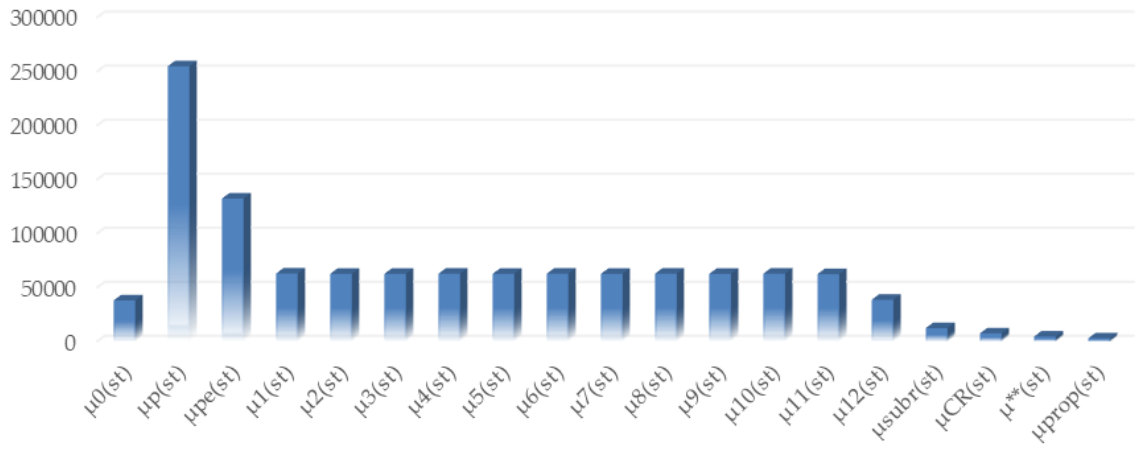

Figure 3: *Optimized MSE integer programming*

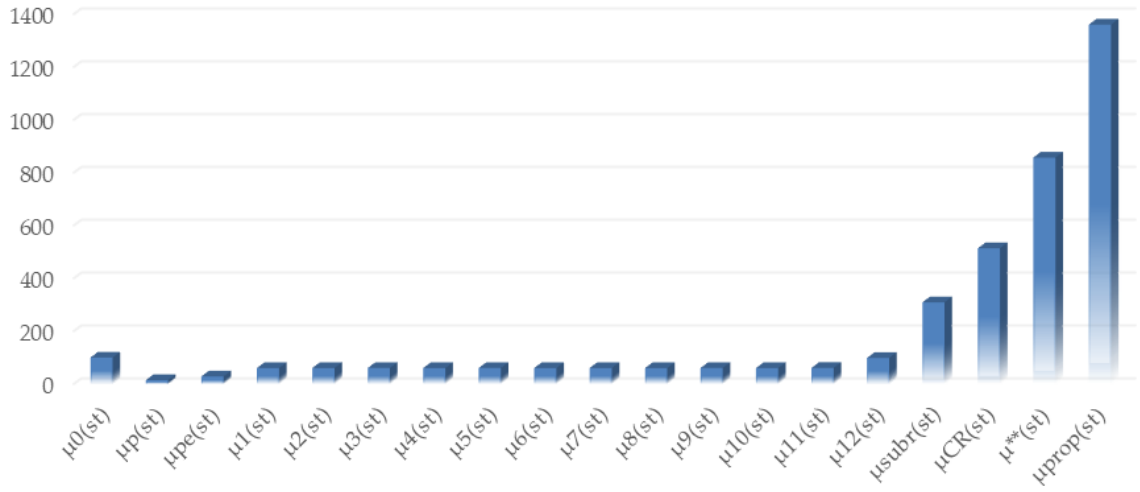

Figure 4: *Optimized PRE integer programming*

In Table 4 the optimized MSE and PRE using integer programing technique is give along with their graphs in Figure 3 and 4.

Table 5: *MSE and PRE comparison of different estimators (standard vs integer)*

Estimators	*MSE*	*Optimized MSE*	*PRE*	*Optimized PRE*
$\mu_{0(st)}$ Stratified	50064.21813	37811.71301	100	100
$\mu_{p(st)}$ Bahl and Tuteja 1991	298413.7926	254647.4931	16.77678	14.84864922
$\mu_{pe(st)}$ Bahl and Tuteja 1991	156446.8056	132124.2009	32.00079	28.6183097
$\mu_{1(st)}$ Kadilar and Cingi 2004	73914.17572	62734.46491	67.7329	60.27263173
$\mu_{2(st)}$ Kadilar and Cingi 2004	73610.17851	62470.09829	68.01263	60.52769892
$\mu_{3(st)}$ Kadilar and Cingi 2004	73581.24647	62444.95167	68.03937	60.55207345
$\mu_{4(st)}$ Kadilar and Cingi 2004	73764.64679	62606.61364	67.87021	60.3957167
$\mu_{5(st)}$ Kadilar and Cingi 2004	73585.78191	62439.33502	68.03518	60.55752034
$\mu_{6(st)}$ Kadilar and Cingi 2004	73794.12042	62632.11269	67.8431	60.37112814
$\mu_{7(st)}$ Kadilar and Cingi 2004	73599.63542	62460.7517	68.02237	60.53675624
$\mu_{8(st)}$ Kadilar and Cingi 2004	73841.96289	62675.90102	67.79914	60.32895002
$\mu_{9(st)}$ Kadilar and Cingi 2004	73572.84621	62437.71155	68.04714	60.55909492
$\mu_{10(st)}$ Kadilar and Cingi 2004	73824.07952	62653.90433	67.81557	60.35013047
$\mu_{11(st)}$ Kadilar and Cingi 2004	73291.58656	62233.04837	68.30827	60.75825305
$\mu_{12(st)}$ Kadilar and Cingi 2004	46271.34602	38419.51093	108.197	98.41799671
$\mu_{subr(st)}$ Subramani 2016	17357.5585	12234.51456	288.4289	309.0577304
$\mu_{CR(st)}$ Cochran estimator 1940	8660.837079	7361.991722	578.0529	513.6071112
$\mu_{**(st)}$ Yadav19	6020.730985	4412.062967	831.5306	857.0075561
$\mu_{prop(st)}$ Estimator	4267.075487	2779.875899	1173.268	1360.194282

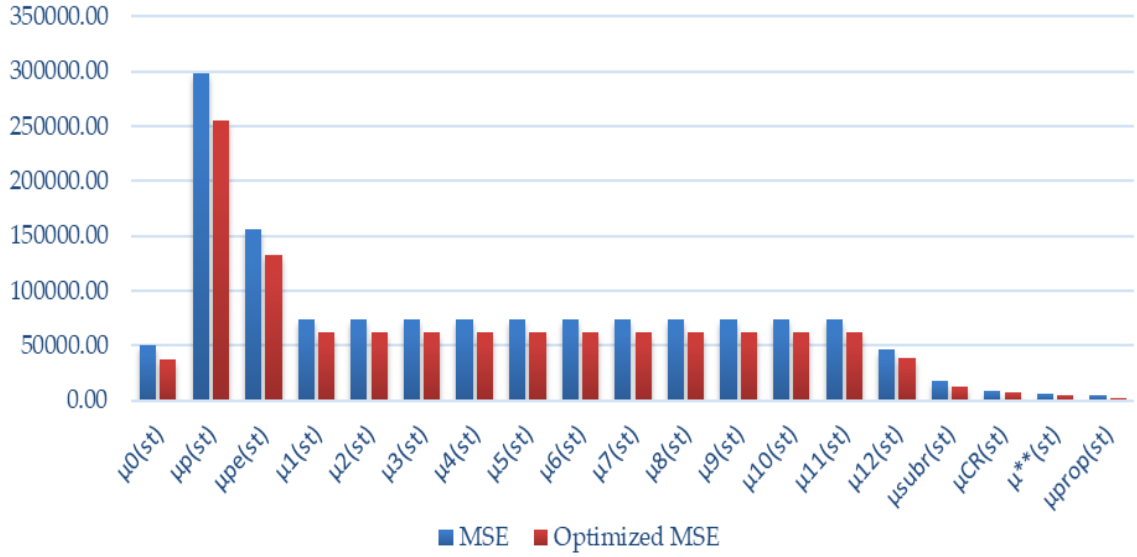

Figure 5: *MSE Comparison (standard vs integer)*

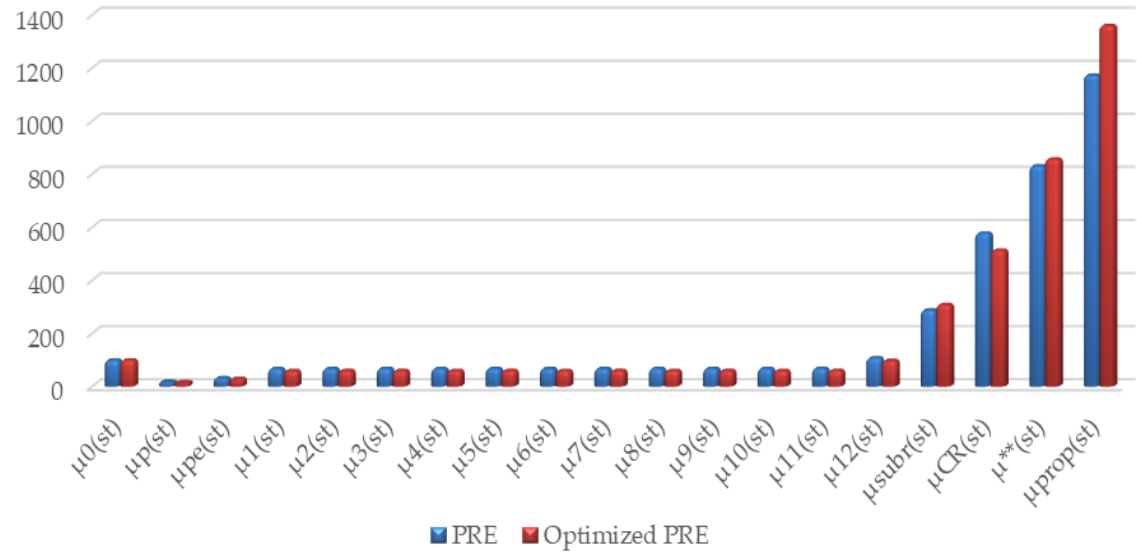

Figure 6: *PRE Comparison (standard vs integer)*

In Table 5 the comparison of MSE and PRE of existing estimator and proposed estimator using integer programming technique is given with their graphs as Figure 5 and 6.

Table 6: *Optimized MSE and PRE of different estimators using lagrange's multiplier technique*

Population $(N,h) = (645,4)$							
Estimators	n_1	n_2	n_3	n_4	n	*MSE*	*PRE*
$\mu_{0(st)}$ Stratified	4	61	24	43	131	37800.05	100
$\mu_{p(st)}$ Bahl and Tuteja 1991	4	6	25	74	110	254555.8	14.8
$\mu_{pe(st)}$ Bahl and Tuteja 1991	4	6	27	73	111	132051	28.6
$\mu_{1(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.7	110	62716.0	60.3
$\mu_{2(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.7	110	62452.1	60.5
$\mu_{3(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.8	110	62427.6	60.6
$\mu_{4(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.7	110	62588.4	60.4
$\mu_{5(st)}$ Kadilar and Cingi 2004	4.3	5.5	26.3	73.9	110	62418.1	60.6
$\mu_{6(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.8	110	62614.9	60.4
$\mu_{7(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.7	110	62442.7	60.5
$\mu_{8(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.7	110	62658.1	60.3
$\mu_{9(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.8	110	62420.4	60.6
$\mu_{10(st)}$ Kadilar and Cingi 2004	4.4	5.7	26.3	73.8	110	62635.5	60.3
$\mu_{11(st)}$ Kadilar and Cingi 2004	4.3	5.7	26.2	73.9	110	62220.1	60.8
$\mu_{12(st)}$ Kadilar and Cingi 2004	3.9	5.7	29.2	71.6	110	38373.7	98.5
$\mu_{subr(st)}$ Subramani 2016	4.0	16.6	35.1	60.4	116	12230.5	309.1
$\mu_{CR(st)}$ Cochran estimator 1940	2.0	3.4	18.0	82.8	106	7360.1	513.6
$\mu_{**(st)}$ Yadav19	5.4	20.5	14.6	73.9	114	4410.8	857.0
$\mu_{prop(st)}$ Estimator	6.8	25.3	16.6	68.8	118	2778.996	1360.2

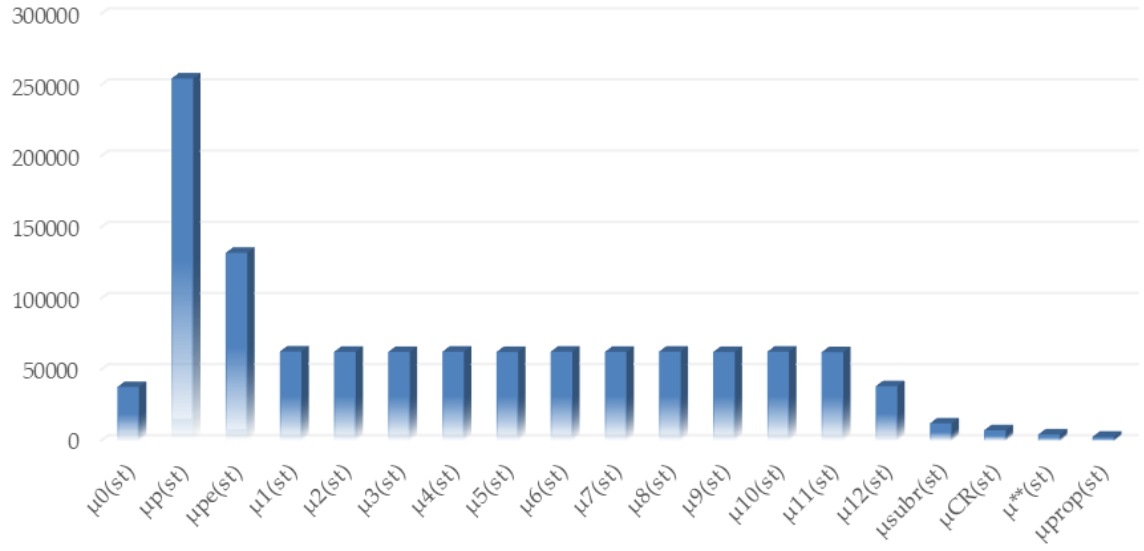

Figure 7: *Optimized MSE lagrange's multiplier*

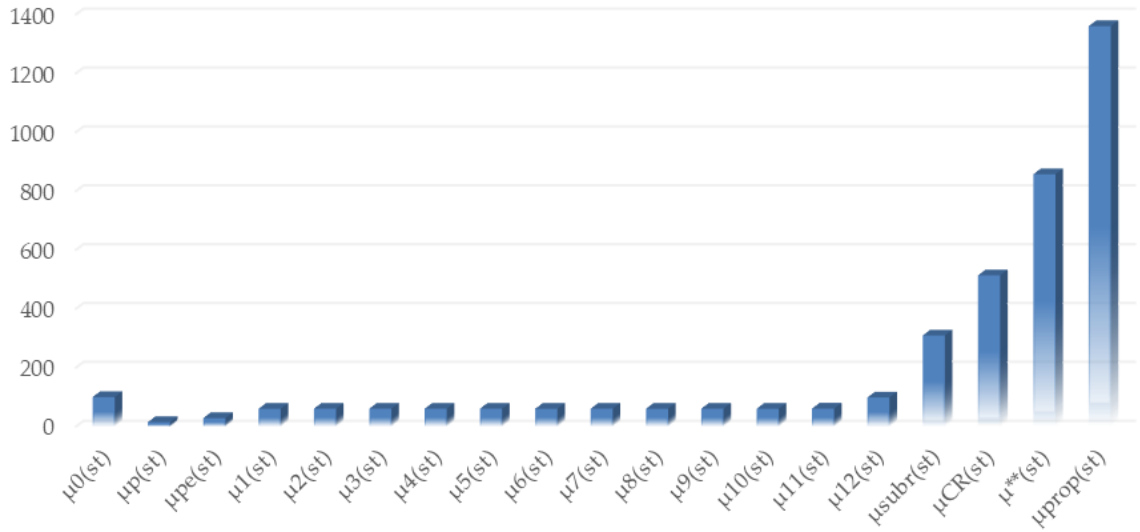

Figure 8: *Optimized PRE lagrange's multiplier*

Similarly in Table 6 the optimized MSE and PRE using language's multiplier technique is give along with their graphs as Figure 7 and 8.

Table 7: *MSE and PRE comparison of different estimators (standard vs lagrange's)*

Estimators	*MSE*	*Optimized MSE*	*PRE*	*Optimized PRE*
$\mu_{0(st)}$ Stratified	50064.21813	37800.05	100	100
$\mu_{p(st)}$ Bahl and Tuteja 1991	298413.7926	254555.8	16.8	14.8
$\mu_{pe(st)}$ Bahl and Tuteja 1991	156446.8056	132050.8	32.0	28.6
$\mu_{1(st)}$ Kadilar and Cingi 2004	73914.17572	62715.97	67.7	60.3
$\mu_{2(st)}$ Kadilar and Cingi 2004	73610.17851	62452.11	68.0	60.5
$\mu_{3(st)}$ Kadilar and Cingi 2004	73581.24647	62427.61	68.0	60.6
$\mu_{4(st)}$ Kadilar and Cingi 2004	73764.64679	62588.4	67.9	60.4
$\mu_{5(st)}$ Kadilar and Cingi 2004	73585.78191	62418.13	68.0	60.6
$\mu_{6(st)}$ Kadilar and Cingi 2004	73794.12042	62614.93	67.8	60.4
$\mu_{7(st)}$ Kadilar and Cingi 2004	73599.63542	62442.74	68.0	60.5
$\mu_{8(st)}$ Kadilar and Cingi 2004	73841.96289	62658.06	67.8	60.3
$\mu_{9(st)}$ Kadilar and Cingi 2004	73572.84621	62420.37	68.0	60.6
$\mu_{10(st)}$ Kadilar and Cingi 2004	73824.07952	62635.47	67.8	60.3
$\mu_{11(st)}$ Kadilar and Cingi 2004	73291.58656	62220.05	68.3	60.8
$\mu_{12(st)}$ Kadilar and Cingi 2004	46271.34602	38373.67	108.2	98.5
$\mu_{subr(st)}$ Subramani 2016	17357.5585	12230.45	288.4	309.1
$\mu_{CR(st)}$ Cochran estimator 1940	8660.837079	7360.138	578.1	513.6
$\mu_{**(st)}$ Yadav19	6020.730985	4410.764	831.5	857.0
$\mu_{prop(st)}$ Estimator	4267.075487	2778.996	1173.3	1360.2

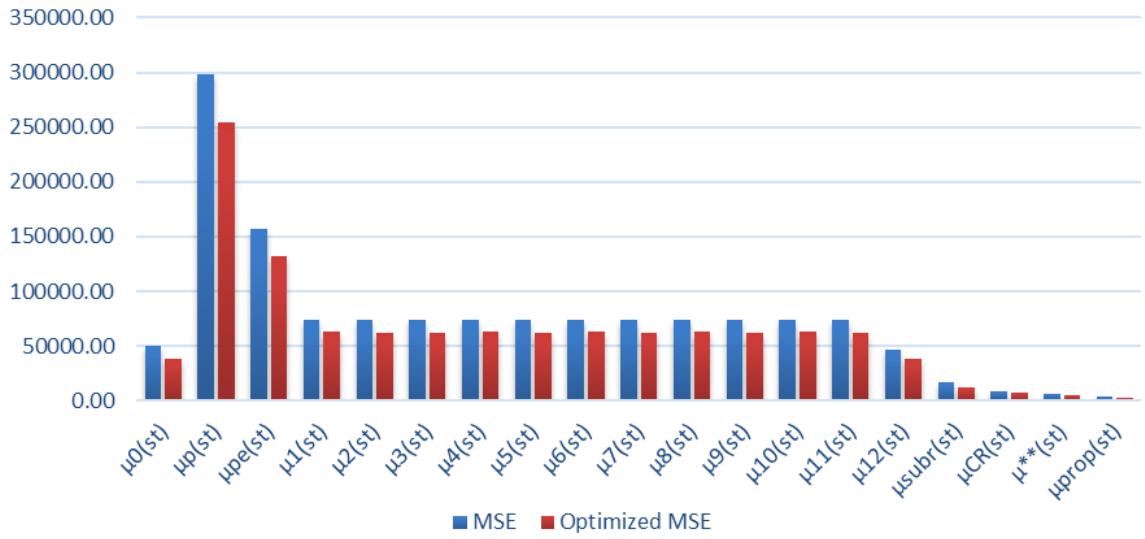

Figure 9: *MSE Comparison (standard vs lagrange's)*

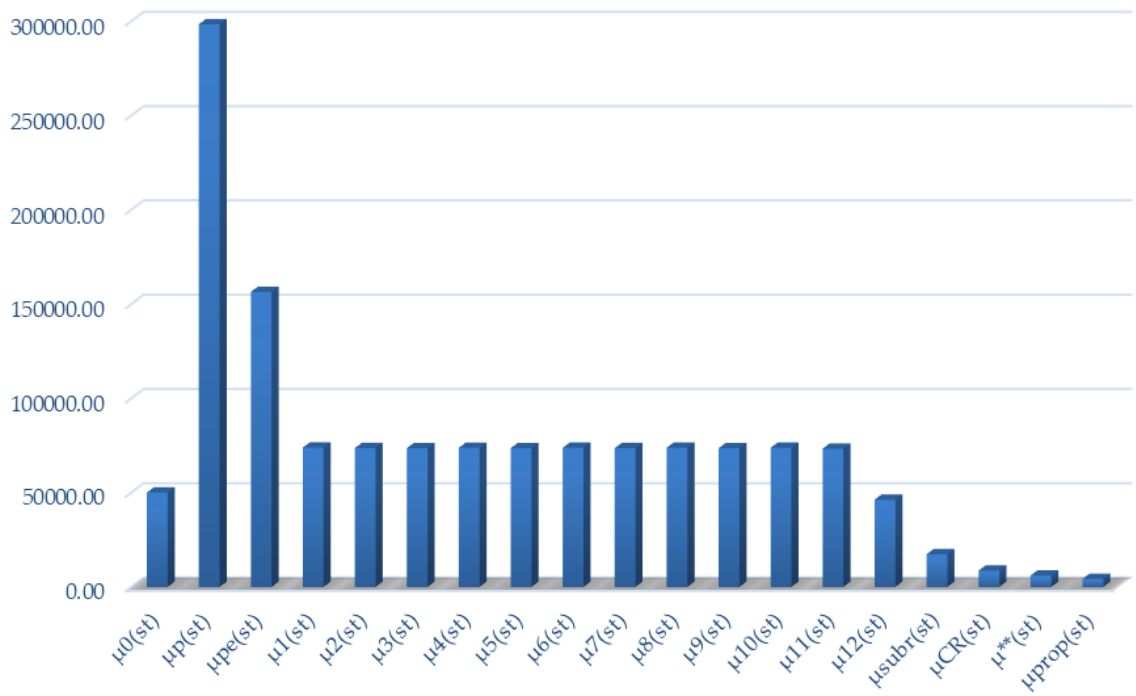

Figure 10: *PRE Comparison (standard vs lagrange's)*

In Table 7 the comparison of MSE and PRE of existing estimator and proposed estimator using langrage's multiplier technique is given with their graphs as Figure 9 and 10.

6. Discussion and Conclusion

In this study, we optimized a new median-based ratio estimator for restricted population means estimation under stratified random sampling. Up to the first level of approximation, bias and MSE formulas are created for the suggested estimators. The suggested estimator was compared theoretically to existing estimators. We determined the conditions in which the suggested

estimator performs better than the traditional estimators. We compare the performance of the proposed estimator quantitatively, considering a real population. The suggested estimator consistently performs better than the existing estimators under stratified random sampling with cost function, both theoretically and numerically. Considering these results, we advise future research to employ the proposed estimator for effective population mean estimation when supplementary data is available. The results indicate a significant reduction in costs through optimal resource allocation. The integer programming and langrage's approach ensures that solutions are both feasible and practical. This methodology can be applied to similar problems in various industries for improved operational efficiency. The problem was successfully solved with the help of LINGO software, which offered a workable solution with either minimizing cost or maximizing precision. To further improve resource allocation tactics, future study might investigate more intricate models and other optimization methodologies.

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RELIABILITY ANALYSIS OF C-SECTION WHERE STRENGTH AND HEAR STRESS ARE NORMALLY DISTRIBUTED

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Abstract

The failure of a component depends on many parameters, such as complexity, time, design, reliability of components, and operating conditions. If failure depends on the stress of a component, such reliability models are called stress dependent models. There are many types of stresses that occur in the body, like tensile, compressive, shear, and bending. Shear stress develops in a body when a pair of opposite forces act across the section tangentially. In structural design, the choice of section shapes for different components is crucial for efficiency, strength, and stability. That's why C –sections are used as purlins. C-sections have a shape that allows for effective load distribution. In this paper, reliability analysis has been conducted over the C-section by applying load and finding the shear stress in the flange and web of C-section. It is observed from the computations that reliability decreases as the load and overall depth of the section increase. Reliability increases as the thickness and width of the web increase.

Keywords: Reliability, Normal distribution, Confidence interval, Shear stress, Hazard rate function.

I. Introduction

Reliability is the ability of the structure to meet the construction requirements set under specified conditions during the service life, according which it is designed. It refers to the capacity, serviceability and durability of construction and according to them different techniques of reliability can be defined.Many researchers have worked on stress-dependent models. Hong and Zhou [1] evaluated the reliability of RC columns. Val [2] studied the deterioration of strength of RC beams due to corrosion and its influence on beam reliability. Abubakar and Edrche [3] analyzed the reliability of simply supported steel beams using FORM. Breccolotti and Materazzi [4] described the structural reliability of eccentrically loaded sections in RC columns constructed by

recycled aggregate concrete. Hariprasadetal.[5] analyzed the reliability for symmetrical columns with eccentric loading from a Lindley distribution. Satyanarayana et al[6] analyzed the reliability over I-Section of Beam due to Uniform Distribution of Load. Yakoob Pasha et al[7] described the reliability comparison of the shafts when shear stress follow the different distributions. Yakoob Pasha et al[8] analysed the reliability values using normal distribution. Wisconsin Electric Power Company [9] discussed the confidence intervals on operating basis earth quake and safe shut down earth quake.

II. Methodology

Reliability function is defined as the probability of success for the intended time t. $R(x) = 1 - F(x) = P(T > x)$ where T is a random variable denoting the failure time and X is random variable.The Hazard function $h(x)$is defined as the limit of the failure rate as the interval approaches zero. Thus, the hazard function is the instantaneous failure rate and is defined as $h(x) = \frac{f(x)}{R(x)} = \frac{f(x)}{1-F(x)}$where $f(x) = \frac{dF(x)}{dx}$. The reliability of an item is defined under stated operating and environmental conditions. This implies that any change in these conditions can effect. The failure rate of almost all components is stress dependent. A component can be influenced by more than one kind of stress. For such cases, a power function model is used $z(t) = h(t)\sigma_1^a \sigma_2^b$, where a, b are positive constants, $\sigma_1 and \sigma_2$are stress ratios for two different kinds of stresses, and z(t) is the failure rate at rated stress conditions.The normal distribution takes the well-known bell shape. This distribution is symmetrical bout its mean value. The probability density function for a normally distributed stress $\chi$and normally distributed strength ξ is given by

$$f_\chi(\chi) = \frac{1}{\sigma_\chi \sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{\chi - \mu_\chi}{\sigma_\chi}\right)^2\right] \quad , for \ -\infty < \chi < \infty$$

$$f_\xi(\xi) = \frac{1}{\sigma_\xi \sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{\xi - \mu_\xi}{\sigma_\xi}\right)^2\right] \quad , for \ -\infty < \xi < \infty$$

Where μ_χ = mean value of the stress

σ_χ = standard deviation of the stress

μ_ξ = mean value for the strength

σ_ξ = standard deviation of the strength

Let us define $y = \xi - \chi$. It is well known that the random variable $y$is normally distributed with mean $\mu_y = \mu_\xi - \mu_\chi$ and standard deviation $\sigma_y = \sqrt{\sigma^2 \xi + \sigma^2 \chi}$

The reliability Rcan be expressed in terms of $y$as

$$R = P(y > 0) = \frac{1}{\sigma_y \sqrt{2\pi}} \int_0^\infty \exp\left[\frac{-1}{2}\left(\frac{y - \mu_y}{\sigma_y}\right)^2\right] dy$$

$$let \ z = \frac{y - \mu_y}{\sigma_y}, \ then \ \sigma_y dz = dy$$

When y = 0, the lower limit of z is given by

And as y tends to ∞ the upper limit of z tends to ∞

$$z = \frac{0 - \mu_y}{\sigma_y} = \frac{-\left(\mu_\xi - \mu_\chi\right)}{\sqrt{\sigma_\xi^2 + \sigma_\chi^2}}$$

And as y tends to ∞ the upper limit of z tends to ∞

Then the reliability is given by

$$R = \frac{1}{\sqrt{2\pi}} \int_{\frac{-\left(\mu_\xi - \mu_\chi\right)}{\sqrt{\sigma_\xi^2 + \sigma_\chi^2}}}^{\infty} \exp\left[\frac{-z^2}{2}\right] dz$$

The random variable $z = \frac{y - \mu_y}{\sigma_y}$ is the standard normal variable

Moment of inertia of the section about NA is

$$I = \frac{BD^3}{12} - \frac{bh^3}{12} = \frac{1}{12}\left(BD^3 - bh^3\right)$$

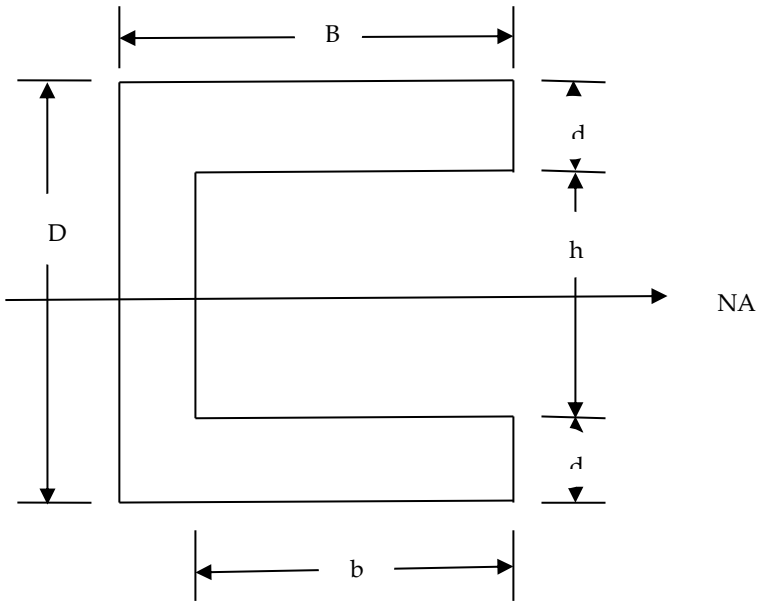

Figure 1:*C-section*

I. Shear stress at top of the Flange

The shear stress at top the flange in C section is

$$\tau_{flange} = \frac{FA\bar{y}}{IB} = \frac{F \times B \times d\left[\frac{D}{2} - \frac{d}{2}\right]}{IB} = \frac{Fd(D-d)}{2I}$$

let

$B = \mu_\xi = mean\ \ strength\ \ of\ Section$

$\tau_{flange} = \mu_\chi = mean\ \ stress\ \ of\ Section$

Then the reliability at top of the flange is

$$R = \frac{1}{\sqrt{2\pi}} \int_{\frac{-(\mu_\xi - \tau_{flange})}{\sqrt{\sigma_\xi^2 + \sigma_\chi^2}}}^{\infty} \exp\left[\frac{-z^2}{2}\right] dz$$

Reliability Computations

The reliability at the top of the flange is computed when B mean strength of the section varies for F=1000KN, D=150mm, d=20mm and depicted in the Figure 2.

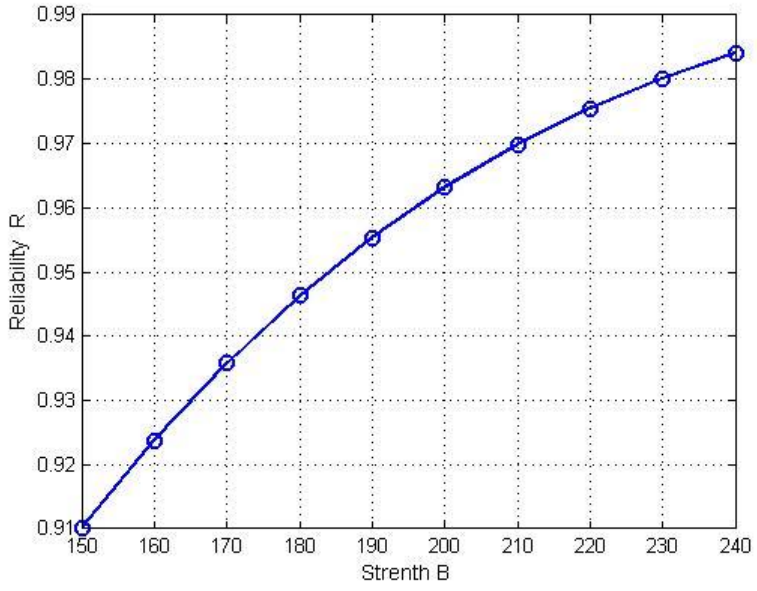

Figure 2: *Reliability Vs Strenth*

The reliability at the top of the flange is computed when d mean strength of the section varies for F=1000KN, B=150mm, D=150mmand depicted in the Figure3.

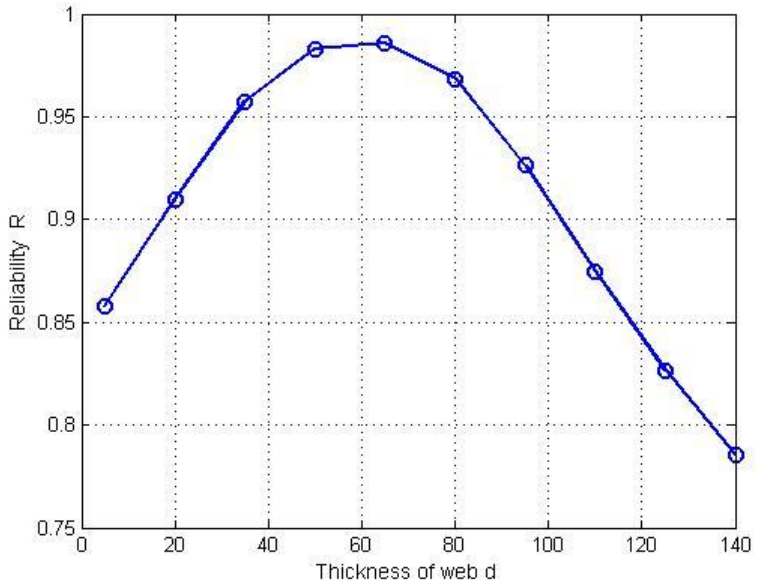

Figure 3: *Reliability Vs Thickness of web*

The reliability at the top of the flange is computed when F mean strength of the section varies for B=150mm, D=150mm, d=20mmand depicted in the Figure 4.

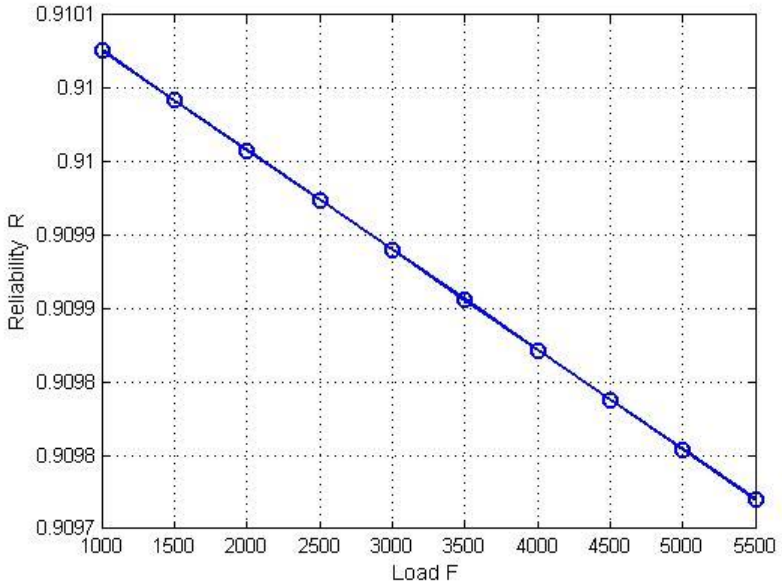

Figure 4: *Reliability Vs Load*

From figure 2, it is observed that if the strength increases then the reliability increases.From figure3, if the thickness of web increases then reliability increases to some extent and after that reliability decreases (i.e, from d=62mm).From figure4, if load increases then reliability decreases.

II. Shear stress at the Bottom of the Flange and Junction of Top of the Web

The Shear stress at the bottom of the flange and junction of top of the web in C section is

$$\tau_{web} = \frac{B}{d}\left[\frac{F \times B \times d\left[\frac{D}{2} - \frac{d}{2}\right]}{IB}\right]$$

$$\tau_{web} = \frac{FB(D-d)}{2I}$$

let

$B = \mu_\xi = mean\ strength\ \ of\ Section$

$\tau_{web} = \mu_\chi = mean\ \ stress\ of\ Section$

Then the reliability at top of the flange is

$$R = \frac{1}{\sqrt{2\pi}} \int_{\frac{-(\mu_\xi - \tau_{web})}{\sqrt{\sigma_\xi^2 + \sigma_\chi^2}}}^{\infty} \exp\left[\frac{-z^2}{2}\right] dz$$

Reliability Computations

The reliability at the top of the flange is computed when B mean strength of the section varies for F=1000KN, D=150mm, d=20mm and depicted in the Figure 5.

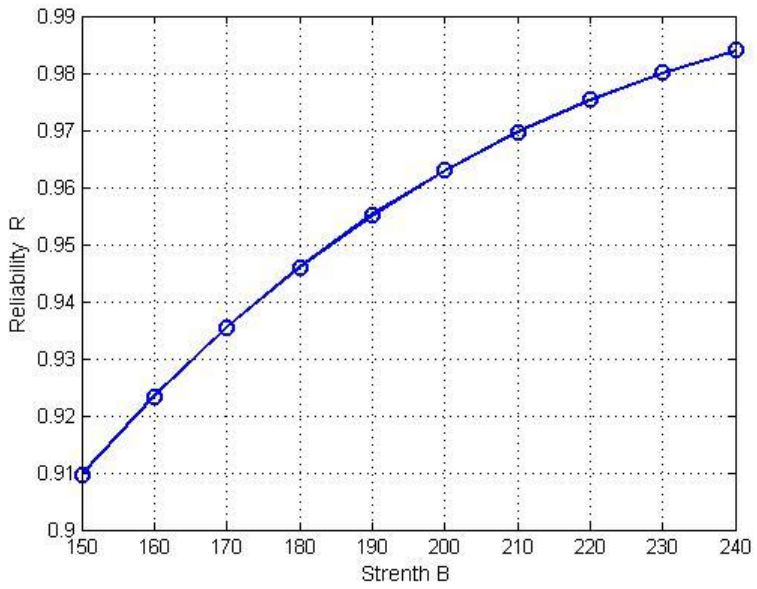

Figure 5: *Reliability Vs Strenth*

The reliability at the top of the flange is computed when d mean strength of the section varies for F=1000KN, B=150mm, D=150mmand depicted in the Figure 6.

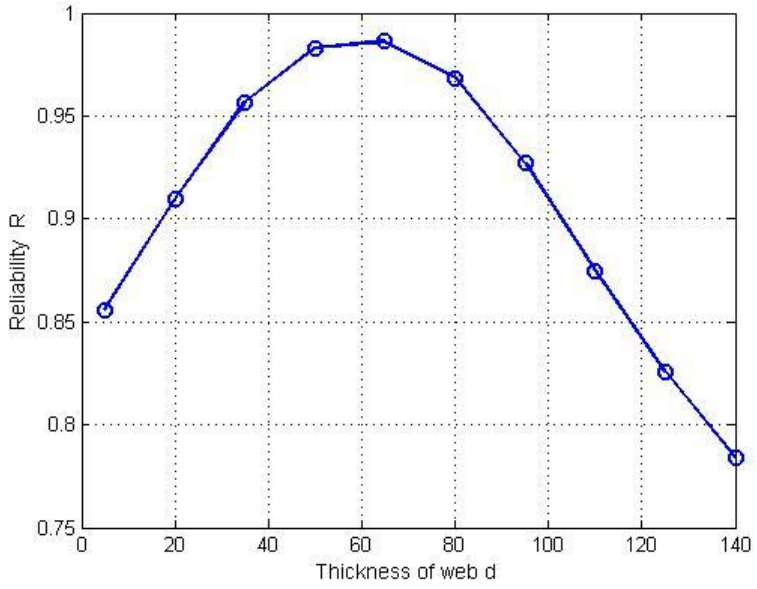

Figure 6: *Reliability Vs Thickness of web*

The reliability at the top of the flange is computed when F mean strength of the section varies for B=150mm, D=150mm, d=20mmand depicted in the Figure 7.

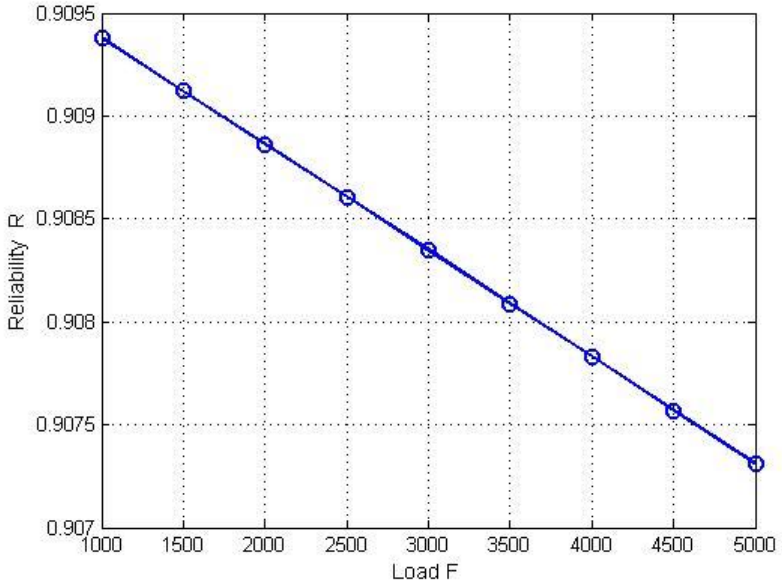

Figure 7: *Reliability Vs Load*

From figure 5, it is observed that if the strength increases then the reliability increases.From figure6, if the thickness of web increases then reliability increases to some extent and after that reliability decreases (i.efrom d=62mm).From figure7, if load increases then reliability decreases.

III. Shear stress at Neutral Axis

$$\tau_{\max} = \frac{F}{Id}\left[B \times d \times \left(\frac{d+h}{2}\right) + \left(d \times \frac{h}{2} \times \frac{h}{4}\right)\right]$$

$$\tau_{\max} = \frac{F\left[4B(d+h)+h^2\right]}{8I}$$

let

$B = \mu_\xi = mean\ \ strength\ \ of\ Section$

$\tau_{\max} = \mu_\chi = mean\ \ stress\ \ of\ Section$

Then the reliability at top of the flange is

$$R = \frac{1}{\sqrt{2\pi}} \int_{\frac{-(\mu_\xi - \tau_{\max})}{\sqrt{\sigma_\xi^2 + \sigma_\chi^2}}}^{\infty} \exp\left[\frac{-z^2}{2}\right] dz$$

Reliability Computations

The reliability at the top of the flange is computed when B mean strength of the section varies for F=1000KN, D=150mm, d=20mm and depicted in the Figure 8.

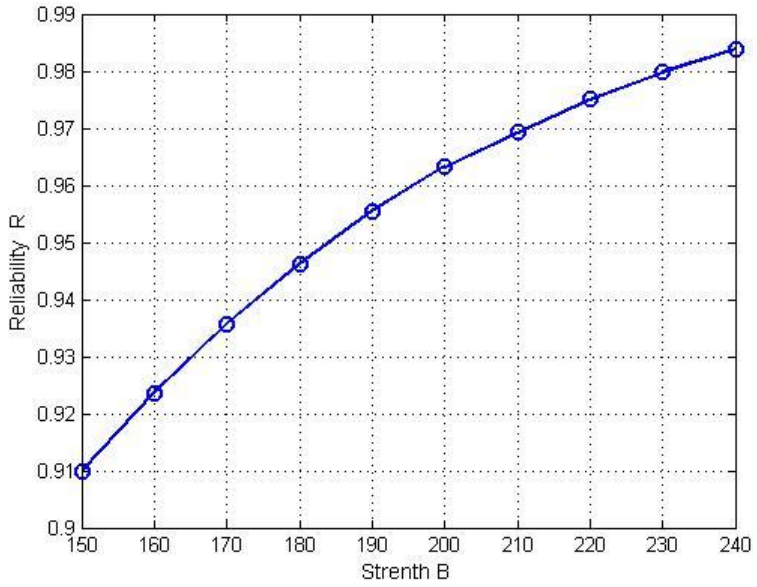

Figure 8: *Reliability Vs Strenth*

The reliability at the top of the flange is computed when d mean strength of the section varies for F=1000KN, B=150mm, D=150mmand depicted in the Figure 9.

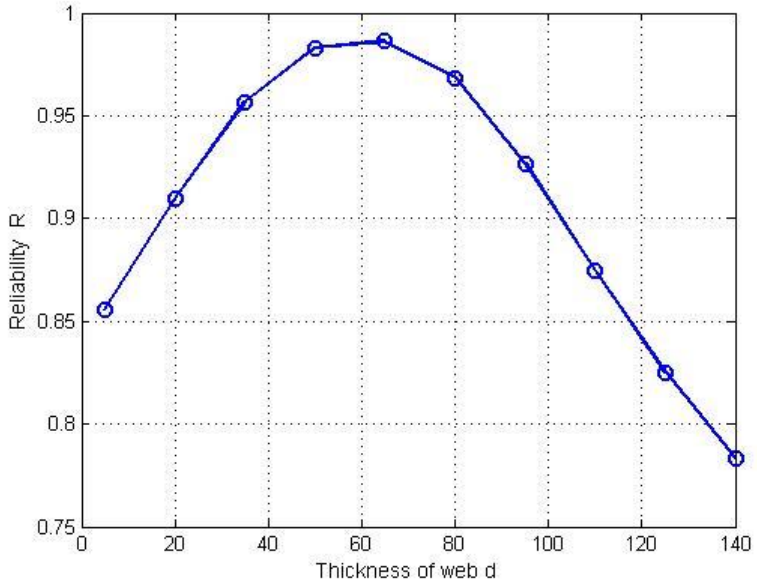

Figure 9: *Reliability Vs Thickness of web*

The reliability at the top of the flange is computed when F mean strength of the section varies for B=150mm, D=150mm, d=20mmand depicted in the Figure 10.

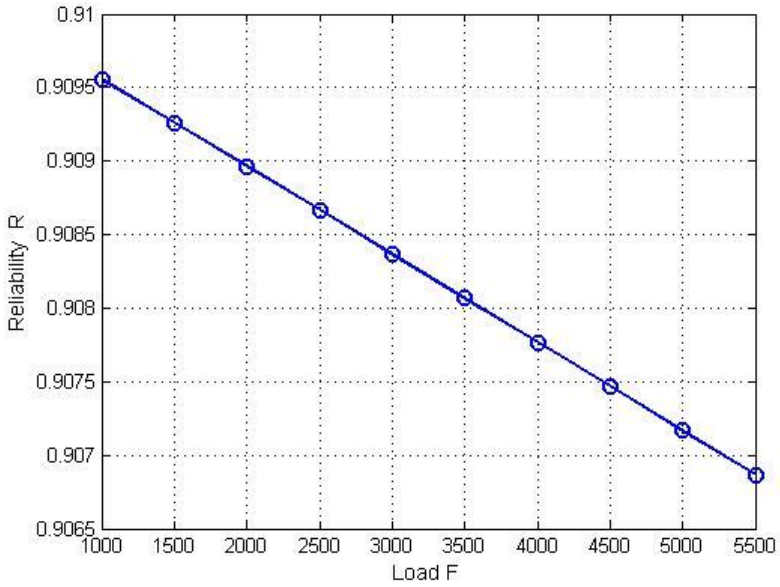

Figure 10: *Reliability Vs Load*

From figure 8, it is observed that if the strength increases then the reliability increases. From figure9, if the thickness of web increases then reliability increases to some extent and after that reliability decreases (i. e, from d=62mm). From figure10, if load increases then reliability decreases.

III. Confidence intervals

The confidence intervals for strength, thickness of web and load applied on C-section are used to observe the maximum reliability. For this, let us consider a large sample of size 40,50 and 50 for strength, thickness of web and load respectively. For this the strength values are considered from 50 mm to 440mm, thickness of web values are from 2mm to 149mm and loadfrom100 KN to 485200 KN. The confidence intervals will be find at different levels, 90%, 95%, 99% for both strength and load. The data never takes negative values and the distribution of data is skewed.

The statistical formula is $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

Here μ=μc mean of population strength, the sample mean is $\bar{x}$

Table 1

	Strength	Thickness of Web	Load
Sample Mean	245	75.5	242650
S.D	116.9045	103.9447	144316.1
Size	40	50	50

The confidence intervals for strength, at90% is (214.5934, 275.4065), at 95% is (208.7710, 281.2290), and at99% is (197.4031, 292.5968). The confidence intervals for thickness of web, at 90%is(51.3185, 99.6815),at 95% is(47.129, 103.8710),and at 99% is (37.6328, 113.3672).The confidence intervals for load, at 90% is (209076.5956,276223.4044), at 95% is (202647.6458, 282652.3542), and at 99% is (189993.8399, 295306.1601).

IV. Conclusion

The reliability is analysed at the top of the flange, bottom of the Flange and Junction of Top of the Web and at neutral axis. It increases with an increase in width at top of the flange, if the thickness of web increases then reliability increases to some extent and after that reliability decreases. The reliability decreases when the load *F*increases. The confidence intervals for both strength, thickness of web and load give a maximum reliability.

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AN ALGORITHM FOR CONDITIONAL EXTREME VALUE THEORY GARCH-EVT TECHNIQUE FOR ESTIMATING VALUE AT RISK

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Abstract

Extreme events in financial time series are characterized by their low probability yet high impact and they pose significant challenges in financial risk management. This study aims to model and forecast extreme events, with a particular emphasis on Value at Risk (VaR) estimation. It explores the concept of conditional Extreme Value Theory (EVT) for modeling volatility series to estimate VaR by integrating Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models with EVT, forming the GARCH-EVT approach. An automated algorithm was developed to optimize both model selection and threshold determination, ensuring accurate estimation of VaR. This automated procedure enhances the model selection process by identifying the optimal GARCH model and the most appropriate EVT threshold, addressing the complexities inherent in modeling extreme events. The comprehensive backtesting procedures are used to assess the effectiveness and precision of the algorithm in forecasting VaR, along with a simulation that evaluates both in-sample and out-of-sample performance of the model and candidate thresholds across various methods. The automated GARCH-EVT approach demonstrates effectiveness in accurately estimating VaR, providing a reliable and efficient method for extreme risk assessment in financial markets. This method streamlines the process of model selection and threshold optimization, contributing to improved risk management in financial markets.

Keywords: Extreme events, Value at Risk (VaR), GARCH models, Threshold selection, Backtesting, Risk management.

I. Introduction

Extreme events in financial time series, such as sudden market crashes or dramatic price movements, pose considerable challenges for risk management strategies. These events are often rare but have significant financial consequences. To effectively manage such risks, accurate Value at Risk (VaR) estimation is critical. VaR is a standard tool for risk management, adopted by financial institutions like banks, investment funds, and corporations worldwide. VaR is determined by the quantile of the gain and loss distribution of the financial positions and it is defined as the maximum possible loss over a time horizon with a given confidence level [22]. Specifically, VaR has emerged as one of the most popular risk management methods. This may also be utilized to estimate the tail probability. The literature also emphasizes the significance of fat tails in calculating and predicting VaR [8], [28]. However, traditional VaR models, which often rely

on normal distribution assumptions, may underestimate the likelihood and impact of extreme events. The limitation of this approach is evident as the assumption of normality for the underlying distribution is unrealistic. In practice, the financial data exhibit the properties of asymmetry and heavy tails. Consequently, there has been growing interest in alternative methods for VaR estimation, particularly for capturing extreme tail behavior and volatility clustering. An alternative way is a non-parametric historical simulation (HS) approach that calculates empirical quantiles from past data without assuming a specific distribution. Parametric models, such as those in the GARCH type model, the entire return distribution under conditional normality, capturing volatility dynamics. On the other hand, the extreme value approach based on VaR estimation is superior to traditional parametric and non-parametric methods in identifying extreme risk [2]. The conventional time series models often assume constant volatility, which fails to adequately account for periods of varying volatility in financial returns. This limitation can lead to misleading conclusions and ineffective risk management strategies.

To address these shortcomings, Engle [15] introduced the Autoregressive Conditional Heteroskedasticity (ARCH) model, which was later extended by Bollerslev [7] into the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model. GARCH models effectively capture essential properties of financial time series, such as volatility clustering, where large price changes tend to occur in clusters, reflecting the time-varying nature of risk. However, while GARCH models allow for dynamic volatility forecasting, they often assume symmetric responses to shocks. This limits their ability to fully capture the asymmetry typically observed in financial returns, where negative shocks have a more significant impact on volatility than positive ones known as the leverage effect. As a result, while GARCH models provide valuable insights into volatility dynamics, their limitations necessitate the exploration of more advanced models that can accommodate asymmetrical volatility behavior and better reflect the complexities of financial markets. The GARCH models with alternative distributions, such as the Student-*t* or skewed-*t*, can offer some improvement, as shown by Giot and Lauren [21]. Nevertheless, these models may still struggle to capture extreme tail events. Recently, EVT has been widely used in VaR estimation for capturing the effect of market behavior under extreme circumstances. EVT has gained popularity in risk management due to its ability to model extreme tail events, which are critical for assessing financial risk. The financial crises of the 1990s and beyond have improved interest in modeling extreme events [18]. Embrechts et al. [14], and Reiss and Thompson [30] provide a theoretical framework for EVT in the context of finance and risk management to model the behavior of extreme events. Beirlant et al. [6] discuss how extreme value models are used to capture tail behavior, while Gilli and Kellezi [19] applied EVT to stock market indices for calculating VaR. Bali [4] demonstrated that EVT outperforms traditional models, such as those based on normal and skewed-*t* distributions, in accurately estimating the VaR of financial assets. However, EVT has two key limitations: it typically assumes independent and identically distributed data, and it does not account for time-varying volatility.

McNeil and Frey [26] proposed the GARCH-EVT approach, or conditional EVT to overcome these limitations, which combines the strengths of both GARCH and EVT models. This two-stage procedure effectively captures both time-varying volatility and tail behavior. In the first stage, GARCH models are used to estimate the conditional volatility and obtain standardized residuals. In the second stage, EVT is applied to the residuals to model extreme tail events. Several studies have demonstrated the superiority of conditional EVT for VaR estimation. Bali and Neftci [3] showed that conditional EVT outperforms GARCH models with skewed distributions when applied to U.S. short-term interest rates. Marimoutou et al. [25] explore the daily Brent oil price and compare the performance of unconditional and conditional EVT models with the conventional GARCH model and historical simulation. Allen et al. [1] found that conditional EVT produced fewer violations in out-of-sample backtesting using stock indices. Karmakar and Shukla [23] confirmed the effectiveness of conditional EVT for estimating VaR for daily stock indices in six

countries. By integrating time-varying volatility with extreme tail modeling, the GARCH-EVT approach offers a more accurate and robust measure of risk compared to traditional methods. Zhang et al. [33] utilized extreme value analysis to investigate the tail risk behavior of the high-frequency returns of the four most popular cryptocurrencies estimating VaR and expected shortfall with varying thresholds.

This study proposes an automated framework for Value at Risk forecasting with conditional extreme value theory. The algorithm automates key steps, including stationarity checks, ARCH effect testing, GARCH model fitting, residual distribution analysis, threshold selection for EVT, and VaR forecasting. Various GARCH models are considered to capture volatility dynamics, while EVT is applied to model extreme tail behavior. A novel dual-phase threshold (DPT) selection technique is introduced to enhance the accuracy of EVT threshold estimation. The framework generates in-sample and out-of-sample VaR forecasts, and performance is validated through backtesting using unconditional and conditional coverage tests. This automated approach provides a robust, data-driven solution for risk management by addressing both volatility clustering and extreme events. The paper is organized as follows: section 2 presents a theoretical framework of conditional extreme value theory, section 3 describes the proposed algorithmic approach for the GARCH-EVT framework, section 4 describes the data analysis of cryptocurrencies, section 5 shows the simulation results, and section 6 provides the summary and conclusion of the study.

II. Methodologies

I. Volatility Models

Volatility models are used to estimate and forecast the variance or volatility of a time series, especially in financial data like stock returns, interest rates, exchange rates, etc. Volatility is a measure of how much the price of an asset fluctuates over time and is commonly used to assess risk. Higher volatility often indicates higher risk, as it increases the likelihood of significant price changes either upward or downward. The Autoregressive Conditional Heteroskedasticity (ARCH) model is designed for modeling time-varying volatility in financial time series. It assumes that the variance of the error term (or the residuals) at time t depends on the squared values of previous error terms. This is particularly useful for capturing volatility clustering, where periods of high volatility are followed by more high volatility, and periods of low volatility are followed by more low volatility. The ARCH model is defined as $r_t = \mu + \epsilon_t$; where, r_t is the observed returns at time t, μ is the constant mean, ϵ_t is the error term or innovation. The conditional variance σ_t^2 at time t depends on past squared residuals ϵ_{t-i}^2 for $i = 1,2, \dots, q$,

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2; \epsilon_t \sim N(0, \sigma_t^2) \quad (1)$$

where q is the order of the ARCH model, $\omega > 0$ is the constant or intercept, $\alpha_i \geq 0$ are the ARCH coefficients, concerning the current volatility to post residuals.

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model extends ARCH model by including lagged conditional variances in the variance equation. It is used to analyze time-series data where the variance of the error term is assumed to be serially auto-correlated. The GARCH models are utilized when the variance of the error term changes, indicating the presence of heteroskedasticity. Let r_t be the return series, μ is the mean and ϵ_t the innovation or error term. The GARCH (p, q) model can be specified in terms of the mean and variance equation as follows

$$r_t = \mu + \epsilon_t, \epsilon_t = \sigma_t z_t$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_j \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2;\ \epsilon_t \sim N(0, \sigma_t^2) \quad (2)$$

where, $\omega > 0$ is the constant or intercept term, $\alpha_i \geq 0$ for $i = 1,2,\ldots,q$ are the ARCH parameters that measure the impact of past squared innovations, $\beta_j \geq 0$ for $j = 1,2,\ldots,p$ are the GARCH parameters that measure the impact of past conditional variances, and σ_t^2 is the conditional variance at time *t*, which is updated based on both the previous squared innovations and lagged variances. In this study, several GARCH-type specifications are considered namely the standard GARCH (SGARCH) by Bollerslev [7], Integrated GARCH (IGARCH) by Engle and Bollerslev [16], Exponentiated GARCH (EGARCH) by Nelson [27], GJR-GARCH by Glosten et al. [20], and Asymmetric Power ARCH (APARCH) by Ding et al., [13] to model the time-varying volatility.

Let r_t be the return at time t and $\epsilon_t = r_t - \mu$, where μ is the conditional mean. The standard GARCH (1,1) model is described as follows

$$r_t = \mu + \sigma_t z_t$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \quad (3)$$

where, $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta < 1$ to ensure stationarity, z_t the innovations are *iid* random variables with zero mean and unit variance, σ_t^2 is the conditional variance at time *t* representing the time-varying volatility, α measures the impact of past residuals ϵ_{t-1}^2 on current volatility, β measures the persistence of volatility from one period to the next. The GARCH (1,1) models tend to be more flexible, efficient, and significant than higher-order models in the out-of-sample analysis. The GARCH model converges to the Integrated GARCH model, where the long-term volatility bears an infinite process.

The IGARCH model is the special version of the SGARCH (1,1) model where the persistence parameter $\alpha + \beta = 1$, implying that volatility follows a unit root GARCH process. Thus, the conditional variance in the IGARCH (1,1) is

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + (1 - \alpha)\sigma_{t-1}^2 \quad (4)$$

by taking $\beta = 1 - \alpha$ in (3) with parameter restriction $\omega > 0$, $\alpha \geq 0$, $1 - \alpha \geq 0$ respectively.

Both the SGARCH and IGARCH models assume that positive and negative shocks affect the conditional variance symmetrically. These models impose non-negative constraints on all coefficients, limiting their ability to account for the negative correlation often observed between returns and volatility. To address these limitations, certain long-memory GARCH-type models have been developed. These models are designed to capture key characteristics such as asymmetry and fat tails in return distributions, which enhance their ability to model volatility and improve the accuracy of Value-at-Risk calculations.

The Exponential GARCH (EGARCH) model allows for asymmetric effects of positive and negative shocks on volatility. The conditional variance equation is logarithmic, ensuring non-negativity without imposing parameter restriction.

$$\ln(\sigma_t^2) = \omega + \alpha \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \gamma \left(\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - E\left[\frac{\epsilon_{t-1}}{\sigma_{t-1}} \right] \right) + \beta \ln(\sigma_{t-1}^2) \quad (5)$$

where, γ captures the asymmetric effect of positive and negative shocks on volatility. If $\gamma \neq 0$, then positive and negative shocks have different impacts on volatility.

The Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model captures leverage effects, where negative shocks increase volatility more than positive shocks of the same magnitude.

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma\epsilon_{t-1}^2 I(\epsilon_{t-1} < 0) + \beta\sigma_{t-1}^2 \quad (6)$$

where, $I(\epsilon_{t-1} < 0)$ is an indicator function that takes the value 1 when ϵ_{t-1} is negative and 0 otherwise; γ represents the additional impact of negative shocks on volatility.

The Asymmetric Power ARCH (APARCH) model generalizes GARCH by allowing for power transformations of the conditional standard deviations and incorporating asymmetry.

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| - \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta \quad (7)$$

where, δ controls the power transformation of volatility and γ captures the asymmetry of positive shocks and negative shocks.

For every GARCH-type model, the innovation process z_t can follow one of several distributions: symmetric, skewed, or heavy-tailed distributions to better capture the characteristics of financial returns, such as symmetry, asymmetry, and fat tails. These distributions include: normal, Student's t distribution, skewed normal, skewed Student's t, generalized error, and skewed generalized error distribution. The parameters for all GARCH-type models can be estimated using maximum likelihood, as it is a reliable and efficient method that produces valid asymptotic standard errors in spite of non-normality. Model selection is performed using information criteria, specifically the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).

II. Extreme value theory

Extreme value theory is the statistical framework for analyzing and modeling extreme events in the tail of the probability distributions. The two main approaches in EVT are block maxima and peaks over threshold approaches. In block maxima, the data is divided into non-overlapping blocks or periods of equal sizes and select the maximum value of each block, which is then modeled using generalized extreme value (GEV) distribution. The peaks over threshold (POT) approach focuses on values that exceed a specified learning threshold and then modeled using a generalized Pareto (GP) distribution. The main challenge in this framework is to select an appropriate threshold for effectively identifying extreme values. The POT method is widely recognized for its effectiveness in characterizing extreme events in the dataset. The cumulative distribution function of the GP distribution with shape parameter ξ and scale parameter σ has the following representation.

$$G_{\xi,\sigma}(y) = \begin{cases} 1 - \left(1 + \xi\left(\frac{y}{\sigma}\right)\right)^{-1/\xi} & ; \; if \; \xi \neq 0 \\ 1 - e^{-\left(\frac{y}{\sigma}\right)} & ; \; if \; \xi = 0 \end{cases} \quad (8)$$

where, i) $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq -\sigma/\xi$ when $\xi < 0$ and ii) $\sigma > 0$ when $\xi = 0$.

The parameter ξ plays a crucial role in characterizing the tail behavior of the distribution. When $\xi = 0$, the distribution simplifies to the exponential distribution (light tail). When $\xi > 0$, the distribution follows the ordinary Pareto distribution (heavy tail). When $\xi < 0$, the distribution is characterized as a short-tailed Pareto distribution.

Let $Y_1, Y_2, \ldots, Y_n$ be the excesses above the sufficient large threshold u, where $Y_i = X_i - u$. Balkema and de Haan [5] and Pickands [29] justify that $F_u(y) \approx G_{\xi,\sigma}(y)$ provided that for large u. By setting $x = u + y$, an approximation of $F(x)$, for $x > u$, can be obtained as

$$F(x) = \big(1 - F(u)\big)G_{\xi,\sigma}(y) + F(u) \quad (9)$$

and here $F(u) = \frac{n-N_u}{n}$; where n is the total number of observations, and N_u the number of observations above the threshold. By using (4) in (5), we get the tail estimator.

$$\hat{F}(x) = 1 - \frac{n}{N_u}\left(1 + \hat{\xi}\left(\frac{x-\hat{u}}{\hat{\sigma}}\right)\right)^{-1/\hat{\xi}} \tag{10}$$

where, $\hat{\xi}$ and $\hat{\sigma}$ are the estimated values obtained using the MLE.

The Value at Risk is calculated by using the (6), we get

$$\widehat{VaR}_p = u + \frac{\hat{\sigma}}{\hat{\xi}}\left(\left[\frac{n}{N_u}(1-p)\right]^{-\hat{\xi}} - 1\right) \tag{11}$$

where, u is the threshold, $\hat{\xi}$ is the estimated shape parameter and $\hat{\sigma}$ is the estimated scale parameter.

The main difficulty of modeling with the POT method is setting the right threshold. It is important to find a good balance in setting the threshold to obtain a suitable balance between the variance and the bias of the model. A high threshold reduces sample size while also increasing uncertainty. At the same time, selecting a small truncation level increases both the sample size and the bias of the results [6].

Method 1: (Threshold or Parameter Stability Method**)** The parameter stability plot, also called the threshold stability plot discussed by Coles [10] is a graphical method to study the stability of the parameter in GP distribution. This method is based on the stability property of the GP distribution. The scale parameter for a GP distribution over a threshold v where $v \geq u$ is specified as $\sigma_v = \sigma_u + \xi(v-u)$, where σ_u is the scale parameter at threshold u, and ξ is the shape parameter. If $\xi \neq 0$, the scale parameter changes as the threshold v varies. To remove the scale parameter dependence on v, it is re-parameterized as $\sigma^* = \sigma_v + \xi u$. In practice, estimates of ξ and σ^* are plotted against different thresholds v, typically with symmetric confidence intervals. The resulting plot is defined by the locus of points: $\{(u, \sigma^*); u < x_{max}\}$ and $\{(u, \xi_u); u < x_{max}\}$. The different thresholds result in different samples of peak magnitudes and times of occurrence. The threshold should be set to the lowest value for which the parameter estimates are approximately stable or constant. The parameter stability plot shows how the shape and modified scale parameters of the GP change over a range of threshold values.

Method 2: (Minimization of Asymptotic Mean Squared Error Method) The minimization of an asymptotic mean squared error (DAMSE) method is an algorithm developed by Cariro and Gomes [9] to identify the tail in data by minimizing the asymptotic mean squared error (AMSE) criterion concerning upper-order statistic k. The optimal number, k_0 corresponds to the unknown threshold u for the tail index in relation to k. The procedure works as follows: Given the observed returns $r_1, \ldots, r_n$, for the tuning parameters $\tau = 0$ and $\tau = 1$, the values of $\hat{\rho}_\tau(k)$ are calculated as:

$$\hat{\rho}_\tau(k) := -\left|\frac{3\left(W_{k,n}^{(\tau)}-1\right)}{\left(W_{k,n}^{(\tau)}-1\right)}\right|, \tag{12}$$

which depend on the statistic:

$$W_{k,n}^{(\tau)} := \begin{cases} \frac{\left(M_{k,n}^{(1)}\right)^{\tau} - \left(M_{k,n}^{(2)}/2\right)^{\tau/2}}{\left(M_{k,n}^{(2)}/2\right)^{\tau/2} - \left(M_{k,n}^{(3)}/6\right)^{\tau/3}} & \text{if } \tau \neq 0 \\ \frac{ln\left(M_{k,n}^{(1)}\right)^{\tau} - ln\left(M_{k,n}^{(2)}/2\right)^{\tau/2}}{(1/2)ln\left(M_{k,n}^{(2)}/2\right) - (1/3)ln\left(M_{k,n}^{(3)}/6\right)} & \text{if } \tau = 0 \end{cases}$$

Here, $M_{k,n}^{(j)}$ is defined as: $M_{k,n}^{(j)} = \frac{1}{k}\sum_{i=1}^{k}(\log r_{n-i+1:n} - \log r_{n-k:n})^j$, $j = 1,2,3$.

To compute the optimal tail parameters:

i. Consider $K = (n^{(0.995)}, n^{(0.999)})$ and compute the median of $\hat{\rho}_\tau(k)$ denoted as K_τ,
ii. Compute $I_\tau = \sum_{k\in K}(\hat{\rho}_\tau(k) - K_\tau)^2$ for $\tau = 0,1$.
iii. Select the tuning parameter, $\tau^* = 0$, if $I_0 \le I_1$, otherwise, select $\tau^* = 1$.

Next, work with $\hat{\rho} = \hat{\rho}_{\tau^*}(k) = \hat{\rho}_{\tau^*}(k_{01})$ and $\hat{\beta} = \hat{\beta}_{\tau^*}(k) = \hat{\beta}_{\tau^*}(k_{01})$ for $k_{01} = n^{0.999}$ and the estimator $\hat{\beta}_{\hat{\rho}}(k)$ is computed as

$$\hat{\beta}_{\hat{\rho}}(k) = \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{d_k(\hat{\rho})D_k(0) - D_k(\hat{\rho})}{d_k(\hat{\rho})D_k(\hat{\rho}) - D_k(2\hat{\rho})} \tag{13}$$

where, $d_k(\alpha) = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha}$, $D_k(\alpha) = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha} U_i$ for any $\alpha \le 0$, with the scaled line spacing or thresholds,

$$U_i = i\sum_{i=1}^{k}(\log r_{n-i+1:n} - \log r_{n-k:n}),\ 1 \le i \le k < n, n^{0.999}. \tag{14}$$

Finally, based on the estimators $\hat{\beta}$ and $\hat{\rho}$ compute: $\hat{k}_0 = \left(\frac{(1-\hat{\rho})^2 n^{-2\hat{\rho}}}{-2\hat{\rho}\hat{\beta}^2}\right)^{\frac{1}{1-2\hat{\rho}}}$ and estimate the shape parameter $\hat{\xi} = \hat{\xi}_{k_0,n}$.

Method 3: (Dual-Phase Threshold Selection - A Proposed Method) The dual-phase threshold (DPT) method can be used to find the optimum threshold based on the two-phase procedure (Sakthivel and Nandhini, [31] and [32]). The procedure is described as follows:

Phase 1: Let $X_1, X_2, \dots, X_n$ be an independent and identically distributed random sample of size n. The non-extremes are trimmed from X and sequential testing of the hypothesis is used to select the most appropriate threshold. The null hypothesis is: $H_0^{(i)}$: The distribution of exceedances n_i above the chosen threshold follows the GP distribution. The sequence of the null hypothesis $H_0^{(1)}, H_0^{(2)}, \dots, H_0^{(k)}$ is tested using goodness of fit tests. For instance, the Kolmogorov-Smirnov (K-S) test and Cramer von Mises (CvM) test with significance level $\alpha = 0.05$ have been performed for this case. The test statistic ω_{ij} and its p-values $p_{ij} \in [0,1]$ for $i \in 1,2,\dots,k$, $j \in 1,2,\dots,l$ denotes the k hypothesis and l test criteria are evaluated. If the p-value $p_{ij} > \alpha$, then $H_0^{(i)}$ is accepted. Otherwise, it is rejected for any $p_{ij} < \alpha$ can be represented as $H_0^{(r)}; r \in 1,2,\dots,k-1$ correspond to the threshold. If $H_0^{(r)}$ is rejected, then the threshold u_r is excluded and the values below u_r are considered to be non-extremes. The refined threshold sequence $u_{r+1} < u_{r+2} < \cdots < u_k$ is tested iteratively until all the null hypotheses are accepted, indicating the exceedances follow the GP distribution. To remove the non-extremes, if both the KS and CvM test yield, $p_{ij} < \alpha$ at different thresholds, the trimming point δ is set as $\delta = \{u_i; \max((p_{CvM}, p_{KS}) < \alpha)\}$. The values $X_i < \delta$ are excluded, and only $X_i > \delta$ are used for selecting an appropriate threshold in the next phase.

Phase 2: Consider a set of threshold values, starting from the trimming point δ obtained in phase 1, as the initial threshold and evaluated up to the 99th percentile with 0.01 increments. For each threshold A_i, where $k = 1,2,\dots,m$, there exists an n_k exceedances, and the p-value for each threshold is calculated based on multiple test criteria. The decision matrix D is created from the p-values of the test criteria evaluated across the threshold range. The matrix $D = (d_{ij})_{m\times n}$ represents the performance values d_{ij} of the i^{th} threshold against the j^{th} criterion, where m is the number of thresholds A_i, and l is the number of test criteria C_j. The matrix D is defined as:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_l \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1l} \\ d_{21} & d_{22} & \dots & d_{2l} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \dots & d_{ml} \end{bmatrix} \end{array} \tag{15}$$

Here, A_j represents the threshold and C_j represents the criteria for $i = 1,2, \dots, m$ and $j = 1,2, \dots, l$. In multiple tests, the p-values can be smoothed to control the overall fluctuation rate of different test criteria. The normalized values are calculated as

$$p_{ij} = \frac{d_{ij}}{m + \sum_{i=1}^{m} (d_{ij})^2}$$

where d_{ij} is the value of the j^{th} criterion for the i^{th} threshold, and m is the number of thresholds. The normalized decision matrix is $p = (p_{ij})_{m \times n}$. The entropy values for each criterion can be calculated with cross-entropy defined as

$$E_j = -\sum_{i=1}^{m} \left(p_{ij} log(p_{ij})\right) - \left(1 - \sum_{i=1}^{l} p_{ij}\right)\left(log\left[1 - \sum_{i=1}^{l} p_{ij}\right]\right) \quad (16)$$

The relative significance of each criterion is given by

$$w_j = \frac{1 - E_j}{\sum_{j=1}^{m} (1 - E_j)}$$

This is the reasonable expression of normalized weighted value, $\sum_{j=1}^{m} w_j = 1$, for $w_j \in [0,1]$. The evaluation indicator (V) can be calculated as

$$V_j = \sum_{j=1}^{m} w_j \, d_{ij} \quad (17)$$

where, w_j is the weight of each criterion d_{ij}. The best threshold is chosen as $u^* = max(V_j)$. This threshold u^* is considered to be optimal, with exceedances above it modeled using the generalized Pareto distribution. The DPT method tests the multiple thresholds, adjusts p-values to control the error rate, and selects the most appropriate threshold.

III. Conditional Extreme Value Theory

The conditional extreme value theory called GARCH-EVT was proposed by McNeil and Frey [26] integrates GARCH and EVT to estimate Value at Risk. By filtering the returns with a GARCH model, it produces an *i.i.d* suitable for the EVT technique, and it captures both conditional heteroskedasticity and extreme tail behavior. The steps for GARCH-EVT VaR estimation:

Step 1: Fit the GARCH-type model to return data by quasi-maximum likelihood. Estimate the one-step ahead forecast of μ_{t+1} and σ_{t+1} from a fitted model and extract the standardized residuals z_t.

Step 2: Consider the standardized residuals computed in step 1, and estimate the tail quantiles of the innovations using EVT. Then construct VaR: The one-step ahead VaR measures for the dynamic volatility model described earlier can be formulated as:

$$VaR_{t+1} = \mu_{t+1} + \sigma_{t+1} VaR_t(z). \quad (18)$$

The backtesting is employed to rigorously evaluate the predictive performance of the GARCH-EVT model used for VaR forecasting. To quantitatively assess the performance of the model, a series of rigorous statistical tests are employed, including the Kupiec Unconditional Coverage (UC) test, and the Christoffersen Conditional Coverage (CC) test.

IV. Rolling Window Method

In the rolling window method, the dataset is divided into overlapping segments, with each segment containing an in-sample and an out-of-sample portion. Initially, the model is trained on

the in-sample data, which consists of a fixed number of observations, and the remaining data is used for out-of-sample forecasting. In this study, 80% of the data might be used for training called in-sample, and the next 20% for testing called out-of-sample. After fitting a GARCH-type model to the in-sample data, it produces one-step-ahead volatility forecasts and VaR estimates for the out-of-sample segment. Then, the window shifts forward by a set number of observations (e.g., one day), removing the earliest observations and adding new ones. The model is re-estimated with the updated in-sample data, and fresh forecasts are made for the new out-of-sample period. This process is repeated continuously, ensuring each forecast is based on previously unseen data. The rolling window approach is effective for evaluating model performance over time, as it mimics real-world forecasting scenarios and prevents over-fitting, leading to more reliable out-of-sample predictions.

III. Automated GARCH-EVT Algorithm

The automated algorithm for GARCH-EVT forecasts Value at Risk by combining GARCH-type models with various advanced threshold selection methods. The procedure is as follows:

Step 1: *Data*: Let Y_t, be the values of time series at the time $= 1,2, \dots, n$.

Step 2: *Test for Normality:* The Jarque-Bera test checks whether a time series follows a normal distribution by measuring skewness and kurtosis. A low p-value suggests non-normality, signaling potential risk from extreme events.

Step 3: *Calculate returns:* The log return series, r_t at time t is $\log\ (r_t) = \log\left(\frac{P_t}{P_{t-1}}\right)$; where, P_t is the price at time t.

Step 4: *Stationarity Check:* The Augmented Dickey-Fuller (ADF) and Kwiatkowski-Phillips-Schmidt-Shin test (KPSS) test are used to check for stationarity in the series. If it shows stationarity then move on to step 5. Otherwise, transform the data and repeat this process.

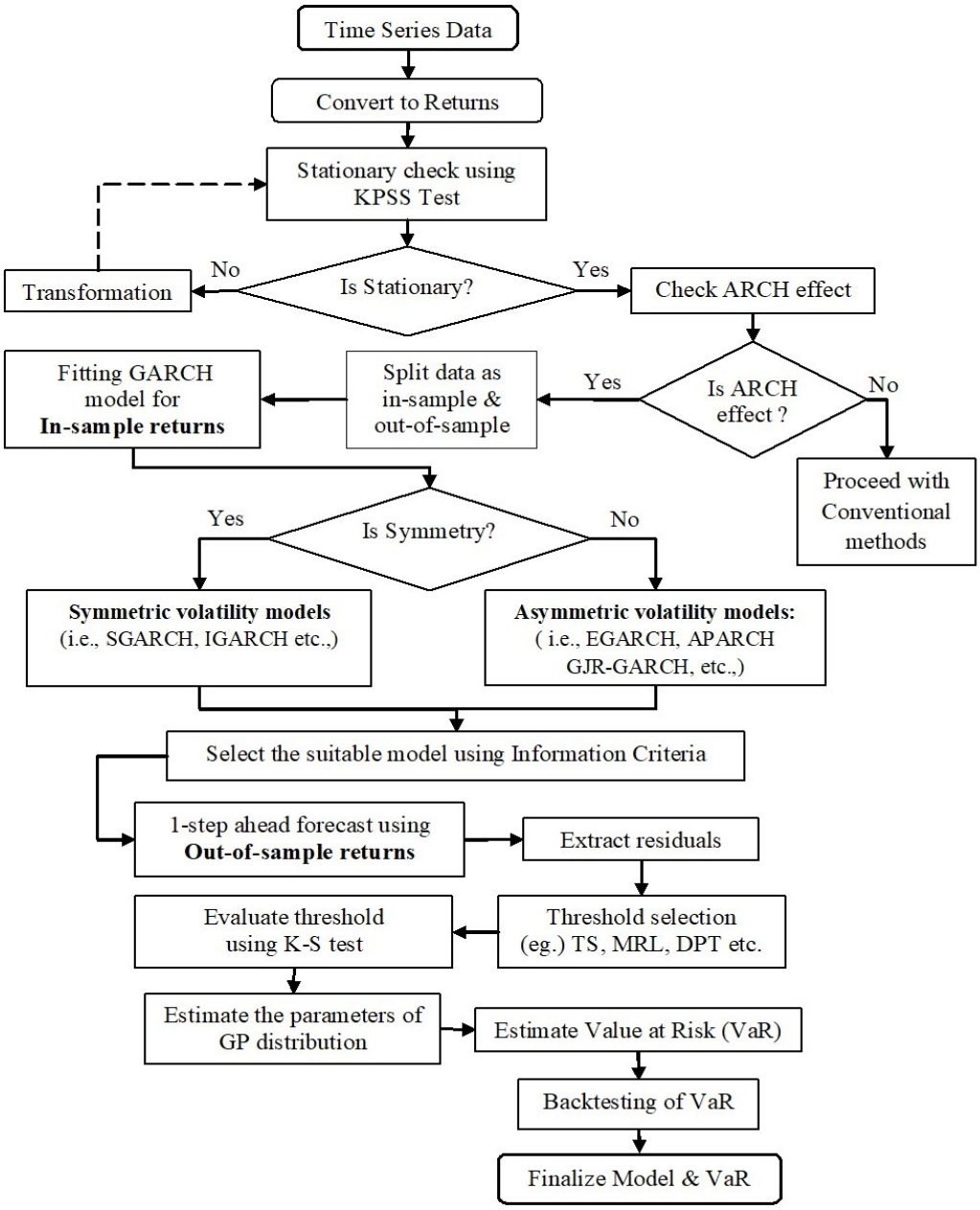
Step 5: *Check for ARCH Effect:* The ARCH-Lagrange Multiplier (ARCH-LM) test is used for testing the auto-correlation in the time series data. If there exists the ARCH effect in the series we proceed to step 6. Otherwise, end this process and proceed with conventional methods.

Step 6: *In-sample and Out-of-sample*: Fixing of In-sample and Out-of-sample proportion for rolling window procedure to obtain better model and VaR forecasting.

$$\text{In-sample: } R_{in} = r_t[1: \lfloor p.k \rfloor]$$
$$\text{Out-of-sample: } R_{out} = r_t[(\lfloor p.k \rfloor + 1): n]$$

where, p is the proportion of the data.

Schematic Representation of GARCH-EVT Algorithm for Volatility Series



Step 7: *Fitting of In-sample returns:* Set In-sample returns as $R_{in} = \{r_1, r_2, \dots, r_k\}$. The iterative procedure through model types and residual distributions is as follows:

For each GARCH model type $M = \{m_1, m_2, \dots, m_i\}$; $m_i \in M$ with each residual distribution is $D = \{d_1, d_2, \dots, d_j\}$; $d_j \in D$, we implement the following procedure for optimal selection.

(i) *Specify the GARCH model:* Create the GARCH specification S_{ij} with the variance model m_i, mean model ARMA(0,0) and distribution d_j Respectively.

(ii) *Fit the GARCH model:* Fit the S_{ij} to the data Y to obtain the best-fitted model F_{ij}. Calculate AIC for F_{ij} to update the best model that is, $F_{best} = arg\min_{F_{ij}} \left(AIC\left(F_{ij}\right)\right)$. If the fit fails, continue the iterative process until selecting the more suitable model.

Step 8: *Out-of-sample forecast:* The rolling window forecast W_i for $i = 1,2, \dots, n_{out}$.

$$W_i = \{r_j | \, j = i, i+1, \ldots, n_{in} + (i-1)\}$$

Fit the Out-of-sample R_{out} returns using the selected best GARCH model from step 7. Then extract residuals e_t, and conditional volatility σ_t.

Step 9: *Threshold Selection:* The threshold selection methods are $u_i = \{u_1, u_2, \ldots, u_n\}$; for $i = 1, 2, \ldots, n$. Fit the GP distribution to the residuals of u_i and to estimate the parameters. The CvM and K-S test can be used to evaluate the threshold-based estimates and choose the best suitable threshold selection method among u_i. The threshold selection methods used in this study are Threshold stability, DAMSE, DPT, and empirical thresholds like 90th percentile, 95th percentile.

Step 10: *Value at Risk Forecast:* The Value at Risk for one step ahead forecast from out-of-sample is defined as

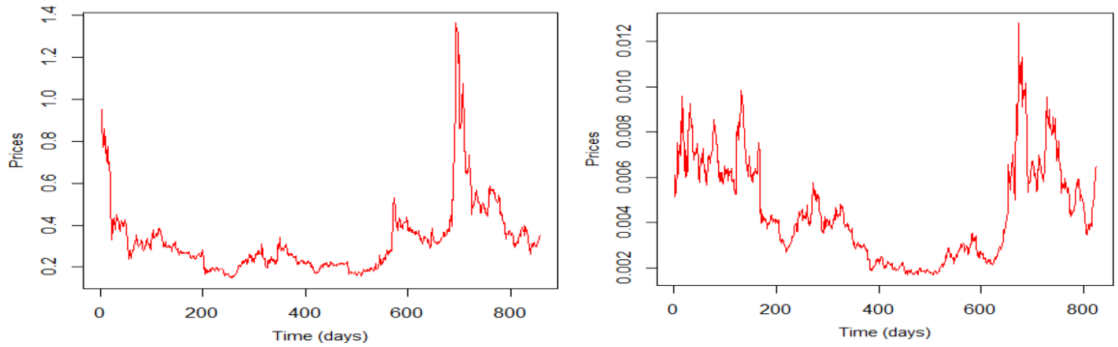
$$VaR_{t+1} = \mu_{t+1} + \sigma_{t+1} VaR_t(z_\alpha);$$

where μ_{t+1} forecasted mean returns and σ_{t+1} forecasted volatility, z_α be the quantile of GP distribution, α is the significance level.

Step 11: *Backtesting:* The Kupiec and Christoffersen test can be used for VaR backtesting. If the *p*-value of the chosen model VaR forecast is greater than the level of significance $\alpha = 0.05$ or 0.01, then finalize the GARCH EVT model. Otherwise, conventional GARCH and EVT techniques can be suitable.

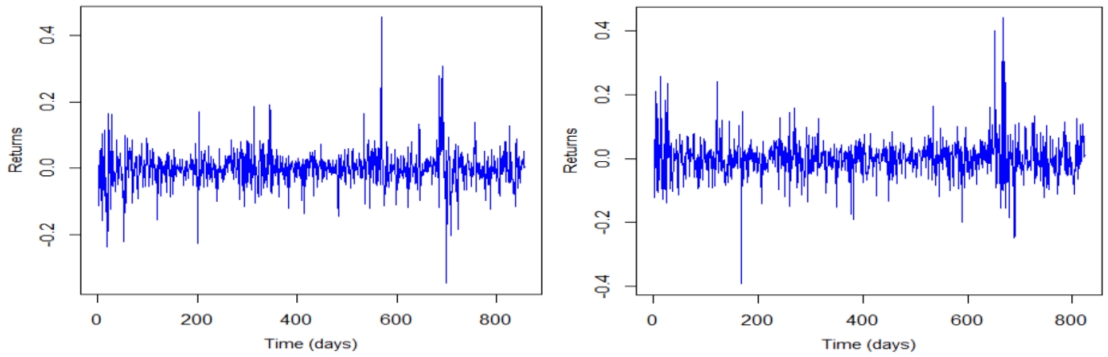
IV. Data Analysis on Real-Time Applications

In this study, the dataset consists of daily closing prices (in dollars) of two cryptocurrencies ZRX token and RSR token from 24 May 2022 to 25 August 2024 (825 observations). The data are available online at marketcap.com and the Kaggle website. Figure 1 shows the time series plots for the daily trading prices of cryptocurrencies. The sample period covers both stable and volatile phases, as well as price fluctuations and extreme jumps. The datasets of cryptocurrencies exhibit clear volatility clustering over time. A data adjustment process is used to achieve stationarity in the cryptocurrency return series, accounting for heteroskedasticity. Figure 2 shows the dynamic behavior of the log returns for all cryptocurrencies, highlighting the characteristic leptokurtosis resulting from time-varying volatility clustering, where high-volatility periods are followed by further high volatility and low-volatility periods are followed by low volatility.



(a) *ZRX Token* (b) *RSR Token*

Figure 1: *Time series plot for the cryptocurrency dataset*



(a) *ZRX Token* (b) *RSR Token*

Figure 2: *Return series for the cryptocurrency dataset*

Table 1 presents summary statistics for the cryptocurrencies and the results of statistical tests. The series shows excess kurtosis, indicating fat tails and non-normal distributions. Table 2 shows the JB test confirms that none of the cryptocurrencies follow a normal distribution. To assess stationarity, the KPSS test was applied, and the results rejected the null hypothesis, indicating that all return series are non-stationary at all levels. Additionally, the presence of significant ARCH effects was confirmed by using the ARCH-LM test and Box-Pierce test in cryptocurrency datasets. The results from these tests confirm the existence of significant ARCH effects in the analyzed datasets, highlighting the importance of using models that account for changing volatility in cryptocurrency datasets.

Table 1: *Descriptive statistics*

Data	Min	Q1	Median	Mean	Q3	Max	Skewness	Kurtosis
ZRX	0.1476	0.2185	0.2887	0.3289	0.3715	1.3634	2.57713	9.0602
RSR	0.0017	0.0026	0.0041	0.0045	0.0061	0.0128	0.5655	7.7823

Table 2: *Preliminary Tests*

Data	JB Test		KPSS test		ARCH-LM test		Box-Pierce test	
	χ^2	p-value	KPSS	p-value	χ^2	p-value	χ^2	p-value
ZRX	3884.4	<0.05	2.1584	<0.05	803.44	<0.05	820.28	<0.05
RSR	62.375	<0.05	2.2168	<0.05	766.85	<0.05	797.48	<0.05

The results from the estimated GARCH-type models are presented in this section. The sample period is divided into two sub-sample periods called the in-sample period; it takes 80% from the starting point and the out-of-sample period covers the last 20% of the dataset. In-sample returns are used to estimate the parameters of the selected models, subject to the assumptions and constraints of each model. The calculated in-sample parameters are applied to forecast the volatilities for both in-sample and out-of-sample periods. We first estimate the SGARCH, EGARCH, GJR-GARCH, APARCH, and IGARCH models for our dataset. Table 3 presents the AIC values of the fitted GARCH type specifications under different types of error distributions such as normal, Student's t, generalized error (GE), skew-normal, skew-t, and skew-generalized error (skew-GE) distribution. The student's t distribution is suitable for both datasets based on the AIC values for all the GARCH-type models. The student t distribution accounts for heavy tails, which allows it to capture the extreme values effectively. The estimated results of GARCH-type models with the selected innovation student's t distribution are presented in Table 4. The diagnostic results like minimum AIC, and BIC reveal that the IGARCH specifications for the ZRX dataset and APARCH specifications for the RSR dataset filter the serial autocorrelation, conditional volatility dynamics, and leverage effects in return series. Therefore we can apply the EVT methods to the *iid* residual series. For the ZRX dataset, we took the IGARCH-EVT approach and for the RSR dataset,

we took the APARCH-EVT approach to compute the one-step-ahead Value at Risk forecast for these cryptocurrencies. The forecast performance of these types of models should be evaluated for the out-of-sample period and using more accurate performance criteria. In this study, optimal POT thresholds are obtained by evaluating the five different threshold methods as 90th percentile, 95th percentile, threshold stability (TS) method, minimization of an asymptotic mean squared error (DAMSE) method, and the proposed dual phase threshold (DPT) selection method and to estimate the GP distribution parameters for both the left and right tails.

Table 3: *In-Sample Estimated Results and Model Selection*

Models		Normal	*t*	GE	Skew-Normal	Skew-*t*	Skew-GE
Data 1: ZRX Token							
SGARCH	AIC	-3.1327	-3.3519	-3.3175	-3.1373	-3.3497	-3.3160
	BIC	-3.1063	-3.3188	-3.2844	-3.1043	-3.3100	-3.2763
EGARCH	AIC	-3.1505	-3.3491	-3.3172	-3.1531	-3.3471	-3.3159
	BIC	-3.1174	-3.3094	-3.2775	-3.1135	-3.3008	-3.2696
GJR-GARCH	AIC	-3.1299	-3.3499	-3.3149	-3.1350	-3.3475	-3.3132
	BIC	-3.0969	-3.3102	-3.2752	-3.0954	-3.3012	-3.2669
APARCH	AIC	-3.1446	-3.3473	-3.3136	-3.1453	-3.3451	-3.3122
	BIC	-3.1049	-3.3010	-3.2673	-3.0990	-3.2922	-3.2593
IGARCH	AIC	-3.1291	-3.3536	-3.3174	-3.1331	-3.3514	-3.3161
	BIC	-3.1093	-3.3272	-3.2909	-3.1067	-3.3183	-3.2830
Data 2: RSR Token							
SGARCH	AIC	-2.8775	-3.0855	-3.0593	-2.8747	-3.0852	-3.0591
	BIC	-2.8503	-3.0514	-3.0252	-2.8406	-3.0446	-3.0182
EGARCH	AIC	-2.9452	-3.0912	-3.0649	-2.9425	-3.0897	-3.0633
	BIC	-2.9111	-3.0503	-3.0240	-2.9016	-3.0420	-3.0156
GJR-GARCH	AIC	-2.9188	-3.0926	-3.0677	-2.9162	-3.0915	-3.0665
	BIC	-2.8848	-3.0517	-3.0268	-2.8754	-3.0438	-3.0188
APARCH	AIC	-2.9125	-3.0941	-3.0697	-2.9105	-3.0938	-3.0689
	BIC	-2.8716	-3.0464	-3.0220	-2.8628	-3.0392	-3.0144
IGARCH	AIC	-2.8595	-3.0824	-3.0505	-2.8578	-3.0821	-3.0508
	BIC	-2.8390	-3.0551	-3.0232	-2.8306	-3.0481	-3.0167

To evaluate the out-of-sample performance of the VaR forecast models using the EVT approach, we implemented a rolling window scheme where 80% of the data was used for in-sample fitting of the GARCH-type model, while the remaining 20% was reserved for out-of-sample forecasting. Within each rolling window, we fitted the chosen best GARCH-type model from in-sample analysis and to extract residuals based on evaluating the AIC. This selection process allowed us to extract the residuals, ensuring that the thresholds for EVT analysis were derived from the most accurate representation of the underlying volatility dynamics. The one-step-ahead VaR is calculated at 95% and 99% confidence levels, which are essential for evaluating the performance of the GARCH-EVT approach in forecasting VaR. We consider both the left and the right tail of the return distribution. The reason is that the left tail represents losses for an investor with a long position on the index, whereas the right tail represents losses for an investor being short on the index.

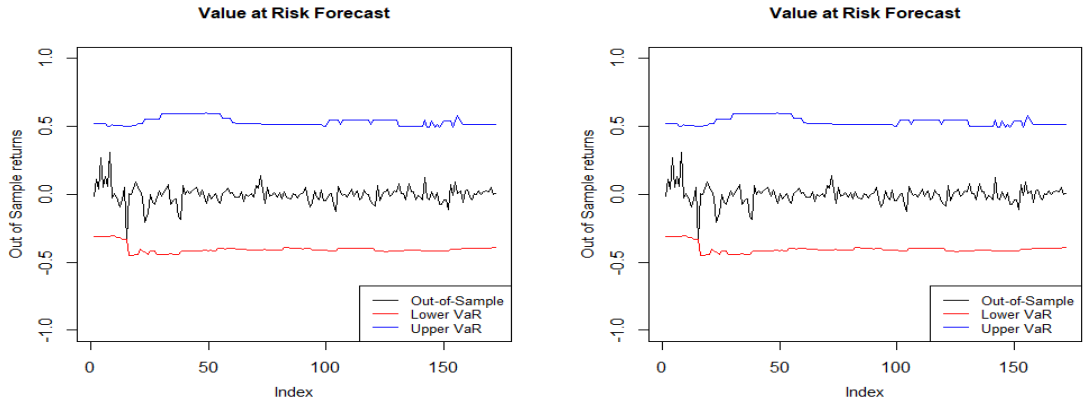
Table 4: *In-Sample: Estimated Values of the Selected Models*

Data 1: ZRX Token- Student t distribution					
Parameters	SGARCH	EGARCH	GJR-GARCH	APARCH	IGARCH
μ	0.0006	0.0008	0.0009	0.0009	0.0006
	(0.0017)	(0.0014)	(0.0014)	(0.0014)	(0.0013)
ω	0.0005	-0.7151	0.0003	0.0006	0.0003
	(0.0001)	(0.2652)	(0.0001)	(0.0010)	(0.0001)
α_1	0.3011	0.0207	0.3451	0.2886	0.3454
	(0.0639)	(0.0495)	(0.1137)	(0.0759)	(0.0756)
β_1	0.5875	0.8818	0.6575	0.6769	0.6545
	(0.0685)	(0.0435)	(0.0755)	(0.0807)	(0.0000)
γ	-	0.4267	-0.0938	-0.0804	-
		(0.0835)	(0.1173)	(0.1023)	
δ	-	-	-	1.7277	-
				(0.5162)	
Shape		4.0116	3.9221	3.9463	3.6521
		(0.5893)	(0.5667)	(0.5741)	(0.4106)
log L	1076.95	1153.05	1153.34	1153.45	1153.61
AIC	-3.1327	-3.3491	-3.3499	-3.3473	-3.3536
BIC	-3.1063	-3.3094	-3.3102	-3.3010	-3.3272
$Q(5)$	0.8911	0.7278	0.7517	0.7591	0.7927
(p-value)	(0.8838)	(0.9175)	(0.9128)	(0.9113)	(0.9045)
$Q^2(5)$	0.2218	0.2890	0.3192	0.3116	0.3907
(p-value)	(0.9909)	(0.9848)	(0.9817)	(0.9825)	(0.9732)
Data 2: RSR Token - Student t distribution					
Parameters	SGARCH	EGARCH	GJR-GARCH	APARCH	IGARCH
μ	0.0011	0.0018	0.0017	0.0015	0.0013
	(0.0017)	(0.0019)	(0.0017)	(0.0017)	(0.0016)
ω	0.0002	-0.1385	0.0001	0.000001	0.0001
	(0.0001)	(0.0255)	(0.0001)	(0.000001)	(0.0001)
α_1	0.0798	0.0753	0.1013	0.0067	0.1095
	(0.0310)	(0.0259)	(0.0368)	(0.0042)	(0.0415)
β_1	0.8623	0.9759	0.9164	0.9307	0.8904
	(0.0533)	(0.0045)	(0.0309)	(0.0180)	(0.0000)
γ	-	0.1235	-0.0862	-0.4294	-
		(0.0522)	(0.0345)	(0.1752)	
δ	-	-	-	3.4999	-
				(0.1193)	
Shape	3.8293	3.9752	3.9360	4.3291	3.1781
	(0.5575)	(0.5136)	(0.5719)	(0.6649)	(0.3396)
log L	1021.66	1024.55	1025.01	1026.52	1019.65
AIC	-3.0855	-3.0912	-3.0926	-3.0941	-3.0824
BIC	-3.0514	-3.0503	-3.0517	-3.0464	-3.0551
$Q(5)$	1.1773	1.4867	1.1772	1.4892	1.1252
(p-value)	(0.8184)	(0.7432)	(0.8184)	(0.7425)	(0.8307)
$Q^2(5)$	0.9295	2.549	0.9816	1.3000	0.7477
(p-value)	(0.7540)	(0.4956)	(0.8638)	(0.7889)	(0.9136)

Table 5: *Parameter estimates of the GP distribution for the selected threshold of returns*

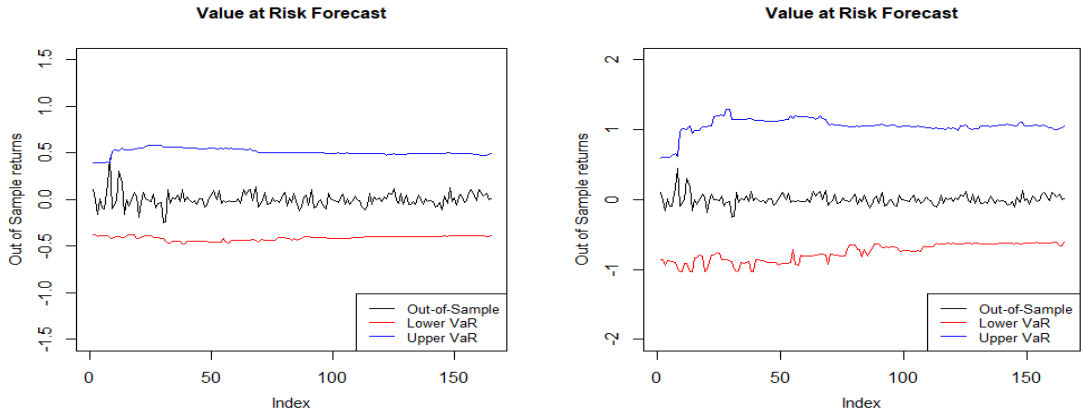
Method	Threshold (Excess)	Estimates		CvM		KS	
		Shape	Scale	Statistic	*p*-value	Statistic	*p*-value
Data 1: Left Tail							
90th Percentile	0.056 (69)	0.2346 (0.1667)	0.0307 (0.0062)	0.0849	0.6650	0.0826	0.7023
95th Percentile	0.081 (35)	0.1466 (0.1899)	0.0386 (0.0097)	0.0561	0.8419	0.1121	0.7296
TS	0.083 (34)	0.1838 (0.2043)	0.0362 (0.0096)	0.0629	0.7989	0.1151	0.6997
DAMSE	0.072 (40)	0.0696 (0.1568)	0.0445 (0.0098)	0.0612	0.8090	0.1089	0.6752
DPT	0.092 (29)	0.2990 (0.2566)	0.0304 (0.0094)	0.0327	0.9686	0.0964	0.9266
Data 1: Right Tail							
90th Percentile	0.053 (69)	0.5141 (0.1872)	0.0232 (0.0049)	0.0611	0.8086	0.0804	0.7328
95th Percentile	0.075 (35)	0.8237 (0.3523)	0.0212 (0.0077)	0.0962	0.6063	0.1233	0.6174
TS	0.074 (36)	0.7726 (0.3315)	0.0223 (0.0077)	0.0303	0.9786	0.10315	0.9848
DAMSE	0.037 (113)	0.3228 (0.1162)	0.0268 (0.0039)	0.0904	0.6346	0.0757	0.5303
DPT	0.087 (18)	0.0064 (0.2848)	0.0913 (0.0337)	0.0275	0.986	0.0963	0.9903
Data 2: Left Tail							
90th Percentile	0.062 (66)	0.1236 (0.1354)	0.0425 (0.0077)	0.0468	0.8967	0.0705	0.8756
95th Percentile	0.090 (31)	0.0476 (0.0129)	0.1308 (0.2119)	0.0306	0.9759	0.0875	0.9430
TS	0.09 (33)	0.1218 (0.2081)	0.0484 (0.0131)	0.0297	0.9784	0.0726	0.958
DAMSE	0.084 (39)	0.1681 (0.2038)	0.0433 (0.0112)	0.0479	0.8915	0.0988	0.7938
DPT	0.033 (162)	0.1903 (0.0945)	0.0318 (0.0038)	0.0164	0.9993	0.0321	0.9963
Data 2: Right Tail							
90th Percentile	0.057 (63)	0.2532 (0.1292)	0.0361 (0.0066)	0.0517	0.8673	0.0666	0.9127
95th Percentile	0.084 (33)	0.0411 (0.0107)	0.2982 (0.1999)	0.1252	0.4768	0.1609	0.3248
TS	0.084 (33)	0.3080 (0.2025)	0.0403 (0.0105)	0.0322	0.9705	0.0869	0.9518
DAMSE	0.077 (40)	0.2948 (0.1876)	0.0393 (0.0095)	0.0798	0.6953	0.1156	0.6170
DPT	0.092 (32)	0.5680 (0.2841)	0.0254 (0.0081)	0.0322	0.9705	0.0869	0.9518

Table 5 presents the estimated parameters of the GP distribution, along with standard errors and goodness-of-fit results, including the CvM and KS tests with their *p*-values. It displays the threshold values and excesses above the threshold for each method. The evaluation of the CvM and K-S test results shows that the excess values from the DPT threshold method yield the best fit for the GP distribution compared to alternative methods like the 95th percentile, 99th percentile, TS method, and DAMSE. Additionally, the positive shape parameter, significantly different from zero for both datasets, indicates a heavy-tailed distribution with finite variance, confirming that the tail distribution of this cryptocurrency data belongs to the Fréchet class.



(a) 95% *VaR* **(b)** 99% *VaR*

Figure 3: *The graph of VaR for IGARCH for the ZRS Token dataset*



(a) 95% *VaR* **(b)** 99% *VaR*

Figure 4: *The graph of VaR for APARCH for the RSR Token dataset*

Table 6: *Backtesting: Kupiec and Christoffersen test Results*

Level of Significance	$\alpha = 0.05$ (95%)		$\alpha = 0.01$ (99%)	
Tails	Left Tail	Right Tail	Left Tail	Right Tail
Data 1	IGARCH- DPT-VaR			
UC: Statistics	0.3584	3.4573	1.8685	0.3442
UC: *p*- value	0.5494	0.0929	0.1716	0.5574
CC: Statistics	0.3702	3.2357	1.8803	0.3528
CC: *p*- value	0.8310	0.1775	0.3906	0.8418
Data 2	APARCH- DPT-VaR			
UC: Statistics	0.3010	3.3166	1.8685	0.3302
UC: *p*- value	0.5832	0.0686	0.1716	0.5656
CC: Statistics	0.3133	3.3256	1.8803	0.3103
CC: *p*- value	0.8549	0.1905	0.3906	0.8478

The graphical representation of the out-of-sample alongside calculated VaR for the return series of the two datasets is in Figures 3 and 4. The x-axis represents the period over which the returns and VaR are measured and the y-axis represents the out-of-sample returns. The black middle line denotes the actual out-of-sample returns of the cryptocurrencies and the fluctuation indicates the performance of the price over time. The red line represents the lower tail VaR indicating that the value below which a certain percent of the returns are expected to fall. The blue line represents the upper tail VaR, indicating that values above which a certain percentage of returns are expected to rise. The results in Table 6 show the performance of the unconditional coverage (UC) and conditional coverage (CC) tests for both IGARCH-EVT for the ZRX dataset and APARCH-EVT for the RSR dataset at the level of significance $\alpha = 0.05$ and $\alpha = 0.01$ indicate that the models perform well in terms of VaR estimation. For both models, the UC test p-values are greater than their significance level, suggesting that the null hypothesis of correct unconditional coverage cannot be rejected. In terms of CC tests, both models yield high p-values, confirming that the model accurately captures the dynamics of the return distributions. Overall, both models corresponding to its datasets demonstrate the performance in estimating VaR concerning both UC and CC across both left and right tails.

V. Simulation Study

The simulation of returns with time-varying volatility is crucial for understanding financial dynamics, particularly in assessing risk. This process allows for the modeling of more realistic return behaviors that account for fluctuations in market conditions. We have set the parameters, the mean return $\mu = 0$, and $\sigma_0 = 1$ is the initial standard deviation. Let n be the number of observations and we have a time index $t = 1,2, \dots, n$, representing each point in time. To introduce time-varying volatility, the standard deviation is calculated at each time step is defined as

$$\sigma_t = \sigma_0 \times \left(1 + 0.5 \sin\left(\frac{2\pi t}{100}\right)\right).$$

This equation can be used to generate a standard deviation that fluctuates over time. The random returns at each time step r_t are then generated from the normal distribution, represented as $r_t \sim N(0, \sigma_t)$. In this case, the mean return $\mu = 0$, and the standard deviation σ_t changes at each time point according to the sinusoidal function. The cumulative returns $R(t)$, representing the sum of returns over time, are calculated as

$$R(t) = \sum_{i=1}^{t} r_i \ .$$

This cumulative process allows us to observe the total gain or loss of the simulated series over time. By simulating random returns with time-varying volatility, we gain insights into volatility clustering in financial markets, where large price movements tend to be followed by similar movements. This simulation is crucial for risk management and financial modeling, as it accurately reflects market behavior compared to constant-volatility models.

We generated two different samples of size n=3000, 5000 respectively. In this simulation of returns, the rolling window procedure of in-sample and out-of-sample techniques was employed to find the best VaR forecast and determine the adequacy and efficiency of the proposed automated GARCH EVT algorithm.

Table 7: *In-Sample Estimated Results and Model Selection*

Models		Normal	*t*	GED	Skew-Normal	Skew-*t*	Skew-GED
Case 1: *n*=3000							
SGARCH	AIC	-2.1057	-2.2148	-2.1938	-2.1095	-2.2169	-2.1968
	BIC	-2.0941	-2.2004	-2.1793	-2.0950	-2.1995	-2.1794
EGARCH	AIC	-2.1450	-2.2543	-2.2246	-2.1464	-2.2579	-2.2290
	BIC	-2.1305	-2.2370	-2.2072	-2.1291	-2.2377	-2.2087
GJR-GARCH	AIC	-2.1429	-2.2394	-2.2177	-2.1477	-2.2429	-2.2227
	BIC	-2.1285	-2.2220	-2.2003	-2.1304	-2.2226	-2.2025
APARCH	AIC	-2.1426	-2.2437	-2.2182	-2.1449	-2.2468	-2.2214
	BIC	-2.1253	-2.2234	-2.1980	-2.1246	-2.2237	-2.1982
IGARCH	AIC	-2.1073	-2.2160	-2.1950	-2.1110	-2.2180	-2.1980
	BIC	-2.0986	-2.2044	-2.1834	-2.0994	-2.2035	-2.1835
Case 2: *n*=5000							
SGARCH	AIC	-2.8250	-2.9005	-2.8883	-2.8361	-2.9041	-2.8926
	BIC	-2.8175	-2.8910	-2.8788	-2.8266	-2.8928	-2.8813
EGARCH	AIC	-2.8622	-2.9225	-2.9116	-2.8717	-2.9278	-2.9173
	BIC	-2.8527	-2.9111	-2.9003	-2.8603	-2.9145	-2.9040
GJR-GARCH	AIC	-2.8567	-2.9158	-2.9062	-2.8672	-2.9211	-2.9123
	BIC	-2.8473	-2.9044	-2.8948	-2.8558	-2.9078	-2.8990
APARCH	AIC	-2.8561	-2.9193	-2.9075	-2.8633	-2.9241	-2.9114
	BIC	-2.8447	-2.9193	-2.8942	-2.8500	-2.9089	-2.8962
IGARCH	AIC	-2.8262	-2.9013	-2.8892	-2.8372	-2.9050	-2.8935
	BIC	-2.8205	-2.8937	-2.8816	-2.8296	-2.8955	-2.8840

Table 7 presents the AIC values of the fitted GARCH-type specifications under different types of error distributions. The skewed student's *t* distribution is suitable for both cases based on the AIC values for all the GARCH-type models. The skewed student *t* distribution accounts for asymmetry and heavy tails, which allows it to capture the extreme values effectively. The estimated results of GARCH-type models with the selected innovation skewed student's t distribution are presented in Table 8. The residuals of the selected models are approximately *iid's* which is the requirement for the further process of applying EVT. For simulated returns, we select the EGARCH-EVT approach to compute the one-step-ahead Value at Risk forecast. The forecast performance of these types of models should be evaluated for the out-of-sample period and using more accurate performance criteria.

The estimated values of parameters of the GP distribution, including their standard errors and the results of goodness-of-fit tests, specifically the CvM and KS tests, along with their *p*-values are shown in Table 9. Our analysis of the CvM and KS test results indicates that the excess values derived from the DPT threshold method yield the best fit for the GP distribution compared to alternative methods. Furthermore, the positive shape parameter indicates that the distribution is heavy-tailed. This means that there is a higher chance of observing extreme values (very large or very small). Heavy-tailed distributions are crucial in risk assessment, particularly in finance and insurance, as they can more accurately reflect the occurrence of rare but significant events.

Table 8: *In-Sample: Estimated Values of the Selected Models*

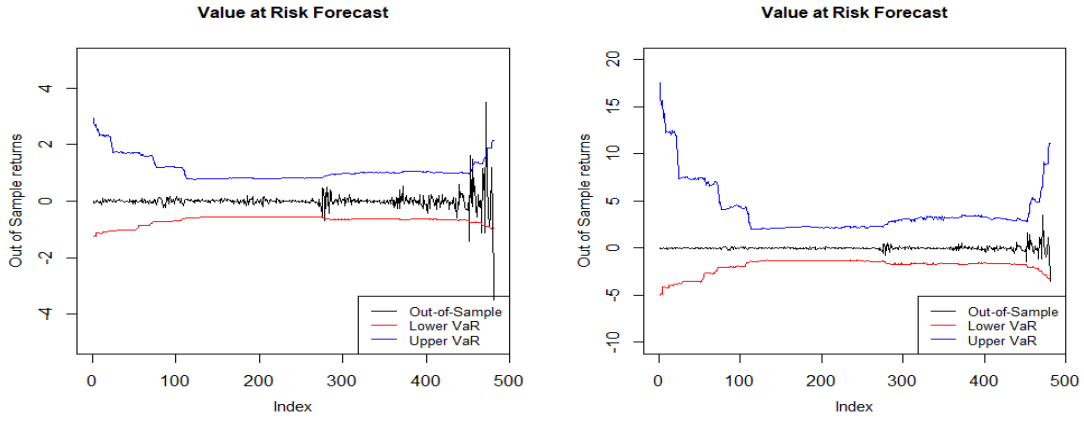
Case 1: n=3000					
Parameters	SGARCH	EGARCH	GJR-GARCH	APARCH	IGARCH
μ	-0.0002	-0.0029	-0.0015	-0.0025	-0.0002
	(0.0012)	(0.0013)	(0.0013)	(0.0013)	(0.0012)
ω	0.0002	-0.1883	0.0002	0.0008	0.0002
	(0.0001)	(0.0317)	(0.0001)	(0.0005)	(0.00003)
α_1	0.2567	-0.1584	0.0832	0.2383	0.2577
	(0.0273)	(0.0201)	(0.0222)	(0.0265)	(0.0215)
β_1	0.7422	0.9624	0.7739	0.7924	0.7422
	(0.0215)	(0.0062)	(0.0192)	(0.0195)	(0.0000)
γ	-	0.3913	0.2943	0.4116	-
		(0.0349)	(0.0421)	(0.0641)	
δ	-	-	-	1.4110	-
				(0.1847)	
Skew	0.9285	0.9066	0.9122	0.9123	0.9285
	(0.0282)	(0.0296)	(0.0284)	(0.0295)	(0.0283)
Shape	5.6529	5.5375	6.0268	5.4114	5.6431
	(0.6078)	(0.6304)	(0.6845)	(0.6179)	(0.5857)
log L	2137.51	2178.01	2163.52	2168.31	2137.60
AIC	-2.2169	-2.2579	-2.2429	-2.2468	-2.2180
BIC	-2.1995	-2.2377	-2.2226	-2.2237	-2.2035
$Q(5)$	2.614	2.886	3.267	4.206	2.620
(p-value)	(0.4819)	(0.4282)	(0.3604)	(0.2295)	(0.4809)
$Q^2(5)$	1.2656	5.0203	4.8132	42.898	1.2502
(p-value)	(0.7972)	(0.1515)	(0.1686)	(6.868e-12)	(0.8009)
Case 2: n=5000					
Parameters	SGARCH	EGARCH	GJR-GARCH	APARCH	IGARCH
μ	-0.0002	-0.0012	-0.0008	-0.0012	-0.0002
	(0.0006)	(0.0005)	(0.0006)	(0.0005)	(0.0006)
ω	0.00003	-0.0975	0.0001	0.0003	0.0001
	(0.00001)	(0.0163)	(0.00001)	(0.0002)	(0.00001)
α_1	0.2159	-0.1410	0.1029	0.2166	0.2169
	(0.0149)	(0.0166)	(0.0162)	(0.0157)	(0.0124)
β_1	0.7831	0.9832	0.7960	0.8162	0.7830
	(0.0149)	(0.0028)	(0.0114)	(0.0132)	(0.0000)
γ	-	0.3663	0.2081	0.3485	-
		(0.0232)	(0.0279)	(0.0547)	
δ	-	-	-	1.4081	-
				(0.1810)	
Skew	0.9173	0.8986	0.9025	0.9039	0.9172
	(0.0215)	(0.0222)	(0.0214)	(0.0222)	(0.0215)
Shape	6.9561	6.5902	7.3866	6.6791	6.9376
	(0.6768)	(0.7198)	(0.7791)	(0.7063)	(0.6579)
log L	4651.17	4689.94	4679.25	4685.02	4651.53
AIC	-2.9041	-2.9278	-2.9211	-2.9241	-2.9050
BIC	-2.8928	-2.9145	-2.9078	-2.9089	-2.8955
$Q(5)$	3.8152	4.4863	0.4004	4.4171	3.8176
(p-value)	(0.2780)	(0.1993)	(0.2536)	(0.2064)	(0.2777)
$Q^2(5)$	1.5662	1.6791	1.6405	0.9894	1.5529
(p-value)	(0.7236)	(0.6959)	(0.7053)	(0.8621)	(0.7269)

Table 9: *Parameter estimates of the GP distribution for a selected threshold of simulated returns*

Method	Threshold (Excess)	Estimates		CVM		KS	
		Shape	Scale	Statistic	p-value	Statistic	p-value
Case 1: Left Tail							
90th Percentile	0.15 (92)	0.3934 (0.1518)	0.1104 (0.0189)	0.0672	0.7702	0.0788	0.5893
95th Percentile	0.24 (46)	0.6624 (0.2878)	0.1033 (0.0318)	0.0988	0.7224	0.3728	0.8746
TS	0.16 (90)	0.4254 (0.1494)	0.1054 (0.1868)	0.0539	0.8536	0.0747	0.6683
DAMSE	0.18 (76)	0.4169 (0.1615)	0.1153 (0.0222)	0.0758	0.7177	0.0935	0.4821
DPT	0.06 (342)	0.5004 (0.0827)	0.0508 (0.0048)	0.0181	0.9985	0.0213	0.9978
Case 1: Right Tail							
90th Percentile	0.14 (101)	0.6917 (0.1663)	0.0835 (0.0151)	0.0355	0.9555	0.0480	0.9740
95th Percentile	0.21 (51)	0.1319 (0.0362)	0.7204 (0.2565)	0.0476	0.8924	0.0809	0.8649
TS	0.15 (95)	0.6819 (0.1693)	0.0886 (0.0164)	0.0443	0.9106	0.0527	0.9415
DAMSE	0.17 (92)	0.7839 (0.2034)	0.0836 (0.0179)	0.0349	0.9583	0.0684	0.8006
DPT	0.11 (173)	0.7229 (0.1319)	0.0543 (0.0077)	0.0156	0.9995	0.0280	0.9992
Case 2: Left Tail							
90th Percentile	0.17 (156)	0.4495 (0.1235)	0.1743 (0.0248)	0.0257	0.9884	0.0325	0.9906
95th Percentile	0.31 (78)	0.3273 (0.1615)	0.2778 (0.0538)	0.0501	0.8774	0.0745	0.7509
TS	0.18 (152)	0.4516 (0.1258)	0.1757 (0.0255)	0.0279	0.9826	0.0342	0.99
DAMSE	0.23 (115)	0.3732 (0.1342)	0.2237 (0.0357)	0.0327	0.9672	0.0481	0.9511
DPT	0.14 (189)	0.4409 (0.1097)	0.1631 (0.0208)	0.0217	0.9952	0.0317	0.9912
Case 2: Right Tail							
90th Percentile	0.15 (165)	0.3371 (0.1166)	0.1869 (0.0257)	0.0970	0.6003	0.0620	0.5494
95th Percentile	0.30 (83)	0.2106 (0.1407)	0.2685 (0.0475)	0.0434	0.8929	0.0618	0.7936
TS	0.16 (160)	0.3258 (0.1170)	0.1922 (0.0267)	0.0941	0.6155	0.0642	0.5242
DAMSE	0.21 (118)	0.2254 (0.1205)	0.2460 (0.0369)	0.0569	0.8339	0.0726	0.5572
DPT	0.22 (113)	0.2062 (0.1195)	0.2567 (0.0387)	0.0345	0.9597	0.0585	0.8377

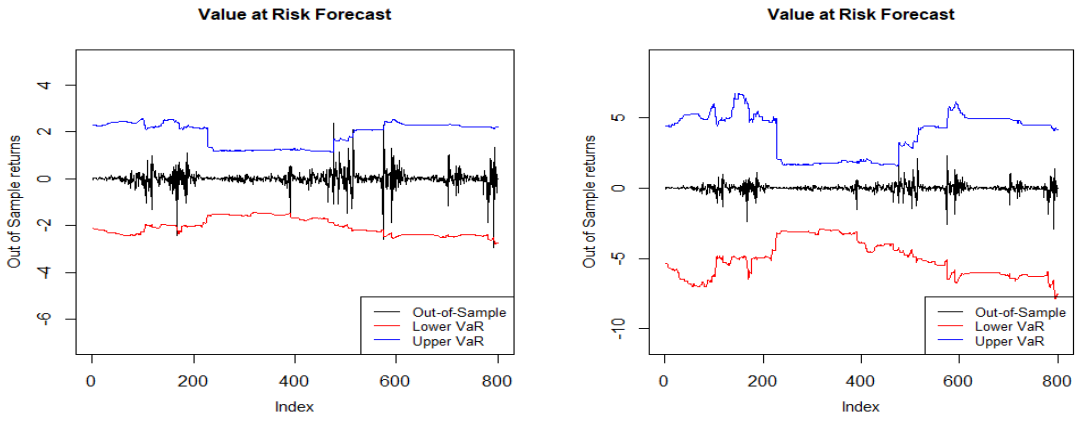
The graphical representation of the out-of-sample returns and corresponding Value at Risk for the two simulated returns series are shown in Figures 5 and 6. The black line shows that the returns exhibit some volatility, with notable fluctuations around the mean. This behavior is typical in financial markets, where returns can vary significantly over time. The red and blue lines illustrate

the estimated Value at Risk levels. The area between these lines indicates the range of potential losses and gains that are considered acceptable within the specified confidence levels (lower and upper VaR). If the black line (out-of-sample returns) crosses below the red line (lower VaR), it indicates a loss exceeding the expected threshold, suggesting that the portfolio is experiencing a significant risk event. Conversely, if the black line crosses above the blue line (upper VaR), it suggests extremely positive returns, indicating potential gains exceeding expectations.



(a) 95% *VaR* **(b)** 99% *VaR*

Figure 5: *The graph of VaR for EGARCH-EVT for n=3000*



(a) 95% *VaR* **(b)** 99% *VaR*

Figure 6: *The graph of VaR for EGARCH-EVT for n=5000*

Table 10: *Backtesting: Kupiec and Christoffersen test Results*

Level of Significance	$\alpha = 0.05$ (95%)		$\alpha = 0.01$ (99%)	
Tails	Left Tail	Right Tail	Left Tail	Right Tail
Case 1: n=3000	Model: EGARCH-EVT-VaR			
UC: Statistics	0.2757	0.7944	0.4263	0.9624
UC: P- value	0.5995	0.3728	0.5138	0.3265
CC: Statistics	0.4022	0.8459	0.4263	0.9626
CC: P- value	0.8179	0.8321	0.8080	0.6180
Case 2: n=5000	Model: EGARCH-EVT-VaR			
UC: Statistics	2.4749	4.1465	1.6008	0.0463
UC: P- value	0.1156	0.0517	0.2057	0.8296
CC: Statistics	2.5152	4.1691	1.6218	0.0488
CC: P- value	0.2843	0.1244	0.4492	0.9758

The UC and CC test results are displayed in Table 10 for the EGARCH-EVT model applied to simulated returns with sample sizes of n=3000 and n=5000 at significance levels of $\alpha = 0.05$ and $\alpha = 0.01$. Specifically, the p-values from the UC tests exceed the significance levels for both sample sizes, indicating that we cannot reject the null hypothesis of correct unconditional coverage which suggests the model accurately estimates VaR. Similarly, the CC tests also yield high p-values, demonstrating that the models effectively capture the dynamics of the return distributions without overestimating or underestimating the risk. Overall, the EGARCH-EVT models show strong reliability and stability in estimating VaR, as evidenced by the favorable outcomes of both UC and CC tests across the left and right tails in the simulated datasets. We observe that the conditional EVT-based models give the best one-step-ahead VaR forecast according to the backtesting results.

VI. Conclusion

This paper developed an algorithm for the GARCH-EVT approach that allows us to model the tails of the time-varying conditional return distribution. In this study, we provide a framework to estimate and forecast the long position as well as short position VaR using this GARCH-EVT algorithm. Modeling the tail behavior of the returns is of utmost importance for both investors and policymakers. The GARCH-EVT approach is implemented in modeling the tail distribution of cryptocurrency returns and forecasting out-of-sample VaR. By employing a rolling window approach, we identified the best GARCH model through in-sample fitting, allowing us to extract reliable residuals for EVT analysis. The DPT method proved to be an effective strategy for selecting appropriate thresholds, significantly improving the fit of the GP distribution to the excess values. The evaluation of goodness-of-fit tests, such as the CvM and KS tests, further confirmed the superiority of the DPT method over alternative threshold selection approaches. Additionally, the positive shape parameter observed in the GP distribution analysis indicates the presence of heavy-tailed behavior, underscoring the potential for extreme events. The backtesting results demonstrate the suitability of the heavy-tailed GARCH EVT models in forecasting out-of-sample VaR. The dual-phase threshold selection procedure is more adaptable in threshold selection for conditional EVT, which has been proved in this paper. Our application and simulation captures the heavy-tailed behavior in daily returns and the asymmetric characteristics in distributions; we treat positive and negative returns separately. Overall, the GARCH EVT with DPT threshold provides a significant improvement in forecasting Value at Risk.

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STRATIFIED RANDOM SAMPLING WITH RISK APPROACH

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Abstract

In stratified random sampling, the sample size allocation is a problem which is tackled by many scientists and survey practitioners. Generally the proportional allocation, Neyman allocation and cost based allocation, are used to conduct sample surveys for gathering information from each strata. One can think of risk imposed on the life of investigators which is yet not considered while sample size allocation to risky strata. In this paper, the risk indicators stratum-wise are defined using police station records and hospital records. Such indicators are used for the determination of sample size allocation. For optimization, the Lagrange multiplier technique is used with two constants whose values need to be determined. An algorithm is proposed and analysed for such using simulation. The outcome of analysis provides that sample size allocation is directly proportional to the strata size and variability but inversely proportional to the square root of risk indicators of the stratum (with varying values of constants). This paper opens a new approach for the consideration of risk based sample size allocation and estimation in the setup of stratified sampling.

Keywords: Simple Random Sampling (SRSWOR), Stratified Random Sampling, Stratum, Sample survey, Lagrange Multiplier, Allocation to Strata, Risk data, Risk Indicators, Optimization, Variance of sample estimate.

1. Introduction

Sample surveys play important role in exploring the hidden characteristics of the whole population without complete enumeration. The sample survey methodologies are used in areas and regions where epidemic occurs. A survey is usually conducted to know about the patients health conditions, rate of spread of disease, estimation of average number of deaths due to disease etc. Where natural disaster happened, to collect facts about casualties, about root causes of natural disaster are further possible field based studies.

Stratification is a sampling technique used in surveys in order to improve upon the precision of the sample estimate. Several authors have developed optimal techniques to allocate sample sizes to stratum (for the single variable under study) using Lagrange multiplier optimization technique. There may be other situations such as war, naxalite movement, dense forest where issue risk is involved on the field officers, who are involved in data collection for the conduct of sample survey. The risk may be on life, infection due to disease or being unhealthy for short duration. In this study, the problem of risk occurrence during the data collection, if so exit, is considered in context to sample size allocation to each strata.

In literature, many studies exist, where the authors have considered the stratum size, variability and stratum cost of the data collection while surveying the population. Yadav and Verma

[5] studied the exponential ratio-type estimators under the linear cost function in the set up of stratified random sampling. Focus of study is on the estimation of population parameter with the help of collected data using proposed method. A linear function is used to determine the relation among sample sizes of each stratum. Yadav et al.[6] worked on the behaviour of ratio-product-cum-exponential-cum-logarithmic type estimators with one auxiliary variable in the stratified random sampling setup and analyzed such with the linear cost function using numerical illustrations. Ghosh[1] suggested a new method of allocation of sample sizes for stratum. In this, the author has taken the average of the optimum allocation for the different characters individually. Khan et al.[2] used compromised allocation in multivariate stratified sampling for an integral solution. Varshney [4] worked on an optimum allocation of sample sizes in the presence of non-response factor under the multivariate stratified double sampling setup. In such, authors have considered sample size allocation problem in stratified random sampling for single character as well as for multiple characters with varying cost functions.

Koyuncu and Kalidar[8] suggested a new family of estimators for stratified random sampling utilizing the information of the coefficient of kurtosis of the population and obtained efficient conditions between the adopted and proposed families. Theoretical finding are supported by the numerical examples with original data. Singh et al.[9] addressed the problem of various types of estimation of the main variable parameter in the presence of non-response and measurement error both incorporating the information of two auxiliary variables. In that, authors derived the optimum strata weights using the suitable calibration technique. Bhushan et al.[10] developed efficient classes of estimators in stratified sampling for combined ratio and separate ratio type estimators. Such estimators are theoretically justified and compared over the conventional estimator, classical ratio estimator and classical regression estimator using the simulation study. Tiwari et al.[11] proposed a general class of estimators for estimating the population mean of study variable using the support variable based correlated information. Members of such proposed class are identified and compared in terms of efficiency. Kadilar and Chingi [16] derived some ratio-type estimators and discussed their properties in the setup of stratified sampling. Aamir et al. [13] suggested a generalised class of exponential-type estimators for population mean by taking the two auxiliary variables for estimating the unknown means with the case of sub-sampling and non-response. In such, authors derived the conditions under which proposed estimators are more efficient as composed to other estimators. Cekim and Kalidar [12] suggested some estimators for estimating the population variances in stratified sampling, in the form of in-function type estimators. Ahnad et al.[17] proposed an improved family of estimators for estimating the population distribution function. The main aim of such contribution was to develop an enhanced family of log ratio-exponential based estimation procedure under stratified sampling. Zaman and Kalidar [15] suggested exponential ratio and product type estimators the mean by considering the two phase sampling setup in stratified sampling.

This paper considers the risk factor exposed on the life of survey workers over different strata. A sample size allocation keeping the method is discussed considering a risk function with computation of allocations variance optimal.

1.1. Risk in Survey Sampling

While data collection, using stratified sampling, some strata may have higher risk on the life of surveyor while others may have a little. For example, a strata of a population is affected by the nuxalite movement, next strata bears high rate of murders and killings, third one is affected by the dangerous epidemics(like malaria, dengue, COVID-19). The strata-wise risk on the life of investigators could be pre-estimated using police record and hospital records (for last one/five years) as below:

Strata I	(1) Deaths due to murder and mass killing=α_{11} (2) Deaths due to communal riots= α_{12} (3) Deaths due to epidemics and community diseases= α_{13}
Strata II	(1) Deaths due to murder and mass killing=α_{21} (2) Deaths due to communal riots= α_{22} (3) Deaths due to epidemics and community diseases= α_{23}

The minimum risk may be assumed as r_0 which includes the normal risk of natural death during the survey work.

1.2. Symbols used for analysis

Let a population of finite size N, divided into L stratum. Each stratum is of size N_i where $N_1 + N_2 + ... + N_L = N$ holds and samples are taken from each strata of size n_i such that $n = (n_1 + n_2 + ... + n_L)$, where n denotes total size of sample.
Notations used for population parameters are:

$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{L}\sum_{j=1}^{N_i} Y_{ij}, \qquad S^2 = \frac{1}{N-1}\sum_{i=1}^{L}\sum_{j=1}^{N_i}(Y_{ij} - \bar{Y})^2 \tag{1.1}$$

$$\bar{Y}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} Y_{ij}, \qquad S_i^2 = \frac{1}{N_i-1}\sum_{j=1}^{L}(Y_{ij} - \bar{Y}_i)^2 \tag{1.2}$$

where Y_{ij} is the j^{th} observation of the i^{th} strata in a population of size N.
Let a sample of size n is drawn by the SRSWOR sampling scheme keeping n_i from each stratum, then sample related notations are:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{L}\sum_{j=1}^{n_i} y_{ij}, \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{L}\sum_{j=1}^{n_i}(y_{ij} - \bar{y})^2 \tag{1.3}$$

$$\bar{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}, \qquad s_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2 \tag{1.4}$$

1.3. Risk Indicators

Define risk indicators r_i as:

$$r_i = (\frac{T_i}{N_i}), \text{ for } i^{th} \text{ strata} \tag{1.5}$$

$$\text{At } i = 1, \qquad r_1(Risk) = \frac{T_1}{N_1}, \ for\ strata\ I \tag{1.6}$$

$$\text{At } i = 2, \qquad r_2(Risk) = \frac{T_2}{N_2}, \ for\ strata\ II \tag{1.7}$$

$$\tag{1.8}$$

where,

$$Total : T_1 = (\alpha_{11} + \alpha_{12} + \alpha_{13}) \tag{1.9}$$

$$Total\ strata\ I\ Population = N_1 \tag{1.10}$$

$$Total : T_2 = (\alpha_{21} + \alpha_{22} + \alpha_{23}) \tag{1.11}$$

$$Total\ strata\ I\ Population = N_2 \tag{1.12}$$

$$\tag{1.13}$$

These indicators are crude measures of the intensity of risk imposed on the life of field investigators who collect primary data through sample survey in a stratified population.

1.4. Motivation

The proportional allocation is stratum size based and Neyman allocation is size + variability based for i^{th} stratum. There is one more method which is cost based allocation per stratum but involvement of stratum risk is yet not considered by any author. In order to utilize the information contained in the risk indicators r_i, the problem of sample size determination is attempted in this paper.

2. Mean Estimation approach in Stratified Sampling

The usual mean estimator under the stratified sampling is:

$$\bar{y}_{st} = \sum_{i=1}^{L} W_i \bar{y}_i \tag{2.1}$$

The variance for stratified random sampling is:

$$V(\bar{y}_{st}) = \sum_{i=1}^{L} W_i^2 \left\{ \frac{1}{n_i} - \frac{1}{N_i} \right\} S_i^2 \tag{2.2}$$

where W_i represents the weight of each stratum on the basis of its size i.e. $W_i = \left(\frac{N_i}{N}\right)$.

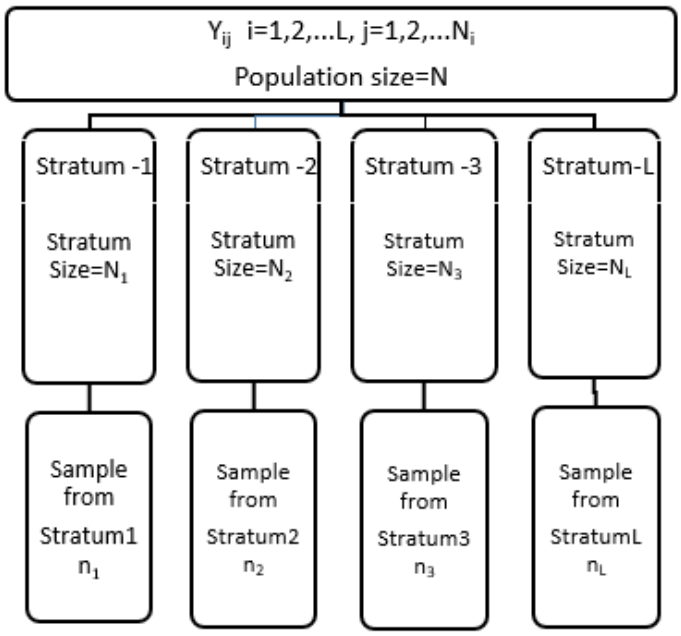

Figure 1: *Sampling structure for L stratum based stratified sampling*

3. Linear risk function

Consider the linear risk function for the stratified sampling:

$$r = r_0 + \sum_{i=1}^{L} n_i r_i \tag{3.1}$$

where,
r_0: Minimum pre fixed risk exposed on life of investigators.
r_i: Risk per unit in a stratum.
r^*: Total risk exposed on investigator while survey of entire population including natural death.
The objective of this paper is to determine the sample size from each stratum using the linear risk function, keeping variance minimum. This objective can be achieved by optimizing following,

$$Minimize\ V(\bar{y}_{st}) \tag{3.2}$$

subject to the conditions,

$$\sum_{i=1}^{L} r_i n_i = r^* \tag{3.3}$$

$$\sum_{i=1}^{L} n_i = n \tag{3.4}$$

For solution using the Lagrange multiplier technique defined and optimize the following function ϕ

$$\phi = V(\bar{y}_{st}) + \lambda_1 \left(\sum_{i=1}^{L} n_i - n \right) + \lambda_2 \left(\sum_{i=1}^{L} r_i n_i - r^* \right) \tag{3.5}$$

where λ_1, λ_2 are constants called Lagrange multipliers. Differentiating ϕ with respect to n_i, λ_1, λ_2 and equating to zero, one can get,

$$n_i = \frac{W_i S_i}{\sqrt{\lambda_1 + \lambda_2 r_i}} \tag{3.6}$$

Summing (3.6) on both sides,

$$n = \sum_{i=1}^{L} n_i = \sum_{i=1}^{L} \left[\frac{W_i S_i}{\sqrt{\lambda_1 + \lambda_2 r_i}} \right] \tag{3.7}$$

From (3.6) and (3.7), one get insights,

$$n_i \propto N_i \tag{3.8}$$

$$n_i \propto S_i \tag{3.9}$$

$$n_i \propto \frac{1}{\sqrt{\lambda_1 + \lambda_2 r_i}} \tag{3.10}$$

where r_i is risk related to i^{th} strata.

4. Computational Algorithm for Optimal Variance along with choice of λ_1 and λ_2

Step I : For given N, n calculate initial values N_i, S_i, $\bar{Y}_i$ and $(N_i S_i)$ and r_i of the population

Step II : Find $V_(\bar{y}_{st})$ using Neyman allocation, which is based on $n_i \propto N_i$ and $n_i \propto S_i$ with expression $n_i = \left\{ \frac{nW_i S_i}{\sum W_i S_i} \right\}$. Find variance $V(\bar{y}_{st})$ using proportional allocation which is based on criteria $n_i \propto N_i$ only with expression $n_i = nW_i$

Step III : Find the risk r_i and use risk function $r^* = \sum r_i n_i$.

Step IV : Set

$$\phi = V(\bar{y_{st}}) + \lambda_1 \left(\sum_{i=1}^{L} n_i - n \right) + \lambda_2 \left(\sum_{i=1}^{L} r_i n_i - r^* \right) \tag{4.1}$$

where λ_1, λ_2 are constants to determine under risk assumption.

Step V : For risk based allocation of sample size n_i,

$$n_i \propto N_i \tag{4.2}$$
$$n_i \propto S_i \tag{4.3}$$
$$n_i \propto \frac{1}{\sqrt{\lambda_1 + \lambda_2 r_i}} \tag{4.4}$$

Step VI : Use simulation procedure to find values of λ_1 and λ_2 to optimize variance $V(\bar{y}_{st})$

(a) Fix the values of λ_1,

(b) Vary λ_2 on x-axis of the graph and plot graph for variance, along with n_1 *and* n_1,

(c) Continue the process of creating graphs for different values of λ_1,

(d) When variance line becomes parallel to x-axis then stop the simulation process.

(i) Choose that input-data set $n_1, n_2, \lambda_1, \lambda_2$ (producing parallel line)

(ii) Use values to get optimal solution.

5. Empirical Study

Consider following data of size N= 244 from 6^{th} Minor Irrigation Census - Village Schedule - Assam[7]. The crime data obtained from police station and hospitals as under(assumed data for a year):

Strata I :

(a) Deaths due to bullet firing = 8

(b) Deaths due to riots = 11

(c) Deaths due to epidemic = 6
Total = 25
Total strata size= 127

Strata II :

(a) Deaths due to bullet firing = 11

(b) Deaths due to riots = 15

(c) Deaths due to epidemic = 10
Total = 36
Total strata size= 135

The basic data and basic computation is as under:

Table 1: *Data for Strata (Source, please see [7])*

i	N_i	W_i	$\bar{Y}_i$	S_i^2	r_i
1	127	0.5205	703.74	883.83	19%
2	135	0.48	413	644.922	26%

Table 2: *The proportional allocation provides*

n_1	n_2	n	$V(\bar{y}_{st})_{prop}$
72	108	180	804.5

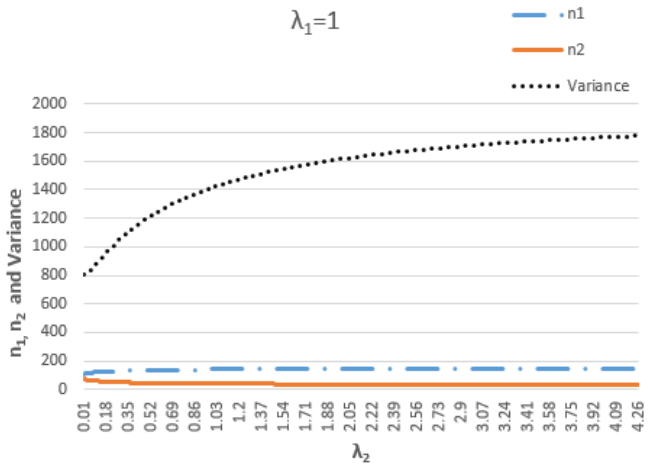

Figure 2: *Variation when λ_1=1 fixed*

Fig.(2) reveals that, for fixed value of $\lambda_1 = 1$, the variance of $V(\bar{y}_{st})$ has growing trend and under risk consideration . It is observed that λ_2 increases for fixed λ_1.Moreover, $V(\bar{y}_{st})$ fluctuates between between 800 to 1800. There is miled increase in n_1 for increasing λ_2.

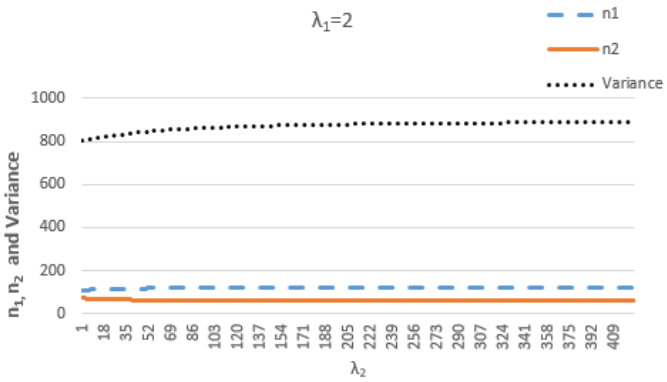

Figure 3: *Variation when λ_1=2 fixed*

Fig(3) is an indicator of the analysis of $V(\bar{y}_{st})$, as the value of λ_2 increases for fixed value of $\lambda_1 = 2$, the value of $V(\bar{y}_{st})$ lies between 800 to 1000.

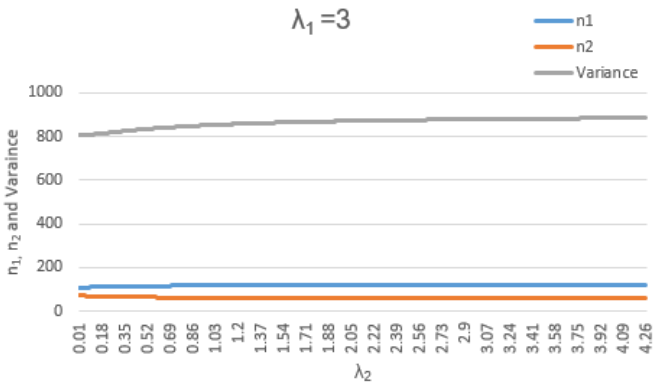

Figure 4: *Variation when λ_1=3 fixed*

Fig.(4) opens starting avenue for decrease in $V(\bar{y}_{st})$ as the value of λ_1 is increases, the $V(\bar{y}_{st})$ reduces.

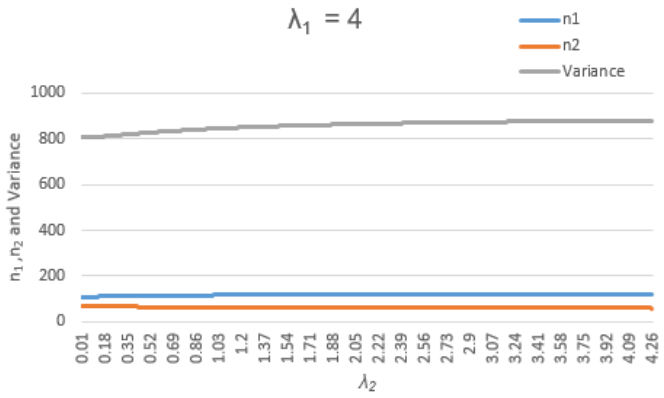

Figure 5: *Variation when λ_1=4 fixed*

Fig.(5) shows that the V($\bar{y}_{st}$) line is tending to become parallel to the x-axis(on higher λ_2 values).

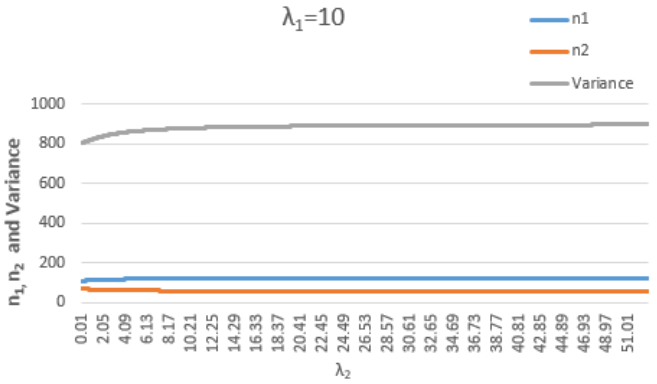

Figure 6: *Variation when λ_1=10 fixed*

Fig.(6) represents the similar pattern as observed in Fig.(5) to get $V(\bar{y}_{st})$.

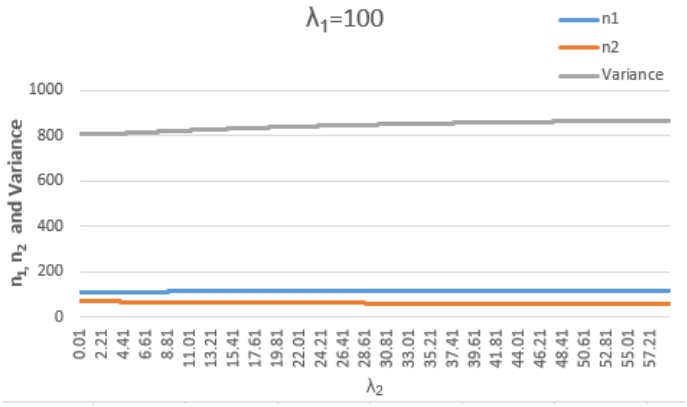

Figure 7: *Variation when λ_1=100 fixed*

Fig.(7) highlights that for higher values of λ_1, the relation between V($\bar{y}_{st}$) over the incrementing values of λ_2 is almost parallel to x-axis. Such indicates for V($\bar{y}_{st}$) being almost independent to the variation of λ_2.

Table 3: *The Neyman allocation provides*

n_1	n_2	n	$V(\bar{y}_{st})_{Ney}$
71	119	180	803.5

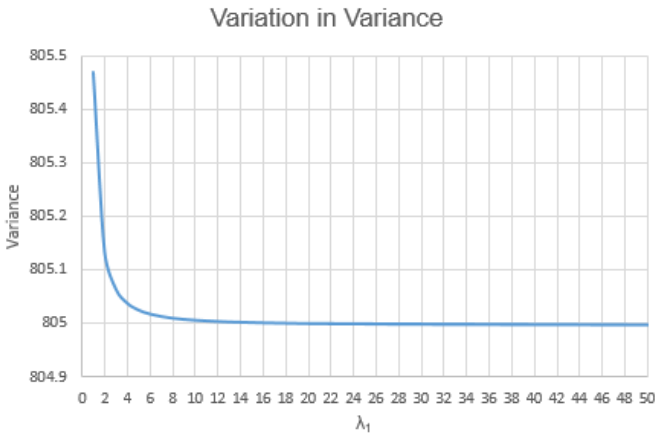

Figure 8: *Variation with respect to parameters*

Fig.(8) depicts the relation between λ_1 and V($\bar{y}_{st}$). The value of V($\bar{y}_{st}$) is gradually decreasing as the values of λ_1 increases from 1 to 19. After $\lambda_2 = 19$ (approximately), there is no significant change in V($\bar{y}_{st}$).

6. Comparison and Discussion

On comparing the different types of allocations (Table 5) it is evident that allocations are very close to each other and providing the optimal variance. The approach aimed at to utilize the

Table 4: *The risk based allocation provides*

n_1	n_2	n	$V(\bar{y}_{st})_{risk}$
72	118	180	805.26

Table 5: *Different allocation methods provides*

Proportional allocation provides	n_1	n_2	n	$V(\bar{y}_{st})_{prop}$
	72	108	180	804.5
Neyman allocation provides	n_1	n_2	n	$V(\bar{y}_{st})_{prop}$
	71	119	180	803.5
Risk based allocation	n_1	n_2	n	$V(\bar{y}_{st})_{risk}$
	72	108	180	805.26

crime record information of police station and hospital records of the strata during sample survey. Such can be useful to determine the sample size allocation n_i from the i^{th} strata ($i = 1, 2, 3, ..., L$), so that $n = \sum_{i=1}^{L} n_i$ remains intact. Risk indicators are suggested and defined using the crime record and hospital records. The Lagrange multiplier technique provides two constants λ_1 and λ_2 whose values need to be computed using the available data. As evident in graphical pattern from [fig(2) to fig(8)], the increasing values of λ_1 provides the solution for best choice of n_1 & n_2 at the situation when variance remain stable (independent of increasing λ_2). An appendix added at the end provides choice of λ_1, λ_2 and n_1 , n_2. At λ_1=300, λ_2=0.01 one gets n_1=108, n_2=72 with lowest

variance 804.99 as displayed in the table(7) of appendix. When λ_1 increases then λ_2 decreases to attain same level of optimality. For any arbitrary choice of λ_1 the table(7) provides the value of λ_2 for quick selection. In general, $21 \leq \lambda_1 \leq 40$ and $0.1 \leq \lambda_2 \leq 0.30$ is the recommended rapid selection of λ-values.

7. Conclusion

This paper presents a new idea of using the regional (strata) risk on the life of survey investigators with the help of risk indicators. In literature, when stratified sampling is used, the problem of sample size allocation appears that it could be resolved as per population strata size or as per population strata variability. The proportional allocation is based on population strata sizes while the Neyman allocation is based on size and variability both. Such allocations do not consider the risk factor imposed on the life of investigator. If risk is high for a particular strata then smaller sample size is required from that strata. The proposed risk based sample size allocation is like $n_i \propto N_i$,$n_i \propto S_i$ and $n_i \propto \frac{1}{\sqrt{\lambda_1+\lambda_2 r_i}}$ incorporating two constants λ_1 & λ_2. An algorithm is proposed in this paper showing how to compute λ_1 and λ_2 constants with minimizing the population variability factor of the mean estimate. If $1 \leq \lambda_1 \leq 10$ then it is suggested to choose $\lambda_2 = \frac{\lambda_1}{200}$ as per table 7. Similarly when $10 \leq \lambda_1 \leq 20$ then recommended to choose $\lambda_2 = \frac{\lambda_1}{100}$ as per table 7, shown in appendix. Various graphs from (fig.(2) to fig.(8)) reveal that when variance line becomes parallel to x-axis for set of values $(\lambda_1, \lambda_2, n_1, n_2)$, such provide the optimal solution for lowest variability due to the risk based sample size allocation. In general, one can work with risk based allocations choosing $21 \leq \lambda_1 \leq 40$ and $0.1 \leq \lambda_2 \leq 0.3$ (table 7) to get nearly optimal result. The crime data of all police stations and health data from hospitals can be utilized for risk computation and accordingly can be used in risk based sample size allocation. The table 7 attached in appendix helps in rapid selection of λ_2 for an arbitrary choice of λ_1

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Appendix

Table 6: *λ_1 varies but λ_2 fixed*

λ_1	λ_2	n_1	n_2	n	$V(\bar{y}_{st})$	λ_1	λ_2	n_1	n_2	n	$V(\bar{y}_{st})$
1	0.01	109	71	180	805.469	30	0.01	108	72	180	804.998
2	0.01	108	72	180	805.134	32	0.01	108	72	180	804.997
3	0.01	108	72	180	805.063	33	0.01	108	72	180	804.997
4	0.01	108	72	180	805.037	34	0.01	108	72	180	804.997
5	0.01	108	72	180	805.024	35	0.01	108	72	180	804.997
6	0.01	108	72	180	805.017	36	0.01	108	72	180	804.997
7	0.01	108	72	180	805.012	37	0.01	108	72	180	804.997
8	0.01	108	72	180	805.009	38	0.01	108	72	180	804.997
9	0.01	108	72	180	805.007	39	0.01	108	72	180	804.997
10	0.01	108	72	180	805.005	40	0.01	108	72	180	804.997
11	0.01	108	72	180	805.004	41	0.01	108	72	180	804.997
12	0.01	108	72	180	805.003	42	0.01	108	72	180	804.997
13	0.01	108	72	180	805.002	43	0.01	108	72	180	804.997
14	0.01	108	72	180	805.001	44	0.01	108	72	180	804.997
15	0.01	108	72	180	805.001	45	0.01	108	72	180	804.997
16	0.01	108	72	180	805.000	46	0.01	108	72	180	804.997
17	0.01	108	72	180	805.000	47	0.01	108	72	180	804.997
18	0.01	108	72	180	805.000	48	0.01	108	72	180	804.997
19	0.01	108	72	180	804.999	49	0.01	108	72	180	804.997
20	0.01	108	72	180	804.999	50	0.01	108	72	180	804.997
21	0.01	108	72	180	804.999	20	0.01	108	72	180	804.999
22	0.01	108	72	180	804.999	25	0.01	108	72	180	804.998
23	0.01	108	72	180	804.998	50	0.01	108	72	180	804.997
24	0.01	108	72	180	804.998	100	0.01	108	72	180	804.996
25	0.01	108	72	180	804.998	150	0.01	108	72	180	804.996
26	0.01	108	72	180	804.998	200	0.01	108	72	180	804.996
27	0.01	108	72	180	804.998	250	0.01	108	72	180	804.996
28	0.01	108	72	180	804.998	300	0.01	108	72	180	804.996
29	0.01	108	72	180	804.998	1000	0.01	108	72	180	804.996

Table 7: *λ_1 and λ_2 are varying*

λ_1	λ_2	n_1	n_2	n	$V(\bar{y}_{st})$
1.00	0.50	118.94	61.06	180.00	863.32
2.00	0.49	116.89	63.11	180.00	843.62
3.00	0.48	115.43	64.57	180.00	832.17
4.00	0.47	114.34	65.66	180.00	824.95
5.00	0.46	113.48	66.52	180.00	820.12
6.00	0.45	112.79	67.21	180.00	816.75
7.00	0.44	112.23	67.77	180.00	814.30
8.00	0.43	111.76	68.24	180.00	812.49
9.00	0.42	111.36	68.64	180.00	811.11
10.00	0.41	111.01	68.99	180.00	810.03
11.00	0.40	110.71	69.29	180.00	809.19
12.00	0.39	110.45	69.55	180.00	808.51
13.00	0.38	110.22	69.78	180.00	807.96
14.00	0.37	110.02	69.98	180.00	807.51
15.00	0.36	109.83	70.17	180.00	807.14
16.00	0.35	109.67	70.33	180.00	806.84
17.00	0.34	109.51	70.49	180.00	806.58
18.00	0.33	109.38	70.62	180.00	806.36
19.00	0.32	109.25	70.75	180.00	806.17
20.00	0.31	109.14	70.86	180.00	806.02
21.00	0.30	109.03	70.97	180.00	805.88
22.00	0.29	108.93	71.07	180.00	805.77
23.00	0.28	108.84	71.16	180.00	805.67
24.00	0.27	108.76	71.24	180.00	805.58
25.00	0.26	108.68	71.32	180.00	805.50
26.00	0.25	108.61	71.39	180.00	805.44
27.00	0.24	108.54	71.46	180.00	805.38
28.00	0.23	108.48	71.52	180.00	805.33
29.00	0.22	108.42	71.58	180.00	805.29
30.00	0.21	108.36	71.64	180.00	805.25
31.00	0.20	108.31	71.69	180.00	805.21
32.00	0.19	108.25	71.75	180.00	805.18
33.00	0.18	108.21	71.79	180.00	805.16
34.00	0.17	108.16	71.84	180.00	805.13
35.00	0.16	108.12	71.88	180.00	805.11
36.00	0.15	108.08	71.92	180.00	805.10
37.00	0.14	108.04	71.96	180.00	805.08
38.00	0.13	108.00	72.00	180.00	805.07
39.00	0.12	107.97	72.03	180.00	805.05
40.00	0.11	107.93	72.07	180.00	805.04

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DIGITAL INVENTORY: REFORMAT RISK OPTIMIZATION MODEL FOR A LAPTOP

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Abstract

In recent times, due to advancements in technologies specially in the computer world, people face problem related to limited digital capacity of a digital devices. Many reasons exist such as unwanted or unnecessary files stored in (a) System digital space (b) ROM space (c) Working space for users and (d) Hard disk space. By the regular use of a laptop, user space and hard disk digital space get occupied because of the creation of new files and new folders at every moment. Such a situation motivates for development of a digital inventory model for digital space. This paper presents a digital inventory model which is a useful tool for laptop reformat risk minimization. Users Categories are defined as per their intensive professional involvements. Several graphs are drawn showing the output analysis and importance of the study. Theoretical findings are supported by the numerical computations. It is found that reformat risk is directly proportional to the growth of file/folder creation in either of categories.

Keywords: Digital Inventory, Mathematical Model, Memory Space, Digital Space Digital Files, Reformat Risk, Digital devices, Risk Optimization, Users Category, Risk Computation, Random Access Memory(RAM), Read Only Memory(ROM).

1. Introduction

The inventory management is a key factor in the supply chain of business for achieving the goal of higher profit. Such is useful to maintain a continuous flow between production and demand. Inventory of items keeps the market stable over the varying demand scenario and investors are aware enough about the cost-profit ratio. With the emergence of Internet in late nineties and drastic spread of the use of digital payments technologies, various companies have started business of producing digital devices. Some most popular digital devices are (i) Mobile phones (ii) Desktop computers (iii) Laptops (iv) tablets etc. Each digital device has memory spaces like RAM, ROM, Hard disk, Memory card, SSD, Flash memory etc. The information in digital form are stored in files and folders in the available digital spaces. A file is a basic unit used for digital inventory. The collection of similar types and similar nature of files constitutes a folder. The basic digital space unit is bit and 8 bits constitute a byte. The file sizes are usually measured in terms of bytes like kilobytes (KB), Megabytes (MB), Gigabits (GB), terabytes (TB), petabytes (PB) etc. In computer sciences literature, there exist basic and well-established rules of classification like (i) System files storage space (ii) Application Program Storage space and (iii) User files storage space. In term of specific terminologies such spaces are called memory spaces like (a) Internal memory space and (b) External memory space. The operating system of a laptop needs a prefixed size memory space allocation which is mandatory and cannot be reformatted unless corrupted. The

software like Browsers, MS Office packages and downloaded programs are the part of application software. These can be deleted or updated or reloaded by users from time to time. The creation of Word files, Excel files, Power Point files, Photo files, Music files and videos are the part of users- files. A digital storage of files in memory of a laptop can be assumed as a digital inventory of files and folders.
A laptop is an excellent tool, but without systematic organization of files (or folders), the same may be troublesome during files search. Every day users of electronic devices create large number of files and delete the same too (if not in use). Following are steps for file management:

- Review of files.
- File size and content evaluation.
- Shorting of files and allocation in appropriate folders.
- Folder indexing and folder organization.
- Safety of storage of files/folders.
- Storage of files/ folders at earmarked location.
  Which file to delete and which file to store is a continuous process. When the creation is faster than deletion, digital storage gets saturated after a duration and situation of reformatting the laptop appears.

2. Literature Review

The problem of dealing with digital inventories have rare academic contributions in terms of models as appeared after web search. The said way of thinking where the storing unit is a digital file (or folder) and digital devices are like a warehouse it is a unique approach attempted in this paper.

Govani[1]proposed a digital supply chain marketing model and operation interactions. Such contributions provide comparisons between static and dynamic solutions of the procedure involved in the supply chain while shifting from traditional to digital platforms.

Al-dulaime et al.[2]suggested inventory management based on the EOQ model for laptop spare parts and analyzed such using XYZ technique. Excess inventory of spare parts and over stocks provide the larger holding cost. The EOQ cost management model is suggested by the author in the form of digital hardware inventory.

Muckstadt and Sapra[8] contributed on the thought provoking idea related to EOQ inventory models advocating that such can be used in practice for prediction and profit maximization. For example, inventory models can be effectively employed in automotive, pharmaceutical and retail sectors of the economy for many years.

Chung[9] advocated for a new EOQ model under the situation of permissible delay in payments. The cost function assumed convex and a theorem is developed for optimality of EOQ. Dhoka and Chaudhary [10] discussed the complexities of managing supply chains that are affected by many external factors like global effect, increase in product portfolio and decentralization. A variety of inventory models and metrics are available today to monitor the supply chains. The main focus of paper[10] is to study the volatility of supply chains and its impact on inventory.

Tersine et al.[11] contributed on the inventory reduction methods and technologies that hold inventory by nature at idle. The intentional and negligent stockpiling inspires reform and heralds the redesign of operating system. If the quest starts with immediate reduction of stocks, it ends with complete system conversion.

Prameswari et al.[13] attempted for product inventory optimization using EOQ model approach by suggesting circular economy model which aims to minimize waste and maximize the use of resources. The EOQ in the content is used as a tool to optimize product order quantities using the circularity index.

Di Nardo et al. [15] worked on stochastic dynamic optimization depending upon logistic environment. Such contains a stock dynamic sizing optimization where the safety stock is conceived to fill up to variation in demand. The contribution aims at to reduce the occurring stock-out events using the link among wear out items rate downstream logistic demand.

Hemant [17] proposed to optimize an inventory model for a company who reduces inventory cost and provides better inventory management system. Such contains quantitative approach as a solution of the problem. Cotribution [17] suggested an idea of computing the appropriate order quantity in dynamic form.

Turkolmez at al. [18] attemped the use of machine learning approch for resolving the pricing of end-of-life remanufactured laptop. Some other useful contributions to the EOQ based inventory models and their applications are due to Khan[7], Taha[12], Vandeput [14], Shah[16], Celik et al.[19], Khedlekar et al.[3],[4],[5]and Shukla et al.[6]

2.1. Motivation

All the EOQ models consider the physical stock of items in the warehouse. The problem to handle is when to fill the stock and when to vacate and sale. with when digital inventory the optimization idea changes quite heavily.

This paper considers the digital inventory of files/folders and presents risk computation of reformatting a laptop over time by presenting a model.

3. Symbols

- S_m- Maximum digital inventory space of laptop,
- S_0- Initial minimum digital storage of laptop (when purchased),
- T - Total time,
- t - Reformat Time of the laptop(when entire digital space occupied),
- R - Rate of filling of digital storage at time t (rate of file/folder creation),
- R_f - Reformat risk,
- ϕ_A - Area of representing unoccupied digital inventory space.

4. Formation of Model

4.1. Assumptions

- Let total digital inventory storage in laptop is S_m at a time $t = 0$ (when purchased from the market).
- Assume that laptop initial storage already (captured by operating system files and necessary user application programs files) is S_0 at a time $t = 0$.
- R is the rate of creating files/folders by users at time t which is assumed constant over $t(R > 0)$.
- It is assumed that there is no deletion of files by laptop user.
- T is the total life time of a laptop.
- t represents time when laptop digital inventory is completely filled and reformat is needed.

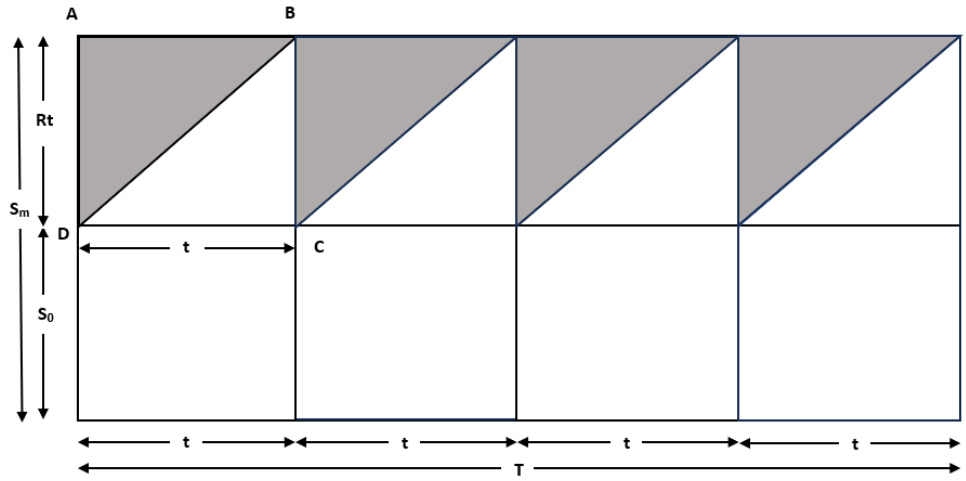

Figure 1: *Working process of occupancy of digital inventory space and reformat at time t*

4.2. Optimization

Consider figure1 who represents the diagram of occupied and free digital inventory space. It is clear that when area of $\triangle ABD$ increases then reformat risk decreases, So

$$\text{Laptop reformat risk } R_f \propto \left[\frac{1}{\text{Area of } \triangle ABD}\right] \tag{1}$$

Maximum (total) digital inventory storage space $= S_m$ at time $t = 0$. therefore, area of entire rectangle $= S_m t$. Internal pre-occupied digital inventory storage $= S_0$. Now $S_0 t$ is area representing the pre-occupied digital inventory space. Area of $\triangle DBC = \frac{1}{2}Rt^2$ which is occupied space. The available digital inventory space is represented by

$$\triangle ABD = \phi_A = [S_m t - S_0 t - \frac{1}{2}Rt^2] \tag{2}$$

for optimizing time of risk of reformat

$$\frac{\partial \phi_A}{\partial t} = 0$$
$$(S_m - S_0 - Rt) = 0$$
$$t_{opt} = \left(\frac{S_m - S_0}{R}\right) \tag{3}$$

Using equation2

$$\phi_{A_{opt}} = S_m\left(\frac{S_m - S_0}{R}\right) - S_0\left(\frac{S_m - S_0}{R}\right) - \frac{1}{2}R\left(\frac{S_m - S_0}{R}\right)^2$$
$$= \left(\frac{S_m - S_0}{R}\right)\left[S_m - S_0 - \frac{1}{2}S_m + \frac{1}{2}S_0\right]$$
$$= \left(\frac{S_m - S_0}{R}\right)\left[\frac{1}{2}S_m - \frac{1}{2}S_0\right]$$

$$[\phi_A]_{opt} = \frac{(S_m - S_0)^2}{2R} \tag{4}$$

Equation4, provides the maximum area of storage space in which the reformat risk at time t is minimum. By equation 2,

$$\frac{\partial^2 \phi_A}{\partial t^2} = -R$$

Since $R > 0$ so,

$$\frac{\partial^2 \phi_A}{\partial t^2} \leq 0 \text{ for all } t$$

Therefore the optimum value of $t_{opt} = \left(\frac{S_m - S_0}{R}\right)$ maximizes the area of $\triangle ABD$.

Now reformat risk

$$R_f = C\left[\frac{2R}{(S_m - S_0)^2}\right] \text{ where } C \text{ is proportionality constant.} \tag{5}$$

4.3. Category of laptop users

Category I: Beginner laptop users.
For this category the value of C will be fixed as $C = 25$ and R varies.

Category II: Moderate laptop users.
The value of C will be fixed as $C = 50$ and R varies.

Category III: Professional laptop users.
For this category, one can choose $C = 75$.

Category IV: Expert IT professional.
For this, it is advised to choose $C = 100$.

Overall the constant C lies between $0 \leq C \leq 100$. It helps to calculate the risk of reformatting category-wise.

5. NUMERICAL SIMULATION

The expression of laptop reformat risk has been computed using simulation at pre-fixed values $S_m = 10$, $S_0 = 2$ keeping variation over $C = [25, 50, 75 \text{ and } 100]$ and $[1, 2, 3, ..., 30]$. Detailed descriptions of R_f are given in tables and graphs attached herewith.

Table 1: *Risk calculation (R_f)*

(a) *Risk calculation (R_f) for Category I*

S_m	S_0	C	R	R_f
10	2	25	1	0.78125
10	2	25	2	1.5625
10	2	25	3	2.34375
10	2	25	4	3.125
10	2	25	5	3.90625
10	2	25	6	4.6875
10	2	25	7	5.46875
10	2	25	8	6.25
10	2	25	9	7.03125
10	2	25	10	7.8125
10	2	25	11	8.59375
10	2	25	12	9.375
10	2	25	13	10.15625
10	2	25	14	10.9375
10	2	25	15	11.71875
10	2	25	16	12.5
10	2	25	17	13.28125
10	2	25	18	14.0625
10	2	25	19	14.84375
10	2	25	20	15.625
10	2	25	21	16.40625
10	2	25	22	17.1875
10	2	25	23	17.96875
10	2	25	24	18.75
10	2	25	25	19.53125
10	2	25	26	20.3125
10	2	25	27	21.09375
10	2	25	28	21.875
10	2	25	29	22.65625
10	2	25	30	23.4375

(b) *Risk calculation (R_f) for Category II*

S_m	S_0	C	R	R_f
10	2	50	1	1.5625
10	2	50	2	3.125
10	2	50	3	4.6875
10	2	50	4	6.25
10	2	50	5	7.8125
10	2	50	6	9.375
10	2	50	7	10.9375
10	2	50	8	12.5
10	2	50	9	14.0625
10	2	50	10	15.625
10	2	50	11	17.1875
10	2	50	12	18.75
10	2	50	13	20.3125
10	2	50	14	21.875
10	2	50	15	23.4375
10	2	50	16	25
10	2	50	17	26.5625
10	2	50	18	28.125
10	2	50	19	29.6875
10	2	50	20	31.25
10	2	50	21	32.8125
10	2	50	22	34.375
10	2	50	23	35.9375
10	2	50	24	37.5
10	2	50	25	39.0625
10	2	50	26	40.625
10	2	50	27	42.1875
10	2	50	28	43.75
10	2	50	29	45.3125
10	2	50	30	46.875

Table 2: *Risk calculation (R_f)*

(a) *Risk calculation (R_f) for category III*

S_m	S_0	C	R	R_f
10	2	75	1	2.34375
10	2	75	2	4.6875
10	2	75	3	7.03125
10	2	75	4	9.375
10	2	75	5	11.71875
10	2	75	6	14.0625
10	2	75	7	16.40625
10	2	75	8	18.75
10	2	75	9	21.09375
10	2	75	10	23.4375
10	2	75	11	25.78125
10	2	75	12	28.125
10	2	75	13	30.46875
10	2	75	14	32.8125
10	2	75	15	35.15625
10	2	75	16	37.5
10	2	75	17	39.84375
10	2	75	18	42.1875
10	2	75	19	44.53125
10	2	75	20	46.875
10	2	75	21	49.21875
10	2	75	22	51.5625
10	2	75	23	53.90625
10	2	75	24	56.25
10	2	75	25	58.59375
10	2	75	26	60.9375
10	2	75	27	63.28125
10	2	75	28	65.625
10	2	75	29	67.96875
10	2	75	30	70.3125

(b) *Risk calculation (R_f) for category IV*

S_m	S_0	C	R	R_f
10	2	100	1	50
10	2	100	2	100
10	2	100	3	150
10	2	100	4	200
10	2	100	5	250
10	2	100	6	300
10	2	100	7	350
10	2	100	8	400
10	2	100	9	450
10	2	100	10	500
10	2	100	11	550
10	2	100	12	600
10	2	100	13	650
10	2	100	14	700
10	2	100	15	750
10	2	100	16	800
10	2	100	17	850
10	2	100	18	900
10	2	100	19	950
10	2	100	20	1000
10	2	100	21	1050
10	2	100	22	1100
10	2	100	23	1150
10	2	100	24	1200
10	2	100	25	1250
10	2	100	26	1300
10	2	100	27	1350
10	2	100	28	1400
10	2	100	29	1450
10	2	100	30	1500

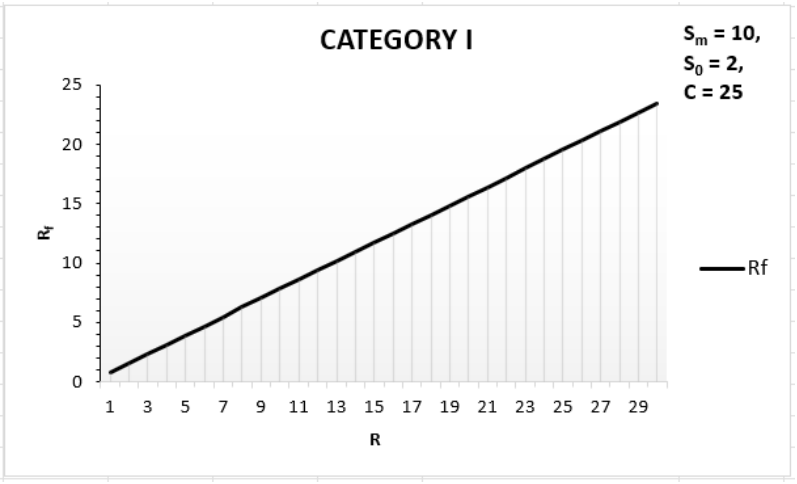

Figure 2: *Risk calculation for category I*

Fig.2 shows laptop reformat risk when $S_m = 10, S_0 = 2, C = 25$.
As the term R increases, the reformat risk increases in a linear manner for the user category of beginners.

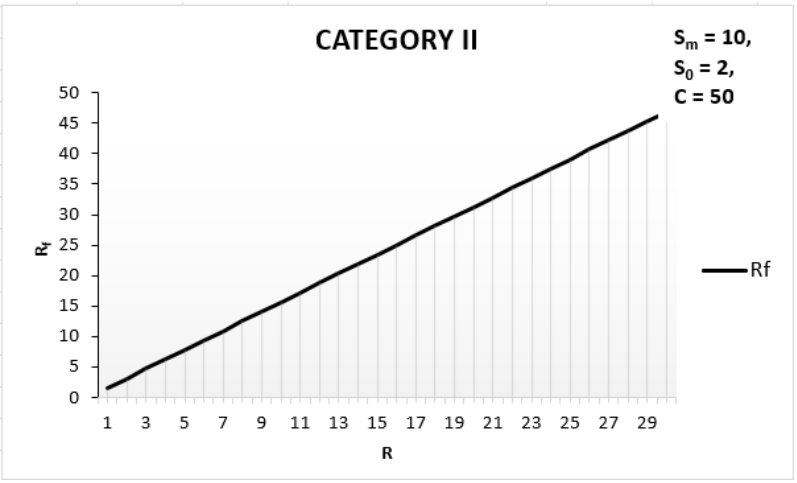

Figure 3: *Risk calculation for category II*

Fig.3 reveals the same pattern (as in fig.2) when $S_m = 10, S_0 = 2, C = 50$ for the category of moderate users.

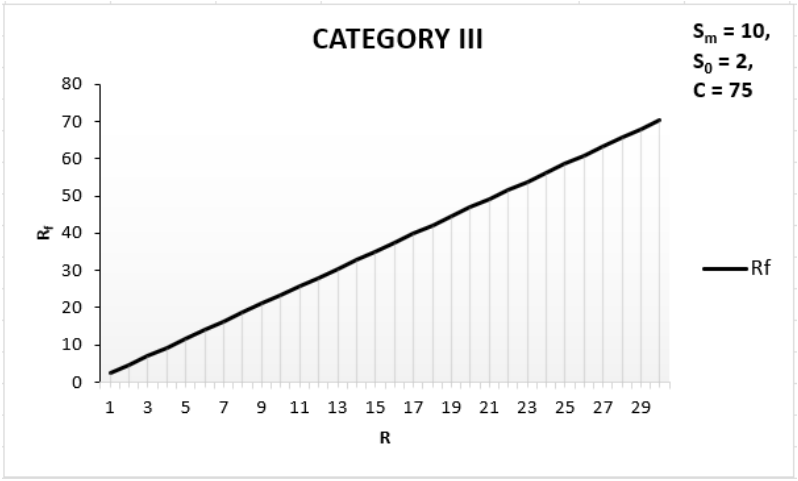

Figure 4: *Risk calculation for category III*

Fig.4 reveals the linear growth of laptop reformat risk when $S_m = 10, S_0 = 2, C = 75$, for the category of IT-professional laptop users.

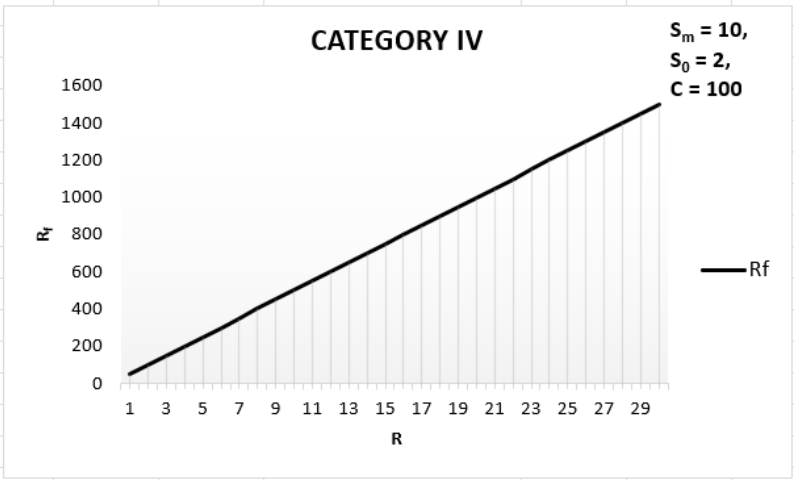

Figure 5: *Risk calculation for category IV*

Fig.5 reflects an increment in risk when $S_m = 10, S_0 = 2, C = 100$, for the category of IT-experts/coders.

6. Conclusion

A digital inventory model is proposed in this paper assuming the digital files/folders creation rate is constant over the time span with with no occurrence of detection of files/folders. The model parameters like S_m (maximum available digital inventory) and S_0 (pre-occupied digital inventory) are kept constant throughout the analysis and computation. An expression for optimal reformat time for a purchased laptop (using the proposed model) is derived. The users vary in terms of their hours spent with laptops. Children spend less, business people provide more, IT-Service providers have high-level associations with their laptops. Four categories with varying constant C values are defined like; beginners, moderate users, professional users, and expert users. The computation of reformat risk of a newly purchased laptop is performed using the proposed digital inventory model.

Figures 2,3,4,5 along with tables 1a,1b, 2a, 2b reveal that compared reformat risk is a linear function over time span in all four categories. The category of IT- Experts bears the highest risk of laptop reformat than the category of beginners.

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ENHANCING PATTERN SEQUENCE-BASED FORECASTS: A MODIFIED STRATEGY RELATIVE TO ELECTRICAL LOAD

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Abstract

A precise forecast of the one-day-ahead load is essential for the efficient management of modern power system operations. This paper proposes a univariate model for short term load forecasting (STLF) that improves the precision of the Pattern sequence forecasting (PSF) algorithm. An analysis was conducted to identify the underlying patterns in the electrical load data using K-means clustering and hierarchical clustering algorithms. The results demonstrate the efficacy of hierarchical clustering. The limitations of the original PSF algorithm, particularly in its clustering and prediction phases are addressed using hierarchical clustering and a new weighted average formula. The proposed method was validated using real-time series datasets and its performance was compared with those of three pattern sequence-based forecasting models. The performance is further evaluated on two electricity demand data sets and compared with bench mark models. The uncertainty and reliability of the forecast model was assessed using an error variance metric. The results show the superior forecast accuracy of the model.

Keywords: short-term load forecasting, hierarchical clustering, pattern sequence, time series, weighted average.

1. Introduction

The growing concerns of society regarding sustainability, decarbonization, and environmental change have spurred technological advancements in electrification, electric vehicles, and renewable energy. These technological breakthroughs present substantial difficulties in the energy supply-demand balance, as electricity storage is difficult [1]. Consequently, electrical load forecasting is crucial for efficient electrical system management. Numerous load forecasting models have been proposed in the literature, depending on the time range of the future values to be predicted: short-term (intraday and day-ahead), medium-term (one week to several months ahead), and long-term (one or more years). Short-term load forecasts are critical for planning power system operations and for bidding strategies in deregulated electricity markets [2]. Load behavior is the fundamental driver of power pricing, therefore the level of accuracy in predicting future loads has a direct impact on the financial performance of energy businesses and other market participants [3].

Over the years, various techniques have been developed for Short Term Load Forecasting (STLF). These models include contemporary computational intelligence, machine learning, and

pattern recognition techniques in addition to traditional methods [4]. Among these, pattern recognition techniques leverage past data to identify load series patterns.

In the short term, load patterns are highly autocorrelated. Univariate models analyse past load patterns to predict future loads and do not depend on external factors. Consequently, univariate models can prevent inaccuracies caused by faulty or noisy exogenous data and produce more reliable and robust forecasts in STLF. This study proposes a univariate model for STLF that relies solely on the historical load series and does not incorporate any other information.

Pattern similarity is crucial in univariate models to ensure precise prediction. Understanding these patterns guarantees that the models capture the essential characteristics of the data, leading to more robust, interpretable, and applicable models across various domains [5]. Unsupervised learning techniques such as clustering reveal hidden patterns in data. This technique groups data points into meaningful clusters based on underlying patterns.

The Pattern Sequence Forecasting (PSF) technique [6] utilized clustering technique to identify patterns in time series data and then applied them to generate predictions. Owing to its efficacy and interpretability, it has gained prominence in a multitude of applications [7-10].

The PSF has certain limitations, although its performance in electrical load forecasting is encouraging. Some previous studies have addressed the limitations of the PSF algorithm and proposed improvements and modifications that are useful in increasing the forecast accuracy of electrical load data and in treating missing values and outliers [11-12]. The current study suggests alterations to the original PSF algorithm in both the clustering and prediction stages to improve the precision of electrical load forecasting. We performed a comprehensive analysis of the proposed methodology using publicly accessible Pennsylvania - New Jersey - Maryland (PJM) market demand data and compared it with benchmark models to ascertain its effectiveness.

The subsequent sections of the paper are organized as follows: Section 2 presents the original PSF algorithm, a literature review of the proposed PSF modifications, and scope for improvement. Section 3 outlines the proposed methodology. Section 4 reports and analyzes the performance of the proposed methodology. Section 5 summarizes the contributions of this study.

2. Pattern Sequence Similarity algorithm: Variations and scope for refinement

2.1. Original PSF algorithm

The PSF algorithm [6] can be divided into two phases: clustering and prediction. Phase one aims to assign each day, or a vector of 24 hours, to a cluster. The cluster pattern sequence prior to the day to be predicted was matched with the historical patterns, and future values were obtained by averaging the subsequent days of the matched patterns.

The different steps involved in both phases of the PSF algorithm are as follows:

The clustering component encompasses several activities, such as data normalization, determination of optimal number of clusters, and obtaining the cluster labels.

- Data normalization: Data normalization was used to smooth the trend from the original data. The transformation used in the original PSF algorithm is $x_j = \frac{x_j}{\frac{1}{N}\sum_{i=1}^{N} x_i}$ where x_j is the demand of the jth hour of the day and N is equal to 24 (the number of hours per day).
- Number of clusters: The optimal number of clusters is determined by the concordance between at least two of the following three indices: the Silhouette index, Dunn index, and the Davies-Bouldin index.
- Clustering/Labelling: K-means clustering was used to label each day with the optimal number of clusters. Clustering reduces the dimensionality of the data from 24 features to a single dimension, which enhances resilience by substituting the actual values with whole numbers (cluster labels).

The prediction phase in PSF consists of activities such as choosing the optimal window size,

identifying matching pattern sequences, and calculating the final forecasts.

Let $X(i) \in \mathbb{R}^{24}$ be a vector composed of 24-hourly demand of day i, and the corresponding cluster label is given by $L_i \in \{1, 2, \dots, K\}$, where K is the number of clusters.

- Selection of optimal window size: The optimal window length (w) of the pattern sequence must be determined prior to prediction. The calculation is performed using n-fold cross validation, and is selected at which prediction error $\sum_{t \in TS} \|\bar{X}(t) - X(t)\|$ is minimum during the training process. Here $\bar{X}(t)$ is the forecasted demand for day t, and TS refers to the testing set.
- Identification of matching pattern sequences: If day d is to be predicted, matchings for a sequence of labels $S_w^{d-1} = [L_{d-w}, L_{d-w+1}, \dots, L_{d-2}, L_{d-1}]$ of window length w, are searched in the labelled data. The search continues until at least one matching pattern sequence of the same length is discovered. If no replicates are identified, the window size is reduced by one unit. This guarantees the presence of at least one duplicate in a labelled sequence, with a minimum w value of 1.
- Forecasting: After identifying the matches, the subsequent 24 values that directly follow all coincidences are extracted to a vector NS. Finally, the values are averaged using the formula given in to anticipate the value of the future load.

$$\bar{X}(t) = \frac{1}{size(NS)} \sum_{j=1}^{size(NS)} NS(j) \quad (1)$$

2.2. Modifications Proposed in the literature

The literature proposes various modifications and improvements to the PSF algorithm. This section discusses some of the variations of the original PSF algorithm. The original PSF algorithm identifies analogous patterns in temporal data, although it had difficulties with specific instances. To address this issue, an enhanced version of the PSF algorithm was developed in [11] to predict anomalies in time series data with high accuracy. The method proceeds by using an additional measure to identify motifs or repetitive patterns in sequences that leads to improved predictions and capacity to identify potential outliers. A modification to PSF was proposed in [12], which uses nonnegative tensor factorization for clustering in PSF and is a promising direction for energy demand prediction. A novel method employing the PSF algorithm is presented in [13] wherein the accuracy of power demand forecasting was enhanced by employing distribution-based predictions and computing the frequency ratios of the cluster patterns. The imputePSF method suggested in [14] is a variation of the PSF algorithm that looks for recurring patterns in observed data to obtain a more accurate estimate of missing values. A novel hybrid algorithm, the funPSF, was designed to forecast functional time series, particularly in the context of electricity demand [15]. This algorithm combines functional data clustering with a forecasting strategy based on pattern sequence similarity. The bigPSF [16] and CUDA-bigPSF [17] algorithms, which build on the PSF method, design big data time series forecasting with notable improvements in scalability and accuracy. An improved version of the algorithm proposed in [10] which makes use of self-organizing maps and artificial neural networks and a genetic algorithm to determine the optimal hyperparameters of the model. MV-bigPSF algorithm [18], was proposed to forecast a multivariate time series. The model leverages the PSF algorithm, showcasing exceptional scalability and effectiveness in handling data sets consisting of millions of samples.

2.3. Scope for Refinement

In the clustering phase, the PSF algorithm employs K-means clustering. Despite its efficiency and simplicity, the K-means clustering algorithm has certain limitations, including sensitivity to the initialization of centroids, scale, and density. Also, ignoring the temporal order of the time series can hinder its efficiency when dealing with time series data or complicated pattern identification.

In the prediction phase, simply averaging the patterns observed immediately after a matched sequence may not be the optimal method. This is because the averaged pattern may not accurately depict the load pattern of the specific day under examination, particularly if the cluster patterns discovered differ from those of the previous working day [13].

3. Proposed Methodology

This section outlines the proposed methodology, which is based on the fundamental PSF algorithm. Below is a summary of the steps of the proposed methodology. Section 3.1 describes the data preprocessing; Section 3.2 details the clustering phase, which includes the determination of the clustering algorithm and the tuning of hyperparameters (k and w); and Section 3.3 presents the prediction phase. Fig. I illustrate the flow of the proposed method.

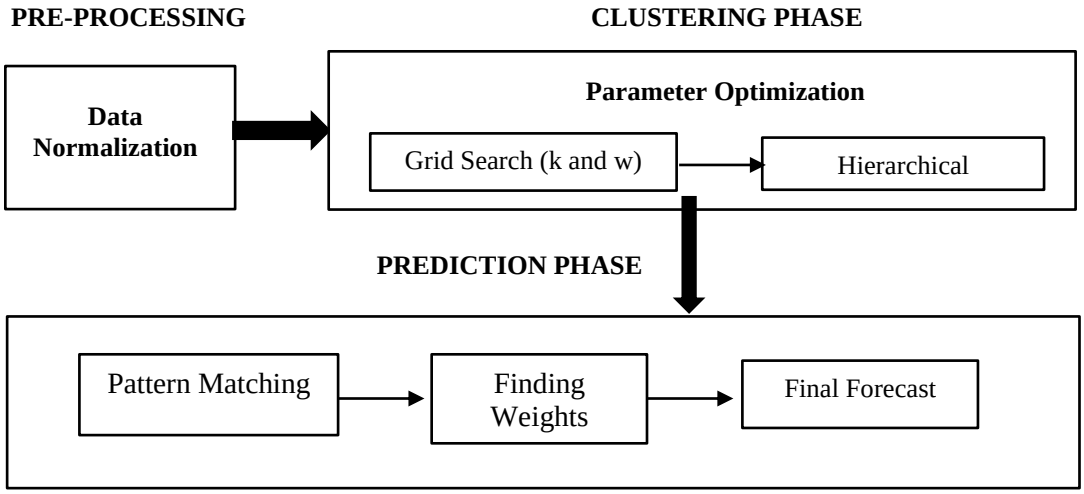

Figure 1: *Flow of the Proposed Methodology*

3.1 Pre-Processing Phase

Data normalization is a crucial technique in data pre-processing, particularly in clustering algorithms, as they rely on distance measurements to determine the similarity between any two data points. When features are not normalized, those with large scales can have a disproportionate impact on the distance calculations, resulting in biased or incorrect cluster labels.

The normalize technique used in this paper is

$$x_j' = \frac{x_j - \min(x_i)}{\max(x_i) - \min(x_i)} \quad (2)$$

where x_j' denotes the normalized value for x_j and $i = 1, 2, \ldots, 24$.

This transformation is called Min-max normalization, which brings all the values into the range $[0,\ 1]$. Min-max normalization ensures that all features contribute equally to the clustering process, prevents any single feature from dominating due to its scale, and can lead to better and more interpretable clustering results [19].

3.2 Clustering Phase

The main objective of the clustering step is to classify the data into groups based on the behavior and underlying patterns in the time series. It provides a representation that preserves the original information and describes the shape of the time series data as accurately as possible.

The clustering phase consists of two steps: finding optimal values for the parameters and the clustering technique.

3.2.1 Parameter Optimization (Optimal values of k and w)

The proposed algorithm has two input parameters: the number of clusters (k) and the length of the window (w) that contains the search patterns. The optimal values of k and w are determined using a grid search over the training set. We split the original data set into training and testing sets. We further divided the training set into two sets, y_T and y_V one for training and the other for validation to fine-tune the hyperparameters. The proposed algorithm was applied to different combinations of k and w for prediction. Among all possible combinations of k and w, the pair that results in the minimum prediction error on y_V i.e., $\sum_{min} \|\overline{y_v}(t) - y_v(t)\|$ is considered as the optimal value of parameters k and w.

3.2.2 Clustering Technique

This study used two different clustering algorithms: K-means and Hierarchical clustering. An analysis of both algorithms was performed to identify the patterns in the historical data.

K-means Clustering:
The primary concept underlying K-means clustering [20] is to establish k centroids, where each centroid represents a distinct cluster with k denoting the predetermined number of clusters. Each point in the given data set was assigned to its closest centroid. The centroids of these new clusters are recalculated and a new binding is performed between the same data points and the new centroids. Consequently, the location of the centroid's changes. This process was repeated until the centroids converged.

Hierarchical Clustering:
Hierarchical clustering generally falls into two types: the agglomerative (bottom-up) and the divisive (top-down). The agglomerative approach is the most common approach for hierarchical clustering. In agglomerative clustering, the clustering algorithm treats each point as an independent cluster and, iteratively merges the two most similar clusters into a single cluster at each step. It creates a tree-like structure called a dendrogram, which records sequences of merges or splits. Fig. 2 depicts a dendrogram with data points on the x-axis and cluster distance on the y-axis. The method of finding similarities between clusters results in the following hierarchical clustering variations: single, average, complete linkages, and Ward's method. Among them is the complete-linkage algorithm, which yields tightly bound clusters [21].

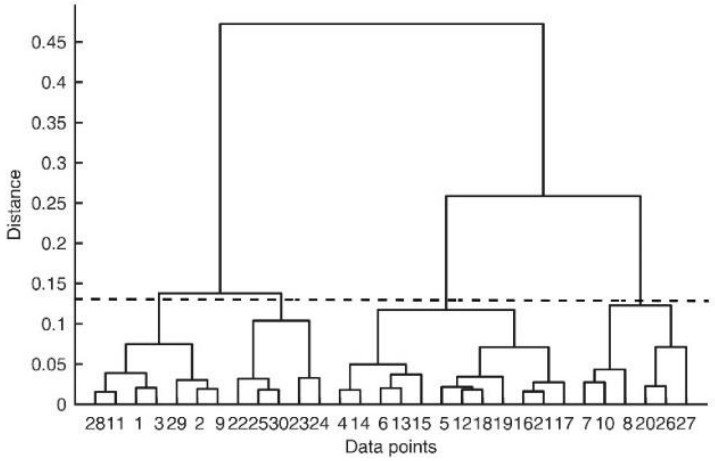

Figure 2: *Dendrogram*

3.3 Prediction Phase

This section proposes a new weighing prediction formula that addresses the limitations of the original PSF algorithm.

Let $O_w^i = [X(i-w+1), X(i-w+2), \dots, X(i-1),\ X(i)]$ be the vector composed of w consecutive days prior to the day $'i'$. The distance between any pair of days i, j is defined as $dist(i,j) = \left\| O_w^i - O_w^j \right\|$, where $\|.\|$ represents the Euclidean norm. The neighbors set of the day $'d-1'$ be $NS = \{q_1, q_2, \dots, q_m\}$ where q_i is the day whose pattern sequence is matched with S_w^{d-1} and q_1 and q_m are the first and m^{th} neighbor in order of distance calculated using the metric $'dist'$. The weighted average of the load for the days following the nearest neighbors provides the prediction, assuming that load profiles that were similar in the past will likely be similar in the future. The prediction is given by Equation. (3)

$$X(d) = \frac{1}{\sum_{i \epsilon NS} \alpha_i} \sum_{i \epsilon NS} \alpha_i X(i+1) \tag{3}$$

where α_i are the weighting coefficients that can be obtained using any of the following schemes given below. The standard method of computing the weighting factors α_i as outlined in [22] is given by means of the Equation (4)

$$\alpha_i = \frac{dist(q_k, d-1) - dist(i, d-1)}{dist(q_k, d-1) - dist(q_1, d-1)} \tag{4}$$

4. Results and Discussion

This section outlines and analyses the performance of the proposed method. Section 4.1 describes the data set used to assess the effectiveness of the proposed method. Section 4.2 outlines the metrics used to measure the quality of the obtained results. Section 4.3 presents an analysis of the clustering techniques. Section 4.4 showcases the performance of the method using PJM market demand data for 2022, while, Section 4.5 outlines a comparative analysis with other methods proposed in the literature.

4.1 Data Set

Electricity demand data of the Pennsylvania - New Jersey - Maryland (PJM) market [23] on hourly basis for the year 2021 is considered to analyse the proposed methodology. The data set comprises 8760 data points with a mean 89.34×10^3 MW. Fig. 3 shows the hourly load data for the year 2021.

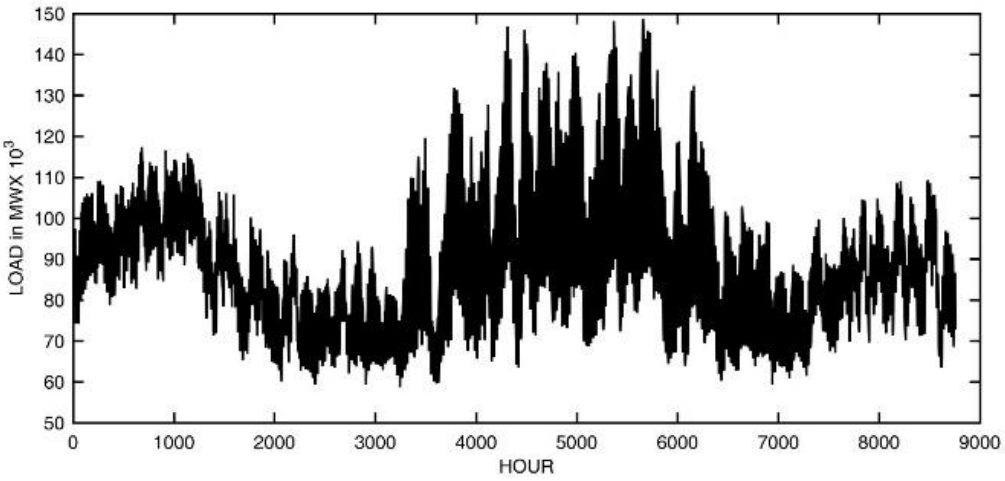

Figure 3: *Hourly demand data of PJM market in the year 2021*

One can observe a high demand during summer months specially from June through August. Before the clustering analysis the data was normalized as mentioned in Section 3.1.

4.2. Performance Metrics

The efficacy of the proposed methodology in obtaining day-ahead forecasts on the considered data was analysed using the forecast error metrics, Mean absolute percentage error (MAPE) in %, Root mean square error (RMSE) and Mean absolute error (MAE).

$$MAPE = \frac{1}{N}\sum_{i=1}^{N}\frac{|l_i-\hat{l}_i|}{l_i}\times 100 \quad (5)$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(l_i-\hat{l}_i\right)^2} \quad (6)$$

$$MAE = \frac{1}{N}\sum_{i=1}^{N}\left|l_i-\hat{l}_i\right| \quad (7)$$

where l_i and $\hat{l}_i$ are the actual load and the forecast load at hour 'i' respectively and N is the number of predictions. In addition, the uncertainty in the forecasts can be estimated through the variance of the forecast error and evaluated using the metric VAR given by [24].

$$VAR = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{|l_i-\hat{l}_i|}{l_i}-MAPE\right)^2 \quad (8)$$

4.3 Analysis of clustering techniques

In the clustering phase prior to the clustering analysis, the important step is to find the best values for the input parameters k and w. We considered the yearly load data of the PJM market from 2021, which spans 365 days, for training, and use the 24-hourly load data from January 1, 2022, for validation. We varied k from 2 to 7 and w from 1 to 12 to measure the forecasting error when predicting the validation set. We found that $k = 4$ and $w = 5$ achieve the minimum RMSE, leading us to choose these values as optimal.

We used a sample data set of PJM load data from March 1, 2021, to May 31, 2021 (spring season) to demonstrate the effectiveness of the clustering techniques (k-means and hierarchical clustering). K-means and Hierarchical clustering were used to label each day in the sample data into 4 clusters. Tables I and II shows the percentage of days classified into four clusters. By observing the tables, we can clearly classify the clusters into two groups: workings days and weekends. From Table 1 and 2 it is evident that Cluster 1 represents a group of weekends and clusters 2, 3 and 4 represents a group of working days. However, from both the tables one can observe that some days are mislabelled owing to the complex behaviour of the load data.

Table 1: *The distribution of days of the week (in%) and clusters using k-means clustering*

Cluster label	Mon	Tue	Wed	Thu	Fri	Sat	Sun
1	7.14	0.00	0.00	0.00	0.00	61.54	69.23
2	64.29	69.23	69.23	53.85	69.23	23.08	15.38
3	21.43	23.08	15.38	30.77	23.08	7.69	7.69
4	7.14	7.69	15.38	15.38	7.69	7.69	7.69

Table 2: *The distribution of days of the week (in%) and clusters using Hierarchical clustering*

Cluster label	Mon	Tue	Wed	Thu	Fri	Sat	Sun
1	7.14	0.00	0.00	0.00	0.00	69.23	84.62
2	64.29	53.85	61.54	46.15	61.54	15.38	0.00
3	0.00	15.38	7.69	7.69	7.69	0.00	0.00
4	28.57	30.77	30.77	46.15	30.77	15.38	15.38

In Table 1, one working day and nine weekends were misclassified, and in Table 2, one working day and six weekends were misclassified. Upon thorough analysis of holidays during the above period, we find that the one mislabelled working day is a Monday, falling on May 31, 2021,

as a holiday. Therefore, out of 92 days (working days and weekends), k-means clustering mislabelled five Saturdays and four Sundays, whereas hierarchical clustering mislabelled four Saturdays and two Sundays. The relative errors for k-means clustering and hierarchical clustering were 9.78% and 6.52%. The above analysis reveals that the hierarchical clustering is effective in labelling the time series data.

4.4 Performance of the Proposed method

A case study is conducted by considering the hourly load data from the PJM market for the year 2022. The methodology is used to forecast day-ahead load data by considering the historical load of one year prior to the day in which the load is to be forecast. We advance the one-year training window for a specific day by one day, resulting in forecasts for the next 24 hours. This process yields forecasts for an entire year. We calculated and presented the monthly MAPE and error variance in Table 3 to evaluate the model's performance across all the months of 2022. The results are compared with the results of the model (K-means) obtained by using K-means clustering in the clustering phase. According to the results in Table 3, it is evident that the proposed methodology performs significantly better than the K-means model. The best and worst predictions occur on 7th of July and 30th of May with 0.2972 and 9.8249 MAPE (%) respectively. Fig. 4 and 5 shows the original day versus the predicted load.

Table 3: *Monthly MAPE (MMAPE) and Error variance (VAR) for all the months of the year 2022*

Month	Proposed Methodology		K-means	
	MMAPE	VAR	MMAPE	VAR
January	2.71	6.46e-4	3.43	10e-4
February	2.41	4.33e-4	3.08	5.92e-4
March	2.54	5.40e-4	3.02	7.59e-4
April	2.46	6.15e-4	2.74	6.02e-4
May	2.22	6.97e-4	2.26	6.57e-4
June	2.18	4.60e-4	2.97	12e-4
July	2.01	4.50e-4	2.58	6.55e-4
August	1.84	4.34e-4	2.27	5.06e-4
September	2.17	3.96e-4	2.42	6.03e-4
October	1.53	1.80e-4	1.51	1.82e-4
November	2.2	4.79e-4	2.08	3.88e-4
December	2.18	5.79e-4	2.97	9.73e-4
Average	**2.20**	**4.92e-4**	**2.61**	**6.76e-4**

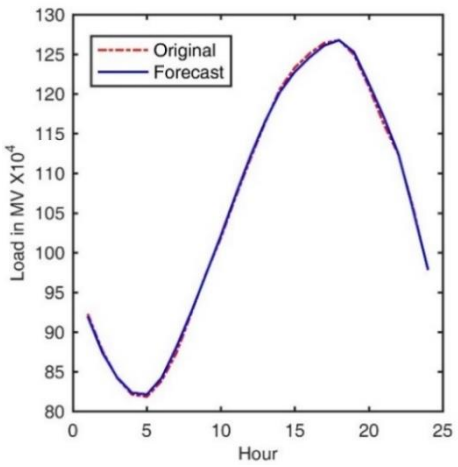

Figure 4: *Best Prediction in PJM load 2022*

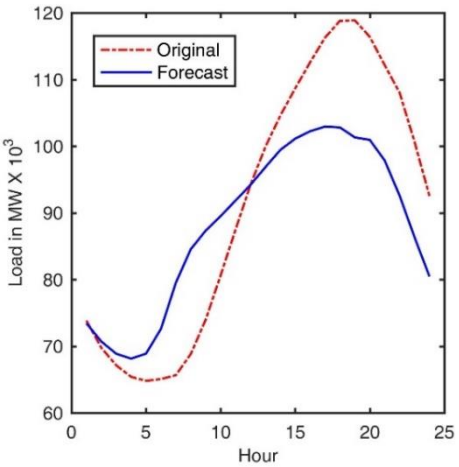

Figure 5: *Worst Prediction in PJM load 2022*

4.5 Comparative Analysis

We compared the proposed methodology to three main approaches: Bokde & Asencio et al.'s R package (2017) [19], which includes the basic PSF function (RPSF) ; Shende et al.'s (2022) Python package [25], which includes the basic PSF function (PPSF) and DPSF functions; and modified PSF algorithm proposed in [13].

First, we validated the proposed methodology using real-time series "nottem" and "CO2" datasets. The "nottem" time series contains the average air temperatures at Nottingham Castle in degrees Fahrenheit over 20 years, and the "CO_2" dataset consists of atmospheric concentrations of CO_2 expressed in parts per million (ppm). We conduct a comparative analysis against RPSF, PPSF, DPSF functions using the metrics root-mean-square error (RMSE), mean absolute error (MAE), and mean absolute percentage error (MAPE). Both the time series data sets were partitioned into training and testing datasets. The training set comprised the time series data, with the exception of the final 12 values. The testing dataset comprised of final 12 values. The values of the error metrics are recorded in Table 4.

Table 4: *A comparative analysis of real-time series forecasting results*

Time Series	Error Metric	RPSF	PPSF	DPSF	Proposed Method
nottem	RMSE	2.24	1.84	5.27	1.81
	MAE	1.94	1.54	4.77	1.35
	MAPE	4.14	3.23	9.43	2.89
CO_2	RMSE	5.93	1.42	0.41	1.21
	MAE	5.91	9.27	0.32	1.13
	MAPE	1.62	2.67	0.08	0.61

From the results, it is evident that the proposed methodology is performed well. The DPSF function yields better results for a data with positive or negative trend.

Secondly, the methodology was compared with the model proposed by Jin et al (2014) [13] using the load data of NYISO market over the year 2006 [26] with the error metric MAE. Load data of 2005 is used as training set and the forecast of 24-hours ahead is calculated. Further forecasts can be found by shifting the window of the training set to next day. The error metric MAE is evaluated and recorded in Table 5 against the measures obtained in [13].

Table 5: ***A*** *comparative analysis of the forecast results using MAE of NYISO load data from 2006*

Month	**Mean Absolute Error (MAE)**				
	RPSF	PPSF	DPSF	Modified PSF	Proposed Method
January	6.71	9.61	6.7	3.45	2.18
February	6.91	9.84	8.07	3.8	2.38
March	5.20	9.01	7.87	3.59	2.58
April	9.15	8.87	12.01	3.32	2.18
May	9.62	12.35	10.37	3.67	2.07
June	8.06	15.27	11.05	4.53	3.35
July	9.60	15.91	10.41	5.84	3.37
August	8.75	13.33	12.83	4.07	2.17
September	8.45	10.39	8.65	2.6	2.18
October	4.22	9.97	7.7	2.92	1.67
November	4.80	9.12	6.87	3.47	2.54
December	7.65	11.73	8.58	3.77	2.52
Average	**7.43**	**11.28**	**9.26**	**3.75**	**2.43**

5. Conclusion

The paper presents a methodology that enhances the forecast capability of the PSF algorithm. The modifications to the PSF algorithm that includes a judicious use of hierarchical clustering algorithm in its clustering phase and a weighted average formula in the prediction phase has led to improved accuracy in day ahead load forecasts. Alongside MAE, RMSE, MAPE, the error variance (VAR) has been used for a comprehensive evaluation of the model's performance. The proposed model outperforms benchmark models in terms of forecasting accuracy, as evidenced by the performance metrics calculated from the real-time series data. The findings highlight the effectiveness of the proposed approach in enhancing the precision of day-ahead load forecasts, making it a valuable tool for efficient power system management and operational planning in electricity markets.

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EXPONENTIATED POISSON-G FAMILY OF DISTRIBUTION: SUB-MODELS, PROPERTIES, ESTIMATION WITH REAL-LIFE APPLICATION

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Abstract

This study proposes a new family of distributions. A study is done on some of its basic characteristics, such as quantile, skewness, kurtosis, hazard rate function, moments, mean deviations, availability and reliability function of successive linear and circular systems, mean time to failure, mean time between failure, and availability, Bonferroni and Lorenz curves, and entropies. Two unique models of the new family are studied in depth once the general class is introduced. The special basis models have been taken from the exponential and Fréchet distributions. The parameters of the model are estimated using maximum likelihood techniques. There is a thorough analysis of percentage points. Three unique real data sets are used to demonstrate the significance of the new family. A comparison is drawn between the suggested distribution family and well-known two-, three-, and four-parameter components. To model actual data, it can be used as an alternative model to various lifetime distributions found in the statistical literature.

Keywords: G-family, Exponentiated model, percentage points, real-life data.

I. Introduction

There are many attempts have been made to introduce new classes in the statistical literature. Several classes have been suggested in the statistical literature by adding one or more factors to construct new distributions. There has been growing interest in introducing and developing more flexible distributions like

Table 1: *Research contributions by various authors*

Contribution	Author(s)	Year of Published
Weibull-G	Bourguignon et. al. [7]	2014
beta Marshall-Olkin family	Alizadeh et. al. [2]	2015
type I half-logistic family	Cordeiro et.al. [9]	2016
The Poisson-G	Hamed & Ibrahim [16]	2017
The generalized transmuted-G	Nofal et.al. [23]	2017
odd flexible Weibull-H	El-Morshedy & Eliwa [13]	2019
odd log-logistic Lindley-G	Alizadeh et. al. [3]	2020
Exponentiated odd Chen-G	Eliwa et.al. [12]	2020
Kumaraswamy Poisson-G	Chakraborty et.al. [8]	2022
Triangle-G	Rahman, H. [25]	2024

The pdf and cdf of the Po-G family of distribution are

$$f_{Po-G}(x,\theta,\xi) = \frac{\theta g(x,\xi)}{e^{\theta}-1}\exp[\theta G(x,\xi)]. \quad (1.1)$$

$$F_{Po-G}(x,\theta,\xi) = \frac{\exp[\theta G(x,\xi)]-1}{e^{\theta}-1}. \quad (1.2)$$

Where $g(x,\xi)$ and $G(x,\xi)$ are the baseline pdf and cdf depending on a parameter vector$\xi$and $\theta > 0$ is a shape parameter. Suppose x is a non-negative random variable that follows an Exponentiated distribution with the following pdf and cdf$f(x)$ and $F(x)$ respectively.

$$g_{\lambda}(x) = \lambda[F(x)]^{\lambda-1}f(x). \quad (1.3)$$

$$G_{\lambda}(x) = [F(x)]^{\lambda}. \quad (1.4)$$

The main goal of this study is to introduce and investigate a generalized family of probability distributions for data modelling with a limited number of parameters but a high degree of flexibility. The remainder of the essay is broken out as follows. We develop a very helpful form for the ExPo-G density function in Section 2. The quantile function (qf), moment generating function (mgf), entropies, and ordinary and incomplete moments are just a few of the general mathematical aspects of the proposed family that are included in Section 3. Section 3 investigates the maximum likelihood estimate of the model parameters. In Section 4, two unique models of this family are provided, along with some plots of their pdfs and hrfs. The results of the proposed models' percentage points are discussed in Section 5. Three applications to actual data sets are made in Section 6 to demonstrate the applicability of two unique models of the proposed family. In Section 7, some last observations are offered.

II. Exponentiated Poisson-G family of Distribution

Let us suppose a random variable $T \in (a,b)$ for $-\infty \le a < b < \infty$ having a probability density function (pdf) and $W[F(x)]$be a function of a cumulative distribution function of the random variable X which satisfies some statistical conditions such as$W[F(x)] \in (a,b)$, $W[F(x)]$ is differentiable and monotonically non-decreasing and $W[F(x)] \to a$ as $x \to -\infty$ and $W[F(x)] \to b$as $x \to \infty$.
Alzaatreh et al. (2013) defined the T-X family cdf by

$$G(x) = \int_a^{W[F(x)]} y(t)dt = Y\{W[F(x)]\}.$$

Where $W[F(x)]$ satisfied all the conditions. The corresponding pdf of the T-X family of distribution is

$$g(x) = \left\{\frac{d}{dx}W[F(x)]\right\} y\{W[F(x)]\}.$$

The distribution function of the Expo-G family of distribution is obtained by substituting (1.1) and (1.2) in (1.3) and (1.4). Thus, the pdf and cdf of the ExPo-G family of distribution are presented as

$$f_{ExPo-G}(x,\theta,\lambda,\xi) = \theta\lambda\frac{1}{(e^{\theta}-1)^{\lambda}}g(x,\xi)\exp[\theta G(x,\xi)]\left[\exp\left(\theta G(x,\xi)\right)-1\right]^{\lambda-1}. \quad (2.1)$$

$$F_{ExPo-G}(x,\theta,\lambda,\xi) = \left[\frac{exp(\theta G(x,\xi))-1}{e^{\theta}-1}\right]^{\lambda}. \quad (2.2)$$

Where $g(x,\theta,\lambda,\xi)$and$G(x,\theta,\lambda,\xi)$ are the baseline pdf and cdf depending on a parameter vector $\xi$and $(\theta,\lambda) > 0$ are shape parameters. Let us denote a random variable X having a density function (2.1). The reliability function (rf), hazard rate function (hrf), and cumulative hazard rate function (chrf) of X are, respectively, given by

$$R_{ExPo-G}(x,\theta,\lambda,\xi) = \left[\frac{e^{\theta}-exp(\theta G(x,\xi))}{e^{\theta}-1}\right]^{\lambda}. \quad (2.3)$$

$$h_{ExPo-G}(x,\theta,\lambda,\xi) = \frac{\theta\lambda g(x,\xi)\,exp[\theta G(x,\xi)][exp(\theta G(x,\xi))-1]^{\lambda-1}}{(e^{\theta}-exp(\theta G(x,\xi)))^{\lambda}}. \quad (2.4)$$

$$H_{ExPo-G}(x,\theta,\lambda,\xi) = -ln\left[\frac{e^{\theta}-exp(\theta G(x,\xi))}{e^{\theta}-1}\right]^{\lambda}. \quad (2.5)$$

III. Statistical Properties

I. Quantile function, Median, Bowley skewness and Moors kurtosis

Let us consider $u \sim U(0,1)$, the u^{th} quantile function of ExPo-G is defined as $Q(u)$ the solution of $G\big(Q(u)\big) = u; Q(u) > 0$. The simplest form can be defined as

$$Q(u) = G^{-1}\left[\frac{1}{\theta} log\left\{1 + u^{\frac{1}{\lambda}}\left(e^{\theta} - 1\right)\right\}\right]. \tag{3.1}$$

where G^{-1} represents the baseline quantile function. The median of the ExPo-G family can be obtained by setting $u = 0.5$. Studying the influence of the shape factors on the skewness and kurtosis using (3.1). The Bowley skewness and Moors kurtosis can be formulated as $B = \frac{Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$ and $M = \frac{Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)+Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$.

II. Ordinary and Incomplete moments, Moment generating Function and Mean Deviation

Let us consider a non-negative random variable$X \sim ExPo - G$, then the r^{th} moment of X is defined as $\mu_r^{'}$and is written as

$$\mu_r^{'} = E(X^r) = \int_0^{\infty} x^r f_{ExPo-G}(x, \theta, \lambda, \xi). \tag{3.2}$$

Using the power series and the generalized binomial expansion, (2.1) can be developed as an infinite mixture of exponential-G (Exp−G) family as

$$f_{ExPo-G}(x, \theta, \lambda, \xi) = \frac{1}{\left(e^{\theta}-1\right)^{\lambda}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^{j+1}\theta^{k+1}}{(j+1)!(k+1)!} h_{jk+1}(x).$$

$$f_{ExPo-G}(x, \theta, \lambda, \xi) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, h_{jk+1}(x). \tag{3.3}$$

Where $h_{jk+1}(x)$ is the Ex-G family of distribution with power parameter $jk + 1$ and

$$\Omega_{jk} = \frac{\lambda^{j+1}\theta^{k+1}}{\left(e^{\theta}-1\right)^{\lambda}(j+1)!(k+1)!}.$$

The r^{th} moment can be obtained by using (3.3) and is given by

$$\mu_r^{'} = E(X^r) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, E\left(Z^r{}_{jk+1}(x)\right). \tag{3.4}$$

$Z_{jk+1}(x)$ is the Ex-G family of distribution with power parameter $jk + 1$.
The mean can be obtained by setting $r = 1$ in (3.4). The i^{th} incomplete moment is defined as $I(x, \theta, \lambda, \xi)$ and is given by

$$I(x, \theta, \lambda, \xi) = \int_{-\infty}^{t} x^i f(x, \theta, \lambda, \xi)\, dx.$$

Using equation (3.3), we get

$$I(x, \theta, \lambda, \xi) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk} \int_{-\infty}^{t} x^i h_{jk+1}(x)\, dx. \tag{3.5}$$

The moment generating function is defined as $M_X(t)$ and given by

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, M_{jk+1}(t); \quad M_{jk+1}(t) \text{ is the mgf of } h_{jk+1}(x).$$

For a random variable $X \sim ExPo-G$, the mean deviations about the mean and median can be written as follows

$$\varepsilon_1 = \int_0^{\infty} \left|x - \mu_1^{'}\right| f_{ExPo-G}(x, \theta, \lambda, \xi) dx = 2\mu_1^{'} F\left(\mu_1^{'}\right) - 2I_{(1)}\left(\mu_1^{'}\right) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, H_{jk+1}(x).$$

where $H_{jk+1}(x)$ is the first incomplete moment of Ex-G family.

$$\varepsilon_2 = \int_0^{\infty} |x - Q(0.5)|\, f_{ExPo-G}(x, \theta, \lambda, \xi) dx = \mu_1^{'} - 2I_{(1)}\big(Q(0.5)\big).$$

III. Reliability function for parallel and series systems

Suppose for $n*$ independent components, each component has the ExPo-G family, the reliability of the parallel system (P) is given by

$$R_p(x,\theta,\lambda,\xi) = \left[\frac{e^{\theta}-exp(\theta G(x,\xi))}{e^{\theta}-1}\right]^{\lambda n^*}.$$

The reliability of the series system (S) is given by

$$R_s(x,\theta,\lambda,\xi) = \left[\left[\frac{e^{\theta}-exp(\theta G(x,\xi))}{e^{\theta}-1}\right]^{\lambda}\right]^{n^*}.$$

IV. Mean time to failure (MTTF), mean time between failure (MTBF) and availability (AvB)

If the MTBF is given as

$$MTBF = \frac{-x}{ln(1-G(x,\theta_1,\lambda_1,\xi_1))};\ x > 0.$$

If $X \sim ExPo - G(\theta_2,\lambda_2,\xi_2)$ then the MTTF is given as

$$MTTF = E(X) = \mu_1^{'}|(\theta_2,\lambda_2,\xi_2);\ x > 0.$$

The AvB is consider the probability that the component is successful at time x, i.e.

$$AvB = MTTF/MTBF = -\mu_1^{'}|(\theta_2,\lambda_2,\xi_2)\frac{ln(1-G(x,\theta_1,\lambda_1,\xi_1))}{x}.$$

V. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves defined for a given probability π is given by

$$B(\pi) = I_1(q)/\pi\mu_1^{'} \text{ and } L(\pi) = I_1(q)/\mu_1^{'}.$$

Where $q = Q(\pi)$is the quantile function of Xat π.

VI. Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$\Upsilon_\gamma(x) = \frac{1}{1-\gamma} log \int_{-\infty}^{\infty} g(x)^\gamma\, dx;\ 0 < \gamma < 1. \quad (3.6)$$

Using equation (3.3) in (3.6), we get

$$\Upsilon_\gamma(x) = \frac{1}{1-\gamma} log\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{jk}\int_{-\infty}^{\infty} h_{jk+1}(x)dx\right]; 0 < \gamma < 1.$$

$$\text{Where } \Psi_{jk} = \frac{(\lambda\gamma)^j(\theta\gamma)^k}{\left(e^{\theta}-1\right)^{\lambda\gamma}(j+1)!(k+1)!}.$$

The Tsalli's Entropy is denoted by $T_\gamma(x)$and given by

$$T_\gamma(x) = \frac{1}{1-\gamma} log\left[1 - \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{jk}\int_{-\infty}^{\infty} h_{jk+1}(x)dx\right\}\right]; 0 < \gamma < 1.$$

VII. Parameter Estimation

Here, we determine the Maximum Likelihood Estimation method to estimate the parameters of the new family of distributions from complete samples only. Let $X_1, \ldots, X_n$be a random sample from the ExPo-G family with parameters (θ,λ,ξ). Let $(\theta,\lambda,\xi^T)^T$be the $(p \times 1)$ parameter vector. Then, the log-likelihood function for Θ, say $l = l(\Theta)$, is given by

$$n\,log\,\theta + n\,log\,\lambda - n\lambda\,log(exp(\theta)-1)^{-(\lambda+1)} + \sum_{i=1}^{n} log\,g\,(x_i;\theta,\lambda,\xi)$$
$$+\theta\sum_{i=1}^{n} G(x_i;\theta,\lambda,\xi) + [\theta\sum_{i=1}^{n} G(x_i;\theta,\lambda,\xi) - 1]^{\lambda-1}. \quad (3.7)$$

Equation (3.7) can be maximized either directly by using the R (optimum function Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating (3.7).

The score vector components, say $U(\Theta) = \frac{\partial l}{\partial \Theta} = \left(\frac{\partial l}{\partial \theta}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \xi}\right)^T = \left(U_\theta, U_\lambda, U_\xi\right)^T$. Setting the nonlinear system of equations $U(\Theta) = 0$and solving them simultaneously yields the MLE $\hat{\Theta} = \left(\hat{\theta}, \hat{\lambda}, \hat{\xi}\right)$ of $\Theta = (\theta, \lambda, \xi)$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\Theta) = \left\{\frac{\partial^3 l}{\partial\theta\partial\lambda\partial\xi}\right\}$, whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\hat{\Theta}$ can be approximated by a multivariate normal $N_p\left(0, J(\hat{\Theta})^{-1}\right)$distribution to obtain confidence intervals for the parameters.

IV. Special ExPo-G models

I. The ExPo-Exponential distribution

Let us consider the pdf and cdf of Exponential distribution with positive parameter α. Then the pdf and cdfof ExPo-Exponential (ExPo-E) distribution is given by

$$f_{ExPo-E}(x, \theta, \lambda, \alpha) = \frac{\alpha\theta\lambda}{(e^\theta - 1)^\lambda} e^{-\alpha x}\, exp[\theta(1 - e^{-\alpha x})]\left[exp\left(\theta(1 - e^{-\alpha x})\right) - 1\right]^{\lambda-1}. \quad (4.1)$$

$$F_{ExPo-E}(x, \theta, \lambda, \alpha) = \left[\frac{exp(\theta(1-e^{-\alpha x})-1}{e^\theta - 1}\right]^\lambda. \quad (4.2)$$

The reliability function is given by

$$R_{ExPo-E}(x, \theta, \lambda, \alpha) = 1 - \left[\frac{exp(\theta(1-e^{-\alpha x})-1}{e^\theta - 1}\right]^\lambda. \quad (4.3)$$

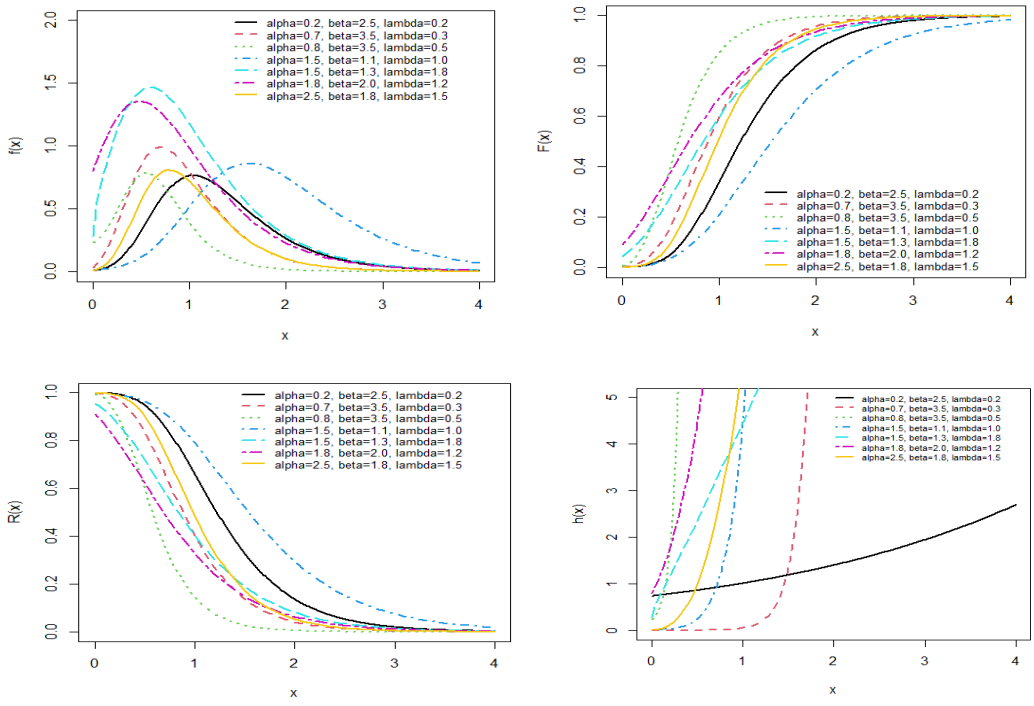

Figure 1: *pdf, cdf, reliability and hrfplots of ExPo-E distribution*

Figure 1 showing the various shapes of the functions with the fluctuation of parameters values. The flexibility of fitting different datasets of the proposed distribution from the increasing or unimodal-bathtub shape of hrf function.

II. The Expo-Frechet (ExPo-Fr) distribution

Let us consider the pdf and cdf of Frechet distribution with positive parameter $\alpha$and β. Then the pdf, cdfand reliability function of ExPo-Frechet (ExPo-Fr) distribution is given by

$$f_{ExPo-Fr}(x,\theta,\lambda,\alpha,\beta) = \frac{\alpha\beta\theta\lambda}{(e^{\theta}-1)^{\lambda}} x^{-(\alpha+1)} e^{-\beta x-\alpha}\, exp\left[\theta\left(e^{-\beta x-\alpha}\right)\right]\left[exp\left(\theta\left(e^{-\beta x-\alpha}\right)\right)-1\right]^{\lambda-1}. \quad (4.4)$$

$$F_{ExPo-Fr}(x,\theta,\lambda,\alpha,\beta) = \left[\frac{exp\left(\theta\left(1-e^{-\beta x-\alpha}\right)-1\right)}{e^{\theta}-1}\right]^{\lambda}. \quad (4.5)$$

$$R_{ExPo-Fr}(x,\theta,\lambda,\alpha,\beta) = 1-\left[\frac{exp\left(\theta\left(1-e^{-\beta x-\alpha}\right)-1\right)}{e^{\theta}-1}\right]^{\lambda}. \quad (4.6)$$

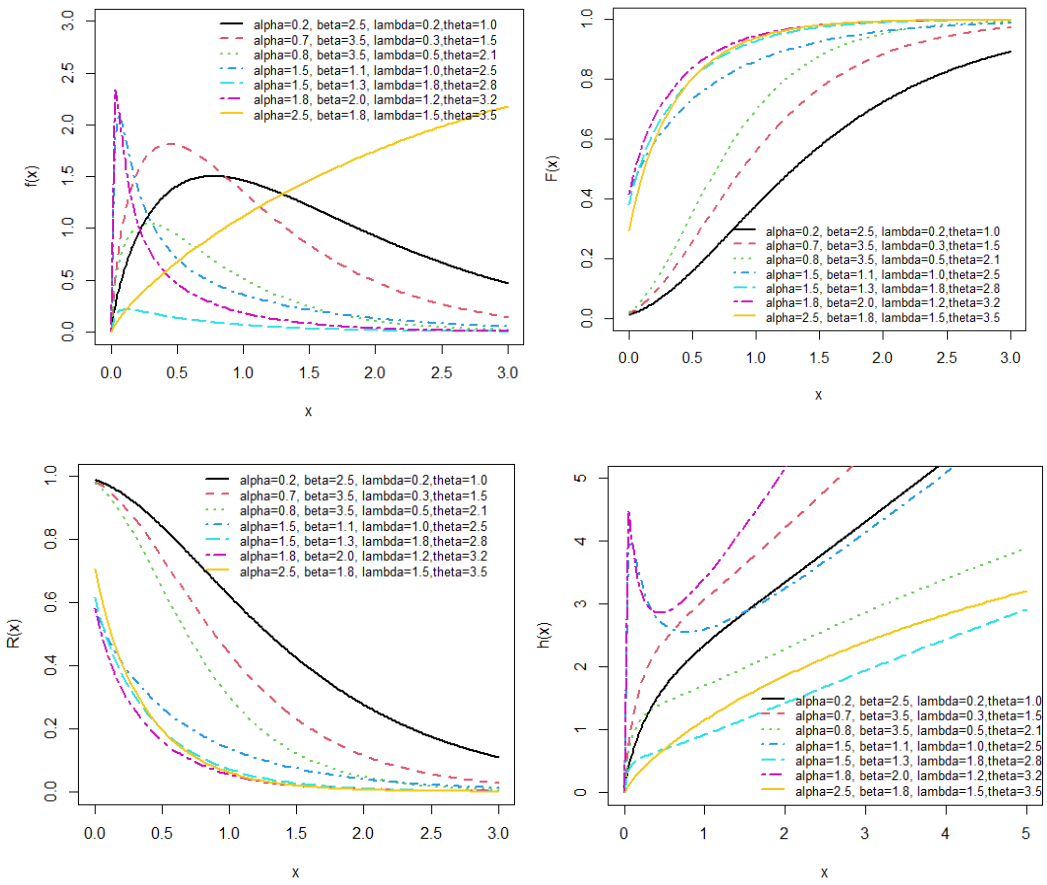

Figure 2: *pdf, cdf, reliability and hrf plots of ExPo-E distribution*

Figure 2 shows the various shapes of the functions with the fluctuation of parameters values. The flexibility of fitting different datasets of the proposed distribution from the increasing shape of hrf function.

V. Percentage Points

A r.v.X is consider a continuous random variable by expecting random values in the interval (a,b) i.e., $a \leq x \leq b$or more specifically it can assume any value like integral or fraction between certain limits. The number of expecting values are uncertain and infinite and for this reason assigning probability to each number is impossible. Therefore, in a continuous probability distribution we assign probabilities to intervals and not to individual values. For a given probability distribution, the specific value which

a random variable X exceeds with a definite probability is called the percentage point of the distribution.

In this part, a discussion of the percentage points of the proposed models ExPo-E and ExPo-Fr has been attempted. Percentage points of the proposed distributions has been computed at a number of different significance levels for different values of the parameters. The calculations in manual are very complicated, so computer programming R has used to calculate the values.

I. Percentage points of ExPo-E model

Suppose$x_1, x_2, \ldots, x_n$are n independent r.v. from ExPo-E with pdf and cdf mentioned in the equation (4.1) and (4.2). The p^{th}percentile equation of ExPo-Eis represented as

$$F_x(x) = P(X \leq x) = p.$$
$$x = F^{-1}(p) = Q(p).$$
$$x = -\frac{1}{\alpha} log\left[-\frac{1}{\theta} log\left\{p^{\frac{1}{\lambda}}\left(e^{\theta}-1\right)\right\}\right]. \quad (5.1)$$

The percent point function of the ExPo-E does not exist in a simple closed form. The numeric computation is not possible in this case. We have used computer programming R to compute the different values for different points. Using the equation (5.1), we compute the percentage points of ExPo-E for $p = 0.01,0.05,0.25,0.50,0.75,0.90,0.95,0.99$ which has been tabulated in table (2). The parameters are varying different the values to compute the p-table in different cases.

II. Percentage points of ExPo-Fr model

Suppose$x_1, x_2, \ldots, x_n$are n independent r.v. from ExPo-Fr with pdf and cdf mentioned in the equation (4.4) and (4.5). The p^{th}percentile equation of ExPo-Fris represented as

$$F_x(x) = P(X \leq x) = p.$$
$$x = F^{-1}(p) = Q(p).$$
$$x = -\frac{\alpha}{\beta} log\left[\frac{1}{\theta} log\left\{p^{\frac{1}{\lambda}}\left(e^{\theta}-1\right)\right\}+1\right]. \quad (5.2)$$

The percent point function of the ExPo-Fr does not exist in a simple closed form. The numeric computation is not possible in this case. We have used computer programming R to compute the different values for different points. Using the equation (5.2), we compute the percentage points of ExPo-E for $p = 0.01,0.05,0.25,0.50,0.75,0.90,0.95,0.99$ which has been tabulated in table (2). The parameters are varying different the values to compute the p-table in different cases.

Table 2: *Percentage points of ExPo-E for different values of parameter*

p	**0.01**	**0.05**	**0.25**	**0.50**	**0.75**	**0.90**	**0.95**	**0.99**
λ	$\alpha = 3.1, \beta = 2.5$							
0.1	-0.4923	-0.3615	-0.1381	0.0432	0.2311	0.3728	0.4304	0.4825
0.2	-0.2737	-0.1455	0.0704	0.2410	0.4099	0.5293	0.5753	0.6154
0.3	-0.1438	-0.0160	0.1987	0.3674	0.5334	0.6496	0.6940	0.7325
0.4	-0.0490	0.0797	0.2974	0.4704	0.6433	0.7673	0.8155	0.8579
0.5	0.0274	0.1584	0.3827	0.5655	0.7559	0.9007	0.9600	1.0140
β	$\alpha = 0.5, \lambda = 0.2$							
0.1	-11.005	-10.2102	-8.8709	-7.8134	-6.7661	-6.0257	-5.7406	-5.4920
0.2	-9.6189	-8.8239	-7.4846	-6.4271	-5.3798	-4.6394	-4.3543	-4.1057
0.3	-8.8080	-8.0130	-6.6737	-5.6161	-4.5689	-3.8285	-3.5433	-3.2947
0.4	-8.2326	-7.4376	-6.0983	-5.0408	-3.9936	-3.2531	-2.9680	-2.7194
0.5	-7.7863	-6.9913	-5.6520	-4.5945	-3.5473	-2.8068	-2.5217	-2.2731

Table 3: *Percentage points of ExPo-Fr for different values of parameter*

p	**0.01**	**0.05**	**0.25**	**0.50**	**0.75**	**0.90**	**0.95**	**0.99**
θ	$\alpha = 1.5, \beta = 1, \lambda = 3.5$							
1.0	2.2337	0.5666	-0.2034	-0.4426	-0.5667	-0.6193	-0.6346	-0.6461
1.1	1.2455	0.2368	-0.3608	-0.5593	-0.6644	-0.7093	-0.7224	-0.7323
1.2	0.7541	0.0179	-0.4736	-0.6440	-0.7354	-0.7748	-0.7863	-0.7950
1.3	0.4454	-0.1386	-0.5579	-0.7077	-0.7888	-0.8239	-0.8342	-0.8420
1.4	0.2298	-0.2563	-0.6229	-0.7568	-0.8299	-0.8617	-0.8710	-0.8781
1.5	0.0698	-0.3479	-0.6743	-0.7956	-0.8622	-0.8913	-0.8998	-0.9063
$\boldsymbol{\beta}$	$\alpha = 0.2, \theta = 1.5, \lambda = 2.3$							
1.0	0.1399	0.1399	-0.0715	-0.0978	-0.1117	-0.1177	-0.1194	-0.1207
1.1	0.1272	0.1271	-0.0650	-0.0889	-0.1015	-0.1069	-0.1086	-0.1097
1.2	0.1166	0.1166	-0.0596	-0.0815	-0.0931	-0.0981	-0.0995	-0.1006
1.3	0.1076	0.1076	-0.0550	-0.0752	-0.0859	-0.0905	-0.0919	-0.0928
1.4	0.0999	0.0999	-0.0510	-0.0699	-0.0798	-0.0841	-0.0853	-0.0862
1.5	0.0933	0.0933	-0.0476	-0.0652	-0.0744	-0.0785	-0.0796	-0.0804

Percentage points of proposed distributions have been presented in Table 2 and 3. For chosen values of the parameters, different values of have been obtained at different significant levels using R-Programme. From the tables of percentage points of these distributions; it is clear that for fixed values of $p = 0.01,0.05,0.25,0.50,0.75,0.90,0.95,0.99$ and for fixed positive values of the parameters. From Table2, it is observed that percentage points increase as the parameter λ increases, decreases when the parameters α and β increases; for fixed positive values of other two parameters. From Table 3, it is observed that percentage points decreases when the value of parameter $\alpha, \beta, \theta, \lambda$ increases; for fixed positive values of other parameters.

VI. Data Analysis

This section goes over the empirical significance of using an application to complete real data with the ExPo-G model. The competitive distributions' best-fitting capabilities are determined using specific analytical metrics. To choose the most suited ones, the values of the Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Corrected Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) were employed. Other goodness-of-fit tests, such as the Cramer-von Mises (W) distance value test, the Kolmogorov-Smirnov (K-S) statistic with accompanying p values, and the loglikelihood function, are also recorded in addition to discriminating tests. The AIC, BIC, CAIC, and HQIC values as well as the W and K-S tests are all lowest for the optimal model. To compare the competitive distributions, the model with the highest p values for the K-S statistics is used. Three data sets have been taken into consideration.
Data Set 1: Bjerkedal (1960) observed and recorded the survival times (in days) of 72 guinea pigs who were infected with virulent tubercle bacilli.

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Data sets-1 have considered fitting Expo-E with other some models like exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E), generalized Marshall-Olkin exponential (GMO-E), Kumaraswamy exponential (Kw-E), Beta exponential (BE), Marshall-Olkin Kumaraswamy exponential (MOKw-E) and Kumaraswamy Marshall-Olkin exponential (KwMO-E) distributions and Beta Poisson-Exponential (BP-E).

Data Set 2: Eliwa et al (2022) recorded 101 observations of the fatigue time of 101 6061-T6 aluminium coupons cut parallel to the direction of rolling and oscillates at 18 cycles per second (cps).
5 ,25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43,43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55,55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64,64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68,69, 69, 69, 69, 71, 71, 72, 73,73, 73, 74, 74, 76,76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83,84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97,98, 98, 99, 101, 103, 105, 109, 136, 147.

Data sets 3 have considered fitting Expo-Frdistribution with some competitive models like EoCh Fr, odd Chen Fr (OChFr), Type I generalized exponential Fr (TIGEFr), odd flexible Weibull Fr (OFWFr), Topp-Leaon Fr (ToLeFr), exponentiated Gompertz Fr (EGoFr), exponentiated transmuted Fr (ETrFr), transmuted Fr (TrFr), Gumbel Fr (GuFr), exponentiated Fr (EFr) and Fr.

Table 4: *MLEs and standard errors values for data set 1*

Models	**ML Estimator**			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
EXPo-E(α,β,λ)	3.100(2.647)	0.210 (0.023)	0.100(0.002)	-
BP-E($\alpha,\beta,\lambda,\theta$)	3.595 (1.031)	1.482(0.516)	0.014(0.010)	0.724(1.590)
KwMo-E($\alpha,\beta,\lambda,\theta$)	0.373(0.136)	0.299(1.112)	3.478(0.861)	3.306(0.779)
MOKw-E($\alpha,\beta,\lambda,\theta$)	0.008 (0.002)	0.099 (0.048)	2.716 (1.316)	1.986 (0.784)
B-E(α,β,λ)	0.807(0.696)	1.331(0.855)	3.461(1.003)	-
Kw-E(α,β,λ)	3.304(1.106)	1.037(0.764)	1.100(0.614)	-
GMO-E(α,β,λ)	47.635 (44.901)	4.465(1.327)	0.179(0.070)	-
MO-E(α,β)	8.778(3.555)	1.379(0.193)	-	-
ME(α)	0.925 (0.077)	-	-	-
Exp(α)	0.540(0.063)	-	-	-

Table 5: *Log-likelihood, AIC, BIC, CAIC, HQIC, W and KS (p-value) values data set 1*

Models	$-2L$	AIC	$CAIC$	BIC	$HQIC$	W^*	$K-S$	$p-value$
EXPo-E(α,β,λ)	**141.52**	**147.52**	**147.87**	**154.35**	**151.12**	**0.17**	**0.03**	**1.00**
BP-E($\alpha,\beta,\lambda,\theta$)	199.42	205.42	206.02	214.50	209.02	0.08	0.09	0.81
KwMo-E ($\alpha,\beta,\lambda,\theta$)	201.82	207.82	208.42	216.94	211.42	0.11	0.08	0.73
MOKw-E($\alpha,\beta,\lambda,\theta$)	203.44	209.44	210.04	218.56	213.04	0.12	0.10	0.44
B-E(α,β,λ)	201.38	207.38	207.73	214.22	210.08	0.15	0.11	0.34
Kw-E(α,β,λ)	203.42	209.42	209.77	216.24	212.12	0.11	0.08	0.50
GMO-E(α,β,λ)	204.54	210.54	210.89	217.38	213.24	0.16	0.09	0.51
MO-E(α,β)	204.36	210.36	210.53	214.92	212.16	0.17	0.10	0.43
ME(α)	204.4	210.40	210.45	212.68	211.30	0.25	0.14	0.13
Exp(α)	228.63	234.63	234.68	236.91	235.54	1.25	0.27	0.06

Table 6: *MLEs and confidence intervals values for data set 2*

Models	ML Estimator				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
EXPo-Fr($\alpha,\beta,\lambda,\theta$)	0.187(0.019)	0.054(0.005)	0.100 (0.214)	0.100(0.025)	-
EOChFr($\alpha,\beta,\lambda,\theta,\gamma$)	0.019	0.257	1.822	4.223	11.500
OChFr($\alpha,\beta,\lambda,\theta$)	49.633	0.629	1.588	444.284	-
TIGEFr($\alpha,\beta,\lambda,\theta$)	16648.994	74.474	5.057	7.531	-
OFWFr($\alpha,\beta,\lambda,\theta$)	9.310	0.380	0.736	312.686	-
ToLeFr($\alpha,\beta,\lambda,\theta$)	35.077	0.783	4.088	59.691	-
EGoFr($\alpha,\beta,\lambda,\theta,\gamma$)	0.013	0.647	1.807	1.158	127.896
TrFr(α,λ,θ)	1.00	-	3.980	136.952	-
GuFr($\alpha,\beta,\lambda,\theta$)	1.968	0.029	0.107	3.457	-
EFr(β,λ,θ)	-	73.221	5.057	51.679	-
Fr(α,β)	5.057	120.782	-	-	-

Table 7: *Log-likelihood, AIC, BIC, CAIC, HQIC, W and KS (p-value) values for the fatigue time of 101 6061-T6 aluminium coupons data set 2*

Models	$-2L$	AIC	$CAIC$	BIC	$HQIC$	$W*$	$K-S$	$p-value$
EXPo-Fr($\alpha,\beta,\lambda,\theta$)	**753.24**	**761.24**	**761.49**	**771.66**	**753.96**	**0.07**	**0.027**	**0.98**
EOChFr($\alpha,\beta,\lambda,\theta,\gamma$)	912.18	920.18	920.43	930.60	912.9	0.08	0.065	0.786
OChFr($\alpha,\beta,\lambda,\theta$)	912.64	920.64	920.89	931.06	913.36	0.07	0.068	0.732
TIGEFr($\alpha,\beta,\lambda,\theta$)	950.38	958.38	958.63	968.80	951.1	0.07	0.133	0.057
OFWFr($\alpha,\beta,\lambda,\theta$)	919.38	927.38	927.63	937.80	920.1	0.15	0.090	0.383
ToLeFr($\alpha,\beta,\lambda,\theta$)	932.70	940.70	940.95	951.12	933.42	0.18	0.121	0.102
EGoFr($\alpha,\beta,\lambda,\theta,\gamma$)	922.60	930.60	930.85	941.02	923.32	0.20	0.107	0.198
TrFr(α,λ,θ)	932.82	940.82	941.07	951.24	933.54	0.20	0.120	0.105
GuFr($\alpha,\beta,\lambda,\theta$)	951.46	959.46	959.71	969.88	952.18	0.25	0.135	0.050
EFr(β,λ,θ)	950.36	958.36	958.61	968.78	951.08	0.27	0.133	0.056
Fr(α,β)	950.36	958.36	958.61	968.78	951.08	0.27	0.133	0.056

Tables 4,5,6 and 7 represent the MLEs with standard errors of the parameters for all the fitted models along with their AIC, BIC, CAIC, HQIC, A, W and KS statistic with p-value from the fitting results of the data sets 4,5 and 6 are presented respectively. The proposed model Expo-E is found to be a better model on the basis of the lowest value different criteria like AIC, BIC, CAIC, HQIC, W and the highest p-value of the KS statistics compared to other introduced models like models Exp, ME, MO-E, GMO-E, Kw-E, BE, MOKw-E, KwMO-E and BP-E considered data set 1.The proposed model Expo-fr is

found to be a better model on the basis of the lowest value different criteria like AIC, BIC, CAIC, HQIC, W and the highest p-value of the KS statistics compared to other introduced models like modelsEoCh Fr, odd Chen Fr (OChFr), Type I generalized exponential Fr (TIGEFr), odd flexible Weibull Fr (OFWFr), Topp-Leaon Fr (ToLeFr), exponentiated Gompertz Fr (EGoFr), exponentiated transmuted Fr (ETrFr), transmuted Fr (TrFr), Gumbel Fr (GuFr), exponentiated Fr (EFr) and Fr considered data set 2.

VII. Conclusion

In this article, we propose and study the family of exponentiated Poisson-G distributions (ExPo-G). The main advantage of the ExPo-G family is that practitioners will have a one-parameter class flexible enough to adapt to real data in applied fields. It can be a good alternative to other four parameter families infected with one, two or three parameters. In some real-world circumstances, nevertheless, it might also outperform other kinds of distributions in terms of model fit, albeit this is not always assured. Additionally, a thorough explanation of several of its mathematical features is given. We demonstrate empirically that the ExPo-G family's unique models can offer a better match than other models produced by the aforementioned classes.

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IMPROVED ADAPTIVE THRESHOLDING LASSO CHART FOR MONITORING DISPERSION OF HIGH-DIMENSIONAL PROCESSES USING GENERALIZED MULTIPLE DEPENDENT STATE SAMPLING

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Abstract

In many applications of multivariate statistical quality control, it is commonly observed that the number of quality characteristics exceeds the sample size. This poses significant challenges in monitoring high-dimensional data. In such conditions, it is challenging to detect sparse changes where an assignable cause leads to the deviation of only a few elements in the covariance matrix. On the other hand, the utilization of the multiple dependent state (MDS) sampling technique to enhance the sensitivity of control charts has recently attracted the attention of researchers. However, to the best of the authors' knowledge, no previous research has been conducted on equipping multivariate dispersion control charting methods with the MDS technique under high dimensionality. Therefore, this article integrates the adaptive thresholding Lasso statistic with the MDS and generalized MDS techniques to track all types of disturbances in the covariance matrix of high-dimensional processes, including diagonal, off-diagonal, and joint diagonal/off-diagonal deviations. The performance of the proposed control charts will be compared through a numerical example under seven out-of-control patterns in terms of three metrics: average, standard deviation, and median of run length. The results clearly indicate that the use of both sampling techniques significantly improves the run length properties of the adaptive thresholding Lasso chart.

Keywords: Adaptive thresholding Lasso control chart; High dimensional process; Multiple dependent state sampling; Run length; Covariance matrix.

I. Introduction

In recent years, increasing customer expectations and technological advancements have resulted in the development of new production processes that require monitoring numerous quality characteristics. In this context, we can mention imaging processes and multi-stage production processes. Conventional multivariate control charts lose their sensitivity to process changes as the number of quality characteristics under study increases. This challenge becomes more significant

when only a limited number of variables are affected by assignable causes [1]. Therefore, in recent years, significant efforts have been devoted to monitoring high-dimensional data, where the number of quality characteristics exceeds the sample size. One effective strategy for monitoring high-dimensional processes involves implementing control charts based on variable selection approach. This method involves identifying a small subset of variables that may deviate from their nominal value and subsequently conducting the monitoring process using this subset. In this connection, Abdella et al. [2] developed a multivariate cumulative sum (MCUSUM) chart using a variable selection approach for the quick detection of small mean disturbances in high-dimensional process monitoring. Sangahn [3] integrated the deviance residual-based multivariate exponentially weighted moving average statistic with a variable selection procedure for Phase II monitoring of high-dimensional multistage processes. Zhang et al. [4] proposed a sensitized variable selection-based control charting method that utilizes a classification algorithm for Phase II monitoring of high-dimensional industrial process.

Although variable selection-based control charts have significant advantages for monitoring high-dimensional processes under sparse conditions, they face challenges in identifying suspicious variables. The efficacy of this control chart is significantly reduced when the suspicious variables are not accurately identified. Specifically, when the shift magnitude in process parameters is small, these monitoring schemes lose their effectiveness because they fail to accurately identify all the variables responsible for the deviation in the selected set. As one of the most effective alternative methods for monitoring the dispersion of high-dimensional processes, the adaptive thresholding Lasso control chart has gained significant attention from researchers in recent years. In this monitoring procedure, the first step involves calculating the difference matrix through the subtraction of the sample covariance from its corresponding target matrix. Then, according to the sparsity assumption, only the values of the difference matrix that exceed a certain threshold value are taken into consideration. In other words, any elements in the difference matrix that are smaller than the threshold value are considered to be equal to zero. Interested readers can refer to important references, such as [5], [6], [7], [8], and [9], for more comprehensive details regarding the adaptive thresholding LASSO control chart.

On the other hand, the performance of control charts significantly depends on the sampling strategy used to collect process observations. This issue becomes even more crucial in situations where any delay in identifying process disturbances, regardless of shift magnitude, results in substantial costs for the production system. In this context, researchers have recently been focusing on new sampling strategies, such as double sampling, ranked set sampling [10], repetitive sampling [11], and multiple dependent state sampling [12], to enhance the power of various control charts. Recently, the multiple dependent state (MDS) sampling strategy has been successfully utilized to enhance the diagnostic capability of control charts. In this regard, Arshad et al. [13] equipped a variability control charting mechanism based on sample variance statistic to MDS sampling strategy in order to improve the detection of increased variance shifts. Aiming to achieve rapid detection of small mean deviations, Naveed et al. [14] have developed a modified version of the exponentially weighted moving average (EWMA) control chart that utilizes MDS strategy. Aslam et al. [15] designed an enhanced version of the $\bar{X}$ control chart based on the generalized multiple dependent state (GMDS) sampling for rapid detection of small mean deviations. Rao et al. [16] studied the efficiency of the GMDS sampling technique in improving the detectability of the coefficient of variation (CV) control chart. Please refer to [17] and [18] for additional information on MDS-based control charts.

The high dispersion of the quality characteristics under investigation will lead to an increase in the production rate of non-conforming products, thereby resulting in an increase in quality loss costs. In many manufacturing environments, product fluctuations often increase due to various

factors, including changes in raw materials, machinery depreciation, operator mistakes, environmental factors, and insufficient calibration of measuring equipment calibration. If these factors are not promptly identified and eliminated, the costs imposed on the system due to the production of non-conforming products will increase greatly. According to the mentioned cases, monitoring the covariance matrix in multivariable production processes is of great importance for maintaining product quality in both industrial and service environments. In recent years, researchers such as [19] and [20] have dedicated their attention to tracing the deviations of the covariance matrix. Most covariance matrix monitoring control charts are commonly designed with the assumption that the sample size exceeds the problem dimension or the number of quality characteristics. However, this assumption does not hold true in numerous statistical quality control applications. Fortunately, there have been recent efforts to monitor the scatter matrix of high-dimensional data. In this context, Kim et al. [21] developed a covariance matrix monitoring scheme based on a ridge penalized likelihood ratio statistic for the identification of general disturbances, without the need for sparsity assumption. A monitoring scheme for Phase I monitoring of high-dimensional variability matrices, utilizing the sparse-leading-eigenvalue statistic was developed by [22]. Safikhani et al. [23] employed an additive covariate model to examine how imprecise observations affect the effectiveness of the ridge penalized likelihood ratio chart in Phase II monitoring of high-dimensional process dispersion. As far as we know, the MDS sampling technique has mainly been used in control charts to monitor univariate processes. Therefore, due to the appropriate performance of the MDS method in improving the sensitivity of control charts. Therefore, this paper aims to utilize the MDS and GMDS sampling methods for Phase II monitoring of high-dimensional process dispersion, as they have been shown to improve the sensitivity of control charts. Specifically, in this article two adaptive thresholding Lasso-based control charting schemes named MDS-ATL and GMDS-ATL are presented to identify the covariance matrix disturbances under high-dimensionality.

The structure of this article is organized as: Section 2 describes the adaptive thresholding Lasso control chart, which is used to detect covariance matrix disturbances in high-dimensional processes. Two variability control charts are developed by integrating the adaptive thresholding Lasso statistic with the multiple dependent state and generalized multiple dependent state sampling strategies in Section 3. In Section 4, firstly, seven out-of-control scenarios for the covariance matrix are defined, based on three patterns: diagonal, off-diagonal, and joint diagonal/off-diagonal patterns. We then conduct extensive simulation studies to demonstrate the effectiveness of the MDS and GMDS sampling strategies in enhancing the sensitivity of the adaptive thresholding Lasso control chart. Section 5 is devoted to the conclusion and recommendations for future research.

II. Adaptive thresholding Lasso monitoring scheme

In this section, we discuss the adaptive thresholding Lasso control chart as an efficient approach in monitoring the dispersion of high-dimensional processes. This control chart has the ability to detect covariance matrix disturbances across all three categories: diagonal, non-diagonal, and diagonal-non-diagonal deviations. In this regard, Table 1 presents the indices, distribution parameters, sample parameters, and chart parameters employed in the development of the adaptive thresholding Lasso chart along with its improved versions based on MDS and GMDS sampling strategies.

Table 1. *Notations*

Indices	Description
t	Index of sample
i	Index of observation
j,k	Indices of quality characteristics
	Distribution parameters
y_j	Quality characteristic j
$\mathbf{Y}_t$	The matrix of observations obtained in the t^{th} sample
$\mathbf{y}_{ti}$	The column vector of quality characteristics in the i^{th} observation of the t^{th}
y_{tij}	The value of the j^{th} quality characteristic in the i^{th} observation of the t^{th} sample
p	Process dimension
$\boldsymbol{\mu}_{ic}$	Target mean vector
μ_j	Target mean parameter of the j^{th} quality characteristic
$\boldsymbol{\Sigma}_{ic}$	Target covariance matrix
σ_j^2	Target variance of the j^{th} quality characteristic
σ_{jk}	Target covariance of quality characteristics j and k
γ_{jk}	Threshold value for the covariance between quality characteristics j and k
θ_{jk}	Variance of $\hat{d}_{t,jk}$
$\boldsymbol{\Sigma}_{oc}$	Off-target covariance matrix
$\boldsymbol{\Psi}$	The matrix of shift magnitudes within the covariance matrix
	Sample parameters
$\bar{\mathbf{y}}_t$	Sample mean vector at the t^{th} sample.
$\mathbf{S}_t$	Sample covariance matrix at the t^{th} sample
S_{tj}^2	Sample variance of quality characteristic j at the t^{th} sample
$S_{t,jk}$	Sample covariance of quality characteristics j and k at the t^{th} sample
$\mathbf{D}_t$	Difference matrix at the t^{th} sample
$\hat{\mathbf{D}}_t$	Shrinkage difference matrix at the t^{th} sample
$d_{t,jk}$	The component of matrix $\mathbf{D}_t$ at row j and column k
$\hat{d}_{t,jk}$	The component of matrix $\hat{\mathbf{D}}_t$ at row j and column k
ATL_t	Adaptive thresholding LASSO statistic at the t^{th} sample
	Chart/Other parameters
α	Probability of type I error
n	Sample size
φ	Regularization parameter
η	Shrinkage parameter
r	Number of additional samples taken
q	Minimum number of additional samples
UCL	Control limit of adaptive thresholding LASSO chart
UCL_{inner}	Inner control limit of the proposed MDS- based adaptive thresholding LASSO charts
UCL_{outer}	Outer control limit of the proposed MDS- based adaptive thresholding LASSO charts
ARL_{ic}	In-control average run length
τ	The time of occurring an assignable cause of deviation
T	The time of alerting an out-of-control signal by the control chart
$(a)_+$	Operator for converting negative values to zero

Consider a process in which a product manufacturer or service provider is interested in concurrently monitoring p correlated quality characteristics $y_1, y_2, ..., y_p$ which all follow a normal distribution. In this case, the observations obtained in the t^{th} sample are described by the matrix $\mathbf{Y}_t = (\mathbf{y}_{t1}, \mathbf{y}_{t2}, ..., \mathbf{y}_{tn_t})_{p \times n_t}$, where $\mathbf{y}_{ti} = \left(y_{ti1}, y_{ti2}, ..., y_{tip}\right)^T ; t = 1, ..., T, i = 1, ..., n_t$. In addition, n_t denotes the number of observations at the t^{th} sample, while T indicates the point at which the out-of-control signal is issued. In this thesis, it is assumed, in accordance with the convention of statistical quality control, that the number of observations in each subgroup remains constant and equal to n between consecutive samples. Furthermore, it is presumed in this study that the observations within a sample are independent from each other. In other words, the columns of the observation matrix $\mathbf{Y}_t$, denoted as $\mathbf{y}_{t1}, \mathbf{y}_{t2}, ..., \mathbf{y}_{tn}$, are considered to be independent. The mean vector and covariance matrix of the study quality characteristics are defined by Equations (1) and (2), respectively, when the process is statistically in-control.

$$\boldsymbol{\mu}_{ic} = \left(\mu_1, \mu_2, ..., \mu_p\right)^T \tag{1}$$

$$\boldsymbol{\Sigma}_{ic} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{pmatrix} \tag{2}$$

Since the purpose of adaptive thresholding Lasso control chart is to identify variations in the covariance matrix in high-dimensional processes, the mean vector is considered to remain constant throughout each production cycle, and the assignable cause solely affects the elements of the covariance matrix. In other words, when an assignable cause occurs at time τ, the covariance matrix of random vectors $\mathbf{y}_{ti}; t = \tau + 1, \tau + 2, ..., T; i = 1, ..., n$ changes from $\boldsymbol{\Sigma}_{ic}$ to $\boldsymbol{\Sigma}_{oc} = \boldsymbol{\Sigma}_{ic} + \boldsymbol{\Psi}$, while the mean vector remains constant at $\boldsymbol{\mu}_{ic}$. The adaptive thresholding Lasso control chart is designed to compare the sample covariance matrix in each sample with its target matrix. In other words, this monitoring procedure conducts the following hypothesis test:

$$\begin{aligned} &H_0 : \boldsymbol{\Sigma} - \boldsymbol{\Sigma}_{ic} = \mathbf{0}_{p \times p} \\ &H_1 : \boldsymbol{\Sigma} - \boldsymbol{\Sigma}_{ic} \neq \mathbf{0}_{p \times p} \end{aligned} \tag{3}$$

To test hypothesis (3) using the adaptive thresholding Lasso method, the difference matrix is calculated based on the difference between the sample and target covariance matrices as:

$$\mathbf{D}_t = \mathbf{S}_t - \boldsymbol{\Sigma}_{ic} \tag{4}$$

By calculating the sample mean vector of the t^{th} subgroup as $\bar{\mathbf{y}}_t = \frac{1}{n}\sum_{i=1}^{n} \mathbf{y}_{ti}$, we can obtain the sample covariance matrix as follows:

$$\mathbf{S}_t = \begin{pmatrix} S_{t1}^2 & S_{t,12} & \cdots & S_{t,1p} \\ S_{t,21} & S_{t2}^2 & \cdots & S_{t,2p} \\ \vdots & \vdots & \ddots & \vdots \\ S_{t,p1} & S_{t,p2} & \cdots & S_{tp}^2 \end{pmatrix} = \frac{1}{n-1}\sum_{i=1}^{n} \left(\mathbf{y}_{ti} - \bar{\mathbf{y}}_t\right)\left(\mathbf{y}_{ti} - \bar{\mathbf{y}}_t\right)^T \tag{5}$$

It is evident that the probability of rejecting the null hypothesis increases as the distance between the elements of matrix $\mathbf{D}_t$ increases from zero. Since $\mathbf{Y}_t$ is a random matrix, the components of the difference matrix exhibit slight deviations from their nominal values even when the covariance

matrix remains stable. Therefore, according to [5], $\hat{\mathbf{D}}_t$ is defined by setting a threshold value based on Equation (6) such that the respective values of the in-control components are equal to zero.

$$\hat{d}_{t,jk} = d_{t,jk}\left(1-\left|\gamma_{jk}/d_{t,jk}\right|^{\eta}\right)_+ ; j,k = 1,...,p \tag{6}$$

In Equation (6), the output of the operator $(.)_+$ is equal to $1-\left|\frac{\gamma_{jk}}{d_{jk}}\right|^{\eta}$ when $\left|\gamma_{jk}/d_{jk}\right|^{\eta}$ is less than 1; otherwise the output will be zero. Moreover, η is a predetermined constant parameter that determines the amount of shrinkage, while γ_{jk} represents an specified threshold which is calculated as follows.

$$\gamma_{jk} = \varphi\sqrt{\frac{\theta_{jk}\log p}{n}} \tag{7}$$

In Equation (7), φ is a non-negative constant known as the regularization parameter, while θ_{jk} represents the variance of $\hat{d}_{t,jk}$ which is calculated in terms of the components of $\boldsymbol{\Sigma}_{ic}$ as:

$$Var(\hat{d}_{t,jk}) = \sigma_j^2\sigma_k^2 + \sigma_{jk}^2 \tag{8}$$

In Equation (8), σ_j^2 and σ_k^2 denote the variances of variables j and k, respectively, whereas σ_{jk} represents the covariance between them. Accordingly, the value of the component in the j^{th} row and k^{th} column of the shrinkage difference matrix at the t^{th} sample can be rewritten as:

$$\hat{d}_{t,jk} = d_{t,jk}\left(1-\left|\frac{\varphi\sqrt{\left(\sigma_j^2\sigma_k^2+\sigma_{jk}^2\right)n^{-1}\log p}}{d_{t,jk}}\right|^{\eta}\right)_+ ; j,k = 1,...,p \tag{9}$$

Finally, the adaptive thresholding Lasso chart statistic for the t^{th} sample based on $\hat{d}_{t,jk}$ and θ_{jk} is calculated according to Equation (10).

$$ATL_t = n^{-1}\sum_{k=1}^{p}\sum_{l=1}^{p}\left(\frac{\hat{d}_{t,jk}^2}{\theta_{jk}}\right) \tag{10}$$

The ideal value of ATL_t is zero, and as the sample covariance matrix deviates further from $\boldsymbol{\Sigma}_{ic}$, the chart statistic increases. As can be observed, each component in Equation (10) is a positive value, and therefore the adaptive thresholding Lasso control chart only has an upper control limit, denoted by UCL. In this study, the value of UCL is determined through 10,000 iterations of Monte Carlo simulation to ensure that the in-control average run length (ARL_{ic}) of the chart is equal to a predetermined value of $\frac{1}{\alpha}$.

III. Proposed MDS-ATL and GMDS-ATL control charting methods

One of the key factors that significantly affects the control chart's ability to detect various shifts in the distribution parameters is the sampling methodology that is employed. This section presents two monitoring approaches, namely MDS-ATL and GMDS-ATL to enhance the detectability of the adaptive thresholding Lasso control chart for high-dimensional process monitoring based on MDS and GMDS sampling strategies. In the proposed control charting methods, the control limit interval is partitioned into three zones of safety, warning, and rejection as follows:

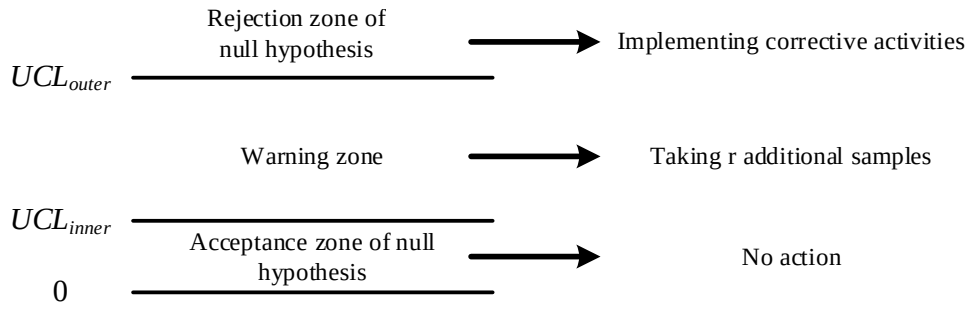

Figure 1. *Safety, warning, and rejection zones*

In the MDS-ATL chart, the covariance matrix is declared to be in-control when the adaptive thresholding Lasso statistic is less than or equal to the internal control limit i.e. $MDS-ATL_t \in [0, UCL_{inner}]$. On the other hand, if the chart statistic exceeds the outer control limit, or equivalently $MDS-ATL_t \in [UCL_{outer}, \infty)$, an out-of-control situation is triggered. Then, the production process is stopped to implement necessary corrective actions. If neither of the two mentioned situations occurs, i.e. $MDS-ATL_t \in (UCL_{inner}, UCL_{outer})$, the final decision regarding the process dispersion is postponed until the additional r samples are taken. In this case, the process variability is deemed to be in-control if the adaptive thresholding Lasso statistic associated with each additional *r* samples fall within the acceptance zone. In other words, if the chart statistic for any of the additional r samples exceeds UCL_{inner}, the process is out-of-control.

The MDS-ATL chart employs a rigorous approach to declare the covariance matrix as being in control when the chart statistic falls within the warning region. To address this issue and enhance chart's flexibility, we propose using a control chart called the GMDS-ATL. In this chart, we define an additional parameter q, as the generalization parameter, alongside the four previous parameters *n*, *r*, UCL_{inner}, and UCL_{outer}. The difference between the GMDS chart and MDS-ATL lies in the condition where the adaptive thresholding Lasso statistic falls within the warning area. In such cases, the process is considered in-control if both of the following conditions are fulfilled: (1) The chart statistic corresponding to at least q samples from r additional samples is less than or equal to UCL_{inner}; (2) the chart statistic for r additional samples are all smaller than UCL_{outer}.

IV. Performance evaluation

In this section, the sensitivity of the proposed MDS-ATL and GMDS-ATL control charts to the covariance matrix deviations of high-dimensional processes is compared with the adaptive thresholding Lasso chart in terms of *ARL*, *SDRL* and *MRL* metrics. To illustrate this, a numerical example based on simulation is presented, wherein the quality of the product or service is assessed through $p=15$ normally distributed variables. When the process is statistically in-control, the mean vector and covariance matrix of the study variables are considered to be equal to $\boldsymbol{\mu}_{ic}=\mathbf{0}_{15\times1}$ and $\boldsymbol{\Sigma}_{ic}=\mathbf{I}_{15}$, respectively. In all ATL, MDS-ATL and GMDS-ATL control charts, the regularization and shrinkage parameters of adaptive thresholding Lasso statistic are set to $\varphi=1$ and $\eta=4$, respectively, and samples of size $n=10$ are used to monitor the process dispersion. Furthermore, the repetition parameter for the MDS-RPLR and GMDS-RPLR charts is set to $r=3$, and two values of $q\in\{1,2\}$ are utilized for the generalization parameter. To ensure a fair comparison, the UCL, UCL_{inner} and UCL_{outer} values of the ATL-based control charting methods are

determined through 20,000 iterations of Monte Carlo simulation in such a way that the in-control *ARL* of all charts is equal to the predefined value of $ARL_{ic} = 200$. The upper control limits for the competing control charting schemes to achieve $ARL_{ic} = 200$ are reported in Table 2.

Table 2. *UCL values*

Chart		UCL_{inner}	UCL_{outer}
ATL		3.6710	3.6710
MDS-ATL		1.7325	3.8002
GMDS-ATL	$q=1$	1.3346	3.9495
	$q=2$	1.0126	3.9910

Next, we will introduce seven out-of-control scenarios for the components of the covariance matrix in order to compare the sensitivity of the competing variability charts in response to process disturbances. It is important to note that the defined out-of-control scenarios are divided into three categories: diagonal, off-diagonal, and diagonal/off-diagonal. The first two scenarios are of the diagonal type, while the third and fourth scenarios belong to the off-diagonal category. Finally, the remaining three scenarios are categorized as diagonal/off-diagonal.

Scenario 1. In this scenario according to Equation (11), the assignable cause has no effect on the covariance components, while the variance of each of the 15 quality characteristics under study deviates equally from their target values.

$$\boldsymbol{\Sigma}_{oc_1} = \begin{bmatrix} 1+\Delta & 0 & \dots & 0 \\ 0 & 1+\Delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+\Delta \end{bmatrix}_{15\times 15} \tag{11}$$

Scenario 2. In this scenario, according to Equation (12), the variance of the first quality characteristic is increased by Δ units from its target value. As a result, scenario 2 will lead to a sparse deviation in the process covariance matrix.

$$\boldsymbol{\Sigma}_{oc_2} = \begin{bmatrix} 1+\Delta & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{15\times 15} \tag{12}$$

Scenario 3. In this off-diagonal scenario, according to Equation (13), the covariance values associated with 70% of the quality characteristics increase from 1 to $1+\Delta$.

$$\boldsymbol{\Sigma}_{oc_3} = \begin{bmatrix} 1 & \Delta & \dots & \Delta & 0 & \cdots & 0 \\ \Delta & 1 & \dots & \Delta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Delta & \Delta & \dots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \dots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \cdots & 1 \end{bmatrix}_{15\times 15} \tag{13}$$

Scenario 4. The main difference between this scenario and scenario 3 is that only the covariance of the first three variables deviate from their nominal value. Given that only 2.66% of all the elements of the covariance matrix change under this scenario, it can be characterized as a sparse pattern.

$$\boldsymbol{\Sigma}_{oc_4}=\begin{bmatrix}1 & \Delta & \Delta & 0 & \cdots & 0\\ \Delta & 1 & \Delta & 0 & \cdots & 0\\ \Delta & \Delta & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}_{15\times 15} \tag{14}$$

Scenario 5. As indicated in Equation (15), the fifth scenario represents a diagonal/off-diagonal out-of-control pattern wherein the variance and covariance components associated with 30% of quality characteristics deviate from their nominal values by Δ units.

$$\boldsymbol{\Sigma}_{oc_5}=\begin{bmatrix}1+\Delta & \Delta & \Delta & \Delta & 0 & \cdots & 0\\ \Delta & 1+\Delta & \Delta & \Delta & 0 & \cdots & 0\\ \Delta & \Delta & 1+\Delta & \Delta & \vdots & \ddots & \vdots\\ \Delta & \Delta & \Delta & 1+\Delta & 0 & \cdots & 0\\ 0 & 0 & \ldots & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 0 & 0 & \cdots & 1\end{bmatrix}_{15\times 15} \tag{15}$$

Scenario 6. In this scenario, the first three variables are impacted by the source of deviation, so that each of the variance and covariance components increase by Δ and Δ^2 units, respectively.

$$\boldsymbol{\Sigma}_{oc_6}=\begin{bmatrix}1+\Delta^2 & \Delta & \Delta & 0 & \cdots & 0\\ \Delta & 1+\Delta^2 & \Delta & 0 & \cdots & 0\\ \Delta & \Delta & 1+\Delta^2 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}_{15\times 15} \tag{16}$$

Scenario 7. Based on Equation (17), the variance and covariance components corresponding to the first three variables, and the last three variables change by the amount of Δ units. However, the components related to the 4th to 12th quality characteristics have remained unchanged.

$$\boldsymbol{\Sigma}_{oc_7}=\begin{bmatrix}1+\Delta & \Delta & \Delta & 0 & \ldots & 0 & 0 & 0 & 0\\ \Delta & 1+\Delta & \Delta & 0 & \ldots & 0 & 0 & 0 & 0\\ \Delta & \Delta & 1+\Delta & 0 & \ldots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & 1+\Delta & \Delta & \Delta\\ 0 & 0 & 0 & 0 & \ldots & 0 & \Delta & 1+\Delta & \Delta\\ 0 & 0 & 0 & 0 & \ldots & 0 & \Delta & \Delta & 1+\Delta\end{bmatrix} \tag{17}$$

The *ARL*, *SDRL*, and *MRL* values for the ATL, competing charts under $\Delta\in\{0,0.1,0.2,0.3,0.4,0.5,0.75\}$ are summarized in Tables 3-9. It is evident that in all scenarios and shift magnitudes, the MDS-ATL and GMDS-ATL charts show significantly better performance than the ATL in timely detecting the covariance matrix disturbances. That is to say, incorporating MDS and GMDS techniques into the ATL chart significantly enhances its performance in terms of *ARL*, *SDRL*, and *MRL* metrics. In particular, the values of *ARL*, *SDRL*, and *MRL* indices under diagonal changes based on 20,000 replicates are reported in Tables 3 and 4. Table 3 shows that the

GMDS-ATL scheme has superior performance compared to the ATL and MDS-ATL charts. As seen, when Δ is equal to 0.1, 0.2, 0.3, 0.4, 0.5, and 0.75, equipping the ATL chart with the MDS technique leads to improvements in its power by 36.30%, 60.57%, 62.70%, 49.92%, 31.84%, and 6.16%, respectively. This means that in the first scenario, the average sensitivity improvement of the ATL chart in terms of the *ARL* index, when using the MDS technique, is 41.25% across different values of Δ. The findings from Table 3 demonstrate that implementing the MDS sampling technique in the design of the ATL control chart enhances its *SDRL* index by 35.63%, 61.57%, 75.82%, 84.03%, 89.11%, and 100% when Δ is equal to 0.1, 0.2, 0.3, 0.4, 0.5, and 0.75, respectively.

Table 3. *RL features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 1*

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	31.3154	8.7568	3.6496	2.1015	1.4789	1.0657
		SDRL	201.2460	30.8788	8.1388	3.1113	1.5094	0.8413	0.2640
		MRL	138	22	6	3	2	1	1
MDS-ATL		*ARL*	199.1255	19.9484	3.4527	1.3611	1.0525	1.0080	1.0000
		SDRL	198.4491	19.8767	3.1274	0.7522	0.2410	0.0916	0
		MRL	138	14	2	1	1	1	1
GMDS-ATL	$q=1$	*ARL*	199.9991	17.8049	2.7309	1.0772	1.0441	1.0000	1.0000
		SDRL	197.4807	17.3989	2.3829	0.5795	0.1678	0.0638	0
		MRL	138	12	2	1	1	1	1
	$q=2$	*ARL*	198.6688	17.6665	2.9177	1.1110	1.0518	1.0000	1.0000
		SDRL	199.5893	17.3772	2.4720	0.5972	0.1628	0.0619	0
		MRL	138	12	2	1	1	1	1

Table 4 shows that in the second scenario, equipping the ATL chart with the MDS technique under $\Delta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.75$ leads to improvements of 3.64, 6.11, 8.59, 11.07, 14.80 and 23.45 percent in the *ARL*. The improvement percentages by employing the GMDS technique, with $q=1$, are 10.86%, 15.15%, 28.51%, 34.12%, 36.89%, and 43.27%, respectively. Hence, as the shift magnitude increases, the percentage of improvement in the ATL scheme from the application of both MDS and GMDS techniques also increases. Furthermore, both the *SDRL* and *MRL* indices demonstrate a similar trend as the *ARL*. The main finding from comparing Tables 3 and 4 is that using MDS and GMDS techniques for detecting non-sparse diagonal shifts (scenario 1) has a considerably greater influence on the performance of the ATL chart compared to sparse diagonal shifts (scenario 2).

Table 4. *RL features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 2*

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	173.9385	147.6309	124.5177	104.2381	88.0053	55.6005
		SDRL	201.2460	175.1907	145.4316	123.1942	104.4754	88.9015	55.0208
		MRL	138	121	103	87	72	60	39
MDS-ATL		*ARL*	199.1255	167.5999	138.6120	113.8206	92.6929	74.9781	42.5623
		SDRL	198.4491	168.7403	139.5496	114.3189	95.0044	75.6889	42.7268
		MRL	138	115	95	79	64	52	30
GMDS-ATL	$q=1$	*ARL*	199.9991	155.0499	125.2636	89.0154	68.6665	55.5435	31.5425
		SDRL	197.4807	147.4571	128.7383	89.7331	65.1677	51.9184	26.8687
		MRL	138	109	88	65	51	41	24
	$q=2$	*ARL*	198.6688	154.9197	125.2593	91.2727	69.2921	56.0495	30.1794
		SDRL	199.5893	157.0294	124.2078	93.0953	66.7335	53.1658	23.6370
		MRL	138	109	88	67	52	42	23

Table 5 indicates that in terms of *ARL* index, the MDS-ATL chart outperforms the ATL at various values of Δ (0.1, 0.2, 0.3, 0.4, 0.5, and 0.75), with percentages of 7.58, 18.45, 30.48, 37.38, 41.21, and 39.89. Additionally, when the GMDS technique is implemented with $q=1$, the performance of the ATL chart improves by 7.92, 21.16, 34.86, 43.39, 46.86, and 55.76 percent, respectively, under the mentioned values of Δ. As can be seen, the positive effect of using the MDS and GMDS techniques on the power of the ATL chart becomes more noticeable with the increase of Δ. A similar trend can be observed in *SDRL* and *MRL* indices produced by the MDS and GMDS charts.

Table 5. *RL* features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 3

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	120.7972	40.3205	15.9755	8.2071	5.1151	2.4923
		SDRL	201.2460	120.7832	39.4855	15.2993	7.6347	4.5666	1.9078
		MRL	138	83	28	11	6	4	2
MDS-ATL		*ARL*	199.1255	111.6419	32.8822	11.1054	5.1396	3.0072	1.4980
		SDRL	198.4491	111.2441	33.2822	11.2720	5.0651	2.7762	1.0197
		MRL	138	78	22	8	3	2	1
GMDS-ATL	$q=1$	*ARL*	199.9991	111.2271	31.7886	10.4066	4.6458	2.7183	1.1026
		SDRL	197.4807	111.1097	31.4771	10.5509	4.6317	2.5387	0.7109
		MRL	138	77	21	8	3	2	1
	$q=2$	*ARL*	198.6688	111.2013	32.0921	10.2317	4.4474	2.6022	1.0711
		SDRL	199.5893	111.2748	31.1282	10.4600	4.3239	2.3699	0.7083
		MRL	138	77	21	7	3	2	1

Table 6 indicates that the GMDS-ATL scheme demonstrates superior performance compared to the MDS-ATL and ATL charts in terms of all *ARL*, *SDRL*, and *MRL* metrics when the assignable cause leads to off-diagonal disturbances according to the fourth out-of-control scenario. Similar to the previous scenario, the performance improvement of the ATL chart through the utilization of MDS and GMDS strategies will be more significant as the shift magnitude increases. For example, when $\Delta=0.1,0.2,0.3,0.4,0.5,0.75$, the MDS-ATL chart demonstrates superiority over ATL in terms of the *ARL* index by 1.15%, 5.47%, 7.92%, 13.93%, 19.36%, and 30.17% respectively. The the *ARL* improvement achieved by implementing the GMDS strategy in the ATL chart for the mentioned shifts, is 2.72%, 17.30%, 24.02%, 47.44%, 50.75%, and 36.34% when $q=1$ while in the case of $q=2$, the *ARL* improvements are 4.16%, 16.92%, 28.19%, 54.93%, 57.78%, and 38.37%. In other words, under this off-diagonal pattern, increasing q from 1 to 2 will lead to improved detectability of the GMDS-ATL chart. By comparing the results in Tables 5 and 6, it is observed that the improvement percentage of all three *RL*-based indices is more tangible when ATL chart is equipped with MDS and GMDS sampling approaches in detection of an off-diagonal non-sparse pattern (as in the third scenario) compared to the off-diagonal sparse pattern (as in the fourth scenario).

Table 6. *RL features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 4*

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	187.9099	155.3939	111.0139	75.9599	51.0764	20.6715
		SDRL	201.2460	186.4131	154.5071	111.4313	75.0029	50.9071	20.2622
		MRL	138	131	108	77	53	35	14
MDS-ATL		*ARL*	199.1255	185.7435	146.8887	102.2206	65.3805	41.1865	14.4354
		SDRL	198.4491	186.5648	147.2405	102.0745	65.2222	41.3985	14.5201
		MRL	138	129	102	70	45	29	10
GMDS-ATL	$q=1$	*ARL*	199.9991	182.8032	128.5112	84.3483	39.9277	25.1525	13.1602
		SDRL	197.4807	185.3521	129.5276	85.2083	37.6359	23.8887	8.4463
		MRL	138	127	90	58	28	18	9
	$q=2$	*ARL*	198.6688	180.0868	129.0974	79.7237	34.2348	21.5663	12.7393
		SDRL	199.5893	176.4070	127.7548	79.4708	32.3020	20.5031	5.7574
		MRL	138	125	90	55	25	16	9

Tables 7 to 9 present the run length properties of the ATL, MDS-ATL, and GMDS-ATL control charting methods for joint diagonal/off-diagonal disturbances in high-dimensional covariance matrices. The results confirm that using the MDS and GMDS control charts, instead of the ATL, will significantly lead to faster detection of co-diagonal/off-diagonal changes in the covariance matrix components. The results presented in Tables 7 to 9 reveal that as the shift magnitudes increase, the superiority of MDS-ATL and GMDS-ATL compared to the ATL control chart becomes more evident.

Table 7. *RL features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 5*

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	98.9288	44.2593	20.4434	11.0304	6.8975	3.1627
		SDRL	201.2460	98.1537	43.3755	20.2059	10.4759	6.3302	2.5983
		MRL	138	69	31	14	8	5	2
MDS-ATL		*ARL*	199.1255	86.9386	32.4176	12.7290	5.8847	3.2767	1.4862
		SDRL	198.4491	86.4191	32.6108	12.6924	5.6984	3.0323	0.9624
		MRL	138	61	22	9	4	2	1
GMDS-ATL	$q=1$	*ARL*	199.9991	86.2533	31.0648	12.5609	5.4242	3.0203	1.2224
		SDRL	197.4807	84.1714	30.9066	12.4237	5.2813	2.8104	0.7855
		MRL	138	61	21	9	4	2	1
	$q=2$	*ARL*	198.6688	86.7379	32.3513	12.8048	5.6068	3.1220	1.2655
		SDRL	199.5893	88.8912	32.1598	12.7473	5.3554	2.8498	0.8444
		MRL	138	61	22	9	4	2	1

Table 8. *RL features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 6*

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	178.3261	128.3747	78.7263	43.6322	24.2064	7.2093
		SDRL	201.2460	178.1352	128.3269	76.9652	43.3895	23.6536	6.7377
		MRL	138	124	89	55	30	17	5
MDS-ATL		*ARL*	199.1255	176.7739	120.3104	66.4980	33.5460	16.5317	3.6357
		SDRL	198.4491	177.6378	120.3849	66.5117	33.6048	16.5764	3.3766
		MRL	138	123	84	46	23	11	2
GMDS-ATL	$q=1$	*ARL*	199.9991	168.6586	108.5676	57.8567	27.8740	13.7365	2.8737
		SDRL	197.4807	169.3584	105.8655	57.3664	29.0077	14.3088	2.5912
		MRL	138	118	76	40	19	9	2
	$q=2$	*ARL*	198.6688	176.2301	114.1713	59.7670	29.2099	14.3948	3.1825
		SDRL	199.5893	166.7157	114.6325	60.3128	29.7811	14.6903	2.7198
		MRL	138	121	79	41	19	9	2

Table 9. *RL features of ATL, MDS-ATL, and GMDS-ATL charts under scenario 7*

Chart		Index	Δ						
			0	0.1	0.2	0.3	0.4	0.5	0.75
ATL		*ARL*	200.6506	78.8354	30.6467	13.7325	7.1715	4.4884	2.1488
		SDRL	201.2460	77.8354	30.0446	13.3228	6.7644	3.9699	1.5554
		MRL	138	55	21	10	5	3	2
MDS-ATL		*ARL*	199.1255	65.0101	20.0844	6.9717	3.1433	1.8793	1.1167
		SDRL	198.4491	64.9520	20.0714	6.8471	2.8302	1.4152	0.3920
		MRL	138	45	14	5	2	1	1
GMDS-ATL	$q=1$	*ARL*	199.9991	64.3264	20.0155	6.7645	3.1169	1.8635	1.0000
		SDRL	197.4807	63.8176	20.2910	6.8467	2.8459	1.4227	0.3619
		MRL	138	44	14	5	2	1	1
	$q=2$	*ARL*	198.6688	65.6486	19.8908	7.1158	3.1635	1.8914	1.0241
		SDRL	199.5893	64.4291	20.7047	6.9845	2.7734	1.3868	0.3694
		MRL	138	45	13	5	2	1	1

V. Conclusion remarks

Manufacturers often prefer using small sample sizes to reduce production costs, while also needing to consider a wide range of quality characteristics in order to enhance their market share. Therefore, in today's competitive world, encountering high-dimensional data where the sample size is smaller than the number of quality characteristics, has become a significant challenge in various industrial and service applications. Due to the singularity of the sample covariance matrix, conventional multivariate statistical methods cannot be used to monitor the process dispersion under high-dimensionality. On the other hand, in recent years, multiple dependent state sampling has been effectively utilized to enhance the effectiveness of control charts, primarily for the purpose of monitoring the mean parameter of univariate processes. As a result, in this paper, we extended two adaptive thresholding Lasso control charts for the rapid detection of covariance matrix disturbances in high-dimensional process monitoring using multiple dependent state and

generalized multiple dependent state sampling techniques. The performance of the improved MDS-ATL and GMDS-ATL control charting methods was compared to the conventional adaptive thresholding Lasso through Monte Carlo simulation, using three metrics of *ARL*, *SDRL* and *MRL*. The simulation results demonstrate that the utilization of MDS and GMDS methods in design of the ATL control chart effectively enhances the detection of covariance matrix disturbances in high-dimensional processes. Particularly when confronted with assignable causes that result in significant deviations in the components of the covariance matrix, the incorporation of MDS and GMDS methods in the design of the ATL chart becomes even more crucial. One assumption of this article is that the assignable cause solely impacts the components of the covariance matrix. However, there are instances where the deviation source can also result in a simultaneous change in the mean vector and the covariance matrix of high-dimensional processes. To overcome this limitation, it is recommended to employ MDS and GMDS techniques to monitor the coefficient of variation in high-dimensional processes.

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A NEUTROSOPHIC FUZZY ACCEPTANCE SAMPLING PLAN BASED ON NEGATIVE BINOMIAL DISTRIBUTION

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Abstract

This paper suggests a novel method for acceptance sampling that integrates neutrosophical fuzzy logic with the negative binomial distribution. The complexity and ambiguity that characterize real-world circumstances are typically overlooked by traditional acceptance sampling methodologies. The neutrosophic Fuzzy Acceptance Sampling Plan (NFASP) incorporates the negative binomial distribution, which is particularly well-suited for count data, to account for circumstances where defect occurrences are important. The efficacy of the methodology is demonstrated by theoretical study and simulations. This innovative method lifts acceptance sampling to a more accurate and sophisticated procedure by dealing with ambiguity and indeterminacy.

Keywords: Neutrosophic fuzzy acceptance sampling, Negative Binomial distribution, Indeterminacy.

I. Introduction

Acceptance sampling is a statistical technique that assess whether a large number of products confirm to particular standards. The goal of acceptance sampling is to minimize both the cost of the inspection and likelihood of approving a deficient batch. By inspecting the sample of the product deciding whether to accept or reject entire lot based on the inspection sample. In this paper our study focusses on a new attribute sampling plan by using a new fuzzy technique neutrosophic concepts. With a help of neutrosophic fuzzy acceptance sampling plan, we demonstrate that our traditional plan is sometimes inadequate when the situations are indeterminacy. We think that this research will significantly add to the body of knowledge on neutrosophic fuzzy acceptance sampling plan.

Acceptance sampling is a crucial quality control method that is applied in a variety of sectors to determine whether to accept or reject a batch of goods after inspecting a sample. Although traditional acceptance sampling plans have been widely used, academics have begun to investigate novel approaches, such as neutrosophic fuzzy acceptance sampling plans, in order to address uncertainty and ambiguity in real-world industrial processes. The purpose of this review of the literature is to examine the material that has been done on the design of fuzzy acceptance sampling plan based on the negative binomial distribution.

According to the neutrosophic statistical interval method, Aslam (2018) [1] suggested a novel acceptance sampling plan for the exponential distribution. The neutrosophic non-linear problem was used to determine the neutrosophic plan parameters of the proposed design. further usage in

industry, tables with different risk values were supplied. The acceptance sampling plan for the binomial distribution based on neutrosophic fuzzy sets was developed by Aslam & Aslam (2020) [2]. Regarding both producer risk and consumer risk, the proposed plan was found to be more effective than conditional acceptance sampling plan. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty. A novel approach of acceptance sampling plan based on neutrosophic fuzzy sets was proposed by Aslam and Aslam (2021) [3]. Regarding both producer risk and consumer risk, the proposed plan was found to be more effective than conditional acceptance sampling plans. Simulated data was used to evaluate the same plan. A variable sampling plan for the Poisson distribution based on neutrosophic fuzzy sets was proposed by Aslam et al. in 2021 [4]. Regarding both the risk to the producer and the risk to the consumer, the proposed plans was shown to be more successful than conditional variable sampling plans. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty. For the binomial distribution with unknown probability of defect, Divya P et al. (2012) [5] suggested a novel type of acceptance sampling plan based on neutrosophic fuzzy sets. The proposed plan was found to be more effective than conditional acceptance sampling plans. For the Poisson distribution with unknown parameters, Aslam et al. (2021) [6] developed a novel acceptance sampling method based on neutrosophic fuzzy sets. Regarding both producer risk and consumer risk, the proposed plan was found to be more effective than conditional acceptance sampling plans. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty.

For the Weibull distribution with unknown parameters, Aslam et al. (2021) [7] developed a novel acceptance sampling method based on neutrosophic fuzzy sets. Regarding both producer risk and consumer risk, the proposed approach was found to be more successful than conditional acceptance sampling plans. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty. For the binomial distribution with unknown parameters, Aslam et al. (2021) [8] developed a novel acceptance sampling method based on neutrosophic fuzzy sets. Regarding both producer risk and consumer risk, the proposed approach was found to be more successful than conditional acceptance sampling plans. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty. A novel kind of double sampling plan based on neutrosophic fuzzy sets was developed by Aslam et al. in 2021 [9] for the Poisson distribution with unknown parameters. Regarding both the risk to the producer and the risk to the consumer, the proposed plans was found to be more successful than conditional double sampling plans. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty. For the Weibull distribution with unknown parameters, Aslam et al. (2021) [10] developed a novel sort of variable sampling plans based on neutrosophic fuzzy sets. Regarding both the risk to the producer and the risk to the consumer, the suggested plans was shown to be more successful than conditional variable sampling plans. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty.

A novel sort of acceptance sampling plan based on intuitionistic fuzzy linguistic concepts was presented by Isik and Kaya in 2022 [11]. Simulated analysis of the plan's performance revealed that it is a promising plan for addressing product quality uncertainty. The effect of neutrosophic statistics on acceptance sampling plans was studied by Raza et al. in 2022 [12]. The study demonstrated several ways in which neutrosophic statistics can be used to enhance the design and analysis of acceptance sampling plans, including addressing product quality uncertainty, enhancing producer and consumer risk, and enhancing the effectiveness of acceptance sampling plans. Furthermore, an acceptance sampling plan for the Weibull distribution based on neutrosophic fuzzy sets was developed by Raza et al. in 2022 [13]. The proposed approach was found to be more successful than conditional acceptance sampling plans. Simulated analysis of the plan's performance

revealed that it is a promising plan for addressing product quality uncertainty. An innovative acceptance sampling approach for attribute data based on neutrosophic fuzzy sets was developed by Raza et al. in 2022 [14]. A kind of double sampling plans for the negative binomial distribution based on neutrosophic fuzzy sets was developed by Sadeghpour Gildeh B et al. in 2022 [15]. For the Weibull distribution with an unknown shape parameter, Aslam et al. (2022) [16] suggested a acceptance sampling plan based on neutrosophic fuzzy sets.

II. Neutrosophic Negative Binomial Distribution (NNBD)

The neutrosophic negative binomial distribution is an extension of the negative binomial distribution that incorporates neutrosophic uncertainty. In this distribution, the parameters of the negative binomial distribution are described with the help of neutrosophic membership degrees, which represent the degree of truth, indeterminacy, and falsity associated with each parameter. The neutrosophic negative binomial distribution allows for modeling uncertainty in the parameter values and is especially useful when dealing with imprecise or incomplete information. The Cumulative Density Function (CDF) is given below:

$$F(x) = 1 - (p_r(S))^{\delta}(p_r(U))^{x-\delta+1}\sum_{m=1}^{\infty}\binom{x-s+m}{m-1}(p_r(I))^{m-1} \tag{1}$$

The Probability Mass Function (PMF) NNBD is given below

$$T_x = (p_r(S))^{s}\sum_{t=0}^{th}\binom{x-s}{t}(p_r(I))^{t}(p_r(U))^{x-s-t} \tag{2}$$

$$U_x = \sum_{\substack{y=s\\ y\neq x}}^{\infty} T_y = (p_r(S))^{s}\sum_{\substack{y=s\\ y\neq x}}^{\infty}\sum^{th}\binom{y-s}{t}(p_r(I))^{t}(p_r(U))^{x-s-t} \tag{3}$$

$$I_x = (p_r(S))^{s}\sum_{z=th+1}^{x-\delta}\binom{x-s}{z}(p_r(I))^{z}\sum_{x-s}^{\infty}\sum_{t=0}^{x-s-z}\binom{x-s-z}{t}(p_r(U))^{x-s-z-t} \tag{4}$$

The mentioned equations namely (1), (2), (3) and (4) gives cumulative density function, probability mass functions of negative binomial distribution. Additionally, Tx provides probability of success, Ux provides probability of failures and Ix provides probability of indeterminacy values.

III. Operating Procedure

The following stages make up the operational method for the neutrosophic negative binomial distribution:

Step 1: Establish the neutrosophic negative binomial distribution's parameters. r: The number of successes (failures before the procedure is stopped), p: The likelihood that each trial will be successful, P(S): The degree of success with uncertainty (degree of success with neutral membership). P(U) is the level of uncertainty for the unknown (neutrosophic membership level), and P(I) is the level of uncertainty for the failure (neutrosophic membership level).

Step 2: Determine the neutrosophic negative binomial distribution's probability mass function (PMF) for a given success rate x. PMF is provided by:

$$T_x = (p_r(S))^s \sum_{t=0}^{th} \binom{x-s}{t} (p_r(I))^t (p_r(U))^{x-s-t} \tag{5}$$

$$U_x = \sum_{\substack{y=s \\ y \neq x}}^{\infty} T_y = (p_r(S))^s \sum_{\substack{y=s \\ y \neq x}}^{\infty} \sum^{th} \binom{y-s}{t} (p_r(I))^t (p_r(U))^{x-s-t} \tag{6}$$

$$I_x = (p_r(S))^s \sum_{z=th+1}^{x-\delta} \binom{x-s}{z} (p_r(I))^z \sum_{x-s}^{\infty} \sum_{t=0}^{x-s-z} \binom{x-s-z}{t} (p_r(U))^{x-s-z-t} \tag{7}$$

Step 3: Determine the Level of Belongingness in the degree of truth, indeterminacy, and falsehood related to each parameter is represented by the neutrosophic membership degrees P(S), P(U), and P(I). The following criteria should be met by these membership degrees: Tx + Ux + Ix = 1.

Step 4: Making Decisions to make judgments in the face of uncertainty, use the neutrosophic negative binomial distribution. You can calculate the likelihood of various outcomes and evaluate the risk involved with certain situations or events.

IV. The Average Outgoing Quality (AOQ)

In order to evaluate the quality of items or services leaving a manufacturing or service process, the Average Outgoing Quality (AOQ) is a crucial metric used in acceptance sampling plans. The predicted quality of the goods or services under the suggested acceptance sampling plans is best understood in the context of the neutrosophic negative binomial distribution. Consider the situation where a random sample of goods or services is chosen for inspection, and a decision is made based on whether the number of successes (k) (e.g., defect-free units, satisfied consumers) in the sample meets a given criterion before a certain number of failures (r) (e.g., defective units, dissatisfied consumers) occur. This will help you understand AOQ in the context of the neutrosophic negative binomial distribution. We seek to ascertain the typical proportion of non-conforming items (defective goods or disgruntled consumers) that will be approved by the acceptance sampling plan across a large number of inspection instances in the context of AOQ for the neutrosophic negative binomial distribution. The formula below can be used to determine

$$AOQ = \frac{k+1}{k+r+1} \tag{8}$$

Where:

- AOQ: Average Outgoing Quality, which denotes the anticipated percentage of non-conforming products in lots that have been accepted.
- k: The predetermined number of accomplishments (such as non-defective products or pleased clients) necessary for acceptance.
- r: The predetermined number of errors permitted prior to rejection (examples: damaged goods, angry consumers).
- An elevated AOQ score means that the acceptance sampling plan is successful in preserving a high standard of quality in the delivered goods or services. A low AOQ number, on the other hand, denotes that the acceptance plan could require modifications in order to better assure the

quality of the delivered goods or services.

Case Study – 1

A company manufactures yarn. The company has a quality control inspector who randomly inspects 10 yarns from each lot of 100 yarns. The inspector defines 20% of the yarns as successes, 60% of the yarns as failures, and 20% of the yarns as indeterminacies. The inspector rejects the lot if he finds more than 3 failures.

Here, X = 0,1, 2…10, th = 2 and s = 3
P(S) = 20% P(U) = 60% P(I) = 20%

Case Study – 2

A company manufactures batteries. The company has a quality control inspector who randomly inspects 15 batteries from each lot of 100 batteries. The inspector defines 10% of the batteries as successes, 80% of the batteries as failures, and 10% of the batteries as indeterminacies. The inspector rejects the lot if he finds more than 4 failures.

Here, X = 0,1, 2,15, th = 2 and s = 4
P(S) = 10% P(U) = 80% P(I) = 10%

Case Study – 3

A company manufactures toys. The company has a quality control inspector who randomly inspects 3 toys from each lot of 50 toys. The inspector defines 80% of the toys as successes, 15% of the toys as failures, and 5% of the toys as indeterminacies. The inspector rejects the lot if he finds more than 1 failure.

Here, X = 0,1, 2,50, th = 2 and s = 4
P(S) = 80% P(U) = 15% P(I) = 05%

V. Conclusion

This work presents a unique discrete neutrosophic negative binomial probability distribution based on neutrosophic logic. We have looked at a variety of case studies throughout the study to show how the suggested distribution works in real-world settings. The suggested neutrosophic negative binomial distribution has proven useful for modeling situations where one wants to know how many successes there will be in a series of independent trials before there are a certain number of failures. This study makes a significant addition to the area of probability theory by broadening the applicability of the negative binomial distribution using neutrosophic logic. It also creates new opportunities for managing uncertainty in a variety of real-world scenarios.

Table 1: *Consisting of n, Ux, Ix and Tx value when c = 3, th = 2, P(S) = 0.35, P(U) = 0.7 and P(I) = 0.15*

n	U_x	I_x	T_x	$T_x+I_x+U_x$	AOQ	n	U_x	I_x	T_x	$T_x+I_x+U_x$	AOQ
2	0.0002	0.0292	0.9706	1.0000	0.0116	49	0.0000	0.0008	0.9992	1.0000	0.0003
3	0.0002	0.0331	0.9666	1.0000	0.0131	50	0.0000	0.0007	0.9993	1.0000	0.0003
4	0.0003	0.0325	0.9673	1.0000	0.0129	51	0.0000	0.0006	0.9994	1.0000	0.0002
5	0.0003	0.0290	0.9707	1.0000	0.0115	52	0.0000	0.0006	0.9994	1.0000	0.0002
6	0.0003	0.0244	0.9752	1.0000	0.0097	53	0.0000	0.0005	0.9995	1.0000	0.0002
7	0.0003	0.0199	0.9798	1.0000	0.0079	54	0.0000	0.0005	0.9995	1.0000	0.0002
8	0.0003	0.0159	0.9838	1.0000	0.0063	55	0.0000	0.0004	0.9996	1.0000	0.0002
9	0.0003	0.0127	0.9869	1.0000	0.0050	56	0.0000	0.0004	0.9996	1.0000	0.0001
10	0.0003	0.0103	0.9894	1.0000	0.0041	57	0.0000	0.0003	0.9997	1.0000	0.0001
11	0.0003	0.0086	0.9911	1.0000	0.0034	58	0.0000	0.0003	0.9997	1.0000	0.0001
12	0.0003	0.0074	0.9924	1.0000	0.0029	59	0.0000	0.0003	0.9997	1.0000	0.0001
13	0.0003	0.0065	0.9932	1.0000	0.0026	60	0.0000	0.0002	0.9998	1.0000	0.0001
14	0.0003	0.0060	0.9937	1.0000	0.0024	61	0.0000	0.0002	0.9998	1.0000	0.0001
15	0.0002	0.0057	0.9941	1.0000	0.0022	62	0.0000	0.0002	0.9998	1.0000	0.0001
16	0.0002	0.0054	0.9943	1.0000	0.0021	63	0.0000	0.0002	0.9998	1.0000	0.0001
17	0.0002	0.0053	0.9945	1.0000	0.0021	64	0.0000	0.0002	0.9998	1.0000	0.0001
18	0.0002	0.0052	0.9946	1.0000	0.0020	65	0.0000	0.0001	0.9999	1.0000	0.0001
19	0.0002	0.0051	0.9947	1.0000	0.0020	66	0.0000	0.0001	0.9999	1.0000	0.0000
20	0.0002	0.0050	0.9948	1.0000	0.0019	67	0.0000	0.0001	0.9999	1.0000	0.0000
21	0.0001	0.0049	0.9950	1.0000	0.0019	68	0.0000	0.0001	0.9999	1.0000	0.0000
22	0.0001	0.0048	0.9951	1.0000	0.0019	69	0.0000	0.0001	0.9999	1.0000	0.0000
23	0.0001	0.0047	0.9952	1.0000	0.0018	70	0.0000	0.0001	0.9999	1.0000	0.0000
24	0.0001	0.0045	0.9954	1.0000	0.0018	71	0.0000	0.0001	0.9999	1.0000	0.0000
25	0.0001	0.0044	0.9955	1.0000	0.0017	72	0.0000	0.0001	0.9999	1.0000	0.0000
26	0.0001	0.0042	0.9957	1.0000	0.0016	73	0.0000	0.0001	0.9999	1.0000	0.0000
27	0.0001	0.0040	0.9959	1.0000	0.0016	74	0.0000	0.0000	1.0000	1.0000	0.0000
28	0.0001	0.0038	0.9961	1.0000	0.0015	75	0.0000	0.0000	1.0000	1.0000	0.0000
29	0.0001	0.0036	0.9963	1.0000	0.0014	76	0.0000	0.0000	1.0000	1.0000	0.0000
30	0.0001	0.0035	0.9965	1.0000	0.0013	77	0.0000	0.0000	1.0000	1.0000	0.0000

Table 2: *Consisting of n, Ux, Ix and Tx value when c = 3, th = 2, P(S) = 0.95, P(U) = 0.05 and P(I) = 0.15*

n	U_x	I_x	T_x	$T_x+I_x+U_x$	AOQ	n	U_x	I_x	T_x	$T_x+I_x+U_x$	AOQ
11	0.0000	0.6405	0.3595	1.0000	0.1248	68	0.0000	0.3681	0.6318	1.0000	0.2068
12	0.0000	0.0885	0.9114	1.0000	0.3161	69	0.0000	0.3728	0.6272	1.0000	0.2050
13	0.0000	0.0106	0.9894	1.0000	0.3428	70	0.0000	0.3771	0.6228	1.0000	0.2034
14	0.0000	0.0044	0.9956	1.0000	0.3446	71	0.0000	0.3813	0.6187	1.0000	0.2018
15	0.0000	0.0055	0.9945	1.0000	0.3439	72	0.0000	0.3851	0.6148	1.0000	0.2003
16	0.0000	0.0075	0.9925	1.0000	0.3429	73	0.0000	0.3888	0.6112	1.0000	0.1989
17	0.0000	0.0099	0.9901	1.0000	0.3417	74	0.0000	0.3921	0.6078	1.0000	0.1976
18	0.0000	0.0127	0.9873	1.0000	0.3404	75	0.0000	0.3953	0.6047	1.0000	0.1964
19	0.0000	0.0160	0.9840	1.0000	0.3389	76	0.0000	0.3981	0.6019	1.0000	0.1953
20	0.0000	0.0196	0.9804	1.0000	0.3373	77	0.0000	0.4007	0.5993	1.0000	0.1942
21	0.0000	0.0236	0.9764	1.0000	0.3356	78	0.0000	0.4031	0.5969	1.0000	0.1932
22	0.0000	0.0281	0.9719	1.0000	0.3337	79	0.0000	0.4052	0.5948	1.0000	0.1923
23	0.0000	0.0329	0.9671	1.0000	0.3317	80	0.0000	0.4071	0.5929	1.0000	0.1915
24	0.0000	0.0381	0.9619	1.0000	0.3296	81	0.0000	0.4087	0.5912	1.0000	0.1908
25	0.0000	0.0437	0.9563	1.0000	0.3273	82	0.0000	0.4101	0.5898	1.0000	0.1901
26	0.0000	0.0496	0.9504	1.0000	0.3250	83	0.0000	0.4113	0.5887	1.0000	0.1896
27	0.0000	0.0558	0.9441	1.0000	0.3225	84	0.0000	0.4122	0.5877	1.0000	0.1890
28	0.0000	0.0624	0.9376	1.0000	0.3200	85	0.0000	0.4130	0.5870	1.0000	0.1886
29	0.0000	0.0693	0.9307	1.0000	0.3173	86	0.0000	0.4135	0.5865	1.0000	0.1882
30	0.0000	0.0764	0.9236	1.0000	0.3145	87	0.0000	0.4137	0.5863	1.0000	0.1880
31	0.0000	0.0838	0.9162	1.0000	0.3117	88	0.0000	0.4138	0.5862	1.0000	0.1877
32	0.0000	0.0914	0.9085	1.0000	0.3088	89	0.0000	0.4136	0.5863	1.0000	0.1876
33	0.0000	0.0993	0.9007	1.0000	0.3058	90	0.0000	0.4133	0.5867	1.0000	0.1875
34	0.0000	0.1073	0.8926	1.0000	0.3027	91	0.0000	0.4128	0.5872	1.0000	0.1874
35	0.0000	0.1156	0.8844	1.0000	0.2996	92	0.0000	0.4120	0.5880	1.0000	0.1875
36	0.0000	0.1240	0.8760	1.0000	0.2965	93	0.0000	0.4111	0.5889	1.0000	0.1876
37	0.0000	0.1325	0.8675	1.0000	0.2933	94	0.0000	0.4100	0.5900	1.0000	0.1877
38	0.0000	0.1411	0.8589	1.0000	0.2901	95	0.0000	0.4087	0.5913	1.0000	0.1879
39	0.0000	0.1498	0.8502	1.0000	0.2868	96	0.0000	0.4072	0.5928	1.0000	0.1882
40	0.0000	0.1586	0.8414	1.0000	0.2836	97	0.0000	0.4056	0.5944	1.0000	0.1885
41	0.0000	0.1674	0.8326	1.0000	0.2803	98	0.0000	0.4038	0.5962	1.0000	0.1888

42	0.0000	0.1763	0.8237	1.0000	0.2771	99	0.0000	0.4019	0.5981	1.0000	0.1892
43	0.0000	0.1851	0.8148	1.0000	0.2738	100	0.0000	0.3998	0.6002	1.0000	0.1897
44	0.0000	0.1940	0.8060	1.0000	0.2705	101	0.0000	0.3975	0.6024	1.0000	0.1902
45	0.0000	0.2029	0.7971	1.0000	0.2673	102	0.0000	0.3952	0.6048	1.0000	0.1907
46	0.0000	0.2117	0.7883	1.0000	0.2640	103	0.0000	0.3927	0.6073	1.0000	0.1913
47	0.0000	0.2204	0.7796	1.0000	0.2608	104	0.0000	0.3900	0.6100	1.0000	0.1919
48	0.0000	0.2291	0.7709	1.0000	0.2577	105	0.0000	0.3873	0.6127	1.0000	0.1926
49	0.0000	0.2376	0.7623	1.0000	0.2545	106	0.0000	0.3844	0.6156	1.0000	0.1933
50	0.0000	0.2461	0.7539	1.0000	0.2515	107	0.0000	0.3814	0.6186	1.0000	0.1940
51	0.0000	0.2544	0.7455	1.0000	0.2484	108	0.0000	0.3783	0.6217	1.0000	0.1947
52	0.0000	0.2627	0.7373	1.0000	0.2454	109	0.0000	0.3751	0.6249	1.0000	0.1955
53	0.0000	0.2707	0.7292	1.0000	0.2425	110	0.0000	0.3718	0.6282	1.0000	0.1963
54	0.0000	0.2786	0.7213	1.0000	0.2396	111	0.0000	0.3684	0.6316	1.0000	0.1972
55	0.0000	0.2864	0.7136	1.0000	0.2368	112	0.0000	0.3649	0.6351	1.0000	0.1980
56	0.0000	0.2940	0.7060	1.0000	0.2340	113	0.0000	0.3614	0.6386	1.0000	0.1989
57	0.0000	0.3013	0.6986	1.0000	0.2313	114	0.0000	0.3578	0.6422	1.0000	0.1998
58	0.0000	0.3085	0.6914	1.0000	0.2287	115	0.0000	0.3541	0.6459	1.0000	0.2008
59	0.0000	0.3155	0.6845	1.0000	0.2261	116	0.0000	0.3503	0.6497	1.0000	0.2017
60	0.0000	0.3223	0.6777	1.0000	0.2237	117	0.0000	0.3465	0.6535	1.0000	0.2027
61	0.0000	0.3288	0.6711	1.0000	0.2213	118	0.0000	0.3426	0.6574	1.0000	0.2036
62	0.0000	0.3352	0.6648	1.0000	0.2190	119	0.0000	0.3386	0.6614	1.0000	0.2046
63	0.0000	0.3413	0.6587	1.0000	0.2167	120	0.0000	0.3346	0.6654	1.0000	0.2056
64	0.0000	0.3471	0.6529	1.0000	0.2146	121	0.0000	0.3306	0.6694	1.0000	0.2066
65	0.0000	0.3527	0.6472	1.0000	0.2125	122	0.0000	0.3265	0.6735	1.0000	0.2077
66	0.0000	0.3581	0.6419	1.0000	0.2105	123	0.0000	0.3224	0.6776	1.0000	0.2087

Table 3: *Consisting of n, Ux, Ix and Tx value when c = 3, th = 2, P(S) = 0.8, P(U) = 0.4 and P(I) = 0.15*

n	U_x	I_x	T_x	$T_x+I_x+U_x$	AOQ	n	U_x	I_x	T_x	$T_x+I_x+U_x$	AOQ
6	0.0020	0.0926	0.9053	1.0000	0.4041	54	0.0000	0.0000	1.0000	1.0000	0.4252
8	0.0019	0.0626	0.9356	1.0000	0.4172	55	0.0000	0.0000	1.0000	1.0000	0.4248
9	0.0017	0.0351	0.9632	1.0000	0.4291	56	0.0000	0.0000	1.0000	1.0000	0.4243
10	0.0015	0.0190	0.9795	1.0000	0.4359	57	0.0000	0.0000	1.0000	1.0000	0.4239
11	0.0014	0.0113	0.9874	1.0000	0.4390	58	0.0000	0.0000	1.0000	1.0000	0.4234
12	0.0012	0.0080	0.9908	1.0000	0.4401	59	0.0000	0.0000	1.0000	1.0000	0.4230
13	0.0010	0.0068	0.9922	1.0000	0.4402	60	0.0000	0.0000	1.0000	1.0000	0.4225
14	0.0009	0.0064	0.9928	1.0000	0.4401	61	0.0000	0.0000	1.0000	1.0000	0.4221
15	0.0007	0.0061	0.9931	1.0000	0.4398	62	0.0000	0.0000	1.0000	1.0000	0.4216
16	0.0006	0.0059	0.9935	1.0000	0.4395	63	0.0000	0.0000	1.0000	1.0000	0.4212
17	0.0005	0.0057	0.9938	1.0000	0.4392	64	0.0000	0.0000	1.0000	1.0000	0.4207
18	0.0004	0.0054	0.9942	1.0000	0.4389	65	0.0000	0.0000	1.0000	1.0000	0.4203
19	0.0003	0.0050	0.9946	1.0000	0.4386	66	0.0000	0.0000	1.0000	1.0000	0.4198
20	0.0003	0.0046	0.9951	1.0000	0.4384	67	0.0000	0.0000	1.0000	1.0000	0.4194
21	0.0002	0.0043	0.9955	1.0000	0.4381	68	0.0000	0.0000	1.0000	1.0000	0.4189
22	0.0002	0.0038	0.9960	1.0000	0.4379	69	0.0000	0.0000	1.0000	1.0000	0.4185
23	0.0001	0.0035	0.9964	1.0000	0.4376	70	0.0000	0.0000	1.0000	1.0000	0.4180
24	0.0001	0.0031	0.9968	1.0000	0.4373	71	0.0000	0.0000	1.0000	1.0000	0.4176
25	0.0001	0.0027	0.9972	1.0000	0.4371	72	0.0000	0.0000	1.0000	1.0000	0.4171
26	0.0001	0.0024	0.9975	1.0000	0.4368	73	0.0000	0.0000	1.0000	1.0000	0.4167
27	0.0001	0.0021	0.9978	1.0000	0.4365	74	0.0000	0.0000	1.0000	1.0000	0.4162
28	0.0000	0.0018	0.9981	1.0000	0.4361	75	0.0000	0.0000	1.0000	1.0000	0.4158
29	0.0000	0.0016	0.9984	1.0000	0.4358	76	0.0000	0.0000	1.0000	1.0000	0.4153
30	0.0000	0.0013	0.9986	1.0000	0.4354	77	0.0000	0.0000	1.0000	1.0000	0.4149
31	0.0000	0.0012	0.9988	1.0000	0.4351	78	0.0000	0.0000	1.0000	1.0000	0.4144
32	0.0000	0.0010	0.9990	1.0000	0.4347	79	0.0000	0.0000	1.0000	1.0000	0.4140
33	0.0000	0.0008	0.9992	1.0000	0.4343	80	0.0000	0.0000	1.0000	1.0000	0.4135
34	0.0000	0.0007	0.9993	1.0000	0.4339	81	0.0000	0.0000	1.0000	1.0000	0.4131
35	0.0000	0.0006	0.9994	1.0000	0.4335	82	0.0000	0.0000	1.0000	1.0000	0.4126
36	0.0000	0.0005	0.9995	1.0000	0.4331	83	0.0000	0.0000	1.0000	1.0000	0.4122
37	0.0000	0.0004	0.9996	1.0000	0.4327	84	0.0000	0.0000	1.0000	1.0000	0.4117

38	0.0000	0.0003	0.9996	1.0000	0.4323	85	0.0000	0.0000	1.0000	1.0000	0.4113
39	0.0000	0.0003	0.9997	1.0000	0.4319	86	0.0000	0.0000	1.0000	1.0000	0.4108
40	0.0000	0.0002	0.9998	1.0000	0.4314	87	0.0000	0.0000	1.0000	1.0000	0.4104
41	0.0000	0.0002	0.9998	1.0000	0.4310	88	0.0000	0.0000	1.0000	1.0000	0.4099
42	0.0000	0.0002	0.9998	1.0000	0.4306	89	0.0000	0.0000	1.0000	1.0000	0.4095
43	0.0000	0.0001	0.9999	1.0000	0.4301	90	0.0000	0.0000	1.0000	1.0000	0.4090
44	0.0000	0.0001	0.9999	1.0000	0.4297	91	0.0000	0.0000	1.0000	1.0000	0.4086
45	0.0000	0.0001	0.9999	1.0000	0.4293	92	0.0000	0.0000	1.0000	1.0000	0.4081
46	0.0000	0.0001	0.9999	1.0000	0.4288	93	0.0000	0.0000	1.0000	1.0000	0.4077
47	0.0000	0.0001	0.9999	1.0000	0.4284	94	0.0000	0.0000	1.0000	1.0000	0.4072
48	0.0000	0.0001	0.9999	1.0000	0.4279	95	0.0000	0.0000	1.0000	1.0000	0.4068
49	0.0000	0.0000	1.0000	1.0000	0.4275	96	0.0000	0.0000	1.0000	1.0000	0.4063
50	0.0000	0.0000	1.0000	1.0000	0.4270	97	0.0000	0.0000	1.0000	1.0000	0.4059
51	0.0000	0.0000	1.0000	1.0000	0.4266	98	0.0000	0.0000	1.0000	1.0000	0.4054
52	0.0000	0.0000	1.0000	1.0000	0.4261	99	0.0000	0.0000	1.0000	1.0000	0.4050
53	0.0000	0.0000	1.0000	1.0000	0.4257	100	0.0000	0.0000	1.0000	1.0000	0.4050

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OPTIMIZING INVENTORY OF DETERIORATING PRODUCTS WITH PRICE-DEPENDENT DEMAND USING QUANTUM-BEHAVED AGTO VARIANTS

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Abstract

Preservation of a product is an important issue in the inventory control system. It prevents the deterioration effect of the products while these are stored in the warehouse/showroom. Considering deterioration effect of the product and preservation technology, an inventory model of non-instantaneous deteriorating items is developed with the demand dependent on the selling price of the product. Two different preservation rates are considered. Shortages are allowed partially with two different backlogging rates. Due to consideration of three-parameter Weibull distributed deterioration and preservation facility, the corresponding optimization problems are highly nonlinear. So, these problems cannot be solved analytically due to nonlinearity. To overcome this situation, different variants of quantum-behaved Artificial Gorilla Troops Optimizer (AGTO) are used. To illustrate and validate the proposed model, a numerical example is considered and solved for each case, and compared the results with the different variants of AGTO algorithms. Finally, a sensitivity analysis is performed to study the effect of changes of different parameters of the model on the optimal policy.

Keywords:AGTO;Quantum-Behaved;Deteriorating Products;Price-Dependent Demand.

1. Introduction

Preserving product quality during storage is a critical issue in inventory management, especially for items susceptible to deterioration [1]. When setting up supplies, systems with two-level assembly and unpredictable lead timing are taken into account. Most likely, the demand for the finished product and its deadline are known. As soon as all necessary components arrive, each level's assembly process begins [2]. Shortages are permitted, but only partially, and are handled with two separate backlogging rates, allowing flexibility in managing stockouts [3]. The product has a set shelf life and is perishable. Expenses include fixed ordering and inventory holding expenses[4]. The deterioration process of the items follows a three-parameter Weibull distribution, capturing the complexity of product degradation over time [5]. Managing stochastic multi-state production and distribution systems can make it difficult to figure out how much safety stock to have on hand. [6]. Due to the inclusion of this deterioration model and the preservation measures, the resulting optimization problem becomes highly nonlinear, making it challenging to solve using traditional analytical methods. To tackle this, various versions

of the Quantum-behaved AGTO are employed, which are metaheuristic algorithms inspired by gorilla troops' natural behaviours [7]. These AGTO variants are particularly well-suited for handling the nonlinear, multi-parameter nature of the inventory problem. A numerical example is provided to demonstrate and validate the effectiveness of the model, with results compared across different AGTO variants [8] . Additionally, a sensitivity analysis is conducted to understand the impact of changing key parameters, such as demand rate, deterioration rate, and preservation efficiency, on the optimal inventory policy. This analysis provides valuable insights into how different factors influence the overall cost and performance of the inventory system, guiding decision-makers in applying the best preservation strategies to maximize profitability and minimize product waste [9]. This study focuses on developing an advanced inventory model for non-instantaneously deteriorating items, considering both the deterioration effects and preservation technologies to minimize losses during storage in warehouses or showrooms. In this model, product demand is driven by its selling price, reflecting real-market dynamics. The inventory system also incorporates two distinct preservation rates, addressing different levels of protection against product degradation.

2. Assumptions and Notation

The following assumptions and notation are considered to develop the model:

1. The inventory system contains only one item with an infinite time horizon.

2. The item is a non-instantaneously deteriorating item. Deterioration occurs after time $h = \gamma$ with the rate $\theta(h)$, which follows a three-parameter Weibull distribution. That is,

$$\theta(h) = \frac{s(h)}{1 - S(h)} = \alpha\beta(h - \gamma)^{\beta-1},$$

   where α, β, and γ (> 0) are the parameters of the Weibull distribution, with $s(h)$ and $S(h)$ as the probability density function and distribution function, respectively.

3. To reduce the deterioration effect, preservation technology is used in the inventory control system. Let $n(\zeta) < 0$ be the preservation technology function, which is an increasing function with $n''(\zeta) < 0$. Here, we consider $n(\zeta)$ as:

$$n(\zeta) = \frac{y_1\zeta}{1 + y_1\zeta}, \quad y_1 > 0,$$
$$n(\zeta) = 1 - e^{-y_1\zeta}, \quad y_1 > 0.$$

4. Lead time is constant and known.

5. Shortages are considered partially with a rate $Z(H - h)$, where h represents the length of waiting time for the customers. In this model, two partial backlogging rates are considered:

$$Z(H - h) = \frac{1}{1 + \delta(H - h)},$$
$$Z(H - h) = e^{-\delta(H-h)}.$$

6. The demand of the item is dependent linearly on the selling price, i.e.,

$$X(l) = y - zl, \quad l < \frac{y}{z},\ y, z > 0.$$

Notation	Description
W_l	Purchase cost per unit item (in \$)
l	Selling price per unit item (decision variable) (in \$), $l > W_l$
$X(l) = y - zl$	The demand function, $l < \frac{y}{z},\ y, z > 0$
W_0	Replenishment cost per order (in \$)
W_t	Holding cost per unit item per unit time (in \$)
W_g	Backordering cost per unit item per unit time (in \$)
W_{pg}	Lost sale cost per unit item per unit time (in \$)
γ	Starting time of deterioration (in month)
h_1	Time of zero ending inventory (decision variable) (in month)
H	Cycle length (decision variable) (in month)
I	Initial ordering quantities (unit)
J	Maximum shortage level (unit)
$Q(h)$	Inventory level at time h (unit)
$\theta(h)$	Deterioration rate per unit time
$Z(H-h)$	Backorder rate at time h, $h_1 < h \leq H$ (in month)
δ	Backlogging parameter
ζ	Preservation cost per unit item per unit time (decision variable) (in \$)
$B(l, \zeta, h_1, H)$	Average profit function (in \$)

3. Model formulation

Let us suppose that a retailer places an order of $(I + J)$ units of a product at time h = 0. After that, the deterioration starts at time $h = \gamma$ and inventory level reaches to zero at time $h = h_1$ due to the combined effect of demand and deterioration [10]. Then partially backlogged shortages are allowed with the backlogging rates. $Z(H-h) = \frac{1}{1+\delta(H-h)}$ and $Z(H-h) = e^{-\delta(H-h)}$: During the time period $[0, \gamma]$, there is no deterioration and after that deterioration starts at $h = \gamma$ and continues upto the time $h = h_1$. To reduce the rate of deterioration, we have considered preservation facility with the rates $n(\zeta) = \frac{y_1\zeta}{1+y_1\zeta}$ and $n(\zeta) = 1 - e^{-y_1\zeta}$ where $z_1 > 0$. Hence during the time interval $[0, H]$; the inventory levels are governed by the differential equations as follows:

$$\frac{\mathrm{d}Q(h)}{\mathrm{d}h} = -X(l), \quad 0 < h \leq \gamma \tag{1}$$

$$\frac{\mathrm{d}Q(h)}{\mathrm{d}h} = -X(l) - \theta(h)[1 - n(\zeta)]Q(h), \quad \gamma < h \leq h_1 \tag{2}$$

$$\frac{\mathrm{d}Q(h)}{\mathrm{d}h} = -X(l)Z(H-h), \quad h_1 < h \leq H \tag{3}$$

With the boundary conditions $Q(0) = I$ and $Q(H) = -J$. Also $Q(h)$ is continuous at $h = \gamma$ and $h = h_1$.

$$Q(h) = -X(l)h + I, \quad 0 < h \leq \gamma \tag{4}$$

$$Q(h) = -X(l)e^{-(h-\gamma)^{\beta}(1-n(\zeta))} \int_h^{h_1} e^{\alpha(h-\gamma)^{\beta}(1-n(\zeta))}\,\mathrm{d}h, \quad \gamma < h \leq h_1 \tag{5}$$

$$Q(h) = X(l) \int_{h_1}^{h} Z(H-h)\,\mathrm{d}h, \quad h_1 < h \leq H \tag{6}$$

Now applying the continuity condition of $Q(h)$ at $h = \gamma$ and h_1, we have:

$$I = X(l)\gamma + X(l) \int_{h_1}^{j} e^{\alpha(h-\gamma)^{\beta}(1-n(\zeta))}\,\mathrm{d}h \tag{7}$$

And

$$J = X(l) \int_{h_1}^{H} Z(H-h)\,\mathrm{dh} \tag{8}$$

Total sales revenue $(\mathrm{GJ}) = \mathrm{l}X(l)h_1 + \mathrm{l}J$

Total purchase cost $(\mathrm{LW}) = W_l(I + J)$

$$\begin{aligned} &= W_l X(l)\gamma + W_l X(l) \int_{h_1}^{j} e^{\alpha(h-\gamma)^{\beta}(1-n(\zeta))}\, \mathrm{d}h \\ &\quad + W_l X(l) \int_{h_1}^{H} Z(H-h)\, \mathrm{d}h \end{aligned} \tag{9}$$

The total inventory holding cost

$$\mathrm{TW} = W_t \int_0^{\gamma} Q(h)\, \mathrm{d}h + W_t \int_{\gamma}^{h_1} Q(h)\, \mathrm{d}h \tag{10}$$

$$= W_t I_{\gamma} - \frac{W_t X(l)\gamma^2}{2} + W_t X(l) \int_{\gamma}^{h_1} e^{-\alpha(h-\gamma)^{\beta}\{1-n(\zeta)\}} \times \int_{h}^{h_1} e^{\alpha(u-\gamma)^{\beta}} \mathrm{dudh} \tag{11}$$

The total shortage cost and lost sale cost are

$$W_g X(l) \int_{h_1}^{H} \int_{h_1}^{h} Z(H-h)\, \mathrm{d}u\, \mathrm{d}h$$

And

$$W_{\mathrm{pg}} X(l)(H-h_1) - W_{\mathrm{pg}} J,$$

Respectively, The preservation technology cost is given by ζH. Therefore, the average profit per cycle is given by...

$$\begin{aligned} B(l,\zeta,h_1,H) = \frac{1}{H} [&\text{Sales revenue} - \text{Purchase cost} - \text{Holding cost} \\ &- \text{Shortage cost} - \text{Lost sale cost} - \text{Ordering cost} - \text{Preservation technology cost}] \end{aligned} \tag{12}$$

$$i.e., B(l,\zeta,h_1,H) = \frac{1}{H} \begin{bmatrix} \mathrm{l}X(l)h_1 + \mathrm{l}J - \left\{ W_l X(l)\gamma + W_l X(l) \int_{\gamma}^{h1} e^{\alpha(h-\gamma)^{\beta}\{1-n(\zeta)\}\mathrm{dh}} + W_l J \right\} \\ - \left\{ W_t Ifl - \frac{W_t X(l)\gamma^2}{2} + W_h X(l) \int_{\gamma}^{h1} e^{-\alpha(h-\gamma)^{\{1-n(\zeta)\}}} \mathrm{dudh} \right\} \\ -W_g X(l) \int_{h1}^{H} \int_{h1}^{h} Z(H-h)\mathrm{dhdh} - W_{\mathrm{pg}} \{X(l)(H-h_1) - J\} - \imath H - W_0 \end{bmatrix} \tag{13}$$

Based on the partial backlogging rate $Z(H-h)$ and also the preservation facility rate $n(\zeta)$, four possible cases may arise:

Case 1: $Z(H-h) = \frac{1}{1+\delta(H-h)}$ and $n(\zeta) = \frac{y_1\zeta}{1+y_1\zeta},\ y_1 > 0$

Case 2: $Z(H-h) = \frac{1}{1+\delta(H-h)}$ and $n(\zeta) = 1 - e^{-y_1\zeta},\ y_1 > 0$

Case 3: $Z(H-h) = e^{-\delta(H-h)}$ and $n(\zeta) = \frac{y_1\zeta}{1+y_1\zeta},\ y_1 > 0$

Case 4: $Z(H-h) = e^{-\delta(H-h)}$ and $n(\zeta) = 1 - e^{-y_1\zeta},\ y_1 > 0$

Now we have discussed each case separately.

Case 1: $Z(H-h) = \frac{1}{1+\delta(H-h)}$ and $n(\zeta) = \frac{y1\imath}{1+y1\imath}, y1 > 0$. In this case,

$$I = X(l)\gamma + X(l) \int_{j}^{h_1} e^{\frac{\alpha(h-\gamma)^{\beta}}{1+\zeta}}\, dh \tag{14}$$

And

$$J = \frac{X(l)}{\delta} \log(l + \delta(H - h_1)) \tag{15}$$

The total purchase cost is given by

$$\text{LW} = W_l(I + J)$$

$$= W_l X(l)\gamma + W_l X(l) \int_j^{h_1} e^{\frac{\alpha(h-\gamma)^\beta}{1+\zeta}}\, dh + W_l \times \frac{X(l)}{\delta} \log|1 + \delta(H - h_1)| \tag{16}$$

The total inventory holding cost is

$$\text{TW} = W_t \int_0^{\gamma} Q(h)\, dh + W_t \int_{\gamma}^{h_1} Q(h)\, dh$$

$$= W_t I_\gamma - \frac{W_t X(l)\gamma^2}{2} + W_t X(l) \int_{\gamma}^{h_1} e^{\frac{\alpha(h-\gamma)^\beta}{1+\zeta}} \int_h^{h_1} e^{\frac{\alpha(U-\gamma)^\beta}{e^{1+\zeta}}}\, du\, dh \tag{17}$$

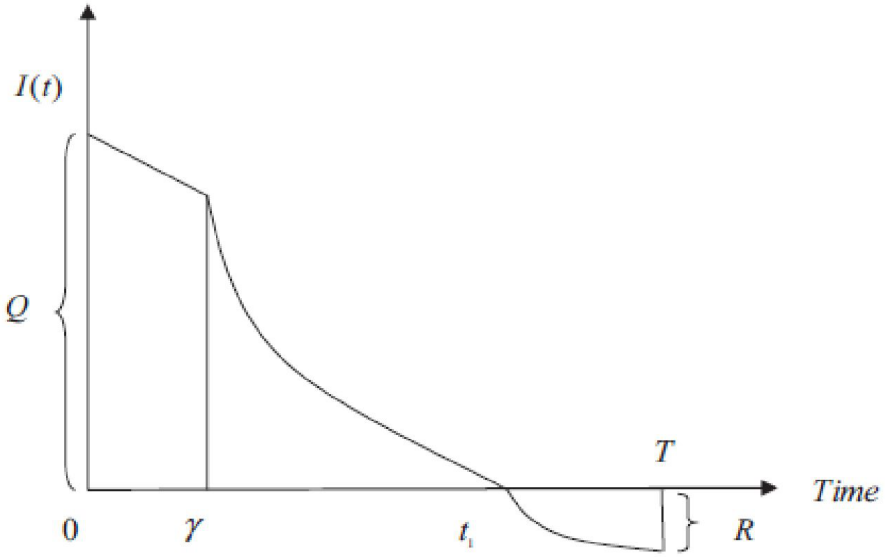

Figure 1: *Pictorial representation of inventory level during the cycle*

$$I = X(l)\gamma + X(l) \int_{\gamma}^{h_1} e^{\alpha(h-\gamma)^\beta e^{-y_1}\zeta}\, dh \tag{18}$$

And

$$J = \frac{X(l)}{\delta} \log(1 + \delta(H - h_1)) \tag{19}$$

Now, the total average profit per cycle is given by

$$B(l,\zeta,h_1,H)=\frac{1}{H}\Bigg[lX(l)h_1+lJ-\Big\{W_lX(l)\gamma+W_lX(l)\int_{\gamma}^{h_1}e^{\alpha(h-\gamma)^{\beta}}e^{-y_1\zeta}\,dh + W_l\frac{X(l)}{\delta}\log(1+\delta(H-h_1))\Big\} - \Big\{W_tI_{\gamma}-\frac{W_tX(l)\gamma^2}{2}+W_tX(l)\int_{\gamma}^{h_1}e^{\alpha(h-\gamma)^{\beta}}e^{-y_1\zeta}\,du\,dh\Big\} - \frac{W_g}{\delta}\Big(X(l)(H-h_1)-J\Big) - W_{\mathrm{pg}}\Big(X(l)(H-h_1)-J\Big)-\zeta H-W_0\Bigg] \tag{20}$$

Shortage cost = $\frac{W_g}{\delta}\left[X(l)\,(H-h1)-J\right]$
Lost sale cost = $C_{\mathrm{pg}}\left[X(l)\,(H-h1)-J\right]$
Therefore, the average profit per cycle is given by
In this case, the corresponding optimization problem is as follows:

$$B(l,\zeta,h_1,H)=\frac{1}{H}\Bigg[lX(l)h_1+lJ-\Big\{W_lX(l)\int_{\gamma}^{h_1}e^{\frac{\alpha(h-\gamma)^{\beta}}{1+\zeta}}\,dh+W_l\frac{X(l)}{\delta}\log\left(1+\delta(H-h_1)\right)\Big\} - \Big\{W_tI_{\gamma}-\frac{W_tX(l)\gamma^2}{2}+W_tX(l)\int_{\gamma}^{h_1}e^{-\frac{\alpha(h-\gamma)^{\beta}}{1+\zeta}}\int_{h}^{h_1}e^{\frac{\alpha(u-\gamma)^{\beta}}{1+\zeta}}\,du\,dh\Big\} - \frac{W_g}{\delta}\left(X(l)(H-h_1)-J\right)-W_{\mathrm{pg}}\left(X(l)(H-h_1)-J\right)-\zeta H-W_0\Bigg] \tag{21}$$

The corresponding optimization problem is as follows:
Maximize $B(l,\zeta,h_1,H)$
Subject to:

$$l>\frac{y}{z},\zeta>0,H>h_1>0$$

Case 2: $Z(H-h)=\frac{1}{1+\delta(H-h)}$ and $n(\zeta)=1-e^{-y_1\zeta}$, $y_1>0$
In this case, the corresponding optimization problem is as follows:
Maximize $B(l,\zeta,h_1,H)$
Subject to:

$$l>\frac{y}{z},\zeta>0,H>h_1>0 \tag{22}$$

Case 3: When $Z\,(H-h)=e^{-\delta(H-h)}$ and $n(\zeta)=\frac{y_1\zeta}{1+y_1\zeta}$, $y_1>0$ When

$$I=X(l)\gamma+X(l)\int_{\gamma}^{h_1}e^{\frac{\alpha(h-\gamma)^{\beta}}{1+\zeta}}\,dh \tag{23}$$

And

$$J=\frac{X(l)}{\delta}\left[1-e^{\delta(H-h_1)}\right] \tag{24}$$

The total purchase cost $(\mathrm{Purcos}t)=W_l(I+J)$

$$=W_lX(l)\gamma+W_lX(l)\int_{\gamma}^{h1}e^{\frac{\alpha(h-\gamma)\beta}{1+\zeta}}\mathrm{dh}+W_l\frac{X(l)}{\delta}\left[1-e^{-\delta(H-h1)}\right] \tag{25}$$

Shortage cost = $\frac{W_g}{\delta}X(l)(H-h1)e^{-\delta(H-h1)} - \frac{W_g}{\delta}J$ Lost sale cost = $W_{\text{pg}}\{X(l)(H-h1)-J\}$ Here, the average profit per cycle is given by

$$B(l,\zeta,h_1,H) = \frac{1}{H}\Bigg[lX(l)h_1 + lJ - \left\{W_lX(l)\gamma + W_lX(l)\int_\gamma^{h_1} e^{\frac{\alpha(h-\gamma)^\beta}{1+\zeta}}dh + W_lJ\right\} - \left\{W_tI_\gamma - \frac{W_tX(l)\gamma^2}{2} + W_tX(l)\int_\gamma^{h_1} e^{\frac{\alpha(h-\gamma)^\beta}{1+\zeta}}\int_h^{h_1} e^{\frac{\alpha(u-\gamma)^\beta}{1+\zeta}}du\,dh\right\} - \frac{W_g}{\delta}X(l)(H-h_1)e^{-\delta(H-h_1)} + \frac{W_g}{\delta}J - W_{pg}X(l)(H-h_1) - \zeta H - W_0\Bigg] \tag{26}$$

The corresponding optimization problem is as follows:
Maximize
$B(l,\zeta,h1.H)$
subject to $l > \frac{y}{z}, \zeta > 0, H > h1 > 0$
Case 4: When $Z(H-h) = e^{-\delta(H-h)}$ and $n(\zeta) = 1 - e^{-y1\iota}, y1 > 0$ In this case,

$$I = X(l)\gamma + X(l)\int_\gamma^{h1} e^{\alpha(h-\gamma)^\beta e^{-y_1\zeta}}\text{dh} \tag{27}$$

And $J = \frac{X(l)}{\delta}\left[1 - e^{-\delta(H-h1)}\right]$
The total purchase cost $(\text{Purcos}t) = W_l(I+J)$

$$= W_lX(l)\gamma + W_lX(l)\int_\gamma^{h1} e^{\frac{\alpha(h-\gamma)\beta}{1+\zeta}}\text{dh} + W_l\frac{X(l)}{\delta}\left[1 - e^{-\delta(H-h1)}\right] \tag{28}$$

Shortage cost = $\frac{W_g}{\delta}X(l)(H-h1)e^{-\delta(H-h1)} - \frac{W_g}{\delta}J$
Lost sale cost = $W_{\text{pg}}\{X(l)(H-h1)-J\}$
Here, the average profit per cycle is given by

$$B(l,\zeta,h_1,H) = \frac{1}{H}\Bigg[IX(l)h_1 + IJ - \left\{W_lX(l)\gamma + W_lX(l)\int_\gamma^{h_1} e^{\frac{\alpha(h-\gamma)^\beta}{1+\zeta}}dh + W_fJ\right\} - \left\{W_tI_\gamma - \frac{W_tX(l)\gamma^2}{2} + W_tX(l)\int_\gamma^{h_1} e^{-(h-\gamma)^\beta}e^{-y_1\zeta}\int_h^{h_1} e^{\alpha(h-\gamma)^\beta}e^{-y_1\zeta}du\,dh\right\} - \frac{W_g}{\delta}X(l)(H-h_1)e^{-\delta(H-h_1)} + \frac{W_\epsilon}{\delta}J - \frac{W_{pg}X(l)(H-h_1)-J}{S} - \zeta H - W_0\Bigg] \tag{29}$$

The corresponding optimization problem is as follows:
Maximize $B(l,\zeta,h_1,H)$ subject to $l > \frac{y}{z}, \quad \zeta > 0, \quad H > h_1 > 0$

4. Solution procedure

In classical mechanics, a particle is depicted by its position and velocity vectors which determine the trajectory of the particle. This means that a particle moves along a determined trajectory [5]. However, this is not true in quantum mechanics. In quantum world, the term trajectory is meaningless, as the position and velocity of a particle cannot be determined simultaneously according to uncertainty principle [6]. Hence, if a particle in AGTO system has quantum behaviour, the AGTO algorithm is bound to work in a different fashion´. Proposed a technique for sparse representation based image steganography by AGTO algorithm. Considering quantum behaviour, first proposed an improved version of AGTO algorithm known as quantum-behaved AGTO (Q AGTO). In this Q AGTO, particles™ state equations were structured by wave function and each particle state was described by the local attracter p and the characteristic length L of d-trap which is determined by the mean optimal position (MP). As MP enhances the cooperation between particles and particles™ waiting with each other, Q AGTO can prevent particles trapping into local minima. However, the speed and accuracy of convergence are also slow.

5. Hyperparameter tuning using Artificial Gorilla Troops Optimizer (AGTO)

Motivated by the collective wisdom of natural phenomena, meta heuristics play a significant role in addressing optimization problems [13]. The revolutionary meta heuristic algorithm known as the artificial gorilla troops optimizer (AGTO) was inspired by the social intelligence of gorilla troops found in the wild. In this study, the social life of gorillas is mathematically defined and new approaches to investigating and profiting from them are developed. A gorilla troop is made up of many adult female gorillas and their offspring as well as an adult male or silverback gorilla bunch [14].

$$\mathrm{ICX}(t+1) = \begin{cases} (\mathrm{UB} - \mathrm{LB}) \times \mathrm{ra}_1 + \mathrm{LB}, & \text{if rand} < p, \\ (\mathrm{ra}_2 - C) \times X_{\mathrm{ra}}(t) + L \times H, & \text{if rand} \geq 0.5, \\ X(i) - L \times (L \times (X(t) - \mathrm{ICX}_{\mathrm{ra}}(t)) + \mathrm{ra}_3 \times (X(t) - \mathrm{ICX}_{\mathrm{ra}}(t))), & \text{if rand} < 0.5. \end{cases} \tag{30}$$

$\mathrm{ICX}(t+1)$ is the upcoming iteration, t indicate the data location. $X(t)$ This position's vector as of right now. $\mathrm{ra}_1, \mathrm{ra}_2, \mathrm{ra}_3$ more than, rand are randomized values updated every cycle, ranging from 0 to 1.is a parameter with a range of 0 to 1 the fact that must provide a value prior to the optimization procedure; UB and LB indicate, accordingly, the parameter's upper and lower boundaries. X_{ra} Out of all the data, is one distance chosen at randomness and $\mathrm{ICX}_{\mathrm{ra}}$. Finally, C, L and H are obtained using the following equations:

$$C = V \times (1 - \frac{\mathrm{It}}{\mathrm{MaxIt}}) \tag{31}$$

Where, MaxIt is the aggregate value corresponding to the optimization operation's iterations.

$$V = \cos(2 \times r_4) + 1 \tag{32}$$

Here, cos denotes the cosine function, r_4 and is changed every iteration with random values between 0 and 1.

$$L = C \times l \tag{33}$$

wherein is an arbitrary number between -1 and 1. To generate the simulated data, apply the equation above.

$$\mathrm{ICX}(t+1) = L \times M \times (X(t) - X_{\mathrm{best}_{\mathrm{distance}}}) + X(t) \tag{34}$$

$X(t)$ is a current vector representing the data's location with respect to $X_{\text{best}_{\text{distance}}}$

$$M = \left(|\frac{1}{N}\sum_{i=1}^{N} \text{ICX}_i(t)|^s \right)^{\frac{1}{s}} \tag{35}$$

Where, $\text{ICX}_i(t)$ represents every place in the loop t. N show the whole amount of information. s further calculated with the equation below,

$$s = 2^L \tag{36}$$

Even if the final solution might not be practical due to distance constraints, it is still possible to make it workable by reordering the factors based on the fitness of the offspring alternatives $\min(\text{ICX}(t))$ is assigned. To minimize the total distance between data points, an adaptive evolutionary algorithm is employed [15]. Assume that the likelihood of crossover and mutation for reducing the distance variable are,

$$\min(\text{ICX}(t)) = \begin{cases} H + \frac{H_{\max} - H_{\min}}{1+\exp\left(\frac{G-G_{\text{avg}}}{G_{\max}-G_{\text{avg}}}\right)}, & \text{if } G \geq G_{\text{avg}}, \\ H_{\max}, & \text{if } G < G_{\text{avg}}. \end{cases} \tag{37}$$

To determine the least amount of distance required to be fit, $H_{\min}$indicate the lowest likelihood of traveling a distance, $H_{\max}$indicates the likelihood of receiving the chosen data, the distance parameter's fitness, G_{avg}shows the average of the chosen data, $G_{\max}$is the data's maximum fitness value.

Do the following steps until the stopping criterion is satisfied:

(a) Calculate the mean best (mbest) position.

(b) Compare each particle™s position with the particle™s pbest position according to their fitness value. Store better one as pbest.

(c) Compare current gbest position with earlier gbest position according to their fitness value. Store better one as gbest. (d) Update the position of each.

(e) Print the position and fitness of global best particle.

(f) End.

Table 1: *Pseudocode for AGTO (Adaptive Gorilla Troop Optimization)*

1. Initialize Parameters: a) Set the population size (gorilla troop) N. b) Set the number of iterations max_iter. c) Define the upper and lower boundaries of the feature space, $Upper\_Bound$, $Lower\_Bound$. d) Set the crossover probability P_c and mutation probability P_m. e) Randomly initialize the positions of gorillas (features) X_i where $i = 1$ to N.
2. Begin Optimization Process: a) For each iteration $t = 1$ to max_iter.
3. Fitness Evaluation: a) For each gorilla position X_i, evaluate its fitness based on the feature selection problem's objective function. b) Identify the best solution X_{best} with the highest fitness.
4. Update Gorilla Positions: a) Update Parameters: - Calculate cosine function parameter $\alpha = \cos\left(\frac{\pi \cdot t}{max\_iter}\right)$. - Generate random numbers r_1, r_2, and r_3 in the range [0,1]. b) Position Update Rule: i) If $r_1 < \alpha$: - Perform an exploitation phase. - Update the gorilla™s position using: $X_i(t+1) = X_i(t) + r_2 \cdot (X_{best} - X_i(t)) + r_3 \cdot Distance(X_i(t), X_{best})$. ii) If $r_1 \geq \alpha$: - Perform an exploration phase. - Randomly select a distance D from the dataset and update using: $X_i(t+1) = X_i(t) + r_2 \cdot (X_{best} - D) + r_3 \cdot Random\_Vector$.
5. Crossover and Mutation: a) Crossover: With probability P_c, perform crossover between two gorilla positions to exploit better solutions: $X_i(t+1) = Crossover(X_i(t), X_j(t))$. b) Mutation: With probability P_m, perform mutation to introduce diversity in the gorilla troop: $X_i(t+1) = Mutation(X_i(t))$.
6. Check Constraints: a) Ensure that each updated position $X_i(t+1)$ remains within the feature boundaries $Upper\_Bound$ and $Lower\_Bound$. b) If violated, reassign the position to the nearest boundary.
7. Update Best Solution: a) If a new position has a better fitness than X_{best}, update X_{best}.
8. End of Iteration: a) Repeat steps 3 to 7 until max_iter is reached.
9. Return Final Solution: a) Return X_{best} as the optimal set of selected features.

For other two algorithms, viz. AQAGTO and WQAGTO, the details are given

Table 2: *Best Found Solution Obtained from GQAGTO*

	Case 1	**Case 2**	**Case 3**	**Case 4**
Z (in $)	303.99	310.15	303.37	309.61
n (in $)	6.05	7.98	6.07	8.09
t1 (in months)	2.427	2.637	2.429	2.639
T (in months)	2.599	2.809	2.587	2.797
R (units)	6.56	6.36	6.09	5.86
Q (units)	103.09	110.39	103.26	110.56
p (in $)	30.89	30.99	30.86	30.96

The table 2 presents the best solutions obtained from the GQAGTO optimization algorithm across four cases, with each case featuring different outcomes for key variables. The objective value, **Z**(in $), represents the main cost or profit metric, with values ranging from $303.37 to $310.15. The secondary metric, **n**(in $), which could indicate resource utilization or additional costs, varies between $6.05 and $8.09. The parameter **t1**(in months), possibly representing the time to achieve a specific milestone, is slightly over 2 months for all cases, with values between 2.427 and 2.639 months. The total time, **T**(in months), ranges from 2.587 to 2.809 months, showing minor differences across cases. The **R**(units) likely indicates a rate or quantity (e.g., production rate or resource output), and it decreases slightly from 6.56 to 5.86 units. Finally, the **Q**(units) represents another quantity, possibly inventory or production units, with values between 103.09 and 110.56 units. The parameter **p**(in $) appears to be a price or cost per unit, showing small variations between $30.86 and $30.99 across the cases.

Table 3: *Best Found Solution Obtained from AQAGTO*

	Case 1	**Case 2**	**Case 3**	**Case 4**
Z (in $)	303.98	310.16	303.38	309.64
n (in $)	6.06	7.99	6.09	8.19
t1 (in months)	2.429	2.639	2.439	2.649
T (in months)	2.699	2.819	2.588	2.797
R (units)	6.57	6.37	6.19	5.87
Q (units)	103.19	110.49	103.27	110.57
p (in $)	30.99	31.99	30.87	30.97

Table 3 displays the optimal solutions found using the AQAGTO algorithm across four cases. The objective value, Z(in $), representing the total cost or profit, ranges from $303.38 to $310.16, showing slight variations across cases. The secondary cost or resource utilization measure, n(in $), fluctuates between $6.06 and $8.19, reflecting small differences in additional expenses or resource use. The parameter t1(in months), likely denoting the time to reach a specific milestone, is consistent across cases, ranging from 2.429 to 2.649 months. The total time, T(in months), shows slight variations, with values between 2.588 and 2.819 months. The R(units) variable, which could represent a production or resource rate, decreases from 6.57 to 5.87 units. The Q(units), possibly indicating inventory or production quantities, remains relatively stable, ranging from 103.19 to 110.57 units. Finally, p(in $), representing a price or cost per unit, varies slightly between $30.87 and $31.99 across the four cases.

Table 4: *Best Found Solution Obtained from WQAGTO*

	Case 1	**Case 2**	**Case 3**	**Case 4**
Z (in $)	304.98	311.16	304.38	309.65
n (in $)	6.07	8.99	6.19	8.29
t1 (in months)	2.439	2.649	2.449	2.659
T (in months)	2.799	2.919	2.688	2.897
R (units)	6.57	6.37	6.19	5.87
Q (units)	103.19	111.49	103.37	110.57
p (in $)	32.99	31.99	30.87	30.97

Table 4 provides the best solutions obtained using the WQAGTO optimization algorithm across four cases. The objective value, **Z**(in $), representing the overall cost or profit, ranges from $304.38 to $311.16, with slightly higher values compared to previous tables. The secondary cost or resource utilization parameter, **n**(in $), varies between $6.07 and $8.99, indicating differences in additional resource expenses across cases. The milestone time, **t1**(in months), shows minimal fluctuation, ranging from 2.439 to 2.659 months, while the total time, **T**(in months), ranges from 2.688 to 2.919 months, slightly longer than in previous tables. The rate, **R**(units), which may represent production or resource output, remains stable, with values between 5.87 and 6.57 units. The quantity, **Q**(units), possibly indicating inventory or production amounts, ranges from 103.19 to 111.49 units, similar to previous tables. Lastly, the **p**(in $) parameter, representing unit price or cost, ranges from $30.87 to $32.99, with Case 1 showing the highest unit price ($32.99), while other cases have values closer to $30.87 to $31.99. This suggests that the unit price may be slightly higher in some cases under this algorithm.

6. Numerical example

To validate the proposed model, a numerical example is considered and solved by different algorithms. The values of different system parameters are given below: W_l= 15:00; W_t =3:00; W_g= 14:00; W_{pg}=16:00; $W_0 = 500{:}0$; $\alpha = 0{:}01$; $\beta = 3{:}5$; $\gamma = 0{:}21$; $\delta = 1{:}48$; $b = 150$; $z = 3{:}5$; $y1 = 0{:}3$ Due to high nonlinearity of the objective function of the optimization problems in different cases, the problem cannot be solved the problem analytically. In this context, we have used soft computing optimization technique (three variants of QAGTO namely AQAGTO, GQAGTO, WQAGTO). We have used three variants of quantum behaved particle swarm optimization technique in order to compare the best found solutions. Clear that GQAGTO gives better result than AQAGTO, WQAGTO algorithms. Also, the average profit of the system of Case 2 is higher than other cases.

7. Conclusion and remarks

In this study, we developed an inventory model for deteriorating items, taking into account preservation technology and price-dependent demand. The model incorporates two different preservation rates and allows partially backlogged shortages with varying backlogging rates. Due to the inclusion of three-parameter Weibull-distributed deterioration, the optimization problem was nonlinear and complex, which we addressed by utilizing different variants of the Quantum-behaved Artificial Gorilla Troops Optimizer (AGTO). Numerical examples were provided for each case, and the results were compared across the GAGTO, WAGTO, and AAGTO algorithms. Sensitivity analysis was performed to graphically demonstrate the effects of various parameter changes on the optimal policy.It was observed that the GAGTO algorithm outperforms both WAGTO and AAGTO in terms of solution quality and computational efficiency. Moreover, the analysis revealed that faster sales lead to reduced preservation costs, subsequently increasing profits. These findings highlight the importance of efficient preservation and pricing strategies

in managing deteriorating inventory. Future research could focus on extending the model to multi-item scenarios and exploring other advanced optimization techniques.

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APPLICATION OF FUZZY LOGIC IN AGRICULTURAL NETWORK ANALYSIS FOR OPTIMIZING CROP PRODUCTION

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Abstract

This study investigates the application of fuzzy logic and fuzzy set theory in agricultural networking to identify the optimal paths for different crop production activities. Traditionally networking methods often face challenges with incomplete and uncertain data, which are prevalent in agriculture. Fuzzy logic using decagonal fuzzy number offers a more versatile method of handing imprecise data. In this study decagonal fuzzy numbers are defuzzified by rolling averages with a window of three to determine the optimal path. The solution of the formulated mathematical programming model is obtained using R software which enabling accurate computation of the best routes in agricultural networks and three different examples were taken and the network diagram is also shown. This paper further shows the scope of agriculture especially network path analysis in agriculture which can enhance decision making, which in turn can rise crop yields and improve agriculture productivity.

Keywords: Fuzzy sets, Fuzzy logics, Network path analysis, Optimization

1. Introduction

Fuzzy logic and fuzzy set theory have gained significant attention in fields of scientific investigations, particularly in agriculture. It is due to their ability to manage uncertainty and deal with incomplete data sets. These techniques have been explored to various aspects of agriculture like decision making processes, precision farming, wireless sensor networks for precision agriculture and project management. The goal is to increase crop yields and overall agricultural productivity, fizzy expert system has been used in various agriculture tasks which includes seed selection, scheduling, transportation, pesticide management and soil preparations. This shows the adaptability of fuzzy logic and potential in solving agriculture problems which are challenging in nature. Lin [3] introduces a technique to solve shortest path problems in fuzzy environment using triangular fuzzy numbers and signed distance ranking. Nagoorgani and Begam [6] propose an alternative approach to determine the optimal shortest path using graded mean integration representation and triangular fuzzy numbers. An algorithm for fuzzy critical path analysis in project networks is developed by Liana and Han [4] using trapezoidal fuzzy numbers to enhance decision-

making in project management. Kung et al. [2] proposed a method for discrete fuzzy networks.

A dynamic programming approach for obtaining shortest paths in fuzzy networks presented in Mahdavi et al. [5]. De and Bhincher [1] used triangular and trapezoidal fuzzy numbers to compare linear programming approach and dynamic programming for fuzzy shortest path problems. Pal et.al [7] apply an Ai based approach to solve intuitionistic fuzzy assignment problems in agriculture through dynamic programming approach. Bose & Tarafdar [8] used fuzzy logic to optimize supply chain management in agriculture with crop transportation models. Singh & Joshi [9] emphasizes fuzzy models to predict crop yield influenced by different factors for improving calibration. Chen, Zhang, & Wang [10] optimizes irrigation schedules based on weather data and to contingent crop planning with weather parameters. Zadeh [11] outlines fuzzy logic principles widely applicable in agriculture decision making particularly network path analysis under uncertainty.

Sivakumar [12] enhance agriculture metrical application in essential for calibrating crop production with whether data. used for basic system providing insights into how to system can manage in preside agriculture. Mendel [13] provides insights how fuzzy system can manage imprecise agriculture data for optimizing crop production. Sharma & Dhillon [14] demonstrates the use of fuzzy logic for predicting crop diseases based on weather patterns. Qin & Liu [15] used fuzzy decision -making methods applied for crop planning for agriculture optimization problems. Pandey [16] use of fuzzy logic for improving agriculture yield by analyzing network paths. Rahman & Sarker, [17] Used fuzzy decision systems to optimize water usage in irrigation, under uncertain weather conditions. Solimun *et al*. [18] demonstrates that certain combinations to provide more accurate path models, for interpreting relationships between categorical variables. Sanjana and Ramesh [19] presents an innovative approach to handle complex assignment problems with imprecise data using the Hungarian method. Behera *et al*. [20] focused on new computational methods to discourse linear programming problems under triangular fuzzy uncertainty.

2. Basic Preliminary

Fuzzy set: Let A be a classical set, $\mu_{\bar{A}}(x)$ be a function from A to $[0,1]$. A fuzzy set $\bar{A}$ with the membership function $\mu_{\bar{A}}(x)$ is defined by $\bar{A} = \{(x, \mu_{\bar{A}}(x)); x \in A, \mu_{\bar{A}}(x) \in [0,1]\}$.

Membership Function $\mu_{\bar{A}}(x)$: Describes the degree of belonging of x to the fuzzy set $\bar{A}$.

Non- Membership Function $\nu_{\bar{A}}(x)$: Describes the degree to which x does not belonging to the fuzzy set $\bar{A}$. That is $\nu_{\bar{A}}(x) = 1 - \mu_{\bar{A}}(x)$.

Fuzzy Number: is a specific type of fuzzy set defined on real numbers having membership function that assigns a value between 0 and 1 for each real number. This value indicates the degree of membership of that number which belongs to the fuzzy set. Fuzzy numbers help to represent uncertain or imprecise values, typically having a peak when membership function having value 1 and gradually decreases to 0 on either side.

Triangular Fuzzy Number: A triangular fuzzy number having triangular shaped membership function with three parameters (a_1, a_2, a_3) where $a_1 \le a_2 \le a_3$. The function rises linearly from a_1 and a_2, and reaches its peak at a_2 and then falls linearly to a_3. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x<a_1\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{(a_3-x)}{a_3-a_2}, & a_2\le x\le a_3\end{cases} \quad (1)$$

Here a_1 and a_3 are the points where membership function starts and ends at 0, while a_2 is the peak where membership value is 1.

Trapezoidal Fuzzy Number: A trapezoidal fuzzy number having trapezoid shaped membership function with four parameters (a_1, a_2, a_3, a_4) where $a_1 \le a_2 \le a_3 \le a_4$. The function rises linearly from a_1 and a_2, and reaches its peak at a_2 and then falls linearly to a_3. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x<a_1 \text{ or } x>a_4\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ 1, & a_2\le x\le a_3\\ \dfrac{(a_4-x)}{a_4-a_3}, & a_3\le x\le a_4\end{cases} \quad (2)$$

Here a_1 and a_4 are the points where membership function starts and ends at 0, while a_2 and a_3 is the flat top of the trapezoid where membership value is 1.

Pentagonal Fuzzy Number: A pentagonal fuzzy number having five parameters $(a_1, a_2, a_3, a_4, a_5)$ where $a_1 \le a_2 \le a_3 \le a_4 \le a_5$. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x<a_1 \text{ or } x>a_5\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{(x-a_2)}{a_3-a_2}, & a_2\le x\le a_3\\ \dfrac{(a_4-x)}{a_4-a_3}, & a_3\le x\le a_4\\ \dfrac{(a_5-x)}{a_5-a_4}, & a_4\le x\le a_5\end{cases} \quad (3)$$

Hexagonal Fuzzy Number: A Hexagonal fuzzy number having six parameters $(a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6$. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x<a_1 \text{ or } x>a_6\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{(x-a_2)}{a_3-a_2}, & a_2\le x\le a_3\\ 1, & a_3\le x\le a_4\\ \dfrac{(a_5-x)}{a_5-a_4}, & a_4\le x\le a_5\\ \dfrac{(a_6-x)}{a_6-a_5}, & a_5\le x\le a_6\end{cases} \tag{4}$$

Heptagonal Fuzzy Number: A heptagonal fuzzy number having seven parameters $(a_1,a_2,a_3,a_4,a_5,a_6,a_7)$ where $a_1\le a_2\le a_3\le a_4\le a_5\le a_6\le a_7$. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x<a_1 \text{ or } x>a_7\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{(x-a_2)}{a_3-a_2}, & a_2\le x\le a_3\\ \dfrac{(x-a_2)}{a_4-a_3}, & a_3\le x\le a_4\\ 1, & a_4\le x\le a_5\\ \dfrac{(a_6-x)}{a_6-a_5}, & a_5\le x\le a_6\\ \dfrac{(a_7-x)}{a_7-a_6}, & a_6\le x\le a_7\end{cases} \tag{5}$$

Octagonal Fuzzy Number: A Octagonal fuzzy number having seven parameters $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8)$ where $a_1\le a_2\le a_3\le a_4\le a_5\le a_6\le a_7\le a_8$. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x<a_1 \text{ or } x>a_8\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{(x-a_2)}{a_3-a_2}, & a_2\le x\le a_3\\ \dfrac{(x-a_2)}{a_4-a_3}, & a_3\le x\le a_4\\ 1, & a_4\le x\le a_5\\ \dfrac{(a_6-x)}{a_6-a_5}, & a_5\le x\le a_6\\ \dfrac{(a_7-x)}{a_7-a_6}, & a_6\le x\le a_7\\ \dfrac{(a_7-x)}{a_8-a_7}, & a_7\le x\le a_8\end{cases} \tag{6}$$

Nonagonal Fuzzy Number: A Nonagonal fuzzy number having seven parameters $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9)$ where $a_1\le a_2\le a_3\le a_4\le a_5\le a_6\le a_7\le a_8\le a_9$. The membership function denoted by $\mu_{\bar{A}}(x)$ is given :

$$\mu_{\bar{A}}(x)=\begin{cases}0, & x\le a_1\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1<x\le a_2\\ \dfrac{(x-a_2)}{a_3-a_2} & a_2<x\le a_3\\ \dfrac{(x-a_2)}{a_4-a_3}, & a_3<x\le a_4\\ \dfrac{(x-a_4)}{a_5-a_4}, & a_4<x\le a_5\\ \dfrac{(x-a_5)}{a_6-a_5}, & a_5<x\le a_6\\ \dfrac{(x-a_6)}{a_7-a_6}, & a_6<x\le a_7\\ \dfrac{(x-a_7)}{a_8-a_7}, & a_7<x\le a_8\\ \dfrac{(x-a_8)}{a_9-a_8}, & a_8<x\le a_9\\ 0, & x\le a_9\end{cases} \tag{7}$$

Decagonal Fuzzy Number: A Decagonal fuzzy number having seven parameters $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})$ where
$a_1\le a_2\le a_3\le a_4\le a_5\le a_6\le a_7\le a_8\le a_9\le a_{10}$. The membership function denoted by

$\mu_{\overline{A}}(x)$ is given :

$$\mu_{\overline{A}}(x)=\begin{cases}0, & x<a_1 \text{ or } x>a_{10}\\ \dfrac{(x-a_1)}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{(x-a_2)}{a_3-a_2} & a_2\le x\le a_3\\ \dfrac{(x-a_3)}{a_4-a_3}, & a_3\le x\le a_4\\ \dfrac{(x-a_4)}{a_5-a_4}, & a_4\le x\le a_5\\ 1, & a_5\le x\le a_6\\ \dfrac{(a_7-x)}{a_7-a_6}, & a_6\le x\le a_7\\ \dfrac{(a_8-x)}{a_8-a_7}, & a_7\le x\le a_8\\ \dfrac{(a_9-x)}{a_9-a_8}, & a_8\le x\le a_9\\ \dfrac{(a_{10}-x)}{a_{10}-a_9}, & a_9\le x\le a_{10}\end{cases} \tag{8}$$

Defuzzified Decagonal Fuzzy Number: The rolling average is used to Defuzzifying Decagonal fuzzy number $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})$ with three as window width taken.

$$\begin{aligned}RAvg(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})=&\left[Average\left(\frac{a_1+a_2+a_3}{3}\right)=A_1\right]+\left[Average\left(\frac{a_2+a_3+a_4}{3}\right)=A_2\right]\\&+\left[Average\left(\frac{a_3+a_4+a_5}{3}\right)=A_3\right]+\left[Average\left(\frac{a_4+a_5+a_6}{3}\right)=A_4\right]\\&+\left[Average\left(\frac{a_5+a_6+a_7}{3}\right)=A_5\right]+\left[Average\left(\frac{a_6+a_7+a_8}{3}\right)=A_6\right]\\&+\left[Average\left(\frac{a_7+a_8+a_9}{3}\right)=A_7\right]+\left[Average\left(\frac{a_8+a_9+a_{10}}{3}\right)=A_8\right]\end{aligned}$$

$$RAvg(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})=\frac{A_1+A_2+A_3+A_4+A_5+A_6+A_7+A_8+A_9}{9}$$

3. Formulation of Network Path Problem

1. Formulate the Problem: To minimize the transportation duration from agriculture.
2. Set up the Network: Develop network diagram with nodes representing farms and markets and transportation path by edges.
3. Define Constraints: setting of constraints like transportation vehicles (maximum Capacity), Market demand and farm production capacity.
4. Develop an Optimization Model: Formulate a Mathematical Model.

5. Implement the Model in Software: R software is used to solve the optimization model.

4. Numerical illustration

Example1: Let us consider an agriculture data set, where each path represents time/duration required for various agriculture processes between different stages of crop production like soil preparation, planting, irrigation etc. used as fuzzy durations.

Table 1: *Duration required for various agriculture processes*

Nodes (i-j)	Fuzzy Duration(hrs)	Defuzzified Duration(hrs)
S1-S2	0.77, 0.79,0.81,0.82,0.84,0.86,0.88,0.9,0.92,0.94	0.85
S1-S3	0.19, 0.2,0.2,0.2,0.21,0.21,0.21,0.22,0.22,0.23	0.21
S2-S4	2.92, 2.95,2.98,3.01,3.04,3.06,3.07,3.12,3.15,3.18	3.05
S2-S5	0.79, 0.8,0.81,0.82,0.83,0.84,0.85,0.86,0.87,0.88	0.84
S3-S5	0.59, 0.6,0.6,0.61,0.61,0.62,0.63,0.63,0.64,0.65	0.62
S3-S6	0.96, 0.98,0.99,1,1.02,1.03,1.04,1.06,1.07,1.08	1.02
S4-S7	0.17,0.17,0.17,0.17,0.17,0.17,0.17,0.18,0.18,0.18	0.17
S5-S7	1.04,1.06,1.09,1.11,1.14,1.16,1.18,1.21,1.23,1.25	1.15
S5-S8	3.22,3.25,3.28,3.31,3.34,3.37,3.39,3.42,3.45,3.48	3.35
S6-S8	2.24, 2.32,2.4,2.48,2.56,2.64,2.72,2.8,2.88,2.96	2.60

Where S1: Initial Soil Preparation, S2: Fertilization, S3: Seed Planting, S4: First Irrigation, S5: Weed Control, S6: Pest Management, S7: Final Irrigation, S8: Harvesting
The network Diagram is shown below

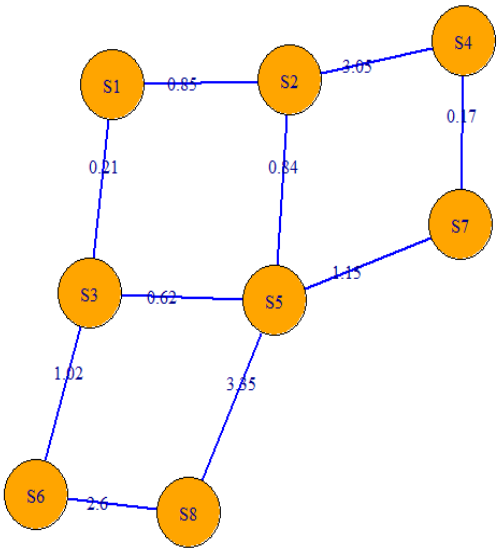

Figure 1: Network diagram

Example2: Here is another example for an agriculture network path problem which represents different agriculture processes between various stages of crop production through fuzzy duration.

Table 2: *Duration required for various agriculture processes*

Nodes (i-j)	Fuzzy Duration(hrs)	Defuzzified Duration(hrs)
P1-P2	0.902,0.922,0.975,0.985,1.005,1.025,1.045,1.065,1.085,1.105	1.03
P1-P3	0.289,0.332,0.365,0.365,0.375,0.375,0.375,0.385,0.385,0.395	0.37
P2-P4	3.019,3.082,3.145,3.175,3.205,3.225,3.235,3.285,3.315,3.345	3.23
P2-P5	0.889,0.932,0.975,0.985,0.995,1.005,1.015,1.025,1.035,1.045	1.00
P3-P5	0.689,0.732,0.765,0.775,0.775,0.785,0.795,0.795,0.805,0.815	0.78
P3-P6	1.059,1.112,1.155,1.165,1.185,1.195,1.205,1.225,1.235,1.245	1.19
P4-P7	0.269,0.302,0.335,0.335,0.335,0.335,0.335,0.345,0.345,0.345	0.33
P5-P7	1.139,1.192,1.255,1.275,1.305,1.325,1.345,1.375,1.395,1.415	1.33
P5-P8	3.319,3.382,3.445,3.475,3.505,3.535,3.555,3.585,3.615,3.645	3.54
P6-P8	2.339,2.452,2.565,2.645,2.725,2.805,2.885,2.965,3.045,3.125	2.83

Interpretation of Nodes: P1: Land Clearing, P2: Soil Testing and Preparation, P3: Seed Selection, P4: Planting and Seeding, P5: Nutrient Management, P6: Pest and Disease Contro, P7: Water Management P8: Harvest and Post-Harvest Processing.

The network Diagram is shown below:

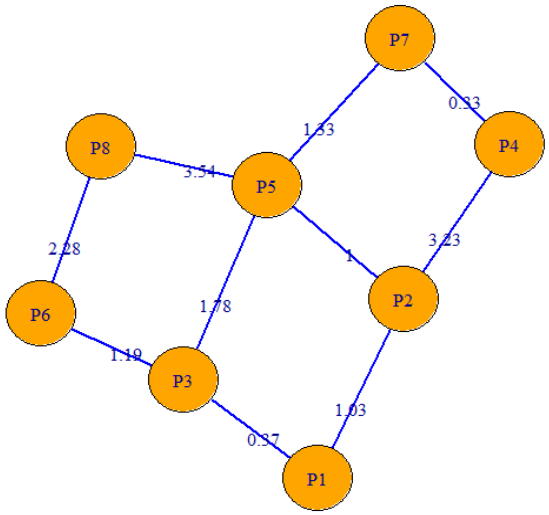

Figure 2: Network diagram

Example 3: Let's consider another network path problem in agriculture. In this data set the aim to is determining the optimal path of transportation for supplying items from agriculture farms to the specified markets.

Table 3: *Duration required transportation for supplying items from agriculture farms to the specified markets.*

Nodes (i-j)	Fuzzy Duration (hrs)	Defuzzified Duration (hrs)
F1-M(A)	(1.5, 1.6, 1.7, 1.8, 1.9, 2, 2.1, 2.4, 2.6, 2.7)	2.01
F1-M(B)	(2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3, 3.2, 3.5)	2.77
F1-M(C)	(3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 4, 4.1)	3.65
F2-M(A)	(1.7, 1.8, 1.9, 2, 2.1, 2.2, 2.3, 2.5, 2.7 ,3)	2.20
F2-M(B)	(1.4, 1.5, 1.6, 1.7,1.8,1.9,2,2.3,2.5, 2.8)	1.92
F2- M(C)	(2.7, 2.8, 2.9, 3, 3.1, 3.2, 3.3, 3.3, 3.4, 3.7)	3.13
F 3- M(A)	(2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3, 3.2)	2.65
F3- M(B)	(2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 3)	2.45

F3- M(C)	(1.3, 1.4, 1.5,1.6,1.7,1.8, 2, 2.3, 2.4, 2.7)	1.85

Here Farms denoted as F1, F2 and F3 and Markets as M(A), M(B), M(C).
The network Diagram is shown below:

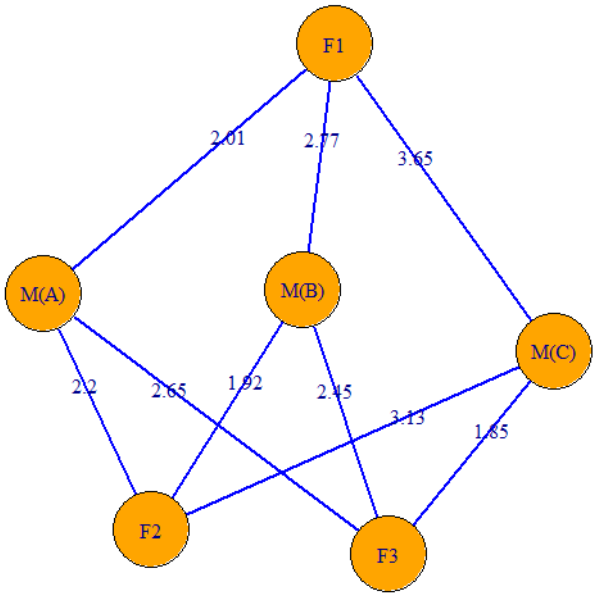

Figure 3: Network diagram

5. Conclusion

This study shows applications of different agriculture process through fuzzy set theory and fuzzy logic. In this study a reliable technique for agricultural logistics optimization of decagonal fuzzy number transforms into crisp values by defuzzifying through rolling regression. In this study if we change the width of the window based rolling average can also slightly change. This is a useful tool for decision-making in a variety of industries and it can be expanded to other domains where uncertainties in path durations must be taken into consideration.

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A GENERALIZED POWER SUJATHA DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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Abstract

This paper introduces a generalized power Sujatha distribution as an extension of the two-parameter generalization of Sujatha distribution, initially proposed for analyzing and modeling lifetime data in medical and engineering fields. The existing generalization of Sujatha distribution, being two-parameter, may not always provide a satisfactory fit for certain lifetime data from both theoretical and practical perspectives. The generalized power Sujatha distribution is presented as a comprehensive model, encompassing both the Generalization of Sujatha distribution and the Sujatha distribution as particular cases, specifically for the analysis of data in medical and engineering domains. The paper delves into the statistical properties of the proposed distribution, examining the behavior of its probability density function and cumulative distribution function across varying parameter values. Additionally, the first four raw moments of the distribution are derived and provided. The expressions for the hazard rate function and mean residual life function are obtained, and their behaviors under different parameter values are discussed. Stochastic ordering, a valuable tool for comparing stochastic nature, is also explored. The method of maximum likelihood is discussed for parameter estimation, and a simulation study is conducted to assess the performance of maximum likelihood estimates as sample sizes increase. To validate the applicability of the distribution, two real lifetime data sets from medical and engineering fields are analyzed. The goodness of fit of the generalized power Sujatha distribution is evaluated using the Akaike Information criterion and Kolmogorov-Smirnov statistic. The results demonstrate that the proposed distribution offers a closer fit compared to three-parameter power Quasi Lindley distribution, Three-parameter Sujatha distribution, Generalized gamma distribution, and two-parameter Generalizations of Sujatha distribution, as well as Weibull distribution and one-parameter Sujatha distribution. Given its superior fit over Power Quasi Lindley and Weibull distributions, particularly in the context of modeling and analyzing data from medical and engineering fields, the paper concludes by recommending the generalized power Sujatha distribution as the preferred choice over the considered distributions for such applications.

Keywords: generalization of Sujatha distribution, statistical properties, maximum likelihood estimation, mean residual life function, application.

I. Introduction

Lindly distribution introduced by [1], a one-parameter lifetime distribution obtained through the convex combination of the exponential distribution and gamma distribution. Subsequently, its statistical properties and goodness of fit conducted and examined by [2]. Their findings revealed that the Lindley distribution outperforms the exponential distribution in terms of fit. Another one-parameter lifetime distribution, the Sujatha distribution (SD), was proposed by [3] using a similar convex combination approach. SD is a convex combination of exponential (η) distribution, gamma $(2,\eta)$ distribution and gamma $(3,\eta)$ distribution, which provides better fit on some dataset as compared to exponential and Lindley distribution. The probability density function (pdf) and the cumulative distribution function (cdf) of SD are given by

$$f(x;\eta)=\frac{\eta^3}{\eta^2+\eta+2}\left(1+x+x^2\right)e^{-\eta x};\ x>0,\eta>0 \tag{1}$$

$$F(x;\eta)=1-\left[1+\frac{\eta x(\eta x+\eta+2)}{\eta^2+\eta+2}\right]e^{-\eta x};\ x>0,\eta>0 \tag{2}$$

and comprehensively explored the statistical and mathematical properties of the Sujatha distribution (SD). This included a detailed discussion on moments and their associated measures, hazard function, mean residual life function, Bonferroni and Lorenz curves, stochastic ordering, mean deviation about the mean and the median, as well as stress-strength reliability and delved into the estimation of parameters using the maximum likelihood method and provided insights into the practical applications of SD in modelling lifetime data. Despite the improved fit of the Sujatha distribution (SD) compared to the exponential and Lindley distributions, limitations arise in situations where these one-parameter distributions fail to provide a good fit. Recognizing this, [4] addressed the issue by introducing a generalized version, termed the Generalization of Sujatha distribution (GSD). This extension involves the incorporation of an additional parameter into the probability density function (pdf) of the SD. The introduction of this extra parameter enhances the flexibility of the GSD, making it better suited to accommodate a broader range of data patterns as compared to its one-parameter counterpart, the SD. The GSD is characterized by its pdf and cdf

$$f(x;\eta,\omega)=\frac{\eta^3}{\eta^2+\eta+2\omega}\left(1+x+\omega x^2\right)e^{-\eta x};x>0,\eta>0,\omega>0 \tag{3}$$

$$F(x;\eta,\omega)=1-\left[1+\frac{\eta x\left(\omega\eta x+\eta+2\omega\right)}{\eta^2+\eta+2\omega}\right]e^{-\eta x};\ x>0,\eta>0\ ,\omega>0 \tag{4}$$

In their study, [4] applied the GSD to four lifetime datasets and demonstrated that the GSD provides a superior fit compared to the SD, Lindley distribution, Aradhana distribution, and exponential distribution. It is important to highlight that the GSD reverts to the SD under specific conditions $\omega=1$. Additionally, [5] introduced the power Sujatha distribution (PSD) by employing power transformation on the SD. The PSD is defined by its pdf and cdf.

$$f(x;\eta,\tau)=\frac{\tau\eta^3}{\eta^2+\eta+2}x^{\tau-1}\left(1+x^{\tau}+x^{2\tau}\right)e^{-\eta x^{\tau}};x>0,\eta>0,\tau>0 \tag{5}$$

$$F(x;\eta,\tau)=1-\left[1+\frac{\eta x^{\tau}\left(\eta x^{\tau}+\eta+2\right)}{\eta^{2}+\eta+2}\right]e^{-\eta x^{\tau}};x>0,\eta>0,\tau>0 \tag{6}$$

Using power transformation in quasi Lindley distribution (QLD) discussed by [6], a Power quasi Lindley distribution (PQLD) has been proposed by [7] and defined by its pdf and cdf as

$$f\left(x;\eta,\omega,\tau\right)=\frac{\tau\eta}{\omega+1}x^{\tau-1}\left(\omega+\eta x^{\tau}\right)e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{7}$$

$$F(x;\eta,\omega,\tau)=1-\left[1+\frac{\eta x^{\tau}}{\omega+1}\right]e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{8}$$

It can be readily confirmed that the power Lindley distribution (PLD), as presented by [8], and the Lindley distribution proposed by [1] are specific cases of the PQLD for $\omega=\eta$ and $\omega=\eta,\tau=1$.

A generalized gamma distribution (GGD) proposed by [9] defined by its pdf and cdf as

$$f\left(x;\eta,\omega,\tau\right)=\frac{\tau\eta^{\omega}}{\Gamma(\omega)}x^{\tau\omega-1}e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{9}$$

$$F(x;\eta,\omega,\tau)=1-\frac{\Gamma\left(\omega,\eta x^{\tau}\right)}{\Gamma\left(\omega\right)};x>0,\eta>0,\omega>0,\tau>0 \tag{10}$$

Recently, a three-parameter Sujatha distribution (ThPSD) proposed by [10], and later its various statistical properties and applications has been studied by [11]. Its pdf and cdf define as

$$f\left(x;\eta,\omega,\tau\right)=\frac{\eta^{2}}{2\left(\eta^{2}+\tau+\omega\right)}\left(2\eta+2\tau x+\eta\omega x^{2}\right)e^{-\eta x};x>0,\eta>0,\omega>0,\tau>0 \tag{11}$$

$$F\left(x;\eta,\omega,\tau\right)=1-\left[1+\frac{\eta^{2}\omega x^{2}+2\eta\left(\tau+\omega\right)x}{2\left(\eta^{2}+\tau+\omega\right)}\right]e^{-\eta x};x>0,\eta>0,\omega>0,\tau>0 \tag{12}$$

The main motivation for proposing a Generalized Power Sujatha distribution is due to the fact that PQLD provides much closer fit than PLD, and the PSD provides much better fit than the PLD, it is expected and hoped that GPSD would provide better fit than PQLD. We discussed various important statistical properties of GPSD, estimation of parameters using maximum likelihood methods and applications to two lifetime data.

II. Generalized Power Sujatha distribution

Considering the power transformation $X = Y^{\frac{1}{\tau}}$ in the pdf (3), the pdf of a Generalized Power Sujatha distribution (GPSD) can be obtained as

$$f(x;\eta,\omega,\tau) = \frac{\tau\eta^3}{\eta^2+\eta+2\omega}\left(1+x^{\tau}+\omega x^{2\tau}\right)x^{\tau-1}e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{13}$$

$$= p_1 f_1(x;\eta,\tau) + p_2 f_2(x;\eta,\tau) + (1-p_1-p_2) f_3(x;\eta,\tau)$$

where $p_1 = \frac{\eta^2}{\eta^2+\eta+2\omega}$, $p_2 = \frac{\eta}{\eta^2+\eta+2\omega}$

$$f_1(x;\eta,\tau) = \tau\eta x^{\tau-1}e^{-\eta x^{\tau}},\ f_2(x;\eta,\tau) = \frac{\tau\eta^2}{\Gamma(2)}x^{2\tau-1}e^{-\eta x^{\tau}},\ f_3(x;\eta,\tau) = \frac{\tau\eta^3}{\Gamma(3)}x^{3\tau-1}e^{-\eta x^{\tau}}$$

This means that GPSD is a convex combination of Weibull (η,τ), Generalized Gamma $(2,\eta,\tau)$ and Generalized Gamma $(3,\eta,\tau)$ distributions. The corresponding cdf of GPSD can be obtained as

$$F(x;\eta,\omega,\tau) = 1-\left[1+\frac{\eta x^{\tau}\left(\omega\eta x^{\tau}+\eta+2\omega\right)}{\eta^2+\eta+2\omega}\right]e^{-\eta x^{\tau}};x>0,\omega>0,\eta>0,\tau>0 \tag{14}$$

The behaviour of the pdf and the cdf of GPSD are presented in figures 1 and 2 respectively.

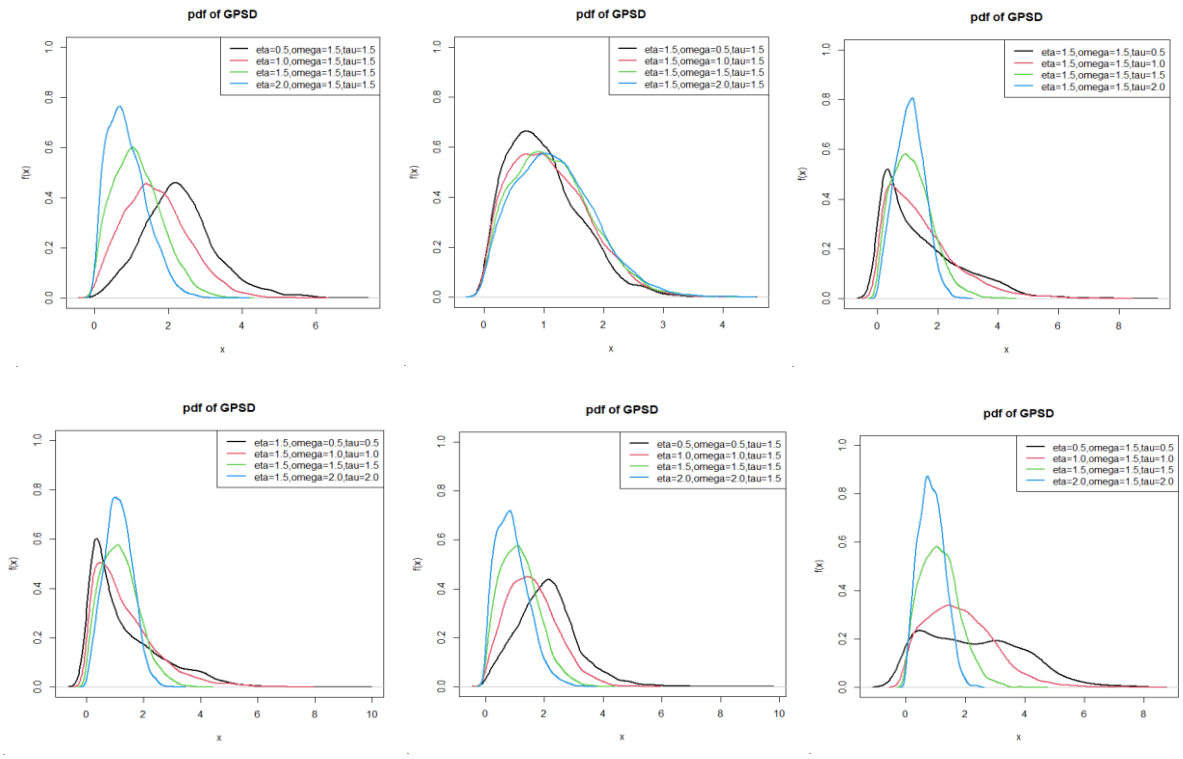

Figure 1: *pdf of GPSD for different values of the parameters*

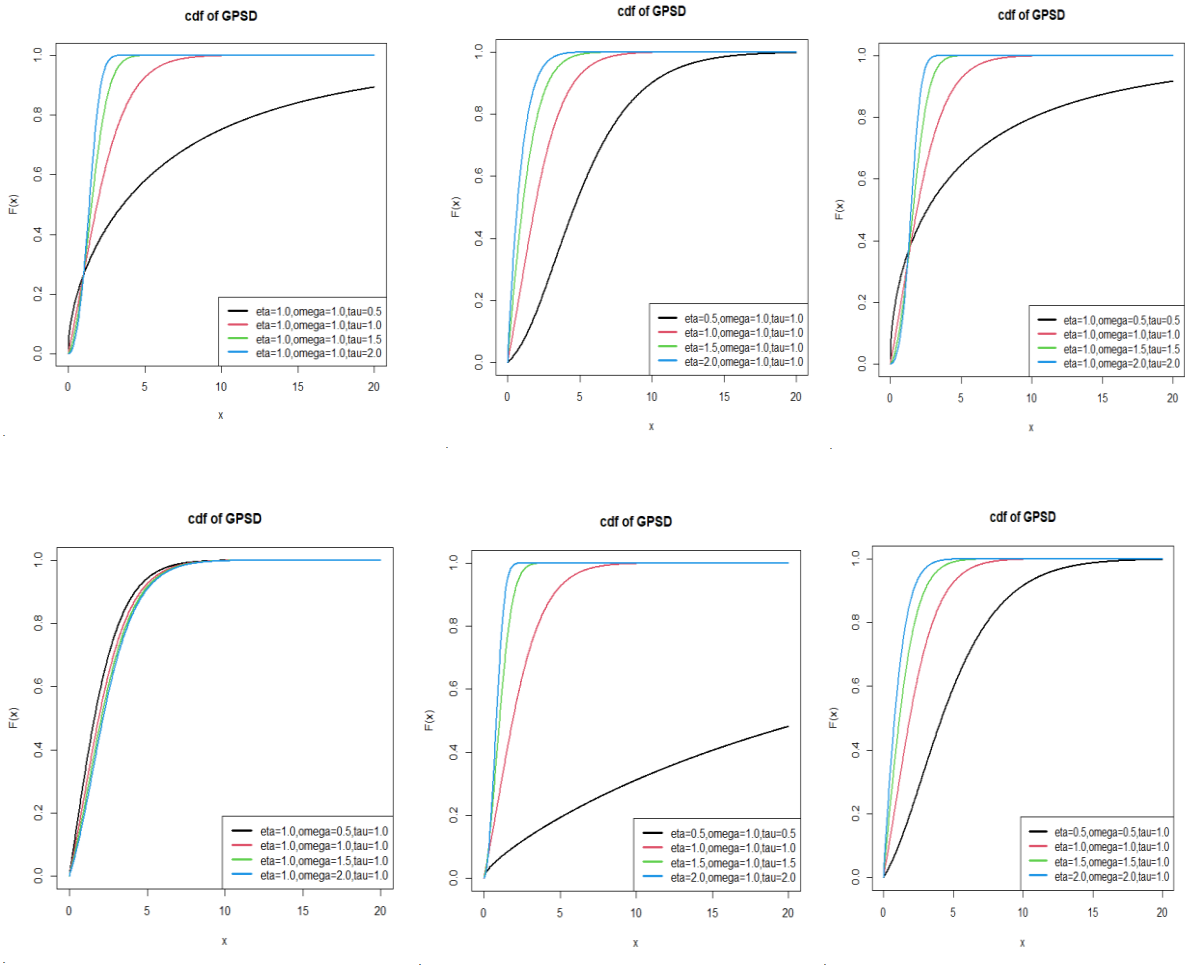

Figure 2: *cdf of GPSD for different values of the parameters*

III. Statistical Properties of GPSD

I. Survival Function

The Survival function of GPSD can be expressed as

$$S(x;\eta,\omega,\tau)=\left[\frac{\left(\eta^2+\eta+2\omega\right)+\eta x^{\tau}\left(\omega\eta x^{\tau}+\eta+2\omega\right)}{\eta^2+\eta+2\omega}\right]e^{-\eta x^{\tau}};x>0,\omega>0,\eta>0,\tau>0 \tag{15}$$

II. Reverse Hazard Function

The reverse hazard function of GPSD can be obtained as

$$r(x;\eta,\omega,\tau)=\frac{\tau\eta^3\left(1+x^{\tau}+\omega x^{2\tau}\right)x^{\tau-1}e^{-\eta x^{\tau}}}{\left(\eta^2+\eta+2\omega\right)-\left[\left(\eta^2+\eta+2\omega\right)+\eta x^{\tau}\left(\omega\eta x^{\tau}+\eta+2\omega\right)\right]e^{-\eta x^{\tau}}};x>0,\omega>0,\eta>0,\tau>0 \tag{16}$$

III. Hazard Function

The hazard function of GPSD can be obtained as

$$h(x;\eta,\omega,\tau)=\frac{\tau\eta^3\left(1+x^\tau+\omega x^{2\tau}\right)x^{\tau-1}}{\left(\eta^2+\eta+2\omega\right)+\eta x^\tau\left(\omega\eta x^\tau+\eta+2\omega\right)} \tag{17}$$

The behaviours of the hazard function of GPSD are explained in the following figure 3.

IV. Cumulative Hazard Function

The cumulative hazard function of GPSD can be obtained as

$$H(x;\eta,\omega,\tau)=-\log\left[\frac{\left\{\left(\eta^2+\eta+2\omega\right)+\eta x^\tau\left(\omega\eta x^\tau+\eta+2\omega\right)\right\}e^{-\eta x^\tau}}{\left(\eta^2+\eta+2\omega\right)}\right] \tag{18}$$

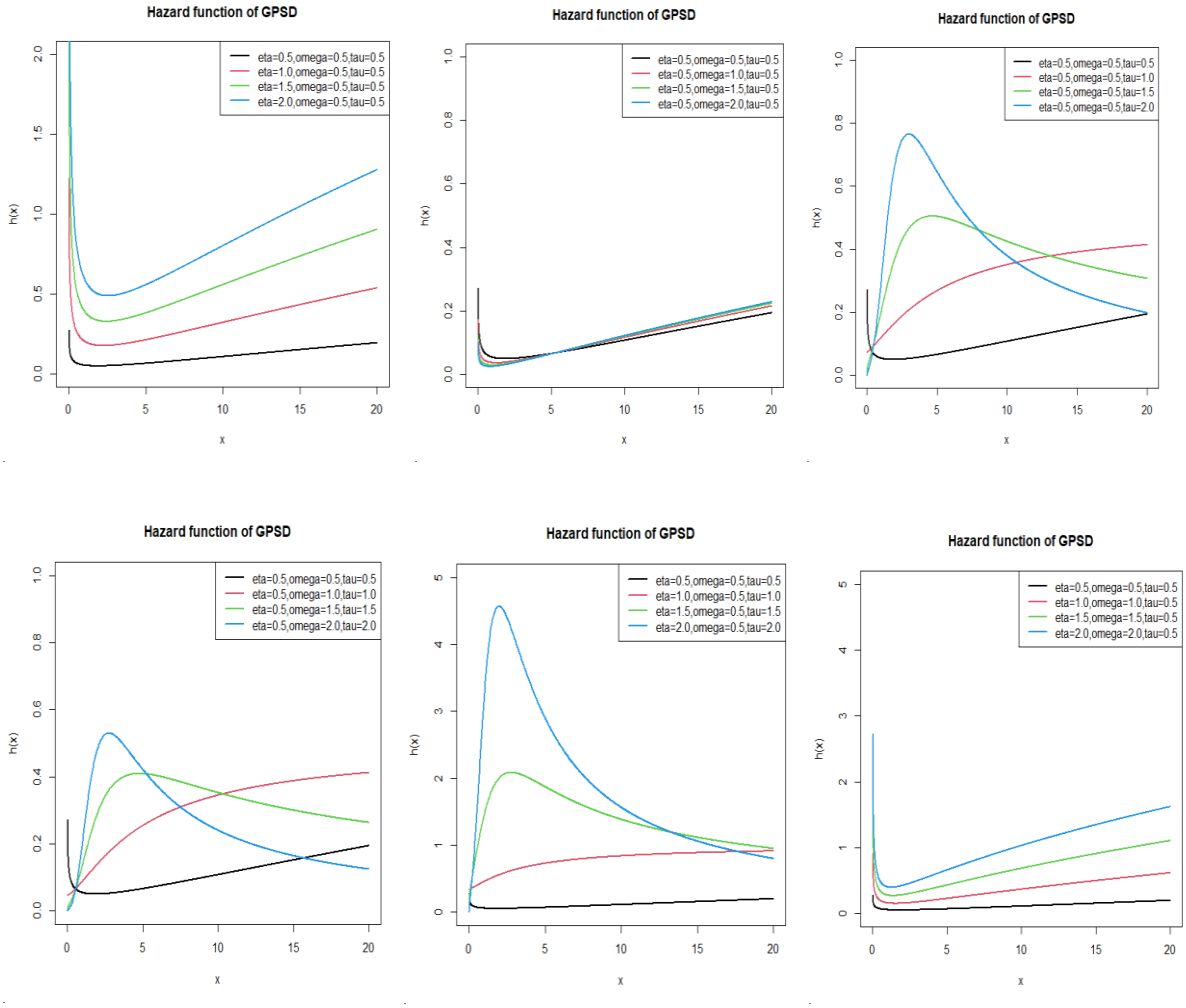

Figure 3: *Hazard function of GPSD for different values of the parameters*

V. Stochastic Ordering

In probability theory and statistics, a stochastic order quantifies the concept of one random variable being bigger than another. A random variable X is said to be smaller than a random variable Y in the:

i. Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(y)$ for all x

ii. Hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_X(x) \geq h_Y(y)$ for all x

iii. Mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_X(x) \geq m_Y(y)$ for all x

iv. Likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\frac{f_X(x)}{f_Y(y)}$ decrease in x

The following results due to [12] are well known for establishing stochastic ordering of distributions

$$X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{mrl} Y$$
$$\Downarrow$$
$$X <_{st} Y$$

Theorem: Let $X \sim$ GPSD $\left(\eta_1, \omega_1, \tau_1\right)$ and $Y \sim$ GPSD$\left(\eta_2, \omega_2, \tau_2\right)$. If $\omega_1 = \omega_2, \tau_1 = \tau_2$ and $\eta_1 > \eta_2$ or $\omega_1 = \omega_2, \eta_1 = \eta_2$ and $\tau_1 < \tau_2$ or $\eta_1 = \eta_2, \tau_1 = \tau_2$ and $\omega_1 < \omega_2$ then $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

Proof: We have $$\frac{f_X(x)}{f_Y(x)} = \frac{\tau_1 \eta_1^3 \left(\eta_2^2 + \eta_2 + 2\omega_2\right)}{\tau_2 \eta_2^3 \left(\eta_1^2 + \eta_1 + 2\omega_1\right)} \left(\frac{1 + x^{\tau_1} + \omega_1 x^{2\tau_1}}{1 + x^{\tau_2} + \omega_2 x^{2\tau_2}}\right) x^{\tau_1 - \tau_2} e^{-\left(\eta_1 x^{\tau_1} - \eta_2 x^{\tau_2}\right)}$$

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left[\frac{\tau_1 \eta_1^3 \left(\eta_2^2 + \eta_2 + 2\omega_2\right)}{\tau_2 \eta_2^3 \left(\eta_1^2 + \eta_1 + 2\omega_1\right)}\right] + \log\left(\frac{1 + x^{\tau_1} + \omega_1 x^{2\tau_1}}{1 + x^{\tau_2} + \omega_2 x^{2\tau_2}}\right) + \left(\tau_1 - \tau_2\right)\log x - \left(\eta_1 x^{\tau_1} - \eta_2 x^{\tau_2}\right)$$

$$\frac{d}{dx}\left[\log\left(\frac{f_X(x)}{f_Y(x)}\right)\right] = \frac{\tau_1 x^{\tau_1 - 1} + 2\omega_1 \tau_1 x^{2\tau_1 - 1}}{1 + x^{\tau_1} + \omega_1 x^{2\tau_1}} - \frac{\tau_2 x^{\tau_2 - 1} + 2\omega_2 \tau_2 x^{2\tau_2 - 1}}{1 + x^{\tau_2} + \omega_2 x^{2\tau_2}} + \frac{\tau_1 - \tau_2}{x} - \left(\eta_1 \tau_1 x^{\tau_1 - 1} - \eta_2 \tau_2 x^{\tau_2 - 1}\right)$$

Thus, for $\omega_1 = \omega_2, \tau_1 = \tau_2$ and $\eta_1 > \eta_2$ or $\omega_1 = \omega_2, \eta_1 = \eta_2$ and $\tau_1 < \tau_2$ or $\eta_1 = \eta_2, \tau_1 = \tau_2$ and $\omega_1 < \omega_2$, $\frac{d}{dx}\left[\log\left(\frac{f_X(x)}{f_Y(x)}\right)\right] < 0$. This means that $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

VI. Mean Residual Life Function

The mean residual life function of GPSD can be obtained as

$$m(x,\eta,\omega,\tau)=\frac{\left(\eta^2+\eta+2\right)\Gamma\left(\frac{1}{\tau},\eta x^{\tau}\right)+\left(\eta+2\omega\right)\Gamma\left(\frac{1}{\tau}+1,\eta x^{\tau}\right)+\omega\Gamma\left(\frac{1}{\tau}+2,\eta x^{\tau}\right)}{\tau\eta^{\frac{1}{\tau}}\left[\left(\eta^2+\eta+2\omega\right)+\eta x^{\tau}\left(\omega\eta x^{\tau}+\eta+2\omega\right)\right]e^{-\eta x^{\tau}}} \tag{19}$$

The behaviours of the mean residual life function of GPSD are explained in the following figure 4.

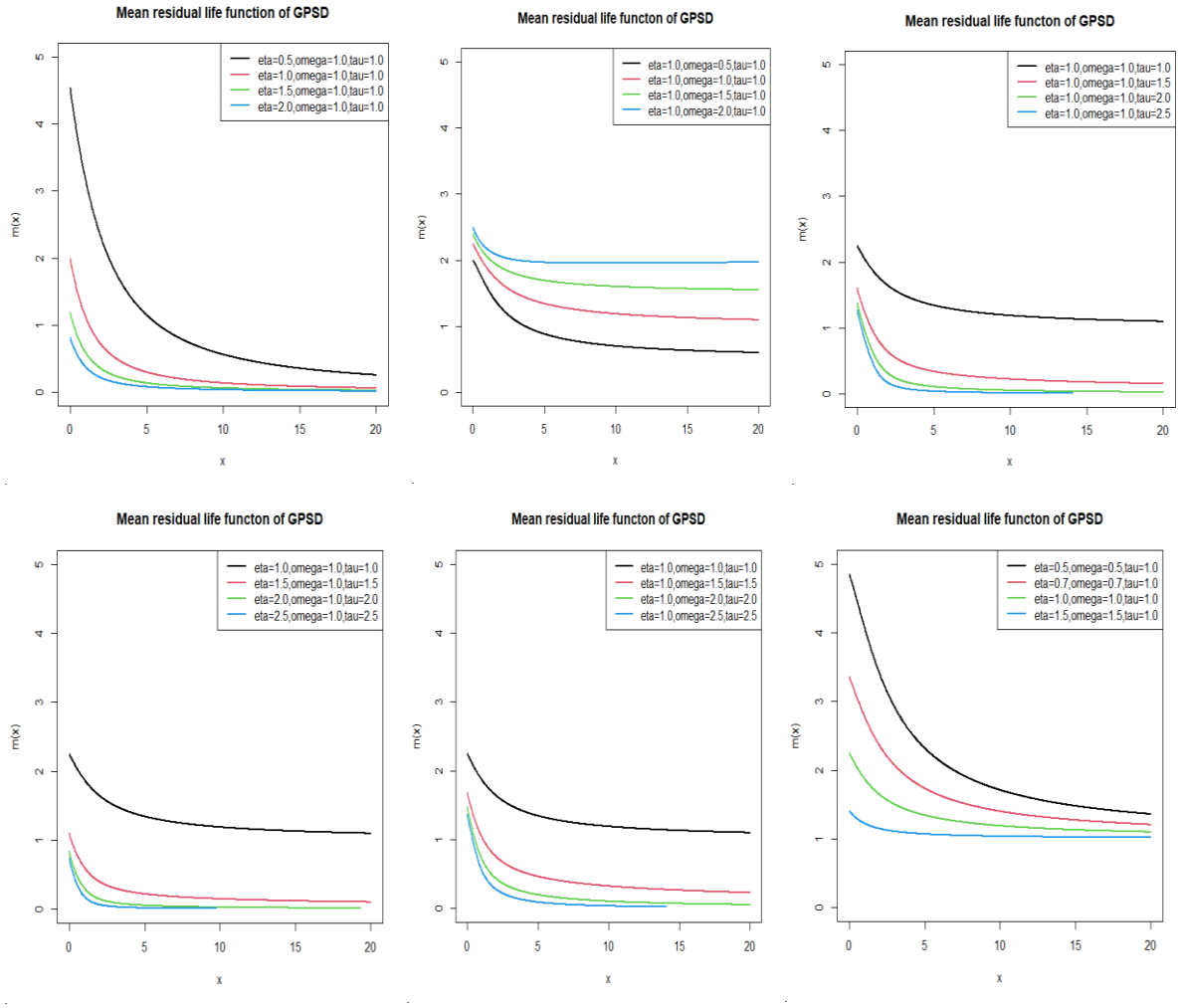

Figure 4: *Mean residual life function of GPSD for different values of the parameters*

IV. Moments Based Measures

The rth moments about origin of GPSD can be obtained as

$$\mu_r' = \int_0^\infty x^r \ f(x,\eta,\omega,\tau)dt = \frac{r\Gamma\left(\frac{r}{\tau}\right)\left[\eta^2\tau^2+\eta\tau(r+\tau)+\omega(r+\tau)(r+2\tau)\right]}{\tau^3\eta^{\frac{r}{\tau}}\left(\eta^2+\eta+2\omega\right)} \tag{20}$$

Putting $r=1,2,3,4$ in (20) we get

$$\mu_1' = \frac{\Gamma\left(\frac{1}{\tau}\right)\left[\eta^2\tau^2+\eta\tau(1+\tau)+\omega(1+\tau)(1+2\tau)\right]}{\tau^3\eta^{\frac{1}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

$$\mu_2' = \frac{2\Gamma\left(\frac{2}{\tau}\right)\left[\eta^2\tau^2+\eta\tau(2+\tau)+\omega(2+\tau)(2+2\tau)\right]}{\tau^3\eta^{\frac{2}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

$$\mu_3' = \frac{3\Gamma\left(\frac{3}{\tau}\right)\left[\eta^2\tau^2+\eta\tau(3+\tau)+\omega(3+\tau)(3+2\tau)\right]}{\tau^3\eta^{\frac{3}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

$$\mu_4' = \frac{4\Gamma\left(\frac{4}{\tau}\right)\left[\eta^2\tau^2+\eta\tau(4+\tau)+\omega(4+\tau)(4+2\tau)\right]}{\tau^3\eta^{\frac{4}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

V. Maximum Likelihood Estimation of the Parameters

Let's as $\left(x_1,x_2,...,x_n\right)$ be a random sample of size n taken from GPSD$\left(\eta,\omega,\tau\right)$. The likelihood function is defined as

$$L = \left(\frac{\tau\eta^3}{\left(\eta^2+\eta+2\omega\right)}\right)^n \prod_{i=1}^n \left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)x^{\tau-1}e^{-\eta x_i^{\tau}}$$

The log-likelihood function is obtained as

$$\log L = n\left[3\log\eta+\log\tau-\log\left(\eta^2+\eta+2\omega\right)\right]+\sum_{i=1}^n \log\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)+(\tau-1)\sum_{i=1}^n \log x_i-\eta\sum_{i=1}^n x_i^{\tau}$$

Now, the log-likelihood equations are given by

$$\frac{\partial \log L}{\partial \eta} = \frac{3n}{\eta} - \frac{n(2\eta+1)}{\eta^2+\eta+2\omega} - \sum_{i=1}^{n} x_i^{\tau} = 0$$

$$\frac{\partial \log L}{\partial \omega} = -\frac{2n}{\eta^2+\eta+2\omega} + \sum_{i=1}^{n} \frac{x_i^{2\tau}}{1+x_i^{\tau}+\omega x_i^{2\tau}} = 0$$

$$\frac{\partial \log L}{\partial \tau} = \frac{n}{\tau} - + \sum_{i=1}^{n} \frac{x_i^{\tau}\log x_i + \omega x_i^{2\tau}\log x_i}{1+x_i^{\tau}+\omega x_i^{2\tau}} + \sum_{i=1}^{n} \log x_i - \eta \sum_{i=1}^{n} x_i^{\tau}\log x_i = 0$$

Solving these three log-likelihood equations directly may not be straightforward. Utilizing maximization techniques within R software is necessary to iteratively solve the likelihood function until sufficiently close values of the parameters are achieved. This methodology is crucial for addressing the intricacies inherent in solving log-likelihood equations and ensuring the accuracy and reliability of the parameter estimates in statistical analyses.

Fisher's scoring method is widely used in statistical software packages like R for optimizing likelihood-based models, especially in cases where direct analytical solutions are challenging. It is a powerful tool for estimating parameters in a variety of statistical models, contributing to the robustness and efficiency of the estimation process.

$$\frac{\partial^2 \log L}{\partial \eta^2} = -\frac{3n}{\eta^2} - \frac{2n\left(\eta^2+\eta+2\omega\right) - n(2\eta+1)^2}{\left(\eta^2+\eta+2\omega\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \omega^2} = \frac{4n}{\left(\eta^2+\eta+2\omega\right)} - \sum_{i=1}^{n} \frac{x_i^{4\tau}}{\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)^2} = 0$$

$$\frac{\partial^2 \log L}{\partial \tau^2} = -\frac{n}{\tau^2} - + \sum_{i=1}^{n} \frac{\left(\log x_i\right)^2\left(x_i^{\tau}+\omega x_i^{2\tau}\right) - \left(x_i^{\tau}\log x_i + \omega x_i^{2\tau}\log x_i\right)^2}{1+x_i^{\tau}+\omega x_i^{2\tau}} + \eta \sum_{i=1}^{n} x_i^{\tau}\left(\log x_i\right)^2$$

$$\frac{\partial^2 \log L}{\partial \eta \partial \omega} = \frac{2n(2\eta+1)}{\left(\eta^2+\eta+2\omega\right)^2} = \frac{\partial^2 \log L}{\partial \omega \partial \eta}$$

$$\frac{\partial^2 \log L}{\partial \eta \partial \tau} = \sum_{i=1}^{n} x_i^{\tau}\log x_i = \frac{\partial^2 \log L}{\partial \tau \partial \eta}$$

$$\frac{\partial^2 \log L}{\partial \tau \partial \omega} = \sum_{i=1}^{n} \frac{x_i^{2\tau}\log x_i\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right) - \left(x_i^{\tau}\log x_i + \omega x_i^{2\tau}\log x_i\right)x_i^{2\tau}}{\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)^2} = \frac{\partial^2 \log L}{\partial \omega \partial \tau}$$

For finding the MLEs $(\hat{\eta}, \hat{\omega}, \hat{\tau})$ of parameters (η, ω, τ) of GPSD, following equations can be solved

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \eta^2} & \frac{\partial^2 \log L}{\partial \eta \partial \omega} & \frac{\partial^2 \log L}{\partial \eta \partial \tau} \\ \frac{\partial^2 \log L}{\partial \omega \partial \eta} & \frac{\partial^2 \log L}{\partial \omega^2} & \frac{\partial^2 \log L}{\partial \omega \partial \tau} \\ \frac{\partial^2 \log L}{\partial \tau \partial \eta} & \frac{\partial^2 \log L}{\partial \tau \partial \omega} & \frac{\partial^2 \log L}{\partial \tau^2} \end{bmatrix}_{\substack{\hat{\eta}=\eta_0 \\ \hat{\omega}=\omega_0 \\ \hat{\tau}=\tau_0}} \begin{bmatrix} \hat{\eta}-\eta_0 \\ \hat{\omega}-\omega_0 \\ \hat{\tau}-\tau_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \eta} \\ \frac{\partial \log L}{\partial \omega} \\ \frac{\partial \log L}{\partial \tau} \end{bmatrix} \tag{21}$$

where η_0, ω_0 and τ_0 are the initial values of η, ω and τ . These equations are solved iteratively till close estimates of parameters are obtained.

VI. Simulation Studies

In this section, a simulation study was conducted to assess the performance of maximum likelihood estimators for the GPSD. The investigation involved the examination of mean estimates, biases (B), mean square errors (MSEs), and variances of the Maximum Likelihood Estimates (MLEs). These metrics were computed using the following formulas:

$$Mean = \frac{1}{n}\sum_{i=1}^{n} \hat{H}_i a,\ B = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right),\ MSE = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right)^2,\ Variance = MSE - B^2$$

where $H = \eta, \omega, \tau$ and $\hat{H}_i = \hat{\eta}_i, \hat{\omega}_i, \hat{\tau}_i$

the simulation results for different parameter values of GPSD are presented in tables 1 and 2 respectively. The steps for simulation study are as follows:

a. Data is generated using the acceptance-rejection method in simulation studies, a commonly employed approach to generate random samples from a target distribution when the inverse transform method of simulation is impractical or inefficient. The acceptance-rejection method for generating random samples from GPSD involves the following steps:

   i. Generate a random variable Y distributed as Gamma (η, ω)
   ii. Generate U distributed as Uniform $(0,1)$
   iii. If $U \le \frac{f(y)}{M\, g(y)}$, then set $X = Y$ ("accept the sample"); otherwise ("reject the sample") and if reject then repeat the process: step (i-iii) until getting the required samples. Where M is a constant.

b. The sample sizes are taken as $n = 25, 50, 100, 200, 300$
c. The parameter values are set as values $\eta = 1.0$, $\omega = 0.1$, $\tau = 1.0$. and $\eta = 1.5$, $\omega = 1.5$, $\tau = 1.5$
d. Each sample size is replicated 10000 times

The findings from Tables 1 and 2 demonstrate that with an increase in sample size, biases, MSEs, and variances of the MLEs for the parameters decrease correspondingly. This result is in line with the first-order asymptotic theory.

Table-1: *The Mean values, Biases, MSEs and Variances of GPSD for parameter values* $\eta = 1.0$, $\omega = 0.1$, $\tau = 1.0$

Parameters	Sample size (n)	Mean	Bias	MSE	Variance
η	25	1.036569	0.03656	0.00220	0.00087
	50	1.035792	0.03579	0.00204	0.00076
	100	1.02854	0.02854	0.00140	0.00059
	200	1.02554	0.02554	0.00113	0.00048
	300	1.02470	0.02470	0.00106	0.00045
ω	25	0.12499	0.02499	0.00084	0.00021
	50	0.12213	0.02213	0.00065	0.00016
	100	0.12105	0.02105	0.00058	0.00014
	200	0.11927	0.01927	0.00050	0.00013
	300	0.11691	0.01691	0.00041	0.00012
τ	25	1.13596	0.13596	0.01984	0.00135
	50	1.13276	0.13276	0.01862	0.00099
	100	1.13212	0.13212	0.01828	0.00082
	200	1.13025	0.13025	0.01757	0.00060
	300	1.50287	0.00287	0.00030	0.00030

Table-2: *The Mean values, Biases, MSEs and Variances of GPSD for parameter values* $\eta = 1.5$, $\omega = 1.5$, $\tau = 1.5$

Parameters	Sample size (n)	Mean	Bias	MSE	Variance
η	25	1.50935	0.50935	0.00106	0.00097
	50	1.50575	0.00575	0.00082	0.00078
	100	1.50525	0.00525	0.00074	0.00071
	200	1.50299	0.00299	0.00062	0.00061
	300	1.50217	0.00217	0.00055	0.00055
ω	25	1.50832	0.00832	0.00087	0.00080
	50	1.50765	0.00765	0.00063	0.00057
	100	1.50558	0.00558	0.00055	0.00052
	200	1.50406	0.00406	0.00042	0.00041
	300	1.50355	0.00355	0.00038	0.00037
τ	25	1.50703	0.00703	0.00041	0.00036
	50	1.50388	0.00388	0.00037	0.00036
	100	1.50355	0.00355	0.00033	0.00031
	200	1.50304	0.00304	0.00031	0.00030
	300	1.50287	0.00287	0.00030	0.00030

VII. Applications Of GPSD

To assess the goodness of fit of the Generalized Power Sujatha distribution (GPSD) compared to other three-parameter, two-parameter, and one-parameter lifetime distributions, the following real lifetime datasets have been examined

Data set-1: The COVID-19 data set has the following source,

http://covid.gov.pk/stats/pakistan. It contains the daily recovered cases of COVID-19 inPakistan from 24 March to 28 April 2020 (36 days). The considered values are given:

2, 2, 3, 4, 26, 24, 25, 19, 4, 40, 87, 172, 38, 105, 155, 35, 264, 69, 283, 68, 199, 120, 67, 36,102, 96, 90, 181, 190, 228, 111, 163, 204, 192, 627, 263.

Data set-2: The following set of complete left skewed data, discussed by [13], reports the failure times of 20 components. The values are:

0.481, 1.196, 1.438, 1.797, 1.811, 1.831, 1.885, 2.104, 2.133, 2.144, 2.282, 2.322, 2.334,2.341, 2.428, 2.447, 2.511, 2.593, 2.715, 3.218.

Table 3: *MLEs,* $-2\log L$ *, AIC, K-S and p-value of the considered distributions for the data set-1*

Distribution	MLE			$-2\log L$	AIC	K-S	p-value
	$\hat{\eta}$	$\hat{\omega}$	$\hat{\tau}$				
GPSD	0.1553	0.1000	0.6031	**416.84**	**422.84**	**0.10**	**0.88**
PQLD	0.1000	0.1648	0.6383	417.00	423.00	0.16	0.34
TPSD	0.1000	12.4100	0.6929	815.27	821.27	0.86	0.00
GGD	0.1000	1.7828	0.6282	417.41	423.41	0.19	0.16
GSD	0.0251	7.2040	…	470.51	474.51	0.20	0.11
WD	0.1000	…	0.5483	429.56	433.56	0.68	0.00
SD	0.0250	…	…	467.13	469.13	0.27	0.01

Table 4: *MLEs,* $-2\log L$ *, AIC, K-S and p-value of the considered distributions for the data set-2*

Distribution	MLE			$-2\log L$	AIC	K-S	p-value
	$\hat{\eta}$	$\hat{\omega}$	$\hat{\tau}$				
GPSD	0.2587	0.3236	2.8755	**33.40**	**39.40**	**0.22**	**0.25**
PQLD	0.1000	0.3464	3.5053	33.49	39.49	0.57	0.00
TPSD	1.4280	14347.6730	0.1000	50.23	56.23	0.410	0.00
GGD	0.1000	1.4004	3.2768	35.61	41.61	0.60	0.00
GSD	1.4280	9294.9234	…	50.23	54.23	0.38	0.00
WD	0.1000	…	3.0368	38.15	42.15	0.57	0.00
SD	1.0538	…	…	60.20	62.20	0.51	0.00

From Table-3 and Table -4 we observed that GPSD has the lower $-2\log L$, AIC, and K-S values and higher p-values as compared to PQLD, TPSD, GGD, GSD, WD and SD. Hence, we may conclude that GPSD give the better fit than PQLD, TPSD, GGD, GSD, WD and SD. Further, it is also clear from the fitted plot of distributions in figure 5 that GPSD provide much closer fit over PQLD, TPSD, GGD, GSD, WD and SD.

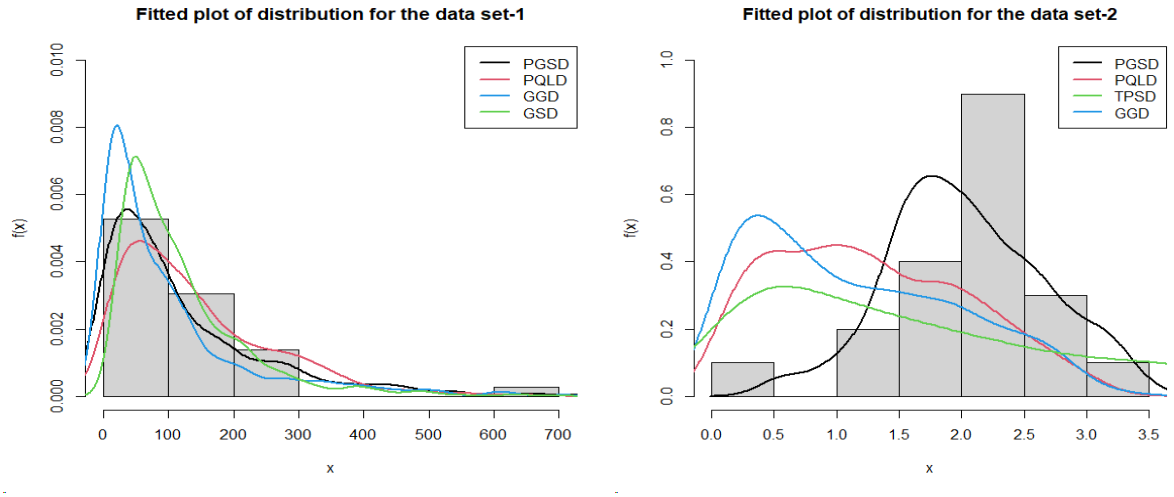

Figure 5: *Graph of the fitted plot of distributions for the data set-1 and data set-2*

VIII. Conclusion

This paper introduces the Generalized Power Sujatha distribution (GPSD) and explores its statistical properties, including moments, survival function, hazard function, reversed hazard function, mean residual life function, and stochastic ordering. The study employs maximum likelihood estimation to estimate the distribution's parameters and evaluates their performance through a simulation study. Additionally, the paper applies GPSD to two real lifetime datasets, comparing its fit with other distributions such as PQLD, TPSD, GGD, GSD, WD and SD. Results indicate that GPSD offers a superior fit compared to PQLD, TPSD, GGD, GSD, WD and SD.

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RELAY CONTACTOR SYSTEM AS A MEANS OF CONTROLLING A LINEAR ELECTRIC DRIVE

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Abstract

The energy sector is currently undergoing rapid change as a result of advances in technology, changes in consumer demand and the desire for more sustainable and efficient energy sources. Against the background of these changes, the problems of process management and optimization in the energy system are particularly relevant. One of the main directions in this field is the application of control systems through different-purpose control apparatus that can effectively react to changes in the environment and dynamically adapt to new conditions.The future development of the theory and practice of automatic control is related to the determination of the maximum possibilities of the systems and their construction, which are the best according to any technical and economic indicator. It is the research and development of control systems through apparatus in the energy sector, taking into account modern requirements and technological possibilities. Control systems are widely used in various fields of technology, they are applied in the automation of production processes and calculations. Positive results are obtained when simulating the system using different parameter values for different types of interference signals. Management systems with the use of hardware can be successfully applied in the real working conditions of energy enterprises and can ensure optimal use of resources, reduction of operating costs and minimization of negative effects on the environment. This article discusses the characteristics of relay-contactor control systems. Relay contactor equipment controls electric drives powered by electric motors from a network with a constant voltage, which are widely used in all industries. Relay-contactor control systems are control systems built on a relay-contactor element base and designed to automate the operation of engines. With the help of such control systems, operations such as turning the engine on and off, choosing the direction and speed of rotation, starting and braking the engine, creating temporary pauses in movement, protective shutdown of the engine and stopping the mechanism are automated. These operations are necessary to perform the movement of the working body of the mechanism according to technological conditions. An electric drive, made on the basis of a relay-contactor control system, is a simple, unregulated electric drive of direct or alternating current, mainly for general industrial use, for example, electric drive of cranes, elevators, conveyors, fans, pumps, some transport devices, etc.

Key words: relay, contactor, equipment, characteristics, control system, electric drive, network, input, output, voltage, load.

I. Introduction

Technological progress in the field of industrial development and research set the task of creating systems of extremely high accuracy and minimal complexity. Such automatic systems must find conditions for highly efficient process management in a certain environment without the presence of an operator. The energy sector is currently undergoing rapid change as a result of advances in technology, changes in consumer demand and the desire for more sustainable and efficient energy sources. Against the background of these changes, the problems of process management and optimization in the energy system are particularly relevant.

One of the main directions in this field is the application of management systems with equipment that can effectively respond to changes in the environment and dynamically adapt to new conditions. Currently, the main and most promising method of automated management of complex dynamic systems is control and data collection systems. Based on control principles, large-scale automated systems are established through control equipment in the industrial and energy, transport, space and military fields, and in various state institutions [13].

The unity of control theory methods enables the synthesis of control systems, they have the possibility of changing the parameters of the regulator or depending on the change of the parameters of the control object or the structure of the regulator depending on the external influences on the control object. Additional great opportunities for improving control processes, depending on the size and signs of the input values, which enter the control device from the measuring device, enable non-linear control of the object's activity by changing the structure of the control device. In this case, combinations of linear control laws can be used.

The direct control object for relay-claw control systems is a motor powered from the network. In relay-claw control systems, two parts can be distinguished according to their functional purpose: the control part (forming the control algorithm, including various relays) and the executive part, which directly carries out control actions on motor (contactors, magnetic starters). Relay-claw control systems include standard units that perform specific functions. In addition to them, non-standard units are involved to solve a specific technological problem, for example, a unit for protecting the working element from slipping of the drive pulley, overspeeding, etc [1]..

The most important typical function of relay-claw control systems is the protection of the electrical and mechanical parts of the electric drive from emergency modes. The task of the protection unit is to disconnect the engine from the power source and stop the working part of the production machine. The number of emergency modes in the mechanical part of the electric drive includes: exceeding the permissible torque in the mechanical transmission (jamming of the mechanism); disengagement of the working body from the engine shaft; exceeding the permissible speed of the engine or working body; exit of the working body beyond the permissible movement zone. The advantages of relay-claw control systems include: the presence of galvanic isolation of power circuits from control circuits; significant switching power (up to several kA); high noise immunity.The disadvantages of relay-claw control systems include: contact switching, which requires appropriate care of the equipment and limits its service life; limited performance; increased weight and size indicators and energy consumption [2]. Table 1 shows the parameters characterizing modern relay contactor equipment [3]..

Relay contactor circuits are presented in the form of finished products - control stations, which contain standard circuits for controlling the movement of an electric drive, as well as the necessary protections. Electric drives are controlled by relay contactor equipment using an electric motor powered from a constant voltage network, and are also widely used in automating the operating principles of electromechanical devices.

Table 1. Technical characteristics of relay contactor equipment

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Operating period $t_{op.}$, sec.	0.005 ÷ 0.4
Number included (in time), N	600.....1200 for contactors 1200....3600 for relay
Power S, VA	5.....50 for contactors 0.2....5 for relay
Service life (total number of starts per hour)	10^6.....10^7
Weight m, kg.	0.03....5

II. Formulation of the problem

The low current relay can be controlled as an electromagnetic switch to perform overcurrent on and off operations. A relay made of two isolated circuits: a control circuit used to control the switch and another circuit from the switch. When voltage is applied to the control circuit, current flows through the coil and creates a magnetic field, which is used to turn the switch on and off. This magnetic field through a wire (current) is created by a flow of electrons and the flow of electrons when passing through the coil is enhanced [4].. A relay in a low voltage circuit allows control of a large electrical load. Because relays consist of two isolated circuits, low-voltage components are protected from extremely high electrical loads because the circuits are physically isolated [5].

This in turn, due to the increased power rating of low-voltage components, prevents any errors compared to high-voltage components. In addition, the relay also contributes to the control of systems via an electrical signal and this shows that it is also possible to control a linear electric drive using a sensor or microcontroller [10-15]. To control a linear electric drive using a relay, it is necessary to achieve the ability to switch the polarity of the input voltage of the electric drive. In this case, it is possible to use a DPDT relay or two SPDT relays.

The DPDT type relay consists of 8 connectors: 2 for the coil, 4 for the switch input and 2 for the output. As on a DPDT switch by switching positive or negative terminals, it is necessary to connect the drive to input jack 4 or to 2 output jacks and supply power to all 4 input jacks. Due to the use of only one relay, only one input signal is needed to control it. When voltage is applied to the coil, the drive moves, and when the supply stops, it comes together [6-9].

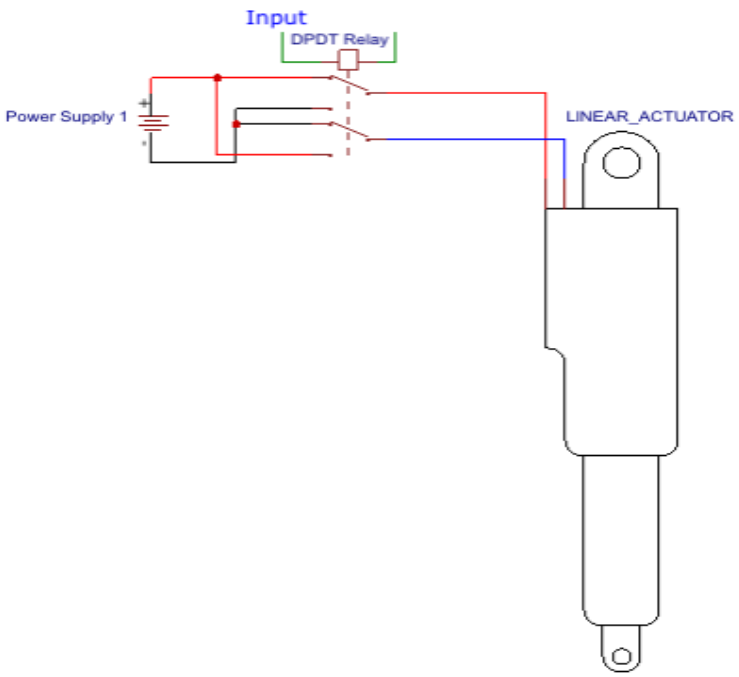

Figure 1. *Control of a linear electric drive using a relay*

In these relays, normal closed connections are connected to the grounding of the power supply, which ensures that in the event of failure of the control system, the actuator is immobilized. With this installation, to control the electric drive, power is supplied to one relay, and for reverse action it is necessary to strengthen the second relay - as shown in the figure (figure 2). It is necessary to ensure that both coils are supplied with power at the same time. Similar 4 SPST relays can be used to ground two relays and power two relays, but instead of a relay configuration with 2 SPDT relays there is no reason to use such a configuration, particularly if a relay module is available [18].

This means that there is no trip position and when certain limits are reached, a linear actuator with internal tip switches is used to trip the actuator. With this configuration, it is necessary to ensure that in the event of a control failure or loss of energy, the system can stop the movement (figure1). Otherwise, various other configurations may be used. If there is a need for a linear actuator for distinctive positions, then two positions can be used: a single pole relay configuration. In this configuration, 2 relays are used to change the polarity of the linear drive voltage, as well as to stop the power supply to the drive [16-17].

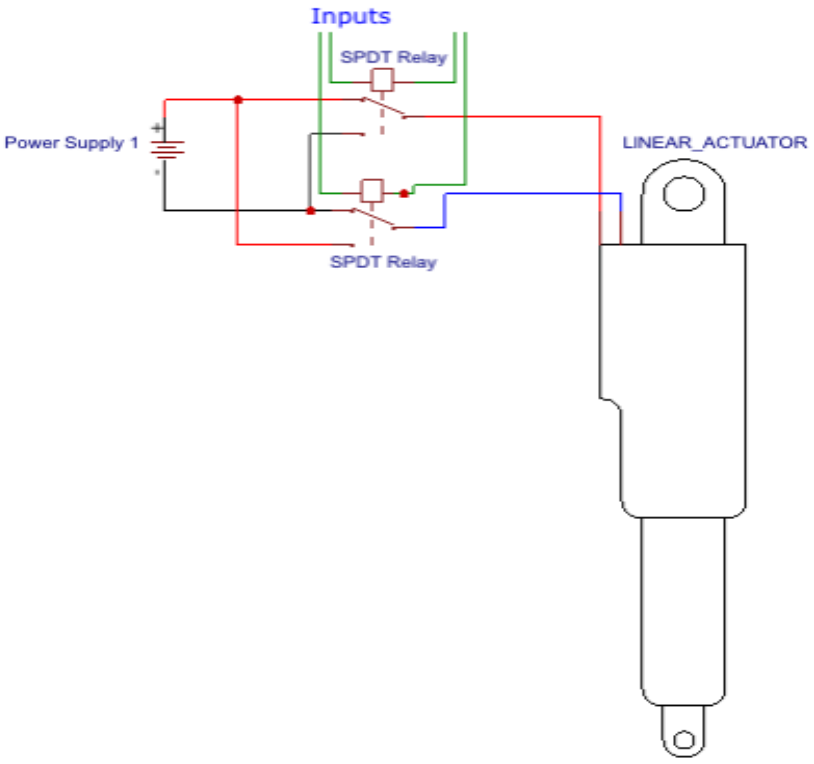

Figure 2. *Control of linear electric drive via relay (other configuration)*

III. Problem solution

Contactors have found wide applications for remote switching of power circuits. Such devices perform multiple reboots normally. Low-power relays with contactors, buttons, switches, etc. through which the remote control scheme of various loads can be realized. For example, in production, various electric motors and other loads are started, they are also used in powerful household equipment.

A contactor is an electromagnetic two-position switching device used for multiple distance switching of power circuits in normal operating modes. Its connection is performed due to electromagnetic transmission. The return (opening) of the contacts is carried out due to the effect of the spring, the mass of the moving structure or the combined effect of these factors. In all cases, the control circuit is connected the same as the coil. In order for the contactor to be connected, current must be applied to the winding, and then the armature is drawn towards the core, pulling the moving contacts towards itself, and they connect with the stationary contacts, closing the circuit. A simple connection diagram of the contactor is shown in figure 3. According to the working principle of the scheme, after the automatic switch QF is connected, the voltage is supplied to the SB 1 button and the upper contacts of the contactor KM 1 according to the scheme. When the SB 1 button is pressed, the control circuit is closed and the current flows through the coil KM 1, the contactor is connected, the contacts KM 1 are closed. Motor M (or any other connected load) is started. Coil and power circuits can be fed from different sources, different types of voltage and current. When the current from the coil is interrupted - the contactor stops, and its contacts open. That is, holding down the SB 1 button, the motor rotates. When you release the button or disconnect the voltage - the QF 1 button opens, the contactor is tripped and the motor stops [19].
Contactors are variously classified according to a number of parameters, one such parameter being the control circuit voltage or winding nominal voltage [20].

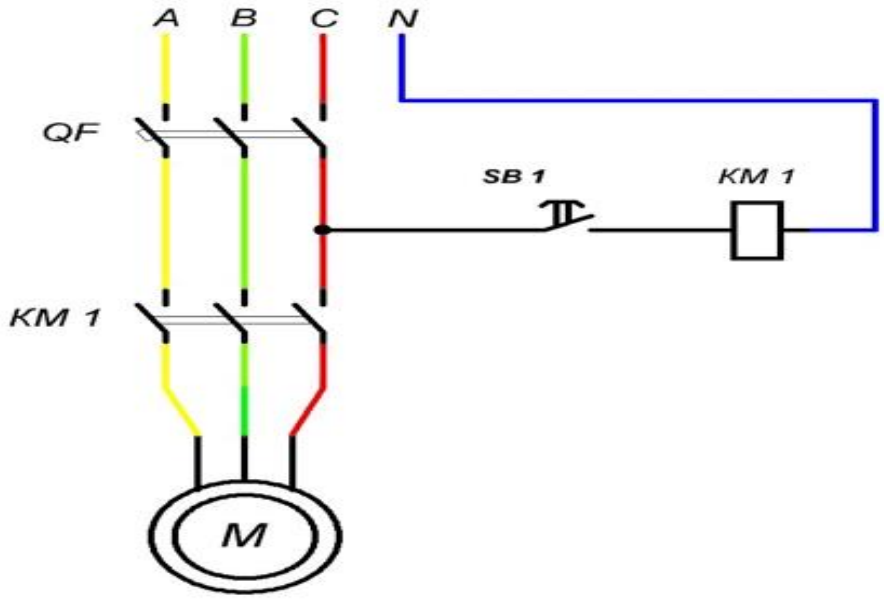

Figure 3. *Contactor coil connection scheme (220V)*

As an electromagnetic inverter, a relay can be controlled at low current to perform high current switching and opening operations. A relay consisting of two isolated circuits: a control circuit used to control the inverter and another circuit consisting of the inverter. When voltage is applied to the control circuit, a current flows through the coil and a magnetic field is created, which is used to open and close the inverter (1). This magnetic field is created by the flow of electrons (current) through the wire (2) and is amplified when the flow of electrons passes through the coil (3) (figure 4) [22]..

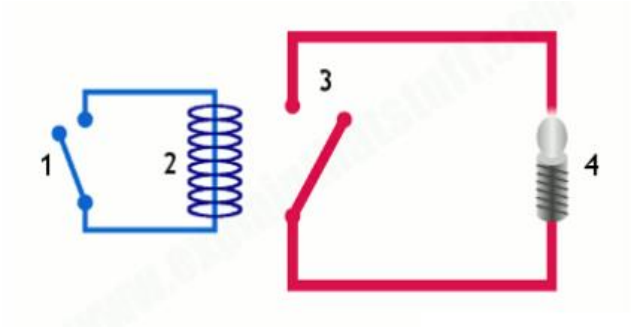

Figure 4. *Electromagnetic relay*

In a low-voltage circuit, the relay has the ability to handle a large electrical load. Because relays consist of two isolated circuits, small voltage components are protected from high electrical loads because the two circuits are physically isolated. This also avoids any errors due to the nominal power increase of the low-voltage components compared to the high-voltage components. In addition, the relay also allows the control of systems through an electrical signal, which indicates that the line can be controlled by electrical transmission through a transmitter or a microcontroller [21].

Variable frequency transmission control is divided into two types: vector and scalar. Scalar controls are used in the transmission of equipment, machines, fans, pumps, which do not require precise adjustment of torque and control of torque and speed at the same time (figure 5). Vector control is the regulation of quantity, frequency and phase of supply voltage. This method allows for virtually inertialess rotational speed and torque changes (figure 6). In the scalar method, control of the supply voltage value and its frequency is carried out. In the vector method, in addition to the value and frequency, the control of the phase is also carried out, that is, the control of the angle and value of the spatial vector is carried out [9]. Compared to the scalar method, the vector method has higher efficiency, adjustment accuracy and range. Any method is selected depending on the requirements, which are given for the technological process - the depth and accuracy of regulation, torque control, the state of the transmission during transition processes - starting, stopping, braking, etc.

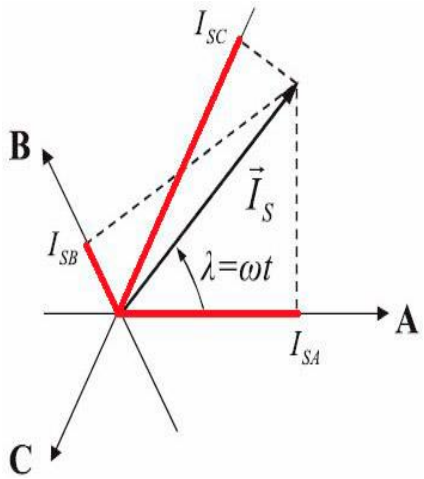

Figure 5. *Vector control method*

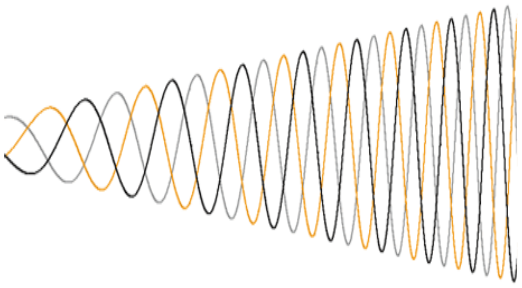

Figure 6. *Scalar control method (variation of feed voltage)*

The scalar method is used for small and medium power transmissions with fan loads. In the application of this method, there is the possibility of using multi-engine transmissions. The stability of the static characteristics of the transmission is practically close to the natural characteristic [23].

The range of the scalar control principle is no more than 1:10 with the lossless resistance moment. For this reason, a constant load capacity of the motor is achieved, and the applied voltage does not depend on the frequency, but at low frequencies the motor may overheat and the generated torque decreases. To prevent this, restrictions are placed on the minimum value of the output frequency [11]. Different analog and pulse speed transmitters are used in order to expand the stiffness of the characteristic and the adjustment limits. Therefore, discrete-analog controlled inputs are available in the frequency converters. Compared to the scalar method, the vector control method has the following advantages:

- high speed accuracy and a wide adjustment range;
- smooth regulation of engine rotation speed in the entire frequency range;
- the ability to maintain speed stability in the load change of electric transmission;
- reduction of losses in transition processes in transmission (increased engine efficiency).

In addition to the advantages, it should also be noted that the computational complexity of the vector control method is high, and the number of parameters must be taken into account in the calculation of the optimal operating modes of the transmission. But only to ensure a wide range and accuracy of regulation, especially at low frequencies of rotation, a vector frequency converter becomes indispensable [18].

IV. Conclusions

An electric motor fed from a constant voltage network through a relay-contactor apparatus provides control of electric transmissions and is also widely used in automating the working principles of electromechanical devices. Relay-contactor circuits are presented as finished products of control stations, on which typical circuits and necessary protections are assembled for the control of electric transmission movement. In many cases, it is required to connect any powerful load through a low-power relay. Sometimes they are needed for direct switching. In such cases, the contactor is simply installed and the output of the relay is connected to its coil. This can be implemented not only with a single voltage relay, but also with other devices of automation, for example, through a low-power intermediate relay or with another voltage type and value from the circuit (for example, in automation in different controllers.

As a reference to the literature reviewed on the basis of the scientific and technical base, it can be noted that in practice, in the choice of methods of adjusting the rotation speed of the electric transmission, it is necessary to evaluate the requirements for the control object, which is the range and accuracy of the adjustment of technological quantities, the need to maintain the torque on the motor shaft (especially small rotation frequencies), are the requirements for traffic control in emergency conditions.

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KERNEL SMOOTHING OF THE MEAN PERFORMANCE FOR HOMOGENEOUS CONTINUOUS TIME SEMI-MARKOV PROCESS

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Abstract

The main goal of the present paper is to propose a systematic approach to model performance measurements within the context of continuous-time semi-Markov processes with a finite state space. Specifically, the mean performance is estimated using the kernel method. The uniform strong consistency and the asymptotic normality of the proposed estimator is investigated. Furthermore, a non-parametric kernel estimation of the expected cumulative operational time is addressed. The constructed estimator is proved to be consistent and to converge to a normal random variable as the time of observation becomes large. As an illustration example, a simulation study has been conducted in order to highlight the efficiency as well as the superiority of our method to the standard empirical method.

Keywords: Semi-Markov processes, Kernel estimator, Mean performance, Reliability, Cumulative operational time, Consistency, Asymptotic normality.

1. Introduction and motivation

Stochastic process theory can be seen as an extension of probability theory that allows modeling the evolution of a system through the time. In addition, any serious study of renewal processes would be impossible without using the powerful tool of Markov processes. Markov processes are significant because, in addition to modeling a wide range of interesting phenomena, their lack of memory property allows for the computation of probabilities and expected values that quantify the behavior of the process and the prediction of its potential behavior. Therefore, the starting step of many attempts to model continuous-time processes has been the Markov process. However, the Markov property has its limitations. It imposes restrictions on the distribution of the sojourn time in a state, which is exponentially distributed in the continuous case, though this is not realistic in general. It is adequate to assume that the probability of leaving a state depends on the time already spent there. More precisely, it has become clear that the propensity to move from one state to another often depends strongly on the length of stay in that state. Therefore, any adequate model must incorporate this feature and the semi-Markov process meets the case.

The study of the semi-Markov process is related to the theory of Markov renewal processes (MRP) which can be considered as an extension of the classical renewal theory (see for instance,

Pyke [23], [24] and Limnios and Oprisan [13]). More precisely, the semi-Markov processes generalize the renewal processes as well as the Markov jump processes and have numerous applications, especially in reliability (see the pioneer work of Janssen [9] which has made a state of the art for the area of the semi-Markov theory and its applications). Furthermore, there are numerous real-world scenarios where semi-Markov models are relevant, such as in fault-tolerant systems, computer systems and networks, manufacturing systems, healthcare systems, and others.

There is a growing need to assess reliability and performance measure (see, for example, Smith et al. [27]). Meyer [16] created a conceptual framework for performability, defining it as the probability that a system achieves a certain level of accomplishment during a utilization interval $[0, t]$. In other words, for the semi Markov process (Z_t), that describes the evolution of the system through a set of states E, the performance level, or a reward rate, $L(j)$ is associated with each state $j \in \mathbb{E}$, where the reward function $L : E \to \mathbb{R}$ is proposed to be a measure of performance per unit time. The resulting semi-Markov reward process is then able to capture not only the failure and repair of the system components, but the degradable performance as well. Therefore, the development of a performance model is badly needed when we are interested in the level of productivity of a system.

The accumulated reward until time t will be $\phi(t) = \int_0^t L(Z_u)du$ which is an integral functional of the process (Z_t). Integral functionals are very important from theoretical as well as practical point of view. Indeed, in martingale theory, integral functionals are very useful since they are used as compensators, see Koroliuk and Limnios [12]. In statistics, they are used as empirical estimators for stationary distributions in semi-Markov processes [15]. In stochastic applications, they are crucial in some reliability studies, in performance evaluation of computer systems [26] and so on.

Since the work of [2] in which the author has defined combined measures of performance and reliability, many researchers have focused on evaluating performability, particularly in cases where (Z_t) is a Markov process with a finite number of states. Meyer [17] has studied the case of Markov process when the function L is monotonic. Beaudry [3] introduced an algorithm to compute performability until absorption over an infinite interval. Additionally, Donatiello and Iyer [5], have proposed an algorithms for computing performability that do not require the function L to be monotonic.

The semi-Markov case with a finite number of states has been examined by Iyer et al. [7]. The authors demonstrated that the distribution function of $\phi(t)$ satisfies a Markov renewal-type equation and proposed approach to solve it. Fore years later, Ciardo et al. [4] have offered an extension of Beaudry's approach to semi-Markov processes.
Since then, several research papers have been published on the development of estimators and the investigation of their asymptotic properties. In [14], the authors have presented a statistical study of the nonparametric estimation of performability of a finite state space semi-Markov system by using empirical estimator and give consistency and asymptotic normality results for such a system.

A specific scenario occurs when a reward of 1 is assigned to all operational states and 0 to all non-operational states. This case was studied by [21], where the expected reward rate at time t, $\mathbb{E}(\phi(t))$, is known as the instantaneous or point availability $A(t)$. In this case, $\phi(t)$ represents the total time spent in operational states during the interval $[0, t]$.

To the best of our knowledge, there are no existing works on nonparametric kernel estimators of the performance and performability. The main goal of the present paper is to propose a systematic approach to model performance measurements within the context of continuous-time semi-Markov processes with a finite state space. Specifically, the mean performance is estimated using the kernel method. The uniform strong consistency and the asymptotic normality of the

proposed estimator is investigated. Furthermore, a non-parametric kernel estimation of the expected cumulative operational time is addressed. The constructed estimator is proved to be consistent and to converge to a normal random variable as the time of observation becomes large. As an illustration example, a simulation study has been conducted in order to highlight the efficiency as well as the superiority of our method to the standard empirical method.

The remainder of the paper is organized as follows. Section 2 presents some definitions and notations of the semi-Markov processes in a countable state space, and these are needed in the work's sequel. An explicit expression of the mean performance of a finite state space semi-Markov system is given in Section 3. The basic elements of statistical estimation are given in Section 4. In Section 5, we introduce and discuss in detail the necessary conditions for establishing the asymptotic properties of the proposed estimator. In Section 6, we illustrate these concepts, measures and estimators through the cumulative operational time as well as a numerical study. Some concluding remarks are given in Section 7.

2. Semi-Markov system and related quantities

Definition 1. (Markov renewal process) Let $E = \{1, \ldots, s\}$ be the state space.
A Markov renewal process is a bivariate stochastic process (J_n, T_n) where J_n is the system state at the nth time, and T_n is the nth jump time, we set $T_0 = 0$. The process has to satisfy the following formula:

$$\mathbb{P}\left(J_{n+1} = j, T_{n+1} - T_n \leq t \mid J_0, J_1, \ldots, J_n, T_0, T_1, \ldots, T_n\right) = \mathbb{P}\left(J_{n+1} = j, T_{n+1} - T_n \leq t \mid J_n\right), \tag{1}$$

for all $j \in E$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$.
Moreover, if Equation (1) is independent of n, (J_n, T_n) is considered to be time homogeneous.

Definition 2. (Continuous-time semi-Markov process) Consider a Markov-renewal process $\{(J_n, T_n) : n \in \mathbb{N}\}$ defined on a complete probability space and with state space E. The stochastic process $\{Z_t; t \in \mathbb{R}_+\}$ defined by

$$Z_t = J_{N(t)}, \tag{2}$$

is called a Semi-Markov Process (SMP) where $N(t) := \sup\{n \geq 0 \mid T_n \leq t\}$ is the counting process of the SMP up to time t. Let us also define $X_n = T_n - T_{n-1}, n \geq 1$, the inter-jump times of $Z = (Z_t)_{t \in \mathbb{R}^+}$.

Let us also introduce some functions associated with the process Z:

- The semi-Markov kernel $\mathbf{Q}(t) = \left\{Q_{ij}(t), i, j \in E\right\}, t \geq 0$ of Z, is given by

  $$Q_{ij}(t) = \mathbb{P}\left(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i\right),$$

  and $p_{ij} := Q_{ij}(\infty) = \mathbb{P}\left(J_{n+1} = j \mid J_n = i\right)$ with $\mathbf{p} = (p_{ij})_{i,j \in E}$ is the transition matrix of the process (J_n) which is called the embedded Markov chain (EMC) of Z.

- The conditional sojourn time distribution in state i, given that the next state to be visited is j, denoted F_{ij}, is defined by

  $$F_{ij}(t) := \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j\right).$$

  Meanwhile, the sojourn time distribution in state i, denoted H_i, is defined, for every $t \in \mathbb{R}_+$, by

  $$H_i(t) = \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i\right) = \sum_{j \in E} Q_{ij}(t),$$

  and its corresponding survival function is defined by $\overline{H}_i(t) = 1 - H_i(t)$.

- If we consider g to be a locally bounded function and G to be a real right continuous nondecreasing function both defined on $\mathbb{R}_+$, the Stieltjes convolution of the function g with the function G is defined, for every $t \in \mathbb{R}_+$, by

$$g * G(t) = \int_{\mathbb{R}} g(t-x)dG(x) = \int_0^t g(t-x)dG(x).$$

Furthermore, when G and F are cumulative distribution functions, we have

$$G * F(t) = \int_0^t G(t-x)dF(x) = \int_0^t F(t-x)dG(x) = F * G(t).$$

- Now, consider the n-fold convolution of Q by itself, for any $i, j \in E$,

$$Q_{ij}^{(n)}(t) = \begin{cases} \mathbf{1}_{\{i=j, t\geq 0\}} & \text{if } n = 0, \\ Q_{ij}(t) & \text{if } n = 1, \\ \sum\limits_{k\in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s) & \text{if } n \geq 2. \end{cases}$$

- The Markov renewal function denoted $\Psi_{ij}(\cdot)$, is defined, for every $i, j \in E, t \geq 0$, by

$$\begin{aligned} \Psi_{ij}(t) &= E_i[\sum_{n=0}^{\infty} \mathbf{1}_{\{J_n=j, T_n\leq t\}}] \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i\left(J_n = j, T_n \leq t\right) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \end{aligned}$$

The matrix renewal function is given by

$$\mathbf{\Psi}(t) = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}(t),$$

where $\mathbf{\Psi}(t) = [\Psi_{ij}(t)]$.

The matrix renewal function $\mathbf{\Psi}(t)$ is the solution of the following Markov renewal equation

$$\mathbf{\Psi}(t) = \mathbf{I}(t) + \mathbf{Q} \star \mathbf{\Psi}(t)^1,$$

where $\mathbf{I}(t) = \mathbf{I}$ when $t \geq 0$ and $\mathbf{I}(t) = 0$ when $t < 0$.

- The transition matrix function $\mathbf{P}(t) = [P_{ij}(t)]$ of the semi-Markov process is defined, for every $i, j \in E, t \geq 0$, by

$$P_{ij}(t) = \mathbb{P}\left(Z_t = j \mid Z_0 = i\right) = \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right).$$

It is known, cf. [23], that

$$P_{ij}(t) = \mathbf{1}_{\{i=j\}} \left(1 - \sum_{k\in E} Q_{ik}(t)\right) + \sum_{k\in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds).$$

By solving the above Markov renewal equation, cf. [13], it is seen that, in matrix notation, we have

$$\mathbf{P}(t) = (\mathbf{\Psi} \star (\mathbf{I} - \mathbf{H}))(\mathbf{t}),$$

where $(\mathbf{I} - \mathbf{H})(\mathbf{t}) = diag[1 - H_i(t)]$.

Definition 3. For a semi-Markov process $(Z_t)_{t\in\mathbb{R}^+}$, the limit distribution $\pi = (\pi_1, \ldots, \pi_s)^t$ is defined, when it exists, for every $i, j \in E$, by

$$\pi_j = \lim_{t\to\infty} P_{ij}(t).$$

[1] $\star$ stands for the matrix-Stieltjes convolution of an $n \times r$ matrix function, $\mathbf{A}$, by an $m \times n$ matrix function, $\mathbf{B}$, denoted $\mathbf{B} \star \mathbf{A}$, which can be defined by $(\mathbf{B} \star \mathbf{A})_{ij}(t) = \sum\limits_{k=1}^{n} B_{ik} * A_{kj}(t), \quad i = 1, \ldots, m, j = 1, \ldots, r.$

3. Mean performance

The performance process at time $t \geq 0$, denoted $\Phi(t)$ is the real-valued integral functional of a homogeneous semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$, cf. [14], defined by

$$\Phi(t) = \int_0^t L(Z_u)\, du = \sum_{j\in E} L(j) \int_0^t \mathbf{1}_{\{z_u=j\}} du, \tag{3}$$

where L is a real-valued function defined on E.

The mean performance at time $t \geq 0$, denoted $\overline{\Phi}(t)$, is defined by

$$\overline{\Phi}(t) := \mathbb{E}[\Phi(t)] = \sum_{i\in E} \alpha_i \overline{\Phi}_i(t) = \sum_{i\in E}\sum_{j\in E} \alpha_i L(j) \int_0^t P_{ij}(u)du, \tag{4}$$

where $\overline{\Phi}_i(t) = \sum_{j\in E} L(j) \int_0^t P_{ij}(u)du$, $\alpha_i = P(J_0 = i)$ and the row vector $\boldsymbol{\alpha} = (\alpha_i : i \in E)$ defines the initial distribution of Z.

4. Elements of statistical estimation

Consider a sample path of the Markov renewal process $(J_n, T_n)_{n\in\mathbb{N}}$

$$\mathcal{Y}(M) := \left(J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M\right), \quad M \in \mathbb{R}_+,$$

where $N(M)$ is defined in Definition 2 and $u_M := M - T_{N(M)}$.

For all $i, j \in$ E, we define:

- $N_i(M) := \sum_{n=1}^{N(M)} 1_{\{J_{n-1}=i\}}$, the number of visits to state i, up to time M.
- $N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}}$, the number of transitions from i to j, up to time M.

For all $i, j \in E$, $t > 0$, we define the kernel estimator of Q_{ij} and H_i respectively (cf. [1]), by

$$\widehat{Q}_{ij}(t, M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} G\left(\frac{t - X_l}{h_{ij,M}}\right) \mathbf{1}_{\{J_{l-1}=i, J_l=j\}},$$

and

$$\widehat{H}_i(t, M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} G\left(\frac{t - X_l}{h_{i,M}}\right) \mathbf{1}_{\{J_{l-1}=i\}};$$

where $G(t) = \int_{-\infty}^t K(t)dt$, with K is a bounded kernel function.

For fixed states i and j, it should be noted that the smoothing parameter of the previous estimators depends on the sample size, so we should write $h_{ij,N_{ij}(M)} = h_{ij,M}$ (resp. $h_{i,N_i(M)} = h_{i,M}$), however we prefer to use a simpler notation.

The kernel estimator of the Markov renewal function $\Psi_{ij}(t)$, in matrix form, is given by

$$\widehat{\boldsymbol{\Psi}}(t, M) = \sum_{n=0}^{\infty} \widehat{\mathbf{Q}}^{(n)}(t, M). \tag{5}$$

The kernel estimator of the transition matrix function $\mathbf{P}(t)$ at time $t \geq 0$ of the SMP, is given by

$$\widehat{\mathbf{P}}(t, M) = \left(\widehat{\boldsymbol{\Psi}}(\cdot, M) \star \left(\mathbf{I} - \widehat{\mathbf{H}}(., M)\right)\right)(t). \tag{6}$$

Based on Equation (4), an nonparametric kernel estimator of the mean performance $\widehat{\widehat{\Phi}}(t, M)$ is given by the following expression:

$$\widehat{\widehat{\Phi}}(t, M) = \sum_{i \in E} \sum_{j \in E} \alpha_i L(j) \int_0^t \widehat{P}_{ij}(u, M) du. \tag{7}$$

5. Asymptotic properties

The following assumptions are necessary to derive the asymptotic behaviour of the kernel estimator defined in (7).

5.1. Assumptions

First, we will assume the following two assumptions:

(H.1) The EMC $(J_n)_{n \in \mathbb{N}}$ is an ergodic Markov chain, with stationary distribution ν.

(H.2) The SMP is regular, with finite mean sojourn times m.

Second, the following assumptions are required in order to establish all the asymptotic properties in this paper:

(H.3) i) $Q_{ij}(t)$ and $q_{ij}(t)$ are continuously differentiable with respect to the Lebesgue measure, and let $q_{ij}(t)$ and $q'_{ij}(t)$ be respectively their corresponding Radon-Nikodym derivatives.

ii) The derivative q'_{ij} is bounded.

(H.4) The function G is a distribution function, where its derivative is K.

(H.5) The kernel K is a density function of bounded variation such that $\lim_{x \to \infty} |xK(x)| = 0$ and $\left| \int t^j K^n(t) dt \right| < \infty$ for $j = 0, 1$, and $n = 1, 2$.

(H.6) The smoothing parameter $h_{ij,M}$ satisfies

$$\lim_{M \longrightarrow \infty} h_{ij,M} = 0 \quad \text{and} \quad \lim_{M \longrightarrow \infty} M h_{ij,M} = \infty.$$

5.2. Comments on the assumptions

The structural assumptions **(H.1)** and **(H.2)** are the same as those used classically for the semi-Markov processes framework (see, for instance [1] and [6]). More precisely, the recurrence and the positivity of the EMC $(J_n)_{n \in \mathbb{N}}$ in **(H.1)** ensure that the stationary distribution π_j defined in Definition 3 is strictly positive and unique. Furthermore, since the EMC $(J_n)_{n \in \mathbb{N}}$ is irreducible and aperiodic, the limit in Definition 3 always exists and it is independent of the distribution in the initial state. **(H.2)** means that the counting process $\{N(t) : t \geq 0\}$ has a finite number of jumps in a finite period with probability 1. In addition, under this hypothesis we have $T_n < T_{n+1}$, for any $n \in \mathbb{N}$, and $T_n \to \infty$ as n goes to infinity. Assumption **(H.3)** as imposed on $Q_{ij}(t)$ and $q_{ij}(t)$ is a regularity type hypothesis. Whereas, assumption **(H.3)(i)** is a regularity constraint using to get the strong consistency. the second derivative hypothesis **(H.3)(ii)** establishes more restrictive constraints when going through to state the asymptotic normality of our estimators. **(H.5)**-**(H.6)** are technical constraints; they are imposed for the sake of the proof's simplicity and brevity.

Before stating our main result, we introduce the following technical lemma which will be necessary to prove our second result.

Lemma 1. For $n = 1, 2$. If **(H.3)-(H.5)** hold, we have

$$\frac{1}{h_{ij,M}} \int_0^{+\infty} K^n \left(\frac{v-x}{h_{ij,M}} \right) dQ_{ij}(x) \leqslant q_{ij}(v) \int_{-\infty}^{+\infty} K^n(z)\, dz + o(h_{ij,M}).$$

Proof of Lemma 1 By using a change of variable followed by Taylor's expansion, we have

$$\begin{aligned}
\frac{1}{h_{ij,M}} \int_0^{+\infty} K^n \left(\frac{v-x}{h_{ij,M}} \right) dQ_{ij}(x) &= \int_{-\infty}^{\frac{v}{h_{ij,M}}} K^n(z)\, q_{dr}(v - h_{ij,M} z) dz \\
&= \int_{-\infty}^{\frac{v}{h_{ij,M}}} K^n(z) \left[q_{ij}(v) - h_{ij,M} z q'_{ij}(v^*) \right] dz \\
&\leqslant q_{ij}(v) \int_{-\infty}^{+\infty} K^n(z)\, dz + o(h_{ij,M}),
\end{aligned}$$

where $v - h_{ij,M} z \leq v^* \leq v$.

5.3. Main Results

Our first result concerns the uniform strong consistency of the proposed estimator.

Theorem 1. For any fixed $0 \leq t \leq M$ and $i \in E$, under **(H.1)-(H.6)**, the estimator $\widehat{\overline{\Phi}}(t, M)$ of $\overline{\Phi}(t)$ is uniformly strongly consistent, that is

$$\max_{i \in E} \sup_{t \in [0,M]} | \widehat{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) | \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

Proof of Theorem 1 The proof of this theorem is based on (5), (6), (7) and the following inequality:

$$\begin{aligned}
\max_{i \in E} \sup_{t \in [0,M]} | \widehat{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) | \leq & \sum_{j \in E} L(j) t \sum_{n=0}^{\infty} \max_{i \in E} \sup_{t \in [0,M]} \left| \widehat{Q}_{ij}^{(n)}(t, M) - Q_{ij}^{(n)}(t) \right| \\
& + \sum_{j \in E} L(j) t \sum_{n=0}^{\infty} \max_{i \in E} \sup_{t \in [0,M]} \left| \widehat{Q}_{ij}^{(n)}(t, M) - Q_{ij}^{(n)}(t) \right| * \widehat{H}_j(t, M) \\
& + \sum_{j \in E} L(j) t \max_{i \in E} \sup_{t \in [0,M]} \left| \widehat{H}_j(t, M) - H_j(t) \right| * \Psi_{ij}(t).
\end{aligned}$$

For all $i, j \in E$, $n \in \mathbb{N}^*$ and $M \in \mathbb{R}_+$, based on a straightforward adaptation of the proof of Lemma 1 in [19], we get that the estimator $\widehat{Q}_{ij}^{(n)}(t, M)$ is uniformly strong consistent in $[0, M]$. In addition, the uniform strong consistency of the kernel estimator $\widehat{H}_j(t, M)$ is stated in Theorem 4.1 of [1]. Then,

$$\max_{i \in E} \sup_{t \in [0,M]} | \widehat{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) | \xrightarrow{a.s.} 0, \quad as \quad M \longrightarrow \infty.$$

Before stating our second result, let us consider the renewal process $\left(T_n^i\right)_{n \geq 0}$ of successive times of visits to state i, then $N_i(t)$ is the counting process of renewals. Let μ_{ii} and μ_{ii}^* denote the mean first passage times of the state i in the MRP and in the corresponding Markov chain $\{J_n; n \geq 0\}$, respectively. Furthermore, μ_{ii} is the mean interarrival times of the eventual delayed renewal process $\left(T_n^i\right), n \geq 0$, i.e., $\mu_{ii} = \mathbb{E}\left[T_2^i - T_1^i\right]$, and $\mu_{ii}^* = \mathbb{E}\left[S_i^* | J_0 = i\right]$ with $S_i^* = \min\{k \geq 1, J_n = i\}$ is the first visit time to the state i.

Theorem 2. For any fixed $0 \leq t \leq M$, if **(H.1)-(H.6)** hold, we have

$$\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right] \xrightarrow{D} N\left(0,\sigma^2_{\overline{\Phi}}(t)\right), \quad as \quad M \longrightarrow \infty,$$

with $h_M = \min\limits_{i,j\in E}\{h_{ij,M}\}$ and the asymptotic variance

$$\sigma^2_{\overline{\Phi}}(t) \leqslant \sum_{i\in E}\sum_{j\in E}\mu_{ii}\int_0^t\left[\left(R_{ij}-D_i\right)^2 * \left(Q_{ij}(\cdot)\int_{-\infty}^{+\infty}K^2(z)\,dz\right)\right](u)du, \tag{8}$$

where

$$R_{ij} = \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\left(\Psi_{di} * \Psi_{jr} * \overline{H}_r\right), \tag{9}$$

and

$$D_i = \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\mathbf{1}_{\{i=r\}}\Psi_{dr}. \tag{10}$$

Proof of Theorem 2 We have,

$$\begin{aligned}\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right] &= \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t \hat{P}_{dr}(u,M)du - \int_0^t P_{dr}(u)du\right]\\ &= \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t\left[\left(\hat{\Psi}_{dr}(\cdot,M) * \left(I-\widehat{H}_r(\cdot,M)\right)\right)(u)\right.\right.\\ &\left.\left. - \left(\Psi_{dr} * (I-H_r)\right)(u)\right]du\right].\end{aligned}$$

Note that the last right side can be written as follows:

$$\begin{aligned}&\sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t\left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right) * \left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du\right.\\ &\left.+\int_0^t\left(\Psi_{dr}(\cdot) * \left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du + \int_0^t\left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right) * \overline{H}_r(\cdot)\right)(u)du\right].\end{aligned}$$

According to [1] and by following the same arguments as [18], the first term converges to zero as M tends to infinity.
Consequently, by applying Slutsky's Theorem, we deduce that $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ converges in distribution to the same limit as

$$\begin{aligned}&\sqrt{Mh_M}\sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\left[\int_0^t\left(\Psi_{dr}(\cdot) * \left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du\right.\\ &\left.+\int_0^t\left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right) * \overline{H}_r(\cdot)\right)(u)du\right].\end{aligned}$$

By combining Theorem 4.3 (i) of [1] and Theorem 4 (b)[18], along with arguments akin to those employed in [9] p. 214, we deduce that $\sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)-\mathbf{\Psi}(\cdot)\right]_{dr}(t)$ has the same limit in distribution as $\sqrt{Mh_M}[\mathbf{\Psi}(\cdot)\star\Delta\mathbf{Q}\star\mathbf{\Psi}(\cdot)]_{dr}(t)$,
where $\Delta\mathbf{Q} = (\widehat{\mathbf{Q}}-\mathbf{Q})$, for every $t \geq 0, t \leq M$, and for every $d, r \in E$, which is written as follows:

$$\Delta Q_{dr}(\cdot) = \frac{1}{N_d(M)}\sum_{l=1}^{N(M)}\left[G\left(\frac{\cdot - X_l}{h_{dr,M}}\right)\mathbf{1}_{\{J_{l-1}=d,J_l=r\}} - Q_{dr}(\cdot)\mathbf{1}_{\{J_{l-1}=d\}}\right].$$

Furthermore,

$$\Psi_{dr} * \left(\widehat{\overline{H}}_r - \overline{H}_r\right) = -\sum_{k\in E}\Psi_{dr} * \Delta Q_{rk} = -\sum_{k\in E}\sum_{m\in E}\mathbf{1}_{\{m=r\}}\Psi_{dr} * \Delta Q_{mk}.$$

Then, $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ has the same limit as

$$\frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)}\sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-X_l}{h_{mk,M}}\right)\mathbf{1}_{\{J_{l-1}=m,J_l=k\}}\right.\right.\right.$$
$$\left.\left.\left.-Q_{mk}(.)\mathbf{1}_{\{J_{l-1}=m\}}\right)\right](u)du\right],$$

where R_{mk} and D_m are given in (9) and (10).
Apply central limit theorem related to semi-Markov processes (see [25]) to the function $W_f(t)$ such that

$$\begin{aligned}W_f(t)&=\sum_{l=1}^{N(M)}f(J_{l-1},J_l,X_l)\\&=\sum_{l=1}^{N(M)}\sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-X_l}{h_{mk,M}}\right)\mathbf{1}_{\{J_{l-1}=m,J_l=k\}}\right.\right.\right.\\&\left.\left.\left.-Q_{mk}(.)\mathbf{1}_{\{J_{l-1}=m\}}\right)\right](u)du\right],\end{aligned}$$

where, for any fixed $t>0$, we have defined the function $f:E\times E\times\mathbb{R}\to\mathbb{R}$ by

$$f(i,j,x)=\sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-x}{h_{mk,M}}\right)\mathbf{1}_{\{i=m,j=k\}}-Q_{mk}(.)\mathbf{1}_{\{i=m\}}\right)\right](u)du\right].$$

In order to apply the Pyke and Schaufele's CLT, we need to compute the quantities $A_{ij}, A_i, B_{ij}, B_i, r_i, m_f, \sigma_d^2$ and then $\sigma^2_{\overline{\Phi}}(t)$, using Lemma 1 with assumptions **(H.3)-(H.5)**. We have

$$\begin{aligned}A_i&=\sum_{j\in E}A_{ij}\\&=\sum_{j\in E}\int_0^{+\infty}f(i,j,x)\,dQ_{ij}(x)\\&=\sum_{j\in E}\sum_{k\in E}\sum_{m\in E}\int_0^{+\infty}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-x}{h_{mk,M}}\right)\mathbf{1}_{\{i=m,j=k\}}\right.\right.\right.\\&\left.\left.\left.-Q_{mk}(.)\mathbf{1}_{\{i=m\}}\right)\right](u)du\right]dQ_{ij}(x)\\&=\sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\left[\int_0^t\int_0^u(R_{ij}-D_i)(u-v)\left(\frac{1}{h_{ij,M}}\int_0^{+\infty}K\left(\frac{v-x}{h_{ij,M}}\right)dQ_{ij}(x)\right)dvdu\right.\\&\left.-\sum_{k\in E}\int_0^t\int_0^u(R_{ik}-D_i)(u-v)q_{ik}(v)\int_0^{+\infty}dQ_{ij}(x)dvdu\right].\end{aligned}$$

Then

$$A_i\leqslant\sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\int_0^t\left[\left(R_{ij}-D_i\right)*\left(o\left(h_{ij,M}\right)\right)\right](u)du.$$

For B_i and by using Jensen's inequality and Lemma 1, we have

$$\begin{aligned}
B_i &= \sum_{j\in E} B_{ij} \\
&= \sum_{j\in E} \int_0^{+\infty} [f(i,j,x)]^2 dQ_{ij}(x) \\
&= \sum_{j\in E} \int_0^{+\infty} \left[\frac{M}{N_i(M)} \sqrt{h_M} \left[\int_0^t \left[\left(R_{ij} - D_i\right) * G\left(\frac{.-x}{h_{ij,M}}\right) \right.\right.\right. \\
&\left.\left.\left. - \sum_{k\in E} \left(R_{ik} - D_i\right) * Q_{ik}(.) \right] (u) du \right] \right]^2 dQ_{ij}(x) \\
&\leqslant \sum_{j\in E} \int_0^{+\infty} \left(\frac{M}{N_i(M)} \right)^2 h_M \left[\int_0^t \left[\int_0^u \left(R_{ij} - D_i\right)(u-v) \left(\frac{1}{h_{ij,M}} K\left(\frac{v-x}{h_{ij,M}}\right) \right) dv \right.\right. \\
&\left.\left. - \sum_{k\in E} \int_0^u \left(R_{ik} - D_i\right)(u-v) q_{ik}(v) dv \right]^2 du \right] dQ_{ij}(x).
\end{aligned}$$

Then

$$\begin{aligned}
B_i \leqslant \sum_{j\in E} \left(\frac{M}{N_i(M)} \right)^2 h_M \int_0^{+\infty} \left[\int_0^t \int_0^u \left[\left(R_{ij} - D_i\right)^2 (u-v) \left(\frac{1}{h_{ij,M}^2} K^2\left(\frac{v-x}{h_{ij,M}}\right) \right) \right.\right. \\
+ \sum_{k\in E} \left(R_{ik} - D_i\right)^2 (u-v) q_{ik}^2(v) \\
\left.\left. -2 \sum_{k\in E} \left(R_{ik} - D_i\right)\left(R_{ij} - D_i\right)(u-v) q_{ik}(v) \frac{1}{h_{ij,M}} K\left(\frac{v-x}{h_{ij,M}}\right) \right] dv du \right] dQ_{ij}(x).
\end{aligned}$$

Since $\frac{N_i(M)}{M} \overset{a.s.}{\to} \frac{1}{\mu_{ii}}$ (see [13]), when $M \to +\infty$ and from the assumption **(H.6)**, the second and the third term in the last inequality converge to zero, we get

$$B_i \leqslant \sum_{j\in E} \mu_{ii}^2 \int_0^t \left[\left(R_{ij} - D_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right] (u) du.$$

Furthermore,

$$\begin{aligned}
r_d &= \sum_{i\in E} A_i \frac{\mu_{dd}^*}{\mu_{ii}^*} = 0 \qquad \text{as} \qquad M \to \infty, \\
m_f &= \frac{1}{\mu_{dd}} r_d = 0 \qquad \text{as} \qquad M \to \infty, \\
\sigma_\Phi^2(t) &= \frac{1}{\mu_{dd}} \sigma_d^2(t),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_d^2(t) &= \sum_{i\in E} B_i \frac{\mu_{dd}^*}{\mu_{ii}^*} \\
&\leqslant \mu_{dd}^* \sum_{i\in E} \sum_{j\in E} \frac{\mu_{ii}^2}{\mu_{ii}^*} \int_0^t \left[\left(R_{ij} - D_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right] (u) du.
\end{aligned}$$

Then, since $\mu_{ii}^* = \frac{1}{v_i}$ (see [10]) and $\mu_{ii} = \frac{\bar{m}}{v_i}$ (see [13]), where $\bar{m} = \sum_{i\in E} m_i v_i$ is the mean sojourn time of the MRP; we have

$$\sigma_\Phi^2(t) \leqslant \sum_{i\in E} \sum_{j\in E} \mu_{ii} \int_0^t \left[\left(R_{ij} - D_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right] (u) du.$$

We obtain from the CLT that $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ converges in distribution, as M tends to infinity, to a zero-mean normal random variable, of variance $\sigma^2_{\overline{\Phi}}(t)$ given in (8).

6. Applications

The cumulative operational time is considered as one of the most relevant performance measures for the reliability. In this section, we propose a non-parametric kernel estimator of the expected cumulative operational time for the semi-Markov system. Then, we investigate the asymptotic properties of the proposed estimator, namely, the strong consistency and the asymptotic normality. As an illustration example, we apply the previous results to three-state continuous time SMP.

6.1. The cumulative operational time

The state space E is often split into two subsets for reliability research. The first one, let's say U, is made up of up states, whereas the second one, let's say D, is made up of down states. The start of an essential event, such as a component failure related to some reason or a complete repair, might well be associated with the transition into a state. Since we suppose that the system can be fixed, the process alternates between U and D.

The cumulative operational time is defined by

$$W(t)=\int_0^t \mathbf{1}_{\{Z_u\in U\}}du.$$

It represents the total time that the semi-Markov process Z spends in the set of up states U over the interval $[0,t]$.

Making use of the assumptions **(H.1) - (H.2)**, along with the aid of the arguments used in [8], we obtain the following result:

$$\lim_{t\to+\infty}\frac{W(t)}{t}=\frac{\sum_{j\in U}\nu_j m_j}{\sum_{k\in E}\nu_k m_k},$$

where $m_j=\int_0^\infty(1-H_j(t))dt$ is the mean sojourn time in state j.

The quantity we aim to analyze is the expected cumulative operational time of a semi-Markov system, denoted by $\overline{W}(t):=\mathbb{E}[W(t)]$. Which is given by

$$\overline{W}(t)=\sum_{i\in E}\alpha_i\overline{W}_i(t)=\sum_{i\in E}\sum_{j\in U}\alpha_i\int_0^t P_{ij}(u)du, \tag{11}$$

where $\overline{W}_i(t)=\sum_{j\in U}\int_0^t P_{ij}(u)du$.

The expected cumulative operational time serves as a crucial indicator in maintenance studies, facilitating the calculation of average system availability, cf. [21], which is expressed as

$$\overline{A}(t)=\frac{1}{t}\overline{W}(t)=\frac{1}{t}\sum_{i\in E}\sum_{j\in U}\int_0^t P_{ij}(u)du.$$

From the definition of the expected cumulative operational time $\overline{W}(t)$ given in Equation (11), and based on a sample path truncated to the time interval $[0,M]$ of the process, the nonparametric kernel estimator $\widehat{\overline{W}}(t,M)$ is given by

$$\widehat{\overline{W}}(t,M)=\sum_{i\in E}\sum_{j\in U}\alpha_i\int_0^t\widehat{P}_{ij}(u,M)du. \tag{12}$$

The asymptotic properties of the proposed estimator are gathered in the following two corollaries.

Corollary 1. For any fixed $0 \leq t \leq M$, under the same assumptions of Theorem 1, the estimator of the expected operational time, $\widehat{\overline{W}}(t,M)$ is strongly consistent, that is

$$\sup_{t\in[0,M]} \mid \widehat{\overline{W}}(t,M) - \overline{W}(t) \mid \xrightarrow{a.s.} 0 \quad \text{as} \qquad M \to \infty.$$

Proof of Corollary 1
This corollary is a particular case of Theorem 1 and then the proof is omitted.

The following result concerns the asymptotic normality of the proposed estimator.

Corollary 2. For any fixed $0 \leq t \leq M$, we have $\sqrt{Mh_M}\left[\widehat{\overline{W}}(t,M) - \overline{W}(t)\right]$ converges in law, as M tends to infinity, to a zero mean normal random variable with the asymptotic variance

$$\sigma^2_{\overline{W}}(t) \leqslant \sum_{i\in E}\sum_{j\in U} \mu_{ii} \int_0^t \left[\left(Y_{ij} - C_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right](u)du,$$

where

$$Y_{ij} = \sum_{d\in E}\sum_{r\in U} \alpha_d \left(\Psi_{di} * \Psi_{jr} * \overline{H}_r\right) \quad \text{and} \quad C_i = \sum_{d\in E}\sum_{r\in U} \alpha_d \mathbf{1}_{\{i=r\}} \Psi_{dr}.$$

Proof of Corollary 2
The proof of this result is based on the same arguments as in the proof of Theorem 2.

6.2. Confidence interval

The main purpose of the confidence interval is to supplement the estimate at a point with information about the uncertainty in this estimate. It is considered as a direct application of the Central Limit Theorem. In order to provide a confidence interval for the expected cumulative operational time $\overline{W}(t)$, we need first to propose a consistent estimator of the variance $\sigma^2_{\overline{W}}(t)$. A natural consistent estimator of this variance, denoted by $\widehat{\sigma}^2_{\overline{W}}(t,M)$, is obtained by estimating the parameters involved in this quantity such as $Q_{mk}(t), \overline{H}_j(t)$ and $\Psi_{im}(t)$.
Indeed, from the strong consistency of $\widehat{Q}_{mk}(t,M), \widehat{\overline{H}}_j(t,M)$ and $\widehat{\Psi}_{im}(t,M)$, (see the proof of Theorems 1 and 2 as well as Theorem 4.1 and Theorem 4.2 (v) in [1]), we deduce the strong consistency of $\widehat{\sigma}^2_{\overline{W}}(t,M)$.
Consequently, from Corollary 2, we get

$$\sqrt{Mh_M}\left[\widehat{\overline{W}}(t,M) - \overline{W}(t)\right] \xrightarrow{D} N\left(0, \widehat{\sigma}^2_{\overline{W}}(t,M)\right).$$

Then

$$\frac{\sqrt{Mh_M}}{\widehat{\sigma}_{\overline{W}}(t,M)}\left[\widehat{\overline{W}}(t,M) - \overline{W}(t)\right] \xrightarrow{D} N(0,1).$$

Hence, for $\alpha \in (0,1)$, an asymptotic $100(1-\alpha)\%$ confidence interval for $\widehat{\overline{W}}(t,M)$ can be straightforwardly computed:

$$I = \left(\widehat{\overline{W}}(t,M) \pm z_{\frac{\alpha}{2}} \frac{\widehat{\sigma}_{\overline{W}}(t,M)}{\sqrt{Mh_M}}\right)$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

6.3. Numerical application

To validate our results, we consider a three state system whose state transition diagram is given in Figure 1. States 1 and 2 are up states and state 3 is a down state.
We have two exponential and two Weibull distribution functions as conditional transitions, for all $x \geq 0$, say $H_{12}(x) = 1 - \exp(-\lambda_1 x)$, $H_{31}(x) = 1 - \exp(-\lambda_2 x)$,
$H_{23}(x) = 1 - \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_2}\right]$, $H_{21}(x) = 1 - \exp\left[-\left(\frac{x}{a_1}\right)^{\beta_1}\right]$.
The parameters of these distributions are: $\lambda_1 = 0.1, \lambda_2 = 0.2, \alpha_1 = 0.3, \beta_1 = 2, \alpha_2 = 0.1, \beta_2 = 2$.

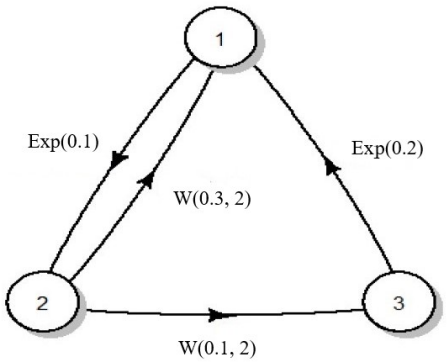

Figure 1: *A three state semi-Markov system.*

The transition probability matrix of the embedded Markov chain (J_n) is:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}$$

Where the system is defined by the initial distribution $\alpha = (1/3, 1/3, 1/3)$.

To construct the kernel estimator for the mean performance of a continuous-time semi-Markov process. The smoothed function $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{4}\left(1 - u^2\right)$ for $|u| \leq 1$ and the cumulative distribution function $G(u) = \int_{-\infty}^{u} \frac{3}{4}\left(1 - z^2\right) \mathbf{1}_{[-1,1]}(z)\mathrm{d}z$. The bandwidth h_M has been obtained by the "PBbw" method, which computes the plug-in bandwidth of the Polansky and Baker method, cf. [22]. We have considered that the observation period is the interval $[0, M]$ with $M = 20000$.

Figure 2 gives a comparison between the kernel estimator of the mean performance for different sample sizes ($M = 2000, M = 10000$ and $M = 20000$). We observe that this estimator converges to the true value of the mean performance as M increases.

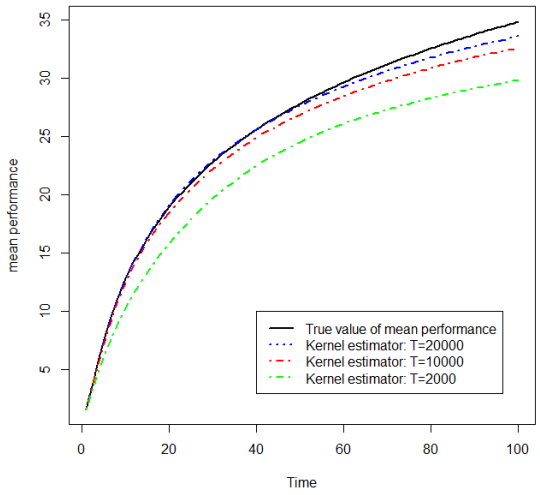

Figure 2: *Comparison between the kernel estimator of the mean performance for different sample sizes and the true value.*

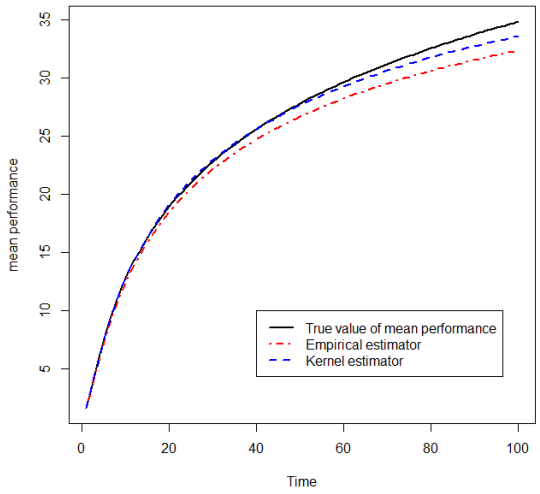

Figure 3: *Comparison between the true values of the mean performance and their estimators (empirical and kernel).*

Figure 3 gives a comparison between the empirical estimator (see [14]) and our kernel estimator of the mean performance. We remark easily that, our method provides better results than the empirical one.

7. Concluding remarks

The application of the nonparametric kernel approach to estimate the mean performance of a continuous-time semi-Markov process is the main element of the work described in this paper. We have proposed a kernel estimator for this quantity then we have provided its asymptotic properties, such as the uniform strong consistency, as well as the asymptotic normality.
Compared to the empirical estimator, the use of this kernel technique approach has a number of benefits. Since the empirical function is always a discontinuous function, the kernel smoothing in particular prevents discontinuities in this function. As a result, the empirical distribution may be considered a poor approximation when knowing that the underlying distribution is continuous. To the best of our knowledge, no limit theorems have been obtained for functionals of homogeneous semi-Markov processes, such as the performance and the related quantities, by using the kernel approach. In particular, we have made an important connection of our results with the reliability theory by focusing on the cumulative operational time of the semi-Markov systems. This crucial indicator is the total time spent by the process in the set of operational states during a specific time interval. It is used to minimize the expected cost of the maintenance process. The uniform strong consistency and asymptotic normality have been stated. In addition, a confidence interval has been constructed. Moreover, a simulation study has been conducted in order to highlight to the efficiency as well as the superiority of our method to the standard empirical method.

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Conflict of interests

The authors declare that there is no conflict of interest.

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CHARACTERIZATION OF SOME GENERALIZED DISTRIBUTIONS USING ORDER STATISTICS

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Abstract

The Lindley distribution has been useful for fitting lifetime data. In recent times, several authors studied the extension of the original Lindley distribution. In this paper, we introduced the two general classes of distributions, which include all earlier versions of Lindley distributions. These general classes are characterized using conditional expectations of order statistics. Further, there results are applied to characterize several known distributions like Lindley, X-Lindley, power Lindley, Lindley-Pareto, Ailamujia, power Ailamujia, Lindley-Weibull, length-biased exponential, inverse Lindley, inverse power Lindley and inverted length biased exponential distributions.

Keywords: Order statistics, characterization, conditional expectation, continuous distributions.

1. Introduction

Lindley distribution is a combination of exponential and gamma distributions. The one parameter Lindley distribution, initially introduced by [1]. This distribution has been made extensively due to its practical applications, ease of implementation, and suitability for modelling lifetime data. Characterization of distribution plays a crucial role in understanding the behaviour of the distributions. The theoretical aspect of probability distributions is the core and significant concept of statistics, as it is the basis for various statistical models. There are several methods to characterize the probability distributions. Some are the moment-generating function, entropy function, conditional expectation, recurrence relation, etc. The literature is rich with studies focusing on characterization via conditional expectations. For instance, [2] characterized a generalized class of distributions using the conditional expectation of order statistics, while [3] explored conditioning on pairs of order statistics. Franco and Ruiz [4] gave some general results of characterization based conditional expectation of adjacent order statistics, whereas [5] studied the characterizations based on conditional expectations of the doubled truncated distribution. Balasubramanian and Dey [6] characterized both absolutely continuous random variables and discrete random variables using conditional expectation. Su and Huang [7] obtained a relationship between failure and conditional expectations, along with characterization results based on conditional expectations. Khan and Abouammoh [8] extended the results of [2] to cases where the conditional order statistic may not be adjacent. Moreover, [9] characterized distributions through linear regression of non-adjacent generalized order statistics. Gupta and Ahsanullah [10] developed characterization results based on the conditional expectation of a function of non-adjacent order statistics. Noor and Athar [15] characterized two general classes of distributions through conditional expectation of power of difference of two record statistics.

Ahsanullah et al. [17] established some new characterization results of continuous distributions by truncated moments, while [21] established characterization results for two general forms of distributions by truncated moments. Furthermore, [22] examined characterization results based on conditional expectations of function of random variables when truncation is from both the left and right sides, whereas [24] developed characterization results for three general classes of continuous distributions using double truncated moments.

In this paper, we present two generalized classes of distributions, which include several known distributions such as the one-parameter Lindley, X-Lindley, power Lindley, Lindley pareto, Ailamujia, power Ailamujia, Lindley-Weibull, length-biased exponential distribution, and others. We aim to characterize these classes through conditional expectations of order statistics, contributing to the ongoing research in statistical distribution characterization. This approach not only enriches the theoretical understanding of these distributions but also enhances their applicability in various fields.

1.1. Conditional Distribution of Order Statistics

Let $X_i, i = 1, 2, 3, \ldots, n$ be the independent and continuous random variables $(r.vs.)$ having probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Then, the conditional distribution of $X_{s:n}$ given $X_{r:n} = x$ is given by

$$\frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{(F(y)-F(x))^{s-r-1}(1-F(y)^{n-s})f(y)}{(1-F(x))^{n-r}}, x < y \tag{1}$$

and the conditional pdf of $X_{r:n}$ given $X_{s:n} = y$, $1 \leq r < s \leq n$, is

$$\frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{(F(x))^{r-1}(F(y)-F(x))^{s-r-1}f(x)}{(F(y))^{s-1}}, x < y \tag{2}$$

See [12].

1.2. Proposed Generalized Classes of Distributions

The Lindley distribution has been useful for its simplicity and ability to model lifetime data effectively. Ghitany *et al*. [11] investigated the properties of the one-parameter Lindley distribution and demonstrated its flexibility in fitting lifetime data better than the exponential distribution. Their work highlighted the practical utility of the Lindley distribution in real-world applications. Ghitany *et al*. [13] extended the Lindley distribution to a two-parameter form, enhancing its flexibility and fitting ability. Moreover, [14] introduced the two-parameter extension, offering additional parameters to capture more complex data behaviors. Further, [20] introduced a new two-parameter version of the Lindley distribution, which showed an improvement to fit skewed real data compared to the inverse Lindley distribution, introduced by [16]. In this paper, we introduced two generalized classes of distributions, named Haseeb generalized Lindley (HGL) and Haseeb Generalized inverse Lindley (HGIL) class of distributions. These generalized classes include all the distributions belonging to Lindley and inverse Lindley families.

1.2.1 Haseeb Generalized Lindley (HGL) Class of Distributions

Let $X \in (\alpha, \beta)$ be a $r.v.$ having pdf $f(x)$ and cdf $F(x)$, then X is said to follow HGL class of distribution, if cdf is given as

$$F(x) = 1 - (b + ah(x))e^{-ch(x)}, x \in (\alpha, \beta) \tag{3}$$

and the corresponding pdf is

$$f(x) = \{c(b + ah(x)) - a\}h'(x)e^{-ch(x)}, x \in (\alpha, \beta) \tag{4}$$

Table 1: *Sub-model of HGL class of distribution*

Distribution	$F(x)$	a	b	c	$h(x)$
One Parameter Lindley	$1-\big(1+\frac{\theta}{1+\theta}x\big)e^{-\theta x};\theta,x>0$	$\frac{\theta}{1+\theta}$	1	θ	x
X-Lindley	$1-\big(1+\frac{\theta x}{(1+\theta)^2}\big)e^{-\theta x};\theta,x>0$	$\frac{\theta}{(1+\theta)^2}$	1	θ	x
Power Lindley	$1-\big(1+\frac{\theta x^{\alpha}}{1+\theta}\big)e^{-\theta x^{\alpha}};\theta,x>0$	$\frac{\theta}{1+\theta}$	1	θ	x^{α}
Lindley Pareto	$1-\frac{\alpha^p+x^p\theta}{(1+\theta)\alpha^p}e^{-\theta\frac{(x^p-\alpha^p)}{\alpha^p}};\theta,\alpha>0,x>0$	$\frac{\theta e^{\theta}}{1+\theta}$	$\frac{e^{\theta}}{1+\theta}$	θ	$\big(\frac{x}{\alpha}\big)^p$
Power Ailamujia	$1-(1+\theta x^{\beta})e^{-\theta x^{\beta}};\theta>0,x>0$	θ	1	θ	x^{β}
Lindley-Weibull	$1-\big(1+\frac{\theta(\alpha x)^{\beta}}{1+\theta}\big)e^{-\theta(\alpha x)^{\beta}};\alpha,\theta,\beta>0,x>0$	$\frac{\theta}{1+\theta}$	1	θ	$(\alpha x)^{\beta}$
Length-Biased Exponential	$1-\big(1+\frac{x}{\theta}\big)e^{-\frac{x}{\theta}};\theta>0,x>0$	$\frac{1}{\theta}$	1	$\frac{1}{\theta}$	x

1.2.2 Haseeb Generalized Inverse Lindley (HGIL) Class of Distributions

Let $f(x)$ and $F(x)$ be the pdf and cdf of a continuous $r.v.$ X respectively, then X is said to follow HGIL class of distribution, if its cdf is given by

$$F(x)=(b+ah(x))e^{-ch(x)},x\in(\alpha,\beta) \tag{5}$$

and corresponding pdf is

$$f(x)=\{a-c(b+ah(x))\}e^{-ch(x)}h'(x),x\in(\alpha,\beta) \tag{6}$$

Table 2: *Sub-model of HGIL class of distribution*

Distribution	$F(x)$	a	b	c	$h(x)$
Inverse Lindley	$\big(1+\frac{\theta}{(1+\theta)x}\big)e^{-\frac{\theta}{x}};\theta>0,x>0$	$\frac{\theta}{1+\theta}$	1	θ	$\frac{1}{x}$
Inverse Power Lindley	$\big(1+\frac{\theta}{(1+\theta)x^{\alpha}}\big)e^{-\frac{\theta}{x^{\alpha}}};\alpha,\theta>0,x>0$	$\frac{\theta}{1+\theta}$	1	θ	$\frac{1}{x^{\alpha}}$
Inverted LBE	$\big(1+\frac{1}{\theta x}\big)e^{-\frac{1}{\theta x}};\theta>0,x>0$	$\frac{1}{\theta}$	1	$\frac{1}{\theta}$	$\frac{1}{x}$

2. Characterization Results

Theorem 1. Let X be the continuous $r.v.$ having cdf $F(x)$ and pdf $f(x)$. Suppose $F(x)$ is defined for all $x\in(\alpha,\beta)$, with boundary conditions $F(\alpha)=0$ and $F(\beta)=1$, then for $1\leq r<r+1\leq$n

$$E[h(X_{r+1:n})|X_{r:n}=x]=g_{r+1|r}(x)=h(x)+\left[\frac{ae^{\frac{cb}{a}}}{1-F(x)}\right]^{n-r}\frac{\Gamma[n-r+1,\frac{c(n-r)}{a}(b+ah(x))]}{[c(n-r)]^{n-r+1}}, \tag{7}$$

if and only if for $n-r>0$

$$1-F(x)=(b+ah(x))e^{-ch(x)}\ ,x\in(\alpha,\beta). \tag{8}$$

where, $h(x)$ is a continuous and differentiable function of x on (α,β), $a\neq 0$ and $\Gamma(n,x)=\int_x^{\infty}u^{n-1}e^{-u}du$ be the upper incomplete gamma function.

Proof. Necessary part: In view of (1), we have

$$E[h(X_{r+1})|X_{r:n}=x]=\frac{(n-r)}{(1-F(x))^{n-r}}\int_x^{\beta}h(y)\big(1-F(y)\big)^{n-r-1}f(y)dy.$$

Integrating right hand side of the above expression by parts, then we get

$$= h(x) + \frac{1}{(1-F(x))^{n-r}} \int_x^\beta h'(y)(b+ah(y))^{n-r} e^{-c(n-r)h(y)} dy$$

$$= h(x) + \frac{1}{(1-F(x))^{n-r}} \times I(x) \tag{9}$$

where,

$$I(x) = \int_x^\beta h'(y)(b+ah(y))^{n-r} e^{-c(n-r)h(y)} dy.$$

Let, $t = b + ah(y)$, implies $dt = ah'(y)dy$, then we have

$$I(x) = \frac{e^{c(n-r)b}}{a} \int_t^\infty t^{n-r} e^{-\frac{c(n-r)t}{a}} dt.$$

Again, substitute $u = \frac{c(n-r)t}{a}$, then

$$I(x) = \frac{e^{\frac{c(n-r)b}{a}}}{c(n-r)} \times \left[\frac{a}{c(n-r)}\right]^{n-r} \int_{\frac{c(n-r)(b+ah(x))}{a}}^\infty u^{n-r+1-1} e^{-u} du$$

$$= \frac{e^{\frac{c(n-r)b}{a}}}{c(n-r)} \times \left[\frac{a}{c(n-r)}\right]^{n-r} \Gamma[n-r+1, \frac{c(n-r)}{a}(b+ah(x))]. \tag{10}$$

Now, substitute (10) in (9) to prove the necessary part.

Sufficient part: From (7), we have

$$(1-F(x))^{n-r} g_{r+1|r}(x) = (1-F(x))^{n-r} h(x) + \frac{(ae^{\frac{bc}{a}})^{n-r}}{[c(n-r)]^{n-r+1}} \times \int_{\frac{c(n-r)(b+ah(x))}{a}}^\infty u^{n-r} e^{-u} du. \tag{11}$$

Differentiating both sides of (11) w.r.t. x, then we have

$$\begin{aligned} g'_{r+1|r}(x)(1-F(x))^{n-r} - (n-r)g_{r+1|r}(x)(1-F(x))^{n-r-1} f(x) &= h'(x)(1-F(x))^{n-r} \\ -(n-r)h(x)(1-F(x))^{n-r-1} f(x) - e^{-c(n-r)h(x)} h'(x) & \end{aligned} \tag{12}$$

Also, we have

$$g_{r+1|r}(x) = \frac{(n-r)}{(1-F(x))^{n-r}} \int_x^\beta h(y)(1-F(y))^{n-r-1} f(y) dy$$

or

$$g_{r+1|r}(x)(1-F(x))^{n-r} = (n-r) \int_x^\infty h(y)(1-F(y))^{n-r-1} f(y) dy \tag{13}$$

Now, differentiate both the sides of (13) w.r.t. x to get

$$\begin{aligned} g'_{r+1|r}(x)(1-F(x))^{n-r} - (n-r)g_{r+1|r}(x)(1-F(x))^{n-r-1} f(x) &= -(n-r)h(x) \times \\ & (1-F(x))^{n-r-1} f(x). \end{aligned} \tag{14}$$

Now on comparison of (12) with (14), we get

$$(1-F(x))^{n-r} = (b+ah(x))^{n-r} e^{-c(n-r)h(x)}$$

or

$$\ln(1-F(x)) = \ln(b+ah(x)) - ch(x)$$

Differentiating both sides of the above expression w.r.t. x, we get

$$-\frac{f(x)}{1-F(x)} = \frac{ah'(x)}{b+ah(x)} - ch'(x)$$

or

$$1-F(x) = \frac{b+ah(x)}{h'(x)\{c(b+ah(x))-a\}}f(x)$$

or

$$\frac{f(x)}{1-F(x)} = \{ch'(x) - \frac{ah'(x)}{b+ah(x)}\}. \tag{15}$$

Now integrating both sides of (15), leads to the sufficiency part. ■

Corollary 1. Under the similar conditions as stated in Theorem 1

$$E[h(x)|X \geq x] = h(x) + \frac{c(b+ah(x))+a}{c^2(b+ah(x))}. \tag{16}$$

Proof. Corollary can be proved by substituting $r = n-1$ in (7). ■

Theorem 2. Let X be the continuous *r.v.* having *cdf* $F(x)$ and *pdf* $f(x)$. Suppose $F(x)$ is defined over $x \in (\alpha, \beta)$, with boundary conditions $F(\alpha) = 0$ and $F(\beta) = 1$. Then for $1 \leq r < r+1 \leq n$

$$E[h(X_{r:n})|X_{r+1:n}] = g_{r|r+1}(x) = h(x) - \frac{1}{rc}\left[\frac{ae^{bc/a}}{rcF(x)}\right]^r \gamma\left(r+1, \frac{rc}{a}(b+ah(x))\right), \tag{17}$$

if and only if

$$F(x) = [b+ah(x)]e^{-ch(x)}; x \in (\alpha, \beta), \tag{18}$$

where, h(x) is a continuous and differentiable function of x on (α, β) and $a \neq 0$ and $\gamma(a,x) = \int_0^x u^{a-1}e^{-u}du$ be the lower incomplete gamma function.

Proof. To prove necessary part, in view of (2), we have

$$E(h(X_{r:n})|X_{r+1:n} = x) = \frac{r}{F^r(x)}\int_\alpha^x h(y)F^{r-1}(y)f(y)dy,$$

$$= h(x) - \frac{1}{F^r(x)}\int_\alpha^x h'(y)(b+ah(y))^r e^{-rch(y)}dy$$

$$= h(x) - \frac{1}{F^r(x)}I(x), \tag{19}$$

where,

$$I(x) = \int_\alpha^x h'(y)(b+ah(y))^r e^{-rch(y)}dy.$$

Let $t = b+ah(y)$, which implies $dt = ah'(y)dy$. Then

$$I(x) = \frac{1}{a}e^{\frac{rbc}{a}}\int_0^{b+ah(x)} t^r e^{-\frac{rct}{a}}.$$

Again suppose $u = \frac{rct}{a}$, then we get

$$I(x) = \frac{1}{rc}\left[\frac{ae^{bc/a}}{rc}\right]^r \gamma(r+1, \frac{rc}{a}(b+ah(x))). \tag{20}$$

Now, substitute (20) into (19) and simplify, this proves the necessary parts.

To prove sufficiency part, in view of (17), we have

$$g_{r|r+1}(x) = h(x) - \frac{1}{rc}\left[\frac{ae^{bc/a}}{rc}\right]^r \gamma(r+1, \frac{rc}{a}(b+ah(x)))$$

or

$$F^r(x)g_{r|r+1}(x) = F^r(x)h(x) - \frac{1}{rc}\left[\frac{ae^{bc/a}}{rc}\right]^r \int_0^{\frac{rc(b+ah(x))}{a}} u^r e^{-u} du. \tag{21}$$

Differentiate both the sides of (21) w.r.t x,

$$rF^{r-1}(x)f(x)g_{r|r+1}(x) + g'_{r|r+1}(x)F^r(x) = h'(x)F^r(x) - (b+ah(x))^r e^{-rch(x)}h'(x) + rF^{r-1}(x)f(x)h(x). \tag{22}$$

Again

$$g_{r|r+1}(x) = \frac{r}{F^r(x)} \int_\alpha^x h(y)F^{r-1}(y)f(y)dy$$

or

$$F^r(x)g_{r|r+1}(x) = r\int_\alpha^x h(y)F^{r-1}(y)f(y)dy \tag{23}$$

Now differentiate both sides of (23) w.r.t x, then we get

$$rF^{r-1}(x)f(x)g_{r|r+1}(x) + F^r(x)g'_{r|r+1}(x) = rh(x)F^{r-1}(x)f(x) \tag{24}$$

Now after comparing (22) with (24), we have

$$F^r(x) = (b+ah(x))^r e^{-rch(x)}$$

Taking log on both the sides of above expression, we have

$$r\ln F(x) = r\ln(b+ah(x)) - rch(x) \tag{25}$$

Differentiating (25) w.r.t. x, we get

$$\frac{f(x)}{F(x)} = \frac{ah'(x)}{b+ah(x)} - ch'(x) \tag{26}$$

Hence, the cdf as given in (18). ■

Corollary 2. For the condition as stated in Theorem 2 and $r = 1$,

$$E[h(x)|X \le x] = h(x) + \frac{c(b+ah(x)) + a - ae^{\frac{c(b+ah(x))}{a}}}{c^2(b+ah(x))}. \tag{27}$$

3. Applications

3.1. Applications of Characterization Theorem 1

Lindley Distribution

Corollary 3. Let X be a continuous *r.v.* with *cdf* $F(x)$ and *pdf* $f(x)$. Further, suppose that $E(X)$ exists. Then

$$E(X_{r+1:n}|X_{r:n} = x) = x + \frac{\Gamma[n-r+1,(n-r)(1+\theta+\theta x)]}{\theta(n-r)^{n-r+1}(1+\theta+\theta x)^{n-r}e^{-(n-r)(1+\theta+\theta x)}}, \tag{28}$$

if and only if

$$F(x) = 1 - \left(1 + \frac{\theta}{1+\theta}x\right)e^{-\theta x}; \theta > 0, x > 0. \tag{29}$$

Proof. To prove the necessary part, we compare (29) with (8), and get $a = \frac{\theta}{1+\theta}, b = 1, h(x) = x, c = \theta$. Now, in view of (7) we get (28).

To prove the sufficiency part, using (15), we get

$$\frac{f(x)}{1-F(x)} = \left\{\theta - \frac{\frac{\theta}{1+\theta}}{1+\frac{\theta x}{1+\theta}}\right\} \tag{30}$$

Now, integrate both the sides of (30), which leads to the *cdf* given in (29). ■

Similar result was obtained by [18].

Remark 1: In view of Corollary 1, we get the characterization result for truncated moment. That is

$$E(X|X \geq x) = x + \frac{2+\theta+\theta x}{\theta(1+\theta+\theta x)}.$$

The similar result was also obtained by [19].

X-Lindley Distribution

Corollary 4. Under the conditions as stated in Corollary 2

$$E(X_{r+1:n}|X_{r:n} = x) = x + \frac{\Gamma[n-r+1,(n-r)((1+\theta)^2+\theta x)]}{\theta((1+\theta)^2+\theta x)^{n-r}e^{-(\theta x+(1+\theta)^2)(n-r)}(n-r)^{n-r+1}}, \tag{31}$$

if and only if

$$F(x) = 1 - \left(1 + \frac{\theta}{(1+\theta)^2}x\right)e^{-\theta x}, \theta > 0, x > 0. \tag{32}$$

Proof. First, we shall prove the necessary part. On comparison of (32) with (8), we get $a = \frac{\theta}{(1+\theta)^2}, b = 1, c = \theta, h(x) = x$. Now, in view of (7), we get (31). Hence, the necessary part is true.

To prove sufficient part, in view of (15) we have

$$\frac{f(x)}{1-F(x)} = \left\{\theta - \frac{\frac{\theta}{(1+\theta)^2}}{1+\frac{\theta}{(1+\theta)^2}x}\right\}. \tag{33}$$

This implies the cdf of X-Lindley distribution as given in (32). ■

Remark 2: In view of Corollary 1, we have

$$E(X|X \geq x) = x + \frac{(1+\theta)^2 + \theta x + 1}{\theta[(1+\theta)^2 + \theta x]}. \tag{34}$$

This characterizing result was also obtained by [23].

Power Lindley Distribution

Corollary 5. Let X be a continuous $r.v$ having cdf $F(x)$ and pdf $f(x)$. Let $E(X^\alpha)$ exists, then

$$E(X^\alpha_{r+1:n}|X_{r:n} = x) = x^\alpha + \frac{\Gamma[n-r+1, (n-r)(1+\theta+\theta x^\alpha)]}{\theta(n-r)^{n-r+1}e^{-(n-r)(1+\theta x^\alpha+\theta)}(1+\theta+\theta x^\alpha)^{n-r}}, \tag{35}$$

if and only if

$$F(x) = 1 - \left(1 + \frac{\theta}{1+\theta}x^\alpha\right)e^{-\theta x^\alpha}, \theta > 0, x > 0. \tag{36}$$

Proof. Necessary part: By comparing (36) with (8) we get

$a = \frac{\theta}{1+\theta}, b = 1, c = \theta, h(x) = x^\alpha$. Now on application of (7), we get (35).

Sufficient part: In view of (15), we have

$$\frac{f(x)}{1-F(x)} = \left\{\alpha\theta x^{\alpha-1} - \frac{\frac{\theta\alpha x^{\alpha-1}}{1+\theta}}{1+\frac{\theta}{1+\theta}x^\alpha}\right\}, \tag{37}$$

which gives cdf of power Lindley distribution as given in (36). ■

Remark 3: The result for characterization based on truncated moment can be seen in view of Corollary 1 as below:

$$E(X|x \geq x) = x^\alpha + \frac{2+\theta+\theta x^\alpha}{1+\theta+\theta x^\alpha}.$$

Lindley Pareto Distribution

Corollary 6. Under the similar condition as given in Corollary 5,

$$E(X^p_{r+1:n}|X_{r:n} = x) = x^p + \frac{\alpha^p[(\alpha+\theta x^p)e^{\frac{\theta(x^p-\alpha^p)}{\alpha^p}+\theta+1}]^{n-r}\Gamma[n-r+1, \frac{(n-r)(\alpha^p+\theta x^p)}{\alpha^p}]}{\theta(n-r)^{n-r+1}}, \tag{38}$$

if and only if

$$F(x) = 1 - \frac{(\alpha^p + x^p\theta)}{(1+\theta)\alpha^p}e^{-\theta\left(\frac{x^p}{\alpha^p}-1\right)}. \tag{39}$$

Proof. Corollary 6 can be proved on the lines of Corollary 5. ■

Remark 4: The characterization result for truncated moment can be obtained on application of Corollary 1.

$$E(X^p|X \geq x) = x^p + \frac{(\alpha^p+\theta x^p)(2\alpha^p+\theta x^p)}{\theta}.$$

Power Ailamujia Distribution

Corollary 7. Let X be a continuous *r.v.* having $pdf\ f(x)$ and $cdf\ F(x)$, then

$$E(X^{\beta}_{r+1:n}|X_{r:n}=x)=x^{\beta}+\frac{\Gamma[n-r+1,(n-r)(1+\theta x^{\beta})]}{(n-r)^{n-r+1}\theta(1+\theta x^{\beta})^{n-r}e^{-(n-r)\theta x^{\beta+1}}}, \tag{40}$$

if and only if

$$F(x)=1-(1+\theta x^{\beta})e^{-\theta x^{\beta}};\theta>0,x>0. \tag{41}$$

Proof. Necessary part: On comparing of (41) with (8), we get $a=c=\theta, b=1, h(x)=x^{\beta}$. Now on application of (7), we get (40).

To prove sufficient part, In view of (15), we have

$$\frac{f(x)}{1-F(x)}=\theta\beta x^{\beta-1}-\frac{\theta\beta x^{\beta-1}}{1+\theta x^{\beta}}. \tag{42}$$

Now integrate both the sides of (42) *w.r.t.* x to get (41). ■

Remark 5: For $\beta=1$ in (40). we get the result for Ailamujia distribution.

$$E(X_{r+1:n}|X_{r:n}=x)=x+\frac{\Gamma[n-r+1,(n-r)(1+\theta x)]}{\theta(1+\theta x)^{n-r}e^{-(n-r)(1+\theta x)}(n-r)^{n-r+1}}. \tag{43}$$

Further, the characterization based on truncated can be obtained using (16) as below:

$$E(X^{\beta}|X\geq x)=x^{\beta}+\frac{2+\theta x^{\beta}}{\theta(1+\theta x^{\beta})}.$$

Lindley-Weibull Distribution

Corollary 8. Let X be the continuous *r.v* having $pdf\ f(x)$ and $cdf\ F(x)$, and $E(X^k)$ exist, then

$$E(X^{\beta}_{r+1:n}|X_{r:n}=x)=x^{\beta}+\frac{\Gamma[n-r+1,(n-r)(1+\theta+\theta(\alpha x)^{\beta})]}{\theta\alpha^{\beta}(1+\theta+\theta(\alpha x)^{\beta})^{n-r}(n-r)^{n-r+1}e^{-(n-r)(1+\theta+\theta(\alpha x)^{\beta})}}, \tag{44}$$

if and only if

$$F(x)=1-\left(1+\frac{\theta}{1+\theta}(\alpha x)^{\beta}\right)e^{-\theta(\alpha x)^{\beta}};\alpha,\beta,\theta>0,x>0. \tag{45}$$

Proof. Corollary 8 can be proved easily on the lines of Corollary 7. ■

Remark 6: The result for characterization using truncated moments based on Corollary 1 is given as

$$E(X^{\beta}|X\geq x)=x^{\beta}+\frac{2+\theta+\theta(\alpha x)^{\beta}}{\theta\alpha^{\beta}(1+\theta+\theta(\alpha x)^{\beta})}.$$

Length-Biased Exponential (LBE) Distribution

Corollary 9. Let X be the continuous *r.v.* having $pdf\ f(x)$ and $cdf\ F(x)$, and $E(X)$ exist, then

$$E(X_{r+1:n}|X_{r:n}=x)=x+\left(\frac{\theta}{n-r}\right)^{n-r+1}\frac{\Gamma[n-r+1,\frac{(n-r)}{\theta}(\theta+x)]}{((\theta+x)e^{-\frac{x+\theta}{\theta}})^{n-r}}, \tag{46}$$

if and only if

$$F(x)=1-\left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}};\theta>0,x>0. \tag{47}$$

Proof. To prove necessary part we compare of (47) with (8) and get

$a = \frac{1}{\theta} = c, b = 1, h(x) = x$. Now using (7), we get (46).

To prove sufficient part, in view of (15), we have,

$$\frac{f(x)}{1-F(x)} = \frac{1}{\theta} - \frac{\frac{1}{\theta}}{1+\frac{x}{\theta}}, \tag{48}$$

which gives the cdf as given in (47). ■

Remark 7: The result for characterization using truncated moment can be seen using (16) as:

$$E(X|x \geq x) = x + \frac{x+2\theta}{x+\theta}.$$

Similarly several other distributions that belong to this class can be characterized.

3.2. Applications of characterization Theorem 2

Inverse Lindley Distribution

Corollary 10. Let X be a continuous $r.v.$ having cdf $F(x)$ and $E(X^{-1}) < \infty$, then

$$E(X_{r:n}^{-1}|X_{r+1:n} = x) = x^{-1} - \frac{1}{r\theta}\left[\frac{xe^{\frac{\theta+x(1+\theta)}{x}}}{r\{\theta+x(1+\theta)\}}\right]^r \gamma(r+1, \frac{r\{\theta+x(1+\theta)\}}{x}), \tag{49}$$

if and only if

$$F(x) = \left(1 + \frac{\theta}{1+\theta}\frac{1}{x}\right)e^{-\frac{\theta}{x}}; \theta > 0, x > 0. \tag{50}$$

Proof. First we shall prove the necessary part. On comparison of (50) with (18), we get $a = \frac{\theta}{1+\theta}, b = 1, c = \theta, h(x) = \frac{1}{x}$. Now in view of (17), we get (49). Hence the necessary part is true.

To prove sufficient part, from (26), we have

$$\frac{f(x)}{F(x)} = -\frac{\frac{\theta}{(1+\theta)}\frac{1}{x^2}}{1+\frac{\theta}{(1+\theta)x}} + \frac{\theta}{x^2} \tag{51}$$

which give cdf as given in (50). ■

Remark 8: The characterization result for truncated moment can be obtained on application of Corollary 9 as below:

$$E(X^{-1}|X \leq x) = x^{-1} + \frac{\theta + x(2+\theta-e^{\frac{\theta+x(1+\theta)}{x}})}{\theta\{\theta+x(1+\theta)\}}.$$

Inverse power Lindley Distribution

Corollary 11. Let X be a continuous $r.v.$ having cdf $F(x)$ and $E(X^{-\alpha})$ exists, then

$$E(X_{r:n}^{-\alpha}|X_{r+1:n} = x) = x^{-\alpha} - \frac{1}{r\theta}\left[\frac{x^\alpha e^{\frac{\theta+x^\alpha(1+\theta)}{x^\alpha}}}{r\{x^\alpha(1+\theta)+\theta\}}\right]^r \gamma(r+1, \frac{r}{x^\alpha}(\theta+x^\alpha(1+\theta))), \tag{52}$$

if and only if

$$F(x) = \left(1 + \frac{\theta}{1+\theta}\frac{1}{x^\alpha}\right)e^{-\frac{\theta}{x^\alpha}}; \alpha, \theta > 0, x > 0. \tag{53}$$

Proof. Corollary can be proved on the lines of Corollary 10. ■

Remark 9: Under the similar condition as stated under Corollary 11, the result for truncated moment is given as

$$E(X^{-\alpha}|X \leq x) = x^{-\alpha} + \frac{\theta + x^{\alpha}(2 + \theta - e^{\frac{x^{\alpha}(1+\theta)+\theta}{x^{\alpha}}})}{\theta(\theta + x^{\alpha}(1+\theta))}.$$

Inverted LBE Distribution

Corollary 12. Under the conditions as stated in Corollary 11.

$$E(X_{r:n}^{-1}|X_{r+1:n} = x) = x^{-1} - \frac{\theta e^{r}}{r^{r+1}}\gamma(r+1, \frac{r(1+x\theta)}{\theta x}), \tag{54}$$

if and only if

$$F(x) = \left(1 + \frac{1}{\theta x}\right)e^{-\frac{1}{\theta x}}, \theta > 0, x > 0. \tag{55}$$

Proof. Proof is straight forward. ■

Remark 10: The characterization result for truncated moment using Corollary 9 is given as:

$$E(X^{-1}|X \leq x) = x^{-1} + \frac{\theta(1 + x(\theta + 1 - e^{\frac{1+\theta x}{\theta x}}))}{1 + \theta x}.$$

4. Conclusion

In this study, we defined the HGL and HGIL classes of distributions, and characterized these using the conditional expectation of adjacent order statistics. This approach has demonstrated efficiency in differentiating the characteristics of the HGL and HGIL classes of distributions. Moreover, we obtained characterization results for right and left truncated moments for HGL and HGIL classes of distributions. Further, main results are applied to characterize several well known continuous distributions, such as the one-parameter Lindley, X-Lindley, power Lindley, Lindley Pareto, Ailamujia, power Ailamujia, Lindley-Weibull, and length-biased exponential. Our findings unify the earlier results obtained in the literature.

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CONSTRUCTION OF GAMMA ZERO-INFLATED POISSON DOUBLE SAMPLING PLANS

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Abstract

In a well-supervised production framework, non-conformities occur seldom, resulting in a more number of zeros in the count of non-conformities. The zero-inflated Poisson (ZIP) distribution is a suitable model for handling zero inflation. Double sampling plan (DSP) is a precise quality inspection method where a decision on the approval or rejection of a lot is made after reviewing two samples, providing stronger conclusions than single sampling plan (SSP). In practice, decision-making for submitted lots requires a consistent assessment of both within-lot and between-lot variations, which can be addressed using Bayesian methodology. A Bayesian approach integrates prior knowledge and provides more information for making decisions about the approval or rejection of a lot. This article focuses on the designing of Bayesian DSPs; employing a Gamma prior to the parameter in the Poisson component of ZIP distribution the operating characteristic (OC) function is derived. Examples are provided to assess Gamma-ZIP (GZIP) DSPs. The significance of GZIP DSPs over conventional ZIP DSPs is also presented.

Keywords: Sampling inspection by attributes, Double sampling plan, Prior distribution, Zero-inflated Poisson distribution, Average quality level, Limiting quality level, Operating characteristics function.

I. Introduction

Sampling inspection is a method employed to assess the quality of items by examining a sample rather than inspecting every individual item. This approach is widely applied in manufacturing, service industries, and other sectors where full inspection would be too expensive or time-consuming. Acceptance sampling is a strategy that helps determine whether entire batches can be approved or declined based on sample inspection. Sampling inspection is categorized into two main types: sampling inspection by attributes and variables, both of which help to assess the quality standard of a batch.

In a SSP, the judgment to approve or decline a batch is based on inspecting only one sample. However, there are situations where a single sample may not provide sufficient information for a conclusive decision. In such cases, a DSP is implemented, where the decision is made based on the inspection of two samples. DSP functions as an extension of SSP, offering more reliable decision-making in quality control. Designing DSP parameters offers enhanced decision-making accuracy and provides better protection to both producer and consumer.

A Bayesian approach to acceptance sampling integrates prior knowledge with observed data to improve decision-making. When items are manufactured in lots, quality variations can occur due to within-lot and between-lot variation. Conventional acceptance sampling often assumes that between-lot variation is less significant than within-lot variation, leading to the assumption that the fraction of nonconforming items in a lot remains constant. In reality, decisions about submitted lots should consider both within-lot and between-lot variations. In such cases, Bayesian methods can be employed to design effective sampling plans based on predictive distributions.

The ZIP distribution is particularly effective in situations where non-conformities are rare. It is suitable for processes where there is a high occurrence of zero non-conformities, though occasional non-conformities are still possible. Loganathan and Shalini [5, 6] pioneered the determination of ZIP SSPs. Later, Uma and Ramya [17], Rao and Aslam [13], and Fu-Kwun and Sharew [3] discussed the construction of sampling plans in different perspectives. The Bayesian approach to developing ZIP SSPs has been explored by Suresh and Latha [15], Vijayaraghavan *et al.*,[19], Shalini *et al.*, [11], Palanisamy and Latha [7, 8] and Kaviyarasu and Sivakumar [4].

The designing of ZIP DSPs has been addressed by Shalini and Sheik [12], Pramote and Wimonmas [9], Wimonmas and Pramote [20]. The integration of Bayesian principles into the designing of DSPs have been discussed by Vijayaraghavan and Sakthivel [18], Balamurali *et al.*,[1], and Suresh and Usha [16].

According to the literature, there has been no research conducted on developing GZIP DSPs. This article focuses on the determination of GZIP DSPs. The OC function of the GZIP DSP is derived in Section 2. Designing GZIP DSPs is discussed in Section 3. Numerical examples and the significance of GZIP DSPs over the conventional ZIP DSPs are given in Section 4. Results are summarized in the concluding section.

II. OC function of GZIP DSPs

DSPs offer more flexibility than SSPs by reducing the risk of making premature decisions. This approach is widely used in industries where cost and time efficiency are critical, providing a balance between minimizing inspection efforts and ensuring product quality.

A DSP is structured around five specific parameters: n_1, n_2, c_1, c_2 and c_3, where $c_1 < c_2$ and $c_2 \leq c_3$ (the third acceptance number*)*. When c_2 is taken in equal to c_3 *(i.e.,* $c_2 = c_3$*)*, the DSP is described through its parameters n_1, n_2, $c_1$and c_2, which represent the sizes of the first and second samples and the first and second acceptance numbers, respectively (Duncan [2] and Stephens [14]).

Upon inspecting all items in the sample, the number of nonconformities (d_1) is established. If $d_1 \leq c_1$, the lot is approved; if $d_1 > c_2$, it is declined. When $c_1 < d_1 \leq c_2$, the initial sample fails, a second sample of size n_2 is taken and the nonconformities (d_2) are counted. The cumulative count, $D = d_1 + d_2$, is compared to c_2: the lot is approved if $D \leq c_2$ and declined if $D > c_2$. This two-stage process enhances the reliability of quality assessment, allowing for more effective decision-making regarding lot approval or rejection based on observed nonconformities (Schilling and Neubauer [10]).

The effectiveness of a sampling plan can be evaluated through its OC function. In various industrial environments, careful monitoring of production processes often results in the frequent occurrence of zero non-conformities. In such scenarios, the ZIP distribution is the fitting probability distribution for nonconformities. The probability mass function (*pmf*) of the ZIP distribution is defined as follows

$$P\,(X = d|\varphi, \lambda) = \begin{cases} \varphi + (1-\varphi)e^{-\lambda}, & when\ d = 0 \\ (1-\varphi)\frac{e^{-\lambda}\lambda^{d}}{d!}, & when\ d = 1,2,3,\ldots \end{cases} \quad (1)$$

In this model, φ and λ are parameters, where φ $(0 < \varphi < 1)$ denotes the mixing proportion, which is assumed to be known.

When the variation in lot quality is significant from lot-to-lot, it indicates an unstable production process. In such cases, the process parameter λ is assumed to vary randomly between lots and follows Gamma (a, m) distribution. This gamma distribution is the natural conjugate prior to $\lambda = np$ in the Poisson component of the ZIP distribution.

Shalini et al. [11] derived the probability distribution of d under the conditions of a ZIP distribution and a gamma prior distribution for λ, where the shape parameter of the gamma distribution is m.

$$f\,(d|\varphi,n,p,m) = \begin{cases} \varphi + (1-\varphi)\left(\frac{m}{np+m}\right)^m, & when\ d = 0 \\ (1-\varphi)\binom{m+d-1}{m-1}\left(\frac{np}{np+m}\right)^d\left(\frac{m}{np+m}\right)^m, & when\ d = 1,2,3,\ldots \end{cases} \quad (2)$$

The OC function of the GZIP DSP can be described as:

$$P_a(p) = F(d_1 \le c_1|n_1) + \sum_{d_1=c_1+1}^{c_2} f\left(d_1|n_1,p,\varphi,m\right)F(d_2 \le c_3 - d_1|n_2) \quad (3)$$

As proposed by Vijayaraghavan and Sakthivel [18], the prior knowledge about the production process must be used to estimate the value of m. The moment estimator $\hat{m} = \frac{s_1^2}{s_2}$ can be used for m, where $s_1 = \frac{1}{k}\sum_{i=1}^{k}\hat{\lambda}_i = \bar{\lambda}$ and $s_2 = \frac{1}{k}\sum_{i=1}^{k}\left(\hat{\lambda}_i - \bar{\lambda}\right)^2$; $\hat{\lambda}_i$, i = 1, 2, 3, … , k.

III. Designing GZIP DSPs

GZIP DSPs are designed by determining the optimum parameters n_1, n_2, $c_1$and c_2 based on the prescribed points $(p_1, 1-\alpha)$ and (p_2, β) on the OC curve so that the determined GZIP DSPs provide adequate protection to both producer and consumer.

The plan must satisfy the following requirements:

(i) $P_a\,(p_1) \ge 1-\alpha$

(ii) $P_a\,(p_2) \le \beta$

The values of the plan parameters n_1, n_2, c_1 and c_2 can be derived for each set of φ, m, p_1, α, p_2 and β by applying the unity value approach. The values np_1 and np_2 satisfying respectively equations *(i)* and *(ii)* are termed as unity values (Schilling and Neubauer [10]). The plan parameters can be arranged in tables for different combinations of $(p_1, \alpha, p_2, \beta)$. The use of an operating ratio $R = \frac{np_2}{np_1}$, reduces the number of tables.

The plan parameters are determined for specific sets of values of φ, m, p_1, α, p_2 and β under the condition of GZIP distribution. The unity values are computed for various values combinations of (φ, m, c, $P_a(p)$) by solving the OC function of GZIP DSPs for each combination of c_1 and c_2 with $n_1 = n_2 = n$. The values taken for m are 5 and 10 and for $Pa\,(p)$ are 0.99, 0.90, 0.50, 0.20 and 0.10. The value considered for c_1 and c_2 in these combinations are 1(1)9 and 2(1)10 respectively. The values taken for φ are 0.03 and 0.07 and are given in Table 1. The operating ratio values calculated corresponding to (α = 0.05, β = 0.10), (α = 0.10, β = 0.20), φ = 0.03 and 0.07, m = 5 and 10, c_1 = 1(1)9 and c_2 = 2(1)10 are listed in Table 2.

For specified strength $(p_1, \alpha, p_2, \beta)$ and values of φ and m these tables can be used to determine the plan parameters by implementing the following procedure:

First, we compute the operating ratio $R = \frac{p_2}{p_1}$ .Next select the unity value np_1 and the acceptance numbers (c_1, c_2) from Table 2 corresponding to the value of φ, m, α, β associated with an operating ratio closest to R. Then, determine the unity values np_1 from Table 1 and calculate the sample size n

as $\frac{np_1}{p_1}$. Thus, the acceptance numbers and the sample size determined together with φ and m are the parameters of the desired plan.

Table 1: *Unity Value of GZIP DSPs*

φ	m	c_1	c_2	$P_a(p)$					
				0.99	0.95	0.9	0.5	0.2	0.1
0.03	5	1	2	0.3488	0.6455	0.8704	2.2795	4.2836	6.2789
		1	3	0.5684	0.9343	1.1976	2.7349	4.7742	6.7438
		1	4	0.791	1.2209	1.5219	3.2119	5.3513	7.3541
		1	5	1.0182	1.5102	1.8489	3.7034	5.9797	8.0607
		1	6	1.2504	1.8033	2.1795	4.2049	6.6399	8.8299
		1	7	1.4872	2.1002	2.5138	4.7136	7.3208	9.6402
		1	8	1.728	2.4004	2.8512	5.2275	8.0156	10.4782
		1	9	1.9723	2.7034	3.1912	5.745	8.7201	11.3353
		1	10	2.2194	3.0087	3.5333	6.2654	9.4313	12.2058
		2	3	0.6147	1.0308	1.3368	3.2234	5.9143	8.5762
		2	4	0.8351	1.3024	1.6345	3.5798	6.2375	8.8573
		2	5	1.0569	1.5775	1.9398	3.9877	6.6701	9.2693
		2	6	1.2833	1.8588	2.2535	4.4292	7.1839	9.7977
		2	7	1.515	2.1461	2.5744	4.8934	7.755	10.4186
		2	8	1.7515	2.4385	2.9012	5.3734	8.3665	11.1092
		2	9	1.9921	2.7352	3.2328	5.8649	9.0067	11.8514
		2	10	2.2362	3.0355	3.5682	6.3648	9.6678	12.6316
		3	4	0.9132	1.4502	1.8397	4.2225	7.5954	10.8922
		3	5	1.1254	1.6984	2.1037	4.4942	7.8127	11.082
		3	6	1.3418	1.9581	2.3863	4.8329	8.1275	11.3654
		3	7	1.5646	2.2284	2.6835	5.2197	8.5307	11.7496
		3	8	1.7936	2.5073	2.9918	5.6407	9.006	12.2295
		3	9	2.028	2.7932	3.3088	6.0863	9.5374	12.7925
		3	10	2.2669	3.0847	3.6324	6.55	10.1113	13.4239
		4	5	1.2365	1.8964	2.3715	5.2571	9.2958	13.2063
		4	6	1.4357	2.1188	2.6012	5.4625	9.4506	13.3504
		4	7	1.6438	2.3606	2.8589	5.7376	9.6809	13.5583
		4	8	1.8607	2.6176	3.1373	6.0692	9.9911	13.8426
		4	9	2.0853	2.8862	3.4308	6.4439	10.3759	14.2093
		4	10	2.3161	3.1639	3.7359	6.8511	10.8249	14.6568
		5	6	1.5793	2.364	2.9258	6.3131	11.0028	15.5146
		5	7	1.7633	2.5604	3.123	6.4695	11.121	15.6345
		5	8	1.961	2.7829	3.3551	6.6905	11.2948	15.7986
		5	9	2.1704	3.0249	3.613	6.9699	11.5339	16.0185
		5	10	2.3891	3.2817	3.8902	7.298	11.8413	16.3033
		6	7	1.9377	2.8482	3.4973	7.3819	12.7115	17.8174
		6	8	2.1057	3.02	3.6652	7.5033	12.808	17.9234
		6	9	2.2918	3.2226	3.872	7.6804	12.9446	18.0614
		6	10	2.4923	3.4487	4.1088	7.9135	13.1325	18.2403
		7	8	2.3088	3.3455	4.0817	8.4583	14.4201	20.1156

φ	m	c_1	c_2	$P_a(p)$					
				0.99	0.95	0.9	0.5	0.2	0.1
		7	9	2.4608	3.4947	4.224	8.555	14.5031	20.2132
		7	10	2.6346	3.6778	4.4068	8.6978	14.6156	20.3348
0.03	5	8	9	2.6901	3.8529	4.676	9.5394	16.128	22.4102
		8	10	2.8268	3.9821	4.7965	9.6186	16.2024	22.5023
		9	10	3.0796	4.3683	5.2775	10.6234	17.8349	24.7018
		1	2	0.3675	0.6684	0.8907	2.192	3.8343	5.2986
		1	3	0.6025	0.9714	1.2293	2.6331	4.2825	5.709
		1	4	0.843	1.2746	1.568	3.0994	4.8159	6.2568
		1	5	1.0908	1.5836	1.9123	3.5825	5.3982	6.8918
		1	6	1.3466	1.8991	2.2628	4.0768	6.0096	7.5801
		1	7	1.6097	2.2207	2.6189	4.5787	6.6386	8.3008
		1	8	1.879	2.5474	2.9796	5.0858	7.2784	9.0415
		1	9	2.1537	2.8783	3.3442	5.5963	7.9249	9.7945
		1	10	2.4329	3.2129	3.7119	6.1093	8.5754	10.5551
		2	3	0.6554	1.0761	1.3762	3.0882	5.2284	7.1066
		2	4	0.8908	1.3588	1.6808	3.4237	5.5103	7.3413
		2	5	1.1303	1.649	1.9979	3.817	5.9025	7.703
		2	6	1.378	1.9494	2.3277	4.2483	6.3765	8.1781
		2	7	1.6342	2.2591	2.6681	4.7053	6.9071	8.7405
		2	8	1.8981	2.5768	3.017	5.18	7.4764	9.3656
		2	9	2.1685	2.9009	3.3727	5.6669	8.0719	10.0347
		2	10	2.4444	3.2302	3.7337	6.1625	8.6853	10.7341
		3	4	0.9843	1.5265	1.9068	4.0436	6.6651	8.9182
		3	5	1.209	1.7809	2.1715	4.2873	6.8423	9.0682
0.03	10	3	6	1.442	2.0528	2.4618	4.6048	7.115	9.3052
		3	7	1.6857	2.3403	2.7723	4.977	7.4781	9.6411
		3	8	1.9393	2.6406	3.0984	5.3883	7.9157	10.0719
		3	9	2.2014	2.9512	3.4365	5.8276	8.41	10.5838
		3	10	2.4706	3.27	3.7839	6.287	8.9462	11.1598
		4	5	1.3465	2.0131	2.4756	5.0383	8.1145	10.7165
		4	6	1.5546	2.2357	2.6989	5.2109	8.2333	10.8275
		4	7	1.7771	2.4852	2.9588	5.4576	8.4208	10.9938
		4	8	2.0134	2.7563	3.2463	5.7678	8.6878	11.2317
		4	9	2.2617	3.0441	3.5545	6.1278	9.0324	11.5508
		4	10	2.5197	3.3449	3.8786	6.5254	9.4443	11.9512
		5	6	1.7368	2.5297	3.0754	6.0563	9.5646	12.5006
		5	7	1.925	2.7199	3.2593	6.1786	9.6524	12.5929
		5	8	2.1339	2.9448	3.487	6.3653	9.7858	12.7213
		5	9	2.3604	3.1967	3.7488	6.616	9.98	12.8984
		5	10	2.601	3.4693	4.0364	6.9226	10.243	13.1364
		6	7	2.1506	3.0708	3.6995	7.0875	11.0118	14.2731
		6	8	2.3177	3.2302	3.8481	7.1763	11.083	14.3555
		6	9	2.511	3.4293	4.0438	7.3159	11.1839	14.4633
		6	10	2.7254	3.6602	4.2784	7.5139	11.328	14.6044

φ	m	c_1	c_2	P_a (p)					
				0.99	0.95	0.9	0.5	0.2	0.1
0.03	10	7	8	2.5841	3.6314	4.3422	8.126	12.4553	16.0363
		7	9	2.7301	3.7629	4.4607	8.1931	12.5175	16.1132
		7	10	2.9065	3.9365	4.6261	8.2982	12.5992	16.2091
		8	9	3.0339	4.2073	4.9987	9.1684	13.8953	17.7925
		8	10	3.1598	4.3147	5.0928	9.2219	13.9523	17.8657
		9	10	3.4971	4.795	5.6656	10.213	15.3323	19.543
0.07	5	1	2	0.3554	0.66	0.8925	2.3944	4.8306	8.5599
		1	3	0.5786	0.9547	1.2272	2.8707	5.3748	9.1613
		1	4	0.8047	1.2465	1.5583	3.3685	6.013	9.946
		1	5	1.0351	1.5407	1.8917	3.8806	6.7086	10.8654
		1	6	1.2702	1.8385	2.2284	4.4028	7.4406	11.8792
		1	7	1.5099	2.1399	2.5686	4.9321	8.1966	12.9589
		1	8	1.7534	2.4444	2.9117	5.4664	8.9689	14.085
		1	9	2.0003	2.7516	3.2573	6.0045	9.7525	15.2443
		1	10	2.2501	3.0611	3.605	6.5453	10.5441	16.4278
		2	3	0.6249	1.0517	1.3677	3.376	6.631	11.531
		2	4	0.8491	1.3291	1.6727	3.7531	7.0123	11.9703
		2	5	1.0741	1.6092	1.9844	4.1799	7.5013	12.556
		2	6	1.3036	1.895	2.3041	4.64	8.0733	13.2756
		2	7	1.5381	2.1866	2.6307	5.1228	8.7063	14.1084
		2	8	1.7772	2.4833	2.963	5.6217	9.3835	15.0324
		2	9	2.0205	2.7842	3.3	6.1323	10.093	16.0281
		2	10	2.2671	3.0885	3.6408	6.6514	10.8262	17.0794
		3	4	0.9273	1.4775	1.8794	4.4115	8.4709	14.4839
		3	5	1.1431	1.7314	2.1507	4.7059	8.756	14.8475
		3	6	1.3627	1.9959	2.4394	5.0632	9.1321	15.32
		3	7	1.5883	2.2705	2.7422	5.4671	9.5924	15.9023
		3	8	1.8199	2.5535	3.0558	5.905	10.1245	16.589
		3	9	2.0569	2.8434	3.3779	6.3677	10.7143	17.3698
		3	10	2.2983	3.1388	3.7067	6.8489	11.3496	18.2324
		4	5	1.2543	1.9301	2.4198	5.4807	10.3229	17.4184
		4	6	1.4573	2.1584	2.6573	5.712	10.5536	17.742
		4	7	1.6684	2.4048	2.9209	6.0069	10.8534	18.1493
		4	8	1.888	2.6658	3.2045	6.3553	11.226	18.6439
		4	9	2.115	2.9383	3.503	6.7457	11.6683	19.2263
		4	10	2.3483	3.2196	3.8129	7.1685	12.173	19.8938
		5	6	1.6007	2.4037	2.9824	6.5697	12.1774	20.339
		5	7	1.7887	2.6065	3.1881	6.7554	12.3767	20.6388
		5	8	1.9894	2.8335	3.4261	6.9985	12.6274	21.0059
		5	9	2.2013	3.0794	3.6891	7.2957	12.9365	21.4438
		5	10	2.4223	3.3398	3.9708	7.6391	13.3066	21.9543
		6	7	1.9627	2.8938	3.5619	7.6702	14.0314	23.2495
		6	8	2.1349	3.0725	3.7391	7.824	14.2118	23.5338
		6	9	2.324	3.2798	3.9521	8.0262	14.4309	23.8743

φ	m	c_1	c_2	P_a (p)					
				0.99	0.95	0.9	0.5	0.2	0.1
		6	10	2.5269	3.5097	4.1939	8.2787	14.6957	24.2737
		7	8	2.3372	3.3968	4.154	8.7776	15.8839	26.1528
		7	9	2.4938	3.5536	4.3066	8.9092	16.0525	26.426
0.07	5	7	10	2.6706	3.7416	4.4961	9.08	16.2509	26.7478
		8	9	2.7219	3.9098	4.7557	9.889	17.7348	29.0506
		8	10	2.8635	4.0472	4.8875	10.0052	17.8956	29.3156
		9	10	3.1147	4.4305	5.3644	11.0028	19.5843	31.9442
		1	2	0.3744	0.6832	0.9129	2.2942	4.2581	6.8443
		1	3	0.6134	0.9921	1.2588	2.7535	4.7481	7.3537
		1	4	0.8574	1.3007	1.6042	3.2377	5.3299	8.0376
		1	5	1.1086	1.6146	1.9546	3.7384	5.9662	8.8447
		1	6	1.3675	1.9346	2.3108	4.2501	6.6352	9.734
		1	7	1.6334	2.2605	2.6724	4.7692	7.3243	10.6776
		1	8	1.9055	2.5913	3.0384	5.2933	8.0258	11.6572
		1	9	2.1828	2.9262	3.4081	5.8206	8.7349	12.6611
		1	10	2.4645	3.2646	3.7808	6.35	9.4486	13.6816
		2	3	0.6663	1.0975	1.4071	3.2217	5.7651	9.0165
		2	4	0.9057	1.386	1.7188	3.5751	6.0952	9.3774
		2	5	1.1485	1.681	2.042	3.9842	6.5325	9.8823
		2	6	1.3992	1.9858	2.3772	4.4308	7.0528	10.5184
		2	7	1.6583	2.2997	2.7228	4.903	7.6328	11.2631
		2	8	1.9249	2.6213	3.0767	5.393	8.2547	12.0916
		2	9	2.1978	2.9492	3.4373	5.8954	8.9057	12.983
		2	10	2.4762	3.2823	3.8031	6.4065	9.5767	13.9204
		3	4	0.9993	1.5547	1.9465	4.2066	7.3015	11.1529
0.07	10	3	5	1.228	1.8147	2.2183	4.4706	7.5386	11.4466
		3	6	1.4641	2.091	2.514	4.8033	7.8632	11.8479
		3	7	1.7105	2.3824	2.8293	5.1888	8.2725	12.3593
		3	8	1.9666	2.6864	3.1601	5.6132	8.7546	12.9752
		3	9	2.2312	3.0006	3.5027	6.0657	9.2948	13.6839
		3	10	2.5028	3.3229	3.8546	6.5388	9.8795	14.4708
		4	5	1.3657	2.0478	2.5239	5.2283	8.8418	13.2582
		4	6	1.5777	2.2761	2.7546	5.4251	9.0295	13.5176
		4	7	1.803	2.5298	3.0196	5.6884	9.2796	13.8605
		4	8	2.0419	2.8044	3.3113	6.0113	9.6	14.2917
		4	9	2.2924	3.0954	3.6237	6.3826	9.9908	14.8122
		4	10	2.5527	3.3994	3.9518	6.7914	10.4459	15.4192
		5	6	1.7601	2.5708	3.132	6.2712	10.378	15.3406
		5	7	1.9523	2.7671	3.324	6.4218	10.5396	15.5798
		5	8	2.1639	2.9959	3.5567	6.6279	10.7447	15.8875
		5	9	2.3926	3.2509	3.8223	6.8922	11.0024	16.2675
		5	10	2.6353	3.5264	4.1135	7.2093	11.3191	16.722
		6	7	2.1779	3.118	3.7639	7.3255	11.9085	17.4062
		6	8	2.3492	3.2841	3.9216	7.4462	12.0559	17.6321

φ	m	c_1	c_2	$P_a(p)$					
				0.99	0.95	0.9	0.5	0.2	0.1
0.07	10	6	9	2.545	3.4872	4.1225	7.609	12.2343	17.9166
		6	10	2.7615	3.7209	4.3607	7.8226	12.4509	18.2623
		7	8	2.6153	3.6844	4.4139	8.3859	13.4337	19.4595
		7	9	2.7658	3.8236	4.5429	8.4879	13.5728	19.6757
		7	10	2.9447	4.0012	4.714	8.6197	13.7352	19.9435
		8	9	3.0688	4.2658	5.0773	9.4496	14.9545	21.5035
		8	10	3.1997	4.3818	5.1832	9.5399	15.0883	21.7122
		9	10	3.5356	4.8587	5.7507	10.515	16.4715	23.5402

Table 2: *Operating Ratio of GZIP DSPs*

φ	m	c_1	c_2	R	
				$\alpha = 0.05, \beta = 0.10$	$\alpha = 0.10, \beta = 0.20$
0.03	5	1	2	9.7272	0.2222
		1	3	7.218	4.9214
		1	4	6.0235	3.9865
		1	5	5.3375	3.5162
		1	6	4.8965	3.2342
		1	7	4.5901	3.0465
		1	8	4.3652	2.9122
		1	9	4.193	2.8113
		1	10	4.0568	2.7325
		2	3	8.3199	2.6693
		2	4	6.8008	4.4242
		2	5	5.8759	3.8162
		2	6	5.271	3.4386
		2	7	4.8547	3.1879
		2	8	4.5558	3.0124
		2	9	4.3329	2.8838
		2	10	4.1613	2.786
		3	4	7.5108	2.7094
		3	5	6.525	4.1286
		3	6	5.8043	3.7138
		3	7	5.2727	3.4059
		3	8	4.8776	3.1789
		3	9	4.5799	3.0102
		3	10	4.3518	2.8824
		4	5	6.9639	2.7836
		4	6	6.3009	3.9198
		4	7	5.7436	3.6332
		4	8	5.2883	3.3862
		4	9	4.9232	3.1846
		4	10	4.6325	3.0243
		5	6	6.5629	2.8975
		5	7	6.1063	3.7606

φ	m	c_1	c_2	R	
				$\alpha = 0.05, \beta = 0.10$	$\alpha = 0.10, \beta = 0.20$
		5	8	5.677	3.561
		5	9	5.2955	3.3665
		5	10	4.9679	3.1923
		6	7	6.2557	3.0439
		6	8	5.9349	3.6347
		6	9	5.6046	3.4945
0.03	5	6	10	5.289	3.3431
		7	8	6.0127	3.1962
		7	9	5.784	3.5329
		7	10	5.5291	3.4335
		8	9	5.8164	3.3166
		8	10	5.6509	3.4491
		9	10	5.6548	3.378
		1	2	7.9273	3.3794
		1	3	5.8771	4.3048
		1	4	4.9088	3.4837
		1	5	4.352	3.0714
		1	6	3.9914	2.8229
		1	7	3.7379	2.6558
		1	8	3.5493	2.5349
		1	9	3.4029	2.4427
		1	10	3.2852	2.3697
		2	3	6.604	2.3102
		2	4	5.4028	3.7992
		2	5	4.6713	3.2784
		2	6	4.1952	2.9544
		2	7	3.869	2.7394
		2	8	3.6346	2.5888
0.03	10	2	9	3.4592	2.4781
		2	10	3.323	2.3933
		3	4	5.8423	2.3262
		3	5	5.0919	3.4954
		3	6	4.5329	3.151
		3	7	4.1196	2.8902
		3	8	3.8142	2.6974
		3	9	3.5863	2.5548
		3	10	3.4128	2.4473
		4	5	5.3234	2.3643
		4	6	4.843	3.2778
		4	7	4.4237	3.0506
		4	8	4.0749	2.846
		4	9	3.7945	2.6762
		4	10	3.573	2.5411
		5	6	4.9415	2.435

φ	m	c_1	c_2	R	
				$\alpha = 0.05, \beta = 0.10$	$\alpha = 0.10, \beta = 0.20$
0.03	10	5	7	4.6299	3.11
		5	8	4.3199	2.9615
		5	9	4.0349	2.8064
		5	10	3.7865	2.6622
		6	7	4.648	2.5377
		6	8	4.4442	2.9766
		6	9	4.2176	2.8801
		6	10	3.9901	2.7657
		7	8	4.416	2.6477
		7	9	4.2821	2.8684
		7	10	4.1176	2.8062
		8	9	4.229	2.7235
		8	10	4.1407	2.7798
		9	10	4.0757	2.7396
0.07	5	1	2	12.9695	2.7062
		1	3	9.596	5.4124
		1	4	7.9791	4.3797
		1	5	7.0522	3.8587
		1	6	6.4614	3.5463
		1	7	6.0558	3.339
		1	8	5.7622	3.1911
		1	9	5.5402	3.0803
		1	10	5.3666	2.994
		2	3	10.9642	2.9249
		2	4	9.0063	4.8483
		2	5	7.8026	4.1922
		2	6	7.0056	3.7801
		2	7	6.4522	3.5039
		2	8	6.0534	3.3095
		2	9	5.7568	3.1669
		2	10	5.53	3.0585
		3	4	9.803	2.9736
		3	5	8.5754	4.5072
		3	6	7.6757	4.0712
		3	7	7.0039	3.7436
		3	8	6.4966	3.4981
		3	9	6.1088	3.3132
		3	10	5.8087	3.1719
		4	5	9.0246	3.0619
		4	6	8.22	4.266
		4	7	7.5471	3.9716
		4	8	6.9937	3.7158
		4	9	6.5433	3.5032
		4	10	6.179	3.3309

φ	m	c_1	c_2	R	
				$\alpha = 0.05, \beta = 0.10$	$\alpha = 0.10, \beta = 0.20$
		5	6	8.4615	3.1926
		5	7	7.9182	4.0831
		5	8	7.4134	3.8822
		5	9	6.9636	3.6856
		5	10	6.5735	3.5067
		6	7	8.0342	3.3511
		6	8	7.6595	3.9393
0.07	5	6	9	7.2792	3.8009
		6	10	6.9162	3.6515
		7	8	7.6992	3.5041
		7	9	7.4364	3.8238
		7	10	7.1488	3.7274
		8	9	7.4302	3.6144
		8	10	7.2434	3.7292
		9	10	7.2101	3.6615
		1	2	10.018	3.6508
		1	3	7.4123	4.6644
		1	4	6.1794	3.7719
		1	5	5.478	3.3225
		1	6	5.0315	3.0524
		1	7	4.7236	2.8714
		1	8	4.4986	2.7407
		1	9	4.3268	2.6415
		1	10	4.1909	2.563
		2	3	8.2155	2.4991
		2	4	6.7658	4.0972
		2	5	5.8788	3.5462
		2	6	5.2968	3.1991
		2	7	4.8976	2.9669
0.07	10	2	8	4.6128	2.8033
		2	9	4.4022	2.683
		2	10	4.2411	2.5909
		3	4	7.1737	2.5181
		3	5	6.3077	3.7511
		3	6	5.6661	3.3984
		3	7	5.1878	3.1278
		3	8	4.83	2.9239
		3	9	4.5604	2.7704
		3	10	4.3549	2.6536
		4	5	6.4744	2.563
		4	6	5.9389	3.5032
		4	7	5.4789	3.278
		4	8	5.0962	3.0731
		4	9	4.7852	2.8992

φ	m	c_1	c_2	R	
				$\alpha = 0.05, \beta = 0.10$	$\alpha = 0.10, \beta = 0.20$
0.07	10	4	10	4.5359	2.7571
		5	6	5.9672	2.6433
		5	7	5.6304	3.3135
		5	8	5.3031	3.1708
		5	9	5.004	3.021
		5	10	4.7419	2.8785
		6	7	5.5825	2.7517
		6	8	5.3689	3.1639
		6	9	5.1378	3.0742
		6	10	4.908	2.9677
		7	8	5.2816	2.8553
		7	9	5.1459	3.0435
		7	10	4.9844	2.9877
		8	9	5.0409	2.9137
		8	10	4.9551	2.9454
		9	10	4.845	2.911

IV. Numerical Examples

This section outlines the process for choosing GZIP DSPs for a defined strength, along with numerical examples.

When solving the OC function of the GZIP DSPs conditions, the unity values for various combinations of $(\varphi, m, c, P_a(p))$ are calculated by considering each combination of c_1 and c_2 with the condition $n_1 = n_2 = n$. Then, the plan parameters for specific values of φ, m, p_1, α, p_2 and β are determined GZIP DSPs.

I. Example 1

Suppose that $\varphi = 0.07$, $m = 10$, and the strength of the plan is specified as $p_1 = 0.01$, $\alpha = 0.05$, $p_2 = 0.06$, $\beta = 0.10$, the operating ratio R corresponding to these specifications is computed as 6. The acceptance number can be determined from Table 2 as $c_1 = 5$ and $c_2 = 6$ with an R value of 5.9672, which is close to 6. The unity values corresponding to the φ, m, p_1, α, $p_2$and β parameters are obtained from Table 1 as $np_1 = 2.5708$.

Since $n = \frac{np_1}{p_1} = \frac{2.5708}{0.01} \approx 257$. Based on these calculations, the optimum DSP is $n_1 = 257$, $c_1 = 5$, $n_2 = 257$ and $c_2 = 6$.

II. Example 2

Assuming $\varphi = 0.03$, $m = 10$, and the plan's strength is set at $p_1 = 0.01$, $\alpha = 0.05$, $p_2 = 0.06$, $\beta = 0.10$, the operating ratio R can be calculated as 6. The acceptance number can be determined from Table 2 as $c_1 = 1$ and $c_2 = 3$ with an R value of 5.8771, which is close to 6. Consequently, the unity values for φ, m, p_1, α, $p_2$and β from Table 1 are $np_1 = 0.9714$.

Since, $n = \frac{np_1}{p_1} = \frac{0.9714}{0.01} \approx 97$. the optimal GZIP DSPs for the specified specifications is $n_1 = 97$, $c_1 = 1$, $n_2 = 97$, $c_2 = 3$, $\varphi = 0.03$ and $m = 10$.

The ZIP DSP for the specified strength is $n_1 = 102$, $c_1 = 1$, $n_2 = 102$, $c_2 = 3$, $\varphi = 0.03$.

This indicates that GZIP DSP requires smaller sample compared to ZIP DSP.

III. Significance of GZIP DSPs over non-Bayesian ZIP DSPs.

The producer's risk for the ZIP DSP is 4.63% when the lot fraction non-conforming is $p = 0.009$. On the other hand, the producer's risk for the GZIP DSPs for the given values of $m = 5$ and 10 are 3.32% and 3.56%, respectively. When the lot fraction non-conforming is $p = 0.059$, the consumer's risk for the ZIP DSP is 10.35%. In contrast, the consumer's risk for the GZIP DSPs for the specified values of $m = 5$ and 10 is 10.36% and 9.94%, respectively. For the ZIP DSP, the combined producer and consumer risk is 14.98%. On the other hand, the GZIP DSP has a total risk for $m = 5$ and 10 are 13.68% and 13.50% respectively.

Table 3: *Values of OC function of DSP ZIP and GZIP DSPs for ($p_1 = 0.005$, $\alpha = 0.05$, $p_2 = 0.05$, $\beta = 0.10$)*

Model	Parameters					Lot Fraction non-conforming (p)			Producer's Risk (%)	Consumer's Risk (%)	Total Risk (%)
	φ	m	c_1	c_2	n	0	0.009	0.059			
ZIP DSP	0.03	-	1	2	75	1	0.9537	0.1035	4.63	10.35	14.98
GZIP DSP	0.03	5	7	8	335	1	0.9668	0.1036	3.32	10.36	13.68
		10	1	3	97	1	0.9643	0.0994	3.56	9.94	13.50

V. Conclusion

The fundamental assumption in the theory of sampling inspection procedures by attributes is that the fraction of nonconforming items in a lot remains constant. However, in real-world scenarios, lots produced from a process often exhibit quality variations due to random fluctuations, leading to a random variation in the fraction of nonconforming units across lots. In such situations, Bayesian acceptance sampling plans (BASP), which incorporate prior information about process variability when making decisions on submitted lots, can offer an advantage over conventional plans. In this paper, Bayesian double sampling plans by attributes are determined for the two specified points on the OC curve based on gamma-ZIP distribution. The GZIP DSPs requires fewer sample units for inspection compared to non-Bayesian ZIP DSPs. As a result, GZIP DSPs effectively lower both producer's and consumer's risks, offering better protection for both parties by minimizing the chances of rejecting good-quality lots and accepting poor-quality ones. By implementing GZIP DSPs, optimal sample sizes and acceptance numbers are achieved, reducing overall risk and delivering benefits such as enhanced customer satisfaction, increased productivity, and sustained market competitiveness.

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A STUDY ON COMPARISON OF VARIOUS CONTINUOUS SAMPLING AND SKIP-LOT SAMPLING PLAN PROCEDURES

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Abstract

This paper explains the brief review of skip-lot sampling plan procedures followed by continuous sampling plan procedures. Also, various types of skip-lot sampling plans are compared with continuous sampling plans. The efficiency of SkSP-T is tested on comparison with various skip-lot sampling plans using Single Sampling Plan. A new system of skip-lot sampling plan of type SkSP-T is compared with other skip-lot sampling plans. Different types of skip-lot sampling plans namely SkSP-2, SkSP-3, SkSP-V and SkSP-R. The tables are constructed for various combinations of various parameters using various numerical methods.

Keywords: Continuous Sampling Plan, Skip Lot Sampling Plan, Skip-lot Sampling Plan of Type SkSP-T, Single Sampling Plan.

I. Introduction

Acceptance sampling is a major tool in statistical quality control. Various Attributes Acceptance sampling procedures has been developed by several authors since 1940. Acceptance sampling plan has four broad categorized. It includes Continuous sampling plans, special purpose plans. Special purpose plan includes skip lot sampling procedure. In this paper explains the brief review of continuous sampling plan procedures and skip-lot sampling plan procedures.

II. Comparison of CSP-1 and SkSP-1

Continuous sampling plan of type CSP-1 is introduced by [5]. CSP-1 plan is a continuous flow of discrete products. The operating procedure of CSP-1 is executed in two stages of inspections namely 100% inspection and sampling inspection. These two types of inspection are controlled by the parameters i (clearance interval used for 100% inspection) and f (sampling frequency used for sampling inspection). CSP's intended for applied only individual units. The average number of total production Pa (p) accepted are passed on a sampling basis is given by

$$P_a(p) = \frac{q^i}{f+(1-f)q^i} \quad (1)$$

Where f is the Sampling Frequency ($0<f<1$), i is the Clearance Interval on 100% inspection. The first skip-lot sampling plan of type SkSP-1 is introduced by [8] it is based on the concept of continuous sampling plan of type CSP-1. Skip-lot sampling plan is a bulk materials or products produced in successive lots. The SkSP-1 sampling plan was proposed without considering the concept of reference plan. Skip-lot sampling plan of type SkSP-1 is followed by continuous sampling plan of type CSP-1. SkSP-1 is implemented by the following procedures are developed by [8]:

a) "Units" are be transformed into "lots" (clubbing of units).
b) After the inspection processes lots are conforming or non-conforming. Suppose accepting a non-conforming lot and rejecting a conforming lot in this situation used AOQL (Average Outgoing Quality Limit).
c) Using 2% AOQL table. If AOQL=2% then the standard plan parameter values are fixed. Since *f=1/2*, *i=14* and *f=1/2*, *i=15*.

Operating Procedure for SkSP-1

a) Each nonconforming lots corrected or replaced by a conforming lot.

When *i=14*, *f=1/2*.

b) Each nonconforming lots rejected and not replaced by a conforming lot.

When *i=15*, *f=1/2*.

Under the plan is executed individually to each characteristic under inspection. During some specific characteristics are elaborated, try to analyze partially single characteristic within per lot.

Step1: Necessarily to test each lot, 15 lots are received and successive lots are found to be conforming.

Step2: During 15 successive lots are found conforming, the lots are selected at randomly and to test half of a lots. In this case accepted lots are not tested.

Step3: During a lot is rejected, go back to step-2.

III. Comparison of CSP-2 and SkSP-2

CSP-2 is proposed by [6] it is a reconstruction of CSP-1 plan. Continuous sampling plan of type CSP-2 is different from CSP-1 plan. In this new plan once the sampling inspection is initiate, stop to 100% inspection. When defect is found in sampling inspection it will use only the sampling frequency (*f*) of a succeeding of two defects occurs in the other *k* (successive sampling units) or less sample units. Suppose two defects occurs in *k* successive sampling units or less sampling units then the system immediately go to 100% inspection otherwise sampling inspection is ongoing. CSP-2 plan is maintained with three parameters *i*, *f* and *k*. The average number of total production Pa(p) accepted are passed on a sampling basis is given by

$$P_a(p) = \frac{q^i(2-q^k)}{f(1-q^k)(1-q^i)+q^i(2-q^k)} \quad (2)$$

Where *f*- Sampling Frequency, *i*- Clearance Interval on 100% inspection and *k*- Clearance Interval on sampling inspection. Skip-lot Sampling Plan of type SkSP-2 is developed by [7]. It is an extension of skip-lot sampling plan of type SkSP-1 and based on the origin of continuous sampling plan of CSP-1. In [15] has provided some operating characteristic of SkSP-2 plan using markov chain techniques. Skip-lot plan of type SkSP-2 is used an attribute inspection plan it is called as "reference plan". The reference plan comprises certain rules, based on the record of lot acceptance or rejections, for switching back and forth between 'normal' and 'skipping' inspection. Every lot has received on the normal inspection and skipping inspection randomly selected fraction *f* of the lots and skipped lots are automatically accepted. Probability of acceptance under SkSP-2 plan is

$$P_a(p) = \frac{fP+(1-f)P^i}{f+(1-f)P^i} \quad (3)$$

Where *f*- Sampling frequency, *i*- clearance interval and P- Probability of acceptance under the reference plans.

IV. Comparison of CSP-3 and SkSP-3

The new concept of continuous sampling plan of type CSP-3 is designed by [6]. It is also moderation of CSP-1 and CSP-2 plans. CSP-3 is filtration of CSP-2 plan to provide additional production against highly defective quality. Four defects occur in k successive sampling units or less sampling units

then the system immediately go to 100% inspection otherwise sampling inspection is ongoing. It supplies additional manufacture on "Spotty quality". The average fraction of total production accepted on a sampling basis is

$$P_a(p) = \frac{q^i[1+q^4(1-q^k)}{f(1-q^{k+4})(1-q^i)+q\left(1+q^{4i}(1-q^k)\right)+4fpq^i} \quad (4)$$

Where *f*- Sampling Frequency, *i*- Clearance Interval on 100% inspection and *k*- Clearance Interval on sampling inspection. Skip-lot sampling plan of type SkSP-3 is developed by [17] using markov chain technique. It is a basic concept of continuous sampling plans and skip-lot sampling plans. Probability of acceptance under SkSP-3 plan is

$$P_a(p) = \frac{fP(1-P^i)(1-P^k)+(1-f)P^i(2-P^k)+fP^{i+1}(2-P^k)}{f(1-P^i)(1-P^k)+P^i(2-P^k)} \quad (5)$$

Where *f*- Sampling Frequency, *i*- Clearance Interval on normal inspection and *k*- Clearance Interval on skipping inspection.

V. Comparison of CSP-V and SkSP-V

Exceptional study of continuous sampling plan of type CSP-V introduced by [2]. In CSP-V plan *f* (sampling frequency) is never control to minimizing; using a smaller clearance interval to reducing inspection. CSP-V plan clearance number *x* (<*i*) can be used when 100% inspection only. And another clearance number *i* can be used in both 100% inspection and sampling inspection. The average fraction of total production accepted on a sampling basis (the operating characteristic function) is

$$P_a(p) = \frac{q^i}{q^i+f[1-q^i+q^k(q^i-q^x)]} \quad (6)$$

Where *f*- Sampling Frequency, *i*- Clearance Interval on 100% inspection, *k*- Clearance Interval on sampling inspection and *x*-Reduced clearance interval for 100% inspection. A new system of skip-lot sampling plan of type SkSP-V is introduced by [14]. The important feature of SkSP-V is the lot is rejected on skipping inspection then the system go to normal inspection with reduced clearance number. The probability of acceptance under the SkSP-V plan is

$$P_a(p) = \frac{fP+(1-f)P^i+fP^{k+1}(P^i-P^x)}{(1+P^{i+k}-P^{2k})+(1-f)P^i} \quad (7)$$

Where, *f*- frequency, *i*- clearance interval, *k*- number of lots that consecutively were accepted under skip-lot inspection, *x*- number of reduced lots that were accepted under normal inspection

VI. Comparison of CSP-R and SkSP-R

CSP-R plan which introduced by [12] the normal-tightened-reduced inspection concept of MIL-STD-105D [13], the US standards for attributes sampling. Monograph for the selection of CSP-R plan is developed by [1]. Skip-lot sampling plan of type Resampling SkSP-R is introduced by [18]. In resampling plan, the producer maintains the quality of product. In SkSP-R sampling plan, m number of time the lots are submitted for resampling. The probability of acceptance under the SkSP-R plan is

$$P_a(p) = \frac{fP+(1-f)P^i+fP^k(P^i-P)(1-Q^m)}{f(1-P^i)[1-P^k(1-Q^m)]+P^i(1+fQP^k)} \quad (8)$$

Where *Q = 1-P*, *P* is the probability of acceptance of a single lot under the reference plan, *f*-the fraction of lots inspected in skipping inspection mode, *i*-clearance number of normal inspection, *k*-the clearance number of sampling inspection, *m*-the number of times the lots are submitted for resampling.

VII. Brief explanation of SkSP-T and compared with CSP-T

The three-level continuous sampling plan assign as CSP-T (T-tightened) is designed by [9]. CSP-T is

derived an Average Outgoing Quality (AOQ) and Average Fraction Inspection (AFI) functions for continuous sampling plan of type CSP-T plan using Markov Chain Techniques. In this sampling plan is modified form of [10] and [4] following the methodology of as a multilevel plan CSP-T allows a reduction in sampling frequency (*f*) as quality improves, reducing the amount of sampling necessary. The sampling frequency is cut in every level then the quality is improved. Many authors study the properties and designing methodologies of continuous sampling plans of type CSP-T. Markov Chain model of CSP-T plan developed by [11], and Modified Tightened three level continuous sampling plan id derived by [3].

Skip-lot sampling plan of type SkSP-T (T-tightened) is introduced by [16]. It is based on the concept of continuous sampling plan of type CSP-T, continuous sampling plan of type CSP-M, modified tightened three level continuous sampling plan and skip-lot sampling plan of type SkSP-2. Sampling level is fixed by using CSP-M procedure; sampling fractions are taken from the CSP-T procedure and other concepts are taken by modified CSP-T and SkSP-2 procedures. The main advantage of skip lot sampling plan of type SkSP-T sampling plan is that the process moves from one level of skipping inspection to another without going back to normal inspection and various references plan are adopted for running normal and skipping inspections. In SkSP-T sampling plan the sampling frequency (*f*) is minimized by every skipping inspection level. This procedure may face the challenges of producer and consumer in terms of risks experienced by them during product and process control. SkSP-T plan contains two inspections, namely normal and skipping inspection. Skip-lot sapling plan is starts with normal inspection using various reference plans. In skipping inspection entire lots in the structure of construction and the skipping inspections are continued. The number of consecutive conforming lots or batches reaches some pre-specified clearance number *i* continue to normal inspection. If *i* consecutive lots are cleared with normal inspection, using skipping inspection with fraction f appear; if another *i* consecutive conforming lots are passed under fractional inspection, the fraction (*f*) is bisecting to *f*/2, and then to *f*/4 provided no non-conforming is found. Then the non-conforming is found in skipping inspection the system goes to normal inspection. SkSP-T plan based on Burr type XII distribution and SkSP-T based on fuzzy logic techniques are developed [19,20].

Operating Procedure for SkSP-T

Step 1: Start with the normal inspection using reference plans.

Step 2: When *i* consecutive lots are accepted on normal inspection, discontinue the normal inspection and started as skipping inspection.

Step 3: Under skipping inspection mode, inspect only fraction *f* of the lots selected randomly, mention first level.

Step 4: After *i* lots in succession have been found without a non-conforming at first level, the system then switches to skipping inspection with a fraction of *f*/2, mention second level.

Step 5: After *i* lots in succession have been found without a non-conforming at second level, the system then switches to skipping inspection with a fraction of *f*/4, called third level.

Step 6: If a non-conforming lot is found on either skipping level, the system reverts to normal inspection.

Step 7: Exchange all non-conforming lots with conforming ones.

The probability of acceptance under SkSP-T plan is

$$\text{Pa (p)} = \frac{P^i(f_2f_3(1-P^i)+f_1f_3P^i(1-P^i)+f_1f_2P^{2i})}{f_1f_2f_3(1-P^i)+P^i(f_2f_3(1-P^i)+f_1f_3P^i(1-P^i)+f_1f_2P^{2i})} \quad (9)$$

Where, P = Probability of Acceptance, *f* is the sampling frequency and *i* is the clearance number.

VIII. Designing plans for given AQL, LQL, α and β

1. Specify p_1 = Acceptable Quality Level at α = 0.05 or 0.01.
2. Specify p_2 = Limiting Quality Level at β = 0.10 or 0.05.

3. Obtain the corresponding ratio $OR = p_2 / p_1$ at different combination of α and β.
4. The actual np_1 and np_2 values corresponding to the OR value has been noted.
5. Determine the sample size $n = np_1 / p_1$. Round up the value for determining the sample Size. Thus, the plan consists of the parameter n, i, f_1, f_2, f_3, and c.

Table 1: *Comparison of Operating Characteristics Values*

P	CSP-2	SkSP-2	CSP-3	SkSP-3	CSP-V	SkSP-V	CSP-T	SkSP-T
0.001	0.9930	0.9976	0.9944	0.9999	0.9965	0.9999	0.9997	0.9999
0.002	0.9748	0.9911	0.9937	0.9996	0.9960	0.9996	0.9987	0.9997
0.003	0.9484	0.9811	0.9921	0.9986	0.9951	0.9990	0.9980	0.9992
0.004	0.9158	0.9682	0.9896	0.9963	0.9934	0.9983	0.9974	0.9990
0.005	0.8784	0.9527	0.9860	0.9921	0.9909	0.9974	0.9967	0.9987
0.006	0.8375	0.9350	0.9810	0.9859	0.9870	0.9962	0.9960	0.9985
0.007	0.7940	0.9155	0.9742	0.9773	0.9813	0.9949	0.9953	0.9982
0.008	0.7488	0.8942	0.9650	0.9663	0.9727	0.9934	0.9946	0.9979
0.009	0.7027	0.8716	0.9528	0.9529	0.9601	0.9916	0.9939	0.9977
0.01	0.6562	0.8477	0.9367	0.9372	0.9418	0.9897	0.9931	0.9974

IX. Comparative study

In nature skip-lot sampling plans are followed by continuous sampling plans. Continuous sampling plan is a series of lots or batches of materials. Skip-lot sampling plan is a bulk materials or products produced in successive lots. In this paper, various types of skip-lot sampling plans (SkSP) are compared with continuous sampling plans (CSP). Skip-lot sampling plan of types SkSP-2, SkSP-3, SkSP-V and SkSP-T using Single Sampling plan as reference plan. Compared to other skip-lot and continuous sampling plans, SkSP-T has more parameters.

From table 1 compared with probability of acceptance (Pa (p)) values of continuous sampling plans and skip-lot sampling plans for using various parameters and its values. It concludes that skip-lot sampling plan has high probability of acceptance (Pa (p)) compared with continuous sampling plans. Hence Skip-lot sampling plans have more efficient than continuous sampling plans. It concludes that SkSP-T is superior to other skip-lot and continuous sampling plans.

X. Conclusion

In this paper, various types of skip-lot sampling plans are compared to Continuous sampling plans. In general skip lot sampling plans are reducing the frequency of sampling inspection and overall inspection cost. Comparison of continuous sampling plans and skip-lot sampling plans the new plan of skip-lot sampling plan of type SkSP-T has high probability of acceptance and good quality level. The new proposed plan of skip-lot sampling plan of type SkSP-T is better protection to the producer and the consumer. The main advantage of skip lot sampling plan of type SkSP-T sampling plan is that the process moves from one level of skipping inspection to another without going back to normal inspection.

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DECISION SUPPORT SYSTEM OF EVAPORATING SYSTEM OF SUGAR PLANT

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Abstract

This paper addresses an analysis methodology for assessing the efficacy of a evaporating system in a sugar industry. A stochastic Petri nets technique is employed to simulate the interactions between the subsystems. A software package, "Petri module," from GRIF, was licensed. The performability of subsystems has been evaluated, and fluctuations in repair and failure rates have been observed. The maintenance order priority was assigned to the subsystems of the evaporating system based on the criticality of failure. Finally, a decision support system is implemented to assist maintenance personnel in making more informed decisions during the development of maintenance policies. It has been noted that the evaporator is an essential component that requires the complete attention of the plant manager.

Keywords: availability analysis, evaporating system, sugar industry, petri nets.

I. Introduction

In the current scenario, modern process facilities must be run at high levels of availability due to the high cost of installation, operation, and maintenance [1]. As a result, achieving high levels of availability is crucial for ensuring their efficiency and economic viability. In this regard, a reliability study should evaluate its availability and provide a method for identifying possible event combinations that might lead to disastrous failures and assess the prospect that they would occur [2]. The assessment process should include cost-related elements and be able to examine other performance metrics [3]. The state-transition diagram (Markov models), Petri nets (PNs), fault tree analysis, event tree analysis (FTA and ETA), and network models are some of the key modeling techniques used in reliability analysis [4]. Models of networks are function-oriented. These models may address structural defects that impair system performance. Maintenance procedures, human and software errors, and other cost-related factors are complicated to include in network models [5]. Trees of faults are event-oriented. Including the repair processes and component reliance in the model is complex [6]. The static structure of failure trees makes it difficult to simulate dynamic behaviour such as standby redundancies and time-delay situations. The inflation of state space is the main disadvantage of Markov models. State-space explosion restricts the use of this formulation, despite the fact that it may describe the dynamic behaviour and reliance between components [7]. When creating a Markov model of a complicated system, it may be challenging to ensure that every conceivable outcome in a subsystem has been considered [8]. State-transition diagrams are also highly challenging to employ for model validation. In this research, a complicated system that is

concurrent, asynchronous, distributed, parallel, and non-deterministic is studied using PN, a graphical and mathematical modeling tool. Using PNs for reliability analysis makes the modeler's job much more manageable. The process involves sketching a net that describes a system model and labeling it with the appropriate transition firing timings [9], [10]. If tools could be developed to automate the process of determining the probability of markings and the algorithms for constructing the set of all reachable markings of a Petri net (PN), analysts would be able to focus more on addressing reliability concerns, rather than spending time writing and solving the equations for the underlying stochastic process. PNs provide for a systems approach as they employ a common vocabulary to represent human behaviour, software, and hardware. Safety and fault tolerance standards may also be included.

II. Literature Review

Malik and Tiwari [11] conducted an assessment of the Coal Ash Handling System's performance at a subcritical thermal power plant. This model combines State probabilities using a normalizing condition. Parkash and Tiwari [12] created performance modeling and suggested a DSS to prioritize repair activities for an assembly line system. Kumar [13] highlighted the importance of many practical units in his Decision Support Priorities framework. Sheikh and Tewari [14] examined the applicability of Reliability, Availability, Maintainability, and Safety ideas in several process sectors to improve performability. Stochastic processes address unpredictability in systems, particularly about unpredictable temporal changes. Performability analysis enhances the understanding of the system's performance behaviour. This enables us to make better-informed decisions on the design and operation of systems to enhance their reliability and efficacy. Mehta et al. [15] examined the steel plant's sheet production unit's dependability, availability, and maintainability. The extractor, conveyors, de-scaling unit, furnace, roughing mill, Steckel mill, strapping machine, and down coiler were among the eight systems that were part of the unit. Simpson's 3/8 rule and the Runge–Kutta fourth-order approach have been used in MATLAB to assess the system's availability. Kumar and Ram [16] used the Markov approach to examine a number of reliability metrics, including availability, reliability, and MTTF for the sugar factory. Apart from the bagasse carrying mechanism, the system's MTTF declined for all failure scenarios. With the exception of the bagasse carrying method, dependability appears physically the same in all failure kinds. Aly et al. [17] used a model based on the Markov technique and state probability to assess systems' availability, dependability, and maintainability for the oil and gas sector. Jalal et al. [18] used SPNs modelling to offer a power production facility's reliability, availability, and maintainability (RAM) analysis. This model assumed that the maintenance and repair crews would always be accessible, regardless of the number of components that failed at any one moment. They showed that, in comparison to the reliability block diagram simulation, SPN modeling produced a smoother exponential curve when analyzing the system's mean availability.

Most models for estimating complex system availability and reliability are based on FMECA, FTA, Markov, and Bayesian network approaches, and few process plants use PNs to consider subsystems and component dependencies.

III. System Description

The evaporation system (ES) removes water from the juice received from the clarifier system. All cells in the ES are in series. A three-subsystem ES is presented below. Fig. 1 shows the ES flowchart.

Evaporator (EV): The system consists of a single unit, and its failure results in the complete failure of the entire system. Under this subsystem, a low vacuum is used to heat the juice.

Syrup Sulphitor (SY): The system consists of two units: one operational unit and one standby unit. If one unit fails, the system continues to operate at a reduced capacity. However, if both units fail, the system fails entirely. This subsystem heats juice at a high vacuum and passes SO2 gas.

Sulphited Syrup (SU): The system consists of three units. If one or two units fail, the system continues to operate at a reduced capacity. However, if all units fail, the system fails entirely.

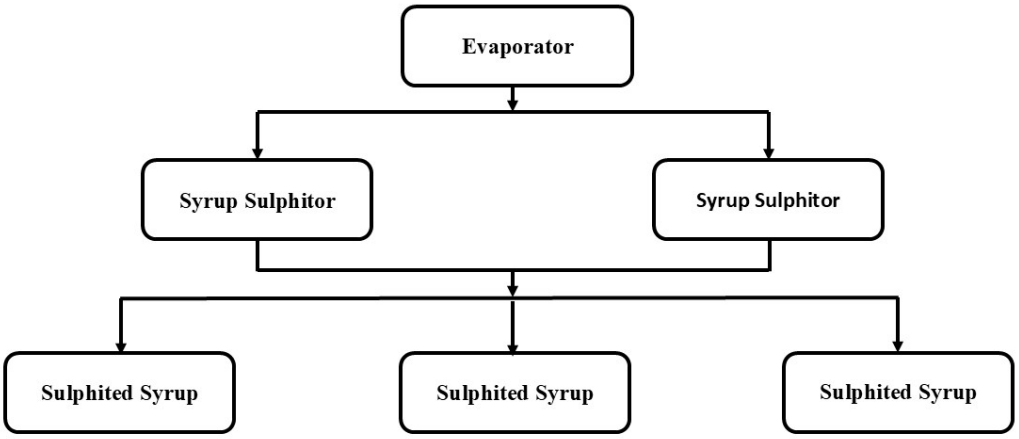

Figure 1. *Flow diagram of ES*

IV. Petri Nets Modelling

This section assesses several components of ES in the sugar industry. The maintenance workers and supervisors helped us extract component FRs and RRs from maintenance and repair manuals. FR and RR values were considered to be Weibull-distributed. The Monte Carlo Simulation Approach-based MOCA-Computation engine modelled ES performance. This was accomplished using the SPN, as seen in Figure 2.

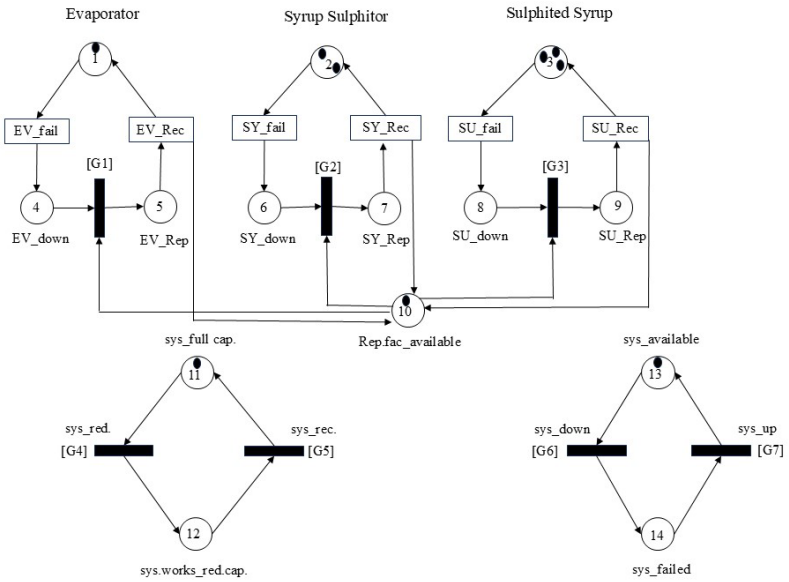

Figure 2. *PNs modelling of ES*

I. Assumptions Notations

The following assumptions were used to employ PNs for availability analysis:

- The FRs and RRs of a number of components follow exponential distributions.
- Individual component failures occur.
- The functionality of repaired components is equivalent to that of new ones.
- Repairs comprise both component replacements and repairs.
- It is not anticipated that two or more components will fail simultaneously.
- Active systems are functionally equivalent to standby components.
- The patterns of FRs and RRs remain statistically independent and stable over time.

II. Places

sys_available: signifies that the entire system is operational and available for use.
sys. works_full cap.: denotes the system's condition when operating at full capacity.
sys.works_red.cap: signifies that the system is operating at a reduced capacity.
sys_failed: denotes the system's state during downstate.
rep. facilities_available: denotes the facility's capacity to undergo swift repairs.
EV_up, SY_up, and SU_up: denote the operational condition, which is the status of the EV, SY, and SU systems, respectively.
EV_down, SY_down, and SU_down: denote the non-operational condition of the EV, SY, and SU systems, respectively.
EV_Rep, SY_Rep, and SU_Rep: represent the restored conditions of the EV, SY, and SU systems, respectively.

III. Transitions

EV_fail, SY_fail, and SU_fail represent timed transitions related to the failure patterns of EV, SY, and SU systems, respectively.
EV_OK, SY_OK, and SU_OK represent cautious transitions associated with the repair patterns of the EV, SY, and SU systems, respectively.
Rep. avail_ EV, Rep. avail_ SY, Rep. avail_ SU: These instantaneous transitions imply that the EV, SY, and SU systems are all instantly available.
sys_ red, sys_ recovered, sys_ fail, and sys_ ok: These are instantaneous transitions that are fire without delay.

IV. Guard Function (GF)

[G1]: = (#4>0 and #10>0) rep. avail_ EV transition was initiated when the GF ensured.

[G2]: = (#6>0 and #10>0) rep. avail_ SY transition was initiated when the GF ensured.

[G3]: = (#8>0 and #10>0) rep. avail_ SU transition was initiated when the GF was ensured.

[G4]: = (#2<2 and #2>0 or #3<3 and #4>0) sys_ red transition initiated when the GF ensured.

[G5]: = (#2>1 and #3>2) blocks transition from firing sys_ recovered.

[G6]: = (#1<1 or #2<1 or #3<1) sys_ fail transition was initiated when the GF ensured.

[G7]: = (#1>0 and #2>0 and #3>0) blocks transition from firing sys_ ok.

V. Performance Analysis

The system's dynamic behavior was evaluated using variables to determine its performance characteristics. After consulting with plant maintenance experts, the acceptable failure and repair rates for subsystems (Table 1) were established. These parameters are also examined about repairman availability. The findings are shown in Tables 2-11 and discussed further below.

Table 1: *Availability matrix for EV*

σ \ Ψ	0.25	0.35	0.45	0.55	0.65
0.021	0.8269	0.8375	0.8442	0.8490	0.8529
0.026	0.8181	0.8277	0.8344	0.8407	0.8456
0.031	0.8061	0.8179	0.8265	0.8332	0.8374
0.036	0.7954	0.8078	0.8175	0.8249	0.8309
0.041	0.7837	0.7973	0.8083	0.8173	0.8240

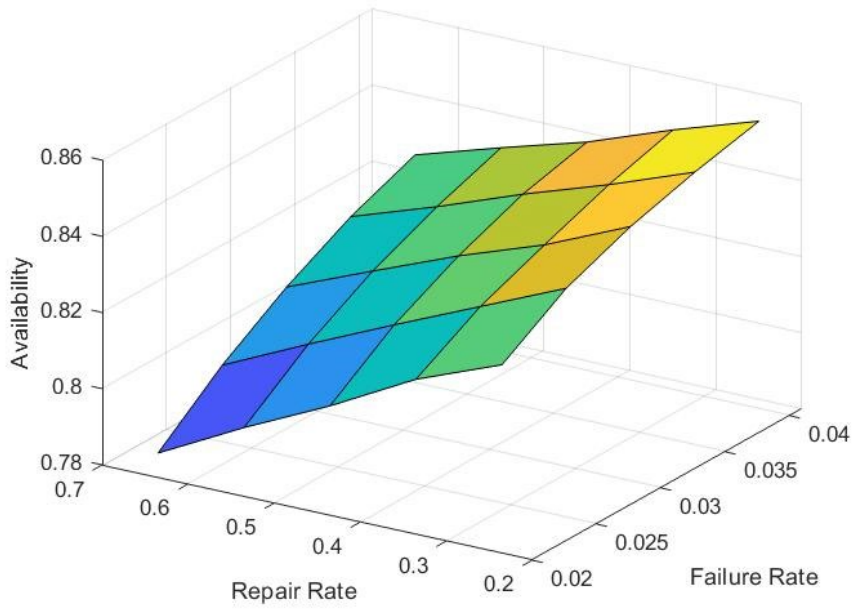

Figure 3. *Influence of varying FR and RR of EV on the availability*

Table 1 and Fig. 3 illustrate the influence of FRs and RRs in the EV on the ES's overall availability. With an increase in the FR from 0.021 to 0.041, there is a significant decrease in system availability, equivalent to a 5.22% decrease. In contrast, the increase in RRs from 0.25 to 0.65 resulted in a slight improved system availability of 3.04%.

Table 2: *Availability matrix for SY*

σ \ Ψ	0.47	0.57	0.67	0.77	0.87
0.040	0.8287	0.8327	0.8365	0.8379	0.8397
0.045	0.8227	0.8270	0.8303	0.8328	0.8345
0.050	0.8179	0.8223	0.8265	0.8279	0.8304
0.055	0.8124	0.8180	0.8207	0.8237	0.8259
0.060	0.8066	0.8121	0.8152	0.8194	0.8232

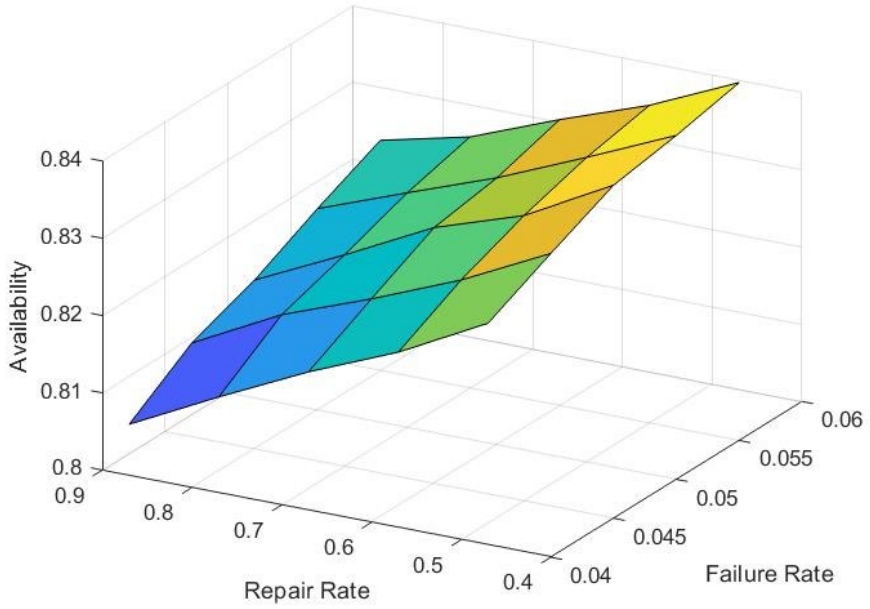

Figure 4. *Influence of varying FR and RR of SY on the availability*

Table 2 and Fig. 4 illustrate the impact of FRs and RRs on the availability of the ES, with a particular emphasis on the SY. When the SY's FR is increased from 0.040 to 0.060, there is a slight decrease in system availability of 2.66%. Conversely, the system's availability is marginally enhanced by 1.3% as a result of the RR being increased from 0.47 to 0.87.

Table 3: *Availability matrix for SU*

σ \ Ψ	0.10	0.20	0.30	0.40	0.50
0.017	0.8258	0.8328	0.8362	0.8397	0.8427
0.022	0.8201	0.8251	0.8299	0.8335	0.8374
0.027	0.8117	0.8188	0.8265	0.8294	0.8303
0.032	0.8019	0.8089	0.8164	0.8209	0.8259
0.037	0.7945	0.8015	0.8091	0.8141	0.8189

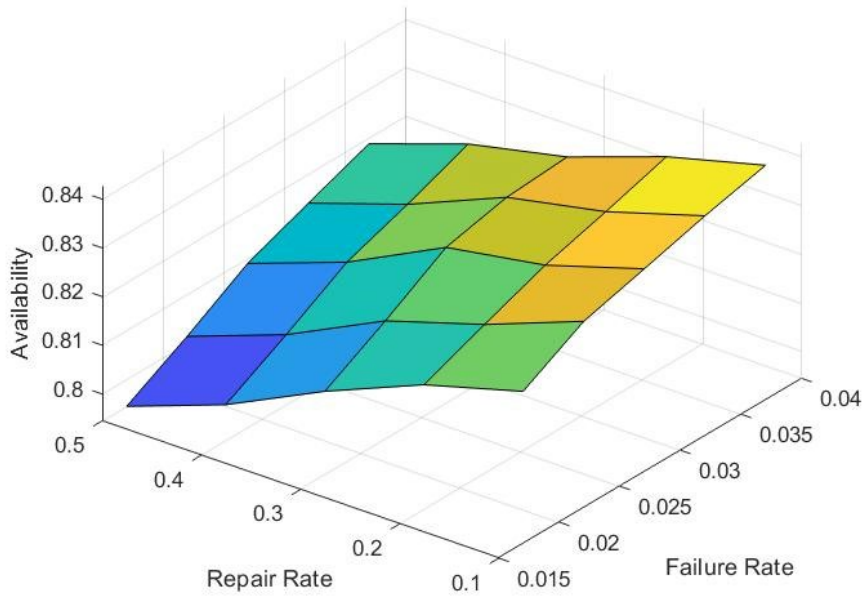

Figure 5. *Influence of varying FR and RR of SU on the availability*

The availability of the ES is influenced by FRs and RRs, as illustrated in Table 3 and Fig. 5, with a particular focus on the SU. The SU's FR increases from 0.019 to 0.039, resulting in a minor decrease in system availability of 3.8%. When the RR is increased from 0.37 to 0.57, the system's availability rises by 2.01%.

According to the study's findings, the EV failure immediately affected total system availability by 5.22%. Thus, the EV is shown to be the most crucial part of the ES, with an FR of 0.031. Likewise, the SY is the least significant element, with an FR of 0.050. Therefore, according to the ES system's ideal FRs and RRs, the maintenance priorities need to be allocated in the following order (as shown in Table 4).

Table 4: *List of maintenance priority*

Component	FR	RR	Reduction in Av due to FR	Elevation in Av due to RR	Maintenance priority No.
EV	0.021 -0.041	0.25-0.65	5.22	3.04	I
SY	0.040- 0.060	0.47-0.87	2.66	1.3	III
SU	0.019 - 0.039	0.37-0.57	3.8	2.01	II

Table 5: *Influence of variation in the RF on availability of CSPS*

No. of RF	1	2	3	4
Availability	0.8265	0.8520	0.8621	0.8621

Figure 6. *Influence of variation in RF on the availability of CSPS*

Table 5 and Fig. 6 show the effect of higher RF on the ES's total availability. From 82.65% to 85.20%, the system's overall performance improves considerably as the number of RF increases from one to two. When RF goes up from 2 to 3, availability goes up from 85.20% to 86.21%, which is a significant improvement. The performance becomes stable at three RF, indicating that the service is consistently available.

VI. Conclusion

The model demonstrates efficacy in evaluating the performance of various ES components, facilitating maintenance decision-making. The study enabled us to assess the impact of several factors, namely FRs and RRs, on the unit's availability. The system's availability diminishes as the failure rate rises. Conversely, higher RRs result in increased system availability. Consequently, increasing the RRs and decreasing the FRs across all four subsystems is essential to improve the ES's performance. Therefore, optimizing FRs and RRs data is essential for attaining high efficiency. The suggested model is effectively used to evaluate the performance of ES in the sugar industry, facilitating decision-making about maintenance measures. Performance metrics indicate that the EV component is crucial for maintenance. Immediate care is necessary since the failure rates of the EV significantly impact system availability relative to other subsystems. Further, the SY component emerges as the least crucial, as the FRs of the SY have minimal influence on the system availability. As industries progress, the insights derived from these evaluations are essential for sustaining system performance and minimizing downtime.

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MODELING RELIABILITY IN k-OUT-OF-m SYSTEMS WITH UNEQUAL LOAD SHARING USING PROPORTIONAL CONDITIONAL REVERSE HAZARD RATE

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Abstract

This paper explores a load-sharing model within a k-out-of-m system, where multiple components work together to handle a shared load. Such systems are prevalent in various engineering and industrial applications.While previous studies have focused on equal load-sharing rules, this research emphasizes systems operating under an unequal load-sharing rule, which has a significant impact on the system's reliability and performance. Specifically, the paper examines a k-out-of-m load-sharing system modeled using the proportional conditional reverse hazard rate model, incorporating unequal load sharing. We have derived expressions for the probability density function and cumulative distribution function of system failure. To illustrate the model, they use a 2-out-of-4 configuration with Weibull baseline distributions. The maximum likelihood estimation method is employed to estimate the model parameters, and the performance of these estimates is evaluated through a simulation study, assessing both bias and mean square errors. Additionally, the practical applicability of the model is demonstrated through the analysis of two real datasets.

Keywords: k-out-of-m system, Load sharing phenomenon, Order statistics, Proportional hazard rate, Reverse hazard rate, Unequal load share rule.

1. Introduction

A k-out-of-m system with m components fails when $(m-k+1)$ or more of its components fail. This setup includes series systems (where $k=m$) and parallel systems (where $k=1$) as specific instances. Generally, the failure times of the first $(m-k+1)$ components are modeled as the first $(m-k+1)$ order statistics from a set of m independent and identically distributed (i.i.d.) random variables. In k-out-of-m load-sharing systems, when one component fails, the load of the failed component is distributed on the remaining components. This creates a dependency among the lifetimes of the components, making these systems dynamic in terms of reliability. Failure of component may increase of release the load of reaming components. Examples of k-out-of-m load-sharing systems include fibrous composite materials, power plants, automobiles, and the two jet engines of an airplane. A similar pattern is observed in the human body, where the failure of one organ (e.g., a kidney) typically increases the failure rate of the surviving organ [1]. Conversely, in scenarios like food scarcity within a letter, the death of some offspring can improve the survival and growth of the remaining ones by increasing their food supply [10]. Similarly, in software development, detecting one bug can help in finding others, thereby reducing the detection time.

[12] and Kvam and Pena [13] have emphasized the importance of modeling the load-sharing phenomenon in various contexts. Besides these works, load-sharing systems have been explored extensively, beginning with Daniels [1], and continuing through the studies of Birnbaum and Saunders [2], Coleman [3], Rosen [4], [5], Singpurwalla [6], Hollander and Pena [7], Cramer and Kamps [8, 12], Lynch [11], Durham and Lynch [12], McCool [13], Pena [16], and Deshpande et al. [17], among others. A comprehensive review can be found in Dewan and Naik-Nimbalkar [18]. Then after Deshpande et al. [17], Jain and Gupta [19], Sutar and Naik-Nimbalkar [20], Sutar and Naik-Nimbalkar [21], Wang et al. [22], Zhao et al. [23], Xu et al. [24], Zhang et al. [25], Sutar and Naik-Nimbalkar [26], Choudhary et al. [27], Park et al. [28], Sutar [29], Zhang et al. [30], Rykov et al. [31], Pesch et al. [32], Sutar et al. [33], Biswas et al. [34], and Pesch et al. [35] all contributed to the study of load-sharing systems.

Sutar and Naik-Nimbalkar [21] explored load-sharing systems within the framework of a k-out-of-m system using a proportional conditional reverse hazard rate model. They concentrated on equal load sharing, particularly examining a two-component parallel system where the components' initial lifetimes followed Weibull and linear failure rate distributions as baseline models. Their research also extended to developing and applying inference techniques for analyzing this system. However, the assumption of equal load sharing does not always hold true, making it essential to create models that account for unequal load distribution. In these situations, it is necessary to develop models that accurately represent the varying loads each component bears to more effectively predict system reliability. An example of this is as follows.

Consider a power grid with multiple generators providing electricity to a city. If one generator fails, the remaining generators must absorb the additional load, but they may not equally distribute this increased demand due to differences in capacity and efficiency. In such a scenario, modeling the system with an unequal load-sharing approach is vital for accurately predicting the grid's reliability and preventing potential outages.

In this article, we present a model for the load-sharing phenomenon in a k-out-of-m system, utilizing proportional conditional reverse hazard rate (PCRHR) with an unequal load-sharing rule. We aim to capture the complexities of real-world systems where components do not equally share the load after failures occur. The structure of this article is as follows:

Section 2 introduces the model for a k-out-of-m unequal load-sharing system. Illustration of the model is given in Section 3. In Section 4, we examine parameter estimation for k-out-of-m and in particular 2-out-of-4, 2-out-of-3, and 1-out-of-4 load sharing systems, assuming that lifetimes are independently distributed according to a Weibull distribution. Section 5 reports a simulation study that assesses the proposed estimation method's performance based on bias and mean square error. Section 6 applies the model to a real dataset, and the final section summarizes the conclusions.

2. Proposed PCRHR based Load Sharing Model for k-out-of-$m$ystem

Let $U_1, U_2, \ldots U_m$ are the components of a k-out-of-m system $(1 \leq k \leq m)$. Let use assume that the lifetimes of components are independent and identical distributed (i.i.d.) with baseline probability density function (pdf) $f(\cdot)$, Cumulative distribution function (cdf) $F(\cdot)$ and survival function (sf) $\bar{F}(\cdot)$ and reverse hazard rate function $r(\cdot)$. The system is to be functioning as long as $(m-k+1)$ components is not failed. Let $X_{(j)}$ be the time of j^{th} failure in the system, $j = 1, 2, \ldots, (m-k+1)$. That is $X_{(j)}$ is minimum of the failure times of remaining $(m-j+1)$ surviving components and system failure is occurs at time point $X_{(m-k+1)}$. In the i.i.d. set up, the failure of a component does not affect the lifetime of the surviving component and the joint density of $X_{(j)}$ and $X_{(j+1)}$ can be written as

$$\begin{aligned} g_{X_{(j)},X_{(j+1)}}(x_j, x_{j+1}) = & \frac{m!}{(j-1)!(m-j-1)!} \left(F(x_j)\right)^{j-1} \\ & \left(\bar{F}(x_{j+1})\right)^{m-j-1} f(x_j) f(x_{j+1}), \quad j = 1, 2, \ldots, (m-k). \end{aligned} \tag{1}$$

The conditional density function of $X_{(j+1)}$ given $X_{(j)} = x_j$ is given by

$$g_{X_{(j+1)}|X_{(j)}=x_j}(x_{j+1}) = (m-j)\left\{\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right\}^{m-j}\frac{f(x_{j+1})}{\bar{F}(x_{j+1})},\ \ j = 1,2,\ldots,(m-k). \tag{2}$$

Thus the conditional distribution function of $X_{(j+1)}$ given $X_{(j)} = x_j$ is given by

$$G_{X_{(j+1)}|X_{(j)}=x_j}(x_{j+1}) = 1 - \left(\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right)^{m-j},\ \ j = 1,2,\ldots,(m-k). \tag{3}$$

Assuming that there exists load sharing effect in model, then conditional distribution function with load sharing effect is given by

$$\begin{aligned} H_{X_{(j+1)}|X_{(j)}=x_j}(x_{j+1}) &= \left\{G_{X_{(j+1)}|X_{(j)}=x_j}(x_{j+1})\right\}^{\beta_j} \\ &= \left\{1 - \left(\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right)^{m-j}\right\}^{\beta_j}, \end{aligned} \tag{4}$$

$$\beta_j > 0,\ \ x_{j+1} \geq x_j,\ \ j = 1,2,\ldots,(m-k).$$

Remark (1): From the above equation (4), we observe that if $\beta_j < 1$ then $H_{X_{(j+1)}|X_{(j)}=x_j}(x_{j+1}) > G_{X_{(j+1)}|X_{(j)}=x_j}(x_{j+1}), \forall\ j = 1,2,\ldots,(m-k)$, The lifetime of the surviving component under the distribution H is stochastically smaller than under the i.i.d. setup G. and if $\beta_j > 1$ then the residual life is stochastically larger. For $\beta_j = 1, \forall\ j = 1,2,\ldots,(m-k)$ means residual lifetimes are same as that under i.i.d. setup.
We also assume that the conditional distribution of the residual lifetime of a surviving component depends only on the last failure time, that is the failure epochs form a Markov process.

Theorem 1. If the conditional distribution of $X_{(j+1)}$ given $X_{(j)} = x_j$ is as given in equation (4), then we have the following

(i) The joint p.d.f. of $(X_{(1)}, X_{(2)}, ..., X_{(m-k+1)})$ is given by,

$$\begin{aligned} h(x_1, x_2, \ldots, x_{m-k+1}) =& \frac{m!}{(k-1)!}\prod_{j=1}^{(m-k)}\beta_j\left\{1 - \left(\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right)^{m-j}\right\}^{\beta_j-1} \\ & \left(\bar{F}(x_{m-k+1})\right)^{k-1}\prod_{j=1}^{m-k+1} f(x_j). \end{aligned} \tag{5}$$

$$0 < x_1 < x_2 < ... < x_{m-k+1},\ \beta_j > 0.$$

(ii) The marginal density of the system failure time X_{m-k+1}, is given by

$$\begin{aligned} h_{X_{(m-k+1)}}(x_{m-k+1}) =& m\prod_{j=1}^{m-k}\beta_j\left(\bar{F}(x_{m-k+1})\right)^{(m-1)} f(x_{m-k+1}) \\ & \int_{x_{m-k}=0}^{1-a_{m-k}} u_{m-k}^{\beta_{m-k}-1}(1-u_{m-k})^{\frac{-m}{k}} \cdots \int_{x_j=0}^{1-a_j} u_j^{\beta_j-1}(1-u_j)^{\frac{-m}{m-j}} \\ & \cdots \int_{x_1=0}^{1-a_1} u_1^{\beta_1-1}(1-u_1)^{\frac{-m}{m-1}} du_1 du_2 \cdots du_{(m-k)}. \end{aligned} \tag{6}$$

$0 < x_{m-k+1} < \infty,\ \beta_j > 0.\ \forall\ j = 1,2,\ldots,(m-k)$, where, $a_{m-k} = \left(\bar{F}(x_{m-k+1})\right)^k$ and

$$a_j = \frac{\left(\bar{F}\left(x_{m-k+1}\right)\right)^{m-j}}{\left(1-u_{j+1}\right)^{\frac{m-j}{m-j-1}}\left(1-u_{j+2}\right)^{\frac{m-j}{m-j-2}}\cdots\left(1-u_{m-k}\right)^{\frac{m-j}{m-(m-k)}}}, \quad 1 \le j \le (m-k-1).$$

Proof. (a) Using (4), the conditional density of $X_{(j+1)}$ given $X_{(j)} = x_j$, is given by

$$h_{X_{(j+1)}|X_{(j)=x_j}}\left(x_{j+1}\right) = (m-j)\beta_j\left\{1-\left(\frac{\bar{F}\left(x_{j+1}\right)}{\bar{F}\left(x_j\right)}\right)^{m-j}\right\}^{\beta_j-1}\frac{f\left(x_{j+1}\right)\left(\bar{F}\left(x_{j+1}\right)\right)^{m-j-1}}{\left(\bar{F}\left(x_j\right)\right)^{m-j}}, \tag{7}$$

$$\beta_j > 0, \;\; x_{j+1} \ge x_j, \;\; j = 1,2,\ldots,(m-k).$$

Using the Markov assumption, the joint density of $(X_{(1)}, X_{(2)}, ..., X_{(m-k+1)})$ can be written

$$h\left(x_1, x_2, \ldots, x_{m-k+1}\right) = \prod_{j=1}^{(m-k)} h_{X_{(j+1)}|X_{(j)=x_j}}\left(x_{j+1}\right) h(x_1),\; 0 < x_1 < x_2 < ... < x_{m-k+1},\; \beta_j > 0.$$

Using expression (7) and the fact that $h(x_1) = m\left(\bar{F}(x_1)\right)^{m-1} f(x_1), \;\; x_1 > 0$, we get (5).
Proof. (b) The marginal density function of $X_{(m-k+1)}$ is obtained by integrating equation (5) with respect to $x_1, x_2, \ldots, x_{m-k}$ over th region defined by $0 \le x_1 \le x_2 \le \cdots \le x_{m-k+1}$.
That is,

$$\begin{aligned} h_{X_{(m-k+1)}}(x_{m-k+1}) &= \int_{x_{m-k}=0}^{x_{m-k+1}}\int_{x_{m-k-1}=0}^{x_{m-k}}\cdots\int_{x_1=0}^{x_2} h\left(x_1, x_2, \ldots, x_{m-k+1}\right) dx_1 dx_2 \cdots dx_{(m-k)} \\ &= \frac{m!}{(k-1)!}\left(\bar{F}(x_{m-k+1})\right)^{k-1} f(x_{m-k+1}) \int_{x_{m-k}=0}^{x_{m-k+1}}\int_{x_{m-k-1}=0}^{x_{m-k}}\cdots \\ &\quad \int_{x_1=0}^{x_2}\prod_{j=1}^{(m-k)}\left\{1-\left(\frac{\bar{F}\left(x_{j+1}\right)}{\bar{F}\left(x_j\right)}\right)^{m-j}\right\}^{\beta_j-1} f(x_j) dx_1 dx_2 \cdots dx_{(m-k)}. \end{aligned}$$

Let

$$I_1 = \int_{x_1=0}^{x_2}\left\{1-\left(\frac{\bar{F}\left(x_2\right)}{\bar{F}\left(x_1\right)}\right)^{m-1}\right\}^{\beta_1-1} f(x_1) dx_1.$$

Putting $u_1 = 1-\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{m-1}$ in above expression and simplifying, we get

$$I_1 = \frac{\bar{F}(x_2)}{m-1}\int_{u_1=0}^{1-\left(\bar{F}(x_2)\right)^{(m-1)}} u_1^{\beta_1-1}(1-u_1)^{\frac{-m}{m-1}} du_1.$$

Similarly, let

$$\begin{aligned} I_2 =& \frac{1}{m-1}\int_{x_2=0}^{x_3}\left\{1-\left(\frac{\bar{F}\left(x_3\right)}{\bar{F}\left(x_2\right)}\right)^{m-2}\right\}^{\beta_2-1}\bar{F}(x_2) f(x_2) \\ & \int_{u_1=0}^{1-\left(\bar{F}(x_2)\right)^{(m-1)}} u_1^{\beta_1-1}(1-u_1)^{\frac{-m}{m-1}} du_1 dx_2. \end{aligned}$$

After putting $u_2 = 1-\left(\frac{\bar{F}(x_3)}{\bar{F}(x_2)}\right)^{m-2}$ and simplifying we get

$$\begin{aligned} I_2 =& \frac{\left(\bar{F}(x_2)\right)^2}{(m-1)(m-2)}\int_{u_1=0}^{1-\left(\bar{F}(x_3)\right)^{(m-2)}} u_2^{\beta_2-1}(1-u_2)^{\frac{-m}{m-2}} \\ & \int_{u_1=0}^{1-\left(\bar{F}(x_3)\right)^{(m-1)}(1-u_2)^{-\frac{(m-1)}{(m-2)}}} u_1^{\beta_1-1}(1-u_1)^{\frac{-m}{m-1}} du_1 du_2. \end{aligned}$$

Proceeding in the same manner and by letting $I_3, I_4, \cdots, I_{(m-k)}$, we get (6).

Remark 2. Under the model defined by (4), the conditional reversed hazard rate (CRHR) $r^*_{X_{(j+1)}|X_{(j)}=s}(t)$ of $X_{(j+1)}$ given $X_{(j)} = s$ is proportional to the CRHR $r_{X_{(j+1)}|X_{(j)}=s}(t)$ under the i.i.d. setup. That is,

$$r^*_{X_{(j+1)}|X_{(j)}=s}(t) = \beta_j r_{X_{(j+1)}|X_{(j)}=s}(t), \;\; \beta_j > 0, \; t \geq s, \; j = 1, 2, ..., (m-k). \tag{8}$$

Thus we refer to the model given in (5) as proportional conditional reversed hazard rate (PCRHR) model.

3. Illustration

3.1. The PCRHR based Load Sharing Model for 2-out-of-4 System

Let us consider the system involve four components with component lifetimes U_1, U_2, U_3, U_4 and are being iid with common lifetime distribution $f_{\underline{\theta}}(.)$, $\underline{\theta}$ may be scale or vector valued parameter. $f_{\underline{\theta}}(.)$ is called the baseline distribution and $\underline{\theta}$ is known as baseline parameter. Let $F_{\underline{\theta}}(.)$ and $\bar{F}_{\underline{\theta}}(.)$ are the distribution function (d.f.) and survival function (s.f.).

The 2-out-of-4 system will work until 3 components work and system fails after the tird failure. Let $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ be the first, second and third failure times. Therefore the $p.d.f.$ of $X_{(1)}$ is given by

$$h_{X_{(1)}}(x_1) = 4 f_{\underline{\theta}}(x_1) \left(\bar{F}_{\underline{\theta}}(x_1)\right)^3, \quad x_1 > 0.$$

From equation (7) the conditional density of $X_{(2)}$ given $X_{(1)} = x_1$ is given by

$$h_{X_{(2)}|X_{(1)}=x_1}(x_2) = 3\beta_1 \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_2)}{\bar{F}_{\underline{\theta}}(x_1)}\right)^3\right\}^{\beta_1 - 1} \frac{f_{\underline{\theta}}(x_2)\left(\bar{F}_{\underline{\theta}}(x_2)\right)^2}{\left(\bar{F}_{\underline{\theta}}(x_1)\right)^3},$$

$$0 \leq x_1 \leq x_2 \leq \infty, \;\; \beta_1 > 0,$$

and the conditional density of $X_{(3)}$ given $X_{(2)} = x_2$ is given by

$$h_{X_{(3)}|X_{(2)}=x_2}(x_3) = 2\beta_2 \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_3)}{\bar{F}_{\underline{\theta}}(x_2)}\right)^2\right\}^{\beta_2 - 1} \frac{f_{\underline{\theta}}(x_3)\bar{F}_{\underline{\theta}}(x_3)}{\left(\bar{F}_{\underline{\theta}}(x_2)\right)^2},$$

$$0 \leq x_2 \leq x_3 \leq \infty, \;\; \beta_2 > 0,$$

Therefore the joint distribution of $(X_{(1)}, X_{(2)}, X_{(3)})$ is given by

$$h(x_1, x_2, x_3) = 4!\beta_1\beta_2\bar{F}_{\underline{\theta}}(x_3) \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_2)}{\bar{F}_{\underline{\theta}}(x_1)}\right)^3\right\}^{\beta_1 - 1} \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_3)}{\bar{F}_{\underline{\theta}}(x_2)}\right)^2\right\}^{\beta_2 - 1} \prod_{j=1}^{3} f_{\underline{\theta}}(x_j). \tag{9}$$

$$0 \leq x_1 \leq x_2 \leq x_3 \leq \infty, \;\; \beta_1, \beta_2 > 0$$

The marginal distribution of system failure, i.e. distribution of $X_{(3)}$ is given by

$$\begin{aligned} h_{X_{(3)}}(x_3) =& 4\beta_1\beta_2 f_{\underline{\theta}}(x_3) \left(\bar{F}_{\underline{\theta}}(x_3)\right)^3 \int_{u_1=0}^{1-(\bar{F}(x_3))^2} u_2^{\beta_2 - 1}(1-u_2)^{-2} \\ & \int_{u_1=0}^{1-(\bar{F}(x_3))^3(1-u_2)^{\frac{-3}{2}}} u_1^{\beta_1 - 1}(1-u_1)^{\frac{-4}{3}} du_1 du_2. \end{aligned} \tag{10}$$

$$0 \leq x_3 \leq \infty, \;\; \beta_1, \beta_2 > 0.$$

In the following subsection, we demonstrate the 2-out-of-4 load-sharing system using a Weibull baseline distribution.

3.2. The PCRHR based Load Sharing Model for 2-out-of-4 System with Weibull Baseline Distribution

Let us consider a 2-out-of-4 load sharing system, with the component lifetimes U_1, U_2, U_3 and U_4 being i.i.d. Weibull random variables. The p.d.f. of Weibull with shape parameter a and scale parameter b is

$$f(u_i) = \left(\frac{a}{b}\right)\left(\frac{u_i}{b}\right)^{a-1} \exp\left\{-\left(\frac{u_i}{b}\right)^{a}\right\}, \quad u_i > 0, \;\; a, b > 0, \;\; i = 1, 2, 3, 4.$$

Therefore the p.d.f of first failure $X_{(1)}$ is given by

$$h_{X_{(1)}}(x_1) = 4\left(\frac{a}{b}\right)\left(\frac{x_1}{b}\right)^{a-1} \exp\left\{-4\left(\frac{x_1}{b}\right)^{a}\right\}, \quad x_1 > 0, \;\; a, b > 0, \;\; i = 1, 2, 3, 4.$$

The conditional p.d.f of $X_{(2)}$ given $X_{(1)} = x_1$ is given by

$$\begin{aligned} h_{X_{(2)}|X_{(1)}=x_1}(x_2) =& 3\beta_1\left(\frac{a}{b}\right)\left(\frac{x_2}{b}\right)^{a-1} \exp\left\{-3\left[\left(\frac{x_2}{b}\right)^{a} - \left(\frac{x_1}{b}\right)^{a}\right]\right\} \\ & \left[1 - \exp\left\{-3\left[\left(\frac{x_2}{b}\right)^{a} - \left(\frac{x_1}{b}\right)^{a}\right]\right\}\right]^{\beta_1 - 1}, \end{aligned} \tag{11}$$

$$0 \le x_1 \le x_2 \le \infty, \;\; \beta_1 > 0, \;\; a, b > 0.$$

and conditional p.d.f of $X_{(3)}$ given $X_{(2)} = x_2$ is given by

$$\begin{aligned} h_{X_{(3)}|X_{(2)}=x_2}(x_3) =& 2\beta_2\left(\frac{a}{b}\right)\left(\frac{x_3}{b}\right)^{a-1} \exp\left\{-2\left[\left(\frac{x_3}{b}\right)^{a} - \left(\frac{x_2}{b}\right)^{a}\right]\right\} \\ & \left[1 - \exp\left\{-2\left[\left(\frac{x_3}{b}\right)^{a} - \left(\frac{x_2}{b}\right)^{a}\right]\right\}\right]^{\beta_2 - 1}, \end{aligned}$$

$$0 \le x_2 \le x_3 \le \infty, \;\; \beta_2 > 0, \;\; a, b > 0.$$

From (9) the joint density function of $(X_{(1)}, X_{(2)}, X_{(3)})$ is given by

$$\begin{aligned} h(x_1, x_2, x_3) =& 4!\beta_1\beta_2\left(\frac{a}{b}\right)^{3}\left(\frac{x_1}{b}\right)^{a-1}\left(\frac{x_2}{b}\right)^{a-1}\left(\frac{x_3}{b}\right)^{a-1} \exp\left\{-4\left(\frac{x_1}{b}\right)^{a} - 3\left[\left(\frac{x_2}{b}\right)^{a} - \left(\frac{x_1}{b}\right)^{a}\right]\right. \\ & \left.-2\left[\left(\frac{x_3}{b}\right)^{a} - \left(\frac{x_2}{b}\right)^{a}\right]\right\}\left[1 - \exp\left\{-2\left[\left(\frac{x_3}{b}\right)^{a} - \left(\frac{x_2}{b}\right)^{a}\right]\right\}\right]^{\beta_2 - 1} \\ & \left[1 - \exp\left\{-3\left[\left(\frac{x_2}{b}\right)^{a} - \left(\frac{x_1}{b}\right)^{a}\right]\right\}\right]^{\beta_1 - 1}, \end{aligned} \tag{12}$$

$$0 \le x_1 \le x_2 \le x_3 \le \infty, \;\; \beta_1, \beta_2 > 0, \;\; a, b > 0.$$

In the next subsequent section we discuss the parameter estimation procedures for the proposed model.

4. Parameter Estimation

Let $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)$ be the baseline parameters and $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_{(m-k)})$ be the load sharing parameters involved in the proposed PCRHR based load sharing model. The maximum likelihood estimation procedure is used to estimate the parameters.

4.1. For k-out-of-m Load Sharing System

Let $r = m - k + 1$, x_{ij} be the j^{th} failure $X_{(j)}$ of i^{th} k-out-of-m load sharing system and x_{ij+1} is the $(j+1)^{\text{th}}$ failure $X_{(j+1)}$ of i^{th} k-out-of-m load sharing system with $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m - k$.

The likelihood function of $(\underline{\theta}, \underline{\beta})$ based on n independent k-out-of-m load sharing systems is given by

$$L\left(\underline{\theta}, \underline{\beta}\right) = \left(\frac{m!}{(k-1)!}\right)^n \left(\prod_{j=1}^{(m-k)} \beta_j\right)^n \prod_{i=1}^{n} \bar{F}_{\underline{\theta}}\left(x_{ir}\right)^{k-1} \prod_{i=1}^{n}\prod_{j=1}^{r} f_{\underline{\theta}}\left(x_{ij}\right)$$
$$\prod_{i=1}^{n}\prod_{j=1}^{(m-k)} \left[1 - \left\{\frac{\bar{F}_{\underline{\theta}}\left(x_{ij+1}\right)}{\bar{F}_{\underline{\theta}}\left(x_{ij}\right)}\right\}^{m-j}\right]^{\beta_j - 1}.$$

The corresponding log-likelihood function can be written as,

$$\log L = n \log\left(\frac{m!}{(k-1)!}\right) + n\sum_{j=1}^{r} \log \beta_j + (k-1)\sum_{i=1}^{n} \log \bar{F}_{\underline{\theta}}\left(x_{ir}\right)$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{r} \log f_{\underline{\theta}}\left(x_{ij}\right) + \sum_{i=1}^{n}\sum_{j=1}^{m-k} (\beta_j - 1) \log\left[1 - \left\{\frac{\bar{F}_{\underline{\theta}}\left(x_{ij+1}\right)}{\bar{F}_{\underline{\theta}}\left(x_{ij}\right)}\right\}^{m-j}\right].$$

To obtain maximum likelihood estimates (MLEs) of unknowns parameters $\underline{\theta}$ and $\underline{\beta}$ differentiate the above log-likelihood partially w.r.t. unknown parameters and equate to zero, we have likelihood equations to estimate as

$$\frac{\partial \log L}{\partial \theta_l} = 0, \quad l = 1, 2, 3, \ldots, p, \tag{13}$$

$$\frac{\partial \log L}{\partial \beta_j} = 0, \quad j = 1, 2, \ldots, (m-k). \tag{14}$$

The above $p + (m-k)$ equations are not in closed form so are solved by iterative procedures. We could use the 'optim' function in *R* software to solve these equations.

4.2. For k-out-of-m Load Sharing System with Weibull Baseline

The likelihood function based on n i.i.d. k-out-of-m load systems with component lifetimes as Weibull (a, b) is given by

$$L(a, b, \underline{\beta}) = \left(\frac{m!}{(k-1)!}\right)^n \left(\frac{a}{b}\right)^{nr} \left(\prod_{j=1}^{m-k} \beta_j\right)^n \prod_{i=1}^{n}\prod_{j=1}^{r} \left(\frac{x_{ij}}{b}\right)^{a-1} \exp\left[-k\sum_{i=1}^{n}\left(\frac{x_{ir}}{b}\right)^a\right]$$
$$\exp\left[-\sum_{i=1}^{n}\sum_{j=1}^{(m-k)}\left(\frac{x_{ij}}{b}\right)^a\right] \prod_{i=1}^{n}\prod_{j=1}^{m-k}\left\{1 - \exp\left(-\frac{(m-j)}{b^a}\left[\left(x_{ij+1}\right)^a - \left(x_{ij}\right)^a\right]\right)\right\}^{\beta_j - 1}.$$

Hence, the log-likelihood function is given by

$$\log L(a, b, \underline{\beta}) = n \log\left(\frac{m!}{(k-1)!}\right) + nr \log\left(\frac{a}{b}\right) + n\sum_{j=1}^{(m-k)} \log \beta_j + (a-1)\sum_{i=1}^{n}\sum_{j=1}^{r} \log\left(\frac{x_{ij}}{b}\right) - k\sum_{i=1}^{n}\left(\frac{x_{ir}}{b}\right)^a$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{(m-k)}\left(\frac{x_{ij}}{b}\right)^a + \sum_{i=1}^{n}\sum_{j=1}^{m-k} (\beta_j - 1) \log\left\{1 - \exp\left(-\frac{(m-j)}{b^a}\left[\left(x_{ij+1}\right)^a - \left(x_{ij}\right)^a\right]\right)\right\}.$$

The MLEs of a, b and β_j, $j = 1, 2, \ldots, (m-k)$ are obtained by maximizing above log-likelihood function. The likelihood or log-likelihood function is maximized by differentiating it partially with respect to unknown parameters and equating to zero. The likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_j} = 0, \quad j = 1, 2, \ldots, (m-k).$$

We observed that likelihood equations are not in closed form, so iterative procedures are used to estimate the unknown parameters.

4.3. For 2-out-of-4 Load Sharing System with Weibull Baseline

The likelihood function for the 2-out-of-4 load sharing system with Weibull (a, b) baseline distribution is given by

$$L(a,b,\beta_1,\beta_2) = (4!)^n \left(\frac{a}{b}\right)^{3n} (\beta_1\beta_2)^n \prod_{i=1}^{n} \left(\frac{x_{i1}}{b}\frac{x_{i2}}{b}\frac{x_{i3}}{b}\right)^{(a-1)} \exp\left[-2\sum_{i=1}^{n}\left(\frac{x_{i3}}{b}\right)^a\right]$$
$$\exp\left[-\sum_{i=1}^{n}\left\{\left(\frac{x_{i1}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a\right\}\right] \prod_{i=1}^{n}\left\{1-\exp\left(-\frac{3}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\}^{\beta_1-1}$$
$$\prod_{i=1}^{n}\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i3})^a - (x_{i2})^a\right]\right)\right\}^{\beta_2-1}.$$

Thus the log-likelihood function is given by

$$\log L(a,b,\beta_1,\beta_2) = n\log(4!) + 3n\log\left(\frac{a}{b}\right) + n\log\beta_1 + n\log\beta_2 + (a-1)\sum_{i=1}^{n}\left(\log\left(\frac{x_{i1}}{b}\right)\right.$$
$$\left. + \log\left(\frac{x_{i2}}{b}\right) + \log\left(\frac{x_{i3}}{b}\right)\right) - \sum_{i=1}^{n}\left\{2\left(\frac{x_{i3}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\}$$
$$+ (\beta_1-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{3}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\}$$
$$+ (\beta_2-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i3})^a - (x_{i2})^a\right]\right)\right\},$$

and the likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_1} = 0, \quad \frac{\partial \log L}{\partial \beta_2} = 0.$$

These equation solved simultaneously to get MLEs of a, b, β_1 and β_2. It is seen that the likelihood equations not in closed form, iterative procedures are used.

4.4. For 2-out-of-3 Load Sharing System with Weibull Baseline

The likelihood function for the 2-out-of-3 load sharing model with Weibull (a, b) baseline distribution is given by

$$L(a,b,\beta_1) = (3!)^n \left(\frac{a}{b}\right)^{2n} \beta_1^n \prod_{i=1}^{n}\left(\frac{x_{i1}}{b}\frac{x_{i2}}{b}\right)^{(a-1)} \exp\left[-\sum_{i=1}^{n}\left\{\left(2\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\}\right]$$
$$\prod_{i=1}^{n}\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\}^{\beta_1-1}.$$

Hence, the log-likelihood function is given by

$$\log L(a,b,\beta_1) = n\log(3!) + 2n\log\left(\frac{a}{b}\right) + n\log\beta_1 + (a-1)\sum_{i=1}^{n}\left\{\log\left(\frac{x_{i1}}{b}\right) + \log\left(\frac{x_{i2}}{b}\right)\right\}$$
$$- \sum_{i=1}^{n}\left\{2\left(\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\} + (\beta_1-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\}.$$

The likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_1} = 0.$$

These equations are solved simultaneously to obtain the MLEs of a, b, and β_1. Since the likelihood equations do not have a closed-form solution, iterative procedures are required to solve them.

4.5. For 1-out-of-3 Parallel Load Sharing System with Weibull Baseline

Under the 1-out-of-3 load sharing model with Weibull (a, b) baseline Distribution the likelihood function is given by

$$L(a,b,\beta_1,\beta_2) = (3!)^n \left(\frac{a}{b}\right)^{3n} (\beta_1\beta_2)^n \prod_{i=1}^{n} \left(\frac{x_{i1}}{b}\frac{x_{i2}}{b}\frac{x_{i3}}{b}\right)^{(a-1)} \exp\left[-\sum_{i=1}^{n}\left\{\left(\frac{x_{i3}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a\right\}\right]$$
$$\prod_{i=1}^{n}\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\}^{\beta_1-1} \prod_{i=1}^{n}\left\{1-\exp\left(-\frac{1}{b^a}\left[(x_{i3})^a - (x_{i2})^a\right]\right)\right\}^{\beta_2-1}.$$

The log-likelihood function is given by

$$\begin{aligned}\log L(a,b,\beta_1,\beta_2) =& n\log(3!) + 3n\log\left(\frac{a}{b}\right) + n\log\beta_1 + n\log\beta_2 + (a-1)\sum_{i=1}^{n}\left\{\log\left(\frac{x_{i1}}{b}\right)\right.\\ &\left.+\log\left(\frac{x_{i2}}{b}\right)+\log\left(\frac{x_{i3}}{b}\right)\right\} - \sum_{i=1}^{n}\left\{\left(\frac{x_{i3}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\}\\ &+(\beta_1-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a-(x_{i1})^a\right]\right)\right\}\\ &+(\beta_2-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{1}{b^a}\left[(x_{i3})^a-(x_{i2})^a\right]\right)\right\}\end{aligned}$$

The likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_1} = 0, \quad \frac{\partial \log L}{\partial \beta_2} = 0.$$

These equation solved simultaneously to get MLEs of a, b, β_1 and β_2. It is seen that the likelihood equations not in closed form, iterative procedures are used.

5. Simulation Study

In this section, we carry out the simulation study for 2-out-of-4 load sharing model with Weibull (a, b) baseline distribution. The MLEs, Bias and Mean Square Estimates (MSE) are obtained for various combination of sample size, baseline parameters and load sharing parameters. The performance of estimates are accessed to bias and MSE. We consider sample sizes $30, 50, 100$ and 200, the baseline parameters are considered to be $a = 1, 1.5, 2$ and $b = 0.5, 0.7, 1, 2$, and load share parameters $\beta_j = 1, 1.5, 2, \quad j = 1, 2$. The parameters are estimated using 1000 samples for each sample size and parameter combinations. The results, including the MLEs, bias, and MSE, are detailed in Tables 1 to 4 and illustrated in Figures 1 to 4. These tables and figures provide a comprehensive overview of the estimation accuracy and performance metrics associated with the model.

Table 1: *MLE's Bias and MSE's for* 2*-out-of-*4 *system with n=30*

n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
30	1	0.5	1	1	1.0268	0.4931	1.0589	1.0666	0.0748	0.05	0.1792	0.1813	0.0092	0.0039	0.06	0.0647
30	1	0.5	1	1.5	1.0201	0.495	1.046	1.5878	0.0711	0.0484	0.1694	0.2763	0.0081	0.0038	0.0499	0.1323
30	1	0.5	1	2	1.0108	0.4889	1.0613	2.1738	0.0645	0.0481	0.1742	0.4075	0.0065	0.0036	0.0557	0.3271
30	1	0.5	1.5	1	1.0178	0.4912	1.5951	1.0784	0.0698	0.05	0.2743	0.1909	0.0079	0.0038	0.134	0.0678
30	1	0.5	1.5	1.5	1.018	0.4913	1.6154	1.6238	0.0676	0.047	0.2773	0.2865	0.0076	0.0035	0.1397	0.1599
30	1	0.5	1.5	2	1.0154	0.491	1.6017	2.2001	0.0645	0.046	0.2854	0.4067	0.0066	0.0033	0.1505	0.3141
30	1	0.5	2	1	1.0205	0.4917	2.1517	1.0744	0.0698	0.0472	0.3935	0.1758	0.0077	0.0035	0.3023	0.054
30	1	0.5	2	1.5	1.0144	0.4894	2.1618	1.624	0.0669	0.0461	0.3895	0.2769	0.0072	0.0033	0.2934	0.1457
30	1	0.5	2	2	1.0148	0.4931	2.1526	2.1655	0.0661	0.0461	0.3874	0.404	0.0071	0.0033	0.2859	0.3107
30	1	0.7	1	1	1.0146	0.6869	1.0624	1.0735	0.0724	0.0671	0.1725	0.184	0.0083	0.0073	0.052	0.0621
30	1	0.7	1	1.5	1.023	0.6904	1.045	1.6113	0.0694	0.0679	0.1664	0.2789	0.0081	0.0074	0.0487	0.15
30	1	0.7	1	2	1.0167	0.6893	1.0636	2.1653	0.0654	0.0696	0.1756	0.4029	0.0074	0.0075	0.0566	0.2987
30	1	0.7	1.5	1	1.0181	0.6894	1.5874	1.0652	0.0696	0.0662	0.2766	0.1751	0.008	0.0068	0.1353	0.057
30	1	0.7	1.5	1.5	1.0154	0.6907	1.6015	1.5991	0.0667	0.0685	0.2911	0.2789	0.0072	0.0072	0.1428	0.1463
30	1	0.7	1.5	2	1.0118	0.6914	1.5864	2.1605	0.0646	0.065	0.2757	0.39	0.007	0.0068	0.1341	0.2808
30	1	0.7	2	1	1.0169	0.6898	2.1319	1.0763	0.0677	0.0683	0.3823	0.1817	0.0073	0.0074	0.2684	0.0593
30	1	0.7	2	1.5	1.0086	0.6908	2.1502	1.5954	0.0639	0.0659	0.393	0.2727	0.0064	0.0068	0.2875	0.1303
30	1	0.7	2	2	1.0132	0.6875	2.121	2.1564	0.0644	0.0625	0.3673	0.3911	0.0067	0.006	0.237	0.2704
30	1	1	1	1	1.0164	0.9878	1.0532	1.0703	0.0705	0.0994	0.1676	0.1735	0.0082	0.0152	0.0526	0.0552
30	1	1	1	1.5	1.0234	0.9811	1.0652	1.6203	0.0703	0.0974	0.1778	0.283	0.0083	0.0147	0.0575	0.142
30	1	1	1	2	1.0166	0.9886	1.0675	2.2175	0.0665	0.0982	0.1734	0.4385	0.0075	0.0148	0.0551	0.3774
30	1	1	1.5	1	1.0144	0.9875	1.6024	1.062	0.0685	0.0969	0.2817	0.1739	0.0076	0.0146	0.146	0.0547
30	1	1	1.5	1.5	1.0118	0.9774	1.5964	1.6388	0.0694	0.0934	0.2795	0.2952	0.0078	0.0137	0.1357	0.1639
30	1	1	1.5	2	1.0138	0.9764	1.6034	2.1774	0.0659	0.093	0.2766	0.3923	0.0069	0.0135	0.1408	0.3079
30	1	1	2	1	1.0158	0.9817	2.1508	1.0642	0.0709	0.0964	0.3795	0.1763	0.0083	0.0141	0.2644	0.0553
30	1	1	2	1.5	1.0149	0.975	2.1838	1.6278	0.067	0.0931	0.3964	0.2881	0.0073	0.0137	0.3241	0.1543
30	1	1	2	2	1.0162	0.9871	2.1375	2.1845	0.0677	0.0948	0.3796	0.4103	0.0073	0.0138	0.2725	0.3222
30	1	2	1	1	1.0136	1.9619	1.0629	1.0621	0.0715	0.2029	0.1802	0.1736	0.0082	0.0636	0.0591	0.0545
30	1	2	1	1.5	1.0176	1.9727	1.0535	1.6167	0.0722	0.1984	0.1709	0.288	0.0085	0.0605	0.0525	0.1493
30	1	2	1	2	1.0167	1.9602	1.0503	2.1844	0.0686	0.1968	0.1629	0.4129	0.0075	0.0593	0.0485	0.314
30	1	2	1.5	1	1.0161	1.9704	1.6143	1.0541	0.0715	0.1895	0.285	0.1701	0.0085	0.0544	0.1511	0.0507
30	1	2	1.5	1.5	1.0152	1.9627	1.6089	1.6237	0.0676	0.1843	0.282	0.2824	0.0076	0.0544	0.1519	0.1466
30	1	2	1.5	2	1.0165	1.9822	1.597	2.1522	0.0672	0.1937	0.2893	0.3881	0.0073	0.0577	0.1505	0.2959
30	1	2	2	1	1.0157	1.9759	2.1385	1.0594	0.0685	0.1868	0.3799	0.1748	0.0075	0.0554	0.2744	0.0576
30	1	2	2	1.5	1.0169	1.9666	2.1583	1.636	0.0686	0.1845	0.4054	0.2894	0.0073	0.0535	0.3028	0.1555
30	1	2	2	2	1.0134	1.9694	2.1574	2.1781	0.0645	0.1853	0.395	0.3899	0.0069	0.0533	0.2934	0.2832
30	1.5	0.5	1	1	1.5264	0.4921	1.0647	1.0691	0.1092	0.0339	0.1774	0.1764	0.0198	0.0018	0.0546	0.0572
30	1.5	0.5	1	1.5	1.5228	0.4931	1.0622	1.6364	0.1022	0.0326	0.1748	0.2978	0.017	0.0016	0.0545	0.1737
30	1.5	0.5	1	2	1.526	0.4931	1.0446	2.1664	0.1049	0.0315	0.1651	0.4004	0.0173	0.0015	0.0468	0.2951
30	1.5	0.5	1.5	1	1.5337	0.4953	1.5892	1.0754	0.1088	0.0318	0.2717	0.1832	0.0187	0.0015	0.1356	0.0606
30	1.5	0.5	1.5	1.5	1.519	0.4931	1.5938	1.597	0.1032	0.0322	0.2758	0.2807	0.0178	0.0016	0.1465	0.149
30	1.5	0.5	1.5	2	1.525	0.495	1.5924	2.1785	0.1013	0.0305	0.2674	0.4083	0.016	0.0014	0.1241	0.3183
30	1.5	0.5	2	1	1.525	0.4926	2.1701	1.0778	0.1044	0.0317	0.405	0.1787	0.0174	0.0016	0.2946	0.0578
30	1.5	0.5	2	1.5	1.5221	0.4948	2.1594	1.6119	0.0999	0.0321	0.4105	0.2809	0.0162	0.0016	0.3054	0.1384
30	1.5	0.5	2	2	1.5308	0.4959	2.1456	2.1656	0.0988	0.0305	0.4034	0.4034	0.0155	0.0015	0.2903	0.3198
30	1.5	0.7	1	1	1.5217	0.6894	1.0594	1.0626	0.1104	0.0457	0.1739	0.1783	0.0194	0.0033	0.0547	0.0586
30	1.5	0.7	1	1.5	1.5298	0.6894	1.0705	1.6159	0.1068	0.0435	0.1799	0.2843	0.0184	0.0029	0.0577	0.153
30	1.5	0.7	1	2	1.524	0.6894	1.0697	2.202	0.1023	0.0458	0.1801	0.4265	0.0168	0.0033	0.0608	0.3443
30	1.5	0.7	1.5	1	1.5227	0.6913	1.5908	1.0733	0.1028	0.0463	0.2727	0.1866	0.0172	0.0033	0.1341	0.0641
30	1.5	0.7	1.5	1.5	1.5244	0.6923	1.591	1.6344	0.1068	0.0432	0.2738	0.2965	0.018	0.003	0.1312	0.1661
30	1.5	0.7	1.5	2	1.5311	0.6923	1.5875	2.1931	0.0975	0.0415	0.2755	0.4173	0.0156	0.0028	0.137	0.3263
30	1.5	0.7	2	1	1.5215	0.6915	2.1586	1.0702	0.1028	0.0459	0.3918	0.1805	0.0167	0.0033	0.2861	0.0579
30	1.5	0.7	2	1.5	1.5286	0.6932	2.1551	1.6275	0.0967	0.0437	0.3857	0.2958	0.0157	0.003	0.3045	0.1644
30	1.5	0.7	2	2	1.5196	0.6891	2.1493	2.2065	0.0961	0.0433	0.3904	0.4354	0.0155	0.003	0.2791	0.3526
30	1.5	1	1	1	1.5298	0.9896	1.0581	1.0668	0.1123	0.0656	0.1701	0.178	0.0207	0.0067	0.0505	0.0594
30	1.5	1	1	1.5	1.524	0.9882	1.0607	1.5935	0.1043	0.0633	0.1753	0.279	0.0176	0.0064	0.0541	0.1455
30	1.5	1	1	2	1.528	0.9902	1.0533	2.1783	0.1013	0.065	0.1682	0.4156	0.0164	0.0066	0.0501	0.3123
30	1.5	1	1.5	1	1.5226	0.9889	1.5983	1.0665	0.1021	0.0651	0.2743	0.178	0.0172	0.0066	0.1441	0.059
30	1.5	1	1.5	1.5	1.5191	0.989	1.6092	1.5996	0.0958	0.0643	0.2735	0.2878	0.0149	0.0065	0.1428	0.1494
30	1.5	1	1.5	2	1.5205	0.9901	1.6032	2.1649	0.0994	0.0634	0.28	0.4113	0.016	0.0064	0.1512	0.3202
30	1.5	1	2	1	1.5289	0.9884	2.16	1.0743	0.1068	0.0637	0.3926	0.1881	0.0185	0.0064	0.2772	0.0653
30	1.5	1	2	1.5	1.5216	0.9879	2.1723	1.6159	0.0969	0.0657	0.414	0.281	0.0152	0.0067	0.3224	0.1448
30	1.5	1	2	2	1.5173	0.9848	2.1531	2.1837	0.0952	0.0635	0.4059	0.4079	0.0148	0.0064	0.2972	0.3037
30	1.5	2	1	1	1.524	1.976	1.0745	1.0682	0.1097	0.134	0.1843	0.177	0.0202	0.0273	0.0596	0.0563
30	1.5	2	1	1.5	1.5267	1.9729	1.0566	1.6222	0.1066	0.1262	0.182	0.2857	0.0184	0.0249	0.0576	0.162
30	1.5	2	1	2	1.5246	1.9786	1.0604	2.1575	0.099	0.1265	0.1821	0.3777	0.0159	0.0248	0.0551	0.2765
30	1.5	2	1.5	1	1.52	1.9707	1.5948	1.0654	0.1082	0.1281	0.2678	0.1775	0.0189	0.0255	0.1306	0.0586
30	1.5	2	1.5	1.5	1.5285	1.9681	1.6067	1.6337	0.1027	0.1315	0.2797	0.3078	0.0173	0.0269	0.143	0.1856
30	1.5	2	1.5	2	1.5258	1.9811	1.5839	2.1726	0.1013	0.131	0.2762	0.4167	0.0163	0.0268	0.1423	0.3165
30	1.5	2	2	1	1.5203	1.97	2.1675	1.0626	0.1016	0.1237	0.4025	0.1739	0.0165	0.0244	0.2863	0.0544
30	1.5	2	2	1.5	1.5213	1.983	2.1255	1.609	0.0998	0.1229	0.3853	0.2927	0.0157	0.0236	0.2591	0.1622
30	1.5	2	2	2	1.5247	1.9748	2.1301	2.1947	0.0963	0.1185	0.3689	0.4202	0.0151	0.0228	0.2574	0.3442
30	2	0.5	1	1	2.0398	0.4955	1.0594	1.081	0.1454	0.0243	0.1744	0.1852	0.0346	0.001	0.0549	0.0667
30	2	0.5	1	1.5	2.0421	0.4963	1.0407	1.604	0.1407	0.0233	0.1621	0.2868	0.0324	0.0009	0.0487	0.1608
30	2	0.5	1	2	2.0297	0.4948	1.0567	2.1828	0.1325	0.024	0.1721	0.4023	0.028	0.0009	0.0573	0.3122
30	2	0.5	1.5	1	2.0302	0.4943	1.6357	1.0626	0.143	0.0241	0.3003	0.175	0.0333	0.0009	0.1702	0.0574
30	2	0.5	1.5	1.5	2.0275	0.4943	1.6087	1.6032	0.1424	0.0232	0.2747	0.2727	0.0326	0.0008	0.1413	0.1399
30	2	0.5	1.5	2	2.0409	0.4951	1.6065	2.1743	0.1377	0.0226	0.2706	0.392	0.0301	0.0008	0.1429	0.2928
30	2	0.5	2	1	2.0252	0.4953	2.1554	1.0713	0.1359	0.0239	0.3936	0.1839	0.029	0.0009	0.2879	0.0619
30	2	0.5	2	1.5	2.0293	0.4942	2.1786	1.6413	0.1339	0.0236	0.3977	0.2919	0.029	0.0009	0.3124	0.1589
30	2	0.5	2	2	2.0367	0.4961	2.1458	2.1796	0.1277	0.0242	0.3861	0.4068	0.0268	0.0009	0.2804	0.3141
30	2	0.7	1	1	2.0289	0.6957	1.0527	1.0514	0.1532	0.0345	0.1728	0.1735	0.0378	0.0019	0.0533	0.0527
30	2	0.7	1	1.5	2.0371	0.6936	1.0569	1.6267	0.1423	0.0342	0.1781	0.2863	0.0335	0.0018	0.0564	0.1603
30	2	0.7	1	2	2.0348	0.6913	1.0651	2.1986	0.1355	0.0344	0.1761	0.4282	0.0305	0.0019	0.0567	0.3639
30	2	0.7	1.5	1	2.0335	0.6924	1.6073	1.078	0.1455	0.0341	0.2667	0.1911	0.0342	0.0018	0.1298	0.0671
30	2	0.7	1.5	1.5	2.0282	0.6925	1.6076	1.6236	0.1301	0.033	0.2864	0.2876	0.0278	0.0017	0.1462	0.155
30	2	0.7	1.5	2	2.0305	0.6937	1.5913	2.166	0.1357	0.0315	0.2788	0.3829	0.0294	0.0016	0.1491	0.2747
30	2	0.7	2	1	2.0296	0.6901	2.1985	1.0755	0.1414	0.0331	0.426	0.1885	0.0322	0.0017	0.3442	0.0631
30	2	0.7	2	1.5	2.0212	0.6922	2.1931	1.6337	0.1286	0.0339	0.4167	0.2974	0.0265	0.0018	0.3243	0.1741
30	2	0.7	2	2	2.0286	0.6948	2.136	2.2045	0.1285	0.0321	0.39	0.4388	0.0265	0.0016	0.2715	0.365
30	2	1	1	1	2.0441	0.9905	1.0588	1.0774	0.1479	0.0514	0.17	0.1893	0.0356	0.0041	0.052	0.0667
30	2	1	1	1.5	2.0291	0.9889	1.0631	1.6235	0.1418	0.0474	0.1777	0.2989	0.0325	0.0035	0.0565	0.162
30	2	1	1	2	2.0295	0.9892	1.0647	2.1925	0.1346	0.0478	0.1737	0.4314	0.0286	0.0035	0.0531	0.3347
30	2	1	1.5	1	2.0325	0.9866	1.6182	1.0814	0.1411	0.0493	0.2832	0.1857	0.031	0.0037	0.1516	0.0637
30	2	1	1.5	1.5	2.0386	0.989	1.6044	1.6333	0.1351	0.0468	0.2801	0.2951	0.0296	0.0034	0.1405	0.1698
30	2	1	1.5	2	2.0247	0.9934	1.5908	2.1767	0.1282	0.0474	0.2733	0.4004	0.026	0.0035	0.1386	0.3143
30	2	1	2	1	2.0329	0.9903	2.1331	1.0593	0.1414	0.0494	0.3966	0.1762	0.0313	0.0038	0.2839	0.0568
30	2	1	2	1.5	2.0321	0.9914	2.1564	1.6206	0.1377	0.0471	0.3945	0.2914	0.0307	0.0035	0.3034	0.1549
30	2	1	2	2	2.0209	0.9906	2.1356	2.1618	0.1276	0.0463	0.4067	0.3998	0.0261	0.0033	0.3058	0.3029
30	2	2	1	1	2.0334	1.9786	1.0532	1.0659	0.1456	0.1039	0.1776	0.1771	0.0339	0.0166	0.0557	0.0555
30	2	2	1	1.5	2.0334	1.9744	1.0604	1.6298	0.1433	0.1022	0.1735	0.3014	0.0323	0.0161	0.0541	0.1643
30	2	2	1	2	2.0278	1.981	1.0616	2.1682	0.1331	0.0978	0.1802	0.4014	0.0276	0.015	0.057	0.3047
30	2	2	1.5	1	2.0278	1.9821	1.614	1.0695	0.1386	0.0971	0.2774	0.1794	0.0314	0.0144	0.1404	0.0555
30	2	2	1.5	1.5	2.0368	1.9795	1.6109	1.6258	0.1379	0.0944	0.2919	0.2974	0.0313	0.0139	0.1594	0.1691
30	2	2	1.5	2	2.0209	1.985	1.6262	2.1672	0.1287	0.0926	0.2862	0.4002	0.0267	0.0134	0.1548	0.3059
30	2	2	2	1	2.0328	1.9779	2.1479	1.0774	0.145	0.0954	0.3935	0.1776	0.0344	0.014	0.3015	0.059
30	2	2	2	1.5	2.0299	1.982	2.1556	1.587	0.136	0.0973	0.3988	0.2725	0.0289	0.0147	0.2886	0.1319
30	2	2	2	2	2.0312	1.9853	2.1534	2.1765	0.128	0.0926	0.3944	0.398	0.0263	0.0136	0.2958	0.3017

Table 2: *MLE's Bias and MSE's for* 2*-out-of-*4 *system with n=50*

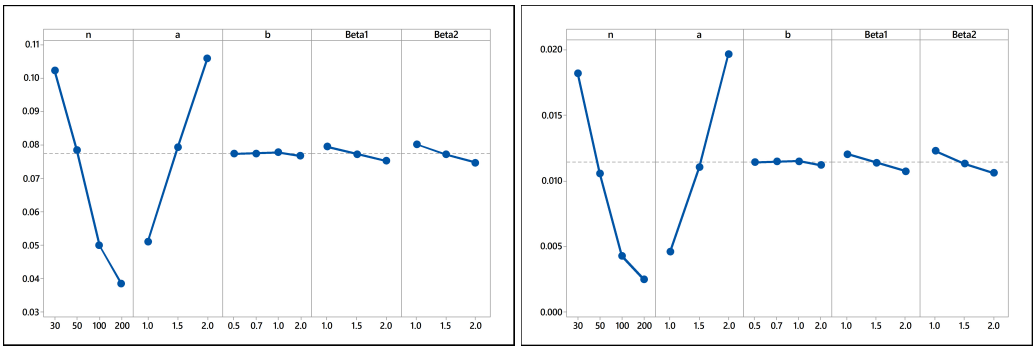
n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta}_1$)	Bias($\hat{\beta}_2$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta}_1$)	MSE($\hat{\beta}_2$)
50	1	0.5	1	1	1.0133	0.4936	1.0305	1.0455	0.0542	0.0378	0.1267	0.1337	0.0046	0.0022	0.0279	0.0311
50	1	0.5	1	1.5	1.0129	0.4941	1.0436	1.5596	0.0548	0.0368	0.1295	0.2088	0.005	0.0021	0.03	0.0739
50	1	0.5	1	2	1.0081	0.497	1.0249	2.1013	0.051	0.0377	0.1286	0.2897	0.004	0.0022	0.0276	0.1463
50	1	0.5	1.5	1	1.0105	0.4947	1.5642	1.0469	0.0542	0.0381	0.2101	0.1347	0.0047	0.0022	0.0742	0.0306
50	1	0.5	1.5	1.5	1.0085	0.4974	1.5487	1.5621	0.0527	0.0368	0.1996	0.2146	0.0045	0.0021	0.067	0.0787
50	1	0.5	1.5	2	1.0093	0.4951	1.5611	2.1001	0.0507	0.0357	0.2009	0.2956	0.0042	0.002	0.0663	0.1423
50	1	0.5	2	1	1.0094	0.4957	2.0719	1.037	0.053	0.0363	0.28	0.1364	0.0045	0.0021	0.1361	0.0298
50	1	0.5	2	1.5	1.0063	0.4962	2.088	1.5732	0.0512	0.0367	0.2904	0.2161	0.0042	0.0021	0.148	0.0828
50	1	0.5	2	2	1.0061	0.4955	2.0713	2.0885	0.0484	0.0358	0.2811	0.2937	0.0037	0.002	0.1384	0.1443
50	1	0.7	1	1	1.012	0.6937	1.0278	1.0383	0.058	0.0523	0.1298	0.1273	0.0054	0.0043	0.0283	0.0276
50	1	0.7	1	1.5	1.0087	0.691	1.0408	1.5579	0.0541	0.053	0.1313	0.2155	0.0047	0.0044	0.0298	0.0801
50	1	0.7	1	2	1.0093	0.6935	1.035	2.097	0.0509	0.0519	0.1288	0.2921	0.0042	0.0042	0.028	0.1498
50	1	0.7	1.5	1	1.0111	0.6931	1.5656	1.0337	0.0547	0.0529	0.2093	0.1327	0.0047	0.0043	0.0743	0.0306
50	1	0.7	1.5	1.5	1.012	0.6956	1.5627	1.568	0.0509	0.052	0.2078	0.214	0.0041	0.0043	0.0775	0.0783
50	1	0.7	1.5	2	1.0103	0.6928	1.5507	2.1132	0.0519	0.0508	0.1977	0.3028	0.0043	0.004	0.0682	0.156
50	1	0.7	2	1	1.007	0.6912	2.0763	1.0376	0.0511	0.0543	0.2792	0.1366	0.0041	0.0045	0.1368	0.0303
50	1	0.7	2	1.5	1.01	0.6933	2.1062	1.5658	0.0515	0.0505	0.3049	0.2062	0.0042	0.004	0.1633	0.0752
50	1	0.7	2	2	1.0072	0.695	2.0728	2.0819	0.0489	0.0511	0.2902	0.2899	0.0038	0.0041	0.1397	0.1471
50	1	1	1	1	1.0112	0.986	1.0442	1.0458	0.0553	0.0768	0.1349	0.1348	0.0048	0.0094	0.0317	0.0318
50	1	1	1	1.5	1.0095	0.9889	1.039	1.5793	0.0516	0.0735	0.1301	0.2141	0.0043	0.0083	0.0295	0.0816
50	1	1	1	2	1.0081	0.9916	1.0303	2.106	0.0506	0.0717	0.1284	0.2981	0.0041	0.0083	0.0282	0.1549
50	1	1	1.5	1	1.0079	0.9909	1.5515	1.0354	0.0547	0.0745	0.2043	0.1305	0.0047	0.0087	0.069	0.0286
50	1	1	1.5	1.5	1.0126	0.9894	1.562	1.5688	0.0537	0.072	0.2109	0.2052	0.0046	0.0081	0.0792	0.0739
50	1	1	1.5	2	1.008	0.9909	1.5716	2.1045	0.0507	0.0725	0.2174	0.2934	0.0041	0.0082	0.0813	0.1501
50	1	1	2	1	1.0134	0.9964	2.0557	1.0397	0.0535	0.0735	0.278	0.1334	0.0047	0.0086	0.1374	0.0305
50	1	1	2	1.5	1.0081	0.9909	2.0845	1.5651	0.0501	0.075	0.2845	0.2085	0.0041	0.0088	0.1368	0.0734
50	1	1	2	2	1.0068	0.9896	2.0872	2.1097	0.0499	0.0729	0.2919	0.2972	0.004	0.0083	0.1499	0.1513
50	1	2	1	1	1.0132	1.9823	1.0231	1.0364	0.0559	0.1562	0.1299	0.1287	0.0049	0.0378	0.0275	0.0278
50	1	2	1	1.5	1.008	1.9865	1.0232	1.5618	0.0537	0.1525	0.132	0.2167	0.0045	0.0365	0.0287	0.0784
50	1	2	1	2	1.0111	1.9849	1.0334	2.0933	0.0497	0.1464	0.1321	0.2926	0.004	0.0327	0.0296	0.1519
50	1	2	1.5	1	1.0069	1.9834	1.5697	1.0356	0.0535	0.1523	0.2151	0.135	0.0045	0.036	0.08	0.0327
50	1	2	1.5	1.5	1.0099	1.9783	1.5591	1.5738	0.0505	0.1425	0.2041	0.2142	0.004	0.0314	0.0694	0.0793
50	1	2	1.5	2	1.0105	1.9774	1.5507	2.1054	0.051	0.1493	0.2059	0.3047	0.0041	0.034	0.0715	0.1562
50	1	2	2	1	1.0118	1.9925	2.0623	1.0397	0.0538	0.1478	0.2863	0.1347	0.0046	0.0352	0.1448	0.0316
50	1	2	2	1.5	1.007	1.9793	2.0973	1.5526	0.0492	0.1427	0.2886	0.2071	0.0038	0.0319	0.1453	0.0746
50	1	2	2	2	1.0094	1.9829	2.0889	2.0993	0.0499	0.1406	0.2874	0.2916	0.004	0.0313	0.1437	0.1455
50	1.5	0.5	1	1	1.5198	0.4964	1.0407	1.0461	0.0846	0.0252	0.1333	0.1363	0.0108	0.001	0.0308	0.0321
50	1.5	0.5	1	1.5	1.5147	0.496	1.0255	1.5712	0.0814	0.0249	0.1231	0.2151	0.0102	0.001	0.0249	0.0783
50	1.5	0.5	1	2	1.5142	0.4959	1.0389	2.0926	0.0772	0.0243	0.1358	0.2913	0.0098	0.0009	0.031	0.1526
50	1.5	0.5	1.5	1	1.5119	0.4964	1.5453	1.0413	0.0818	0.0251	0.2027	0.1341	0.0105	0.001	0.0723	0.031
50	1.5	0.5	1.5	1.5	1.5201	0.4962	1.5629	1.5678	0.0778	0.0258	0.2106	0.2053	0.0101	0.001	0.075	0.0737
50	1.5	0.5	1.5	2	1.516	0.4974	1.5642	2.0999	0.0741	0.0245	0.2135	0.2967	0.0088	0.0009	0.0771	0.1614
50	1.5	0.5	2	1	1.5162	0.4975	2.0883	1.0427	0.082	0.0242	0.2908	0.1296	0.0108	0.0009	0.1467	0.0289
50	1.5	0.5	2	1.5	1.514	0.4961	2.1021	1.5932	0.0759	0.0246	0.2941	0.2171	0.0092	0.0009	0.1501	0.085
50	1.5	0.5	2	2	1.5129	0.4966	2.0801	2.0923	0.0742	0.0235	0.2879	0.283	0.0088	0.0009	0.1433	0.1422
50	1.5	0.7	1	1	1.5149	0.6948	1.0342	1.0343	0.0861	0.0361	0.1305	0.1286	0.0115	0.002	0.0294	0.0284
50	1.5	0.7	1	1.5	1.5193	0.6935	1.0351	1.5778	0.0843	0.034	0.1314	0.2178	0.0113	0.0018	0.0304	0.0827
50	1.5	0.7	1	2	1.5188	0.6927	1.0378	2.1083	0.0785	0.0347	0.1326	0.3028	0.0099	0.0019	0.0298	0.1586
50	1.5	0.7	1.5	1	1.5204	0.695	1.547	1.0448	0.0816	0.036	0.2015	0.1354	0.0105	0.002	0.0708	0.0316
50	1.5	0.7	1.5	1.5	1.5117	0.6947	1.5663	1.5638	0.0775	0.0343	0.2113	0.2127	0.0095	0.0018	0.0766	0.08
50	1.5	0.7	1.5	2	1.5124	0.694	1.5551	2.0891	0.0765	0.0341	0.203	0.3042	0.0097	0.0018	0.0705	0.1648
50	1.5	0.7	2	1	1.5145	0.6929	2.0955	1.0447	0.0803	0.0351	0.303	0.1275	0.01	0.0019	0.1641	0.0278
50	1.5	0.7	2	1.5	1.5099	0.6929	2.1013	1.5583	0.0754	0.0352	0.3049	0.2111	0.009	0.0019	0.1614	0.0787
50	1.5	0.7	2	2	1.5111	0.6942	2.0835	2.099	0.0715	0.0329	0.2897	0.291	0.0084	0.0017	0.1453	0.1513
50	1.5	1	1	1	1.5214	0.9894	1.0494	1.0557	0.0815	0.0515	0.1344	0.1299	0.0112	0.0041	0.0314	0.0309
50	1.5	1	1	1.5	1.5116	0.9915	1.0326	1.5616	0.0818	0.0479	0.1291	0.2025	0.0108	0.0036	0.0267	0.075
50	1.5	1	1	2	1.5156	0.9924	1.0299	2.1147	0.0791	0.0504	0.1254	0.3049	0.0101	0.004	0.0261	0.1585
50	1.5	1	1.5	1	1.515	0.9918	1.56	1.0331	0.0775	0.0504	0.2066	0.1287	0.01	0.0039	0.0721	0.0274
50	1.5	1	1.5	1.5	1.5206	0.9938	1.5535	1.5658	0.0815	0.0492	0.2022	0.2067	0.0104	0.0038	0.0692	0.0729
50	1.5	1	1.5	2	1.5101	0.9933	1.5457	2.0914	0.0735	0.0492	0.2109	0.2805	0.0087	0.0037	0.0767	0.1392
50	1.5	1	2	1	1.5131	0.9937	2.0884	1.0412	0.082	0.0505	0.2935	0.1361	0.0107	0.0041	0.1482	0.0317
50	1.5	1	2	1.5	1.5079	0.9932	2.0776	1.5625	0.0717	0.0482	0.2755	0.2124	0.0085	0.0036	0.1284	0.0753
50	1.5	1	2	2	1.5086	0.9905	2.0892	2.1054	0.0727	0.0467	0.2893	0.294	0.0085	0.0035	0.1456	0.1478
50	1.5	2	1	1	1.5154	1.9786	1.0374	1.0383	0.0831	0.1027	0.1292	0.1348	0.0108	0.0168	0.0289	0.0312
50	1.5	2	1	1.5	1.5193	1.9805	1.0358	1.5872	0.0822	0.1043	0.136	0.2312	0.0106	0.0167	0.0306	0.0916
50	1.5	2	1	2	1.515	1.9795	1.0415	2.1138	0.0787	0.0993	0.1313	0.297	0.0103	0.0152	0.0291	0.1511
50	1.5	2	1.5	1	1.5139	1.9771	1.5613	1.0467	0.0838	0.0993	0.209	0.1323	0.0113	0.0158	0.0739	0.0309
50	1.5	2	1.5	1.5	1.5155	1.9836	1.5592	1.5715	0.0778	0.0958	0.2011	0.2147	0.0098	0.0142	0.0689	0.0789
50	1.5	2	1.5	2	1.5123	1.9803	1.5583	2.1148	0.0759	0.0965	0.1998	0.2913	0.0093	0.0144	0.0696	0.1496
50	1.5	2	2	1	1.5105	1.9804	2.0973	1.0419	0.0789	0.1016	0.3075	0.1346	0.0096	0.0159	0.1648	0.0313
50	1.5	2	2	1.5	1.5157	1.9873	2.0769	1.5705	0.0754	0.0963	0.2893	0.2111	0.0088	0.0147	0.1404	0.0795
50	1.5	2	2	2	1.5135	1.9834	2.0869	2.0957	0.0758	0.0988	0.2874	0.2906	0.009	0.015	0.1358	0.152
50	2	0.5	1	1	2.025	0.497	1.0331	1.0456	0.1121	0.0192	0.1269	0.1378	0.0197	0.0006	0.0275	0.0323
50	2	0.5	1	1.5	2.0184	0.4977	1.0279	1.5588	0.1041	0.0186	0.1341	0.2098	0.0178	0.0006	0.0293	0.08
50	2	0.5	1	2	2.0134	0.4967	1.0338	2.1003	0.0996	0.0183	0.1288	0.2976	0.0158	0.0005	0.028	0.156
50	2	0.5	1.5	1	2.0223	0.4963	1.5636	1.0285	0.1113	0.0195	0.2192	0.1313	0.0197	0.0006	0.0854	0.0301
50	2	0.5	1.5	1.5	2.0148	0.4975	1.5511	1.5597	0.0994	0.0192	0.2079	0.2126	0.0158	0.0006	0.0722	0.0797
50	2	0.5	1.5	2	2.0222	0.4961	1.5579	2.1214	0.1085	0.0184	0.2007	0.3121	0.0185	0.0005	0.0698	0.1623
50	2	0.5	2	1	2.0148	0.4973	2.0912	1.0352	0.105	0.0176	0.2831	0.131	0.0176	0.0005	0.1428	0.0297
50	2	0.5	2	1.5	2.0138	0.497	2.0865	1.5705	0.1012	0.0184	0.2919	0.2091	0.0163	0.0005	0.1415	0.078
50	2	0.5	2	2	2.0158	0.4972	2.0773	2.1211	0.0971	0.0172	0.2756	0.3005	0.0153	0.0005	0.1293	0.1557
50	2	0.7	1	1	2.0187	0.6943	1.0462	1.0351	0.1072	0.0276	0.1346	0.132	0.0185	0.0012	0.0315	0.0291
50	2	0.7	1	1.5	2.0193	0.6957	1.0372	1.5715	0.1018	0.0263	0.1341	0.2153	0.0173	0.0011	0.0313	0.0815
50	2	0.7	1	2	2.0182	0.6954	1.0359	2.1108	0.1045	0.0258	0.1245	0.295	0.017	0.001	0.027	0.1539
50	2	0.7	1.5	1	2.0308	0.6962	1.567	1.0466	0.1121	0.0254	0.2106	0.1367	0.0203	0.001	0.0732	0.0326
50	2	0.7	1.5	1.5	2.0167	0.6954	1.5625	1.559	0.1032	0.0261	0.209	0.2052	0.0172	0.001	0.078	0.0745
50	2	0.7	1.5	2	2.0188	0.6963	1.5583	2.1124	0.1013	0.0253	0.2204	0.3077	0.0162	0.001	0.0783	0.1661
50	2	0.7	2	1	2.0179	0.6978	2.0763	1.0375	0.103	0.0252	0.284	0.1322	0.0167	0.001	0.1509	0.0299
50	2	0.7	2	1.5	2.0216	0.696	2.1024	1.5755	0.0986	0.025	0.2877	0.2149	0.0164	0.001	0.1394	0.079
50	2	0.7	2	2	2.0188	0.6966	2.0867	2.1149	0.0983	0.0254	0.293	0.2952	0.0157	0.001	0.1522	0.1604
50	2	1	1	1	2.0203	0.9935	1.0323	1.0448	0.1105	0.0401	0.1267	0.1341	0.0197	0.0025	0.0278	0.031
50	2	1	1	1.5	2.0158	0.9951	1.0312	1.5555	0.1063	0.0373	0.129	0.2037	0.0181	0.0022	0.0272	0.0692
50	2	1	1	2	2.0165	0.9939	1.0186	2.0898	0.1066	0.037	0.1244	0.2801	0.0177	0.0021	0.0258	0.1434
50	2	1	1.5	1	2.0242	0.9942	1.5634	1.0327	0.1114	0.0384	0.2046	0.1344	0.0201	0.0023	0.0742	0.0306
50	2	1	1.5	1.5	2.0181	0.9952	1.5371	1.5745	0.1017	0.0373	0.2081	0.2118	0.0163	0.0022	0.0712	0.0763
50	2	1	1.5	2	2.0147	0.9933	1.5585	2.1206	0.1019	0.0381	0.1995	0.3049	0.0168	0.0022	0.0671	0.1626
50	2	1	2	1	2.0123	0.9938	2.0907	1.0306	0.1063	0.0366	0.2901	0.131	0.0179	0.002	0.1436	0.0293
50	2	1	2	1.5	2.0245	0.9942	2.093	1.5647	0.1002	0.0375	0.2923	0.21	0.0161	0.0022	0.1483	0.0749
50	2	1	2	2	2.0087	0.9971	2.0817	2.0914	0.0983	0.0357	0.2847	0.2955	0.0155	0.002	0.1385	0.1516
50	2	2	1	1	2.0192	1.9871	1.0295	1.033	0.1095	0.076	0.1304	0.1291	0.019	0.0092	0.0295	0.0297
50	2	2	1	1.5	2.0214	1.9898	1.0322	1.5646	0.1056	0.0723	0.1313	0.2108	0.0179	0.0084	0.0299	0.0811
50	2	2	1	2	2.0167	1.9836	1.0375	2.1223	0.1021	0.0788	0.1349	0.3135	0.0173	0.0098	0.0309	0.1749
50	2	2	1.5	1	2.0238	1.9965	1.5528	1.0331	0.1068	0.0721	0.2093	0.128	0.0183	0.0083	0.0734	0.0289
50	2	2	1.5	1.5	2.0159	1.9857	1.5642	1.5644	0.1023	0.0716	0.2124	0.2094	0.0167	0.0083	0.0773	0.0712
50	2	2	1.5	2	2.0197	1.9879	1.5591	2.1024	0.1031	0.0693	0.202	0.2964	0.0165	0.0077	0.0677	0.1602
50	2	2	2	1	2.0227	1.9887	2.0887	1.0379	0.1045	0.0719	0.2903	0.1392	0.0173	0.0084	0.1439	0.0326
50	2	2	2	1.5	2.0133	1.9911	2.0853	1.5666	0.0991	0.0747	0.2876	0.2182	0.0161	0.0088	0.1411	0.0817
50	2	2	2	2	2.0151	1.9819	2.117	2.1113	0.1004	0.0715	0.3016	0.2958	0.0158	0.0083	0.1586	0.1529

Table 3: *MLE's Bias and MSE's for* 2*-out-of-*4 *system with n=100*

n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
100	1	0.5	1	1	1.0044	0.4961	1.019	1.0199	0.0384	0.0266	0.089	0.0893	0.0023	0.0011	0.0131	0.013
100	1	0.5	1	1.5	1.0037	0.4973	1.0171	1.5267	0.0369	0.0266	0.0917	0.1416	0.0021	0.0011	0.0138	0.033
100	1	0.5	1	2	1.0027	0.4974	1.0165	2.0491	0.0362	0.0257	0.0904	0.1946	0.0021	0.001	0.0135	0.063
100	1	0.5	1.5	1	1.0066	0.4971	1.5287	1.0194	0.0373	0.0268	0.1406	0.0868	0.0022	0.0011	0.0319	0.0122
100	1	0.5	1.5	1.5	1.0024	0.4969	1.5289	1.5294	0.0363	0.0255	0.1377	0.1453	0.0021	0.001	0.0313	0.0348
100	1	0.5	1.5	2	1.0024	0.4986	1.5277	2.0465	0.0345	0.026	0.1458	0.2003	0.0019	0.0011	0.034	0.067
100	1	0.5	2	1	1.0052	0.4984	2.0357	1.0181	0.037	0.0244	0.1977	0.0919	0.0022	0.0009	0.0647	0.0135
100	1	0.5	2	1.5	1.0052	0.5009	2.0343	1.5121	0.0369	0.0253	0.1944	0.1392	0.0021	0.001	0.0609	0.0307
100	1	0.5	2	2	1.0042	0.4962	2.0524	2.0673	0.0355	0.0248	0.1989	0.2018	0.002	0.001	0.0648	0.069
100	1	0.7	1	1	1.0046	0.6969	1.017	1.0186	0.0382	0.0382	0.0891	0.0906	0.0023	0.0023	0.0127	0.0138
100	1	0.7	1	1.5	1.0042	0.6967	1.0152	1.5289	0.0361	0.0366	0.0885	0.141	0.002	0.0021	0.0135	0.0331
100	1	0.7	1	2	1.0069	0.6971	1.012	2.0511	0.0366	0.0369	0.0857	0.2052	0.0021	0.0021	0.0115	0.07
100	1	0.7	1.5	1	1.0056	0.6967	1.5262	1.0187	0.0373	0.0381	0.1428	0.0892	0.0022	0.0023	0.0332	0.0131
100	1	0.7	1.5	1.5	1.0072	0.6938	1.5352	1.5361	0.0366	0.0376	0.1429	0.1446	0.0022	0.0022	0.034	0.0334
100	1	0.7	1.5	2	1.0036	0.6957	1.5298	2.055	0.0353	0.0356	0.1346	0.1964	0.002	0.002	0.0306	0.0659
100	1	0.7	2	1	1.0049	0.6977	2.0355	1.0158	0.0375	0.0372	0.1961	0.0885	0.0022	0.0022	0.0648	0.0131
100	1	0.7	2	1.5	1.0029	0.6978	2.056	1.5329	0.0345	0.0358	0.2051	0.1377	0.002	0.002	0.0694	0.0323
100	1	0.7	2	2	1.0036	0.6993	2.0392	2.0338	0.035	0.0357	0.1912	0.1983	0.0019	0.002	0.0601	0.0634
100	1	1	1	1	1.006	0.9934	1.0127	1.0244	0.0397	0.0544	0.0876	0.0903	0.0025	0.0046	0.0122	0.0134
100	1	1	1	1.5	1.0052	0.9949	1.0117	1.5402	0.0379	0.0532	0.0886	0.1473	0.0023	0.0046	0.0127	0.0376
100	1	1	1	2	1.004	0.9943	1.0189	2.0531	0.0348	0.051	0.0877	0.2018	0.002	0.004	0.0127	0.0661
100	1	1	1.5	1	1.0042	0.9948	1.5375	1.0241	0.0379	0.0543	0.1461	0.0904	0.0023	0.0046	0.0356	0.0133
100	1	1	1.5	1.5	1.0046	0.9979	1.5177	1.5281	0.0363	0.0525	0.1427	0.1394	0.002	0.0043	0.0342	0.0326
100	1	1	1.5	2	1.0044	0.9981	1.5142	2.0407	0.035	0.0526	0.1434	0.1971	0.0019	0.0043	0.0341	0.0633
100	1	1	2	1	1.0041	0.9982	2.042	1.0192	0.0373	0.0517	0.1971	0.0902	0.0022	0.0041	0.064	0.0134
100	1	1	2	1.5	1.0052	0.9944	2.0545	1.5383	0.0358	0.0508	0.2045	0.1384	0.002	0.004	0.0684	0.0319
100	1	1	2	2	1.0035	0.9937	2.0391	2.0546	0.0332	0.0514	0.2028	0.2072	0.0017	0.0042	0.0671	0.0713
100	1	2	1	1	1.0083	1.9845	1.0233	1.0236	0.0391	0.111	0.0934	0.0931	0.0025	0.0193	0.0147	0.0144
100	1	2	1	1.5	1.0037	1.9841	1.0181	1.5355	0.0352	0.1049	0.0897	0.1459	0.002	0.0175	0.0134	0.0352
100	1	2	1	2	1.0052	1.9914	1.0134	2.0471	0.0368	0.1053	0.092	0.2053	0.0021	0.0175	0.0138	0.0731
100	1	2	1.5	1	1.005	1.9951	1.5276	1.0174	0.0389	0.1051	0.138	0.089	0.0024	0.0173	0.031	0.013
100	1	2	1.5	1.5	1.0036	1.9884	1.5306	1.5302	0.035	0.102	0.1382	0.1466	0.002	0.0164	0.0319	0.0354
100	1	2	1.5	2	1.0044	1.9939	1.5344	2.0389	0.0355	0.1033	0.1425	0.1874	0.002	0.0166	0.034	0.0572
100	1	2	2	1	1.0069	1.9917	2.0367	1.0205	0.0368	0.1042	0.1978	0.0892	0.0021	0.0175	0.0642	0.0132
100	1	2	2	1.5	1.0052	1.9941	2.0546	1.528	0.0343	0.1055	0.2003	0.1421	0.0019	0.0171	0.0688	0.033
100	1	2	2	2	1.0063	1.9906	2.0382	2.0515	0.0344	0.101	0.1995	0.1929	0.0019	0.0159	0.0653	0.0647
100	1.5	0.5	1	1	1.5084	0.4974	1.0138	1.0204	0.056	0.0172	0.0881	0.0939	0.005	0.0005	0.0126	0.0141
100	1.5	0.5	1	1.5	1.5074	0.498	1.015	1.534	0.056	0.0173	0.0881	0.1401	0.0049	0.0005	0.0125	0.0315
100	1.5	0.5	1	2	1.5083	0.4987	1.0183	2.0514	0.0545	0.0172	0.0883	0.1969	0.0048	0.0005	0.0128	0.0659
100	1.5	0.5	1.5	1	1.5122	0.4983	1.532	1.0214	0.0571	0.0177	0.1411	0.0916	0.0052	0.0005	0.0335	0.0138
100	1.5	0.5	1.5	1.5	1.5055	0.4978	1.5322	1.5367	0.057	0.0179	0.1402	0.1463	0.0051	0.0005	0.0316	0.0345
100	1.5	0.5	1.5	2	1.5099	0.4974	1.5321	2.0439	0.053	0.0172	0.1432	0.1989	0.0045	0.0005	0.0334	0.0675
100	1.5	0.5	2	1	1.5108	0.4982	2.0424	1.0207	0.0554	0.0176	0.1963	0.092	0.005	0.0005	0.0655	0.0137
100	1.5	0.5	2	1.5	1.51	0.5	2.0346	1.5222	0.053	0.0163	0.1921	0.1382	0.0046	0.0004	0.0619	0.0318
100	1.5	0.5	2	2	1.5041	0.4973	2.051	2.0532	0.0533	0.0166	0.1948	0.2024	0.0044	0.0004	0.0622	0.0694
100	1.5	0.7	1	1	1.5058	0.6972	1.0194	1.0193	0.0589	0.0255	0.0893	0.0907	0.0056	0.001	0.0134	0.0133
100	1.5	0.7	1	1.5	1.5062	0.6965	1.0161	1.5375	0.0549	0.0248	0.0879	0.1433	0.0047	0.001	0.0124	0.0351
100	1.5	0.7	1	2	1.5052	0.6979	1.0151	2.0436	0.0531	0.0248	0.0885	0.2023	0.0045	0.001	0.0127	0.0662
100	1.5	0.7	1.5	1	1.5086	0.6971	1.5275	1.0162	0.0556	0.0248	0.1413	0.0858	0.005	0.001	0.0321	0.0122
100	1.5	0.7	1.5	1.5	1.5029	0.6956	1.5433	1.5349	0.0543	0.0254	0.149	0.1483	0.0045	0.001	0.037	0.0365
100	1.5	0.7	1.5	2	1.5075	0.6976	1.5269	2.042	0.0527	0.0241	0.1445	0.1921	0.0044	0.0009	0.0335	0.0625
100	1.5	0.7	2	1	1.5056	0.6969	2.0374	1.0249	0.0524	0.0249	0.1911	0.0931	0.0045	0.001	0.062	0.0147
100	1.5	0.7	2	1.5	1.5041	0.6966	2.0505	1.5324	0.0529	0.0234	0.1951	0.1369	0.0045	0.0008	0.0632	0.0308
100	1.5	0.7	2	2	1.5058	0.697	2.0354	2.0473	0.0517	0.0229	0.1881	0.1955	0.0044	0.0008	0.0572	0.061
100	1.5	1	1	1	1.508	0.9958	1.0204	1.0119	0.058	0.0383	0.0914	0.0876	0.0053	0.0023	0.0138	0.0124
100	1.5	1	1	1.5	1.5124	0.9984	1.0177	1.534	0.0565	0.0374	0.0928	0.1438	0.005	0.0022	0.0142	0.0338
100	1.5	1	1	2	1.5033	0.9941	1.0216	2.0522	0.0533	0.0359	0.0898	0.1963	0.0046	0.0019	0.0135	0.0637
100	1.5	1	1.5	1	1.5068	0.9977	1.5296	1.0188	0.0563	0.0357	0.1428	0.0903	0.0051	0.002	0.0333	0.0127
100	1.5	1	1.5	1.5	1.508	0.9969	1.5257	1.5383	0.0543	0.0346	0.1432	0.1492	0.0048	0.0019	0.0332	0.0365
100	1.5	1	1.5	2	1.5081	0.996	1.5329	2.0603	0.0525	0.0334	0.1398	0.2053	0.0044	0.0017	0.0313	0.0714
100	1.5	1	2	1	1.5026	0.9973	2.0422	1.0135	0.0534	0.0341	0.198	0.0897	0.0046	0.0018	0.0633	0.013
100	1.5	1	2	1.5	1.5065	0.9927	2.0546	1.5428	0.0545	0.0343	0.1979	0.1417	0.0047	0.0018	0.0653	0.0342
100	1.5	1	2	2	1.507	0.9971	2.0402	2.0543	0.0533	0.0335	0.2013	0.2121	0.0043	0.0018	0.0659	0.0734
100	1.5	2	1	1	1.5089	1.9884	1.0215	1.0222	0.0582	0.0727	0.0903	0.091	0.0053	0.0084	0.013	0.0141
100	1.5	2	1	1.5	1.5093	1.9923	1.0162	1.5404	0.0575	0.0717	0.0912	0.1444	0.0052	0.0082	0.0137	0.0356
100	1.5	2	1	2	1.51	1.9924	1.0163	2.0383	0.0549	0.0701	0.0887	0.1991	0.0047	0.008	0.0127	0.0652
100	1.5	2	1.5	1	1.5079	1.9908	1.5317	1.0182	0.0554	0.0694	0.1402	0.0974	0.005	0.0076	0.0318	0.0154
100	1.5	2	1.5	1.5	1.5063	1.9885	1.5387	1.5344	0.0561	0.0681	0.1382	0.1353	0.0048	0.0073	0.0327	0.0304
100	1.5	2	1.5	2	1.5079	1.9883	1.5328	2.0683	0.0526	0.0687	0.1458	0.206	0.0044	0.0073	0.0341	0.0692
100	1.5	2	2	1	1.5074	1.9913	2.0529	1.0206	0.0564	0.0683	0.1963	0.0919	0.0051	0.0073	0.0654	0.0135
100	1.5	2	2	1.5	1.5079	1.9955	2.0392	1.527	0.0534	0.0694	0.1973	0.1408	0.0047	0.0074	0.0646	0.0327
100	1.5	2	2	2	1.5076	1.9938	2.0398	2.0397	0.0518	0.0662	0.1958	0.2042	0.0042	0.007	0.0649	0.0679
100	2	0.5	1	1	2.0141	0.4981	1.0176	1.02	0.0793	0.0129	0.0911	0.0878	0.0101	0.0003	0.0135	0.0131
100	2	0.5	1	1.5	2.0077	0.4977	1.0197	1.5334	0.074	0.0135	0.0865	0.1436	0.0087	0.0003	0.0122	0.0337
100	2	0.5	1	2	2.0084	0.4986	1.0156	2.0431	0.0733	0.0127	0.0907	0.2034	0.0084	0.0003	0.0137	0.0667
100	2	0.5	1.5	1	2.0169	0.4983	1.5341	1.0237	0.0749	0.0134	0.145	0.089	0.009	0.0003	0.0352	0.0131
100	2	0.5	1.5	1.5	2.0065	0.4973	1.5277	1.533	0.0729	0.0126	0.1452	0.1429	0.0085	0.0002	0.035	0.0337
100	2	0.5	1.5	2	2.0103	0.4986	1.528	2.045	0.0722	0.0126	0.1424	0.2048	0.0081	0.0002	0.0343	0.0697
100	2	0.5	2	1	2.0089	0.4977	2.0321	1.0244	0.0719	0.0132	0.1961	0.0938	0.0082	0.0003	0.0625	0.0146
100	2	0.5	2	1.5	2.0079	0.4983	2.0534	1.5229	0.0717	0.0133	0.1993	0.141	0.0081	0.0003	0.0638	0.0327
100	2	0.5	2	2	2.0097	0.4982	2.0336	2.0464	0.0663	0.0134	0.1928	0.2018	0.0069	0.0003	0.0621	0.0654
100	2	0.7	1	1	2.01	0.6978	1.0153	1.0186	0.0778	0.0192	0.0928	0.0932	0.0097	0.0006	0.0142	0.0145
100	2	0.7	1	1.5	2.0043	0.6982	1.014	1.5226	0.0731	0.0175	0.0913	0.1345	0.0085	0.0005	0.0138	0.0293
100	2	0.7	1	2	2.0095	0.7003	1.0153	2.0401	0.0738	0.0183	0.0855	0.1979	0.0081	0.0005	0.0123	0.0645
100	2	0.7	1.5	1	2.006	0.6968	1.5426	1.0249	0.0727	0.0185	0.1466	0.0906	0.0087	0.0005	0.0345	0.0136
100	2	0.7	1.5	1.5	2.0061	0.6973	1.5409	1.5424	0.0698	0.0182	0.1462	0.1486	0.0077	0.0005	0.0358	0.037
100	2	0.7	1.5	2	2.0102	0.6986	1.5165	2.0377	0.0721	0.0172	0.1356	0.2004	0.0086	0.0005	0.0296	0.0673
100	2	0.7	2	1	2.0079	0.6983	2.0437	1.0188	0.0743	0.0176	0.19	0.0924	0.0087	0.0005	0.059	0.0139
100	2	0.7	2	1.5	2.0054	0.6976	2.0369	1.5228	0.0693	0.0176	0.1892	0.1408	0.0077	0.0005	0.0576	0.0322
100	2	0.7	2	2	2.0115	0.6986	2.0441	2.0439	0.0682	0.0176	0.1936	0.1982	0.0074	0.0005	0.0608	0.0645
100	2	1	1	1	2.0089	0.9964	1.0171	1.0196	0.0803	0.0272	0.0906	0.0905	0.0102	0.0012	0.0136	0.0133
100	2	1	1	1.5	2.008	0.9964	1.0172	1.5272	0.0738	0.0262	0.088	0.1422	0.0086	0.0011	0.0127	0.0349
100	2	1	1	2	2.0103	0.9969	1.0189	2.0424	0.0755	0.027	0.0906	0.2073	0.0087	0.0011	0.0134	0.068
100	2	1	1.5	1	2.0031	0.9953	1.5348	1.029	0.0743	0.0269	0.1429	0.0896	0.0089	0.0011	0.033	0.0136
100	2	1	1.5	1.5	2.0064	0.9977	1.5238	1.5306	0.0728	0.0252	0.1405	0.1368	0.0083	0.001	0.0318	0.0316
100	2	1	1.5	2	2.0086	0.9968	1.5263	2.0525	0.0696	0.0253	0.1372	0.2008	0.0078	0.001	0.0319	0.0676
100	2	1	2	1	2.0108	0.999	2.0305	1.0136	0.0724	0.0253	0.1974	0.0868	0.0084	0.001	0.0633	0.0121
100	2	1	2	1.5	2.0055	0.9948	2.0442	1.5397	0.0681	0.0257	0.2014	0.1419	0.0075	0.001	0.067	0.0319
100	2	1	2	2	2.0083	0.9994	2.0303	2.0373	0.0691	0.0244	0.1893	0.1894	0.0072	0.001	0.0572	0.0581
100	2	2	1	1	2.007	1.9939	1.024	1.0144	0.0785	0.0538	0.0957	0.0873	0.0094	0.0045	0.0151	0.0124
100	2	2	1	1.5	2.005	1.9916	1.0117	1.5323	0.0755	0.0558	0.0879	0.1444	0.0089	0.0048	0.0123	0.0353
100	2	2	1	2	2.0071	1.9969	1.0151	2.041	0.0692	0.0494	0.085	0.1913	0.0079	0.0038	0.0116	0.0604
100	2	2	1.5	1	2.0135	1.9936	1.5266	1.0227	0.0762	0.0523	0.1312	0.0884	0.0092	0.0042	0.028	0.013
100	2	2	1.5	1.5	2.006	1.9939	1.5276	1.535	0.0717	0.0516	0.1448	0.1453	0.0082	0.0041	0.0333	0.0341
100	2	2	1.5	2	2.0068	1.9928	1.5265	2.0507	0.0666	0.0501	0.1401	0.2026	0.0071	0.0039	0.0322	0.0664
100	2	2	2	1	2.0091	1.9951	2.0468	1.0185	0.0738	0.0528	0.2041	0.0912	0.0084	0.0045	0.0707	0.0135
100	2	2	2	1.5	2.0059	1.9921	2.0596	1.5272	0.0751	0.0519	0.1969	0.136	0.0087	0.0041	0.0676	0.0303
100	2	2	2	2	2.0058	1.9966	2.0459	2.0429	0.0686	0.0505	0.1928	0.1952	0.0074	0.0039	0.0612	0.0631

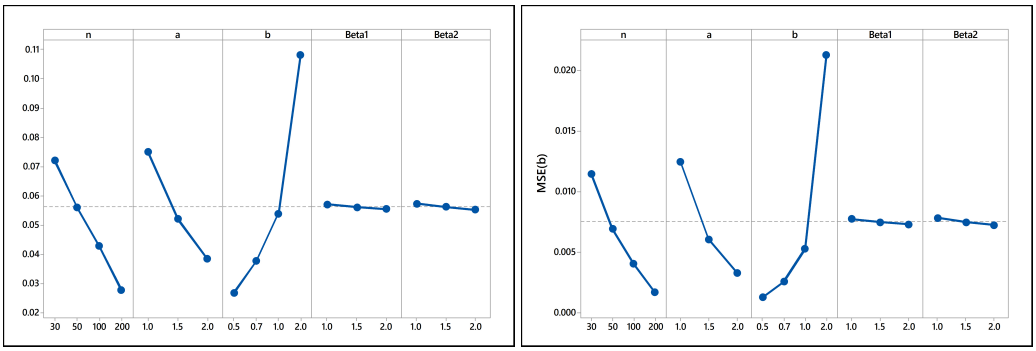
Table 4: *MLE's Bias and MSE's for* 2*-out-of-*4 *system with n=200*

n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
200	1	0.5	1	1	1.0019	0.4982	1.0096	1.0066	0.0278	0.0191	0.0616	0.062	0.0012	0.0006	0.0062	0.0063
200	1	0.5	1	1.5	1.0048	0.4996	1.0062	1.5151	0.0279	0.019	0.0631	0.1	0.0012	0.0006	0.0064	0.0157
200	1	0.5	1	2	1.0008	0.4982	1.0095	2.0232	0.0255	0.0179	0.0637	0.1363	0.001	0.0005	0.0064	0.0316
200	1	0.5	1.5	1	1.0022	0.4985	1.5094	1.011	0.0268	0.0191	0.0981	0.0626	0.0011	0.0006	0.015	0.0062
200	1	0.5	1.5	1.5	1	0.4987	1.512	1.5104	0.0252	0.0183	0.0971	0.0969	0.001	0.0005	0.0149	0.0153
200	1	0.5	1.5	2	1.0031	0.499	1.5127	2.0196	0.0245	0.0178	0.0982	0.1408	0.0009	0.0005	0.0153	0.0318
200	1	0.5	2	1	1.0034	0.4996	2.0098	1.0098	0.0261	0.0189	0.1372	0.0649	0.0011	0.0006	0.0306	0.0068
200	1	0.5	2	1.5	1.0013	0.4986	2.019	1.5198	0.0249	0.0177	0.133	0.1024	0.001	0.0005	0.028	0.0165
200	1	0.5	2	2	1.0009	0.4993	2.0244	2.0149	0.024	0.0182	0.1422	0.1311	0.0009	0.0005	0.0316	0.0282
200	1	0.7	1	1	1.0024	0.6975	1.0076	1.0108	0.0272	0.0272	0.0629	0.0631	0.0012	0.0012	0.0063	0.0065
200	1	0.7	1	1.5	1.0035	0.6992	1.0064	1.5128	0.0262	0.0276	0.0638	0.0991	0.0011	0.0012	0.0066	0.0156
200	1	0.7	1	2	1.0016	0.6981	1.0068	2.0196	0.0261	0.0252	0.0641	0.1421	0.0011	0.001	0.0064	0.0322
200	1	0.7	1.5	1	1.0011	0.6989	1.5137	1.0095	0.0264	0.0263	0.0989	0.0621	0.0011	0.0011	0.0157	0.0061
200	1	0.7	1.5	1.5	1.0019	0.6982	1.5185	1.5191	0.0253	0.0269	0.0981	0.1014	0.001	0.0011	0.015	0.0168
200	1	0.7	1.5	2	1.0028	0.6985	1.5105	2.0229	0.0248	0.0248	0.0952	0.1394	0.001	0.001	0.0146	0.0306
200	1	0.7	2	1	1.0037	0.6988	2.0135	1.0096	0.0265	0.0254	0.1332	0.0636	0.0011	0.001	0.0284	0.0066
200	1	0.7	2	1.5	1.0026	0.6974	2.0215	1.5158	0.0245	0.0259	0.135	0.0978	0.0009	0.001	0.0296	0.0154
200	1	0.7	2	2	1.0007	0.6991	2.0202	2.0166	0.0253	0.0263	0.1356	0.1332	0.001	0.0011	0.0297	0.0292
200	1	1	1	1	1.0026	0.9974	1.0072	1.0103	0.0271	0.0383	0.0608	0.063	0.0012	0.0023	0.0061	0.0062
200	1	1	1	1.5	1.0021	0.9956	1.0139	1.5282	0.0265	0.0387	0.0618	0.1002	0.0011	0.0024	0.0063	0.0163
200	1	1	1	2	1.0018	0.9954	1.0107	2.0352	0.0265	0.036	0.0626	0.1382	0.0011	0.0021	0.0063	0.032
200	1	1	1.5	1	1.0027	0.994	1.5196	1.0107	0.0271	0.0374	0.1033	0.061	0.0012	0.0022	0.0169	0.006
200	1	1	1.5	1.5	1.0038	0.9984	1.5109	1.5169	0.027	0.0352	0.0996	0.0988	0.0011	0.002	0.0156	0.016
200	1	1	1.5	2	1.0014	0.998	1.5124	2.015	0.025	0.037	0.1017	0.1396	0.001	0.0021	0.0164	0.0323
200	1	1	2	1	1.0023	0.9984	2.0106	1.0072	0.0259	0.0357	0.1371	0.0621	0.0011	0.002	0.0297	0.0061
200	1	1	2	1.5	1.0022	0.9991	2.0154	1.5085	0.0251	0.0365	0.1388	0.0975	0.001	0.0021	0.0308	0.0156
200	1	1	2	2	1.0013	0.9973	2.0261	2.0268	0.0243	0.035	0.1364	0.1367	0.0009	0.002	0.03	0.0302
200	1	2	1	1	1.004	1.9956	1.0095	1.0094	0.0277	0.0789	0.0632	0.065	0.0012	0.0097	0.0063	0.0069
200	1	2	1	1.5	1.0034	1.9946	1.0065	1.5194	0.0255	0.0742	0.0626	0.0984	0.001	0.0088	0.0061	0.0164
200	1	2	1	2	1.0005	1.9899	1.0057	2.0317	0.0254	0.0738	0.0602	0.1385	0.001	0.0086	0.0056	0.0312
200	1	2	1.5	1	1.0043	1.9973	1.5143	1.0122	0.0258	0.0748	0.1018	0.0613	0.001	0.0088	0.0169	0.006
200	1	2	1.5	1.5	1.0043	1.9974	1.5121	1.5184	0.0251	0.0763	0.0996	0.1005	0.001	0.0092	0.0161	0.0163
200	1	2	1.5	2	1.0034	2.001	1.5053	2.0175	0.0245	0.0764	0.099	0.1356	0.001	0.009	0.0157	0.0297
200	1	2	2	1	1.0015	1.9973	2.0171	1.0105	0.025	0.0708	0.1322	0.0631	0.001	0.0081	0.0288	0.0063
200	1	2	2	1.5	1.0009	1.9963	2.0176	1.5114	0.0255	0.0745	0.1364	0.098	0.001	0.0085	0.0303	0.0148
200	1	2	2	2	1.002	1.9915	2.0262	2.0187	0.0252	0.0694	0.1347	0.1346	0.001	0.0077	0.0289	0.0305
200	1.5	0.5	1	1	1.5056	0.4992	1.0053	1.0091	0.0418	0.0123	0.0616	0.0627	0.0027	0.0002	0.0058	0.0064
200	1.5	0.5	1	1.5	1.5009	0.5002	1.0085	1.5077	0.0374	0.0127	0.0595	0.102	0.0022	0.0003	0.0057	0.0163
200	1.5	0.5	1	2	1.5058	0.499	1.0099	2.0262	0.0395	0.0123	0.0618	0.1436	0.0024	0.0002	0.0061	0.0325
200	1.5	0.5	1.5	1	1.505	0.4999	1.5109	1.0086	0.0416	0.0123	0.1015	0.0672	0.0026	0.0002	0.016	0.0071
200	1.5	0.5	1.5	1.5	1.503	0.4994	1.512	1.5208	0.0391	0.0121	0.0992	0.0962	0.0024	0.0002	0.0157	0.0153
200	1.5	0.5	1.5	2	1.5027	0.4999	1.5143	2.0128	0.0354	0.0115	0.1	0.1403	0.002	0.0002	0.0159	0.0317
200	1.5	0.5	2	1	1.5034	0.4994	2.0198	1.0096	0.0394	0.0127	0.1404	0.0642	0.0025	0.0002	0.0305	0.0064
200	1.5	0.5	2	1.5	1.5033	0.4992	2.0186	1.5211	0.0374	0.0124	0.1407	0.0997	0.0023	0.0002	0.0307	0.017
200	1.5	0.5	2	2	1.5042	0.5003	2.0139	2.0168	0.0359	0.0121	0.1321	0.1324	0.002	0.0002	0.0282	0.028
200	1.5	0.7	1	1	1.5026	0.699	1.0051	1.0051	0.0411	0.0182	0.0647	0.0618	0.0028	0.0005	0.0066	0.006
200	1.5	0.7	1	1.5	1.502	0.6982	1.0112	1.5224	0.0395	0.0174	0.0631	0.0968	0.0024	0.0005	0.0065	0.0155
200	1.5	0.7	1	2	1.5033	0.6993	1.0088	2.0169	0.0382	0.0174	0.0639	0.1279	0.0023	0.0005	0.0064	0.0266
200	1.5	0.7	1.5	1	1.5045	0.6995	1.5062	1.0094	0.0413	0.017	0.1003	0.0634	0.0027	0.0005	0.0155	0.0063
200	1.5	0.7	1.5	1.5	1.503	0.6981	1.5139	1.5115	0.0374	0.0171	0.0964	0.0995	0.0022	0.0005	0.0149	0.0158
200	1.5	0.7	1.5	2	1.5007	0.6971	1.5184	2.0358	0.0379	0.0172	0.101	0.1344	0.0022	0.0005	0.0164	0.0295
200	1.5	0.7	2	1	1.5036	0.6984	2.0257	1.0058	0.0385	0.0166	0.1413	0.0629	0.0024	0.0004	0.0317	0.0064
200	1.5	0.7	2	1.5	1.5028	0.6986	2.0217	1.5143	0.0398	0.0167	0.1333	0.0971	0.0025	0.0004	0.0285	0.015
200	1.5	0.7	2	2	1.5033	0.6988	2.0145	2.0163	0.0365	0.0173	0.1379	0.1433	0.0022	0.0005	0.0306	0.0336
200	1.5	1	1	1	1.5035	0.9984	1.0074	1.007	0.041	0.0253	0.0612	0.0612	0.0027	0.001	0.0061	0.006
200	1.5	1	1	1.5	1.5015	0.9994	1.0069	1.5124	0.0381	0.0242	0.0605	0.0991	0.0023	0.0009	0.0057	0.0155
200	1.5	1	1	2	1.5024	0.9984	1.0034	2.0262	0.0384	0.0241	0.0603	0.1381	0.0022	0.0009	0.0059	0.0311
200	1.5	1	1.5	1	1.5046	0.9973	1.5193	1.0103	0.0391	0.0256	0.0989	0.0643	0.0024	0.001	0.0156	0.0066
200	1.5	1	1.5	1.5	1.5016	0.9999	1.5145	1.5096	0.0393	0.0251	0.1005	0.0986	0.0024	0.001	0.0164	0.0154
200	1.5	1	1.5	2	1.5027	0.9991	1.5119	2.0197	0.0374	0.0242	0.0946	0.1393	0.0021	0.0009	0.0145	0.0315
200	1.5	1	2	1	1.5032	0.9987	2.0159	1.0099	0.0387	0.0243	0.1407	0.0612	0.0025	0.0009	0.0314	0.006
200	1.5	1	2	1.5	1.5032	0.9968	2.0193	1.5122	0.0386	0.0247	0.1348	0.1013	0.0023	0.0009	0.029	0.0166
200	1.5	1	2	2	1.5053	0.9986	2.0253	2.0268	0.0364	0.0238	0.1455	0.1334	0.0021	0.0009	0.0337	0.0284
200	1.5	2	1	1	1.5049	1.9978	1.0084	1.0126	0.0407	0.0514	0.0611	0.0649	0.0026	0.0042	0.0061	0.0067
200	1.5	2	1	1.5	1.5019	1.9955	1.0077	1.5205	0.0407	0.0509	0.0622	0.1018	0.0025	0.004	0.0063	0.0166
200	1.5	2	1	2	1.5036	1.9971	1.0069	2.0193	0.0369	0.0484	0.0631	0.142	0.0021	0.0037	0.0062	0.0327
200	1.5	2	1.5	1	1.5037	1.9974	1.5135	1.0117	0.0375	0.0504	0.0988	0.0632	0.0023	0.004	0.016	0.0065
200	1.5	2	1.5	1.5	1.5043	1.9974	1.5161	1.5141	0.0397	0.0483	0.0991	0.0967	0.0025	0.0037	0.0158	0.0149
200	1.5	2	1.5	2	1.5028	1.9978	1.5091	2.0191	0.0388	0.0509	0.098	0.1445	0.0023	0.0039	0.0155	0.0324
200	1.5	2	2	1	1.503	1.9972	2.0199	1.0102	0.0382	0.0512	0.1408	0.0656	0.0023	0.0041	0.0319	0.007
200	1.5	2	2	1.5	1.5038	1.998	2.0252	1.5157	0.0373	0.0467	0.1399	0.0976	0.0022	0.0035	0.0314	0.0153
200	1.5	2	2	2	1.5046	1.9946	2.0122	2.0302	0.0378	0.0496	0.1382	0.1402	0.0022	0.0039	0.0304	0.0319
200	2	0.5	1	1	2.01	0.4986	1.0122	1.0118	0.0579	0.0096	0.065	0.0637	0.0052	0.0001	0.0069	0.0067
200	2	0.5	1	1.5	2.0073	0.4991	1.0115	1.5159	0.0516	0.0091	0.0637	0.1004	0.0043	0.0001	0.0066	0.0159
200	2	0.5	1	2	2.0092	0.4993	1.0045	2.0287	0.0509	0.0093	0.0629	0.1453	0.0042	0.0001	0.0062	0.0345
200	2	0.5	1.5	1	2.0012	0.4991	1.52	1.012	0.0549	0.0091	0.0988	0.0602	0.0047	0.0001	0.0157	0.0058
200	2	0.5	1.5	1.5	2.0003	0.4997	1.5111	1.5048	0.0512	0.0092	0.0977	0.0915	0.0041	0.0001	0.0157	0.0138
200	2	0.5	1.5	2	2.0035	0.4999	1.5126	2.0175	0.0486	0.0095	0.0978	0.1391	0.0038	0.0001	0.015	0.0316
200	2	0.5	2	1	2.0037	0.4993	2.0246	1.0109	0.0516	0.0094	0.1402	0.0623	0.0042	0.0001	0.0313	0.0063
200	2	0.5	2	1.5	2.0094	0.4993	2.0184	1.5125	0.0508	0.0089	0.1369	0.0998	0.004	0.0001	0.0307	0.0156
200	2	0.5	2	2	2.0016	0.499	2.0237	2.0294	0.0482	0.0091	0.1405	0.1352	0.0037	0.0001	0.0319	0.0301
200	2	0.7	1	1	2.0049	0.6995	1.0044	1.0081	0.055	0.0131	0.0609	0.0608	0.0047	0.0003	0.0059	0.0061
200	2	0.7	1	1.5	2.0037	0.6985	1.0078	1.5173	0.0526	0.0135	0.0645	0.0989	0.0044	0.0003	0.0068	0.0163
200	2	0.7	1	2	2.0014	0.6983	1.0111	2.0267	0.0506	0.0132	0.0652	0.1429	0.0041	0.0003	0.0068	0.0331
200	2	0.7	1.5	1	2.0047	0.6988	1.5087	1.0087	0.0527	0.0131	0.0964	0.0627	0.0045	0.0003	0.0151	0.0063
200	2	0.7	1.5	1.5	2.0072	0.699	1.5174	1.5104	0.0536	0.0127	0.1	0.0941	0.0044	0.0002	0.0162	0.0142
200	2	0.7	1.5	2	2.0023	0.6988	1.5131	2.0248	0.0491	0.0129	0.0973	0.1428	0.0038	0.0003	0.0154	0.0319
200	2	0.7	2	1	2.0041	0.6993	2.0308	1.008	0.0524	0.0127	0.1384	0.0577	0.0043	0.0003	0.031	0.0055
200	2	0.7	2	1.5	2.0048	0.6996	2.0201	1.5219	0.0504	0.013	0.1456	0.1005	0.0041	0.0003	0.0341	0.0164
200	2	0.7	2	2	2.004	0.699	2.0207	2.0247	0.0476	0.0121	0.136	0.1361	0.0035	0.0002	0.0292	0.0299
200	2	1	1	1	2.0087	0.9977	1.0098	1.0078	0.0568	0.0195	0.0626	0.0617	0.005	0.0006	0.0062	0.0061
200	2	1	1	1.5	2.0069	0.9986	1.0121	1.522	0.0543	0.019	0.0641	0.0999	0.0046	0.0006	0.0065	0.0162
200	2	1	1	2	2.0065	0.9995	1.0059	2.0222	0.0502	0.0183	0.0609	0.1306	0.004	0.0005	0.006	0.0279
200	2	1	1.5	1	2.0036	0.9978	1.5105	1.0115	0.0538	0.0186	0.099	0.0627	0.0045	0.0006	0.016	0.0063
200	2	1	1.5	1.5	2.0022	0.9992	1.5109	1.5208	0.0511	0.0186	0.097	0.1027	0.0041	0.0006	0.0153	0.0168
200	2	1	1.5	2	2.0021	0.9986	1.5168	2.0172	0.048	0.0182	0.098	0.1391	0.0037	0.0005	0.0154	0.0314
200	2	1	2	1	2.0017	0.9977	2.0237	1.0111	0.0512	0.0179	0.1393	0.0633	0.0041	0.0005	0.0311	0.0064
200	2	1	2	1.5	2.0039	0.9984	2.0227	1.5191	0.0515	0.0183	0.1431	0.1	0.0041	0.0005	0.0323	0.0168
200	2	1	2	2	2.0023	0.9987	2.0221	2.0268	0.0478	0.0177	0.1371	0.1377	0.0036	0.0005	0.0299	0.0306
200	2	2	1	1	2.0045	1.9973	1.0047	1.008	0.0574	0.0374	0.0623	0.0618	0.0051	0.0023	0.0063	0.0061
200	2	2	1	1.5	2.0062	1.9942	1.0054	1.5211	0.052	0.0361	0.0594	0.0975	0.0042	0.002	0.0056	0.0161
200	2	2	1	2	2.0078	1.9969	1.008	2.0209	0.0505	0.0356	0.0652	0.1419	0.0041	0.002	0.0065	0.0319
200	2	2	1.5	1	2.0074	1.9985	1.5165	1.0108	0.0521	0.0354	0.1026	0.0638	0.0043	0.002	0.0171	0.0064
200	2	2	1.5	1.5	2.0076	1.9951	1.5133	1.5237	0.0513	0.0376	0.1048	0.1063	0.0042	0.0022	0.017	0.0183
200	2	2	1.5	2	2.0048	1.9972	1.5109	2.0162	0.0497	0.0368	0.0977	0.1428	0.0038	0.0021	0.0155	0.0323
200	2	2	2	1	2.0063	1.9966	2.02	1.0097	0.052	0.0382	0.1404	0.0605	0.0042	0.0022	0.0314	0.0058
200	2	2	2	1.5	2.0064	1.9975	2.027	1.5173	0.049	0.036	0.1396	0.0993	0.0039	0.002	0.0312	0.016
200	2	2	2	2	2.0042	1.9964	2.0243	2.0203	0.0493	0.0353	0.133	0.1374	0.0038	0.002	0.0284	0.0306



(a) (b)

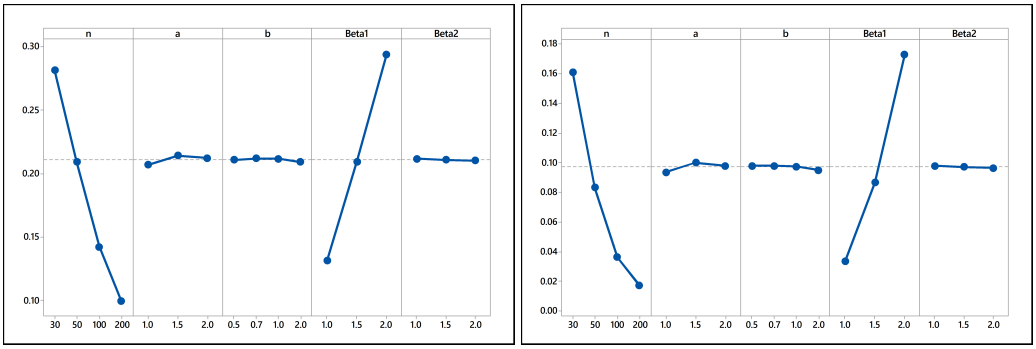
Figure 1: *Main Effect Plots of and MSE of shape parameter : (a) Bias($\hat{a}$) (b) MSE($\hat{a}$)*



(a) (b)

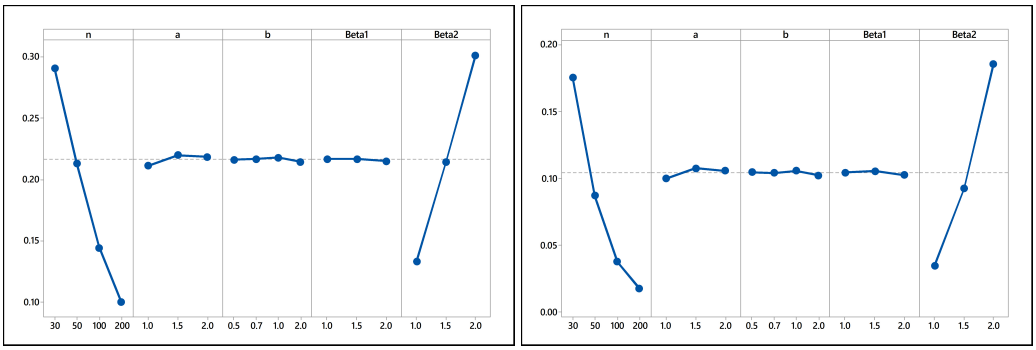
Figure 2: *Main Effect Plots of and MSE of scale parameter: (a)Bias($\hat{b}$) (b) MSE($\hat{b}$)*

Figure 1 is the main effect plot for the bias and MSE of shape parameter a for various sample sizes (n), shape parameters(a), scale parameters (b), and load share parameters (β_1 and β_2). It is observed that as n increases both the bias and MSE decreases. Bias and MSE increase as its own value is increases. Scale parameter (b) is not affecting on the bias and MSE of shape parameter (a). Bias in (a) slightly decreases as load share parameters β_1 and β_2 increases. Figure 2 is the main effect plot for the bias and MSE of scale parameter b for various values of different parameters. From figure 2 observed that bias as well as MSE is decreases as a and n increases and it increases as its own value increases while is not affected by changes in load share parameters.



(a) (b)

Figure 3: *Main Effect Plots of bias of different parameters: (a) Bias($\hat{\beta_1}$) (b) MSE($\hat{\beta_1}$)*



(a) (b)

Figure 4: *Main Effect Plots of MSE of different parameters: (a) Bias($\hat{\beta_2}$) (b) MSE($\hat{\beta_2}$)*

Figure 3 and 4 is the main effect plot for the bias and MSE of load share parameters β_1 and β_2 for various sample sizes (n), shape parameters(a), scale parameters (b), and load share parameters (β_1 and β_2). It is observed that as n increases both the bias and MSE decreases. Bias and MSE increase as its own value is increases. Changes in scale parameter (b) is not affecting on the bias as well as MSE. Also, the bias and MSE of load share parameter in not affected by the changes in other load share parameter.

6. Data Analysis

To illustrate the practical application of the proposed method, we have analyzed two datasets taken from Kim and Kvam [13]. Each datasets consists of 20 observations, specifically capturing the first three failure times of the systems under consideration. We interpret these datasets in the context of two load-sharing configurations: a three-component parallel (1-out-of-3) load-sharing system and a 2-out-of-3 load-sharing system. In our analysis, we utilize Weibull distributions as the baseline distributions for modeling

The failure time data for both datasets are summarized in Table 5, showing the values of the first three failure times (x_1, x_2, x_3) for each dataset.

Table 5: *Failure Time Data*

Data Set 1		
x_1	x_2	x_3
0.37	1.94	6.93
0.06	2.42	7.44
0.14	0.2	0.2
1.62	2.14	2.34
1.91	1.96	5.7
2.25	4.6	8.23
0.09	1.4	2.5
0.79	2.44	7.27
0.06	0.12	0.92
0.73	0.79	8.61
1.38	2.78	7.22
0.85	2.81	5.05
0.52	4.13	8.5
1.11	5.67	12.93
0.96	3.54	4.46
2.38	3.5	7.16
0.32	1.89	19.59
1.54	4.98	7.32
2.58	8.61	10.29
1.22	1.73	2.22

Data Set 2		
x_1	x_2	x_3
0.36	3.85	6.49
0.14	0.32	7.57
0.12	5.98	8.29
0.86	3.43	6.12
1.19	2.42	6
0.2	1.53	6.26
1.01	1.18	2.5
1.19	1.3	9.13
2.08	3.62	4.32
0.49	2.89	6.28
3.25	3.88	6.22
0.03	4.18	17.87
0.46	8.99	27.63
2.17	4.08	15.02
1.93	6.81	10.18
0.37	2.7	5.04
0.34	0.97	2.47
2.64	5.16	5.43
0.1	2.38	4.03
0.16	2.26	3.98

The Table 6 summarizes the estimated parameters, along with the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for the models applied to the given datasets.

Table 6: *Estimates of Parameters, AIC, and BIC for Different Load Sharing Models*

Data	Model	Estimate				AIC	BIC
		$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$		
Dataset 1	1 out of 3	0.948	3.597	0.915	0.931	-194.09	-191.1
	2 out of 3	1.001	3.484	0.926	–	-100.227	-97.25
Dataset 2	1 out of 3	0.978	3.419	1.41	1.35	-209.04	-206.05
	2 out of 3	0.928	3.492	1.395	–	-108.72	-105.74

The table above presents the estimated parameters, along with the AIC and BIC values, for different load-sharing models applied to two datasets. The analysis compares the performance of "1-out-of-3" and "2-out-of-3" load-sharing systems, with parameters $\hat{a}$, $\hat{b}$, $\hat{\beta_1}$, and $\hat{\beta_2}$ estimated for each model. Notably, the "2-out-of-3" model is simpler, as it omits the $\hat{\beta_2}$ parameter.

The comparison of AIC and BIC values reveals that the "1-out-of-3" model consistently outperforms the "2-out-of-3" model in both datasets, as indicated by its lower AIC and BIC values. This suggests that the "1-out-of-3" model provides a better fit, despite its increased complexity. The variability in the estimates of $\hat{\beta_1}$ and $\hat{\beta_2}$ across the models further highlights the different behaviors of the systems under the respective load-sharing rules.

Overall, these findings suggest that while the "2-out-of-3" model is simpler, the "1-out-of-3" model may more accurately represent the underlying load-sharing systems in these datasets. This makes it potentially more effective and practical for engineering and industrial applications where precise modeling is crucial.

7. Conclusions

In this paper, we developed a load-sharing model for k-out-of-m systems using the Proportional Conditional Reverse Hazard Rate approach with an unequal load-sharing rule. Our model addresses the complexities of real-world systems where components share the load unevenly after failures occur. To illustrate the model, a 2-out-of-4 configuration with Weibull baseline distributions is employed. Maximum likelihood estimation is used to estimate the model parameters, and the accuracy of these estimates is assessed through a simulation study that evaluates bias and mean square error. Furthermore, the model's practical applicability is demonstrated by analyzing two real datasets.

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IMPROVING VARIANCE PRECISION IN POPULATION STUDIES: THE ROLE OF POST-STRATIFICATION AND AUXILIARY DATA

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Abstract

In this study, we propose an enhanced estimator for the finite population variance in the context of post-stratified sampling, incorporating an auxiliary variable to improve accuracy. We derive expressions for the bias and mean square error (MSE) of the proposed estimator, providing an approximation accurate up to the first order. The theoretical analysis highlights the conditions under which the proposed estimator yields lower bias and reduced MSE, making it a more efficient alternative to traditional methods. To evaluate the practical performance of this estimator, we apply it to two real-world data sets, where our results demonstrate a marked improvement in efficiency over existing estimators. The numerical findings confirm that, in post-stratified sampling, the proposed estimator significantly enhances the precision of variance estimation, especially when the auxiliary variable is highly correlated with the study variable. This work not only contributes a more efficient estimator but also provides valuable insights into the application of auxiliary information in post-stratified sampling designs.

Keywords: Post stratification, Auxiliary variable, Estimation, Population variance and Mean squared error.

I. Introduction

This paper presents an enhanced estimator for population variance under post-stratification by utilizing auxiliary information. The use of auxiliary information in survey sampling has long been recognized for its ability to improve the efficiency of estimators for various population parameters, such as the mean, variance, median, mode, quartiles, interquartile range, percentiles, coefficient of variation, and proportions. Numerous methods for effectively incorporating auxiliary information have been extensively documented in survey sampling literature. Estimators of the ratio, product, and regression types leverage the correlation between the study variable and the auxiliary variable to achieve better precision. In this study, the mean square error (MSE) and bias of the proposed estimator are derived under large sample conditions, accurate to the first order of approximation. Theoretical comparisons with existing estimators are made, and conditions are established under which the proposed estimator is more efficient than those previously developed.

The application of stratified random sampling (STRS) presumes that the sizes and structure of sampling frames for each stratum are already available. Whereas the total population size and the percentage of the unit that belongs to each stratum may be known in many existing system, it is possible that the sample frame for every stratum is neither available or would be costly and difficult to construct. In social surveys relevant census information, where it is necessary to partition the heterogeneous population into different sub-groups, the sampling frame may not be available. In such types of situations, STRS is not applicable as such. In order to resolve these difficulties, post stratification technique is applied, in which a sample of necessary size is first selected from the population employing simple random sampling with or without replacement, and it is then stratified using the stratification variable

The procedure is identical to the one of stratified sampling and the only difference is that the allocation into strata is made ex-post. The gain in precision is related to the sample size in each stratum and (inversely) to the difference between the sample weights and the population ones. The standard error for the post-stratified mean estimator is larger than the stratified sampling one, because additional variability is given by the fact that the sample stratum sizes are themselves the outcome of a random process.

Initially introduced the concept of post-stratification [1]. Later extended this work by [2] investigating [3] classic ratio estimator in the context of post-stratification. They first considered the sample sizes within each stratum as fixed and then accounted for variations across possible stratum sample sizes, drawing on a result from [4]. Several researchers have made notable contributions to the development of post-stratification techniques. Important groundwork in the field laid by [5], followed by [6], who further advanced the methodology. Significant strides in [7] refining the theoretical foundations, while key modifications that improved estimator efficiency introduced by [8]. The scope of post-stratification by applying it to new contexts expanded by [9], and [10] provided valuable insights, enhancing the understanding and application of these methods in finite population estimation. More recently, characterized post-stratification product and ratio-type exponential estimators by [11]. While the regression estimator has generally been shown to outperform ratio and product estimators, this is not the case when the regression line of the primary variable on the auxiliary variable passes through a region near the origin [12].

In the literature, seldom is known about estimation of population variance under post-stratified sampling. A number of estimators for the limited population variance of the post-stratified sample mean utilizing data from the auxiliary variable developed by [13], a new ratio estimators in stratified random sampling using the information of an auxiliary attributes suggested by [14], An exponential estimator in the stratified random sampling taking an auxiliary attribute proposed by [15], An efficient exponential ratio estimator allows estimating the population mean in stratified random sampling using an auxiliary variables developed by [16], memory-type ratio and product estimators for the estimation of population variance based on exponentially weighted moving averages (EWMA) statistic proposed by [17], the generalized estimator of population mean using auxiliary attributes in stratified two- phase sampling introduced by [18], the estimation of rare and clustered population mean using stratified adaptive clustered sampling proposed by [19]. The difficulty of estimating the population mean in the situation of post-stratification discussed by [20].

II. Methods

Let the population of size N that is finite and partitioned into L strata of sizes $N_1, N_2, \dots, N_L$ such that $\sum_{h=1}^{L} N_h = N$. Simple random sampling without replacement (SRSWOR) is used to select a sample of size n from the whole population. Following the method of selection from the population, the number of units falling under the h^{th} stratum is indicated. Let n_h be the size of the

sample falling in the h^{th} stratum such that $\sum_{h=1}^{L} n_h = n$, here, it is expected that n is large enough so that the probability of n_h being zero is too low. Let $\bar{y}_{hi}\ and\ \bar{x}_{hi}$ are the observed values of y and x respectively on the i^{th} unit of the h^{th} stratum. Let $\bar{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi}$ and $\bar{x}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} x_{hi}$ represent the sample means corresponding to the population means $\bar{Y}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} y_{hi}$ and $\bar{X}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} x_{hi}$ of the study variable (y) and auxiliary variable (x) respectively in the h^{th} stratum.

Let $s_{yh}^2 = \frac{1}{n_h-1}\sum_{i=1}^{n_h}(y_{hi} - \bar{y}_h)^2\ and s_{xh}^2 = \frac{1}{n_h-1}\sum_{i=1}^{n_h}(x_{hi} - \bar{x}_h)^2$represent the sample variances corresponding to the population variances $S_{yh}^2 = \frac{1}{N_h-1}\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)^2 and S_{xh}^2 = \frac{1}{N_h-1}\sum_{i=1}^{N_h}(x_{hi} - \bar{X}_h)^2$ of the study variable (x) and auxiliary variable (x) respectively in the h^{th} stratum. Also, $c_{yh} = \frac{s_{yh}}{\bar{y}_h}, and\ c_{xh} = \frac{s_{xh}}{\bar{x}_h}$ represent the sample coefficient of variation corresponding to the population coefficient of variation $C_{yh} = \frac{S_{yh}}{\bar{Y}_h}\ and\ C_{xh} = \frac{S_{xh}}{\bar{X}_h}$ respectively. Let $r_{yxh} = \frac{s_{yxh}}{s_{yh}s_{xh}}$ represent the sample correlation coefficient corresponding to the population correlation coefficient $\rho_{yxh} = \frac{S_{yxh}}{S_{yh}S_{xh}}$ between their respective subscripts in the h^{th} stratum.

Further $s_{yxh} = \frac{1}{n_h-1}\sum_{i=1}^{n_h}(y_{hi} - \bar{y}_h)(x_{hi} - \bar{x}_h)$ represents the sample covariance corresponding to population variance $S_{yxh} = \frac{1}{N_h-1}\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)(x_{hi} - \bar{X}_h)$, the study variable (y) and auxiliary variable (x).

Let $\sigma_{pst}^2 = \sum_{h=1}^{L} w_h^2 \frac{S_{yh}^2}{n_h}$ and $\hat{\sigma}_{pst}^2 = \sum_{h=1}^{L} w_h^2 \frac{s_{yh}^2}{n_h}$ represent the post-stratified population and sample variances of y, where $w_h = \frac{N_h}{N}$ is the h^{th} stratum weight respectively. Ignoring the finite population correction factor to make calculations easier in post-stratified sampling

The variance of the estimator $\hat{\sigma}_{pst}^2$ is provided as

$$Var(\hat{\sigma}_{pst}^2) = \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S_{yh}^4(\lambda_{400h} - 1) \tag{1}$$

$$where\ \ \lambda_{rsth} = \frac{\mu_{rsh}}{\mu_{200h}^{r/2}\mu_{020h}^{s/2}}\ \ and\ \ \mu_{rsth} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)^r\,(x_{hi} - \bar{X}_h)^s$$

A post-stratified regression estimator $\hat{\sigma}_{pst(reg)}^2$ for the finite population variance of the post-stratified sample mean utilizing data from the study's auxiliary variables developed [13] as:

$$\hat{\sigma}_{pst(reg)}^2 = \sum_{h=1}^{L} \frac{W_h^4}{n_h}\left[(S_{yh}^2 + \hat{\Psi}_{220h}(S_{xh}^2 - s_{xh}^2)\right] \tag{2}$$

Where $\hat{\Psi}_{220h}$ is the sample regression coefficient of y on x with corresponding population regression coefficient $\Psi_{220h} = \frac{S_{yh}^2(\lambda_{220h}-1)}{S_{xh}^2(\lambda_{040h}-1)}$ in h^{th} stratum.

The variance of $\hat{\sigma}_{pst(reg)}^2$ is given by

$$Var(\hat{\sigma}_{pst(reg)}^2) = \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S_{yh}^4(\lambda_{400h} - 1) - \frac{(\lambda_{220h} - 1)}{(\lambda_{040h} - 1)} \tag{3}$$

A conventional ratio estimator $\sigma_{pst(rat)}^2$ for the finite population variance of the post-stratified sample mean utilizing data from the study's auxiliary variables developed by [13] as:

$$\hat{\sigma}_{pst(rat)}^2 = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\left((S_{yh}^2 \frac{S_{xh}^2}{s_{xh}^2}\right) \tag{4}$$

The Bias of $\hat{\sigma}_{pst(rat)}^2$ up to the first order of approximation, is given as

$$Bias\left(\hat{\sigma}^2_{pst(rat)}\right) = \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} S_{yh}^2 (\lambda_{040h} - 1) - (\lambda_{220h} - 1)$$

The MSE of $\hat{\sigma}^2_{pst(rat)}$ up to the first order of approximation, is given by

$$Var\hat{\sigma}^2_{pst(rat)} = \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S_{yh}^4 (\lambda_{400h} - 1) + (\lambda_{040h} - 1) - 2(\lambda_{220h} - 1) \quad (5)$$

The new efficient type estimator of population variance developed [21] as follows:

$$\hat{\sigma}^2_{pst(\alpha,\delta)} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} S_{yh}^2 \left[2 - \left(\frac{s_{xh}^2}{S_{xh}^2}\right)^{\alpha} exp\left\{\frac{\delta(s_{xh}^2 - S_{xh}^2)}{s_{xh}^2 - S_{xh}^2}\right\}\right] \quad (6)$$

Where $\alpha \; and \; \delta$ are unknown constants whose values are to be determined such that the MSE of the proposed estimator is minimum.

The Bias of $\hat{\sigma}^2_{pst(\alpha,\delta)}$ up to the first order of approximation, is given as

$$Bias\left(\hat{\sigma}^2_{pst(\alpha,\delta)}\right) = \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} S_{yh}^2 \frac{(2\alpha + \delta)(2\alpha + \delta - 2)}{8} (\lambda_{040h} - 1) - \frac{(2\alpha + \delta)}{2} (\lambda_{220h} - 1)$$

The MSE of $\hat{\sigma}^2_{pst(\alpha,\delta)}$ up to the first order of approximation, is given as

$$MSE\left(\hat{\sigma}^2_{pst(reg)}\right) = \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S_{yh}^4 (\lambda_{400h} - 1) - \frac{(\lambda_{220h} - 1)^2}{(\lambda_{040h} - 1)} \quad (7)$$

A new ratio type estimator under post-stratified sampling developed [20] as follows:

$$\hat{\sigma}^2_{pst(k)} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} S_{yh}^2 \left[k + (1 - k) exp\left\{\frac{S_{xh}^2 - s_{xh}^2}{S_{xh}^2 - s_{xh}^2}\right\}\right] \quad (8)$$

The Bias of $\hat{\sigma}^2_{pst(k)}$ up to the first order of approximation, is given by

$$Bias\left(\hat{\sigma}^2_{pst(k)}\right) = \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} S_{yh}^2 \frac{3}{8} (\lambda_{040h} - 1)(1 - k) - \frac{1}{2} (\lambda_{220h} - 1)(1 - k)$$

The MSE of $\left(\hat{\sigma}^2_{pst(k)}\right)$ up to the first order of approximation, is given by

$$MSE\left(\hat{\sigma}^2_{pst(reg)}\right) = \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S_{yh}^4 (\lambda_{400h} - 1) - 2\frac{(\lambda_{220h} - 1)}{(\lambda_{040h} - 1)} \quad (9)$$

III. Proposed Estimator

In the line with the direction of study carried out by [22], we proposed an improved estimator for estimating the finite population variance under the method of post stratified sampling. The bias and MSE of the existing and proposed estimator are derived up to the first order of approximation. The performance of the proposed estimator is the best as compared to existing counterparts in terms of efficiency.

$$\hat{\sigma}^2_{pst(YS)} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left[K_{1h} S_{yh}^2 + K_{2h}(S_{xh}^2 - s_{xh}^2)\right] exp\left(\frac{S_{xh}^2 - s_{xh}^2}{S_{xh}^2 - s_{xh}^2}\right) \quad (10)$$

Where K_{1h} and K_{2h} are unknown constants whose values are to be determined such that the MSE of the proposed estimator is minimum.

For examining the large sample characteristics of the developed estimator, $\hat{\sigma}^2_{pst(YS)}$ we define the random variables up to the first order of approximation as:

$$s_{yh}^2 = S_{yh}^2(1 + \varepsilon_{2h}), \qquad s_{xh}^2 = S_{xh}^2(1 + \varepsilon_{2h}) \quad such \; that \; E(\varepsilon_{2h}) = E(\varepsilon_{2h}) = 0$$

Also,

$$E(\varepsilon_{2h}^2) = \frac{1}{n_h}(\lambda_{400h} - 1),\ \ E(\varepsilon_{4h}^2) = \frac{1}{n_h}(\lambda_{040h} - 1) \quad and \quad E(\varepsilon_{2h}\varepsilon_{4h}) = \frac{1}{n_h}(\lambda_{200h} - 1)$$

When we use the values of above terms in eq. (10), we have

$$\hat{\sigma}_{pst(YS)}^2 = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\left[\left(K_{1h}S_{yh}^2(1+\varepsilon_{2h}) - K_{2h}S_{xh}^2 - S_{xh}^2(1+\varepsilon_{4h})\right)\right] exp\left(\frac{S_{xh}^2 - S_{xh}^2(1+\varepsilon_{4h})}{S_{xh}^2 - S_{xh}^2(1+\varepsilon_{4h})}\right)$$

Now expanding the right hand side along with the exponential term of the above equation up to the first degree of approximation, we

$$\hat{\sigma}_{pst(YS)}^2 \approx \sum_{h=1}^{L} \frac{W_h^2}{n_h}\left[\left(K_{1h}S_{yh}^2(1+\varepsilon_{2h}) - K_{2h}S_{xh}^2 - S_{xh}^2(1+\varepsilon_{4h})\right)\right]\left(1 - \frac{1}{2}\varepsilon_{4h} + \frac{3}{8}\varepsilon_{4h}^2\right) \tag{11}$$

Subtracting S_{yh}^2 from both sides of (11), we obtain

$$\begin{aligned}\left(\hat{\sigma}_{pst(YS)}^2 - S_{yh}^2\right) = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\Bigg[\Bigg(&S_{yh}^2 + K_{1h}S_{yh}^2 + K_{1h}S_{yh}^2\varepsilon_{2h} - \frac{1}{2}K_{1h}S_{yh}^2\varepsilon_{4h} \\ &- K_{2h}S_{xh}^2\varepsilon_{4h} - \frac{1}{2}K_{1h}S_{yh}^2\varepsilon_{2h}\varepsilon_{4h} + \frac{3}{8}K_{1h}S_{yh}^2\varepsilon_{4h}^2 - \frac{1}{2}K_{2h}S_{xh}^2\varepsilon_{4h} - S_{xh}^2\Bigg)\Bigg]\end{aligned} \tag{12}$$

By applying expectation on both sides of eq. (12), we get the Bias of $\hat{\sigma}_{pst(YS)}^2$ as

$$\begin{aligned}Bias\left(\hat{\sigma}_{pst(YS)}^2\right) = \sum_{h=1}^{L} \frac{W_h^2}{n_h^2}\Bigg[\Bigg(&S_{yh}^2 \\ &+ K_{1h}S_{yh}^2\left\{1 + \frac{3}{8}(\lambda_{040h} - 1) - \frac{1}{2}(\lambda_{330h} - 1) + \frac{1}{2}K_{2h}S_{xh}^2(\lambda_{040h} - 1)\right\}\Bigg)\Bigg]\end{aligned} \tag{13}$$

Equation (12) can be squared on both sides we get,

$$\begin{aligned}MSE\left(\hat{\sigma}_{pst(YS)}^2\right) = \sum_{h=1}^{L} \frac{W_h^4}{n_h^2}\Bigg[&S_{yh}^4 + K_{1h}^2S_{yh}^4 + K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{400h} - 1) - \frac{1}{4}K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{040h} - 1) \\ &+ K_{2h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{040h} - 1) + \frac{1}{4}K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{220h} - 1)^2 + 2K_{1h}S_{yh}^4 \\ &- 2K_{1h}S_{yh}^4\frac{1}{n_h}(\lambda_{220h} - 1) + \frac{6}{8}K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{040h} - 1) + K_{2h}S_{yh}^2S_{yh}^2\frac{1}{n_h}(\lambda_{040h} - 1) \\ &- K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{220h} - 1) + \frac{6}{8}K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{040h} - 1) + K_{1h}^2S_{yh}^4\frac{1}{n_h}(\lambda_{220h} - 1) \\ &- 2K_{1h}K_{2h}S_{yh}^2S_{yh}^2\frac{1}{n_h}(\lambda_{220h} - 1) \\ &+ K_{1h}^2S_{yh}^2\frac{1}{n_h}(\lambda_{040h} - 1)\Bigg]\end{aligned} \tag{14}$$

In the above equation, K_{1h} and K_{2h} are unknown constants whose values are to be determined such that the MSE of the proposed estimator is minimum and their optimal values are obtained by differentiating partially equation (14) with respect to K_{1h} and K_{2h} and then equating to zero as:

$$\frac{\partial MSE\left(t_{(YS)}/n_h\right)}{\partial K_{1h}} = 0$$

$$K_{1hopt} = \left(\frac{\lambda_{040h}-1}{8}\right)\left[\frac{8-(\lambda_{040h}-1)}{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2}\right]$$

$$\frac{\partial MSE\left(t_{(YS)}/n_h\right)}{\partial K_{2h}} = 0$$

$$K_{2hopt} = \left(\frac{S_{yh}^2}{8S_{xh}^2}\right)\left[\frac{4(\lambda_{220h}-1)^2-(\lambda_{040h}-1)(\lambda_{220h}-1)^2+(\lambda_{040h}-1)}{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2}\right]$$

The reduced MSE of $\hat{\sigma}^2{}_{Pst(YS)}$ at the optimum values of K_{1h} and K_{2h} is obtained as

$$MSE\left(\hat{\sigma}^2{}_{Pst(YS)}\right)_{min} = \sum_{h=1}^{L}\left(\frac{W_h^2}{n_h^2}\right)S_{yh}^4\left\lfloor\frac{64\{(\lambda_{400h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}-(\lambda_{040h}-1)^3}{64\{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}\right.$$
$$\left.\frac{-16(\lambda_{040h}-1)\{(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}{\rfloor}\right. \quad (15)$$

IV. Efficiency Comparison

The developed estimator has been theoretically compared to the competing estimators of the population variance. As a result, when (1), (3), (5), (7), (9), (11) and (17) are compared, it is clear that the suggested estimator $\hat{\sigma}^2{}_{Pst(YS)}$, would be more efficient than [13] usual unbiased estimator$\hat{\sigma}^2{}_{Pst}$, [13] regression estimator $\hat{\sigma}^2{}_{Pst(reg)}$, [13] regression estimator $\hat{\sigma}^2{}_{Pst(rat)}$, [21] efficient type estimator $\hat{\sigma}^2{}_{Pst(\alpha,\delta)}$, [20] new ratio type estimator $\hat{\sigma}^2{}_{Pst(k)}$.

1. By taking eq. (1) and (15), we get

$$\left\{Var\left(\hat{\sigma}^2{}_{Pst}\right)-MSE\left(\hat{\sigma}^2{}_{Pst(YS)}\right)_{min}\right\}>0 \quad (16)$$

$$\hat{\sigma}^2{}_{Pst}\ if:\ \sum_{h=1}^{L}\left(\frac{W_h^2}{n_h^2}\right)S_{yh}^4\left\lfloor(\lambda_{400h}-1)\right.$$
$$\left\lfloor-\frac{64\{(\lambda_{400h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}-(\lambda_{040h}-1)^3}{64\{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}\right.$$
$$\left.\frac{-16(\lambda_{040h}-1)\{(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}{\rfloor}\right\rfloor>0$$

2. By taking eq. (3) and (15), we get

$$\left\{Var\left(\hat{\sigma}^2{}_{Pst(reg)}\right)-MSE\left(\hat{\sigma}^2{}_{Pst(YS)}\right)_{min}\right\}>0 \quad (17)$$

$$\hat{\sigma}^2{}_{Pst(reg)}\ if:\ \sum_{h=1}^{L}\left(\frac{W_h^2}{n_h^2}\right)S_{yh}^4\Bigg|_{(\lambda_{400h}-1)-\frac{(\lambda_{220h}-1)^2}{(\lambda_{040h}-1)}}\Bigg\lfloor -\frac{64\{(\lambda_{400h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}-(\lambda_{040h}-1)^3}{64\{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}\Bigg\rfloor\Bigg\rfloor>0$$

3. By taking eq. (5) and (15), we get

$$\left\{Var\left(\hat{\sigma}^2{}_{Pst(rat)}\right)-MSE\left(\hat{\sigma}^2{}_{Pst(YS)}\right)_{min}\right\}>0 \tag{18}$$

$$\hat{\sigma}^2{}_{Pst(reg)}\ if:\ \sum_{h=1}^{L}\left(\frac{W_h^2}{n_h^2}\right)S_{yh}^4\left[\begin{matrix}(\lambda_{400h}-1)+(\lambda_{040h}-1)\\-2(\lambda_{220h}-1)\end{matrix}\right]\Bigg\lfloor -\frac{64\{(\lambda_{400h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}-(\lambda_{040h}-1)^3}{64\{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}\Bigg\rfloor\Bigg\rfloor>0$$

4. By taking eq. (7) and (15), we get

5. $$\left\{Var\left(\hat{\sigma}^2{}_{Pst(\alpha,\delta)}\right)-MSE\left(\hat{\sigma}^2{}_{Pst(YS)}\right)_{min}\right\}>0 \tag{19}$$

$$\hat{\sigma}^2{}_{Pst(\alpha,\delta)}\ if:\ \sum_{h=1}^{L}\left(\frac{W_h^2}{n_h^2}\right)S_{yh}^4\Bigg|_{(\lambda_{400h}-1)-\frac{(\lambda_{220h}-1)^2}{(\lambda_{040h}-1)}}\Bigg\lfloor -\frac{64\{(\lambda_{400h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}-(\lambda_{040h}-1)^3}{64\{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}\Bigg\rfloor\Bigg\rfloor>0$$

5. By taking eq. (9) and (15), we get

$$\left\{Var\left(\hat{\sigma}^2{}_{Pst(k)}\right)-MSE\left(\hat{\sigma}^2{}_{Pst(YS)}\right)_{min}\right\}>0 \tag{20}$$

$$\hat{\sigma}^2{}_{Pst(\alpha,\delta)}\ if:\ \sum_{h=1}^{L}\left(\frac{W_h^2}{n_h^2}\right)S_{yh}^4\Big|(\lambda_{400h}-1)-2\frac{(\lambda_{220h}-1)}{(\lambda_{040h}-1)}$$
$$\Big\lfloor -\frac{64\{(\lambda_{400h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}-(\lambda_{040h}-1)^3}{64\{(\lambda_{040h}-1)+(\lambda_{040h}-1)(\lambda_{040h}-1)-(\lambda_{220h}-1)^2\}}\Big\rfloor\Big] > 0$$

As the conditions (16)-(20) are always satisfied, it is inferred that the proposed estimator is more efficient than the other existing estimators under all cases in theory.

V. Empirical Study

5. Numerical Analysis

We will take into account the data set to assess the efficiency of the suggested estimators. The characteristics of the population are described as below:

Population 1:

Let y is the output and x is the fixed capital of 80 factories [23]. The data have classified arbitrarily into four strata as $x \le 500$,$500 < x \le 1000$, $1000 < x \le 2000$, and $x>2000$, respectively.

y: Output
x: Fixed capital

Table 1(a): *Statistical Description of the Population:*

Constants	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	s_{yh}^2	λ_{400h}	λ_{040h}	λ_{220h}
Stratum I	20	11	3006.55	65.90	572819.20	3.45	1.55	1.49
Stratum II	31	18	4687.62	141.90	433681.58	1.56	3.09	1.73
Stratum III	13	8	6496.23	392.38	162104.69	1.98	1.49	1.56
Stratum IV	16	8	7795.31	749.50	426528.63	2.35	1.91	2.05

Population 2:

we use the data concerning the number of teachers as study variable and the number of students as auxiliary variable in both primary and secondary schools for 923 districts at six regions (as 1: Marmara 2: Agean 3: Mediterranean 4: Central Anatolia 5: Black Sea 6: East and Southeast Anatolia) in Turkey (Source: Ministry of Education, Republic of Turkey).

Y: number of teachers

X: Number of students

Table 2(a): *Statistical Description of the Population:*

Constants	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	s_{yh}^2	λ_{400h}	λ_{040h}	λ_{220h}
Stratum I	127	31	703.740	20804.59	781163.9	3.94783	6.251589	3.720488
Stratum II	117	21	413.000	9211.79	415924.8	17.33181	19.35622	18.35209
Stratum III	103	29	573.174	14309.30	1068054	15.87136	16.3073	16.09088
Stratum IV	170	38	424.664	9478.85	657047.8	13.60375	11.67999	11.65605
Stratum V	205	22	267.029	5569.94	162936.9	22.31908	23.14865	22.30021
Stratum VI	201	39	393.840	12997.59	506549	21.49882	24.26014	21.79386

Table 3: *Conditional values of different estimators using real data sets:*

Conditional values	**Population I**	**Population II**
Conditional values I	2964740.76	628422.27
Conditional values II	1651010.71	29677.75
Conditional values III	2308528.95	41662.86
Conditional values IV	1254126.31	29677.75
Conditional values V	1651010.71	224246.86
Conditional values VI	6579180.77	706578.08

Table 4: *Bias values of different estimators using real data sets:*

Estimators	**Population I**	**Population II**
$\sigma_{(pst)}^2$	-	-
$\sigma_{pst(Reg)}^2$	-	-
$\sigma_{Pst(Rat)}^2$	248.85	111.00
$\sigma_{Pst(\alpha,\delta)}^2$	-6154.30	-124519.90
$\sigma_{Pst(T)}^2$	-6392.45	-218847.51
$\sigma_{Pst(k)}^2$	-104.74	1024.01
$\sigma_{Pst(YS)}^2$	247404838.9	4.92057E+19

Table 1(b): *MSE and PRE of suggested estimator in relation to* $\hat{\sigma}^2{}_{Pst}$

No.	1	2	3	4	5	6	**7**
Estimators	$\sigma_{(pst)}^2$	$\sigma_{pst(Reg)}^2$	$\sigma_{Pst(Rat)}^2$	$\sigma_{Pst(T)}^2$	$\sigma_{Pst(\alpha,\delta)}^2$	$\sigma_{Pst(k)}^2$	$\boldsymbol{\sigma_{Pst(YS)}^2}$
MSE	3574695.71	2260965.66	2918483.90	1864081.26	2260965.66	7189135.72	**609954.95**
PRE	100.00	157.40	121.28	191.77	157.40	49.72	**586.06**

Table 2(b): *MSE and PRE of suggested estimator in relation to* $\hat{\sigma}^2{}_{Pst}$

No.	1	2	3	4	5	6	**7**
Estimators	$\sigma^2_{(pst)}$	$\sigma^2_{pst(Reg)}$	$\sigma^2_{Pst(Rat)}$	$\sigma^2_{Pst(\alpha,\delta)}$	$\sigma^2_{Pst(T)}$	$\sigma^2_{Pst(k)}$	$\boldsymbol{\sigma^2_{Pst(YS)}}$
MSE	651146.08	52401.56	64386.67	52401.56	246970.67	729301.89	**22723.81**
PRE	100.00	1242.61	1011.31	1242.61	263.65	89.28	**2865.48**

Where

$$PRE = \frac{Var\left(\hat{\sigma}^2{}_{Pst}\right)}{MSE\left(\hat{\sigma}^2{}_{Pst(i)}\right)} \times 100 \;\; : \;\; i = Re, Rat, (\alpha, \delta), T, and\ k.$$

VI. Conclusion

The population variance of the research variable can be effectively estimated using auxiliary data through an improved estimator under post-stratification. The proposed single and combined classes of estimators, such as bias and mean square error (MSE), are derived approximately to the first order of accuracy. Under specific efficiency conditions, the recommended estimator significantly outperforms existing separate and combined estimators. Additionally, empirical research is conducted using both artificially generated symmetric and asymmetric populations, as well as real-world data, to validate the theoretical findings. The results demonstrate that the proposed estimator is more efficient, with a lower MSE and higher percentage relative efficiency (PRE) than the alternatives. This study provides clear evidence supporting the robustness and practicality of the suggested estimator in experimental surveys. Given its superior performance, we strongly recommend its adoption over traditional estimators for post-stratification variance estimation. The integration of theoretical and empirical analyses makes this research highly credible, insightful, and impactful for statistical applications.

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A COMPARATIVE STUDY ON PARAMETER ESTIMATION TECHNIQUES FOR THE DISCRETE INVERSE RAYLEIGH DISTRIBUTION

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Abstract

This article explores into the Discrete Inverse Rayleigh Distribution, a novel discrete analogue of the continuous Inverse Rayleigh distribution, formulated by inverting a continuous Rayleigh random variable. The Discrete Inverse Rayleigh Distribution can effectively capture a range of hazard rate shapes, exhibiting either unimodal or monotonic decreasing behaviors depending on parameter values. To estimate the parameters of this distribution, we examine four distinct methods: a heuristic algorithm, a probability paper plotting technique designed for the Inverse Rayleigh, the method of moments, and the method of proportions. Each method offers unique strengths and presents different computational requirements and precision levels. Through rigorous simulation studies, we assess the accuracy and reliability of these methods, evaluating their performance across a variety of scenarios. Our results indicate that the methods of moments and proportions encounter significant difficulties when estimating parameters for right-skewed Discrete Inverse Rayleigh distributions. These challenges are primarily due to numerical instability and poor convergence properties under certain parameter configurations, which can limit their practical applicability in these cases. In contrast, both the probability paper plotting method and the heuristic algorithm demonstrate robustness and enhanced accuracy, especially in the context of right-skewed distributions. The probability paper plot is notably effective due to its reliance on graphical techniques that simplify parameter estimation in complex, non-monotonic datasets, whereas the heuristic algorithm provides a more computationally efficient solution without sacrificing precision. To validate the utility of the Discrete Inverse Rayleigh Distribution, we compare its performance with the Discrete Rayleigh Distribution by fitting both models to a real-world dataset. The comparative analysis leverages the Akaike Information Criterion (AIC) to quantitatively assess model fit. Our findings underscore the advantages of the Discrete Inverse Rayleigh Distribution, particularly in applications where discrete data exhibits non-monotonic hazard rates, highlighting its superior fit over the traditional Discrete Rayleigh in this context. This study contributes to the growing toolkit for discrete time-to-event data modeling, offering insights into effective parameter estimation strategies and demonstrating the value of the Discrete Inverse Rayleigh Distribution for specialized discrete hazard rate analysis.

Keywords: Akaike Information Criterion, Discrete Inverse Rayleigh Distribution, Inverse Rayleigh Probability Paper Plot, Heuristic Algorithm, method of moments, method of proportions

I. Introduction

In life-testing experiments, measuring a device's lifespan on a continuous scale is often impractical or even infeasible. For example, the lifetime of a device that operates in an on/off mode, such as a switch, is usually a discrete variable, representing the number of cycles or operations until it fails. Many real-world reliability studies record failure data based on discrete occurrences, such as the count of cycles, runs, or shocks a device can withstand before malfunctioning. Similarly, in survival analysis, data like the number of days a lung cancer patient survives post-treatment or the period from remission to relapse is frequently recorded in discrete time intervals, like days.

Historically, discrete analogues of continuous probability distributions have been employed to model such data. For instance, the geometric distribution serves as the discrete counterpart to the exponential distribution, while the negative binomial distribution is analogous to the gamma distribution [9]. However, one limitation of these traditional discrete distributions is that they generally assume a monotonic hazard rate function, which remains either increasing or decreasing. This monotonicity can be restrictive for applications where the hazard rate does not exhibit such a simple pattern, limiting the flexibility of these distributions in accurately capturing the underlying risk dynamics in various scenarios. Fortunately, numerous continuous distributions can be adapted into discrete counterparts. The geometric and negative binomial distributions are well-known discretizations of the exponential and gamma distributions, respectively [7,8]. Additionally, discrete analogues for the Weibull, normal, and Rayleigh distributions have also been developed [20]. Roy introduced the discrete normal and Rayleigh distributions [16, 17], while Krishna and Pundir [13] proposed discrete versions of the Burr XII and Pareto distributions.

In the article, we propose the study on Discrete Inverse Rayleigh Distribution, a similar approach can be employed to model situations where the underlying process follows an inverse Rayleigh-like behavior but data is recorded in discrete units. Estimation of the parameters for such distributions can be performed using several techniques, such as the method of moments, the method of proportions, heuristic algorithms (like Nelder-Mead), or by utilizing probability paper methods[3].

This approach can be useful in fitting the discrete Inverse Rayleigh Distribution to datasets, such as the lifetimes of electronic devices, where discrete time-to-failure data is available [15, 10]. Comparisons between the discrete Inverse Rayleigh and Discrete Rayleigh can be made using model selection criteria like the Akaike Information Criterion (AIC), allowing researchers to determine the most appropriate model for their specific application. The Inverse Rayleigh Distribution is derived from the standard Rayleigh distribution, but it models a different type of relationship between the variable and its probabilities [11]. While the Rayleigh distribution is often used for modeling the magnitude of a two-dimensional vector, the Inverse Rayleigh distribution models scenarios where larger values are less probable, often used to model the time to failure or lifetime of systems.

II. The Theoretical Perspective on the Rayleigh Distribution

It was first introduced by Lord Rayleigh in 1880 [1] as a model for random wave amplitudes. The distribution is a special case of the Weibull distribution with a shape parameter of 2 [14]. The Rayleigh distribution is used for modeling the magnitude of vectors in 2D space or for modeling phenomena where small values are less probable, but larger values occur more frequently up to a certain threshold. It has a probability density function (PDF) is given by

$$f(x;q)=\frac{x}{q^2}\exp\left(-\frac{x^2}{2q^2}\right),\ x\geq 0$$

Where q is the scale parameter. The Inverse Rayleigh distribution is the distribution of the inverse of a Rayleigh-distributed variable. It is used to model the lifetime of devices or systems, where failure becomes less likely as time progresses (i.e., early failures are more probable). Its PDF is given by:

$$f(x;q,\beta)=\frac{q^2}{x^3}\exp\left(-\frac{q^2}{2x^2}\right),\ x\geq 0$$

where q is the scale parameter. This distribution describes a situation where the probability of larger values (longer lifetimes) diminishes rapidly, meaning that failures tend to happen early in the system's lifecycle. On the other hand, the Discrete Weibull Distribution, a discrete version of the continuous Weibull distribution, was introduced by Nakagawa and Osaki [2]. Its cumulative distribution function is defined as:

$$F(t)=1-q^{t^\beta},\ \text{t= 1, 2, 3…}$$

where $\beta > 0$ and $0<q<1$. This distribution has a probability mass function (PMF) and a hazard rate function that depend on the shape parameter β and the scale parameter q. In particular, when $\beta=1$, the distribution reduces to a geometric distribution, which is a discrete analogue of the exponential distribution with a constant hazard rate. For $\beta=2$, the distribution corresponds to the discrete Rayleigh distribution. The hazard rate can be either increasing or decreasing based on the value of β. The Discrete Inverse Rayleigh distribution can be seen as a discrete version of the Inverse Rayleigh distribution, offering a better fit for data sets that require modeling with both monotonic and non-monotonic hazard rates. The discrete inverse Rayleigh model provides flexibility and simplicity, making it a valuable tool for reliability and survival analysis where traditional models fail to provide a suitable fit.

III. Techniques for Parameter Estimation in the Discrete Inverse Rayleigh Distribution

The Discrete Inverse Rayleigh Distribution (DIRD) can be defined as a discrete analogue of the continuous inverse Rayleigh distribution, which has applications in reliability analysis, survival studies, and related fields. The derivation of the DIRD involves transforming the continuous Rayleigh distribution through inversion and discretization.

If X is a discrete random variable that follows the Discrete Inverse Rayleigh Distribution, its probability mass function (PMF) can be defined as:

$$f(X=x)=\frac{q^2}{x^3}\exp\left(-\frac{q^2}{2x^2}\right),\ x=1,2,3,..$$

Where $q>0$ is the scale parameter, x is a discrete integer representing possible values of the random variable. This distribution is derived from the continuous inverse Rayleigh distribution, adapting it for scenarios where the variable of interest can only take discrete values.

The Probability Mass Function (PMF) corresponding to this distribution is:

$$p_n(q,\beta)=\begin{cases} q, & if\ n=1 \\ q^{n^\beta}-q^{(n-1)^\beta}, & if\ n=2,3,\ldots \end{cases}$$

Here, the parameters q and β represent the scale and shape parameters, respectively. The PMF shows that the probability decreases as n increases, with the scale parameter q determining the likelihood at n=1, and the shape parameter β influencing the decay of probability for larger values of n.

The parameter q primarily influences the PMF at $n=1$. When $logq= log(2)/(2-\beta+1)$, the PMF becomes monotone decreasing. For other values, the PMF is unimodal, typically with the mode at n=2. The shape parameter β exerts greater influence on the PMF beyond n=1; as β decreases, the tail of the distribution extends, shifting the probability mass to higher values of n.

Moment: The moments of the distribution can be derived, but often result in infinite series that cannot be expressed in closed form. The first and second moments are defined as:

$$E(X)=\sum_{i=0}^{\infty}\left(1-q^{i^\beta}\right)\ and\ E\left(X^2\right)=2\sum_{x=1}^{\infty}x\left(1-q^{x^\beta}\right)+E(X)$$

Where the sums extend over all possible values of n. The mean of the discrete distribution is bounded between the means of the corresponding continuous inverse Rayleigh distribution, with the discrete mean typically being smaller [5, 6].

Inverse Rayleigh Probability Paper Plot (IRPP): The Inverse Rayleigh Probability Paper (IRPP) plot is a graphical method used to assess the suitability of the inverse Rayleigh model for a given dataset. For the continuous inverse Rayleigh model, [4] Drapella proposed the transformation:

$$x=1n(t), y=-1n\left(-1n\left(F(t)\right)\right)$$

which yields a straight line for the inverse Rayleigh distribution, making it a useful diagnostic tool to assess whether the discrete inverse Rayleigh distribution fits the data well.

Hazard Rate Function: The Hazard Rate Function for the discrete inverse Rayleigh distribution is derived as the conditional probability that a failure occurs at time n, given that no failure has occurred by time n-1. It is defined as:

$$h_n=\frac{p_n}{P(X\geq n)}=\frac{q^{n^\beta}-q^{(n-1)^\beta}}{1-q^{(n-1)^\beta}}, n=1,2,3,\ldots$$

For larger values of q, the hazard rate is monotone decreasing. However, for smaller values of q, it becomes unimodal, showing a non-monotonic behavior.

When estimating the parameters of the Discrete Inverse Rayleigh Distribution (DIRD), several methods can be utilized. The commonly applied methods are (i) Method of Proportions, (ii) Method of Moments, (iii) Heuristic Algorithm, (iv) Inverse Rayleigh Probability Plot (IRPP).

Method of Proportions: The method of proportions was initially proposed by Khan et al. [12] for the discrete Weibull distribution. A similar approach can be adapted for the Discrete Inverse Rayleigh Distribution (DIRD).

Let $x_1, x_2,\ldots,x_n$ be a random sample from the DIRD with the corresponding probability mass function (PMF). Define the indicator function as: $I(x_i) = 1\ if\ x_i = 1$
The sum of these indicator functions $Y = \sum_{i=1}^{n} I(x_i)$ represents the number of ones in the sample. The proportion $\frac{Y}{n}$ gives an estimate of the probability of observing x=1, which corresponds to the parameter q: $\hat{q} = \frac{y}{n}$Where y is the observed number of ones in the sample. Similarly, for higher values of x, the parameter β is estimated by considering the proportion of values of 2, 3, etc., in the sample. For instance, the probability $p_2(q, \beta)$ is estimated using:

$$\hat{\beta} = \frac{1}{log(2)} . \frac{log\left(log\left(\frac{z}{n} + \frac{y}{n}\right)\right)}{log\left(\frac{y}{n}\right)}$$

where z is the number of twos observed in the sample. The method of proportions provides consistent estimates of q and β, making it a suitable approach for the Discrete Inverse Rayleigh Distribution.

Method of Moments: The method of moments requires equating the population moments to the sample moments. For a sample $x_1, x_2... x_n$ from the distribution, we calculate the first and second sample moments M_1 and M_2 as follows:

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \ and\ M_2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2$$

These moments are equated to the population moments of the discrete inverse Rayleigh distribution, and the parameters q and β are then solved simultaneously. However, due to the complex nature of the moments for the DIRD, these equations often require numerical methods to solve. In practice, a pseudo-moment method is used to minimize the difference between the sample moments and the theoretical moments, which is expressed as:

$$S(q, \beta) = \left(M_1 - E(X)\right)^2 + \left(M_2 - E(X^2)\right)^2$$

Where E(X) and $E(X^2)$ are the theoretical moments of the DIRD. Minimizing S(q,β) with respect to q and β provides parameter estimates, though it has been found that this method is not always satisfactory for the DIRD.

Heuristic Algorithm: The heuristic algorithm combines maximum likelihood estimation (MLE) with an optimization method. Since the likelihood function of the DIRD can be challenging to optimize directly, the Nelder-Mead optimization method is used, which iteratively refines the parameter estimates by optimizing the likelihood function. The heuristic algorithm starts with an initial guess for the shape parameter β and iteratively updates the parameters using maximum likelihood estimates. At each step, the likelihood function is maximized with respect to q, and the updated values are used in the next iteration. The process continues until the parameter estimates stabilize and converge to their optimal values.

- Set initial values for the shape parameter $\beta_{1\text{-}1}$.
- Set a value of the variation rte r and the initial variation width z_1.
- After the setting, we compute maximum likelihood estimator of the parameter q with respect to $D = \{\beta_{1,1-z_1}, \beta_{1,1}, \beta_{1,1+z_1}\}$ to get the maximum likelihood estimate $\hat{\beta}$
- $\beta_{m,l+1} = \overset{\arg\max}{D}$ {L(D)} after that we get $\beta_{m,l+1} = \beta_{m,l}$ if yes we get $\beta_m = \beta_{m,l}$ if no then l=l+1

- The heuristic algorithm repeats this loop with different variation rates and widths until there is no significant difference between maximum likelihood estimates and their likelihood functions.
- If the stopping criterion is met ie, $|L(\beta_m) - L(\beta_{m-1})| < \in$ proceed to output the best solution. Otherwise, return to step 4.

Inverse Rayleigh Probability Plot (IRPP): The IRPP plot is a graphical method for estimating the parameters of the Discrete Inverse Rayleigh Distribution. This method involves transforming the data and plotting it in a way that liberalizes the distribution, making it easier to estimate the parameters.

For the inverse Rayleigh distribution, the transformation is:

$$x = -In(t)\ and\ y = In\left(-In\left(F(t)\right)\right)$$

where F(t) is the cumulative distribution function (CDF) of the DIRD. The plot of y versus x results in a straight line, allowing q and β to be estimated using a simple linear regression model. The IRPP plot provides a straightforward method for parameter estimation when the transformed data follows a linear trend. The slope of the line gives the estimate of β, while the intercept provides an estimate for q.

IV. Comparison of Estimation Methods for the Discrete Inverse Rayleigh Distribution

Khan et al. [12] compared the method of proportions with the method of moments in the discrete Weibull distribution based on 100 replications of simulated samples. Here, in this section, we shall compare the four mentioned methods in the preceding section presented for the discrete inverse Rayleigh distributions. Some of replications sizes are less than 100 so that the numerical algorithms can converge faster. We compare the estimates obtained by the method of proportions and the method of IRPP plot. Table 1 shows estimates and their variances by the two methods. These simulation results are based on 50 replications. It is clear that accuracies and precisions of estimates given by the method of proportions are slightly improved as the sample size increases from 20 to 50. From Table 1, the result indicates that Method of Proportions consistently yields smaller variance estimates for both q and β across various sample sizes and parameter combinations compared to the Heuristic Algorithm. This suggests that the Method of Proportions may provide more stable estimates, making it potentially preferable for applications requiring higher precision. For example, in the case of q=0.2 and β=1.5 with a sample size of n=50, the Method of Proportions estimates q as 0.2114 with a variance of 0.0032, while the Heuristic Algorithm gives q as 0.213 with a slightly higher variance of 0.0035. This pattern persists across different parameter settings, indicating that the Method of Proportions often provides tighter bounds around its estimates. Also for shape parameter variance this consistent pattern suggests.

Next, we compare the estimates obtained by the method of proportions and IRPP plot with the heuristic algorithm. Tables 2 and 3 give the estimates and their variances for these methods. These simulation results are based on 10 replications.

Tables 2 and 3 shows both tables indicate that when the initial values are close to the true parameter values, the heuristic algorithm tends to produce slightly better results than the method of moments in terms of the variances of the estimates. However, the variances of estimates from the method of moments are generally comparable to those obtained from the heuristic algorithm. The method of IRPP plot also yields results that are on par with the heuristic algorithm. While there are instances where the heuristic algorithm exhibits better convergence, it may sometimes fail

to reach a solution. In such cases, the estimates from the IRPP plot can serve as reliable initial values for the heuristic algorithm, facilitating convergence and improving the estimation process. Overall results suggest that while the heuristic algorithm may offer slight advantages in specific scenarios, the method of IRPP plot and the method of moments also provide robust estimates. As the sample size increases, the precision and reliability of all methods improve, making them effective tools for parameter estimation in the discrete inverse Rayleigh distribution context.

We now consider the estimates by the method of moments. Table 4 shows that estimates by the method of moments result show larger variances compared to the estimates obtained from other methods such as the heuristic algorithm or the method of proportions. This indicates that the method of moments may introduce more variability in the parameter estimates, particularly for smaller sample sizes. For example, as seen in the table, the variances for q and β increase as the sample size decreases. The accuracies of the estimates using the method of moments are generally lower than those from the previous methods, particularly when considering smaller sample sizes. For instance, the estimated values for q and β show wider discrepancies from the true parameter values as sample sizes reduce, which affects the reliability of the estimates. Despite the initial lower accuracy and higher variance, the results indicate that the estimates improve as sample sizes increase. This is evident from the decreasing variances and increasing closeness of estimates to the true parameters when moving from smaller sample sizes (20) to larger sample sizes (80). For example, the estimates for q for the parameter pair (0.2, 1.5) improved from 0.0513 (with a variance of 0.0216) at a sample size of 20 to 0.0732 (with a variance of 0.0152) at a sample size of 80. These simulation results are based on 100 replications, reinforcing the statistical reliability of the observed trends. The larger sample sizes not only yield more stable estimates but also improve the overall accuracy of the parameters. Khan et al. [12] introduced two methods of estimates, namely the method of proportions and the method of moments to estimate the parameters of the basic discrete Weibull. They used the results of simulation runs to compare the accuracies and precisions of these estimates. The comparison showed that the method of moments performs significantly better than the method of proportions.

For the discrete Rayleigh model, we use simulation runs to compare the accuracies and precisions of the parameter estimates using the four estimation methods discussed in this section.

Table 1: *Method of proportions versus Heuristic algorithm*

q, β	Sample Size n	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$
0.2, 1.5	50	0.213	0.0035	1.4236	0.1245	0.2114	0.0032	1.505	0.1124
	30	0.195	0.004	1.6485	0.2511	0.1922	0.0029	1.689	0.2205
	20	0.184	0.0062	1.789	0.3948	0.1825	0.0039	1.752	0.3509
0.5, 2.5	50	0.512	0.0051	2.442	0.1403	0.5105	0.0049	2.48	0.1352
	30	0.492	0.0078	2.5789	0.2145	0.4897	0.0071	2.617	0.1932
	20	0.478	0.0096	2.7231	0.3517	0.4739	0.0087	2.682	0.3108
0.7, 1.0	50	0.703	0.0028	1.0305	0.1012	0.705	0.0029	1.014	0.092
	30	0.69	0.0036	1.0751	0.1667	0.692	0.0032	1.059	0.1472
	20	0.675	0.0051	1.1604	0.2289	0.67	0.0048	1.132	0.1457
1.0, 3.0	50	1.002	0.0065	2.9985	0.1526	1.004	0.0067	2.93	0.1457
	30	0.988	0.0079	3.0457	0.2045	0.9854	0.0075	3.02	0.1934

	20	0.97	0.0098	3.1906	0.3417	0.9693	0.0091	3.149	0.3008
1.5, 2.0	50	1.478	0.0072	2.134	0.1845	1.482	0.0069	2.09	0.1713
	30	1.452	0.0087	2.2782	0.2529	1.4485	0.0081	2.231	0.2352

Table 2: *Method of IRPP plot versus Heuristic algorithm*

q, β	Sample Size n	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$
0.2, 1.5	50	0.2112	0.0054	1.5037	0.1042	0.2051	0.0047	1.51	0.0912
	30	0.1974	0.0102	1.6241	0.1925	0.1893	0.0083	1.57	0.1681
	20	0.1832	0.0088	1.7123	0.2438	0.1726	0.0095	1.645	0.2155
0.5, 2.5	50	0.5234	0.0058	2.4739	0.1287	0.5081	0.0051	2.515	0.1202
	30	0.4927	0.0087	2.6328	0.1934	0.4819	0.0083	2.581	0.1756
	20	0.4693	0.0114	2.7851	0.2851	0.4582	0.0096	2.712	0.2392
0.7, 1.0	50	0.7145	0.0043	1.0124	0.0835	0.7041	0.0035	1.025	0.0771
	30	0.6892	0.0067	1.0753	0.1487	0.6738	0.0057	1.042	0.1281
	20	0.6632	0.0085	1.1129	0.1896	0.6519	0.0069	1.086	0.1547
1.0, 3.0	50	1.0274	0.0093	2.9852	0.1567	1.0159	0.0088	3.021	0.1476
	30	0.9925	0.0131	3.1156	0.2278	0.9784	0.0109	3.072	0.1974
	20	0.9452	0.0156	3.2931	0.3126	0.9315	0.0123	3.251	0.2712
1.5, 2.0	50	1.5227	0.0082	1.9321	0.1292	1.5091	0.0075	2.015	0.1135
	30	1.4859	0.0119	2.1745	0.2046	1.4728	0.0102	2.11	0.1751
	20	1.4323	0.0136	2.3121	0.2772	1.4162	0.0128	2.256	0.2456

Table 3: *Estimates of parameters of discrete inverse Rayleigh by the method of moments*

q, β	Sample size n	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$
0.2, 1.5	80	0.0732	0.0152	1.4674	0.2675
	50	0.0659	0.0163	1.5246	0.3251
	20	0.0513	0.0216	1.6821	0.5124
0.5, 2.5	80	0.5128	0.1024	2.4576	0.3286
	50	0.4819	0.1311	2.5382	0.4528
	20	0.4542	0.1629	2.7496	0.7682
0.7, 1.0	80	0.6743	0.0891	0.9572	0.1446
	50	0.6487	0.1075	1.0653	0.1836
	20	0.6224	0.1347	1.1234	0.2469
1.0, 3.0	80	0.9886	0.0325	3.1498	0.4152
	50	0.9512	0.0548	3.4132	0.5114
	20	0.9063	0.0795	3.7291	0.8329
1.5, 2.0	80	1.4752	0.0268	1.9672	0.2157
	50	1.4213	0.0432	2.0956	0.2789
	20	1.3539	0.0715	2.2471	0.4328

V. Estimation for real data

Consider the 18 lifetimes (in hours) of certain electronic devices given as 6, 14, 23, 37, 52, 68, 89, 115, 136, 153, 183, 210, 237, 279, 308, 332, 362, and 398.

Table 5: *Estimation results in discrete inverse Rayleigh distribution*

Method	q	β
Heuristic Algorithm	0.0065	0.45
IWPP Plot	2.657312e−10	0.7932
Method of Moments	0.8764	1.12

Table 6: *Estimation results for discrete Rayleigh distribution*

Method	q	β
Heuristic Algorithm	0.985	1.12
IWPP Plot	0.9785	1.0457
Method of Moments	0.9923	1.2034

Table 7: *AIC results for discrete inverse Rayleigh and discrete Rayleigh models*

Model	Heuristic Algorithm	IWPP Plot	Moments
Discrete Inverse Rayleigh	218.4572	232.9821	375.8419
Discrete Rayleigh	220.1345	220.5823	221.4237

Note that the method of proportions is not applicable here because of the nature of the data, which does not contain 1s and 2s. In this context, as previously mentioned, the method of moments tends to be the least preferred approach for the discrete inverse Rayleigh distribution due to its higher variances and lower accuracy. In this analysis, we use parameter estimates from the IRPP plot as the initial values for the heuristic algorithm, which often yields more accurate results. The parameter estimates across the three methods are given in Tables 5 and 6. Table 5 demonstrates that the parameter estimates obtained from the three methods differ significantly. Based on the simulation results, we expect the heuristic algorithm to provide higher accuracy and precision compared to the other methods. On the other hand, Table 6 shows that the parameter estimates from the three methods are more consistent with each other. Previous simulation studies have indicated that the method of moments performs relatively well for the discrete Rayleigh distribution, offering accurate and precise estimates. The Akaike Information Criterion (AIC) values for both the discrete inverse Rayleigh and discrete Rayleigh models, based on the three methods, are presented in Table 7. When comparing the AIC values for the discrete inverse Rayleigh model, the heuristic algorithm produces the best fit, as indicated by the lowest AIC value. In contrast, the method of moments performs poorly, yielding the highest AIC, which suggests it is the least effective for this model. For the discrete Rayleigh model, the AIC values across all methods are relatively close, indicating that all methods perform similarly. However, the heuristic algorithm still provides a slight advantage in terms of model fit.

VI. Conclusion

This article outlines a comparative study of various parameter estimation methods for the discrete inverse Rayleigh distribution, including the Method of Proportions, IRPP plot, heuristic algorithm,

and Method of Moments. Through simulations and replications, the study evaluates these methods' accuracy, precision, and convergence properties across different sample sizes and parameter values. The paper also applies these estimation methods to real data and examines the fit of the discrete inverse Rayleigh and discrete Rayleigh models using Akaike Information Criterion (AIC) values. The results highlight that while the heuristic algorithm often provides the best fit for the discrete inverse Rayleigh model, the Method of Proportions delivers more stable and precise estimates, especially for larger sample sizes.

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DESIGNING SINGLE SAMPLING PLANS BASED ON ZERO-INFLATED BINOMIAL DISTRIBUTION

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Abstract

Statistical Quality Control (SQC) involves the use of st
monitor the quality of a product. Acceptance sampling
whether to approve or decline a batch of products using a sa
helps to manage quality assurance and reduces the need for
items and rejecting good o
involves inspecting a fixed number of it
the entire batch should be accepted
to
the evolution in technology the nu
zeros in
the count data of
nt number
of zeros, the Zero-inflate
uch conditions. The
ZIB distributio
stribution with an
inflation par
e accurate representation of
the qua
his article aims to design SSPs
wh
By adopting the ZIB distribution, the
acceptance or rejection of product
e ZIB sampling plan is derived. Furthermore,
size n and acceptance number c, are derived for
illustrations are presented to demonstrate the application of the
ared with Binomial SSP.

spection by attributes, Single sampling plan, Zero-inflated Binomial characteristic function, Producer's risk, Consumer's risk.

I. Introduction

Acceptance sampling provides a statistically valid way to decide if a product batch should be accepted or rejected using a sample, maintaining quality standards and reducing the need for full inspection. By using this approach, companies can uphold excellent quality standards without the need to inspect each item in a batch, saving time and money. Sampling inspection evaluate the quality of a batch by examining only a portion of the items, enabling effective decision-making while ensuring overall quality assurance. This method reduces the requirement for inspecting the entire batch, helping to save resources and improve the efficiency of the quality control procedure. A single

sampling plan (SSP) is a simple and widely adopted strategy that offers an efficient and effective way to control quality, requiring only one sample to determine the acceptability of a batch. It simplifies the inspection process, reducing the costs and efforts involved in quality control while still delivering reliable outcomes.

The advanced evolution in technology leads to the development of the production and quality of the products which results in the proportion of non-conforming units being very less. Hence, the manufactured products are in a nearly perfect state such that count of the nonconforming will contain more number of zeros. The appropriate probability distribution, to model the count data with a large number of zeros in the count of nonconforming is the Zero-inflated Binomial (ZIB) distribution. ZIB distribution is a mixture distribution, which degenerates at zero, and a binomial distribution.

Several attempts can be found in regression modeling based on the ZIB model [illegible] reviewed the ZIP regression model and introduced the ZIB regression model. He al[illegible] mixed versions of ZIP and ZIB regression models. He described an example [illegible] where count data with both upper and lower bounds and excessive ze[illegible] repeated measures experimental design. Alpha *et. al.,* [1] extended th[illegible] model to include cases with randomly missing covariates.

The ZIB distribution has found widespread applicat[illegible] epidemiology, horticulture, healthcare, industry, an[illegible] using ZIB distribution can be found in the stu[illegible] [7], and Wahyu *et. al.,* [13]. In recent years [illegible] develop control charts. Rassoul *et. al.* [illegible] Vasileious and Christos [11], ha[illegible] ZIB distribution.

If excessive numb[illegible] ung, the ZIB distribution can b[illegible] ipt is found in the literature on s[illegible] determine ZIB SSPs. The ZIB dist[illegible] perating characteristic (OC) fun[illegible] eters are determined in Section 4. [illegible] ed in Section 5. The risk efficiency is [illegible] given in Section 7.

[illegible]iew of ZIB distribution

[illegible]bauer [9], when the lot size and sample size n are limited, the [illegible]or the random variable X is the hypergeometric distribution. If the lot [illegible]conforming p are sufficiently large and $\frac{n}{N} \leq 0.10$, the probability distribution [illegible]ual $B(n, p)$ p distribution. If the occurrences of nonconforming units [illegible]esult in zero counts, the appropriate probability model for X is the ZIB [illegible].. It is often used in modeling processes where the occurrence of zeros is more frequent th[illegible] standard binomial distribution.

The probability mass function of the ZIB (φ, n, p) distribution proposed by Hall [5] is as follows.

$$P(X = x|\varphi, n, p) = \begin{cases} \varphi + (1-\varphi)\, q^n & \text{, when } x = 0 \\ (1-\varphi)\binom{n}{x} p^x\, q^{n-x}, & \text{when } x = 1,2, \dots, n \end{cases} \quad (1)$$

where $q = 1 - p$

The mean and variance of the ZIB distribution with parameters (φ, n, p) are given by, (Athanasios *et. al.,* [2])

$$E(X) = np(1-\varphi),$$
$$V(X) = np(1-p+np\varphi)(1-\varphi). \quad (2)$$

The parameters p and φestimated using method of moments (Rassoul *et. al.*, [8]) are given as follows.

$$\hat{p} = \frac{\sum_{t=1}^{m} x_t^2 - \sum_{t=1}^{m} x_t}{(n-1)\sum_{t=1}^{m} x_t}$$

$$\hat{\varphi} = 1 - \frac{(n-1)(\sum_{t=1}^{m} x_t)^2}{nm(\sum_{t=1}^{m} x_t^2 - \sum_{t=1}^{m} x_t)} \quad (3)$$

III. Designing of Single Sampling Plan

A SSP is a method used to determine the acceptance or rejection of a lot through the inspection of a single sample. A SSP based on attributes is characterized by three key parameters N, represent ng the lot size; n, signifies the sample size; and c, indicating the acceptance number or the m permissible count of nonconforming items within the sampled lot.

The step-by-step procedure for SSP by attributes is given as follows:

Step 1: Select a random sample comprising n items from the lot of size N

Step 2: Each item in the n sample undergoes inspection agains standards.

Step 3: Enumerate the number of nonconforming units, x

Step 4: If $x \leq c$, the lot is approved. If not, the lot is

The process of determining the para producer and consumer is commonly Schilling and Neubauer [9] provid on, determination of sampling p

The OC function o on its quality level. OC functio Plans, helping to visualize, an acceptance or rejection decision

$$\varphi, n, p)$$

$$+ (1-\varphi)q^n + \sum_{x=1}^{c} \binom{n}{x} p^x q^{n-x} \quad (4)$$

nination of plan parameters for ZIB SSP

are determined in such a way that they ensure protection to both producer and operating characteristic curve is a key metric for evaluating the performance of an sampling plan. Thus, the OC curve displays the discriminatory power of the sampling pl (Montgomery, [6]). Typically, optimum plan (n, c) is designed from the OC curve which passes through the two points $(p_1, 1-\alpha)$, and (p_2, β), where p_1 is the Acceptable Quality Level (AQL), α is the producer's risk, p_2 is the Limiting Quality Level (LQL) and β is the consumer's risk.

The plan parameters n and c may be determined subject to:

$$P_a(p_1) = 1-\alpha \text{ and } P_a(p_2) = \beta \quad (5)$$

Due to the discrete nature of the parameters and the need for integers, it is often challenging to find a plan that precisely meets all specified requirements. Hence the above conditions are reformulated as:

$$P_a(p_1) \geq 1-\alpha \text{ and } P_a(p_2) \leq \beta \quad (6)$$

The parameter values can be determined to meet specified conditions while simultaneously minimizing both producer and consumer risks. The optimum plan parameters under ZIB distribution for given $(p_1, 1-\alpha)$ and (p_2, β) can be determined by using the iterative procedure proposed by Guenther [7].

By using the iterative procedure, the optimum sampling plans are determined for some sets of values of p_1, α, p_2, β and φ, fixing α and β at 0.05 and 0.10 respectively. The Python software is used to acquire the plan parameters. Python plays a crucial role in this process because of its strong computational abilities and wide range of libraries for statistical analysis. It enables the effective execution of iterative algorithms for finding the best sampling plans. Python libraries like SciPy and NumPy offer strong resources for computing probabilities and conducting advanced mathematical calculations, necessary for identifying the optimum plans. The plan parameters are presented in Tables I to Table IV. The values of p_1, p_2 and φ are taken as p_1= 0.005(0.005)0.05(0.01)0.10 and φ = 0.01(0.03)0.10.

In certain instances, very large values for n are observed, the optimal sample scenarios are provided as ***. Similarly, the optimum plans can be determined of p_1, α, p_2, β and φ.

An optimum plan (n, c) corresponding to the specified (p_1, α) following tables.

Table 1: *Optimum ZIB SSPs when α =*

AQL		
(p_1)	0.05	0.06
0.005	(108,2)	(66,1)
0.010	(135,3)	
0.015	(213,6)	
0.020	(310	
0.025		
0.030		

					(53,2)	
					(67,3)	
				(80,5)	(80,4)	
			(131,7)	(105,6)		
			(171,10)	(130,8)		
		(211,18)	(211,13)	(154,10)		
		(423,26)	(275,18)	(189,13)		
		(590,38)	(351,24)	(247,18)		

SSPs when α = 0.05, β = 0.10 and φ = 0.04

	LQL (p_2)				
	0.06	0.07	0.08	0.09	0.10
	(73,1)	(63,1)	(55,1)	(48,1)	(43,1)
(174,4)	(122,3)	(104,3)	(73,2)	(65,2)	(58,2)
(252,7)	(167,5)	(124,4)	(91,3)	(81,3)	(72,3)
0.020 (376,12)	(231,8)	(161,6)	(124,5)	(96,4)	(86,4)
0.025 (565,20)	(333,13)	(215,9)	(157,7)	(125,6)	(99,5)
0.030 (953,37)	(470,20)	(285,13)	(203,10)	(153,8)	(125,7)
0.035 (1789,75)	(718,33)	(402,20)	(264,14)	(194,11)	(150,9)
0.040 ***	(1181,58)	(566,30)	(352,20)	(234,14)	(174,11)
0.045 ***	(2207,115)	(870,49)	(481,29)	(312,20)	(222,15)
0.050 ***	***	(1400,83)	(663,42)	(402,27)	(269,19)

Table 3: *Optimum ZIB SSPs when $\alpha = 0.05$, $\beta = 0.10$ and $\varphi = 0.07$*

AQL	LQL (p_2)					
(p_1)	0.05	0.06	0.07	0.08	0.09	0.10
0.005	(136,2)	(113,2)	(74,1)	(64,1)	(57,1)	(51,1)
0.010	(195,4)	(138,3)	(118,3)	(84,2)	(75,2)	(67,2)
0.015	(303,8)	(207,6)	(158,5)	(112,4)	(91,3)	(82,3)
0.020	(430,13)	(273,9)	(196,7)	(155,6)	(122,5)	(96,4)
0.025	(695,24)	(398,15)	(270,11)	(188,8)	(152,7)	(123,6)
0.030	(1134,43)	(559,23)	(358,16)	(251,12)	(181,9)	(150,8)
0.035	(2157,89)	(868,39)	(478,23)	(313,16)	(223,12)	(175,10)
0.040	***	(1429,69)	(694,36)	(418,23)	(291,17)	(2
0.045	***	(2646,137)	(1033,57)	(578,34)	(371,23)	
0.050	***	***	(1694,99)	(805,50)	(48	

Table 4: *Optimum ZIB SSPs when $\alpha = 0.05$, $\beta = 0$*

AQL	LQL (p_2)					
(p_1)	0.05	0.06	0.07			
0.005	(1343,11)	(1043,9)	(8			
0.010	(1915,26)	(1409,20)				
0.015	(2739,51)	(1902				
0.020	(4041,95)	(				-5)
0.025	***					(1108,36)
0.030						(1132,50)
0.035					78,86)	(1604,68)
					(2605,120)	(1936,91)
				***	(3332,169)	(2379,123)
				***	***	(2965,167)

. Illustrations

selection of sampling plans for given φ and specified

ne specified strength of the plan as (0.010, 0.05, 0.10, 0.10) and φ = 0.01, the plan d in Table I. The required ZIB SSP is (53,2) and hence a random selection of 53 items fr ne submitted lot for inspection is recommended. If the count of nonconforming units in the sample exceeds 2, the lot may be rejected; otherwise, the lot may be accepted. In accordance with this strength, the Binomial SSP determined is (52,2) (Guenther [7]).

II. Illustration 2

In the case where φ = 0.0001 and the strength of the plan is defined as (0.005, 0.05, 0.10, 0.10) the ZIB SSP is (38, 1). Corresponding to this strength, the Binomial SSP is (38,1) (Guenther, [7]).

When φ is small, the ZIB SSPs are identical to those under the Binomial SSPs. Hence, if φ is small, the Binomial SSPs becomes special cases of the ZIB SSPs.

The optimum ZIB SSPs with φ = 0.01, 0.04, 0.07 and 0.1 for the same strength are, respectively, (39, 1), (43, 1), (51, 1) and (522, 5). Thus, both the number of products required to be inspected and the acceptance number of nonconforming units in the sample increases as the value of φ increases. Hence, with an increase in the value of φ, both the quantity of products required for inspection and the acceptable number for nonconforming units in the sample also increases.

VI. Risk Efficiency

Values of the OC functions of the Binomial and ZIB plans are calculated and are presented in Table V. If p is small, the acceptance probabilities are high, and the acceptance probabilities decrease as p increases. Each plan ensures a secure level of protection to both the producer and the consum

Table 5: *Values of OC function of Binomial SSP and ZIB SSPs for (p_1=0.005, α =0.05, p_2=0*

	Parameters			Lot fraction nonconforming (p)					
Model	φ	n	c	0.004	0.008	0.04			
Binomial	-	38	1	0.9898	0.9628	0.5			
ZIB	0.0001	38	1	0.9898	0				
	0.01	39	1	0					15.05
	0.04	43						13.41	14.65
	0.07						11.67	12.02	13.69
						10.00	11.77	10.00	11.77

V … SSP. The consumer's risk under ZIB
… 0.0001, 0.01, 0.04, 0.07 and 0.1 respectively.
… distributions offer the same level of consumer
…, 0.04, 0.07, and 0.1, the ZIB SSPs provide greater
…ual SSPs by lowering the risk of accepting poor-quality

…ucer's risk under Binomial SSP is 1.02 %. The producer's risk under
…07 %, 1.24 %, 1.67 % for φ = 0.0001, 0.01, 0.04, 0.07 and 0.1 respectively.
… SSPs under Binomial and ZIB provides same amount of producer's risk.
…ied risk is determined by adding both the producer's risk and the consumer's risk
… Jhavan et al., [12]). When φ = 0.0001, the total risk for both the producer and the consumer
ur… the Binomial SSP is 15.11 %. In contrast, the total risks under ZIB SSPs are 15.11%, 15.11%, 15.05%, 14.65%, 13.69%, and 11.77% for φ values of 0.0001, 0.01, 0.04, 0.07, and 0.1, respectively. The total risk is the same for both Binomial and ZIB SSPs when φ = 0.0001. However, using ZIB SSPs lowers the overall risk for both the producer and the consumer.

VII. Conclusion

In a well-monitored production process, most of the products will satisfy the prescribed quality requirements and hence the count data on the number of non-conforming units will contain more number of zeros. ZIB distribution is the appropriate probability distribution for such manufacturing

processes. ZIB SSP involves specified inspection strategies to accommodate the unique characteristics of the distribution, ensuring effective quality control. This enables better informed decision-making, leading to more reliable acceptance or rejection decisions. The operating characteristics function of the ZIB SSP is derived in this paper. The methods for the selection of SSPs based on ZIB distribution are presented. The optimum ZIB SSP determined for different values of p_1,α, p_2, $\beta$and φ are provided in the tables and numerical illustrations are presented. If φ is small, the Binomial SSPs becomes special cases of the ZIB SSPs. The ZIB SSP reduces the consumer's risk and safeguards the consumer against the acceptance of poor quality lots. The total risk of producer and consumer is also reduced yielding numerous advantages such as increased customer satisfaction, enhanced productivity, cost efficiency and long-term sustainability in the market.

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LEVERAGING RANK SET SAMPLING FOR ENHANCED STRESS-STRENGTH ESTIMATION IN THE CONTEXT OF NAKAGAMI DISTRIBUTION

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Abstract

This study addresses the estimation of the stress-strength reliability model, where stress and strength both following the Nakagami distribution. While conventional approaches have relied on simple random sampling (SRS) for estimating reliability models, recent research suggests that ranked set sampling (RSS) offers a more efficient alternative. RSS yields more informative samples compared to SRS, potentially enhancing the accuracy of reliability estimations. Our investigation focuses on deriving maximum likelihood estimators (MLEs) for stress-strength under both SRS and RSS methodologies. To evaluate the comparative efficacy of these sampling techniques, we conduct a comprehensive Monte Carlo simulation study. The results of this analysis provide compelling evidence that RSS-based estimators outperform their SRS counterparts in terms of efficiency and precision. This research contributes to the growing body of literature supporting the adoption of RSS in reliability engineering. By demonstrating the superior performance of RSS in the context of Nakagami-distributed stress-strength models, we offer valuable insights for researchers and practitioners seeking to optimize their estimation procedures in reliability analysis.

Keywords*:* Stress–strength reliability, simple random sampling, ranked set sampling, Nakagami distribution, maximum likelihood estimation.

1. Introduction

The concept of stress-strength reliability plays a pivotal role in engineering decision-making, design optimization, and risk evaluation, particularly where safety, performance, and longevity are paramount. This analytical approach is indispensable for ensuring that engineered systems, structures, and components not only meet functional requirements but also withstand the challenges posed by fluctuating loads, environmental influences, and operational dynamics. At the core of reliability engineering and statistics lies the stress-strength model, which primarily aims to quantify the probability of system success or failure when both stress and strength are subject to

random variations. This methodology finds applications across diverse sectors, including engineering, materials science, quality control, and even finance. Within the framework of the stress-strength paradigm, the expression "$P_r(Y < X)$" represents the likelihood that a system's stress remains below its inherent strength. Essentially, this metric gauges the probability of system survival in the stress-strength model. Conversely, system failure occurs when the applied stress exceeds the material or component strength.

In the literature, the work on stress-strength model was first done by Birnbaum [2] and Birnbaum and McCarty [3]. The word stress-strength was first used by Church and Harris [4] in their research article, and they done a remarkable work under parametric and non-parametric inference. After that various authors choose different probabilistic models for estimating the stress-strength models. Some of these choices were summarised by Johnson [5]. A summary of all approaches and findings on the stress-strength model during the previous four decades was published by Kotz et al. [6]. The situation where X and Y are independent Type XII Burr random variables was examined by Awad and Gharraf [7]. For the recent development on this topic, one may refer to Chaturvedi and Kumar [8], Kundu and Gupta [9], Kundu and Raqab [10], Krishnamoorthy and Lin [11], Lio and Tsai [12], Barbiero [13], Chaturvedi and Kumari [14]. In all the above studies the authors have used the simple random sampling technique.

The ranked set sampling introduced by McIntyre [1][1], gained importance when Halls and Dell [15] applied ranked set sampling to estimate forage yields under pine-hardwood forest. Takahashi and Wakimoto [16], Dell and Clutter [17], David and Levine [18] focused on the efficiency of the estimators based on RSS and they established that RSS outperforms its counterpart simple random sampling with an identical sample size. Expanding the horizons of RSS, Yu and Lam [19] and Chen [20] explored regression estimation based on this methodology, providing notable examples and results. Additionally, studies on the estimation of distribution functions under various RSS techniques were conducted by Stokes and Sager [21], Kvam and Samaniego [22], and Chen [23]. Zamanzade and Vock [24], Zhang et al. [25] and Ozturk [26] have yields insights into inferential procedures reliant on ranked set sampling.

To delve deeper into this specialized data collection technique, one may refer to the review papers of Kaur et al. [27], Bai and Chen [28], and Wolfe [29]. These review papers include all pertinent references on RSS, including historical development, current status and future research direction. Hassan et al. [30] obtained the point and interval estimators of $P = P_r(Y < X)$ for the case of independent Gompertz random variables with common scale and different shape parameters based on RSS.

Here, we have consider the estimation of $P = P_r(Y < X)$ with a focus on situation where the random stress Y and random strength X are two independent Nakagami random variables with shape parameters (α_1, α_2) and scale parameters (λ, λ), respectively. The point estimator of $P = P_r(Y < X)$, is obtained using the maximum likelihood method based on both SRS and RSS, and the efficiency of this method based on SRS and RSS is compared. In Section 2, we present a brief overview about the Nakagami distribution and its relationship with other probability distributions. Point estimation of the parameters is given in Section 3. Section 4 and Section 5 comprises the point estimation of stress-strength model under SRS and under RSS, respectively. A simulation study employing the Monte Carlo method is discussed in Section 6. Section 7 details an empirical data analysis, and lastly Section 8 provides concluding remarks for the paper.

2. Preliminary

Consider a random variable X that adheres to the Nakagami distribution, denoted as NAD (α, λ). In this distribution, α represents the shape parameter, which is bounded by the condition $\alpha > 0.5$, while λ symbolizes the scale parameter, constrained to be strictly positive ($\lambda > 0$). For this distribution, the probability density function (PDF) and cumulative distribution function (CDF) are characterized as follows:

$$f(x;\alpha,\lambda) = \frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{(2\alpha-1)} exp\left(-\frac{\alpha}{\lambda}x^2\right); x > 0, \alpha > 0.5, \lambda > 0 \quad (1)$$

and

$$F(x) = \frac{1}{\Gamma\alpha}\gamma\left(\alpha, \frac{\alpha}{\lambda}x^2\right); x > 0, \alpha > 0.5, \lambda > 0 \quad (2)$$

Where, $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.

The reliability function of NAD (α,λ) is

$$R(t) = 1 - \frac{1}{\Gamma\alpha}\gamma\left(\alpha, \frac{\alpha}{\lambda}t^2\right); t > 0, \alpha > 0.5, \lambda > 0 \quad (3)$$

The hazard rate of NAD (α,λ) is.

$$h(t) = \frac{\frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha} t^{(2\alpha-1)} exp\left(-\frac{\alpha}{\lambda}t^2\right)}{1-\frac{1}{\Gamma\alpha}\gamma\left(\alpha, \frac{\alpha}{\lambda}t^2\right)}; t > 0, \alpha > 0.5, \lambda > 0 \quad (4)$$

Other distribution relationships

1) If α = 0.5, then Nakagami distribution (α,λ) becomes Half Normal Distribution.
2) For α = 1, then Nakagami distribution (α,λ) reduces to Rayleigh Distribution.
3) If random variable Y ~ Gamma (α,λ) where α is shape parameter and λ is scale parameter, then $\sqrt{Y}$ ~ NAD $(\alpha,\alpha\lambda)$.
4) If Z ~ chi-square (2α) and then $\sqrt{\frac{\lambda}{2\alpha}Z}$ ~ NAD (α,λ) where 2α is integer-valued.

3. Point estimation of the parameters

Let us draw a random sample $X_1, X_2, \ldots, X_n$ from the NAD (α,λ) of size n. The likelihood function of the Nakagami distribution NAD (α,λ) is given by

$$L(x,\alpha,\lambda) = \frac{(2\alpha^{\alpha})^n}{(\Gamma\alpha)^n(\lambda)^{n\alpha}}\prod_{i=1}^{n} x_i^{2\alpha-1} \exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{n} x_i^2\right)$$

Theorem 1. The Maximum Likelihood Estimator of scale parameter λ is

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i^{\,2}}{n}$$

Theorem 2. The Maximum Likelihood Estimator of shape parameter α is

$$\hat{\alpha} = \frac{0.5}{\log\left(\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)}$$

Proof. If we suppose that λ is known, then the likelihood function for the parameter α is given as

$$L(\alpha|\underline{x}) = \frac{(2\alpha^{\alpha})^n}{(\Gamma\alpha)^n(\lambda)^{n\alpha}}\prod_{i=1}^{n} x_i^{2\alpha-1}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{n} x_i^2\right)$$

The log likelihood function is

$$\log L = n\log 2 - n\log(\Gamma\alpha) + n\alpha\log(\alpha) - n\alpha\log(\lambda) + \sum_{i=1}^{n}(2\alpha-1)\log(x_i) - \frac{\alpha}{\lambda}\sum_{i=1}^{n} x_i^{\,2}$$

Partially differentiating with respect to α, and equating it equal to zero, we get

$$\hat{\alpha} = \frac{0.5}{\log\hat{\lambda} - \frac{2}{n}\left(\sum_{i=1}^{n}\log x_i\right) + \frac{1}{\hat{\lambda}}\left(\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}\right) - 1}$$

From theorem 1

$$\hat{\alpha} = \frac{0.5}{\log\left(\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)}$$

4. Point estimation of $P = P_r(Y < X)$ in case of simple random sampling

To derive the stress-strength reliability model $P = P_r(Y < X)$, here we assumed that X is the strength variable and Y is the stress variable, both are following the Nakagami distribution with common scale parameter $\lambda > 0$ and different shape parameters $\alpha_1 > 0.5$ and $\alpha_2 > 0.5$, respectively. By notation $X \sim NAD\ (\alpha_1, \lambda)$ and $Y \sim NAD\ (\alpha_2, \lambda)$, then

$$P = \int_0^{\infty} P\ (Y < X) f(x)dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma\alpha_2}\gamma\left(\alpha_2, \frac{\alpha_2}{\lambda}x^2\right)\frac{2}{\Gamma\alpha_1}\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1} x^{(2\alpha_1-1)}exp\left(-\frac{\alpha_1}{\lambda}x^2\right)dx$$

where $\gamma(n,x) = \Gamma n\left(1 - e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}\right)$ is the lower incomplete gamma function.

$$= 2\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1}\frac{1}{\Gamma\alpha_1}\left(\int_0^{\infty} x^{(2\alpha_1-1)}exp\left(-\frac{\alpha_1}{\lambda}x^2\right)dx - \int_0^{\infty} x^{(2\alpha_1-1)}exp\left(-\frac{\alpha_1}{\lambda}x^2\right)exp\left(-\frac{\alpha_2}{\lambda}x^2\right)\sum_{m=0}^{\alpha_2-1}\frac{\left(\alpha_2 x^2/\lambda\right)^m}{m!}dx\right)$$

$$= 2\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1}\frac{1}{\Gamma\alpha_1}\left(\frac{1}{2}\left(\frac{\lambda}{\alpha_1}\right)^{\alpha_1}\Gamma\alpha_1 - \sum_{m=0}^{\alpha_2-1}\frac{{\alpha_2}^m\,\lambda^{\alpha_1}}{2m!}\left(\frac{\Gamma(\alpha_1+m)}{(\alpha_1+\alpha_2)^{\alpha_1+m}}\right)\right)$$

$$P = 1 - \frac{(\alpha_1)^{\alpha_1}}{\Gamma\alpha_1}\sum_{m=0}^{\alpha_2-1}\frac{(\alpha_2)^m}{m!}\frac{\Gamma(\alpha_1+m)}{(\alpha_1+\alpha_2)^{\alpha_1+\mathrm{m}}} \tag{5}$$

Let two independent random samples X and Y of size n and m are drawn from Nakagami distribution with parameters (α_1, λ) and (α_2, λ), respectively. For known λ, the invariance characteristic of the maximum likelihood estimator provides the maximum likelihood estimator for P. The maximum likelihood estimators of α_1 and α_2 are

$$\hat{\alpha}_{1SRS} = \frac{0.5}{\log\left(\frac{\sum_{i=1}^{n} {x_i}^2}{n}\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)} \quad \text{and} \quad \hat{\alpha}_{2SRS} = \frac{0.5}{\log\left(\frac{\sum_{k=1}^{m} {y_k}^2}{m}\right) - 2\left(\frac{1}{m}\sum_{k=1}^{m}\log y_k\right)}$$

Maximum likelihood estimator of P in case of simple random sampling is given by

$$\hat{P}_{SRS} = 1 - \frac{(\hat{\alpha}_{1SRS})^{\hat{\alpha}_{1SRS}}}{\Gamma\hat{\alpha}_{1SRS}}\sum_{m=0}^{\hat{\alpha}_{1SRS}-1}\frac{(\hat{\alpha}_{2SRS})^m}{m!}\frac{\Gamma(\hat{\alpha}_{2SRS}+m)}{(\hat{\alpha}_{1SRS}+\hat{\alpha}_{2SRS})^{\hat{\alpha}_{1SRS}+\mathrm{m}}}$$

5. Point estimation of $P = P_r(Y < X)$ in case of ranked set sampling

1. Standard ranked set sampling

Ranked set sampling (RSS) represents a cutting-edge approach in statistical sampling, designed to boost the accuracy of parameter estimation, particularly in scenarios where resources are scarce or data collection costs are prohibitive. This method diverges from traditional random sampling by utilizing the ranked order or order statistics of sampled observations, thereby enhancing the quality and efficiency of estimations. The concept of RSS, initially proposed in the mid-20th century, has since gained traction across diverse fields such as environmental science, forestry, and ecology. Its popularity stems from its ability to yield robust statistical insights even when comprehensive population surveys are unfeasible. By offering a pragmatic and economical alternative to conventional sampling techniques, RSS has become an invaluable asset for researchers and statisticians aiming to refine their sampling strategies. The implementation of RSS to generate a sample of size n = r*m involves a series of structured steps, where m denotes the number of sample units selected in each cycle (of fixed size) and r represents the total number of cycles. These steps are executed sequentially as follows:

1. A random subset of the population consisting of m^2 units is selected.
2. The m^2 units are then divided arbitrarily into m sets, each containing m units.
3. The units within each set are ranked based on either professional judgment or correlation

with the variable of interest.

4. An individual quantile sample is constructed by taking the lowest ranked unit from the first set, the second lowest ranked unit from the second set, and continuing in this fashion.
5. To obtain a larger sample of size n = r*m, steps 1 through 4 can be repeated for r cycles.

The ranked set sampling (RSS) method takes only one observation from each set in each cycle. In the first cycle, it chooses the lowest observation $X_{(11)r}$. In later cycles, it independently selects the second lowest $X_{(22)r}$ from a different set of m observations and the highest $X_{(mm)r}$ from the final set of m. Let $X_{(ii)k},\ i = 1, 2, \dots, m;\ k = 1, 2, \dots, r$, be a ranked sample set with set size m and r cycles. For convenience, this paper will use the notation $X_{(i)r}$ in place of the full description.

2. The maximum likelihood estimation of $P = P_r(Y < X)$ in case of RSS

Let $X_{(ij)}, i = 1, 2, \dots, r_1;\ j = 1, 2, \dots, m_1$, denote the ranked set sample of size $n_1 = r_1 m_1$ from Nakagami distribution with parameter (α_1, λ), where m_1 is the set size and r_1 is the number of cycles and $Y_{(kl)},\ k = 1, 2, \dots, r_2;\ l = 1, 2, \dots, m_2$, denote the ranked set sample of size $n_2 = r_2 m_2$ from Nakagami distribution with parameter (α_2, λ), where m_2 is the set size and r_2 is the number of cycles. Then the PDF of $X_{(ij)}$ and $Y_{(kl)}$ are given by

$$f_i(x_{ij}) = \frac{m_1!}{(i-1)!(m_1-i)!}[F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i} f(x_{ij}) \tag{6}$$

$$g_k(y_{kl}) = \frac{m_2!}{(k-1)!(m_2-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k} g(y_{kl}) \tag{7}$$

Now the likelihood function is given as

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k(y_{kl})$$

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} \frac{m_1!}{(i-1)!\,(m_1-i)!}[F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i} f(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} \frac{m_2!}{(k-1)!\,(m_2-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k} g(y_{kl}) \tag{8}$$

Let $u = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} \frac{m_1!}{(i-1)!\,(m_1-i)!}$, $v = \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} \frac{m_2!}{(k-1)!\,(m_2-k)!}$

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} u\left[F_X(x_{ij})\right]^{i-1}\left[1-F_X(x_{ij})\right]^{m_1-i} f(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} v[F_Y(y_{kl})]^{k-1}[1-F_Y(y_{kl})]^{m_2-k} g(y_{kl})$$

$$L = u\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left(\frac{1}{\Gamma\alpha_1}\right)^{m_1-1}\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{i-1}\left[\Gamma\alpha_1 - \gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{m_1-i} \left(\frac{2}{\Gamma\alpha_1}\right)\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1} {x_{ij}}^{2\alpha_1-1}\exp\left(-\frac{\alpha_1}{\lambda}{x_{ij}}^2\right) v\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left(\frac{1}{\Gamma\alpha_2}\right)^{m_2-1}\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{k-1} \left[\Gamma\alpha_2 - \gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{m_2-k}\left(\frac{2}{\Gamma\alpha_2}\right)\left(\frac{\alpha_2}{\lambda}\right)^{\alpha_2} {y_{kl}}^{2\alpha_2-1}\exp\left(-\frac{\alpha_2}{\lambda}{y_{kl}}^2\right) \tag{9}$$

sum of lower incomplete gamma function and upper incomplete gamma function is a gamma function, which implies

$$\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right) + \Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right) = \Gamma\alpha_1$$

gives

$$\Gamma\alpha_1 - \gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right) = \Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)$$

Thus,

$$L = \left(\frac{1}{\Gamma\alpha_1}\right)^{n_1(m_1-1)} \left(\frac{2}{\Gamma\alpha_1}\right)^{n_1} \left(\frac{\alpha_1}{\lambda}\right)^{n_1\alpha_1} \left(\frac{1}{\Gamma\alpha_2}\right)^{n_2(m_2-1)} \left(\frac{2}{\Gamma\alpha_2}\right)^{n_2} \left(\frac{\alpha_2}{\lambda}\right)^{n_2\alpha_2} uv$$
$$\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{i-1} \left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{m_1-i} {x_{ij}}^{2\alpha_1-1} exp\left(-\frac{\alpha_1}{\lambda}{x_{ij}}^2\right)$$
$$\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{k-1} \left[\Gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{m_2-k} {y_{kl}}^{2\alpha_2-1} exp\left(-\frac{\alpha_2}{\lambda}{y_{kl}}^2\right) \tag{10}$$

Taking log on both sides

$$\log L = \log L_1 + \log L_2 \tag{11}$$

where

$$L_1 = u\left(\frac{1}{\Gamma\alpha_1}\right)^{n_1(m_1-1)} \left(\frac{2}{\Gamma\alpha_1}\right)^{n_1} \left(\frac{\alpha_1}{\lambda}\right)^{n_1\alpha_1} \prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{i-1}$$
$$\left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{m_1-i} {x_{ij}}^{2\alpha_1-1} exp\left(-\frac{\alpha_1}{\lambda}{x_{ij}}^2\right) \tag{12}$$

and

$$L_2 = v\left(\frac{1}{\Gamma\alpha_2}\right)^{n_2(m_2-1)} \left(\frac{2}{\Gamma\alpha_2}\right)^{n_2} \left(\frac{\alpha_2}{\lambda}\right)^{n_2\alpha_2} \prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{k-1}$$
$$\left[\Gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{m_2-k} {y_{kl}}^{2\alpha_2-1} exp\left(-\frac{\alpha_2}{\lambda}{y_{kl}}^2\right) \tag{13}$$

This implies,

$$\log L_1 = \log u + n_1(m_1-1)(-\log\Gamma\alpha_1) + n_1(\log 2 - \log\Gamma\alpha_1) + n_1\alpha_1(\log\alpha_1 - \log\lambda)$$
$$+ \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\log\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\log\left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]$$
$$+ (2\alpha_1 - 1)\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij} - \frac{\alpha_1}{\lambda}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}^2 \tag{14}$$

Differentiating Eq.(5.2.9) with respect to α_1 and α_2 respectively, we get

$$\frac{\partial \log L_1}{\partial \alpha_1} = -m_1 \frac{\partial}{\partial \alpha_1} \log \Gamma \alpha_1 + n_1(\log \alpha_1 + 1) - n_1 \log \lambda + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial \alpha_1}\log\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1 - i)\frac{\partial}{\partial \alpha_1}\log\left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right] + 2\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij} - \frac{1}{\lambda}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1} x_{ij}^2 \tag{15}$$

and

$$\frac{\partial \log L_2}{\partial \alpha_2} = -m_2 \frac{\partial}{\partial \alpha_2} \log \Gamma \alpha_2 + n_2(\log \alpha_2 + 1) - n_2 \log \lambda + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\frac{\partial}{\partial \alpha_2}\log\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right] + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2 - k)\frac{\partial}{\partial \alpha_2}\log\left[\Gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right] + 2\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\log y_{kl} - \frac{1}{\lambda}\sum_{k=1}^{r_2}\sum_{l=1}^{m_2} {y_{kl}}^2 \tag{16}$$

Differentiating Eq.(5.2.6) with respect to λ, we get

$$\frac{\partial log L}{\partial \lambda} = -\frac{n_1 \alpha_1}{\lambda} + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial \lambda} log\left[\gamma\left(\alpha_1, \frac{\alpha_1 x^2}{\lambda}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1 - i)\frac{\partial}{\partial \lambda} log\left[\Gamma\left(\alpha_1, \frac{\alpha_1 x^2}{\lambda}\right)\right] + \frac{\alpha_1}{\lambda}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1} x_{ij}^2 - \frac{n_2 \alpha_2}{\lambda} + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\frac{\partial}{\partial \lambda} log\left[\gamma\left(\alpha_2, \frac{\alpha_2 y^2}{\lambda}\right)\right] + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2 - k)\frac{\partial}{\partial \lambda} log\left[\Gamma\left(\alpha_2, \frac{\alpha_2 y^2}{\lambda}\right)\right] + \frac{\alpha_2}{\lambda^2}\sum_{k=1}^{r_2}\sum_{k=1}^{m_2} y_{kl}^2 \tag{17}$$

A numerical approach is utilized to obtain the maximum likelihood estimates for α_1 and α_2, denoted as by $\hat{\alpha}_{1RSS}$ and $\hat{\alpha}_{2RSS}$, from equations 5.2.10 and 5.2.11, respectively using the ranked set sampling method. Applying the invariance property of maximum likelihood estimators, the maximum likelihood estimate of the reliability parameter P based on RSS, denoted $\hat{P}_{RSS}$, can then be derived as

$$\hat{P}_{RSS} = 1 - \frac{(\hat{\alpha}_{1RSS})^{\hat{\alpha}_{1RSS}}}{\Gamma \hat{\alpha}_{1RSS}} \sum_{m=0}^{\hat{\alpha}_{1RSS}-1} \frac{(\hat{\alpha}_{2RSS})^m}{m!} \frac{\Gamma(\hat{\alpha}_{2RSS} + m)}{(\hat{\alpha}_{1RSS} + \hat{\alpha}_{2RSS})^{\hat{\alpha}_{1RSS}+\mathrm{m}}}$$

6. Simulation study

In this section we carried out a simulation study. Bias and mean square error (MSE) for P are provided by $Bias(\hat{P}) = E(\hat{P} - P)$ and $MSE(\hat{P}) = E(\hat{P} - P)^2$, respectively to compare our suggested reliability estimator P based on ranked set sampling RSS with the conventional reliability estimator of P based on SRS. The formula for calculating the relative efficiency RE of the estimator of P is $\frac{MSE(\hat{P}_{SRS})}{MSE(\hat{P}_{RSS})}$. Relative efficiency values greater than one suggest that the $\hat{P}_{RSS}$ is more efficient than the $\hat{P}_{SRS}$. All computations are performed using the R programming language. The simulation study is explained in the following steps.

Step 1: We generate 1000 simple random samples of $X_1, X_2, \ldots, X_{n_1}$, and $Y_1, Y_2, \ldots, Y_{n_2}$ from Nakagami distribution with the sample sizes of (n_1, n_2) = (15, 15), (15, 20), (15, 25), (20, 20), (20, 25), (25, 25) in Case 1 and (20, 20), (20, 30), (20, 40), (30, 30), (30, 40), (40, 40) in Case 2.

Step 2: We generate 1000 ranked set samples of $X_{11}, \ldots, X_{m_1 r_1}$ and $Y_{11}, \ldots, Y_{m_2 r_2}$ from Nakagami distribution for the first case when the number of cycles is taken as $r_1 = r_2 = 5$ with set sizes $m_1 = m_2 = 3, 4, 5$ and for the second case when the number of cycles is taken as $r_1 = r_2 = 10$ with set sizes $m_1 = m_2 = 2, 3, 4$.

Step 3: To generate the simple random samples and ranked set samples for Nakagami distribution,

we consider the true value of the common scale parameter $\lambda = 3$ and the true values of the shape parameter α_x and α_y are (0.5, 0.9), (0.7, 1.2) and (0.9, 1.5), respectively for the strength variable X and the stress variable Y, respectively. For these values, the true value of stress-strength model P is 0.40238, 0.50290 and 0.58635, respectively.

Step 4: The Biases, MSES and relative efficiency are presented in the Table 1.

Table 1: *Biases, MSES and RE of P under SRS and RSS when the common scale parameter $\lambda = 3$*

				SRS			RSS			
	Case-1			$r_1 = r_2 = 5$						
(α_1, α_2)	(n_1, n_2)	(m_1, m_2)	P_{True}	$\hat{P}_{SRS}$	Bias	MSE	$\hat{P}_{RSS}$	Bias	MSE	RE
(0.5,0.9)	(15,15)	(3,3)	0.40238	0.38205	-0.02033	0.007075	0.37479	-0.02759	0.005877	1.2037
	(15,20)	(3,4)		0.37976	-0.02262	0.006309	0.36312	-0.03926	0.005434	1.1609
	(15,25)	(3,5)		0.36858	-0.0338	0.005457	0.35188	-0.05050	0.005283	1.0328
	(20,20)	(4,4)		0.36992	-0.03247	0.005543	0.36500	-0.03738	0.004656	1.1903
	(20,25)	(4,5)		0.36951	-0.03287	0.004886	0.36047	-0.04190	0.004239	1.1522
	(25,25)	(5,5)		0.36761	-0.03477	0.004822	0.35752	-0.04486	0.004346	1.1095
(0.7,1.2)	(15,15)	(3,3)	0.50290	0.48768	-0.01521	0.0081549	0.48285	-0.02004	0.007379	1.1051
	(15,20)	(3,4)		0.48781	-0.01509	0.0071741	0.47443	-0.02846	0.006434	1.1149
	(15,25)	(3,5)		0.48497	-0.01792	0.0064565	0.46760	-0.03529	0.005642	1.1442
	(20,20)	(4,4)		0.48402	-0.01887	0.0065401	0.47210	-0.03079	0.005534	1.1816
	(20,25)	(4,5)		0.48118	-0.02171	0.0058446	0.46763	-0.03527	0.004607	1.2684
	(25,25)	(5,5)		0.47852	-0.02437	0.0056632	0.46007	-0.04282	0.005261	1.0762
(0.9,1.5)	(15,15)	(3,3)	0.58635	0.58608	-0.00026	0.009415	0.56558	-0.02759	0.008915	1.0560
	(15,20)	(3,4)		0.57906	-0.00729	0.008066	0.56648	-0.03926	0.006912	1.1669
	(15,25)	(3,5)		0.57434	0.57434	0.006819	0.55942	-0.05050	0.005731	1.1897
	(20,20)	(4,4)		0.57625	-0.01010	0.007531	0.55986	-0.03738	0.006197	1.2151
	(20,25)	(4,5)		0.56799	-0.01836	0.006249	0.54742	-0.04190	0.005974	1.0460
	(25,25)	(5,5)		0.56607	-0.02027	0.005941	0.55713	-0.04486	0.004836	1.2282
	Case-2			$r_1 = r_2 = 10$						
(α_1, α_2)	(n_1, n_2)	(m_1, m_2)	P_{True}	$\hat{P}_{SRS}$	Bias	MSE	$\hat{P}_{RSS}$	Bias	MSE	RE
(0.5,0.9)	(20,20)	(2,2)	0.40238	0.37497	-0.02741	0.005234	0.37116	-0.03121	0.005073	1.0316
	(20,30)	(2,3)		0.36741	-0.03497	0.004678	0.36359	-0.03879	0.004379	1.0682
	(20,40)	(2,4)		0.36257	-0.03980	0.004350	0.36089	-0.04149	0.003852	1.1290
	(30,30)	(3,3)		0.36183	-0.04055	0.004494	0.36021	-0.04216	0.004249	1.0574
	(30,40)	(3,4)		0.36209	-0.04028	0.004011	0.35657	-0.04581	0.003841	1.0444
	(40,40)	(4,4)		0.35891	-0.04346	0.004020	0.35649	-0.04589	0.003606	1.1146
(0.7,1.2)	(20,20)	(2,2)	0.50290	0.48040	-0.02249	0.007313	0.48090	-0.02200	0.006477	1.1290
	(20,30)	(2,3)		0.47824	-0.02465	0.005291	0.47161	-0.03129	0.004850	1.0908
	(20,40)	(2,4)		0.47323	-0.02966	0.004726	0.46726	-0.03563	0.004282	1.1037
	(30,30)	(3,3)		0.47157	-0.03132	0.004867	0.46328	-0.03961	0.004715	1.0322
	(30,40)	(3,4)		0.46866	-0.03423	0.004614	0.46346	-0.03943	0.004210	1.0958
	(40,40)	(4,4)		0.46736	-0.03553	0.004181	0.46037	-0.04253	0.003984	1.0494
(0.9,1.5)	(20,20)	(2,2)	0.58635	0.57266	-0.01369	0.007225	0.56863	-0.01771	0.006831	1.0576
	(20,30)	(2,3)		0.56358	-0.02276	0.006070	0.55271	-0.03364	0.005913	1.0265
	(20,40)	(2,4)		0.56086	-0.02548	0.004649	0.55686	-0.02948	0.004561	1.0192
	(30,30)	(3,3)		0.55881	-0.02753	0.005109	0.55648	-0.02986	0.005039	1.0137
	(30,40)	(3,4)		0.55972	-0.02662	0.004832	0.55016	-0.03619	0.004452	1.0853
	(40,40)	(4,4)		0.55734	-0.02901	0.004548	0.55338	-0.03296	0.003982	1.1421

It is evident from the Table 1 that the relative efficiency is greater than one in every case; so, we can say that the ranked set sampling is showing more efficient results in comparison to simple random sampling in estimating the stress-strength reliability.

7. Real data application

In order to comprehend and provide a broad illustration of the processes covered in the preceding sections, we now take two real data sets. The first data set is used for the strength variable X and second data set is used for the stress variable Y in the stress-strength model $P = P_r(Y < X)$.

7.1 First Data Set

Lawless (2003, pp. 267) is the source of the data set. The first report on this was published in 1987 by Schat, Staton, Mandel, and Shott. The hours to failure of 59 conductors with a length of 400 micrometres are represented by this data. The specimens are tested at the same temperature and current density, and at a specific high temperature and current density, they all failed. The MLES of the parameters α and λ for this dataset is $\hat{\alpha}_x$= 4.6731 and $\hat{\lambda}_x$ = 51.2823

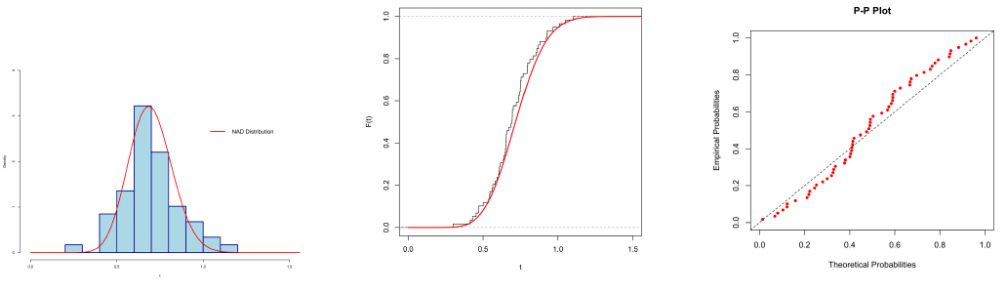
7.2 Second Data Set

The second data set is taken from Murthy et al. (2004, pp.180). This data represents 50 items that are put on use at time t = 0 and failure times are recorded (in weeks). The MLES for the parameters α and λ for this dataset is $\hat{\alpha}_y$= 0.1924 and $\hat{\lambda}_y$ = 144.2292. Both the datasets are shown in Table 2.

Table 2: *Dataset – 1 supposed to be X - Population and Dataset-2 supposed to be Y – Population*

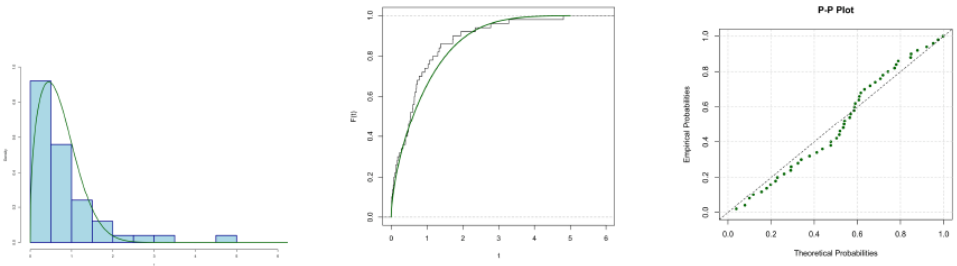
Dataset-1 : X-Population			
6.545	6.522	7.945	7.224
9.289	4.137	6.869	7.365
7.543	7.459	6.352	6.923
6.956	7.495	4.7	5.64
6.492	6.573	6.948	5.434
5.459	6.538	9.254	7.937
8.12	5.589	5.009	6.515
4.706	6.087	7.489	6.476
8.687	5.807	7.398	6.071
2.997	6.725	6.033	10.491
8.591	8.532	10.092	5.923
6.129	9.663	7.496	
11.038	6.369	4.531	
5.381	7.024	7.974	
6.958	8.336	8.799	
4.288	9.218	7.683	

Dataset-2 : Y-Population			
0.013	2.838	7.291	32.795
0.065	3.269	7.087	48.105
0.111	3.977	7.787	
0.111	3.981	8.596	
0.163	4.52	9.388	
0.309	4.789	10.261	
0.426	4.849	10.713	
0.535	5.202	11.658	
0.684	5.291	13.006	
0.747	5.349	13.388	
0.997	5.911	13.842	
1.284	6.018	17.152	
1.304	6.427	17.283	
1.647	6.456	19.418	
1.829	6.572	23.471	
2.336	7.023	27.777	



(a) PDF Plot (b) CDF Plot (c) P-P Plot

Figure 1: *The PDF, CDF and P-P Plots of the Nakagami distribution for First dataset*



(a) PDF Plot (b) CDF Plot (c) P-P Plot

Figure 2: *The PDF, CDF and P-P Plots of the Nakagami distribution for Second dataset*

Before we dive into the core of our investigation, it's crucial to thoroughly examine the key characteristics of our data. To validate the strength of our results, we employ a powerful statistical instrument: the Kolmogorov-Smirnov (K-S) test, along with its corresponding P-value (P-V). This approach enables us to measure how well our empirical observations align with theoretical expectations.

Our analysis yields promising outcomes. For the first dataset, we calculate a K-S distance of 0.06779 and a P-V of 0.99940. The second dataset produces similar results, with a K-S distance of 0.12 and a P-V of 0.86428. These metrics provide compelling evidence that our model closely matches the observed data.

To enhance our understanding and provide visual context, we have created a series of graphical representations. These illustrations, found in the accompanying Figure 1 and 2, offer a comprehensive view of our statistical findings. They include probability-probability (PP) plots, as well as visualizations of the estimated probability density function (PDF) and cumulative distribution function (CDF) for both datasets. These visual aids serve to reinforce and clarify the numerical results of our analysis

We consider these two datasets as our random strength X and random stress Y, respectively. The MLES for α and λ i.e. $\hat{\alpha}_x$= 4.6731, $\hat{\lambda}_x$= 51.2823 and $\hat{\alpha}_y$= 0.1924 and $\hat{\lambda}_y$= 144.2292 is taken as the true value of the parameters for this study. Now if $\hat{\alpha}_x = \alpha_1$ = 4.6731 and $\hat{\alpha}_y = \alpha_2$ = 0.1924 then the true value of the stress-strength model from Eq.(4.1) is P = 0.1718.

In this analysis, we draw simple random samples of size 10 from each dataset and estimate the MLES for α and λ, respectively. The simple random samples and MLES are presented in Table 3 and Table 4, respectively.

Table 3: *MLES of α and λ for each random sample of X-Population*

Simple Random Samples from X-Population												
											$\hat{\alpha}_x$	$\hat{\lambda}_x$
Sample 1	7.489	5.589	7.495	4.137	6.492	8.687	6.538	8.532	7.683	7.459	6.7031	50.8427
Sample 2	4.531	6.956	6.923	10.491	7.937	8.12	4.137	4.7	5.589	6.129	3.1983	46.3841
Sample 3	8.532	6.087	7.489	6.958	6.522	6.545	7.398	8.591	6.923	7.496	7.9133	42.4977
Sample 4	6.033	6.352	8.799	2.997	5.64	6.956	11.038	6.522	7.024	5.009	2.6220	48.2145
Sample 5	5.807	7.398	5.923	7.945	10.092	4.706	11.038	6.369	7.459	5.589	3.7265	55.9926
Sample 6	10.491	6.071	8.591	7.024	7.495	4.706	4.137	4.531	6.369	6.352	3.2477	46.6938
Sample 7	2.997	4.137	4.7	9.254	7.683	6.923	7.974	8.687	6.573	6.545	2.6515	46.5869
Sample 8	5.923	9.663	6.515	5.589	10.491	6.948	7.495	11.038	4.531	6.492	3.3188	60.1161
Sample 9	5.807	6.956	7.459	7.496	8.687	5.64	5.434	4.288	11.038	5.381	3.5093	49.9915
Sample10	6.948	6.538	5.434	7.365	5.589	7.543	4.706	4.7	7.489	9.218	5.7396	44.8218

Table 4: *MLES of α and λ for each random sample of Y-Population*

Simple Random Samples from Y-Population												
											$\hat{\alpha}_y$	$\hat{\lambda}_y$
Sample 1	7.787	6.456	7.087	5.202	5.911	7.023	0.684	23.471	19.418	7.291	0.4860	124.543
Sample 2	5.202	7.023	1.284	6.018	9.388	1.647	4.789	0.684	10.713	0.163	0.3309	34.329
Sample 3	13.842	0.997	5.911	13.006	23.471	0.535	1.284	0.684	5.349	7.787	0.2469	103.923
Sample 4	11.658	17.152	23.471	7.787	8.596	0.163	6.456	13.006	4.789	0.013	0.2078	134.931
Sample 5	5.291	1.284	13.006	8.596	0.426	11.658	9.388	4.789	17.283	7.787	0.4596	87.918
Sample 6	0.426	8.596	2.838	1.647	6.572	0.111	7.291	0.111	1.829	13.388	0.2006	36.379
Sample 7	0.997	3.981	7.087	0.426	6.456	8.596	13.388	6.427	13.006	4.789	0.4675	59.545
Sample 8	23.471	2.336	3.269	6.427	0.163	4.849	4.52	7.787	5.291	5.349	0.3125	76.955
Sample 9	9.388	6.456	17.152	17.283	24.777	6.018	13.388	4.789	0.163	10.713	0.4078	168.979
Sample10	0.747	2.838	0.309	8.596	5.202	10.713	13.006	19.418	23.471	1.647	0.2632	132.424

Now, we draw 10 samples using ranked set sampling technique. We run two cycle (r = 2) of set size m = 5 to get a ranked set sample of size n = r * m = 10.

Table 5: *MLES of α and λ for each ranked set sample of X-Population*

Ranked Set Samples from X-Population												
											$\hat{\alpha}_x$	$\hat{\lambda}_x$
Sample 1	4.137	6.476	6.369	6.515	8.799	5.381	6.948	5.807	8.532	8.687	5.3408	47.8695
Sample 2	4.288	6.492	6.725	6.071	10.491	4.288	6.369	8.591	7.024	9.254	3.3715	52.0406
Sample 3	4.288	6.869	6.071	7.496	11.038	5.009	6.071	7.495	7.495	11.038	3.0263	57.6589
Sample 4	7.683	7.489	6.515	9.218	6.476	6.958	4.288	6.492	7.543	7.543	8.1074	50.7209
Sample 5	4.137	6.958	6.545	6.956	11.038	4.706	6.476	8.12	7.543	9.289	3.5078	55.1790
Sample 6	4.531	6.522	7.683	7.024	10.092	4.288	6.033	7.489	7.937	7.398	4.5176	50.1875
Sample 7	4.137	6.515	6.129	6.129	8.591	2.997	6.869	5.807	7.489	9.218	3.0288	43.9437
Sample 8	4.137	5.807	6.948	7.224	10.092	4.531	4.706	9.289	7.945	10.491	2.5873	55.5291
Sample 9	6.071	6.948	7.224	6.948	9.289	4.531	6.492	5.009	7.398	11.038	3.9824	53.6212
Sample 10	5.589	5.589	6.087	7.945	9.218	5.009	4.531	6.545	7.459	8.799	4.8530	46.9136

Table 6: *MLES of α and λ for each ranked set sample of Y-Population*

Ranked Set Samples from Y-Population												
											$\hat{\alpha}_y$	$\hat{\lambda}_y$
Sample 1	5.291	0.997	4.52	17.152	9.388	0.065	1.284	1.647	13.842	11.658	0.24212	76.3620
Sample 2	0.013	7.787	2.336	6.456	17.152	3.977	0.747	11.658	19.418	17.283	0.21862	123.001
Sample 3	0.535	4.52	2.336	9.388	13.388	0.747	0.065	4.849	13.006	23.471	0.20209	103.7666
Sample 4	3.981	0.426	4.52	4.789	23.471	0.684	4.789	7.023	7.023	48.105	0.19412	304.6421
Sample 5	1.284	3.269	11.658	8.596	10.713	0.065	4.789	6.572	6.456	13.842	0.34995	63.6314
Sample 6	1.647	1.647	5.911	13.842	7.023	0.111	4.789	5.291	7.291	11.658	0.36021	52.1297
Sample 7	0.111	1.829	3.977	23.471	7.787	0.065	6.427	4.789	4.849	48.105	0.14340	303.2548
Sample 8	0.111	3.977	6.018	4.52	17.283	0.163	0.065	3.977	7.023	32.795	0.14298	151.186
Sample 9	0.111	5.202	0.535	4.52	7.023	2.336	0.426	2.336	13.006	5.911	0.25672	31.2303
Sample10	1.304	4.789	8.596	7.023	32.795	0.684	0.997	6.456	32.795	19.418	0.23438	271.9073

To obtain 10 samples of size 10, we conducted 20 cycles. Every pair of consecutive cycles makes up one sample of size 10. The 20 cycles we performed to get the 10 ranked samples from Population X. The ranked set samples from Population X, along with the corresponding maximum likelihood estimates of α_x and λ_x, are shown in Table 5. Similarly, we run the 20 cycles to draw ranked set samples from Population Y and the ranked set samples with maximum likelihood estimates of α_y and λ_y for Y Population are shown in Table 6.

Table 7: *Bias, MSE and Relative efficiency of MLE of stress-strength model P in case of SRS and RSS*

	SRS					RSS					
	$\hat{\alpha}_x$	$\hat{\alpha}_y$	$\hat{P}_{SRS}$	Bias	MSE	$\hat{\alpha}_x$	$\hat{\alpha}_y$	$\hat{P}_{RSS}$	Bias	MSE	RE (%)
Sample 1	6.7031	0.4860	0.2725	0.1007	0.01494	5.3408	0.2421	0.2016	0.0298	0.00359	415.43%
Sample 2	3.1983	0.3309				3.3715	0.2186				
Sample 3	7.9133	0.2469				3.0263	0.2020				
Sample 4	2.6220	0.2078				8.1074	0.1941				
Sample 5	3.7265	0.4596				3.5078	0.3499				
Sample 6	3.2477	0.2006				4.5176	0.3602				
Sample 7	2.6515	0.4675				3.0288	0.1434				
Sample 8	3.3188	0.3125				2.5873	0.1429				
Sample 9	3.5093	0.4078				3.9824	0.2567				
Sample10	5.7396	0.2632				4.8530	0.2343				

An analysis of the statistical outcomes presented in the Table 7. This summary reveals a notable difference in the Mean Square Error (MSE) of the stress-strength model P between two sampling techniques. The Ranked Set Sampling (RSS) method demonstrates a significantly lower MSE compared to that obtained through Simple Random Sampling (SRS). Quantitatively, the relative efficiency (RE) of RSS surpasses SRS by a remarkable 415.43%. This substantial improvement in efficiency underscores the superior performance of RSS in practical applications. The findings strongly suggest that RSS offers more reliable and accurate results in real-world scenarios, outperforming the conventional SRS approach in the context of stress-strength modeling.

8. Conclusion

Delving into the realm of reliability engineering, this study sheds new light on the estimation of stress-strength models, with a particular focus on the intriguing $P_r(Y < X)$ paradigm. Here, we explore the behavior of independent random variables Y and X, both dancing to the tune of the Nakagami distribution. While conventional wisdom has long favored simple random sampling, our research unveils a game-changing approach: ranked set sampling. By deriving maximum likelihood estimators for P under both sampling regimes, we set the stage for a riveting comparison.

Our simulation studies paint a vivid picture of ranked set sampling's superiority, showcasing its ability to outperform its traditional counterpart in efficiency. But we don't stop at theoretical musings - we put our findings to the test in the crucible of real-world data, where ranked set sampling continues to shine brightly.

As we draw the curtain on this investigation, one conclusion stands tall: in the arena of Nakagami stress-strength model estimation, ranked set sampling emerges as the undisputed champion over simple random sampling. Yet, this is not the end of our journey. The horizon beckons with tantalizing possibilities, as we set our sights on exploring the potential of other ranked set sampling methods in this critical field of study. The quest for ever-more efficient estimation techniques in stress-strength modeling continues, promising exciting developments in the future of reliability engineering.

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EVALUATION OF PARAMETRIC ESTIMATION METHODS FOR THE GAMMA DISTRIBUTION USING MAXIMUM LIKELIHOOD AND BAYESIAN APPROACHES IN A CENSORED LIFE-TESTING STRATEGY WITH MARKOV CHAIN MONTE CARLO SIMULATIONS

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Abstract

The goal of this study was to address the computational challenges associated with parametric estimation of the gamma distribution by evaluating the performance of the maximum likelihood and maximum a-posteriori estimation methods within the framework of Markov Chain Monte Carlo simulations. This was done by first assuming a censored life-testing strategy that terminates on the rth failure from a given sample of n electronic devices. Second, we obtained the joint distribution function of the first r-order statistic by arranging the r values in order of magnitude. Finally, we explored through the Markov Chain Monte Carlo framework using the maximum likelihood and maximum a-posteriori to estimate the gamma distribution parameters. The findings of this study suggest that both estimation methods were not significantly different from the actual hypothesized parameter values. Further, we observed that irrespective of the prior distribution used for the Bayesian maximum a-posteriori Markov Chain Monte Carlo estimation, the resulting parametric estimates of the gamma distribution remain the same, confirming the assertion that the Bayesian maximum a-posteriori Markov Chain Monte Carlo approach is a valuable tool for informative posterior analysis. The study's uniqueness lies in adopting a censored life-testing strategy centered on the joint distribution function of the first r-order statistic.

Keywords: bayesian inference, gamma distribution, maximum likelihood estimation, maximum a-posteriori, reliability analysis

I. Introduction

The Gamma distribution has been extensively studied and developed in statistical inference. It is preferred over other probability distributions for its superior applications in insurance and finance. Dickson [1] and Veazie et al. [2] project the two or three parameter Gamma distribution for its ability to model highly skewed positive and or negative data points. Gamma distribution application is most evident in studying random variables such as waiting times, claim size or frequency and investment returns. While every probability distribution offers some distinct advantage in specific contexts, Dey et al. [3] support the gamma distribution for its computational efficiency and memoryless property. Memoryless property means that the occurrences of past events do not influence the probability of future occurrences of the event (Noguchi & Robles [4]; Shore [5]; Tao [6]). In other words, the memoryless property of the gamma distribution makes it easy to study the probability of the occurrence of an event independently of the probability of future occurrences of the event. This property demonstrates the usefulness and versatility of the gamma distribution in several survival and reliability studies.

The use of the gamma distribution in reliability analysis extends to other fields such as engineering, manufacturing and biomedical research. For instance, in modeling time-to-failure of manufactured components, Elsayed [7] employed the gamma distribution alongside the Maximum Likelihood Estimation (MLE) method to estimate the parameters. Similarly, Shipes et al. [8] used the Poisson-Gamma Model in their survival analysis of time-to-event clinical trial data. These studies underscore the flexibility of the gamma distribution in capturing diverse event patterns. More importantly, the parametric estimation of the gamma distribution in these studies offers room for further exploration. Meeker and Escobar [9] explained parametric estimation in both reliability and survival analysis as fitting a specific probability distribution (e.g., gamma, exponential, Weibull) to the observed failure data and estimating its parameters (location, shape and scale parameters).

Literature abounds with different combinations of estimation and simulation methods to estimate the gamma distribution parameters. Several studies (Ghosh & Hamedani [10]; Junmei & Liqin [11]) seem to opt for the MLE method due to its optimal and consistent parameter estimators. Ghosh and Hamedani [10] provided detailed properties of the two-parameter gamma distribution using the MLE method to investigate its moments, hazard function and reliability parameters. When applied to a lifetime data set, the gamma distribution produced a superior fit compared to other models. While the gamma distribution is applauded for its flexibility in model fitting, Ozsoy, Unsal and Orkcu [12] warned of a potential computational complexity when MLE is used for its parameter estimation. They explained that the distribution function (or survival function) of the gamma distribution is not available in a closed form if the shape parameter is not an integer, thereby making the use of MLE a near futile exercise. This notwithstanding, studies (Hamada et al. [13]; Rubinstein & Kroese [14]) have adapted numerical methods to evaluate the parameters of the gamma distribution by exploring a combination of Bayesian estimation and simulation procedures. Hamada et al. [13] found the Bayesian estimation useful in their probabilistic framework for reliability estimation as it incorporates additional information about the distribution known as a prior. The Bayesian framework entails careful elicitations of prior expert information to enhance the data, leading to improved prediction of extreme cases (Coles & Tawn [15]). Hussain et al. [16] and Kohole et al. [17] proffer the Bayesian method, as it at least offers a way around the complexity of the root of the maximum likelihood equation known to exist in MLE. The Bayesian approach, therefore, appears more flexible and informative through its posterior analysis.

In recent decades, the surge in statistical applications has sparked a growing interest in Bayesian parametric simulation, giving rise to the efficient concept of Maximum a-Posteriori (MAP). Serving as the Bayesian counterpart to MLE, MAP estimation entails identifying parameter values

that maximize the posterior distribution and act as estimates for the unknown parameters (Hesse et al., 2016). When there is a noninformative prior in the Bayesian analysis, the MAP estimate is the same as that of the MLE. Due to the computational intensity of MAP resulting from the incorporation of prior information, Hesse et al. [18] turned to Markov chain Monte Carlo (MCMC) to obtain samples from the posterior distribution, enabling the estimation of regression parameters. The concept of MCMC is popular in fields such as manufacturing, physics and finance, and it uses probability distributions to make selections (Benson & Kellner [19]). In reliability assessment, Naess, Leira and Batsevych [20]) noted that the MCMC can check failure criterion, regardless of the distribution or system complexity. Fauzi et al. [21]) relied on the MCMC algorithms for sampling from a posterior distribution, essentially, to simulate system behavior and estimate reliability metrics.

This study aims to tackle the computational challenges associated with parametric estimation of the gamma distribution by evaluating the performance of two estimation procedures: Maximum Likelihood Estimation (MLE) and Maximum a-Posteriori (MAP) estimation. These assessments will be conducted within the framework of Markov Chain Monte Carlo (MCMC) simulations. In this paper, the two MCMC-based estimation techniques are denoted as MLE_MCMC and MAP_MCMC. The study's uniqueness lies in the adoption of a censored life-testing strategy, terminating upon the occurrence of the r^{th} (where $r < n$) failure. This approach diverges from classical life testing, which requires the complete failure of all n samples. The study concentrates on the joint distribution function of the first r-order statistic, precisely the smallest r values, as an alternative to utilizing the complete dataset for estimation. Additionally, we look into the sensitivity of MAP_MCMC by applying various prior distributions. This study is relevant as it illustrates that the joint distribution function of the first r-order statistic proves more suitable for estimating the parameter(s) of the probability density function (pdf) of the time-to-failure random variable for any engineered device.

II. Methods

Consider n samples of manufactured components that were subjected to reliability life tests from a certain population of interest. The random variable T of interest is the time it takes until the component fails. Suppose the underlying failure times are $T_{(1)}, \cdots, T_{(n)}$ where $T_{(i)} \leq T_{(i+1)}$, $i = 1, \cdots, n-1$. And let $F_T(t)$ be the distribution function of T and let $f_T(t)$ be its probability density function (pdf). Assuming further that the reliability life tests conclude at the r^{th} failure, where r is less than or equal to n, the number of failures is treated as a fixed value, while the failure times are regarded as random variables. We employ the gamma distribution to model the time-to-failure random variables in this scenario of life testing, assuming that the failure rate is not constant. The gamma distribution is preferred in this instance because it exhibits a failure rate that follows a bathtub-shaped curve (decreasing failure rates at the initial phase and increasing failure rates at a later phase). Blanksby and Lyons [22] assert that the gamma distribution allows for flexibility in capturing diverse failure rate behaviours and is well-suited for scenarios where the hazard function varies over time. The next subsection presents a synopsis of the gamma distribution.

I. Gamma Time-to-Failure Random Variable

The continuous random variable T, is said to have the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by:

$$f_T(t) = \frac{\beta^{\alpha} t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)}, t > 0 \quad (1)$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ is the gamma function. The cumulative distribution function is the

regularized gamma function:

$$F_T(t) = \frac{\gamma(\alpha,\ \beta t)}{\Gamma(\alpha)}, t > 0. \quad (2)$$

where $\gamma(\alpha,\ \beta t)$ is the lower incomplete gamma function, that is $\gamma(\alpha,\ \beta t) = \int_0^{\beta t} t^{\alpha-1}e^{-t}dt$. This gamma distribution is in the two-parameter family of continuous probability distributions. These are:

- Shape Parameter (α): This parameter determines the shape of the distribution. It is a positive real number.
- Scale Parameter (β): This parameter is associated with the rate of events. It is also a positive real number. The density and cumulative distribution functions are sometimes expressed in terms of the scale parameter, $\theta = 1/\beta$.

Table 1 shows some characteristics of the gamma distribution (Mann et al. [23]).

Table 1: *Properties of the gamma distribution*

Properties	
Measures	Properties
Mean	$\frac{\alpha}{\beta}$
Variance	$\frac{\alpha}{\beta^2}$
Median	No simple closed
Mode	$\frac{\alpha-1}{\beta}$ for $\alpha \geq 1$
Reliability function	$e^{-\beta t}\sum_{n=1}^{\alpha-1}\frac{(\beta t)^n}{n!}$

In the next subsection, we explore two approaches for estimating the parameters of the gamma distribution through the Markov Chain Monte Carlo (MCMC) simulation framework: (a) Maximum Likelihood Estimation (MLE) and (b) Maximum a-Posteriori (MAP).

II. Parametric Estimation of the Gamma Distribution

According to Ofosu and Hesse [24], the likelihood function L of the first r-order statistics, $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(r)}$, of the random variable of interest in this study can be specified as:

$$\begin{aligned}
L &= f_{T_{(1)},\cdots,T_{(r)}}(t_1,\cdots,t_r) \\
&= \frac{n!}{(n-r)!}[1-F_T(t_r)]^{n-r}\prod_{i=1}^{r} f_T(t_i) \\
&= \frac{n!}{(n-r)!}\left[1-\frac{\gamma(\alpha,\ \beta t_r)}{\Gamma(\alpha)}\right]^{n-r}\prod_{i=1}^{r}\left\{\frac{\beta^\alpha t_i^{\alpha-1}e^{-\beta t_i}}{\Gamma(\alpha)}\right\} \\
&= \frac{n!}{(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{r}\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\} \\
&= \frac{n!}{(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{n}\beta^{r\alpha}\left(\prod_{i=1}^{r}t_i^{\alpha-1}\right)e^{-\beta\sum_{i=1}^{r}t_i}.
\end{aligned}$$

(3)

The natural logarithm of the likelihood function gives:

$$\ln L = \ln n! - \ln(n-r)! + (n-r)\ln[\Gamma(\alpha) - \gamma(\alpha,\ \beta t_r)] - n\ln\Gamma(\alpha) + r\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{r}\ln t_i - \beta\sum_{i=1}^{r} t_i. \quad (4)$$

This function yields the following logarithmic likelihood equations:

$$\frac{\partial \ln L}{\partial \beta} = \frac{r\alpha}{\beta} - \sum_{i=1}^{r}\ln t_i + \frac{\beta^{\alpha-1}(n-r)t_r^{\alpha}e^{-\beta t_r}}{[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]} = 0 \quad (5)$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{(n-r)[\Gamma'(\alpha)-\Gamma'(\alpha,\ \beta t_r)]}{[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]} + \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + r\ln\beta + \sum_{i=1}^{r}\ln t_i = 0 \quad (6)$$

Solving Equations (5) and (6) is notably challenging. When a straightforward solution to the likelihood equations is elusive, various procedures are available for MLE. Common methods encompass Iterative Methods, the Expectation-Maximization (EM) Algorithm, Gradient Descent, Quasi-Newton Methods, Monte Carlo Methods, Profile Likelihood, Bootstrapping, and Numerical Optimization (Dempster et al.[25]; Gilks et al. [26]; Nocedal & Wright [27]; Press et al. [28]).
In cases where obtaining a solution to the log-likelihood equations proves to be difficult, we turn to Markov Chain Monte Carlo (MCMC) sampling techniques to generate samples from the likelihood function. Consequently, we applied the MCMC estimation technique and referred to it as MLE_MCMC. The primary objective is to determine parameter estimates that maximize the likelihood function given the sample data. The MLE_MCMC approach identifies the mode of the simulated MCMC sample from the bivariate likelihood function in Equation (3), representing the point estimate of the parameter vector $\theta = (\alpha,\ \beta)$. That is,

$$\theta_{MLE} = \arg\ max\left\{k[\Gamma(\alpha) - \gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{n}\beta^{r\alpha}(\prod_{i=1}^{r} t_i^{\alpha-1})e^{-\beta\sum_{i=1}^{r} t_i}\right\} \quad (7)$$

The following algorithm is the description for the multivariate Metropolis Hastings procedure (Steyvers [36]):

- Set $t = 1$
- Generate an initial value for $\beta \sim U(u_1,\ u_2)$.
- Repeat
  $t = t + 1$
  Do a MH step on α,
  Generate a proposal $\theta^* \sim N(\theta,\ \sigma^2)$;
  Evaluate the acceptance probability $a = min\left[1,\ \frac{L(\theta^*/x)}{L(\theta/x)}\right]$;
  Generate a u from a Uniform(0, 1) Distribution
  If $u \leq a$, accept the proposal and set $\theta = \theta^*$
- Until $t = T$.

Maximum A Posteriori (MAP) estimation is the Bayesian counterpart to Maximum Likelihood Estimation (MLE), incorporating additional information through the prior distribution. Now, the joint pdf of $X_{(1)}, \cdots, X_{(r)}$ and $\theta = (\alpha,\ \beta)$ is given by $g(x_1, \cdots, x_r,\ \theta) = f_{X_{(1)},\cdots,X_{(r)}}(x_1, \cdots, x_r|\theta)\pi(\theta)$, where $\pi(\theta)$ is the prior distribution of the parameter Θ. We assume α and β are independent and exponentially distributed with means a and b, respectively. Thus, $\pi(\theta) = \frac{1}{ab}e^{-(\alpha/a+\beta/b)}$, $\alpha > 0, \beta > 0$.

$$g(x_1,\cdots,x_r,\theta)=\frac{n!}{ab(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}. \quad (8)$$

Thus, the marginal p.d.f. of $X_{(1)},\cdots,X_{(r)}$ is

$$\begin{aligned}g_{X_{(1)},\cdots,X_{(r)}}(x_1,\cdots,x_r)&=\int_0^\infty\int_0^\infty g(x_1,\cdots,x_r,\theta)d\alpha d\beta\\&=\frac{n!}{ab(n-r)!}\int_0^\infty\int_0^\infty[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}d\alpha d\beta.\end{aligned} \quad (9)$$

The conditional p.d.f. of Θ, given $X_{(1)},\cdots,X_{(r)}$, is therefore defined by

$$\begin{aligned}\pi(\theta|x_1,\cdots,x_r)&=\frac{g(x_1,\cdots,x_r,\ \theta)}{g_{X_{(1)},\cdots,X_{(r)}}(x_1,\cdots,x_r)}\\&=\frac{\frac{n!}{ab(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}}{\frac{n!}{ab(n-r)!}\int_0^\infty\int_0^\infty[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}d\alpha d\beta}\\&=K[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}(\textstyle\prod_{i=1}^{r}t_i^{\alpha-1})e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}+\beta\sum_{i=1}^{r}t_i\right)}.\end{aligned} \quad (10)$$

where *K* is independent of α and β. The typical approach in Bayesian estimation is to employ the posterior mean, $E(\theta|x_1,\cdots,x_r)$, as a point estimate for θ (Hesse et al. [18]). The Maximum a posteriori (MAP) estimator of θ is the value that maximizes the posterior distribution. This study uses the Markov Chain Monte Carlo (MCMC) sampling approach to draw samples from the posterior distribution. This specific method of estimation, denoted as MAP_MCMC for the purpose of this study, identifies the mode of the posterior distribution, representing the point estimate for the parameter θ. Thus,

$$\theta_{MLE}=\arg\ max\left\{=K[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}(\textstyle\prod_{i=1}^{r}t_i^{\alpha-1})e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}+\beta\sum_{i=1}^{r}t_i\right)}\right\}. \quad (11)$$

III. Results and Discussion

Consider a scenario where a company specializes in manufacturing a specific device, its components, or equipment. The company is dedicated to assessing the reliability of these items. In this context, reliability denotes the probability of a device successfully fulfilling its intended function. A sample of *n* such devices from a population of interest is placed in an environment closely resembling the conditions the items will encounter in actual use. One or more stresses of constant severity are then applied to simulate real-world scenarios. Given that, 200 of these devices are programmed to undergo reliability life testing with the test truncating when the 100th failed device was observed. The first 100 sampled time-to-failure units were ordered and fitted to the gamma distribution.

Given that *T* is a continuous random variable with a distribution function $F_T(t)$, then, according to the probability integral transformation concept, $U=F_T(t)$, follows the continuous uniform distribution over the interval (0, 1) (Ofosu & Hesse [24]). The inverse transform sampling method is employed subsequently to generate samples from the gamma distribution using the following steps:

- Generate a random number *u* from the standard uniform distribution in the interval (0, 1).
- Find the inverse of the desired cumulative density functions (CDF), denoted as $F^{-1}(u)$.
- Compute $t=F^{-1}(u)$. The computed values of the random variable *X* correspond to the desired distribution with probability density function $f_T(t)$.

There is no closed-form expression for the gamma distribution's inverse cumulative distribution function (CDF). However, various numerical methods and statistical software packages are available to calculate quantiles (inverse CDF values) for the gamma distribution. Commonly employed numerical methods include the Newton-Raphson Method and Brent's Method (Burden & Faires [29]; Kincaid & Cheney [30]; Dahlquist & Björck [31]). In addition, other statistical software packages like R, Python (SciPy), MATLAB and MS. Excel provide functions to compute quantiles for the gamma distribution (Eaton [32]; Hanselman & Littlefield [33]).

We, therefore, use the 'gaminv' function in MATLAB (MathWorks [34]) to calculate quantiles for a given probability, simulated from the uniform distribution over the interval (0, 1). Table 2 displays the first 100 out of 200 ordered data points simulated from the gamma distribution with parameters alpha (α) = 10 and beta (β) = 0.05, resulting in a mean (μ) of 200. It is assumed that these observations represent the outcomes of a reliability life test involving 200 devices until the failure of the 100th device.

Table 2: *Ordered data simulated from the gamma distribution*

73.230	76.643	81.308	85.709	88.578	90.103	91.545	91.884	95.091	97.130
98.564	99.319	101.788	105.722	107.333	107.880	110.057	111.884	113.007	115.471
117.521	121.779	125.079	125.766	128.059	128.199	130.834	132.564	134.648	135.505
135.923	136.019	138.582	141.671	142.111	143.420	144.675	146.128	147.288	147.939
152.135	153.413	153.475	155.564	155.716	156.338	156.353	157.133	157.848	157.903
159.273	159.895	159.900	160.563	160.563	161.096	162.658	162.837	168.294	168.322
169.501	169.539	170.437	170.723	170.831	172.767	173.089	173.164	173.700	175.040
175.844	176.285	176.485	177.148	177.296	177.426	177.463	178.349	178.942	179.484
181.027	181.703	181.777	182.658	182.677	182.807	185.919	188.967	189.609	189.759
189.848	190.031	191.716	192.004	192.051	192.169	192.448	194.546	194.670	195.500

On the MLE_MCMC parameter estimates, we deduce from Equation (3), given that n = 200, r = 100, $t_r = 195.500$ from Table 2, together with the Metropolis-Hastings algorithm, as described above, were employed to draw samples from the likelihood function. The MATLAB code for implementing the component-wise Metropolis sampler for the likelihood function is provided in Listings A1 and A2, in the appendix. The mode of this bivariate sample provides the maximum likelihood estimate for the parameters alpha (α) and beta (β) of the gamma distribution, which are given by $\alpha_{MLE} = 10.4169$ and $\beta_{MLE} = 0.0487$, respectively. Note that the maximum likelihood estimates are not significantly different from the actual value of the parameters, that is α = 10 and β = 0.05. Figure 1 shows the graph of the actual and the estimated gamma function with the given parameters.

The closeness of these estimated parameters of the gamma distribution is consistent with observations made by Saulo et al. [35], who observed that the generalized gamma distribution generated very close values of the log-likelihood function when compared to the Dagum distribution in an MLE procedure for censored remission times of cancer patients. The closeness of the values affirms that the MLE_MCMC approach is as trustworthy as the classical MLE.

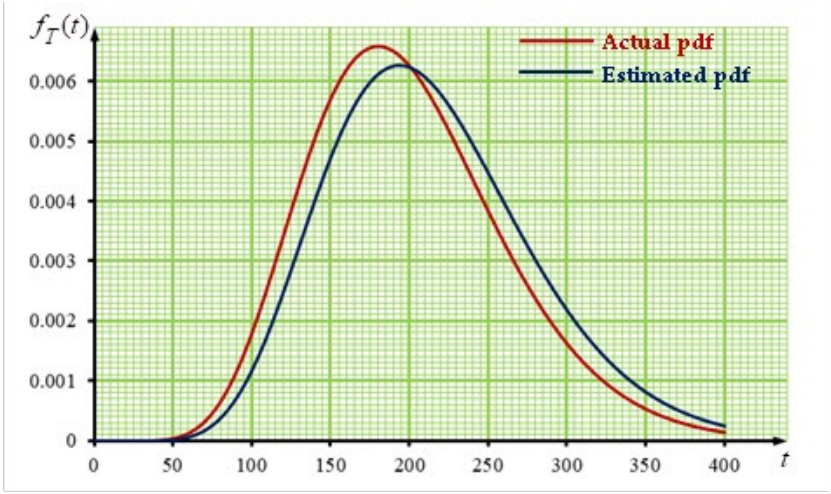

Figure 1: *pdf of the gamma distribution*

Similar to the MLE_MCMC estimates, we utilize the data from Table 2 and the Metropolis-Hastings algorithm to simulate a sample from the Bayesian posterior distribution. We further assumed that α and β are independent and exponentially distributed with means $a = 15$ and $b = 0.08$. The MATLAB code for implementing the Metropolis-Hastings sampler for the posterior distribution in Equation (9) is provided in Listings A3 and A4, in the appendix to obtain the MAP_MCMC parameter estimates of the gamma distribution.

The results reveal that the MAP_MCMC estimates of α and β are $\alpha_{MAP} = 10.4169$ and $\beta_{MAP} = 0.0487$ which precisely match the results obtained when sampling is done directly from the likelihood function. These results remain consistent even when varying the values of a and b, such as $a = 20$ and $b = 1$. In other words, the parameter estimates of the gamma distribution from the MAP_MCMC approach produce precisely the same estimates as the MLE_MCMC approach. The seemingly no difference between the two estimation methods confirms their flexibility in resolving estimation complexities associated with the gamma distribution for time-to-failure variables.

Further, we explore the sensitivity of the MAP_MCMC estimator to different prior distributions by conducting a repeated MCMC simulation assuming that the parameters α and β are independent and follow Lognormal, Pareto, Weibull, and Gumbel priors. The results of the MAP_MCMC estimate for α and β across these four prior distributions are consistent with those obtained using the exponential prior. The implication is that irrespective of the distribution used, the outcome for the two estimation procedures or methods produced the same parameter estimates. We, therefore, concur with previous studies (Hussain et al. [16]; Kohole et al. [17] to conclude that the Bayesian MCMC approach offers a more flexible and informative posterior analysis. In addition, our estimation procedure affirms Fauzi et al. [21] assertion that Bayesian MCMC algorithms for sampling from a posterior distribution essentially simulate system behavior and estimate reliability metrics.

IV. Conclusions

The applicability of the gamma distribution to study the distribution of random variables, such as time-to-failure of an event, has proven to be effective and versatile in many reliability studies. The preoccupation of these studies centers on the parametric estimation and derivation of other properties of the gamma distribution. The challenge, however, is the computational complexities encountered when the classical maximum likelihood estimation (MLE) method is used for the

parameter estimation. It is contested that MLE does not provide a straightforward solution to the log-likelihood equations since the inverse cumulative distribution function (CDF) of the gamma distribution cannot be expressed in a closed form. Consequently, many studies have resorted to numerical methods to evaluate gamma distribution parameters by exploring a combination of Bayesian estimation and other simulation procedures.

Therefore, in this study, we relied on one of the Bayesian parametric simulation methods called Maximum a-Posteriori (MAP) to estimate the gamma distribution parameters. The MAP is considered the Bayesian counterpart to MLE. It entails identifying parameter values that maximize the posterior distribution and act as estimates for the unknown parameters. Subsequently, we used the Markov Chain Monte Carlo (MCMC) simulation procedure to obtain samples from the posterior distribution to obtain the gamma parameters. The MCMC was necessary because of the computational intensity the MAP requires to incorporate the prior information. Our objective was to evaluate the performance of the Maximum Likelihood Estimation (MLE) and Maximum a-Posteriori (MAP) estimation methods under the framework of Markov Chain Monte Carlo (MCMC) simulations. This evaluation was done by first assuming a censored life-testing strategy that terminates on the r^{th} (where $r < n$) failure from a given sample of n electronic devices. Second, we obtained the joint distribution function of the first r-order statistic by arranging the values of r in order of magnitude. Finally, we explored, through the MCMC framework, the parametric estimation of the gamma distribution using the MLE and the MAP.

The finding of the study suggests that both estimation methods yielded the exact parameter estimates of 10.4169 and 0.0487, respectively, for alpha (α) and beta (β) of the gamma distribution. These estimates were not significantly different from the actual hypothesized value of $\alpha = 10$ and $\beta = 0.05$. The seemingly no difference between the two estimation methods confirms their flexibility in resolving estimation complexities associated with the gamma distribution for time-to-failure variables. Further, we observed that irrespective of the prior distribution used for the MAP_MCMC estimation, the resulting parametric estimates of the gamma distribution remain the same (unchanged), confirming the assertion that the Bayesian MCMC approach is a valuable tool for informative posterior analysis.

Declarations

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None

Conflict of interest

The authors declare that they have no competing interests.

Availability of data and materials

The dataset used in the analysis are contained in the study.

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Appendix

Listing A1: *Likelihood function for α and β*

```
1.      function y = MLE_gamma(alpha,beta,x,n,xr,r)
2.   y=(gamma(alpha)-gammainc(alpha,beta*xr))^(n-r)*(1/gamma(alpha))^n*beta^
     (r*alpha)*prod(x.^(alpha-1))*exp(-beta*sum(x));
```

Listing A2: *Implementation of Metropolis Hastings algorithm in MATLAB using the likelihood function*

```
1.  % % Metropolis procedure to sample from the posterior distribution
2.  % Component-wise updating. Use a normal proposal distribution
3.  %% Set up the Import Options and import the data
4.  opts = spreadsheetImportOptions("NumVariables", 1);
5.  % Specify sheet and range
6.  opts.Sheet = "Sheet1";
7.  opts.DataRange = "A1:A100";
8.  % Specify column names and types
9.  opts.VariableNames = "x";
10. opts.VariableTypes = "double";
11. % Import the data
12. x = readtable("C:\Users\USER\OneDrive\New Papers\Life Testing\Gamma.xlsx", opts,
    "UseExcel", false);
13. x=table2array(x);
14. r=length(x);
15. xr=x(100);
16. n=200;
17. % % Initialize the Metropolis sampler
18. T=5000;  % Set the maximum number of iteration
19. propsigma=[0.01,0.0001]; % standard deviation of proposal distribution
20. parametermin=[9,0.04];  % define minimum for alpha and beta
```

```
parametermax=[11,0.06];  % define maximum for alpha and beta
seed=1; rand( 'state' , seed ); randn('state',seed ); %#ok<RAND> % set the random seed
state=zeros(2,T);   % storage space for the state of the sampler
alpha=unifrnd(parametermin(1),parametermax(1)); % Start value for alpha
beta=unifrnd(parametermin(2),parametermax(2)); % Start value for beta
t=1;  % initialize iteration at 1
state(1,t)=alpha;  % save the current state
state(2,t)=beta;
% % Start sampling
while t<T  % Iterate until we have T samples
    t=t+1;
    % % Propose a new value for alpha
    new_alpha=normrnd(alpha,propsigma(1));
    pratio=MLE_gamma(new_alpha,beta,x,n,xr,r)/MLE_gamma(alpha,beta,x,n,xr,r);
    a=min([1 pratio]);  % Calculate the acceptance ratio
    u=rand;  % Draw a uniform deviate from [0 1]
    if u<a  % Do we accept this proposal?
    alpha=new_alpha;  % proposal becomes new value for alpha
    end
    % % Propose a new value for beta
    new_beta=normrnd(beta,propsigma(2));
    pratio=MLE_gamma(alpha,new_beta,x,n,xr,r)/MLE_gamma(alpha,beta,x,n,xr,r);
    a=min([1 pratio]);  % Calculate the acceptance ratio
    u=rand;  % Draw a uniform deviate from [0 1]
    if u<a  % Do we accept this proposal?
        beta=new_beta;  % proposal becomes new value for beta
    end
    % %  Save state
    state(1,t) = alpha;
    state(2,t) = beta;
end
Mean=mean(state,2)
Mode=mode(state,2)
```

Listing A3*: Likelihood function for α and β*

```
    function y = MLE_gamma(alpha,beta,x,n,xr,r,a,b)
 y=(gamma(alpha)-gammainc(alpha,beta*xr))^(n-r)*(1/gamma(alpha))^n*beta^
 (r*alpha)*prod(x.^(alpha-1))* exp(-beta*sum(x)-alpha/a-beta/b);
```

Listing A4: *Implementation of Metropolis Hastings algorithm in MATLAB using the likelihood function*

```
% % Metropolis procedure to sample from the posterior distribution
% Component-wise updating. Use a normal proposal distribution
%% Set up the Import Options and import the data
opts = spreadsheetImportOptions("NumVariables", 1);
% Specify sheet and range
opts.Sheet = "Sheet1";
opts.DataRange = "A1:A100";
% Specify column names and types
opts.VariableNames = "x";
opts.VariableTypes = "double";
```

```
% Import the data
x = readtable("C:\Users\USER\OneDrive\New Papers\Life Testing\Gamma.xlsx", opts, "UseExcel", false);
x=table2array(x);
r=length(x);
xr=x(100);
n=200;
a=15
b=0.08
% % Initialize the Metropolis sampler
T=5000;  % Set the maximum number of iteration
propsigma=[0.01,0.0001]; % standard deviation of proposal distribution
parametermin=[9,0.04];  % define minimum for alpha and beta
parametermax=[11,0.06];  % define maximum for alpha and beta
seed=1; rand( 'state' , seed ); randn('state',seed ); %#ok<RAND> % set the random seed
state=zeros(2,T);   % storage space for the state of the sampler
alpha=unifrnd(parametermin(1),parametermax(1)); % Start value for alpha
beta=unifrnd(parametermin(2),parametermax(2)); % Start value for beta
t=1;  % initialize iteration at 1
state(1,t)=alpha;  % save the current state
state(2,t)=beta;
% % Start sampling
while t<T  % Iterate until we have T samples
    t=t+1;
    % % Propose a new value for alpha
    new_alpha=normrnd(alpha,propsigma(1));
    pratio=MLE_gamma(new_alpha,beta, x,n,xr,r,a,b)/MLE_gamma(alpha,beta, x,n,xr,r,a,b);
    a=min([1 pratio]);  % Calculate the acceptance ratio
    u=rand;  % Draw a uniform deviate from [0 1]
    if u<a  % Do we accept this proposal?
    alpha=new_alpha;  % proposal becomes new value for alpha
    end
    % % Propose a new value for beta
    new_beta=normrnd(beta,propsigma(2));
    pratio=MLE_gamma(alpha,new_beta, x,n,xr,r,a,b)/MLE_gamma(alpha,beta, x,n,xr,r,a,b);
    a=min([1 pratio]);  % Calculate the acceptance ratio
    u=rand;  % Draw a uniform deviate from [0 1]
    if u<a  % Do we accept this proposal?
        beta=new_beta;  % proposal becomes the new value for beta
    end
    % %  Save state
    state(1,t) = alpha;
    state(2,t) = beta;
end
Mean=mean(state,2)
Mode=mode(state,2)
```

ENHANCING LINDLEY DISTRIBUTION PARAMETER ESTIMATION WITH HYBRID BAYESIAN AVERAGE MODEL FOR FUZZY DATA

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Abstract

With the ultimate goal of increasing parameter estimate accuracy, this study will examine and assess a number of estimating techniques used with the Lindley distribution in the context of fuzzy data. Gibbs sampling, Bootstrapping Sampling, MCMC, MH, and a unique hybrid methodology that combines these approaches via Bayesian model averaging were also studied. The research looks at several sample sizes ranging from 15 to 100 and repeats the estimate method 10,000 times for each size. Fuzzy data are created using established fuzzy systems, and the performance of each approach is measured using average values (AV), mean squared errors (MSE), coverage probabilities, and confidence interval lengths. The findings show that the hybrid technique consistently produces estimates closer to the genuine parameter value of one across all sample sizes, with smaller mean squared errors than individual methods. Furthermore, the hybrid method's confidence intervals preserve coverage probabilities that are consistent with the targeted confidence level, demonstrating the method's trustworthiness in statistical inference. Overall, the results show that the hybrid technique improves estimate accuracy and reliability, providing a strong foundation for parameter estimation in the Lindley distribution framework using fuzzy data.

Keywords: MCMC, MH, lindley distribution, gibbs Sampling, bootstrap sampling, bayesian model average technique.

1. Introduction

The Lindley distribution was first presented by [1] as a novel distribution has been developed to simplify lifespan data analysis, particularly in applications that include modeling the dependability of stress strength. Various authors have utilized complete and censored samples to tackle inferential challenges related to the Lindley distribution parameter. For example, a rigorous mathematical methodology to examine the Lindley distribution's properties used by [2] . In addition, through the use of a numerical example, they proved that modeling with the Lindley distribution beats modeling with the exponential distribution in terms of efficiency. The primary subjects of discussion were the estimation of reliability using an expanding type II right censored sample along with the Lindley distribution by [3] . When the Lindley distribution is used to spread the causes of failure, a rival risk model investigated by [4]. The Lindley distribution assumption states that, both traditional and Bayesian techniques to examine the hybrid censored lifespan data by [5]. Recently, numerous research has been published regarding the use of traditional statistical methods for analyzing fuzzy data. Bayesian estimation is used to examine lifetime data [6] in situations that lack clarity. Fuzzy random variable theory was examined in hindsight by Gil, Lopez-Diaz, and Ralescu, focusing on its modeling, interpretation, along with resulting implications. The use of traditional statistical inference techniques to univariate fuzzy data was

investigated by [7]. Regarding complete and refined datasets, investigated the use of ambiguous set theory in Bayesian failure rate along with mean time to failure estimate investigated by [8]. The Lindley distribution is characterized by its probability density function.

$$f(x;\beta) = \frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta\ x}; x > 0; \beta > 0,$$

In this study, we examine several approaches for estimating the Lindley distribution parameter in cases where fuzzy numbers are used to characterize the experimental data. First, how fuzzy data are created from imprecise observations is explained, and then the method for calculating the greatest probability estimate of the parameter β is discussed. When working with fuzzy data, the maximum likelihood estimate can be obtained using the MCMC, MH, Gibbs, and bootstrap methods because the MLE lacks a closed form. Furthermore, the estimated confidence interval of the unknown parameter is calculated using the asymptotic distribution of the maximum likelihood estimators (MLEs). We also look at the Lindley distribution's parameter using the Bayesian average model. Since the Bayes estimate cannot be obtained directly, we Estimate it using several techniques such as Gibbs sampling, Bootstrapping Sampling, Markov Chain Monte Carlo (MCMC), and Metropolis-Hastings (MH). The highest posterior density (HPD) credible interval of the parameter is also calculated using similar approaches.

2. Literature Review

A variety of techniques for model selection and parameter estimation utilizing fuzzy and stochastic data from various distributions are presented in the reviewed literature. Many research concentrate on distributions such as Weibull, Rayleigh, Lindley, and Inverse Lindley, and Maximum Likelihood (ML) and Bayesian techniques are frequently used. For the Weibull distribution, Bayesian and machine learning techniques with Newton-Raphson and Expectation-Maximization algorithms by [9] however, it does not compare with other fuzzy data estimate techniques. Although there are still issues with practical implementation, fuzzy Kullback Leibler (f-KL) divergence, which offers flexibility in parameter estimation presented by [10]. For the Rayleigh distribution, fuzzy parameter estimation but it doesn't outline the estimate process addressed by [11], which restricts its use. In order to accommodate stochastic and fuzzy data, generalized estimators are presented in [12] for a variety of lifespan distributions. This introduces a unique technique that requires additional performance assessment versus current approaches. Although there are little information on network construction, neural networks and compares neural-based estimators for the Weibull distribution with classical approaches examines by [13]. In many research, Bayesian methods are widely used, particularly in Lindley and Generalized Lindley distributions. With innovations like squared error loss functions and novel estimators, respectively, studies [14] and [15] use Bayesian and machine learning techniques to achieve competitive results, however they are limited to particular data circumstances. Maximum Product Spacing (MPS) is presented for the Inverse Lindley distribution in [16]. It works well but has a narrow application. While acknowledging the higher computing requirements, the Bayesian technique to Lindley distributions for hypothesis testing expanded by [17]. Model selection is the primary focus of [18]. Bayesian approaches help identify the best-fitting models for fuzzy data under the Lindley distribution when proper prior selection is used. A hybrid Bayesian-Bootstrap method in [19] enhances the generation of confidence intervals but increases computing costs. For certain fuzzy data situations, the Lindley distribution's benefit is demonstrated by comparative Bayesian analyses in [20], while an Empirical Bayes technique in [21], provides a computationally effective substitute. Further refinements in Bayesian methods include the use of informative priors [22] for improved model selection and parameter estimation in Lindley distributions. Finally, fuzzy logic and Bayesian techniques are combined in [23] for classification problems. This approach shows potential but needs more extensive validation on real-world datasets. Each method presents unique strengths and challenges, with limitations often centered on computational efficiency, distribution-specific applications, and the need for extensive comparison across methodologies.

2.1. Research gap

The literature on parameter estimation with fuzzy data lacks comprehensive comparison and integration of methodologies across different distributions. The hybrid Bayesian average model addresses this gap by integrating Bayesian inference with bootstrap resampling, providing robust parameter estimation and confidence interval construction. This model improves coverage probabilities, adaptability to various distributions, and mitigates computational cost by combining Bayesian inference with bootstrap resampling. It is more feasible for practical applications, especially when dealing with large datasets or complex models. The hybrid Bayesian average model offers improved accuracy, reliability, and computational efficiency compared to existing methods, making it a valuable solution for parameter estimation in fuzzy data analysis.

3. Simulation and Comparison Studies

3.1. Bayesian Average Modelling

This study used the average model to estimate the parameters. Updating previous knowledge based on observed data and combining it coherently is made possible by the Bayesian framework. When working with sparse or ambiguous data, the Bayesian average model comes in handy because it provides reliable parameter estimates.

Let $M = (M_1, \ldots, M_K)$ represent the set of models being studied. A model can be characterized by various attributes, such as the form of the error variance or the specific collection of explanatory factors that are part of the model. The posterior distribution of Δ, which represents the quantity of interest (such as a future observable or a model parameter), is determined by data Z. Additionally, it is as follows.

$$p(\Delta|Z) = \sum_{i=1}^{K} p(\Delta|Z, M_k)p(M_k|Z)$$

This is the average posterior predictive distribution value for Δ across all considered models, considering the likelihood of the associated posterior model. The posterior probability is the likelihood of model M_k based on the data that is already available.

$$p(M_k|Z) = \frac{p(Z|M_k)p(M_k)}{\sum_{i=1}^{K} p(Z|M_i)p(M_i)}$$

where,

$$p(Z|M_k) = \int \cdots \int p(Z|\theta_k, M_k)p(\theta_k|M_k)d\theta_k$$

is Model M_k's parameter vector, θ_k represents the integrated likelihood of the model; $p(\theta_k|M_k)$ represents the prior density of parameters of M_k; $p(Z|\theta_k, M_k)$ $p(M_k)$ stands for the previous probability of M_k™s genuineness, and reflects the likelihood. The collection of all models that are taken into consideration, M implicitly depends on each probability.

Applying the above discussed concepts simply results in parameter estimates and other quantities of interest. A parameter θ may be estimated using the Bayesian model averaging (BMA) method is,

$$\theta_{BMA} = \sum_{k=1}^{K} \hat{\theta}_k(M_k|Z)$$

where, $\hat{\theta}_k$ represents model k's posterior mean. There may also be accessible alternative amounts and variances of these estimations.

Bayesian model averaging (BMA) presents many issues. These tasks encompass determining the prior model probabilities $p(M_k)$, computing for numerous models, and evaluating the integrals that are frequently not solvable using standard mathematical expressions.

Algorithm 1 Hybrid Bayesian Average Model

Require: data, num_iterations, num_bootstrap_iterations, sample_size
Define function MCMC(data, num_iterations):
Initialize $\theta \leftarrow 0$
Initialize chain $\leftarrow$ empty list
FOR each iteration from 1 to num_iterations **DO**
Draw θ_{new} from a normal distribution with mean θ and standard deviation 0.1
Compute acceptance probability $\alpha \leftarrow \min(1, \frac{\text{probability density at } \theta_{new}}{\text{probability density at } \theta})$
IF a random number between 0 and 1 is less than α **THEN**
Set $\theta \leftarrow \theta_{new}$
Add θ to chain
END FOR
RETURN chain
Define function MH(data, num_iterations):
Initialize $\theta \leftarrow 0$
Initialize chain $\leftarrow$ empty list
FOR each iteration from 1 to num_iterations **DO**
Draw θ_{new} from a normal distribution with mean θ and standard deviation 0.1
Compute acceptance probability $\alpha \leftarrow \min(1, \frac{\text{probability density at } \theta_{new}}{\text{probability density at } \theta})$
IF a random number between 0 and 1 is less than α **THEN**
Set $\theta \leftarrow \theta_{new}$
Add θ to chain
END FOR
RETURN chain
Define function Gibbs(data, num_iterations):
Initialize $\theta \leftarrow 0$
Initialize chain $\leftarrow$ empty list
FOR each iteration from 1 to num_iterations **DO**
Draw θ from a normal distribution with mean equal to data mean and standard deviation equal to data std
Add θ to chain
END FOR
RETURN chain
Define function Bootstrapping(data, num_bootstrap_iterations, sample_size):
Initialize bootstrapped_samples $\leftarrow$ empty list
FOR each iteration from 1 to num_bootstrap_iterations **DO**
Randomly select sample_size indices from data with replacement
Create bootstrapped_sample by selecting elements of data at these indices
Add bootstrapped_sample to bootstrapped_samples
END FOR
RETURN bootstrapped_samples
Define function Bayesian_Average(parameters):
RETURN mean of parameters
Define function run_hybrid_model(data, num_iterations, num_bootstrap_iterations, sample_size):
Initialize parameters $\leftarrow$ empty list
FOR each iteration from 1 to num_iterations **DO**
Obtain MCMC chain using MCMC(data, num_iterations)
Obtain MH chain using MH(data, num_iterations)
Add results to parameters list
END FOR
RETURN Bayesian_Average(parameters)

3.2. Gibbs sampling

A series of samples from a joint probability distribution may be created iteratively using the Gibbs sampling technique. When using such a sequence, the goal is to compute an integral (like an anticipated value) or estimate the joint distribution (like a histogram). Gibbs sampling is suitable when the individual conditional distributions of each variable are known, but the joint distribution is not explicitly known. Using the current values of the other variables as a conditioning factor, the Gibbs sampling technique is used to iteratively produce a sample from each variable's distribution. The samples can be demonstrated to constitute a Markov chain, and the stationary distribution of that chain precisely corresponds to the joint distribution that is required. Gibbs sampling is a highly suitable approach for sampling the posterior distribution of a Bayesian network, which is typically characterized as a set of conditional distributions.

3.2.1 The Gibbs sampler

In the context of picture restoration using the MCMC technique, Gibbs sampling is a flexible technique for fitting statistical models [24]. This method is based on the recommendations of Tanner and Wong (1987) for replacement sampling using data augmentation.

Assume that we divide λ into r blocks, such that $\lambda = (1, 2, \ldots, r)$. Considering that λ is now in the state $\lambda^{(t)}$, let's say we make the transition in the manner described below:

$$\begin{aligned}
&\text{Draw } \lambda_1^{(t+1)} \text{ from } h(\lambda_1|\lambda_2^{(t)}, \ldots, \lambda_r^{(t)}),\\
&\text{Draw } \lambda_2^{(t+1)} \text{ from } h(\lambda_2|\lambda_1^{(t+1)}, \lambda_3^{(t)}, \ldots, \lambda_r^{(t)}),\\
&\quad\vdots\\
&\text{Draw } \lambda_r^{(t+1)} \text{ from } h(\lambda_r|\lambda_1^{(t+1)}, \ldots, \lambda_{r-1}^{(t+1)}).
\end{aligned}$$

We call the distributions h as conditional distributions in their whole. By executing one full cycle of the Gibbs sampler, we may update the whole vector λ by changing each of the r blocks in the manner illustrated. The selection of components lower-dimensional blocks has assumed the role.

3.3. Bootstrap sampling

A potent resampling method for estimating a statistic's sampling distribution in statistics is called a bootstrap sample. It is beneficial when standard parametric approaches are not appropriate or when the underlying distribution of the data is unclear. Bootstrapping is the process of repeatedly to generate a large number of bootstrap samples, replacement samples are chosen from the observable data. The distribution of the statistic's sample may be approximated using these samples, including the mean, variance, and regression coefficients. The fundamental tenet of bootstrap sampling is that the data's empirical distribution can be utilized to approximate the actual population distribution.

3.3.1 Procedure for Bootstrap sampling

The sampling distributions of sample statistics are fundamental for statistical inference. Primarily, the bootstrap method enables one to estimate the sampling distribution based on a single sample, although with some degree of approximation. The following is an explanation of how it functions.

Step 1: Perform resampling. Generate several more samples, referred to as bootstrap samples or resamples, by randomly selecting and replacing elements from the first random sample. Each sample is of same magnitude as the initial random sample.
Sampling with replacement involves randomly selecting one observation from the original sample

and returning it before selecting the next observation. This is like to randomly selecting a number from a hat, and then returning it before making another selection. Consequently, any number may be selected once, several times, or not at all. If we conducted a replacement sampling, we would get the identical set of numbers that we initially had, only in a new sequence. Practically, we would begin with the whole original sample, rather than only six observations, and generate several resamples, rather than just three.
Step 2: Compute the bootstrap distribution. Compute the statistical measures for each individual sample. The collection of these resample statistics is referred to as the bootstrap distribution. If we wish to calculate the average repair time of the population, the statistic we use is the sample mean.
Step 3: Utilize the bootstrap distribution. The bootstrap distribution provides insights on the form, central tendency, and the statistic's sample distribution's variability. The bootstrap standard error may be defined as the standard deviation of the bootstrap distribution associated with a given statistic.If the specific statistic being considered is the sample mean, then the bootstrap standard error may be calculated using resamples is

$$SE_{(\text{boot, } \bar{X})} = \sqrt{\frac{1}{B-1}\sum\left(X - \frac{1}{B}\sum \bar{X}^*\right)^2}$$

in this formula, $\bar{X}^*$ represents the average value of a single resample. Simply taking the standard deviation of the B values yields the bootstrap standard error. The asterisk in * distinguishes between the original sample mean and the resample mean.

3.4. MH Algorithm

The algorithm of Metropolis-Hastings [25, 26] is a Samples from the posterior distribution $p(\lambda|X)$ may be generated by the Markov chain Monte Carlo approach. The algorithm™s starting vector value is denoted by λ_0. Afterwards, a sequence of N parameter vectors, λ_i for $i = 1, N$, are produced as follows:

a. Create a proposal distribution $q(\lambda^*|\lambda_{i-1})$, such as a Normal distribution with λ_{i-1} as the mean, and use it to generate a candidate parameter vector λ^*.

b. Calculate
$$T = \frac{p(Y|\lambda^*)\lambda(\lambda^*)q(\lambda_{i-1}|\lambda^*)}{p(Y|\lambda_{i-1})\lambda(\lambda_{i-1})q(\lambda^*|\lambda_{i-1})},$$
where $p(Y|\lambda^*)$ and the prior densities of λ^* and λ_{i-1}, respectively, and the likelihood values of the parameter vectors λ^* and λ_{i-1}, respectively.

c. Assuming that u is chosen at random from a uniform distribution over the interval $(0, 1)$ and if T, is greater than u, then $\lambda_i = \lambda^*$; otherwise, $\lambda_i = \lambda_{i-1}$.

d. Following an initial phase of, let™s say, M iterations, the resultant chain will converge to a chain whose members are randomly selected from the posterior parameter distribution $p(\lambda|X)$. Eliminate the first M iterations as much as possible.

e. The beginning value λ_0, the proposal distribution $q(\lambda^*|\lambda_{i-1})$. Before using the Metropolis-Hastings method, you must ascertain the overall iteration count N as well as the number of rejected iterations M. The criteria for selecting these parts still need more investigation.

f. It is possible to summarize many aspects of the posterior parameter distribution using the collected sample. Using the sample of parameter vectors λ_j with $j = M + 1, \ldots, N$, one may calculate the posterior means to minimize expected quadratic loss,

$$\bar{\lambda} = \frac{1}{N-M}\sum_{j=M+1}^{N}\lambda_j$$

Next, the vector $\bar{\lambda}$ can be viewed as a model parameter estimate. Comparing parameter estimates to the complete posterior probability distribution, one must forfeit some information. The posterior variances, parameter correlations, and model prediction distribution can all be computed using the sample of parameter vectors.

3.5. Markov Chain Monte Carlo Methods (MCMC)

Let $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$ be a vector that includes every variable for which we require probability density functions. This indicates that we want to discover $p(\lambda_i|Y)$, where Y is the collection of measurements, for each variable λ_i. Applying the Bayes theorem, we get

$$p(\lambda_i|Y) = \frac{p(Y|\lambda_i)p(\lambda_i)}{p(Y)} = \frac{p(Y|\lambda_i)p(\lambda_i)}{\int_{\theta_i} p(Y|\lambda_i)p(\lambda_i)d\lambda_i}$$

The term $p(\lambda_i|Y)$ is also referred to as the target distribution or posterior because this is the distribution we are attempting to estimate. The likelihood function, $p(Y|\lambda)$, indicates the likelihood that the data was generated with the given set of parameters. It is represented by the term $p(Y)$, which is the prior distribution of λ. If the denominator is not tractable, it is not possible to give an explicit analytic expression of the probability density function. It is possible that the formula is tractable for some λ_i but not for others, which means that another approach may need to be used for estimating each variable has few different possibilities.

1. The closed analytic form of $p(\lambda|Y)$ can be written as a conventional probability distribution that can be directly sampled, or in the event that this is not possible, sampled via the inverse transform sampling method.

2. Although there are some variables, λ_{new} such that the expressions $p(\lambda_1|\lambda_{new} \ldots Y)$ and $p(\lambda_{new}|\lambda_1, \lambda_i, Y)$ can be in closed analytic form, $p(\lambda|Y)$ cannot be written in a closed analytic form. Drawing λ_i from $p(\lambda_i|\lambda_i)$ and λ_0 from $p(\lambda|\lambda_1, Y)$ can then be used to sample $p(\lambda|Y)$.

3. Although the proportionality of the joint posterior density $p(\lambda_1, \lambda_{new}, \ldots, \lambda_i)$ is known, no closed analytic equation for $p(\lambda|Y)$ has been established. The so-called Metropolis-Hastings algorithm, one specific application of the MCMC method, which stands for Markov Chain Monte Carlo, is applicable here.

4. Performance Evaluation

In order to determine which estimating method is the most effective, performance measurements like average variance (AV), mean squared error (MSE), coverage probabilities, and interval lengths are utilized. This will allow for the identification of any trends or patterns that may exist across a range of sample sizes, as well as the provision of insights into the relative effectiveness of the techniques.

4.1. Average values(AV)

The following mathematical formula can be used to determine the average values (AV) in the context of performance metrics,

$$\text{AV} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where, AV represents the average value.
N represents the overall count of observations.
x_i stands for every single observation.
In the context of the research on estimation techniques for the Lindley distribution with fuzzy

data, the average values (AV) are calculated to represent the central tendency of the estimation results obtained from different techniques. The AV provides insight into how closely the estimates align with the true parameter value, which in this case is 1.

4.2. Mean squared error(MSE)

By averaging the squares of the errors between estimated and true values, the mean squared error (MSE) is a frequently used metric to assess the accuracy of an estimator. This mathematical formula is used to compute the MSE

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x}_i)^2$$

where, Mean squared error is abbreviated as MSE.
N represents the total amount of observations.
x_i is the actual value. $\bar{x}_i$ represents the estimated value.

4.3. Coverage probability

A statistical metric called coverage probability is employed to evaluate the dependability of confidence intervals. It displays the proportion of confidence intervals containing the parameter's true value. Coverage probability is computed as follows in the context of studies on fuzzy data estimate methods for the Lindley distribution.

$$\text{Coverage Probability} = \frac{\text{Number of Confidence Intervals that contain the True Value}}{\text{Total Number of Confidence Intervals}}$$

This measure provides insights into the effectiveness of the estimation techniques in providing confidence intervals that capture the true parameter value. A coverage probability close to the desired confidence level indicates that the estimation technique is providing reliable and accurate confidence intervals.

4.4. Interval Length

The width or range of a confidence interval, which expresses the accuracy of the estimation, is referred to as the interval length. It is computed as the absolute difference between the confidence interval's upper and lower boundaries. In the context of the research on estimation techniques for the Lindley distribution with fuzzy data, interval length can be calculated as follows

$$\text{Interval Length} = \text{Upper Bound} - \text{Lower Bound}$$

This measure provides insights into how narrow or wide the confidence intervals are, which reflects the precision of the estimation. Smaller interval lengths indicate higher precision, while larger interval lengths indicate lower precision.

5. Results and Discussion

5.1. Results

We provide the results of our investigation on the performance of several estimate approaches applied to the Lindley distribution with fuzzy data in the results section. We compare the efficiency of the following techniques: Gibbs sampling, Bootstrapping Sampling, Markov Chain Monte Carlo (MCMC), and Metropolis-Hastings (MH). Our investigation covers a range of sample sizes, from 15 to 100, with each estimating step performed 10,000 times. The *fuzzification*

procedure uses preset fuzzy systems $\{\tilde{x}_1, \ldots, \tilde{x}_8\}$ to alter produced samples to account for dataset uncertainty. Each membership function, $\mu_{\tilde{x}_i}(x)$, is designed to reflect the data's fuzzy properties correctly. This section tries to highlight any trends or patterns discovered across various sample sizes, as well as give insights into the approaches' comparative performance. Furthermore, we take into account minor fluctuations in x-values to guarantee that uncertainties in fuzzy data are accurately represented, improving the integrity of our analysis and aiding researchers in picking the best methodologies for statistical inference.

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1, & \text{if } x \leq 0.13 \\ \dfrac{0.33 - x}{0.2}, & \text{if } 0.13 \leq x \leq 0.33 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \dfrac{x - 0.13}{0.2}, & \text{if } 0.13 \leq x \leq 0.33 \\ \dfrac{0.58 - x}{0.25}, & \text{if } 0.33 \leq x \leq 0.58 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \dfrac{x - 0.33}{0.25}, & \text{if } 0.33 \leq x \leq 0.58 \\ \dfrac{0.83 - x}{0.25}, & \text{if } 0.58 \leq x \leq 0.83 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \dfrac{x - 0.58}{0.25}, & \text{if } 0.58 \leq x \leq 0.83 \\ \dfrac{1.08 - x}{0.25}, & \text{if } 0.83 \leq x \leq 1.08 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \dfrac{x - 0.83}{0.25}, & \text{if } 0.83 \leq x \leq 1.08 \\ \dfrac{1.58 - x}{0.5}, & \text{if } 1.08 \leq x \leq 1.58 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \dfrac{x - 1.08}{0.5}, & \text{if } 1.08 \leq x \leq 1.58 \\ \dfrac{2.08 - x}{0.5}, & \text{if } 1.58 \leq x \leq 2.08 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \dfrac{x - 1.58}{0.5}, & \text{if } 1.58 \leq x \leq 2.08 \\ 3.08 - x, & \text{if } 2.08 \leq x \leq 3.08 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\bar{x}_8}(x) = \begin{cases} \dfrac{x - 2.08}{1}, & \text{if } 2.08 \leq x \leq 3.08 \\ 1, & \text{if } x \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

Table 1: *Averages values and mean squared errors of the GIBBS estimates of $\theta = 1$, coverage probabilities*

n	AV	MSE	Coverage	Length
15	1.0307	0.0405	0.9403	0.7508
20	1.0289	0.0386	0.9457	0.7205
30	1.0289	0.0349	0.9524	0.6501
50	1.0246	0.0258	0.9526	0.4507
70	1.0154	0.0192	0.9538	0.3509
100	1.0102	0.0187	0.9559	0.1804

Table 2: *Averages values and mean squared errors of the Bootstrapping Sampling estimates of $\theta = 1$, coverage probabilities*

n	AV	MSE	Coverage	Length
15	1.0327	0.0426	0.9385	0.7703
20	1.0295	0.0375	0.9447	0.7409
30	1.0255	0.0329	0.9486	0.6305
50	1.0198	0.0247	0.9519	0.4902
70	1.0147	0.0198	0.9538	0.4003
100	1.0098	0.0115	0.9556	0.2106

Table 3: *Averages values and mean squared errors of the MCMC estimates of $\theta = 1$, coverage probabilities*

n	AV	MSE	Coverage	Length
15	1.0316	0.0437	0.9365	0.7559
20	1.0285	0.0389	0.9436	0.7253
30	1.0247	0.0337	0.9489	0.6154
50	1.0184	0.0256	0.9519	0.4708
70	1.0139	0.0209	0.9537	0.3804
100	1.0087	0.0127	0.9553	0.1905

Table 4: *Averages values and mean squared errors of the MH estimates of $\theta = 1$, coverage probabilities*

n	AV	MSE	Coverage	Length
15	1.0309	0.0425	0.9378	0.7653
20	1.0287	0.0384	0.9446	0.7357
30	1.0249	0.0339	0.9485	0.6254
50	1.0175	0.0245	0.9529	0.4956
70	1.0125	0.0195	0.9538	0.3907
100	1.0087	0.0128	0.9552	0.2104

Table 5: *Combined of all methods using Bayesian model average technique*

n	AV	MSE	Coverage	Length
15	1.0305	0.0409	0.9379	0.7559
20	1.0284	0.0379	0.9441	0.7264
30	1.0249	0.0339	0.9490	0.6174
50	1.0193	0.0251	0.9519	0.4761
70	1.0140	0.0204	0.9536	0.3877
100	1.0093	0.0132	0.9552	0.2009

Diagrammatic presentation of the table values obtained in the simulation process

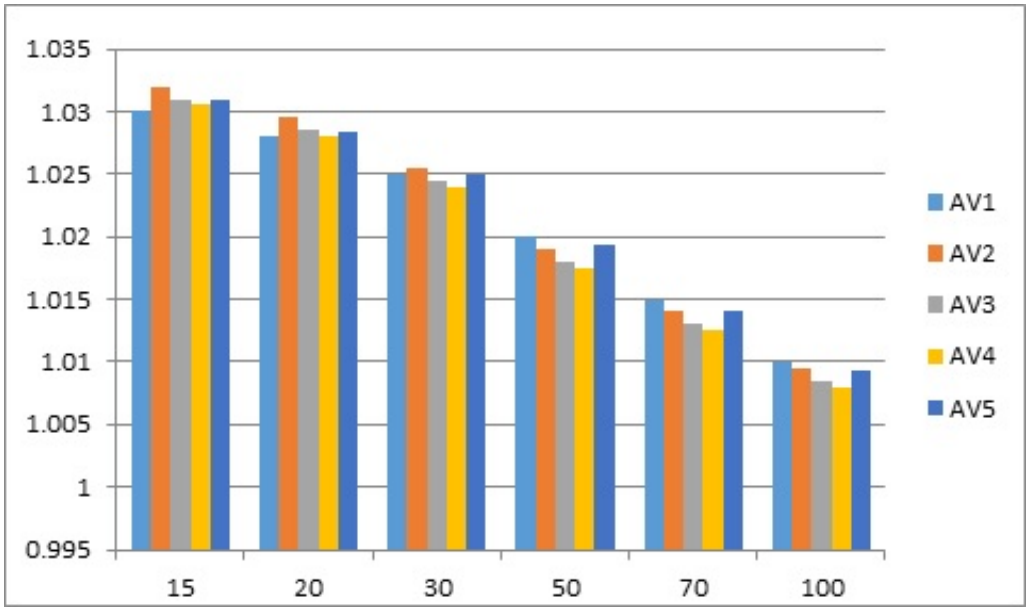

Figure 1: *Comparison of Average Values Across Different Samples sizes*

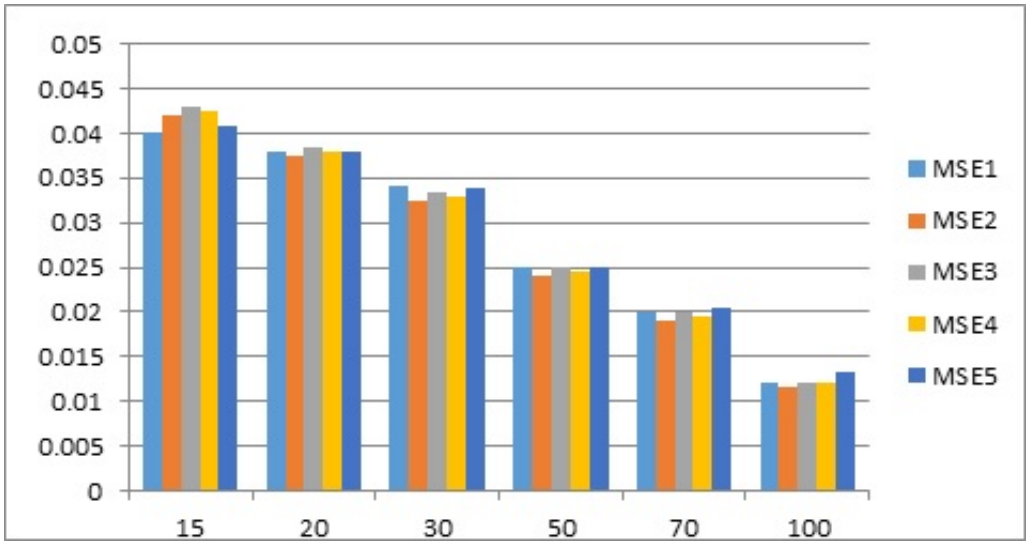

Figure 2: *Comparison of MSE Across Different Samples sizes*

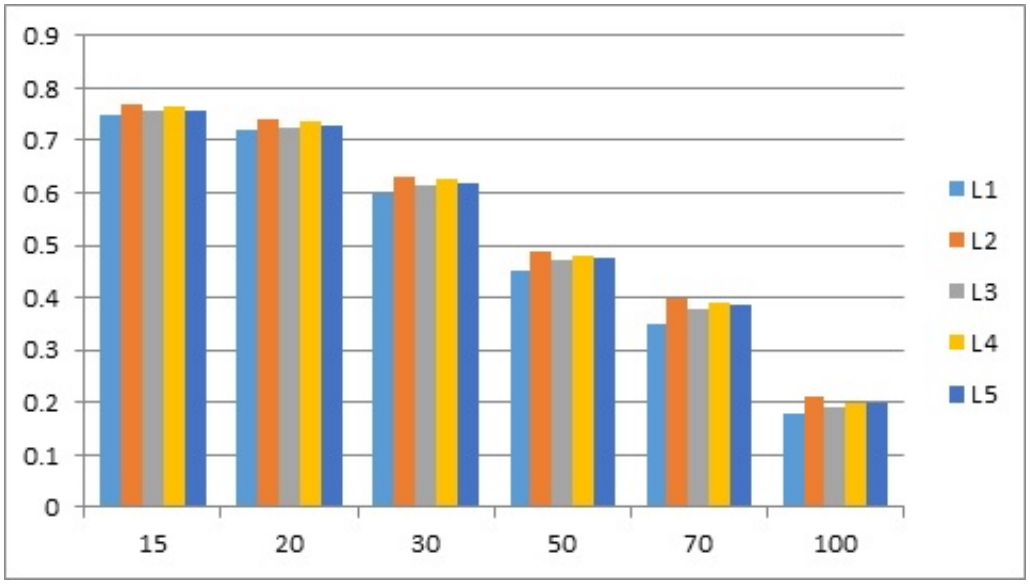

Figure 3: *Comparison of Interval length Across Different Samples sizes*

From Tables 1 to 5, it is observable that the hybrid approach, which combines all methods using the Bayesian model averaging technique, yields more reliable estimates. In Table 5, the average values (AV) are calculated using this Bayesian model averaging technique, combining estimates from all methods. Compared to individual approaches, this hybrid method provides estimates closer to the true value of 1, indicating improved accuracy. Table 5 shows that the modifications applied in the hybrid approach result in values that are even closer to the true value of 1 than those in Tables 1 through 4. Specifically, the Proposed table (Table 5) offers more precise average values, suggesting that this method achieves a superior level of accuracy in estimating the true value. Additionally, it reports lower mean squared error (MSE) values, highlighting improved accuracy in estimating the population parameters. Lower MSE values suggest confidence intervals that are closer to the true values. Furthermore, the coverage probabilities in Table 5 align closely with the desired confidence level, demonstrating the reliability of this approach. The graphical representations in Figures 1 to 4 further support these findings. Over Gibbs Sampling, Bootstrapping Sampling, MCMC, the Metropolis-Hastings (MH) Algorithm, and the unique hybrid strategy, the hybrid method stands out for its accuracy in parameter estimation for the Lindley distribution on fuzzy data. Figures 1 and 2 demonstrate that the hybrid approach yields average values closer to 1, and Figure 3 confirms lower MSE, indicating improved precision. Additionally, Figures 3 and 4 show that coverage probabilities and interval lengths are more favorable, underlining the robustness of the hybrid approach compared to individual methods. In conclusion, the unique hybrid strategy, presented in Table 5, offers balanced and efficient parameter estimation with superior accuracy across multiple metrics. The Bayesian model averaging technique combines estimates from all methods, producing values that consistently approach the true value of 1. This approach enhances mean squared errors, coverage probabilities, and confidence interval lengths across different sample sizes, outperforming Bootstrapping, MCMC, and MH estimates.

5.2. Discussion

The provided research summarizes the performance of Gibbs sampling, Bootstrapping Sampling, MCMC, and MH approaches for estimating parameters for the Lindley distribution on fuzzy data, as well as a unique hybrid strategy that combines these methods using Bayesian model averaging. The combination of these strategies presents a potential opportunity for improving estimate accuracy and dependability. Notably, the Bayesian model average methodology used

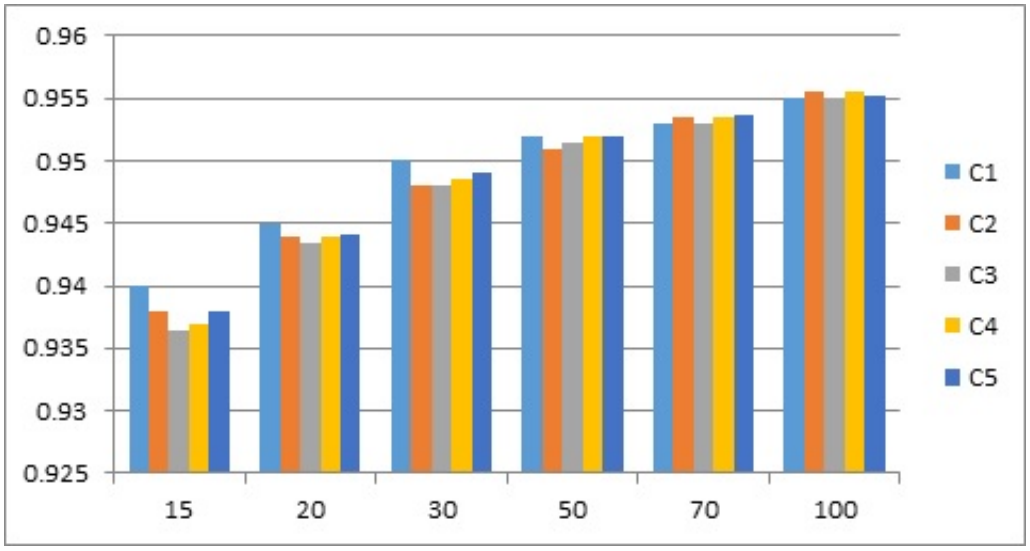

Figure 4: *Comparison of Coverage Values Across Different Samples sizes*

in the proposed hybrid method combines data from many estimating methodologies, resulting in estimates that are consistently closer to the real parameter value of 1 over a range of sample sizes. By exploiting the characteristics of each individual technique, the hybrid strategy yields reduced mean squared errors, indicating greater estimate accuracy. Furthermore, the hybrid method's confidence intervals preserve coverage probabilities that are consistent with the targeted confidence level, demonstrating the method's trustworthiness in statistical inference. Overall, the results from emphasize the hybrid method's usefulness in improving estimate accuracy and reliability, providing a strong foundation for parameter estimation in the Lindley distribution framework with fuzzy data.

6. Conclusion

Ultimately, a novel method for parameter estimation in the Lindley distribution domain is embodied by the combination of Gibbs sampling, Bootstrapping sampling, MCMC, and MH techniques using Bayesian model averaging, especially when dealing with fuzzy data. Examining the hybrid approach closely across a range of sample sizes, it shows itself to be a remarkable performer, always guiding estimates toward the real parameter value of 1. This convergence toward the true value highlights the ability of the hybrid method to strengthen accuracy and avoid estimate mistakes, therefore promoting increased trust in the deduced conclusions. A notable finding is that the hybrid approach tends to provide less mean squared errors than its component individual techniques. This decrease in mean squared errors not only shows improvement in estimating accuracy but also highlights the ability of the hybrid method to extract a plethora of data from many estimation techniques, hence improving the quality of the estimated values. In addition, the confidence intervals produced by the hybrid approach firmly adhere to coverage probabilities that correspond with the target confidence level, therefore confirming the robustness and dependability of the approach in the field of statistical inference. These results have far-reaching consequences that cut across many fields where ambiguity and imprecision are commonplace, even beyond the boundaries of academic study. Researchers and professionals struggling with parameter estimate problems in the face of erratic fuzzy data will find the hybrid approach to be a true miracle. Through the use of complementary strengths of many estimating methods, the hybrid approach causes a paradigm change in statistical analysis, enabling more careful decision-making processes and a higher degree of confidence in the conclusions that are obtained. All things considered, the combination of Gibbs sampling, Bootstrapping sampling, MCMC, and MH techniques via Bayesian model averaging is a fundamental development in the

history of statistical inference techniques. Going ahead, more research and development of the hybrid approach show promise for overcoming the many obstacles related to parameter estimation in a variety of distributional frameworks and datasets, so paving the way for a revolutionary development in the field of statistical methods and their usefulness in many different research fields.

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UNRELIABLE M/G/1 QUEUE WITH GENERAL RETRIAL TIME, WORKING VACATION AND SETUP TIME

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Abstract

In the current article, a retrial queuing system with working vacations, interruptions, setup time, and perfect repair is analyzed. The scenario includes a server taking working vacations during empty periods without a complete halt of servicing customers; however, the rates of service remain reduced. Further, a setup time is included here, implying that if the server remains idle when a new customer enters, the state changes to inactive plus a setup duration before restarting operation. In this phase of setup, the setup failure happens and is replaced immediately before the server can proceed to normal operations. In addition to this, automatic power-off to conserve energy is there when no customer comes while the server is in vacation mode. Customers who find that the server cannot be accessed spend time waiting in retrial orbit instead of entering a normal queue. Here they're encouraged to try again for service after a random time. The steady state probability generating functions for system size and retrial group size are obtained by analyzing the system dynamics through the supplementary variable technique (SVT). Reliability and optimization analyses will be included in what will be studied from the system. Reliability concerns evaluating the chances of the server being available at different failure and repair sites while in the system, while optimization looks at the best configuration of system parameters that will work towards achieving greater efficiency and reduced delays. Explicit mathematical formulations can be obtained under ergodicity conditions describing the system size distribution and sojourn time and state probabilities. For a practical realization of the model, which numerically experiments would be carried out in Python, the theoretical results were validated .Such results therefore hold information on how direct retrials, setup times, service rates, and repair mechanisms affect overall system behavior. They also provide strong evidence for trade-offs between energy conservation on the one hand and reliability together with continuous service on the other. The proposed model together with practical implementation thus produces very significant inferences relevant to real service models in which the optimization of resources and efficiency of operation are critical.

Keywords: Retrial queue; working vacation; setup time; interruption; perfect repair ; general retrial times.

1. Introduction

Queueing theory is a fundamental branch of applied mathematics with widespread applications in various fields such as telecommunications, computer systems, transportation, and manufacturing. When servers are busy, customers often enter a retrial orbit, awaiting service availability according to predefined policies. The groundwork laid by Yang and Falin, Falin and Templeton [7], Yang et al. Extending by Artalejo, Gomez-Corral [8] , Arrar et al [2] [3] [4]. Queueing systems, particularly those involving retrials with vacations play a crucial role in balancing resource utilization and service efficiency. The queueing systems with server vacations introduced by Levy and Yechiali [14] are widely used in manufacturing systems, production systems, service systems, inventory

systems, and other stochastic systems. In traditional scenarios, service stops when servers are on break, many researchers have worked on vacation interruption. Notably, Keilson and Servi [11] conducted significant research. One can refer also Li and Tian [15]. Takagi [21] contributed to the field by studying single-server queueing models with Bernoulli vacations. However, in many particular cases, an alternative approaches like working vacations provide service at a reduced rate during idle times, thereby enhancing the overall performance of the system. Numerous researchers have dedicated their efforts to develop models for queueing systems with working vacation concepts. Pioneering work on this subject was also carried out by Servi and Finn [18] considered an M/M/1 queue with working vacations, wherein vacation times are exponentially distributed. Most work on queueing models with working vacations can be found in Tian et al [19] and Chandrasekaran et al [6]. The M/M/1 retrial queue with working vacations was first studied by Do. Subsequently, Banik et al[5] analyzed a general working vacation queueing model. Arivudainambi et al [1] focused on analyzing a single-server retrial queue with working vacation dynamics. Zhang and Hou [20] studied M/G/1 queueing model with vacation interruption. And Gupta and Kumar [9] considered retrial queueing system with working vacation with breakdown and repair. Furthermore, the integration of setup time becomes essential, especially in the context of energy conservation. Setup time enables power-saving strategies by allowing server deactivation during periods of inactivity. Recognizing the importance of power conservation, several researchers have explored queueing models incorporating setup time. Phung-Duc [16] [17], for instance, integrated the notion of setup time into retrial queueing systems. Gupta and Kumar [10] analyzed retrial queue with feedback, setup time, working vacation perfect repair. For further insights into related research, readers may refer to Manoharan and Jeeva [12] [13]. And Pannom Gupta. The model under investigation is an M/G/1 retrial queue with a working vacation and setup time. This system represents a complex real-life case where customers may encounter delays due to server unavailability, requiring them to reattempt service after a waiting period. The setup times can significantly impact service efficiency and system performance. We extend the results obtained by considering the general law of inter-retrial times and service times. Our study aims to provide a detailed analysis of this type of complex system. We introduce the Markov chain to prove the stability condition of the studied model. Using the method of supplementary variables, we obtain the partial generating functions and the limiting distributions that this type of model can possess. We also present some performance measures of the studied model.

2. The model

In this paper, we consider an M/G/1 unreliable retrial queue with single working vacation , setup time. The model is described in great detail as follows:

1. The arrival of consumers follows a Poisson process with a rate $\lambda > 0$. If a consumer arrives and the server is idle, they begin service immediately . on the other hand, if he finds it busy, on working vacation, on setup, or broken then the consumer leaves the service area and enters a pool of blocked consumers called orbit in accordance with an First Come First Served discipline. While in orbit, the consumer waits a random amount of time before retrying. The inter-retrial times follow an arbitrary probability distribution function $N(x)$ with corresponding LST function $\mathcal{N}^*(\theta)$.

2. The server promptly initiates the normal service for the new or retrial consumers upon their arrival while in an idle state. The normal service time distributed with general distribution function $G(x)$ having LST $\mathcal{G}^*(\theta)$.

3. in the case of the orbit is empty.The server automatically begins a single working vacation, which follow an exponential distribution with rate ω .If a consumers comes up during the working vacation , the consumers are served at a reduced rate. when the working vacation is finished the server resumes normal service . while the working vacation period , The service time distributed with general distribution function $W_v(x)$ having LST $\mathcal{W}_v^*(\theta)$.

4. In the final of working vacation , if no consumers are waiting for their turn , the server is shut down directly to save power .

5. in the off-state of the server, if any client arrives he will wait for his turn in forward position of the server until it is activated (setup time). the setup time is assumed to follow general distribution with probability distribution function $T_s(x)$ with corresponding LST function $\mathcal{T}_s^*(\theta)$. The customers arriving in the setup state, have to join the orbit.

5. whene the server undergoes to the set-up state , this operation may fail with probability $\bar{\alpha} = (1 - \alpha)$. Then the server is sent for repair and The repair time of the server has arbitrary probability distribution function $S_f(x)$ with corresponding LST function $\mathcal{S}_f^*(\theta)$.

2.1. Practical justifications of the suggested model

Consider a manufacturing system consisting of a paper recycling machine , a Foreman(server) and a worker(assistant) to operate the machine . The foreman will operate the machine if the waste paper (customer) is available and produce the products.if this last is not available due to transport issues , then the foreman may go on vacation .During the vacation period of the foreman, if waste paper becomes available then the worker will operate the machine , but the production will be relatively at a slow speed (working vacation) . When a batch of the product is completed, then the worker will call the foreman to resume the production at a higher speed (vacation interruption). In another situation, if the foreman's vacation period completes, he will return to the production to operate the machine . If the waste paper is available then he will manage the production at a higher speed otherwise, if the waste paper is not available, to save power, he may turn off the machine. Again, the availability of new waste paper will initiate the setup of the machine (setup time) and production starts again if setup occurs successfully otherwise the machine will be sent for repair, and during this period there will be no production.

3. Steady-state analysis

Let $\chi_1(t), \chi_2(t), \chi_3(t), \chi_4(t), \chi_5(t)$ be the elapsed time in retrial, time in regular service, working vacation time , repair time and setup time sequentially at time t.
Let supose that :
$N(0) = 0, N(\infty) = 1, G(0) = 0, G(\infty) = 1, W_v(0) = 0, W_v(\infty) = 1, T_s(0) = 0, T_s(\infty) = 1, S_f(0) = 0, S_f(\infty) = 1$ are continuous at $x = 0$.consequently, we specify the hazard rate functions $f(x), \mu_n(x), \mu_w(x), v(x), \delta(x)$, for retrial, normal service, lower rate service, delayed repair and repair, respectively.

$$f(x)dx = \frac{dN(x)}{1 - N(x)}$$
$$\mu_n(x)dx = \frac{dG(x)}{1 - G(x)}$$
$$\mu_w(x)dx = \frac{dW_v(x)}{1 - W_v(x)}$$
$$v(x)dx = \frac{dS_f(x)}{1 - S_f(x)}$$
$$\delta(x)dx = \frac{dT_s(x)}{1 - T_s(x)}$$

The state of the system at time t can be defined by the Markov process $\{N(t); t \geq 0\} = \{D(t), X(t), \chi_1(t), \chi_2(t), \chi_3(t), \chi_4(t), \chi_5(t) t \geq 0\}$, where $X(t)$ denotes the number of customers in the orbit at time t and

$$\text{D(t)=}\begin{cases}0, & \text{if the server is idle in a normal period}\\ 1, & \text{if the server is idle in a working vacation period}\\ 2, & \text{if the server is busy on normal service}\\ 3, & \text{if the server is busy on working vacation period}\\ 4, & \text{if the server is on repair}\\ 5, & \text{if the server is on setup state}\end{cases}$$

respectively. If $D(t) = 0$ and $X(t) > 0$, then $\chi_1(t)$ represents the elapsed retrial time, if $D(t) = 2$, then $\chi_2(t)$ represents the elapsed service time during normal busy period at time t, if $D(t) = 3$ and $X(t) \geq 0$ then $\chi_3(t)$ represents the elapsed working vacation time at time t and if $D(t) = 4$ and $X(t) \geq 1$ then $\chi_4(t)$ represents the elapsed repair time at time t and if $D(t) = 5$ and $X(t) \geq 1$ then $\chi_5(t)$ represents the elapsed setup time at time t.

3.1. Stability and ergodicity Condition

Let $\{t_n; n \in N\}$ be the sequence of epochs of either service completion times or vacation termination time. The sequence of random vectors $Z_n = \{D(t_n+), X(t_n+)\}$ form a Markov chain which is the embedded Markov chain for our queueing system. Its state space is $S = \{0, 1, 2, 3, 4 \text{ and } 5\} \times N$.

Theorem 1. The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\rho = \lambda\left(\frac{1}{\mu_n} + \frac{1}{\mu_w} + \omega \mathcal{W}_v^{*\prime}(\omega)\right) < \mathcal{N}^*(\lambda)$.

3.2. Equations Governing The System

For the Markov process $\{N(t); t \geq 0\}$, we define the probability

$$P_{00}(t) = \{D(t) = 0, X(t) = 0\} Q_0(t) = \{D(t) = 1, X(t) = 0\}$$

and the probability densities

$$\begin{aligned}
P_n(x,t)dx &= \{D(t) = 0, X(t) = n, x \leq \chi_1(t) < x + dx\}, x \geq 0, n \geq 1\\
M_{n,b}(x,t)dx &= \{D(t) = 2, X(t) = n, x \leq \chi_2(t) < x + dx\}, x \geq 0, n \geq 0\\
G_{n,v}(x,t)dx &= \{D(t) = 3, X(t) = n, x \leq \chi_3(t) < x + dx\}, x \geq 0, n \geq 0\\
U_n(x,t)dx &= \{D(t) = 4, X(t) = n, x \leq \chi_4(t) < x + dx\}, x \geq 0, n \geq 1\\
K_n(x,t)dx &= \{D(t) = 5, X(t) = n, x \leq \chi_5(t) < x + dx\}, x \geq 0, n \geq 1
\end{aligned}$$

We assume that the stability condition is fulfilled in the sequel and so that we can set $P_{00} = \lim_{t\to\infty} P_{00}(t)$ and $Q_0 = \lim_{t\to\infty} Q_0(t)$ limiting densities for $x > 0$ and $n \geq 0$

$$M_{n,b}(x) = \lim_{t\to\infty} M_{n,b}(x,t), P_n(x) = \lim_{t\to\infty} P_n(x,t) \text{ and } G_{n,v}(x) = \lim_{t\to\infty} G_{n,v}(x,t)$$

$$\text{and } U_n(x) = \lim_{t\to\infty} U_n(x,t) \text{ and } K_n(x) = \lim_{t\to\infty} K_n(x,t).$$

Based on the above assumptions and notations, our model is governed by the following set of differential difference equations,

$$\lambda P_{00} = \omega Q_0 \tag{1}$$

$$(\lambda + \omega) Q_0 = \int_0^\infty M_{0,b}(x)\mu_n(x)dx + \int_0^\infty G_{0,v}(x)\mu_w(x)dx \tag{2}$$

$$\frac{d}{dx}P_n(x) + (\lambda + f(x))P_n(x) = 0, \quad x > 0, \quad n \geq 1 \tag{3}$$

$$\frac{d}{dx}M_{0,b}(x) + (\lambda + \mu_n(x))\, M_{0,b}(x) = 0, \quad x > 0 \tag{4}$$

$$\frac{d}{dx}M_{n,b}(x) + (\lambda + \mu_n(x))\, M_{n,b}(x) = \lambda M_{n-1,b}(x), \quad x > 0, \quad n \geq 1 \tag{5}$$

$$\frac{d}{dx}G_{0,v}(x) + (\lambda + \omega + \mu_w(x))\, G_{0,v}(x) = 0, \quad x > 0 \tag{6}$$

$$\frac{d}{dx}G_{n,v}(x) + (\lambda + \omega + \mu_w(x))\, G_{n,v}(x) = \lambda G_{n-1,v}(x), \quad x > 0, \quad n \geq 1 \tag{7}$$

$$\frac{d}{dx}U_0(x) + (\lambda + v(x))U_0(x) = 0, \quad x > 0 \tag{8}$$

$$\frac{d}{dx}U_n(x) + (\lambda + v(x))U_n(x) = \lambda U_{n-1}(x), \quad x > 0, \quad n \geq 1 \tag{9}$$

$$\frac{d}{dx}K_0(x) + (\lambda + \delta(x))K_0(x) = 0, \quad x > 0 \tag{10}$$

$$\frac{d}{dx}K_n(x) + (\lambda + \delta(x))K_n(x) = \lambda K_{n-1}(x), \quad x > 0, \quad n \geq 1 \tag{11}$$

The boundary conditions at $x = 0$ include $P_n(0), M_{0,b}(0), M_{n,b}(0), G_{0,v}(0), U_0(0), U_n(0), K_0(0)$. The description of these terms at $x = 0$ are described as follows:

$$P_n(0) = \int_0^\infty G_{n,v}(x)\mu_w(x)dx + \int_0^\infty M_{n,b}(x)\mu_n(x)dx, \quad n \geq 1 \tag{12}$$

$$M_{0,b}(0) = \alpha \int_0^\infty K_0(x)\delta(x)dx + \int_0^\infty P_1(x)f(x)dx + \omega \int_0^\infty G_{0,v}(x)dx + \int_0^\infty U_0(x)v(x)dx \tag{13}$$

$$M_{n,b}(0) = \alpha \int_0^\infty K_n(x)\delta(x)dx + \int_0^\infty P_{n+1}(x)f(x)dx + \omega \int_0^\infty G_{n,v}(x)dx + \lambda \int_0^\infty P_n(x)dx + \int_0^\infty U_n(x)v(x)dx, \quad n \geq 1 \tag{14}$$

$$G_{0,v}(0) = \lambda Q_0 \tag{15}$$

$$G_{n,v}(0) = 0, \quad n \geq 1 \tag{16}$$

$$K_0(0) = \lambda P_{00} \tag{17}$$

$$K_n(0) = 0, \quad n \geq 1 \tag{18}$$

$$U_0(0) = \bar{\alpha} \int_0^\infty K_0(x)\delta(x)dx \tag{19}$$

$$U_n(0) = \bar{\alpha} \int_0^\infty K_n(x)\delta(x)dx, \quad n \geq 1 \tag{20}$$

The normalization condition is given by

$$P_{00} + Q_0 + \sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty}\int_0^\infty M_{n,b}(x)dx + \sum_{n=0}^{\infty}\int_0^\infty G_{n,v}(x)dx + \sum_{n=0}^{\infty}\int_0^\infty U_n(x)dx + \sum_{n=0}^{\infty}\int_0^\infty K_n(x)dx = 1$$

In order to solve the above set of equations we define the generating functions as, $P(x,z) = \sum_{n=1}^{\infty} z^n P_n(x)$ for $|z| \leq 1$ and $x > 0, P(0,z) = \sum_{n=1}^{\infty} z^n P_n(0)$ for $|z| \leq 1, M_b(x,z) = \sum_{n=0}^{\infty} z^n M_{n,b}(x)$ for $|z| \leq 1$ and $x > 0, M_b(0,z) = \sum_{n=0}^{\infty} z^n M_{n,b}(0)$, $G_v(x,z) = \sum_{n=0}^{\infty} z^n G_{n,v}(x)$ for $|z| \leq 1$ and $x > 0, G_v(0,z) = \sum_{n=0}^{\infty} z^n G_{n,v}(0)$, $U(x,z) = \sum_{n=0}^{\infty} z^n U_n(x)$ for $|z| \leq 1$ and $U(0,z) = \sum_{n=0}^{\infty} z^n U_n(0)$ for $|z| \leq 1$, $K(x,z) = \sum_{n=0}^{\infty} z^n K_n(x)$ for $|z| \leq 1$ and $K(0,z) = \sum_{n=0}^{\infty} z^n K_n(0)$ for $|z| \leq 1$. Multiplying equations (2)-(11) by suitable powers of z and summing over n, we obtain the following set of partial differential equations :

$$\frac{\partial P(x,z)}{\partial x} + (\lambda + f(x))P(x,z) = 0, \tag{21}$$

$$\frac{\partial M_b(x,z)}{\partial x} + (\lambda - \lambda z + \mu_n(x))\, M_b(x,z) = 0, \tag{22}$$

$$\frac{\partial G_v(x,z)}{\partial x} + (\lambda - \lambda z + \omega + \mu_w(x))\, G_v(x,z) = 0, \tag{23}$$

$$\frac{\partial U(x,z)}{\partial x} + (\lambda - \lambda z + v(x))U(x,z) = 0, \tag{24}$$

$$\frac{\partial K(x,z)}{\partial x} + (\lambda - \lambda z + \delta(x))K(x,z) = 0. \tag{25}$$

Solving the above partial differential equations (22) to (26)

$$P(x,z) = P(0,z)[1 - N(x)]e^{-\lambda x}, \tag{26}$$

$$M_b(x,z) = M_b(0,z)\,[1 - G(x)]\,e^{-I(z)x}, \tag{27}$$

$$G_v(x,z) = G_v(0,z)\,[1 - W_v(x)]\,e^{-(I(z)+\eta)x}, \tag{28}$$

$$U(x,z) = U(0,z)\left[1 - S_f(x)\right]e^{-I(z)\cdot x}, \tag{29}$$

$$K(x,z) = K(0,z)\,[1 - T_s(x)]\,e^{-I(z)\cdot x}. \tag{30}$$

where $I(z) = \lambda(1-z)$
Multiplying equation (12) by suitable powers of z, summing over n from 1 to ∞ and using equations (1) and (2) after some algebraic manipulations, we get

$$P(0,z) = \int_0^\infty G_v(x,z)\mu_w(x)dx + \int_0^\infty M_b(x,z)\mu_n(x)dx - \left(\frac{\lambda+\omega}{\omega}\right)\lambda P_{00} \tag{31}$$

Similarly, multiplying equations (14)-(21) by suitable powers of z, summing over n and after some algebraic manipulations, we obtain

$$M_b(0,z) = \frac{\alpha}{z}\int_0^\infty K(x,z)\delta(x)dx + \lambda\int_0^\infty P(x,z)dx + \frac{1}{z}\int_0^{+\infty} P(x,z)f(x)dx$$
$$+ \omega\int_0^{+\infty} G_v(x,z)dx + \int_0^{+\infty} U(x,z)v(x)dx, \tag{32}$$

$$G_v(0,z) = G_{0,v}(0), \tag{33}$$

$$U(0,z) = \bar{\alpha}\int_0^\infty K(x,z)\delta(x)dx, \tag{34}$$

$$K(0,z) = K_0(0). \tag{35}$$

inserting equations 28-29 in 32 we get :

$$P(0,z) = G_v(0,z)\mathcal{W}_v^*(I(z)+\eta) + M_b(0,z)G^*(I(z)) - \left(\frac{\lambda+\omega}{\omega}\right)\lambda P_{00} \tag{36}$$

in similary way inserting equations 27-31 in 33 we get :

$$M_b(0,z) = \frac{1}{z}P(0,z)\mathcal{N}^*(\lambda) + \alpha\mathcal{T}_s^*(I(z))K(0,z) + V(z)G_v(0,z) + P(0,z)(1-\mathcal{N}^*(\lambda))$$
$$+U(0,z)\mathcal{S}_f^*(I(z)) \tag{37}$$

where $V(z) = \frac{\omega}{I(z)+\omega}(1 - \mathcal{W}_v^*(I(z)+\omega))$ using equations 16 and 1 :

$$G_v(0,z) = \frac{\lambda^2}{\omega}P_{00} \tag{38}$$

using equation 18 :

$$K(0,z) = \lambda P_{00} \tag{39}$$

using equation 31 in 35 :

$$U(0,z) = \overline{\alpha}\mathcal{T}_s^*(I(z))K(0,z) \tag{40}$$

using equation 40 in 41 :

$$U(0,z) = \overline{\alpha}\lambda P_{00}\mathcal{T}_s^*(I(z)) \tag{41}$$

using equations 39 40 and 42 in 37 and 38 we get

$$P(0,z) = \frac{N_r(z)}{D_r(z)} \tag{42}$$

Where $N_r(z) = \lambda P_{00} z\mathcal{G}^*(I(z))\mathcal{T}_s^*(I(z))[(\alpha-1)\omega\mathcal{S}_f^*(I(z)) - \alpha\omega] + \omega$
$-\lambda\mathcal{G}^*(I(z))V(z) - \lambda(\mathcal{W}_v^*(I(z)+\omega) - 1)$
$D_r(z) = \omega\mathcal{G}^*(I(z))[z(1-\mathcal{N}^*(\lambda)) + \mathcal{N}^*(\lambda)] - z$

$$M_b(0,z) = \frac{-\lambda P_{00}}{D_r(z)} \left\{ \begin{array}{c} \left(\omega z\left(-\alpha\mathcal{S}_f^*(I(z)) + \alpha + \mathcal{S}_f^*(I(z))\right)\mathcal{T}_s^*(I(z))\right) \\ +\lambda z V(z) + z\left(\mathcal{N}^*(\lambda) - 1\right)\left(\omega - \lambda\mathcal{W}_v^*(I(z)+\omega) + \lambda\right) \\ -\left(\omega - \lambda\mathcal{W}_v^*(I(z)+\omega) + \lambda\right)\mathcal{N}^*(\lambda) \end{array} \right\} \tag{43}$$

Substituting equations (39-44) in 27-31 .

4. Steady state results

If the system is in steady state condition $\rho < \mathcal{N}^*(\lambda)$, the PGFs are as follows:
(I) the number of customers in the orbit when the server is idle;

$$P(z) = \int_0^\infty P(x,z)dx = \frac{P(0,z)(1-\mathcal{N}^*(\lambda))}{\lambda} \tag{44}$$

(II) the number of customers in the orbit when the server is regularly busy;

$$M_b(z) = \int_0^\infty M_b(x,z)dx = \frac{M_b(0,z)(1-\mathcal{G}^*(I(z))}{I(z)} \tag{45}$$

(III) the number of customers in the orbit when the server is at a lower speed service;

$$G_v(z) = \int_0^\infty G_v(x,z)dx = \frac{G_v(0,z)(1-\mathcal{W}_v^*(I(z)+\omega)}{I(z)+\omega} \tag{46}$$

(IV) the number of customers in the orbit when the server is at repair time;

$$U(z) = \int_0^\infty U(x,z)dx = \frac{U(0,z)(1-\mathcal{S}_f^*(I(z))}{I(z)} \tag{47}$$

(V) the number of customers in the orbit when the server is at setup time;

$$K(z) = \int_0^\infty K(x,z)dx = \frac{K(0,z)(1-\mathcal{T}_s^*(I(z))}{I(z)} \tag{48}$$

From the above equations, the only unknown is P_{00} which can be obtained by using the normalization condition $P_{00} + Q_0 + P(1) + M_b(1) + G_v(1) + U(1) + K(1) = 1$ as

$$P_{00} = \frac{\omega^2(A^*(\lambda) - \lambda E(G))}{\left\{ \begin{array}{c} \omega^2\left(\lambda\left(E(S_f)(1-\alpha) + E(T_s)\right) + \mathcal{N}^*(\lambda)\right) \\ +\omega\left(\lambda^2\left\{2\mathcal{W}_v^{*'}(\omega)(\lambda E(N_b) - \mathcal{N}^*(\lambda) + 1) - E(N_b)\mathcal{W}_v^*(\omega)\right\}\right) \\ +\lambda\mathcal{N}^*(\lambda) + \lambda^2(1 - \mathcal{W}_v^*(\omega)) \end{array} \right\}} \tag{49}$$

Corollary 1. If the system satisfies the steady state condition, The PGF of the number of customers in the system $(K_s(z))$ is obtained using

$$K_s(z) = P_{00} + Q_0 + P(z) + zM_b(z) + zG_v(z) + U(z) + K(z). \tag{50}$$

The PGF of the number of customers in the orbit $(K_0(z))$ is obtained using

$$K_o(z) = P_{00} + Q_0 + P(z) + M_b(z) + G_v(z) + U(z) + K(z). \tag{51}$$

5. System performance measures

Our analysis is based on the following system characteristics of the retrial queueing system.

5.1. System state probabilities

1. The steady state probability that the server is idle during the retrial time is given by

$$\begin{aligned} I &= P(1) \\ &= \frac{-\lambda P_{00}\left(\mathcal{N}^*(\lambda)-1\right)}{\omega^2(\mathcal{N}^*(\lambda)-\lambda E\left(N_b\right))} \left\{ \begin{array}{c} \omega\left(-E(N_b)\lambda\left(\mathcal{W}_v^*(\omega)-1\right)+\omega\left(-\alpha E(S_f)+E(N_b)+E(T_s)\right)\right) \\ +\omega E(S_f)+\lambda\omega\mathcal{W}_v^{*'}(\omega)+\lambda\left(\omega\mathcal{W}_v^{*'}(\omega)-\mathcal{W}_v^*(\omega)+1\right) \end{array} \right\} \end{aligned}$$

2. The steady state probability that the server is busy on normal service period is given by

$$U = M_b(1) = \frac{\lambda P_{00}E(N_b)}{\omega^2(\mathcal{N}^*(\lambda)-\lambda E\left(N_b\right))} \left\{ \begin{array}{c} \omega^2\left(-\alpha E(S_f)\lambda+E(T_s)\lambda+E(S_f)\lambda+\mathcal{N}^*(\lambda)\right)\right) \\ +\omega\left(2\lambda^2\mathcal{W}_v^{*'}(\omega)-\lambda\mathcal{W}_v^*(\omega)\mathcal{N}^*(\lambda)+\lambda\mathcal{N}^*(\lambda)\right) \\ -\lambda^2\mathcal{W}_v^*(\omega)+\lambda^2 \end{array} \right\}$$

3. The steady state probability that the server is busy on working vacation period is given by

$$V = G_v(1) = \frac{\lambda^2 P_{00}\cdot\left(1-\mathcal{W}_v^*(\omega)\right)}{\omega^2}$$

4. The steady state probability that the server is under repair time is given by

$$S = U(1) = E(S_f)\lambda P_{00}\cdot(1-\alpha)$$

5.The steady state probability that the server is under setup time is given by

$$K = K(1) = E(T_s)\lambda P_{00}$$

5.2. Mean system size and orbit size

(i) The expected number of customers in the orbit $\left(L_q\right)$ is obtained by differentiating equation 52 with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z)$$

(ii) The expected number of customers in the system (L_s) is obtained by differentiating equation 51 with respect to z and evaluating at z $= 1$

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z)$$

(iii) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue $\left(W_q\right)$ are found using Little's formula

$$W_s = \frac{L_s}{\lambda} \quad \text{and} \quad W_q = \frac{L_q}{\lambda}.$$

6. Reliability Measures

In the retrial queueing system with unreliable server, the reliability measures provide the information, which is required for the improvement of the system.
(i) The steady state availability A_v, which is the probability that the server is either working for a positive customer or in an idle period such that the steady state availability of the server is given by

$$A_v = 1 - U(1) = 1 - E(S_f)\lambda P_{00} \cdot (1 - \alpha) \tag{52}$$

(ii) Let F_f be the steady state probability of server failure,

$$F_f = \alpha K(1) = \alpha E(T_s)\lambda P_{00}$$

Theorem 2. Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions, we have

$$E(T_b) = \frac{1}{\omega^2(A^*(\lambda) - \lambda E(N_b))} \left\{ \begin{array}{c} \omega^2\left(-\alpha E(S_f) + E(N_b) + E(T_s) + E(S_f)\right)\Big) \\ \omega\left(-E(N_b)\lambda \mathcal{W}_v^*(\omega) + 2\lambda \mathcal{W}_v^{*'}(\omega)\left(E(N_b)\lambda - \mathcal{N}^*(\lambda) + 1\right)\right. \\ \left. + \mathcal{N}^*(\lambda) - \lambda \mathcal{W}_v^*(\omega) + \lambda \right. \end{array} \right\} \tag{53}$$

$$E(T_c) = \frac{1}{\omega^2\lambda(\mathcal{N}^*(\lambda) - \lambda E(N_b))} \left\{ \begin{array}{c} \omega^2\left(\lambda\left(E(S_f)(1-\alpha) + E(T_s)\right) + \mathcal{N}^*(\lambda)\right)\Big) \\ +\omega\left(\lambda^2\left\{2\mathcal{W}_v^{*'}(\omega)\left(\lambda E(N_b) - \mathcal{N}^*(\lambda) + 1\right)\right\}\right) \\ -\lambda^2\omega E(N_b)\mathcal{W}_v^*(\omega) + \lambda\mathcal{N}^*(\lambda) + \lambda^2\left(1 - \mathcal{W}_v^*(\omega)\right) \end{array} \right\} \tag{54}$$

Proof. The result follows directly by applying the argument of an alternating renewal process which leads to

$$P_{00} = \frac{E(T_0)}{E(T_b) + E(T_0)}; E(T_b) = \frac{1}{\lambda}\left(\frac{1}{P_{00}} - 1\right) \text{ and } E(T_c) = E(T_0) + E(T_b) \tag{55}$$

where T_0 is the time length that the system in empty state. Since the inter-arrival time between two customers follows exponential distribution with parameter λ, we have that $E(T_0) = (1/\lambda)$. Inserting equation 50 into 56 and by direct calculations, we can get 54 and 55.

7. Cost model

In practice, queue managers aim to minimize the operating cost per unit time. In this section of the paper, we begin by formulating a steady-state expected cost function per unit time, where the service rate μ_n is the decision variable. Our objective is to find the optimal value of μ_n to minimize the expected cost function. To reach this, We need to define cost elements as follows:
- $\mathcal{C}_1$: is the cost of each consomer in the system per unit of time.
- $\mathcal{C}_2$: represents the cost per unit time to leave the server functioning
- $\mathcal{C}_3$: Cost per service per unit time.
- $\mathcal{C}_4$: represents the cost per unit time needed to prepar starting up the server.
Let - $\mathcal{T}_c$ be the total expected cost per unit time of the system:

$$\mathcal{T}_c = \mathcal{C}_1 L_s + \mathcal{C}_2(1 - P_{00}) + \mathcal{C}_3\mu_n + \mathcal{C}_4 P_{00}.$$

7.1. Quadratic Fit Search Method

This part considers the cost optimization problem under a given cost structure via quadratic fit search method (QFSM), This method uses a 3-point pattern to fit a quadratic function that ensures a unique optimal solution., see [22] . So, We aim to optimize the service rate μ_n in various cases to minimize the expected cost function $\mathcal{T}_c$ denoted in this part by H. Assume that all system

parameters have fixed values, and the only controlled parameter is the service rate μ_n. Thus, The optimization problem can be mathematically expressed as:

$$\text{Minimize : } H(\mu_n) = \mathcal{C}_1 L_s + \mathcal{C}_2(1 - P_{00}) + \mathcal{C}_3\mu_n + \mathcal{C}_4 P_{00}.$$

As it has been mentioned in Laxmi et al [23], given a 3-point pattern, we may fit a quadratic function via corresponding functional values that has a unique minimum, x^q, for the given objective function $H(x)$. the quadratic fit improves the current 3-point pattern by replacing one of its points with the optimum x^q , using this approximation. The unique optimum x^q of the quadratic function agreeing with $H(x)$ at 3-point operation $\left(x^l, x^m, x^u\right)$ is given as

$$x^q \cong \frac{1}{2}\left[\frac{H\left(x^l\right)\left((x^m)^2 - (x^u)^2\right) + H\left(x^m\right)\left((x^u)^2 - \left(x^l\right)^2\right) + H\left(x^u\right)\left(\left(x^l\right)^2 - (x^m)^2\right)}{H\left(x^l\right)\left(x^m - x^u\right) + H\left(x^m\right)\left(x^u - x^l\right) + H\left(x^u\right)\left(x^l - x^m\right)}\right].$$

For the whole analysis in this numerical part, we fixe $\mathcal{C}_1 = 10, \mathcal{C}_2 = 350, \mathcal{C}_3 = 20, \mathcal{C}_4 = 120,$.

7.2. Optimization analysis

To conduct the numerical analysis for parameter optimization in the queueing system under consideration, we use the following default values for the parameters: $\alpha = 0.7, \lambda = 2, \delta = 8, \mu_w = 2, s = 5, r = 4$ and $\xi = 8$, and the tolerance of QFSM is $\epsilon = 10^{-6}$.

From **Figure** 1, The curves clearly show convexity, which means , the existence of a specific service rate μ_n that minimizes the total expected cost function for the given set of model parameters. By adopting QFSM and choosing the initial 3point pattern as $\left(\mu_n^l, \mu_n^m, \mu_n^u\right) = (3.05, 3.5, 3.75)$, and after finite iterations, we observe that the minimum expected operating cost per unit time converges to the solution $H = 372.29$ at $\mu_n^* = 3.282$,

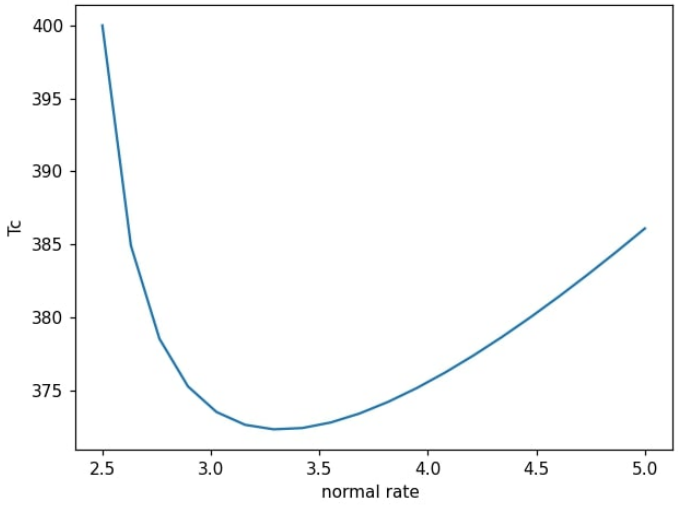

Figure 1: *The optimum service rate μ_n^**

Moreover, we examine the behavior of the expected cost function under different values of the cost parameters. System parameters are fixed as follows: $\alpha = 0.7, \lambda = 2, \delta = 8, \mu_v = 2, s = 5, r = 4$ and $\xi = 8$; Tables 1-3 illustrate the effects of $(\mathcal{C}_1, \mathcal{C}_2)$, $(\mathcal{C}_2, \mathcal{C}_4)$ and $(\mathcal{C}_4, \mathcal{C}_3)$ on the expected cost function, respectively. It can be see that the expected cost function shows a linearly increasing trend with increasing cost parameters.

Table 1: *Effects of* $(\mathcal{C}_4, \mathcal{C}_3)$ *on the expected cost function* $\mathcal{T}_c$ *with* $\mathcal{C}_1 = 10$ *and* $\mathcal{C}_2 = 350$

$(\mathcal{C}_4, \mathcal{C}_3)$	(100,10)	(200,10)	(200,15)	(120,5)	(120,20)
$\mathcal{T}_c$	272.4873	313.8563	323.8563	270.7611	300.7611

Table 2: *Effects of* $(\mathcal{C}_2, \mathcal{C}_4)$ *on the expected cost function* $\mathcal{T}_c$

$(\mathcal{C}_2, \mathcal{C}_4)$	(350,150)	(350,200)	(250,200)	(150,120)	(100,120)
$\mathcal{T}_c$	293.1718	313.8563	255.2253	163.4991	134.1836

Table 3: *Effects of* $(\mathcal{C}_4, \mathcal{C}_3)$ *on the expected cost function* $\mathcal{T}_c$

$(\mathcal{C}_4, \mathcal{C}_3)$	(100,10)	(200,10)	(200,15)	(120,5)	(120,20)
$\mathcal{T}_c$	272.4873	313.8563	323.8563	270.7611	300.7611

8. Sensitivity Analysis and Numerical Examples

In this section, we provide numerical examples using Python to illustrate how different parameters affect system performance measures. We assume that retrial times, service times, lower-speed service times, vacation times, setup times, and repair times all follow exponential distributions. The parameter values are chosen to satisfy the system's stability conditions. The following tables present computed values for various model characteristics, such as the probability that the server is idle (P_{00}), the mean orbit size (L_q), probability that server in working vacation , probability that server in setup time, and probability that server in repair time . The exponential distribution is $k(x) = ve^{-vx}, x > 0$.

In **Figure** 2 , we examine the behavior of the idle probability (P_{00}) increases for increasing the value of the lower service rate (μ_n) and regular service rate (μ_w).

In **Figure** 3 , we examine the behavior of the mean orbit size (L_q) décreases for increasing the value of lower speed service rate μ_w and retrial rate ξ.

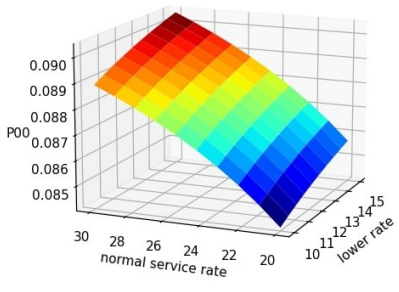

Figure 2: *Variation in* P_{00} *with* μ_n *and* μ_w

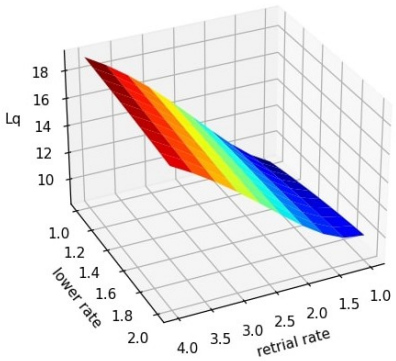

Figure 3: (L_q) *versus* ξ *and* μ_w

In **Figure** 4 , we see that the behavior of the mean orbit size (L_q) decreases as the values of the lower service rate μ_w and regular service rate μ_n increase.

In **Figure** 5 , we examine the behavior of the idle probability (P_{00}) increases with an increase in the setup rate, for a fixed value of repair rate.

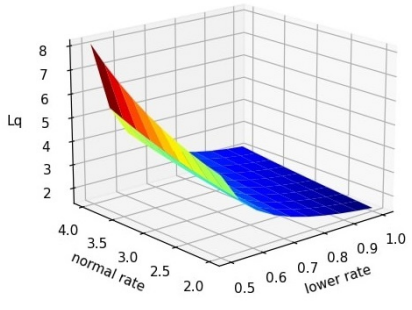

Figure 4: (L_q) *versus* μ_n *and* μ_w

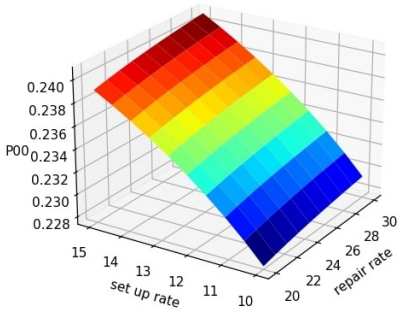

Figure 5: P_{00} *versus setup rate and repair rate*

Figure 6 depicts that with an increase in repair rate, the probability of the server being in repair state decreases.
Figure 7 depicts that with an increase in service rate μ_n , the probability of the server being in setup state increases ;this is due to faster activation of server with reduced setup time.

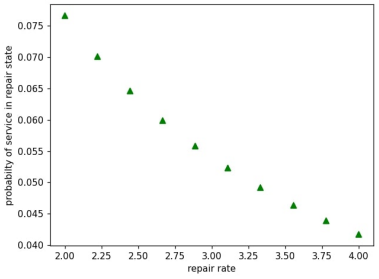

Figure 6: *Effect of repair rate on repair state probability of server*

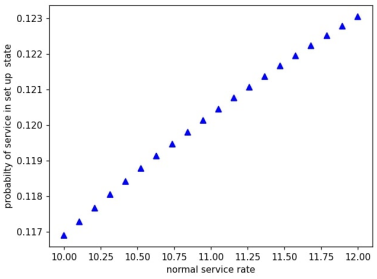

Figure 7: *Probability of server in set up versus* μ_n

We observe from **Figure** 8 that the probability of the server being in vacation state decreases with an increase in the rate of working vacation. The reason behind the observation is a decrease in the duration of vacation with an increase in the vacation rate.

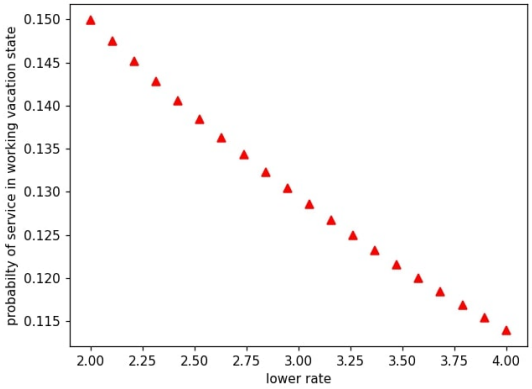

Figure 8: *Probability of server in vacation versus vacation rate* μ_w

9. Conclusion

In this article, we analyzed an Unreliable single-server retrial queue model with general retrial time, working vacation and setup time. if certain required and sufficient conditions are satisfied the system can be stabilized .Using he supplementary variable approach and the Probability Generating Function (PGF) approach to determine The PGF of the no. of clients in the system and its orbit . The performance of the model is illustrated using **PYTHON** . The operating cost of the queuing system is optimized by adjusting the service rate of the server.

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ESTIMATING THE POPULATION MEAN USING STRATIFIED DOUBLE UNIFIED RANKED SET SAMPLING FOR ASYMMETRIC DISTRIBUTIONS

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Abstract

In this study, we propose a novel sampling technique known as Stratified Unified Ranked Set Sampling (SDURSS) and evaluate its efficiency for estimating population means. SDURSS is designed to enhance the estimation accuracy by integrating concepts from ranked set sampling with stratified sampling. Our results demonstrate that the SDURSS estimator generally exhibits superior efficiency compared to SRS, particularly in complex distribution scenarios. While SDURSS often performs more efficiently than SSRS and SRSS, its performance relative to these methods varies depending on the specific distribution and sample size. In several cases, SDURSS outperforms SSRS and SRSS, highlighting its potential benefits in practical applications. The findings suggest that SDURSS is a promising alternative to traditional sampling methods, offering improved efficiency and potentially more accurate estimates of population means. This research underscores the value of exploring advanced sampling techniques to enhance statistical estimation, particularly in scenarios involving asymmetric distributions where traditional methods may be less effective.

Keywords: Simple random sampling, Ranked set sampling, Median Ranked Set Sampling, Unified ranked set sampling, Double Unified ranked set sampling, Stratified Double Unified ranked set sampling

I. Introduction

The ranked set sampling (RSS) method to estimate the population mean of average yields prosed by [1]. Later, RSS was developed and modified by many authors to estimate the population parameters. The mathematical proof for RSS. They proved that the sample mean based on RSS is an unbiased estimator of the population mean, which gave smaller variance than the sample mean based on a simple random sample (SRS) with the same sample size provided [2]. The variance of the sample mean based on RSS is less than or equal to that of the SRS, whether or not there are

errors in ranking demonstrated by [3]. The RSS method of stratified ranked set sampling (SRSS) suggested by [4]. Some nonparametric tests for assessing the assumption of perfect ranking in RSS and powerful rank tests for perfect rankings proposed by [5-9], a Unified ranked sampling (URSS) suggested by [9-10]. RSS is called Double ranked set sampling (DRSS) developed by [1]. The RSS method is efficiency increasing the number of set and the number of cycles.

This study aims to propose the Stratified Double Unified Ranked Set Sampling (SDURSS) for estimating the population mean of asymmetric distributions and to study the efficiency of the empirical mean estimator based on SDURSS. Estimators in literature.

II. Materials and methods

I. Simple Random Sampling (SRS)

SRS is a method of selecting units out of units such that every one of the distinct samples has an equal chance of being drawn.

II. Stratified Sampling

In the stratified sampling method, the population of units is divided into non-overlapping sub-groups known as strata each stratum has units, respectively, such that For full benefit from stratification, the size of the hth strata, denoted by for, must be known. Then the samples are drawn independently from each stratum, producing sample sizes denoted by , such that the total sample size is If a simple random sample is taken from each stratum, the whole procedure is known as a stratified simple random sampling (SSRS).

III. Ranked Set Sampling (RSS)

RSS technique can be described as follows:
Step 1: Select samples of the SRS method from the population of interest.
Step 2: Allocate the selected units as randomly as possible into sets, each of size.
Step 3: Rank the units in each set with respect to the variable of interest.
Step 4: Choose a sample by taking the smallest ranked unit in the first set, the second smallest ranked unit in the second set, continue the process until the largest ranked unit is selected from the last set. Then the taken samples are measured the variable of interest.
Step 5: Repeat step 1 through step 4 for cycles to draw the RSS sample of size. [12]

Example 1: Let $(m=7, r=1)$ be

$$\begin{bmatrix} X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17} \\ X_{21}, X_{22}, X_{23}, X_{24}, X_{25}, X_{26}, X_{27} \\ X_{31}, X_{32}, X_{33}, X_{34}, X_{35}, X_{36}, X_{37} \\ X_{41}, X_{42}, X_{43}, X_{44}, X_{45}, X_{46}, X_{47} \\ X_{51}, X_{52}, X_{53}, X_{54}, X_{55}, X_{56}, X_{57} \\ X_{61}, X_{62}, X_{63}, X_{64}, X_{65}, X_{66}, X_{67} \\ X_{71}, X_{72}, X_{73}, X_{74}, X_{75}, X_{76}, X_{77} \end{bmatrix}$$

Then the measured RSS units are $X_{11}, X_{22}, X_{33}, X_{44}, X_{55}, X_{66}, X_{77}$
The empirical mean estimator of RSS is given by

$$\bar{X}_{RSS} = \frac{1}{mr} \sum_{j=1}^{r} \sum_{i=1}^{m} X_{(i:m)j}$$

and variance can be estimated by

$$Var\left(\bar{X}_{RSS}\right)=\frac{\sigma^2}{mr}-\frac{1}{m^2 r}\sum_{i=1}^{m}\left(\mu_{(i:m)j}-\mu\right)^2.$$

Also

$$\mu=\frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m}\mu_{(i:m)j}$$

Considerations:

1. Note: $m=$ set size, $r=$number of cycles)times(, $n=$sample of size(
2. The RSS use for Population infinite.

IV. Unified Ranked Set Sampling (URSS)

URSS technique can be described as follows:

Step 1: Use a SRS method to select m^2 units from the population of interest and rank them with respect to the variable of interest.

Step 2: Select the sample units for measurement as follow If m is an <u>odd</u> number, the ranked $\left(\frac{m+1}{2}+(i-1)m\right)$ units will be selected for $i=1,2,\cdots,m$. On the other hands, if m is an even number, divide the sample unit into 2 sections from size of the sample unit m^2. Where sections 1 select $\left(\frac{m}{2}+(i-1)m\right)$ units and sections 2 select $\left(\left(\frac{m}{2}+1\right)+(i-1)m\right)$units will be selected, for $i=1,2,\cdots,m$

Step 3: Repeat steps 1 and 2 for r cycles) for $j=1,2,\cdots,r$ (to draw the URSS of size $n=mr$

Define $x_{[i]j}$ be the URSS sampled unit of the i^{th} rank from the j^{th} cycle, where $i=1,2,\cdots,m$ and $j=1,2,\cdots,r$. [9-10]

)even number(

Example 2: Consider the case of $(m=6,r=1)$. Draw a simple random sample of size $m^2=6^2=36$ units as

$X_{11},X_{12},X_{13},X_{14},X_{15},X_{16},X_{17},X_{18},X_{19},X_{10},X_{11},X_{12},X_{13},X_{14},X_{15},X_{16},X_{17},X_{18},$
$X_{19},X_{20},X_{21},X_{22},X_{23},X_{24},X_{25},X_{26},X_{27},X_{28},X_{29},X_{30},X_{31},X_{32},X_{33},X_{34},X_{35},X_{36}$

From the size of the sample unit m^2, divide the sample unit into 2 sections

Where section 1, We select the sample unit from $\left(\frac{m}{2}+(i-1)m\right)$ for $i=1,2,\cdots,m$

$X_{11},X_{12},X_{13},X_{14},\boxed{X_{15}},X_{16},X_{17},X_{18},X_{19},X_{10},\boxed{X_{11}},X_{12},X_{13},X_{14},X_{15},X_{16},\boxed{X_{17}},X_{18}$

Where section 2, We select the sample unit from $\left(\left(\frac{m}{2}+1\right)+(i-1)m\right)$ for $i=1,2,\cdots,m$

$X_{19},X_{20},X_{21},X_{22},X_{23},\boxed{X_{24}},X_{25},X_{26},X_{27},X_{28},X_{29},\boxed{X_{30}},X_{31},X_{32},X_{33},X_{34},X_{35},\boxed{X_{36}}$

Let $X_{13},X_{19},X_{15},X_{22},X_{28},X_{34}$ is DURSS of size 6

)odd number(

Example 3: Consider the case of $(m=7,r=1)$. Draw a simple random sample of size $m^2=7^2=49$ units as

$X_{11},X_{12},X_{13},X_{14},X_{15},X_{16},X_{17},X_{18},X_{19},X_{10},X_{11},X_{12},X_{13},X_{14},X_{15},X_{16},X_{17},X_{18},$
$X_{19},X_{20},X_{21},X_{22},X_{23},X_{24},X_{25},X_{26},X_{27},X_{28},X_{29},X_{30},X_{31},X_{32},X_{33},X_{34},X_{35},$
$X_{36},X_{37},X_{38},X_{39},X_{40},X_{41},X_{42},X_{43},X_{44},X_{45},X_{46},X_{47},X_{48},X_{49}$

We select the sample unit from $\left(\frac{m+1}{2}+(i-1)m\right)$ for $i=1,2,\cdots,m$

$X_{11},X_{12},X_{13},X_{14},X_{15},X_{16},X_{17},X_{18},X_{19},X_{10},X_{11},X_{12},X_{13},X_{14},X_{15},X_{16},X_{17},X_{18},$
$X_{19},X_{20},X_{21},X_{22},X_{23},X_{24},X_{25},X_{26},X_{27},X_{28},X_{29},X_{30},X_{31},X_{32},X_{33},X_{34},X_{35},$
$X_{36},X_{37},X_{38},X_{39},X_{40},X_{41},X_{42},X_{43},X_{44},X_{45},X_{46},X_{47},X_{48},X_{49}$

Let $X_{14},X_{11},X_{18},X_{25},X_{32},X_{39},X_{46}$ is URSS of size 7

V. Double Unified Ranked Set Sampling (DURSS)

In research, the DURSS method is applied from the [11] follows:
Step 1: Use a SRS method to identify m^3 elements from the target population and divide these elements randomly into m sets each of size m^2 elements.
Step 2: Use the usual URSS procedure on each set to obtain m ranked set samples of size m each.
Step 3: Apply the URSS procedure again on step)2(to obtain a DURSS of size m.
The procedure is illustrated for the case of even and odd in the following example.

)even number(

Example 4: Consider the case of $(m=6,r=1)$. Draw a simple random sample of size $m^3=6^3=216$ elements)6 sets of size 36 each(. Assume the elements are

$$X_{(1)}^{(1)},X_{(2)}^{(1)},X_{(3)}^{(1)},\cdots,X_{(36)}^{(1)},X_{(1)}^{(2)},X_{(2)}^{(2)},X_{(3)}^{(2)},\cdots,X_{(36)}^{(2)},X_{(1)}^{(3)},X_{(2)}^{(3)},X_{(3)}^{(3)},\cdots,X_{(36)}^{(3)},$$
$$X_{(1)}^{(4)},X_{(2)}^{(4)},X_{(3)}^{(4)},\cdots,X_{(36)}^{(4)},X_{(1)}^{(5)},X_{(2)}^{(5)},X_{(3)}^{(5)},\cdots,X_{(36)}^{(5)},X_{(1)}^{(6)},X_{(2)}^{(6)},X_{(3)}^{(6)},\cdots,X_{(36)}^{(6)}$$

After ranking the elements of each set obtain 6 ranked set samples of size 6 each $(m^2=6^2=36)$.

$$\begin{bmatrix} X_{(1)}^{(1)},X_{(2)}^{(1)},X_{(3)}^{(1)},X_{(4)}^{(1)},X_{(5)}^{(1)},X_{(6)}^{(1)},X_{(7)}^{(1)},X_{(8)}^{(1)},X_{(9)}^{(1)},X_{(10)}^{(1)},X_{(11)}^{(1)},X_{(12)}^{(1)},X_{(13)}^{(1)},X_{(14)}^{(1)},X_{(15)}^{(1)},X_{(16)}^{(1)},X_{(17)}^{(1)},X_{(18)}^{(1)}, \\ X_{(19)}^{(1)},X_{(20)}^{(1)},X_{(21)}^{(1)},X_{(22)}^{(1)},X_{(23)}^{(1)},X_{(24)}^{(1)},X_{(25)}^{(1)},X_{(26)}^{(1)},X_{(27)}^{(1)},X_{(28)}^{(1)},X_{(29)}^{(1)},X_{(30)}^{(1)},X_{(31)}^{(1)},X_{(32)}^{(1)},X_{(33)}^{(1)},X_{(34)}^{(1)},X_{(35)}^{(1)},X_{(36)}^{(1)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(2)},X_{(2)}^{(2)},X_{(3)}^{(2)},X_{(4)}^{(2)},X_{(5)}^{(2)},X_{(6)}^{(2)},X_{(7)}^{(2)},X_{(8)}^{(2)},X_{(9)}^{(2)},X_{(10)}^{(2)},X_{(11)}^{(2)},X_{(12)}^{(2)},X_{(13)}^{(2)},X_{(14)}^{(2)},X_{(15)}^{(2)},X_{(16)}^{(2)},X_{(17)}^{(2)},X_{(18)}^{(2)}, \\ X_{(19)}^{(2)},X_{(20)}^{(2)},X_{(21)}^{(2)},X_{(22)}^{(2)},X_{(23)}^{(2)},X_{(24)}^{(2)},X_{(25)}^{(2)},X_{(26)}^{(2)},X_{(27)}^{(2)},X_{(28)}^{(2)},X_{(29)}^{(2)},X_{(30)}^{(2)},X_{(31)}^{(2)},X_{(32)}^{(2)},X_{(33)}^{(2)},X_{(34)}^{(2)},X_{(35)}^{(2)},X_{(36)}^{(2)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(3)},X_{(2)}^{(3)},X_{(3)}^{(3)},X_{(4)}^{(3)},X_{(5)}^{(1)},X_{(6)}^{(3)},X_{(7)}^{(3)},X_{(8)}^{(3)},X_{(9)}^{(3)},X_{(10)}^{(3)},X_{(11)}^{(3)},X_{(12)}^{(3)},X_{(13)}^{(3)},X_{(14)}^{(3)},X_{(15)}^{(3)},X_{(16)}^{(3)},X_{(17)}^{(3)},X_{(18)}^{(3)}, \\ X_{(19)}^{(3)},X_{(20)}^{(3)},X_{(21)}^{(3)},X_{(22)}^{(3)},X_{(23)}^{(3)},X_{(24)}^{(3)},X_{(25)}^{(3)},X_{(26)}^{(3)},X_{(27)}^{(3)},X_{(28)}^{(3)},X_{(29)}^{(3)},X_{(30)}^{(3)},X_{(31)}^{(3)},X_{(32)}^{(3)},X_{(33)}^{(3)},X_{(34)}^{(3)},X_{(35)}^{(3)},X_{(36)}^{(3)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(4)},X_{(2)}^{(4)},X_{(3)}^{(4)},X_{(4)}^{(4)},X_{(5)}^{(4)},X_{(6)}^{(4)},X_{(7)}^{(4)},X_{(8)}^{(4)},X_{(9)}^{(4)},X_{(10)}^{(4)},X_{(11)}^{(4)},X_{(12)}^{(4)},X_{(13)}^{(4)},X_{(14)}^{(4)},X_{(15)}^{(4)},X_{(16)}^{(4)},X_{(17)}^{(4)},X_{(18)}^{(4)}, \\ X_{(19)}^{(4)},X_{(20)}^{(4)},X_{(21)}^{(4)},X_{(22)}^{(4)},X_{(23)}^{(4)},X_{(24)}^{(4)},X_{(25)}^{(4)},X_{(26)}^{(4)},X_{(27)}^{(4)},X_{(28)}^{(4)},X_{(29)}^{(4)},X_{(30)}^{(4)},X_{(31)}^{(4)},X_{(32)}^{(4)},X_{(33)}^{(4)},X_{(34)}^{(4)},X_{(35)}^{(4)},X_{(36)}^{(4)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(5)},X_{(2)}^{(5)},X_{(3)}^{(5)},X_{(4)}^{(5)},X_{(5)}^{(5)},X_{(6)}^{(5)},X_{(7)}^{(5)},X_{(8)}^{(5)},X_{(9)}^{(5)},X_{(10)}^{(5)},X_{(11)}^{(5)},X_{(12)}^{(5)},X_{(13)}^{(5)},X_{(14)}^{(5)},X_{(15)}^{(5)},X_{(16)}^{(5)},X_{(17)}^{(5)},X_{(18)}^{(5)}, \\ X_{(19)}^{(5)},X_{(20)}^{(5)},X_{(21)}^{(5)},X_{(22)}^{(5)},X_{(23)}^{(5)},X_{(124)}^{(5)},X_{(25)}^{(5)},X_{(26)}^{(5)},X_{(27)}^{(5)},X_{(28)}^{(5)},X_{(29)}^{(5)},X_{(30)}^{(5)},X_{(31)}^{(5)},X_{(32)}^{(5)},X_{(33)}^{(5)},X_{(34)}^{(5)},X_{(35)}^{(5)},X_{(36)}^{(5)} \end{bmatrix}$$

and

$$\begin{bmatrix} X_{(1)}^{(6)},X_{(2)}^{(6)},X_{(3)}^{(6)},X_{(4)}^{(6)},X_{(5)}^{(6)},X_{(6)}^{(6)},X_{(7)}^{(6)},X_{(8)}^{(6)},X_{(9)}^{(6)},X_{(10)}^{(6)},X_{(11)}^{(6)},X_{(12)}^{(6)},X_{(13)}^{(6)},X_{(14)}^{(6)},X_{(15)}^{(6)},X_{(16)}^{(6)},X_{(17)}^{(6)},X_{(18)}^{(6)}, \\ X_{(19)}^{(6)},X_{(20)}^{(6)},X_{(21)}^{(6)},X_{(22)}^{(6)},X_{(23)}^{(6)},X_{(24)}^{(6)},X_{(25)}^{(6)},X_{(26)}^{(6)},X_{(27)}^{(6)},X_{(28)}^{(6)},X_{(29)}^{(6)},X_{(30)}^{(6)},X_{(31)}^{(6)},X_{(32)}^{(6)},X_{(33)}^{(6)},X_{(34)}^{(6)},X_{(35)}^{(6)},X_{(36)}^{(6)} \end{bmatrix}$$

We select the sample unit from the elements of each set obtain 6 ranked set samples of size 6 each as

$$\begin{bmatrix} X_{(1)}^{(1)}, X_{(2)}^{(1)}, \boxed{X_{(3)}^{(1)}}, X_{(4)}^{(1)}, X_{(5)}^{(1)}, X_{(6)}^{(1)}, X_{(7)}^{(1)}, X_{(8)}^{(1)}, \boxed{X_{(9)}^{(1)}}, X_{(10)}^{(1)}, X_{(11)}^{(1)}, X_{(12)}^{(1)}, X_{(13)}^{(1)}, X_{(14)}^{(1)}, \boxed{X_{(15)}^{(1)}}, X_{(16)}^{(1)}, X_{(17)}^{(1)}, X_{(18)}^{(1)}, \\ X_{(19)}^{(1)}, X_{(20)}^{(1)}, X_{(21)}^{(1)}, \boxed{X_{(22)}^{(1)}}, X_{(23)}^{(1)}, X_{(24)}^{(1)}, X_{(25)}^{(1)}, X_{(26)}^{(1)}, X_{(27)}^{(1)}, \boxed{X_{(28)}^{(1)}}, X_{(29)}^{(1)}, X_{(30)}^{(1)}, X_{(31)}^{(1)}, X_{(32)}^{(1)}, X_{(33)}^{(1)}, \boxed{X_{(34)}^{(1)}}, X_{(35)}^{(1)}, X_{(36)}^{(1)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(2)}, X_{(2)}^{(2)}, \boxed{X_{(3)}^{(2)}}, X_{(4)}^{(2)}, X_{(5)}^{(2)}, X_{(6)}^{(2)}, X_{(7)}^{(2)}, X_{(8)}^{(2)}, \boxed{X_{(9)}^{(2)}}, X_{(10)}^{(2)}, X_{(11)}^{(2)}, X_{(12)}^{(2)}, X_{(13)}^{(2)}, X_{(14)}^{(2)}, \boxed{X_{(15)}^{(2)}}, X_{(16)}^{(2)}, X_{(17)}^{(2)}, X_{(18)}^{(2)}, \\ X_{(19)}^{(2)}, X_{(20)}^{(2)}, X_{(21)}^{(2)}, \boxed{X_{(22)}^{(2)}}, X_{(23)}^{(2)}, X_{(24)}^{(2)}, X_{(25)}^{(2)}, X_{(26)}^{(2)}, X_{(27)}^{(2)}, \boxed{X_{(28)}^{(2)}}, X_{(29)}^{(2)}, X_{(30)}^{(2)}, X_{(31)}^{(2)}, X_{(32)}^{(2)}, X_{(33)}^{(2)}, \boxed{X_{(34)}^{(2)}}, X_{(35)}^{(2)}, X_{(36)}^{(2)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(3)}, X_{(2)}^{(3)}, \boxed{X_{(3)}^{(3)}}, X_{(4)}^{(3)}, X_{(5)}^{(1)}, X_{(6)}^{(3)}, X_{(7)}^{(3)}, X_{(8)}^{(3)}, \boxed{X_{(9)}^{(3)}}, X_{(10)}^{(3)}, X_{(11)}^{(3)}, X_{(12)}^{(3)}, X_{(13)}^{(3)}, X_{(14)}^{(3)}, \boxed{X_{(15)}^{(3)}}, X_{(16)}^{(3)}, X_{(17)}^{(3)}, X_{(18)}^{(3)}, \\ X_{(19)}^{(3)}, X_{(20)}^{(3)}, X_{(21)}^{(3)}, \boxed{X_{(22)}^{(3)}}, X_{(23)}^{(3)}, X_{(24)}^{(3)}, X_{(25)}^{(3)}, X_{(26)}^{(3)}, X_{(27)}^{(3)}, \boxed{X_{(28)}^{(3)}}, X_{(29)}^{(3)}, X_{(30)}^{(3)}, X_{(31)}^{(3)}, X_{(32)}^{(3)}, X_{(33)}^{(3)}, \boxed{X_{(34)}^{(3)}}, X_{(35)}^{(3)}, X_{(36)}^{(3)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(4)}, X_{(2)}^{(4)}, \boxed{X_{(3)}^{(4)}}, X_{(4)}^{(4)}, X_{(5)}^{(4)}, X_{(6)}^{(4)}, X_{(7)}^{(4)}, X_{(8)}^{(4)}, \boxed{X_{(9)}^{(4)}}, X_{(10)}^{(4)}, X_{(11)}^{(4)}, X_{(12)}^{(4)}, X_{(13)}^{(4)}, X_{(14)}^{(4)}, \boxed{X_{(15)}^{(4)}}, X_{(16)}^{(4)}, X_{(17)}^{(4)}, X_{(18)}^{(4)}, \\ X_{(19)}^{(4)}, X_{(20)}^{(4)}, X_{(21)}^{(4)}, \boxed{X_{(22)}^{(4)}}, X_{(23)}^{(4)}, X_{(24)}^{(4)}, X_{(25)}^{(4)}, X_{(26)}^{(4)}, X_{(27)}^{(4)}, \boxed{X_{(28)}^{(4)}}, X_{(29)}^{(4)}, X_{(30)}^{(4)}, X_{(31)}^{(4)}, X_{(32)}^{(4)}, X_{(33)}^{(4)}, \boxed{X_{(34)}^{(4)}}, X_{(35)}^{(4)}, X_{(36)}^{(4)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(5)}, X_{(2)}^{(5)}, \boxed{X_{(3)}^{(5)}}, X_{(4)}^{(5)}, X_{(5)}^{(5)}, X_{(6)}^{(5)}, X_{(7)}^{(5)}, X_{(8)}^{(5)}, \boxed{X_{(9)}^{(5)}}, X_{(10)}^{(5)}, X_{(11)}^{(5)}, X_{(12)}^{(5)}, X_{(13)}^{(5)}, X_{(14)}^{(5)}, \boxed{X_{(15)}^{(5)}}, X_{(16)}^{(5)}, X_{(17)}^{(5)}, X_{(18)}^{(5)}, \\ X_{(19)}^{(5)}, X_{(20)}^{(5)}, X_{(21)}^{(5)}, \boxed{X_{(22)}^{(5)}}, X_{(23)}^{(5)}, X_{(124)}^{(5)}, X_{(25)}^{(5)}, X_{(26)}^{(5)}, X_{(27)}^{(5)}, \boxed{X_{(28)}^{(5)}}, X_{(29)}^{(5)}, X_{(30)}^{(5)}, X_{(31)}^{(5)}, X_{(32)}^{(5)}, X_{(33)}^{(5)}, \boxed{X_{(34)}^{(5)}}, X_{(35)}^{(5)}, X_{(36)}^{(5)} \end{bmatrix}$$

and

$$\begin{bmatrix} X_{(1)}^{(6)}, X_{(2)}^{(6)}, \boxed{X_{(3)}^{(6)}}, X_{(4)}^{(6)}, X_{(5)}^{(6)}, X_{(6)}^{(6)}, X_{(7)}^{(6)}, X_{(8)}^{(6)}, \boxed{X_{(9)}^{(6)}}, X_{(10)}^{(6)}, X_{(11)}^{(6)}, X_{(12)}^{(6)}, X_{(13)}^{(6)}, X_{(14)}^{(6)}, \boxed{X_{(15)}^{(6)}}, X_{(16)}^{(6)}, X_{(17)}^{(6)}, X_{(18)}^{(6)}, \\ X_{(19)}^{(6)}, X_{(20)}^{(6)}, X_{(21)}^{(6)}, \boxed{X_{(22)}^{(6)}}, X_{(23)}^{(6)}, X_{(24)}^{(6)}, X_{(25)}^{(6)}, X_{(26)}^{(6)}, X_{(27)}^{(6)}, \boxed{X_{(28)}^{(6)}}, X_{(29)}^{(6)}, X_{(30)}^{(6)}, X_{(31)}^{(6)}, X_{(32)}^{(6)}, X_{(33)}^{(6)}, \boxed{X_{(34)}^{(6)}}, X_{(35)}^{(6)}, X_{(36)}^{(6)} \end{bmatrix}$$

So, we have 6 DURSS

$$\begin{aligned} &X_{(3)}^{(1)}, X_{(9)}^{(1)}, X_{(15)}^{(1)}, X_{(22)}^{(1)}, X_{(28)}^{(1)}, X_{(34)}^{(1)}, \\ &X_{(3)}^{(2)}, X_{(9)}^{(2)}, X_{(15)}^{(2)}, X_{(22)}^{(2)}, X_{(28)}^{(2)}, X_{(34)}^{(2)}, \\ &X_{(3)}^{(3)}, X_{(9)}^{(3)}, X_{(15)}^{(3)}, X_{(22)}^{(3)}, X_{(28)}^{(3)}, X_{(34)}^{(3)}, \\ &X_{(3)}^{(4)}, X_{(9)}^{(4)}, X_{(15)}^{(4)}, X_{(22)}^{(4)}, X_{(28)}^{(4)}, X_{(34)}^{(4)}, \\ &X_{(3)}^{(5)}, X_{(9)}^{(5)}, X_{(15)}^{(5)}, X_{(22)}^{(5)}, X_{(28)}^{(5)}, X_{(34)}^{(5)}, \\ &X_{(3)}^{(6)}, X_{(9)}^{(6)}, X_{(15)}^{(6)}, X_{(22)}^{(6)}, X_{(28)}^{(6)}, X_{(34)}^{(6)} \end{aligned}$$

We select the sample unit from 6 DURSS

$$\begin{aligned} &X_{(3)}^{(1)}, X_{(9)}^{(1)}, \boxed{X_{(15)}^{(1)}}, X_{(22)}^{(1)}, X_{(28)}^{(1)}, X_{(34)}^{(1)}, \\ &X_{(3)}^{(2)}, X_{(9)}^{(2)}, \boxed{X_{(15)}^{(2)}}, X_{(22)}^{(2)}, X_{(28)}^{(2)}, X_{(34)}^{(2)}, \\ &X_{(3)}^{(3)}, X_{(9)}^{(3)}, \boxed{X_{(15)}^{(3)}}, X_{(22)}^{(3)}, X_{(28)}^{(3)}, X_{(34)}^{(3)}, \\ &X_{(3)}^{(4)}, X_{(9)}^{(4)}, X_{(15)}^{(4)}, \boxed{X_{(22)}^{(4)}}, X_{(28)}^{(4)}, X_{(34)}^{(4)}, \\ &X_{(3)}^{(5)}, X_{(9)}^{(5)}, X_{(15)}^{(5)}, \boxed{X_{(22)}^{(5)}}, X_{(28)}^{(5)}, X_{(34)}^{(5)}, \\ &X_{(3)}^{(6)}, X_{(9)}^{(6)}, X_{(15)}^{(6)}, \boxed{X_{(22)}^{(6)}}, X_{(28)}^{(6)}, X_{(34)}^{(6)} \end{aligned}$$

Let $X_{(15)}^{(1)}, X_{(15)}^{(2)}, X_{(15)}^{(3)}, X_{(22)}^{(4)}, X_{(22)}^{(5)}, X_{(22)}^{(6)}$ is DURSS of size 6

)odd number(

Example 5: Consider the case of $(m=7, r=1)$. Draw a simple random sample of size $m^3 = 7^3 = 343$ elements)7 sets of size 49 each(. Assume the elements are

$$X_{(1)}^{(1)}, X_{(2)}^{(1)}, X_{(3)}^{(1)}, \cdots, X_{(49)}^{(1)}, X_{(1)}^{(2)}, X_{(2)}^{(2)}, X_{(3)}^{(2)}, \cdots, X_{(49)}^{(2)}, X_{(1)}^{(3)}, X_{(2)}^{(3)}, X_{(3)}^{(3)}, \cdots, X_{(49)}^{(3)}, X_{(1)}^{(4)}, X_{(2)}^{(4)}, X_{(3)}^{(4)}, \cdots, X_{(49)}^{(4)},$$
$$X_{(1)}^{(5)}, X_{(2)}^{(5)}, X_{(3)}^{(5)}, \cdots, X_{(49)}^{(5)}, X_{(1)}^{(6)}, X_{(2)}^{(6)}, X_{(3)}^{(6)}, \cdots, X_{(49)}^{(6)}, X_{(1)}^{(7)}, X_{(2)}^{(7)}, X_{(3)}^{(7)}, \cdots, X_{(49)}^{(7)}$$

After ranking the elements of each set obtain 7 ranked set samples of size 7 each) $m^2 = 7^2 = 49$ (.

$$\begin{bmatrix} X_{(1)}^{(1)}, X_{(2)}^{(1)}, X_{(3)}^{(1)}, X_{(4)}^{(1)}, X_{(5)}^{(1)}, X_{(6)}^{(1)}, X_{(7)}^{(1)}, X_{(8)}^{(1)}, X_{(9)}^{(1)}, X_{(10)}^{(1)}, X_{(11)}^{(1)}, X_{(12)}^{(1)}, X_{(13)}^{(1)}, X_{(14)}^{(1)}, X_{(15)}^{(1)}, X_{(16)}^{(1)}, X_{(17)}^{(1)}, X_{(18)}^{(1)}, \\ X_{(19)}^{(1)}, X_{(20)}^{(1)}, X_{(21)}^{(1)}, X_{(22)}^{(1)}, X_{(23)}^{(1)}, X_{(24)}^{(1)}, X_{(25)}^{(1)}, X_{(26)}^{(1)}, X_{(27)}^{(1)}, X_{(28)}^{(1)}, X_{(29)}^{(1)}, X_{(30)}^{(1)}, X_{(31)}^{(1)}, X_{(32)}^{(1)}, X_{(33)}^{(1)}, X_{(34)}^{(1)}, X_{(35)}^{(1)}, X_{(36)}^{(1)}, \\ X_{(37)}^{(1)}, X_{(38)}^{(1)}, X_{(39)}^{(1)}, X_{(40)}^{(1)}, X_{(41)}^{(1)}, X_{(42)}^{(1)}, X_{(43)}^{(1)}, X_{(44)}^{(1)}, X_{(45)}^{(1)}, X_{(46)}^{(1)}, X_{(47)}^{(1)}, X_{(48)}^{(1)}, X_{(49)}^{(1)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(2)}, X_{(2)}^{(2)}, X_{(3)}^{(2)}, X_{(4)}^{(2)}, X_{(5)}^{(2)}, X_{(6)}^{(2)}, X_{(7)}^{(2)}, X_{(8)}^{(2)}, X_{(9)}^{(2)}, X_{(10)}^{(2)}, X_{(11)}^{(2)}, X_{(12)}^{(2)}, X_{(13)}^{(2)}, X_{(14)}^{(2)}, X_{(15)}^{(2)}, X_{(16)}^{(2)}, X_{(17)}^{(2)}, X_{(18)}^{(2)}, \\ X_{(19)}^{(2)}, X_{(20)}^{(2)}, X_{(21)}^{(2)}, X_{(22)}^{(2)}, X_{(23)}^{(2)}, X_{(24)}^{(2)}, X_{(25)}^{(2)}, X_{(26)}^{(2)}, X_{(27)}^{(2)}, X_{(28)}^{(2)}, X_{(29)}^{(2)}, X_{(30)}^{(2)}, X_{(31)}^{(2)}, X_{(32)}^{(2)}, X_{(33)}^{(2)}, X_{(34)}^{(2)}, X_{(35)}^{(2)}, X_{(36)}^{(2)}, \\ X_{(37)}^{(2)}, X_{(38)}^{(2)}, X_{(39)}^{(2)}, X_{(40)}^{(2)}, X_{(41)}^{(2)}, X_{(42)}^{(2)}, X_{(43)}^{(2)}, X_{(44)}^{(2)}, X_{(45)}^{(2)}, X_{(46)}^{(2)}, X_{(47)}^{(2)}, X_{(48)}^{(2)}, X_{(49)}^{(2)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(3)}, X_{(2)}^{(3)}, X_{(3)}^{(3)}, X_{(4)}^{(3)}, X_{(5)}^{(1)}, X_{(6)}^{(3)}, X_{(7)}^{(3)}, X_{(8)}^{(3)}, X_{(9)}^{(3)}, X_{(10)}^{(3)}, X_{(11)}^{(3)}, X_{(12)}^{(3)}, X_{(13)}^{(3)}, X_{(14)}^{(3)}, X_{(15)}^{(3)}, X_{(16)}^{(3)}, X_{(17)}^{(3)}, X_{(18)}^{(3)}, \\ X_{(19)}^{(3)}, X_{(20)}^{(3)}, X_{(21)}^{(3)}, X_{(22)}^{(3)}, X_{(23)}^{(3)}, X_{(24)}^{(3)}, X_{(25)}^{(3)}, X_{(26)}^{(3)}, X_{(27)}^{(3)}, X_{(28)}^{(3)}, X_{(29)}^{(3)}, X_{(30)}^{(3)}, X_{(31)}^{(3)}, X_{(32)}^{(3)}, X_{(33)}^{(3)}, X_{(34)}^{(3)}, X_{(35)}^{(3)}, X_{(36)}^{(3)}, \\ X_{(37)}^{(3)}, X_{(38)}^{(3)}, X_{(39)}^{(3)}, X_{(40)}^{(3)}, X_{(41)}^{(3)}, X_{(42)}^{(3)}, X_{(43)}^{(3)}, X_{(44)}^{(3)}, X_{(45)}^{(3)}, X_{(46)}^{(3)}, X_{(47)}^{(3)}, X_{(48)}^{(3)}, X_{(49)}^{(3)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(4)}, X_{(2)}^{(4)}, X_{(3)}^{(4)}, X_{(4)}^{(4)}, X_{(5)}^{(4)}, X_{(6)}^{(4)}, X_{(7)}^{(4)}, X_{(8)}^{(4)}, X_{(9)}^{(4)}, X_{(10)}^{(4)}, X_{(11)}^{(4)}, X_{(12)}^{(4)}, X_{(13)}^{(4)}, X_{(14)}^{(4)}, X_{(15)}^{(4)}, X_{(16)}^{(4)}, X_{(17)}^{(4)}, X_{(18)}^{(4)}, \\ X_{(19)}^{(4)}, X_{(20)}^{(4)}, X_{(21)}^{(4)}, X_{(22)}^{(4)}, X_{(23)}^{(4)}, X_{(24)}^{(4)}, X_{(25)}^{(4)}, X_{(26)}^{(4)}, X_{(27)}^{(4)}, X_{(28)}^{(4)}, X_{(29)}^{(4)}, X_{(30)}^{(4)}, X_{(31)}^{(4)}, X_{(32)}^{(4)}, X_{(33)}^{(4)}, X_{(34)}^{(4)}, X_{(35)}^{(4)}, X_{(36)}^{(4)}, \\ X_{(37)}^{(4)}, X_{(38)}^{(4)}, X_{(39)}^{(4)}, X_{(40)}^{(4)}, X_{(41)}^{(4)}, X_{(42)}^{(4)}, X_{(43)}^{(4)}, X_{(44)}^{(4)}, X_{(45)}^{(4)}, X_{(46)}^{(4)}, X_{(47)}^{(4)}, X_{(48)}^{(4)}, X_{(49)}^{(4)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(5)}, X_{(2)}^{(5)}, X_{(3)}^{(5)}, X_{(4)}^{(5)}, X_{(5)}^{(5)}, X_{(6)}^{(5)}, X_{(7)}^{(5)}, X_{(8)}^{(5)}, X_{(9)}^{(5)}, X_{(10)}^{(5)}, X_{(11)}^{(5)}, X_{(12)}^{(5)}, X_{(13)}^{(5)}, X_{(14)}^{(5)}, X_{(15)}^{(5)}, X_{(16)}^{(5)}, X_{(17)}^{(5)}, X_{(18)}^{(5)}, \\ X_{(19)}^{(5)}, X_{(20)}^{(5)}, X_{(21)}^{(5)}, X_{(22)}^{(5)}, X_{(23)}^{(5)}, X_{(124)}^{(5)}, X_{(25)}^{(5)}, X_{(26)}^{(5)}, X_{(27)}^{(5)}, X_{(28)}^{(5)}, X_{(29)}^{(5)}, X_{(30)}^{(5)}, X_{(31)}^{(5)}, X_{(32)}^{(5)}, X_{(33)}^{(5)}, X_{(34)}^{(5)}, X_{(35)}^{(5)}, X_{(36)}^{(5)}, \\ X_{(37)}^{(5)}, X_{(38)}^{(5)}, X_{(39)}^{(5)}, X_{(40)}^{(5)}, X_{(41)}^{(5)}, X_{(42)}^{(5)}, X_{(43)}^{(5)}, X_{(44)}^{(5)}, X_{(45)}^{(5)}, X_{(46)}^{(5)}, X_{(47)}^{(5)}, X_{(48)}^{(5)}, X_{(49)}^{(5)} \end{bmatrix},$$

$$\begin{bmatrix} X_{(1)}^{(6)}, X_{(2)}^{(6)}, X_{(3)}^{(6)}, X_{(4)}^{(6)}, X_{(5)}^{(6)}, X_{(6)}^{(6)}, X_{(7)}^{(6)}, X_{(8)}^{(6)}, X_{(9)}^{(6)}, X_{(10)}^{(6)}, X_{(11)}^{(6)}, X_{(12)}^{(6)}, X_{(13)}^{(6)}, X_{(14)}^{(6)}, X_{(15)}^{(6)}, X_{(16)}^{(6)}, X_{(17)}^{(6)}, X_{(18)}^{(6)}, \\ X_{(19)}^{(6)}, X_{(20)}^{(6)}, X_{(21)}^{(6)}, X_{(22)}^{(6)}, X_{(23)}^{(6)}, X_{(24)}^{(6)}, X_{(25)}^{(6)}, X_{(26)}^{(6)}, X_{(27)}^{(6)}, X_{(28)}^{(6)}, X_{(29)}^{(6)}, X_{(30)}^{(6)}, X_{(31)}^{(6)}, X_{(32)}^{(6)}, X_{(33)}^{(6)}, X_{(34)}^{(6)}, X_{(35)}^{(6)}, X_{(36)}^{(6)}, \\ X_{(37)}^{(6)}, X_{(38)}^{(6)}, X_{(39)}^{(6)}, X_{(40)}^{(6)}, X_{(41)}^{(6)}, X_{(42)}^{(6)}, X_{(43)}^{(6)}, X_{(44)}^{(6)}, X_{(45)}^{(6)}, X_{(46)}^{(6)}, X_{(47)}^{(6)}, X_{(48)}^{(6)}, X_{(49)}^{(6)} \end{bmatrix}$$

and

$$\begin{bmatrix} X^{(7)}_{(1)}, X^{(7)}_{(2)}, X^{(7)}_{(3)}, X^{(7)}_{(4)}, X^{(7)}_{(5)}, X^{(7)}_{(6)}, X^{(7)}_{(7)}, X^{(7)}_{(8)}, X^{(7)}_{(9)}, X^{(7)}_{(10)}, X^{(7)}_{(11)}, X^{(7)}_{(12)}, X^{(7)}_{(13)}, X^{(7)}_{(14)}, X^{(7)}_{(15)}, X^{(7)}_{(16)}, X^{(7)}_{(17)}, X^{(7)}_{(18)}, \\ X^{(7)}_{(19)}, X^{(7)}_{(20)}, X^{(7)}_{(21)}, X^{(7)}_{(22)}, X^{(7)}_{(23)}, X^{(7)}_{(24)}, X^{(7)}_{(25)}, X^{(7)}_{(26)}, X^{(7)}_{(27)}, X^{(7)}_{(28)}, X^{(7)}_{(29)}, X^{(7)}_{(30)}, X^{(7)}_{(31)}, X^{(7)}_{(32)}, X^{(7)}_{(33)}, X^{(7)}_{(34)}, X^{(7)}_{(35)}, X^{(7)}_{(36)}, \\ X^{(7)}_{(37)}, X^{(7)}_{(38)}, X^{(7)}_{(39)}, X^{(7)}_{(40)}, X^{(7)}_{(41)}, X^{(7)}_{(42)}, X^{(7)}_{(43)}, X^{(7)}_{(44)}, X^{(7)}_{(45)}, X^{(7)}_{(46)}, X^{(7)}_{(47)}, X^{(7)}_{(48)}, X^{(7)}_{(49)} \end{bmatrix}$$

We select the sample unit from the elements of each set obtain 7 ranked set samples of size 7 each as

$$\begin{bmatrix} X^{(1)}_{(1)}, X^{(1)}_{(2)}, X^{(1)}_{(3)}, \boxed{X^{(1)}_{(4)}}, X^{(1)}_{(5)}, X^{(1)}_{(6)}, X^{(1)}_{(7)}, X^{(1)}_{(8)}, X^{(1)}_{(9)}, X^{(1)}_{(10)}, \boxed{X^{(1)}_{(11)}}, X^{(1)}_{(12)}, X^{(1)}_{(13)}, X^{(1)}_{(14)}, X^{(1)}_{(15)}, X^{(1)}_{(16)}, X^{(1)}_{(17)}, \boxed{X^{(1)}_{(18)}}, \\ X^{(1)}_{(19)}, X^{(1)}_{(20)}, X^{(1)}_{(21)}, X^{(1)}_{(22)}, X^{(1)}_{(23)}, X^{(1)}_{(24)}, \boxed{X^{(1)}_{(25)}}, X^{(1)}_{(26)}, X^{(1)}_{(27)}, X^{(1)}_{(28)}, X^{(1)}_{(29)}, X^{(1)}_{(30)}, X^{(1)}_{(31)}, \boxed{X^{(1)}_{(32)}}, X^{(1)}_{(33)}, X^{(1)}_{(34)}, X^{(1)}_{(35)}, X^{(1)}_{(36)}, \\ X^{(1)}_{(37)}, X^{(1)}_{(38)}, \boxed{X^{(1)}_{(39)}}, X^{(1)}_{(40)}, X^{(1)}_{(41)}, X^{(1)}_{(42)}, X^{(1)}_{(43)}, X^{(1)}_{(44)}, X^{(1)}_{(45)}, \boxed{X^{(1)}_{(46)}}, X^{(1)}_{(47)}, X^{(1)}_{(48)}, X^{(1)}_{(49)} \end{bmatrix},$$

$$\begin{bmatrix} X^{(2)}_{(1)}, X^{(2)}_{(2)}, X^{(2)}_{(3)}, \boxed{X^{(2)}_{(4)}}, X^{(2)}_{(5)}, X^{(2)}_{(6)}, X^{(2)}_{(7)}, X^{(2)}_{(8)}, X^{(2)}_{(9)}, X^{(2)}_{(10)}, \boxed{X^{(2)}_{(11)}}, X^{(2)}_{(12)}, X^{(2)}_{(13)}, X^{(2)}_{(14)}, X^{(2)}_{(15)}, X^{(2)}_{(16)}, X^{(2)}_{(17)}, \boxed{X^{(2)}_{(18)}}, \\ X^{(2)}_{(19)}, X^{(2)}_{(20)}, X^{(2)}_{(21)}, X^{(2)}_{(22)}, X^{(2)}_{(23)}, X^{(2)}_{(24)}, \boxed{X^{(2)}_{(25)}}, X^{(2)}_{(26)}, X^{(2)}_{(27)}, X^{(2)}_{(28)}, X^{(2)}_{(29)}, X^{(2)}_{(30)}, X^{(2)}_{(31)}, \boxed{X^{(2)}_{(32)}}, X^{(2)}_{(33)}, X^{(2)}_{(34)}, X^{(2)}_{(35)}, X^{(2)}_{(36)}, \\ X^{(2)}_{(37)}, X^{(2)}_{(38)}, \boxed{X^{(2)}_{(39)}}, X^{(2)}_{(40)}, X^{(2)}_{(41)}, X^{(2)}_{(42)}, X^{(2)}_{(43)}, X^{(2)}_{(44)}, X^{(2)}_{(45)}, \boxed{X^{(2)}_{(46)}}, X^{(2)}_{(47)}, X^{(2)}_{(48)}, X^{(2)}_{(49)} \end{bmatrix},$$

$$\begin{bmatrix} X^{(3)}_{(1)}, X^{(3)}_{(2)}, X^{(3)}_{(3)}, \boxed{X^{(3)}_{(4)}}, X^{(1)}_{(5)}, X^{(3)}_{(6)}, X^{(3)}_{(7)}, X^{(3)}_{(8)}, X^{(3)}_{(9)}, X^{(3)}_{(10)}, \boxed{X^{(3)}_{(11)}}, X^{(3)}_{(12)}, X^{(3)}_{(13)}, X^{(3)}_{(14)}, X^{(3)}_{(15)}, X^{(3)}_{(16)}, X^{(3)}_{(17)}, \boxed{X^{(3)}_{(18)}}, \\ X^{(3)}_{(19)}, X^{(3)}_{(20)}, X^{(3)}_{(21)}, X^{(3)}_{(22)}, X^{(3)}_{(23)}, X^{(3)}_{(24)}, \boxed{X^{(3)}_{(25)}}, X^{(3)}_{(26)}, X^{(3)}_{(27)}, X^{(3)}_{(28)}, X^{(3)}_{(29)}, X^{(3)}_{(30)}, X^{(3)}_{(31)}, \boxed{X^{(3)}_{(32)}}, X^{(3)}_{(33)}, X^{(3)}_{(34)}, X^{(3)}_{(35)}, X^{(3)}_{(36)}, \\ X^{(3)}_{(37)}, X^{(3)}_{(38)}, \boxed{X^{(3)}_{(39)}}, X^{(3)}_{(40)}, X^{(3)}_{(41)}, X^{(3)}_{(42)}, X^{(3)}_{(43)}, X^{(3)}_{(44)}, X^{(3)}_{(45)}, \boxed{X^{(3)}_{(46)}}, X^{(3)}_{(47)}, X^{(3)}_{(48)}, X^{(3)}_{(49)} \end{bmatrix},$$

$$\begin{bmatrix} X^{(4)}_{(1)}, X^{(4)}_{(2)}, X^{(4)}_{(3)}, \boxed{X^{(4)}_{(4)}}, X^{(4)}_{(5)}, X^{(4)}_{(6)}, X^{(4)}_{(7)}, X^{(4)}_{(8)}, X^{(4)}_{(9)}, X^{(4)}_{(10)}, \boxed{X^{(4)}_{(11)}}, X^{(4)}_{(12)}, X^{(4)}_{(13)}, X^{(4)}_{(14)}, X^{(4)}_{(15)}, X^{(4)}_{(16)}, X^{(4)}_{(17)}, \boxed{X^{(4)}_{(18)}}, \\ X^{(4)}_{(19)}, X^{(4)}_{(20)}, X^{(4)}_{(21)}, X^{(4)}_{(22)}, X^{(4)}_{(23)}, X^{(4)}_{(24)}, \boxed{X^{(4)}_{(25)}}, X^{(4)}_{(26)}, X^{(4)}_{(27)}, X^{(4)}_{(28)}, X^{(4)}_{(29)}, X^{(4)}_{(30)}, X^{(4)}_{(31)}, \boxed{X^{(4)}_{(32)}}, X^{(4)}_{(33)}, X^{(4)}_{(34)}, X^{(4)}_{(35)}, X^{(4)}_{(36)}, \\ X^{(4)}_{(37)}, X^{(4)}_{(38)}, \boxed{X^{(4)}_{(39)}}, X^{(4)}_{(40)}, X^{(4)}_{(41)}, X^{(4)}_{(42)}, X^{(4)}_{(43)}, X^{(4)}_{(44)}, X^{(4)}_{(45)}, \boxed{X^{(4)}_{(46)}}, X^{(4)}_{(47)}, X^{(4)}_{(48)}, X^{(4)}_{(49)} \end{bmatrix},$$

$$\begin{bmatrix} X^{(5)}_{(1)}, X^{(5)}_{(2)}, X^{(5)}_{(3)}, \boxed{X^{(5)}_{(4)}}, X^{(5)}_{(5)}, X^{(5)}_{(6)}, X^{(5)}_{(7)}, X^{(5)}_{(8)}, X^{(5)}_{(9)}, X^{(5)}_{(10)}, \boxed{X^{(5)}_{(11)}}, X^{(5)}_{(12)}, X^{(5)}_{(13)}, X^{(5)}_{(14)}, X^{(5)}_{(15)}, X^{(5)}_{(16)}, X^{(5)}_{(17)}, \boxed{X^{(5)}_{(18)}}, \\ X^{(5)}_{(19)}, X^{(5)}_{(20)}, X^{(5)}_{(21)}, X^{(5)}_{(22)}, X^{(5)}_{(23)}, X^{(5)}_{(124)}, \boxed{X^{(5)}_{(25)}}, X^{(5)}_{(26)}, X^{(5)}_{(27)}, X^{(5)}_{(28)}, X^{(5)}_{(29)}, X^{(5)}_{(30)}, X^{(5)}_{(31)}, \boxed{X^{(5)}_{(32)}}, X^{(5)}_{(33)}, X^{(5)}_{(34)}, X^{(5)}_{(35)}, X^{(5)}_{(36)}, \\ X^{(5)}_{(37)}, X^{(5)}_{(38)}, \boxed{X^{(5)}_{(39)}}, X^{(5)}_{(40)}, X^{(5)}_{(41)}, X^{(5)}_{(42)}, X^{(5)}_{(43)}, X^{(5)}_{(44)}, X^{(5)}_{(45)}, \boxed{X^{(5)}_{(46)}}, X^{(5)}_{(47)}, X^{(5)}_{(48)}, X^{(5)}_{(49)} \end{bmatrix},$$

$$\begin{bmatrix} X^{(6)}_{(1)}, X^{(6)}_{(2)}, X^{(6)}_{(3)}, \boxed{X^{(6)}_{(4)}}, X^{(6)}_{(5)}, X^{(6)}_{(6)}, X^{(6)}_{(7)}, X^{(6)}_{(8)}, X^{(6)}_{(9)}, X^{(6)}_{(10)}, \boxed{X^{(6)}_{(11)}}, X^{(6)}_{(12)}, X^{(6)}_{(13)}, X^{(6)}_{(14)}, X^{(6)}_{(15)}, X^{(6)}_{(16)}, X^{(6)}_{(17)}, \boxed{X^{(6)}_{(18)}}, \\ X^{(6)}_{(19)}, X^{(6)}_{(20)}, X^{(6)}_{(21)}, X^{(6)}_{(22)}, X^{(6)}_{(23)}, X^{(6)}_{(24)}, \boxed{X^{(6)}_{(25)}}, X^{(6)}_{(26)}, X^{(6)}_{(27)}, X^{(6)}_{(28)}, X^{(6)}_{(29)}, X^{(6)}_{(30)}, X^{(6)}_{(31)}, \boxed{X^{(6)}_{(32)}}, X^{(6)}_{(33)}, X^{(6)}_{(34)}, X^{(6)}_{(35)}, X^{(6)}_{(36)}, \\ X^{(6)}_{(37)}, X^{(6)}_{(38)}, \boxed{X^{(6)}_{(39)}}, X^{(6)}_{(40)}, X^{(6)}_{(41)}, X^{(6)}_{(42)}, X^{(6)}_{(43)}, X^{(6)}_{(44)}, X^{(6)}_{(45)}, \boxed{X^{(6)}_{(46)}}, X^{(6)}_{(47)}, X^{(6)}_{(48)}, X^{(6)}_{(49)} \end{bmatrix}$$

and

$$\begin{bmatrix} X^{(7)}_{(1)}, X^{(7)}_{(2)}, X^{(7)}_{(3)}, \boxed{X^{(7)}_{(4)}}, X^{(7)}_{(5)}, X^{(7)}_{(6)}, X^{(7)}_{(7)}, X^{(7)}_{(8)}, X^{(7)}_{(9)}, X^{(7)}_{(10)}, \boxed{X^{(7)}_{(11)}}, X^{(7)}_{(12)}, X^{(7)}_{(13)}, X^{(7)}_{(14)}, X^{(7)}_{(15)}, X^{(7)}_{(16)}, X^{(7)}_{(17)}, \boxed{X^{(7)}_{(18)}}, \\ X^{(7)}_{(19)}, X^{(7)}_{(20)}, X^{(7)}_{(21)}, X^{(7)}_{(22)}, X^{(7)}_{(23)}, X^{(7)}_{(24)}, \boxed{X^{(7)}_{(25)}}, X^{(7)}_{(26)}, X^{(7)}_{(27)}, X^{(7)}_{(28)}, X^{(7)}_{(29)}, X^{(7)}_{(30)}, X^{(7)}_{(31)}, \boxed{X^{(7)}_{(32)}}, X^{(7)}_{(33)}, X^{(7)}_{(34)}, X^{(7)}_{(35)}, X^{(7)}_{(36)}, \\ X^{(7)}_{(37)}, X^{(7)}_{(38)}, \boxed{X^{(7)}_{(39)}}, X^{(7)}_{(40)}, X^{(7)}_{(41)}, X^{(7)}_{(42)}, X^{(7)}_{(43)}, X^{(7)}_{(44)}, X^{(7)}_{(45)}, \boxed{X^{(7)}_{(46)}}, X^{(7)}_{(47)}, X^{(7)}_{(48)}, X^{(7)}_{(49)} \end{bmatrix}$$

so we have 7 DURSS

$$X^{(1)}_{(4)}, X^{(1)}_{(11)}, X^{(1)}_{(18)}, X^{(1)}_{(25)}, X^{(1)}_{(32)}, X^{(1)}_{(39)}, X^{(1)}_{(46)},$$
$$X^{(2)}_{(4)}, X^{(2)}_{(11)}, X^{(2)}_{(18)}, X^{(2)}_{(25)}, X^{(2)}_{(32)}, X^{(2)}_{(39)}, X^{(2)}_{(46)},$$
$$X^{(3)}_{(4)}, X^{(3)}_{(11)}, X^{(3)}_{(18)}, X^{(3)}_{(25)}, X^{(3)}_{(32)}, X^{(3)}_{(39)}, X^{(3)}_{(46)},$$
$$X^{(4)}_{(4)}, X^{(4)}_{(11)}, X^{(4)}_{(18)}, X^{(4)}_{(25)}, X^{(4)}_{(32)}, X^{(4)}_{(39)}, X^{(4)}_{(46)},$$
$$X^{(5)}_{(4)}, X^{(5)}_{(11)}, X^{(5)}_{(18)}, X^{(5)}_{(25)}, X^{(5)}_{(32)}, X^{(5)}_{(39)}, X^{(5)}_{(46)},$$
$$X^{(6)}_{(4)}, X^{(6)}_{(11)}, X^{(6)}_{(18)}, X^{(6)}_{(25)}, X^{(6)}_{(32)}, X^{(6)}_{(39)}, X^{(6)}_{(46)},$$
$$X^{(7)}_{(4)}, X^{(7)}_{(11)}, X^{(7)}_{(18)}, X^{(7)}_{(25)}, X^{(7)}_{(32)}, X^{(7)}_{(39)}, X^{(7)}_{(46)},$$

We select the sample unit from 7 DURSS

$$X^{(1)}_{(4)}, X^{(1)}_{(11)}, X^{(1)}_{(18)}, \boxed{X^{(1)}_{(25)}}, X^{(1)}_{(32)}, X^{(1)}_{(39)}, X^{(1)}_{(46)},$$
$$X^{(2)}_{(4)}, X^{(2)}_{(11)}, X^{(2)}_{(18)}, \boxed{X^{(2)}_{(25)}}, X^{(2)}_{(32)}, X^{(2)}_{(39)}, X^{(2)}_{(46)},$$
$$X^{(3)}_{(4)}, X^{(3)}_{(11)}, X^{(3)}_{(18)}, \boxed{X^{(3)}_{(25)}}, X^{(3)}_{(32)}, X^{(3)}_{(39)}, X^{(3)}_{(46)},$$
$$X^{(4)}_{(4)}, X^{(4)}_{(11)}, X^{(4)}_{(18)}, \boxed{X^{(4)}_{(25)}}, X^{(4)}_{(32)}, X^{(4)}_{(39)}, X^{(4)}_{(46)},$$
$$X^{(5)}_{(4)}, X^{(5)}_{(11)}, X^{(5)}_{(18)}, \boxed{X^{(5)}_{(25)}}, X^{(5)}_{(32)}, X^{(5)}_{(39)}, X^{(5)}_{(46)},$$
$$X^{(6)}_{(4)}, X^{(6)}_{(11)}, X^{(6)}_{(18)}, \boxed{X^{(6)}_{(25)}}, X^{(6)}_{(32)}, X^{(6)}_{(39)}, X^{(6)}_{(46)},$$
$$X^{(7)}_{(4)}, X^{(7)}_{(11)}, X^{(7)}_{(18)}, \boxed{X^{(7)}_{(25)}}, X^{(7)}_{(32)}, X^{(7)}_{(39)}, X^{(7)}_{(46)},$$

Let $X^{(1)}_{(25)}, X^{(2)}_{(25)}, X^{(3)}_{(25)}, X^{(4)}_{(25)}, X^{(5)}_{(25)}, X^{(6)}_{(25)}, X^{(7)}_{(25)}$ is DURSS of size 7

VI. Stratified Unified Ranked Set Sampling)SDURSS(

The population of N units is divided into L non-overlapping sub-groups known as strata each stratum have $N_1, N_2, \cdots, N_L$ units, respectively, such that $N_1 + N_2 + \cdots + N_L = N$. The size of the h^{th} strata denotes by N_h for $h = 1, 2, \cdots, L$. Then the samples are drawn independently from each stratum, producing samples sizes denoted by $n_1, n_2, ..., n_L$, such that the total sample size is $n = \sum_{h=1}^{L} n_h$. If the DURSS technique is applied for each stratum then the whole procedure is called a SDURSS. Define $X^{k(h)}_{[i]j}$ be the SDURSS sampled unit of the i^{th} rank, the j^{th} cycle in the h^{th} stratum, where $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, r$; $k = 1, 2, \cdots, m$; and $h = 1, 2, \cdots, L$. The mean of selected units is used as a population mean estimator.

Example 6: Suppose that we have two strata, i.e. $L = 2$ and $h = 1,2$. Let (m, r) Assume that from the first stratum we select a sample of size $m \times r = 6 \times 2 = 12$ and from the second stratum we want a sample of size $m \times r = 6 \times 2 = 12$ Then the process as illustrates as follows :

Stratum 1: *Now, select 12 samples as follows*:

Consider the case of $(\text{stratum1},\ m = 6,\ r = 1)$ *and* $(\text{stratum1},\ m = 6,\ r = 2)$. *Draw a simple random sample of size* $m^3 = 6^3 = 216$ *elements)6 sets of size 36 each(. Assume the elements are*

Stratum 1	$(r=1)$	$X_{[1]1}^{1(1)}, X_{[2]1}^{1(1)}, X_{[3]1}^{1(1)}, \cdots, X_{[36]1}^{1(1)}, X_{[1]1}^{2(1)}, X_{[2]1}^{2(1)}, X_{[3]1}^{2(1)}, \cdots, X_{[36]1}^{2(1)}, X_{[1]1}^{3(1)}, X_{[2]1}^{3(1)}, X_{[3]1}^{3(1)}, \cdots, X_{[36]1}^{3(1)},$ $X_{[1]1}^{4(1)}, X_{[2]1}^{4(1)}, X_{[3]1}^{4(1)}, \cdots, X_{[36]1}^{4(1)}, X_{[1]1}^{5(1)}, X_{[2]1}^{5(1)}, X_{[3]1}^{5(1)}, \cdots, X_{[36]1}^{5(1)}, X_{[1]1}^{6(1)}, X_{[2]1}^{6(1)}, X_{[3]1}^{6(1)}, \cdots, X_{[36]1}^{6(1)},$
	$(r=2)$	$X_{[1]2}^{1(1)}, X_{[2]2}^{1(1)}, X_{[3]2}^{1(1)}, \cdots, X_{[36]2}^{1(1)}, X_{[1]2}^{2(1)}, X_{[2]2}^{2(1)}, X_{[3]2}^{2(1)}, \cdots, X_{[36]2}^{2(1)}, X_{[1]2}^{3(1)}, X_{[2]2}^{3(1)}, X_{[3]2}^{3(1)}, \cdots, X_{[36]2}^{3(1)},$ $X_{[1]2}^{4(1)}, X_{[2]2}^{4(1)}, X_{[3]2}^{4(1)}, \cdots, X_{[36]2}^{4(1)}, X_{[1]2}^{5(1)}, X_{[2]2}^{5(1)}, X_{[3]2}^{5(1)}, \cdots, X_{[36]2}^{5(1)}, X_{[1]2}^{6(1)}, X_{[2]2}^{6(1)}, X_{[3]2}^{6(1)}, \cdots, X_{[36]2}^{6(1)},$

For $h=1$ we have: $X_{[1]1}^{15(1)}, X_{[2]1}^{15(1)}, X_{[3]1}^{15(1)}, X_{[4]1}^{22(1)}, X_{[5]1}^{22(1)}, X_{[6]1}^{22(1)}, X_{[1]2}^{15(1)}, X_{[2]2}^{15(1)}, X_{[3]2}^{15(1)}, X_{[4]2}^{22(1)}, X_{[5]2}^{22(1)}, X_{[6]2}^{22(1)}$

Stratum 2: *Now, select 12 samples as follows*:

Consider the case of $(\text{stratum2},\ m=6,\ r=1)$ *and* $(\text{stratum2},\ m=6,\ r=2)$. *Draw a simple random sample of size* $m^3=6^3=216$ *elements*)*6 sets of size 36 each*(. *Assume the elements are*

Stratum 2	$(r=1)$	$X_{[1]1}^{1(2)}, X_{[2]1}^{1(2)}, X_{[3]1}^{1(2)}, \cdots, X_{[36]1}^{1(2)}, X_{[1]1}^{2(2)}, X_{[2]1}^{2(2)}, X_{[3]1}^{2(2)}, \cdots, X_{[36]1}^{2(2)}, X_{[1]1}^{3(2)}, X_{[2]1}^{3(2)}, X_{[3]1}^{3(2)}, \cdots, X_{[36]1}^{3(2)},$ $X_{[1]1}^{4(2)}, X_{[2]1}^{4(2)}, X_{[3]1}^{4(2)}, \cdots, X_{[36]1}^{4(2)}, X_{[1]1}^{5(2)}, X_{[2]1}^{5(2)}, X_{[3]1}^{5(2)}, \cdots, X_{[36]1}^{5(2)}, X_{[1]1}^{6(2)}, X_{[2]1}^{6(2)}, X_{[3]1}^{6(2)}, \cdots, X_{[36]1}^{6(2)},$
	$(r=2)$	$X_{[1]2}^{1(2)}, X_{[2]2}^{1(2)}, X_{[3]2}^{1(2)}, \cdots, X_{[36]2}^{1(2)}, X_{[1]2}^{2(2)}, X_{[2]2}^{2(2)}, X_{[3]2}^{2(2)}, \cdots, X_{[36]2}^{2(2)}, X_{[1]2}^{3(2)}, X_{[2]2}^{3(2)}, X_{[3]2}^{3(2)}, \cdots, X_{[36]2}^{3(2)},$ $X_{[1]2}^{4(2)}, X_{[2]2}^{4(2)}, X_{[3]2}^{4(2)}, \cdots, X_{[36]2}^{4(2)}, X_{[1]2}^{5(2)}, X_{[2]2}^{5(2)}, X_{[3]2}^{5(2)}, \cdots, X_{[36]2}^{5(2)}, X_{[1]2}^{6(2)}, X_{[2]2}^{6(2)}, X_{[3]2}^{6(2)}, \cdots, X_{[36]2}^{6(2)},$

For $h=2$ we have: $X_{[1]1}^{15(2)}, X_{[2]1}^{15(2)}, X_{[3]1}^{15(2)}, X_{[4]1}^{22(2)}, X_{[5]1}^{22(2)}, X_{[6]1}^{22(2)}, X_{[1]2}^{15(2)}, X_{[2]2}^{15(2)}, X_{[3]2}^{15(2)}, X_{[4]2}^{22(2)}, X_{[5]2}^{22(2)}, X_{[6]2}^{22(2)}$

)Define: $X_{[i]j}^{k(h)}$, $k=$ number of ranking the elements of each set, $h=$ stratum size, $i=$ number ofeach set,

$r=$ number of cycles)times((

Therefore, the measured SDURSS units are

$X_{[1]1}^{15(1)}, X_{[2]1}^{15(1)}, X_{[3]1}^{15(1)}, X_{[4]1}^{22(1)}, X_{[5]1}^{22(1)}, X_{[6]1}^{22(1)}, X_{[1]2}^{15(1)}, X_{[2]2}^{15(1)}, X_{[3]2}^{15(1)}, X_{[4]2}^{22(1)}, X_{[5]2}^{22(1)}, X_{[6]2}^{22(1)}$

$X_{[1]1}^{15(2)}, X_{[2]1}^{15(2)}, X_{[3]1}^{15(2)}, X_{[4]1}^{22(2)}, X_{[5]1}^{22(2)}, X_{[6]1}^{22(2)}, X_{[1]2}^{15(2)}, X_{[2]2}^{15(2)}, X_{[3]2}^{15(2)}, X_{[4]2}^{22(2)}, X_{[5]2}^{22(2)}, X_{[6]2}^{22(2)}$

where their mean of these units is used as an estimator of the population mean.

To compare the efficiency of the empirical mean estimator based on SDURSS with their counterparts in SRS, SSRS, SRSS, and SMRSS via a simulation in R)Version 4.3.2(under the population of 100,000 units divided into two strata each stratum has 50,000 units with the numbers of set in each stratum $m=2,4,6,10$ and the number of cycles $r=2,5$. Using 5000 replications, estimates of the means, variances and mean square errors.

III. Results and Discussions

I. Estimation of Population Mean

Let $X_1, X_2, ..., X_n$ be n independent random variables from a probability density function $f(x)$, with mean (μ) and variance (σ^2). The empirical mean estimator of DURSS is given by

$$\bar{X}_{DURSS}=\frac{1}{mr}\sum_{i=1}^{m}\sum_{j=1}^{r}X_{\left[l+(i-1)m\right]j} \tag{1}$$

where $l=\frac{m}{2}$ if i is an even number and $l=\frac{m+1}{2}$ if i is an odd number)for $i=1,2,...,m$(.

The DURSS variance can be estimated by

$$S^2_{DURSS} = \frac{1}{mr-1}\left\{\sum_{i=1}^{m}\sum_{j=1}^{r}\left(X_{\left[l+(i-1)m\right]j} - \bar{X}_{DURSS}\right)^2\right\}. \quad (2)$$

The SDURSS estimator of the population mean is given by

$$\bar{X}_{SDURSS} = \sum_{h=1}^{L} W_h\left(\bar{X}^h_{DURSS}\right) \quad (3)$$

Where $W_h = \frac{N_h}{N}$ and $\bar{X}^h_{FRSS}$ is the DURSS mean estimator in the h^{th} stratum.

The variance of $\bar{X}_{SFRSS}$ is given by

$$\begin{aligned} Var\left(\bar{X}_{SDURSS}\right) &= Var\left[\sum_{h=1}^{L}\frac{W_h}{m_h r}\left(\sum_{i=1}^{m_h}\sum_{j=1}^{r} X_{\left[l+(i-1)m_h\right]j}\right)\right] \\ &= \sum_{h=1}^{L}\frac{W_h^2}{m_h^2 r^2}\left(\sum_{i=1}^{m_h}\sum_{j=1}^{r} Var\left(X_{\left[l+(i-1)m_h\right]j}\right)\right) \\ &= \sum_{h=1}^{L}\frac{W_h^2}{m_h^2 r^2}\left(\sum_{i=1}^{m_h}\sum_{j=1}^{r}\sigma^2_{\left[l+(i-1)m_h\right]j,h}\right) \\ &= \sum_{h=1}^{L}\frac{W_h^2}{m_h r}\sigma^2_{\left[l+(i-1)m_h\right]j,h} \end{aligned} \quad (4)$$

IV. Simulation Study

The simulation study is designed to investigate the performance of SDURSS for estimating the population mean compared to their counterparts in SRS, SSRS, and SRSS under parent asymmetric distributions: Exp)1(, Geo)0.5(, Gamma)0.5,1(, Gamma)1,2(, Beta)3,3(, Beta)9,2(, Weibull)0.5,1(, Weibull)1,2(, Log N)0,1(, Logistic)0,1(, CHI)1(. The simulations are done based on the population of 100,000 units is divided into two strata each stratum has 50,000 units, which are conducted for the numbers of set in each stratum $m = 2,4,6,10$ and the number of cycles $r = 2,5$ on 5,000 replications. If the underlying distribution is asymmetric, the efficiencies of SDURSS relative to SRS, SSRS, SRSS, and SMRSS, respectively are given by

$$eff\left(\bar{X}_{SDURSS}, \bar{X}_{SRS}\right) = \frac{MSE\left(\bar{X}_{SRS}\right)}{MSE\left(\bar{X}_{SDURSS}\right)},$$

$$eff\left(\bar{X}_{SDURSS}, \bar{X}_{SSRS}\right) = \frac{MSE\left(\bar{X}_{SSRS}\right)}{MSE\left(\bar{X}_{SDURSS}\right)},$$

$$eff\left(\bar{X}_{SDURSS}, \bar{X}_{SRSS}\right) = \frac{MSE\left(\bar{X}_{SRSS}\right)}{MSE\left(\bar{X}_{SDURSS}\right)},$$

$$eff\left(\bar{X}_{SDURSS}, \bar{X}_{SMRSS}\right) = \frac{MSE\left(\bar{X}_{SMRSS}\right)}{MSE\left(\bar{X}_{SDURSS}\right)},$$

where MSE is the mean square error)MSE(.
The simulation results are shown in Tables 1-3.

Table 1: *The efficiency of SDURSS relative to SRS, SSRS, SRSS and SMRSS for estimating the population mean with* $m=2$ *and* $r=2,5$

Distribution	r	Efficiency			
		SDURSS vs. SRS	SDURSS vs. SSRS	SDURSS vs. SRSS	SDURSS vs. SMRSS
Exp)1(	2	0.7259	0.3542	0.6500	0.6256
	5	0.0409	0.0206	0.6197	0.6194
Geo)0.5(	2	0.7141	0.3609	0.6721	0.6717
	5	0.0426	0.0209	0.6467	0.6479
Gamma)0.5,1(	2	0.6433	0.3343	0.6179	0.6040
	5	0.0383	0.0195	0.6139	0.6088
Gamma)1,2(	2	0.7026	0.3572	0.6709	0.6739
	5	0.0401	0.0201	0.6086	0.6171
Beta)3,3(	2	0.8158	0.4088	0.7422	0.7366
	5	0.0443	0.0219	0.6806	0.6852
Beta)9,2(	2	0.7868	0.3959	0.7208	0.7107
	5	0.0429	0.0213	0.6623	0.6632
Weibull)0.5,1(	2	0.5839	0.2680	0.5658	0.5407
	5	0.0374	0.0186	0.5836	0.5369
Weibull)1,2(	2	0.7080	0.3512	0.6523	0.6644
	5	0.0404	0.0201	0.6165	0.6199
Log N)0,1(	2	0.5840	0.3203	0.5488	0.6177
	5	0.0366	0.0178	0.6153	0.5559
Logistic)0,1(	2	0.7376	0.3687	0.6582	0.6721
	5	0.0399	0.0201	0.6191	0.6138
CHI)1(	2	0.6550	0.3469	0.6387	0.6079
	5	0.0393	0.0200	0.6222	0.6233

Based on Table 1, the numbers of set in each stratum $m=2$ and numbers of cycle $r=2$, it indicates that the SDURSS estimator is less efficient than SRS, SSRS, SRSS and SMRSS estimators all asymmetric distributions.

Table 2: *The efficiency of SDURSS relative to SRS, SSRS, SRSS and SMRSS for estimating the population mean with* $m=4$ *and* $r=2,5$

Distribution	r	Efficiency			
		SDURSS vs. SRS	SDURSS vs. SSRS	SDURSS vs. SRSS	SDURSS vs. SMRSS
Exp)1(	2	0.9332	0.4654	0.4024	0.3737
	5	0.0617	0.0313	0.4720	0.1862
Geo)0.5(	2	0.9068	0.4551	0.3939	0.3768
	5	0.0618	0.0312	0.4675	0.2088
Gamma)0.5,1(	2	0.8045	0.3935	0.3509	0.3304
	5	0.0551	0.0278	0.4217	0.1454
Gamma)1,2(	2	0.9144	0.4566	0.3913	0.3603
	5	0.0611	0.0307	0.4724	0.1862
Beta)3,3(	2	**1.4000**	0.6931	0.5862	0.5552
	5	0.0808	0.0406	0.6180	0.3185

Distribution	r	Efficiency			
		SDURSS vs. SRS	SDURSS vs. SSRS	SDURSS vs. SRSS	SDURSS vs. SMRSS
Beta)9,2(	2	**1.2778**	0.6481	0.5556	0.5185
	5	0.0730	0.0365	0.5587	0.2628
Weibull)0.5,1(	2	0.5858	0.3235	0.2795	0.2676
	5	0.0440	0.0218	0.3354	0.0640
Weibull)1,2(	2	0.9209	0.4507	0.4038	0.3824
	5	0.0610	0.0303	0.4637	0.1839
Log N)0,1(	2	0.6315	0.2918	0.2912	0.2554
	5	0.0442	0.0237	0.3748	0.0823
Logistic)0,1(	2	**1.1039**	0.5582	0.4666	0.4395
	5	0.0649	0.0323	0.5025	0.1992
CHI)1(	2	0.7453	0.3833	0.3540	0.3336
	5	0.0543	0.0273	0.4172	0.1449

Based on Table 2, the numbers of set in each stratum $m = 4$, we can conclude that the SDURSS estimator is less efficient compared to SRS, SSRS, SRSS and SMRSS estimators for the numbers of cycle $r = 2$ underlying all asymmetric distributions.

Table 3: *The efficiency of SDURSS relative to SRS, SSRS, SRSS and SMRSS for estimating the population mean with* $m = 6$ *and* $r = 2,5$

Distribution	r	Efficiency			
		SDURSS vs. SRS	SDURSS vs. SSRS	SDURSS vs. SRSS	SDURSS vs. SMRSS
Exp)1(	2	0.1401	0.7593	0.4259	0.2994
	5	0.0856	0.0427	0.4305	0.1173
Geo)0.5(	2	**1.4798**	0.7489	0.4205	0.4103
	5	0.0868	0.0430	0.4352	0.1515
Gamma)0.5,1(	2	**1.2423**	0.6121	0.3599	0.1710
	5	0.0730	0.0362	0.3663	0.0909
Gamma)1,2(	2	**1.5048**	0.7504	0.4287	0.1875
	5	0.0848	0.0424	0.4296	0.1164
Beta)3,3(	2	**2.4839**	**1.2516**	0.6903	0.2645
	5	0.1263	0.0630	0.6389	0.2194
Beta)9,2(	2	**2.0938**	**1.0469**	0.5781	0.2344
	5	0.1102	0.0547	0.5534	0.1742
Weibull)0.5,1(	2	0.8754	0.4244	0.2435	0.0809
	5	0.0503	0.0245	0.2469	0.0344
Weibull)1,2(	2	**1.5581**	0.7616	0.4327	0.1887
	5	0.0849	0.0430	0.4331	0.1194
Log N)0,1(	2	0.8643	0.4264	0.2589	0.0808
	5	0.0557	0.0273	0.2816	0.0444
Logistic)0,1(	2	**1.8218**	0.9101	0.5095	0.1399
	5	0.0929	0.0466	0.4730	0.1166
CHI)1(	2	**1.2097**	0.5977	0.3581	0.2171
	5	0.0723	0.0367	0.3665	0.0985

Based on Table 3, the numbers of set in each stratum $m=6$, it implies that the SDURSS estimator is less efficient than SRS, SSRS, SRSS and SMRSS estimators for the numbers of cycle $r=2$ based on all asymmetric distributions.

Table 4: The efficiency of SDURSS relative to SRS, SSRS, SRSS and SMRSS for estimating the population mean with $m=10$ and $r=2,5$

Distribution	r	Efficiency			
		SDURSS vs. SRS	SDURSS vs. SSRS	SDURSS vs. SRSS	SDURSS vs. SMRSS
Exp)1(	2	0.0860	0.0432	0.0139	0.0044
	5	0.0843	0.0417	0.2568	0.0522
Geo)0.5(	2	0.0864	0.0430	0.0143	0.0065
	5	0.0846	0.0417	0.2531	0.0799
Gamma)0.5,1(	2	0.0735	0.0371	0.0122	0.0043
	5	0.0717	0.0356	0.2148	0.0462
Gamma)1,2(	2	0.0863	0.0430	0.0142	0.0042
	5	0.0843	0.0414	0.2531	0.0513
Beta)3,3(	2	0.1287	0.0643	0.0205	0.0051
	5	0.1246	0.0623	0.3758	0.0780
Beta)9,2(	2	0.1110	0.0555	0.0179	0.0043
	5	0.1085	0.0538	0.3271	0.0649
Weibull)0.5,1(	2	0.0512	0.0255	0.0086	0.0023
	5	0.0488	0.0244	0.1510	0.0203
Weibull)1,2(	2	0.0877	0.0438	0.0143	0.0044
	5	0.0846	0.0425	0.2560	0.0523
Log N)0,1(	2	0.0540	0.0275	0.0092	0.0022
	5	0.0530	0.0280	0.1627	0.0223
Logistic)0,1(	2	0.0944	0.0473	0.0153	0.0026
	5	0.0919	0.0462	0.2773	0.0382
CHI)1(	2	0.0754	0.0369	0.0126	0.0058
	5	0.0710	0.0360	0.2185	0.0595

Based on Table 4, the numbers of set in each stratum $m=10$, it implies that the SDURSS estimator is less efficient than SRS, SSRS, SRSS and SMRSS estimators for the numbers of cycle $r=2$ based on all asymmetric distributions.

V. Real Data example

In this section, the application of the proposed sampling method is shown by using a real data example. The researcher went to the area to collect data by himself. The data sets used in this example include: There are a total of 5 plots of False pakchoi, with a length of 20 meters and a width of 1 meter. Each plant will have a minimum number of flowers of 3 flowers per plant. If data is collected in batches, it will be 25 plants per batch with 75-150 flowers. Where 1 plot can store 20 data sets from a total of 5 plots, totaling 100 data sets. Figure 1-2 illustrate False pakchoi and Table 5

represent Number set and real data.

Figure 1

Figure 2

Table 5: *Number set and real data for False pakchoi*

Number set	data	Number set	data	Number set	data	Number set	data	Number set	data
Set 1	103	Set 21	125	Set 41	109	Set 61	89	Set 81	97
Set 2	115	Set 22	123	Set 42	141	Set 62	131	Set 82	148
Set 3	103	Set 23	129	Set 43	133	Set 63	123	Set 83	118
Set 4	117	Set 24	118	Set 44	114	Set 64	144	Set 84	90
Set 5	150	Set 25	99	Set 45	138	Set 65	128	Set 85	125
Set 6	110	Set 26	97	Set 46	111	Set 66	149	Set 86	104
Set 7	123	Set 27	92	Set 47	116	Set 67	123	Set 87	90
Set 8	102	Set 28	146	Set 48	146	Set 68	148	Set 88	129
Set 9	143	Set 29	143	Set 49	145	Set 69	105	Set 89	108
Set 10	76	Set 30	115	Set 50	132	Set 70	120	Set 90	110
Set 11	97	Set 31	76	Set 51	129	Set 71	114	Set 91	118
Set 12	135	Set 32	99	Set 52	127	Set 72	125	Set 92	118
Set 13	140	Set 33	117	Set 53	108	Set 73	130	Set 93	126
Set 14	81	Set 34	112	Set 54	107	Set 74	120	Set 94	83
Set 15	99	Set 35	89	Set 55	136	Set 75	137	Set 95	114
Set 16	136	Set 36	130	Set 56	97	Set 76	139	Set 96	121
Set 17	93	Set 37	111	Set 57	99	Set 77	84	Set 97	97
Set 18	103	Set 38	142	Set 58	112	Set 78	115	Set 98	108
Set 19	83	Set 39	136	Set 59	97	Set 79	147	Set 99	94
Set 20	90	Set 40	96	Set 60	123	Set 80	142	Set 100	113

Total Population flower False pakchoi is 11,636, population mean $\bar{X} = 116.36$ to collect a sample of size 8, using set size is $m = 4$ and number of cycles)times(is $r = 2$ in SRS, SSRS, SRSS, and SDURSS designs, DURSS technique can be described as follows:

I. Draw a simple random sample of size $m^3 = 4^3 = 64$ elements)4 sets of size 16 each(.

II. Use the usual URSS procedure on each set to obtain m ranked set samples of size m each.

III. Apply the URSS procedure again on step)2(to obtain a DURSS of size 8.

The measured values in both SRS, SSRS, SRSS, SMRSS, and SDURSS and designs are presented in Table 6.

Table 6: *Sampled units in SRS, SSRS, SRSS, SMRSS, and SDURSS designs*.

	SRS	117	90	97	123	146	145	108	94
SSRS	Stratum 1	115	104	92	132	114	140	114	99
	Stratum 2	111	146	148	97	118	107	84	97
SRSS	Stratum 1	90	99	109	144	76	90	105	129
	Stratum 2	90	114	143	145	81	97	111	120
SMRSS	Stratum 1	76	108	142	148	81	96	114	115
	Stratum 2	97	111	103	144	89	94	104	127
SDURSS	Stratum 1	123	129	146	150	81	83	112	136
	Stratum 2	104	109	125	148	83	97	102	139

$\bar{X}_{SRS} = 115$

$\bar{X}_{SSRS(stratum1)} = 113.75, \bar{X}_{SSRS(stratum2)} = 113.5$

$\bar{X}_{SRSS(stratum1)} = 105.25, \bar{X}_{SRSS(stratum2)} = 112.63$

$\bar{X}_{SMRSS(stratum1)} = 110, \bar{X}_{SMRSS(stratum2)} = 108.63$

$\bar{X}_{SDURSS(stratum1)} = 120, \bar{X}_{SDURSS(stratum2)} = 113.38$

$S^2_{SRS} = 481.15$

$S^2_{SSRS} = 20.7731$

$S^2_{SRSS} = 29.2044$

$S^2_{SMRSS} = 41.1914$

$S^2_{SDURSS} = 37.23332$

VI. Conclusion

In conclusion, the proposed estimator in SDURSS provide efficient their counterparts in SRS, SSRS, SRSS, and SMRSS for eleven parent asymmetric distributions in the case of a larger sample size and small size numbers of cycle. For the small sample size, the proposed estimator in SDURSS still provide less efficient than four methods, but it gives more efficient than SRS and SSRS in some cases.

VII. Acknowledgement

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A SIGNIFICANT STUDY ON ROBUST MEASURE OF LOCATION PARAMETERS USING DATA DEPTH APPROACHES

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Abstract

Data depth procedures are statistical methods used to measure the centrality or depth of a point within a multivariate dataset. These procedures provide a way to quantify how deep or outlying a point is relative to the overall distribution of the data. This study explores various data depth procedures to find reliable location estimations in cases like with and without outliers. In this paper, various depth procedures, such as Mahalanobis depth, Halfspace depth, Euclidean depth, Simplicial depth, and Projection depth, are studied and compared. The efficiency of these depth functions is evaluated using real datasets and simulation studies with R software.

Keywords: data depth, robust procedures, inference, outliers

I. Introduction

Robust statistics is a fundamental branch of statistical theory and methodology designed to address the challenges posed by data containing deviations from standard assumptions. These deviations may include outliers or non-normality in the data. Robust statistics prioritizes methods that are insensitive to small outliers, which are a common occurrence in traditional statistical techniques. It aims to yield precise and reliable results even when the assumptions of classical statistics are not fulfilled. Robust statistical methods have been developed for many common problems such as estimating location, scale and regression parameters. The data depth approach is one of the robust statistical methods that measures the depth of a data point in a multivariate dataset. It determines the depth of a point by its distance from the center of the data, with points closer to the center having a higher depth value. This approach is useful for identifying outliers and robustly estimating location and scatter. These methods can be applied to both univariate and multivariate datasets, providing robust estimates of location and scatter [2].

The rest of the paper is structured as follows. The second section provides a concise overview and definitions of different data depth procedures. In the third section, the findings from numerical study conducted in both real datasets and simulated environments are presented. Finally, the paper concludes with a discussion the last section.

II. Data Depth Procedures

Data depth procedures are an innovative approach in robust statistics designed to measure the centrality of points within a data set, especially in multivariate contexts [3]. Depth assigns an integer to a candidate fit relative to a data set, enabling a center-outward ordering of sample points [6]. Unlike traditional order statistics, which rank data from smallest to largest, depth order statistics start from the center and move outward [8]. This center-outward approach is crucial for multivariate data sets, extending univariate concepts to multivariate analysis and allowing nonparametric methods to be used in multivariate data analysis [9]. This concept is particularly useful when dealing with complex data structures where classical methods may falter due to the presence of outliers or deviations from model assumptions.

The applications of data depth procedures are vast and varied, encompassing robust location estimation, multivariate outlier detection, classification, and data visualization. The data depth procedures used in this study is detailed below.

Mahalanobis Depth (MD)

Mahalanobis (1936) [7] introduced the Mahalanobis depth in robust statistics which measures the centrality of a point within a multivariate data set by using the Mahalanobis distance. Mahalanobis depth of a point x relative to a data set X is inversely related to the Mahalanobis distance from x to the mean of X. Mahalanobis depth function can be written as

$$M_hD\ (\mathrm{x}) = [1 + (x - \bar{x})^T S^{-1} (x - \bar{x})]^{-1} \tag{1}$$

where $\bar{x}$ and S are the mean vector and dispersion matrix.

This function lacks robustness because it relies on non-robust measures like the mean and the dispersion matrix, making it inadequate for handling outliers in a data set.

Halfspace Depth (HD)

In 1975, Tukey (1975) [10] introduced the concept of location depth, also known as halfspace depth or Tukey depth, as a tool for visually describing bivariate data sets. In p dimensions, the halfspace location depth of a point θ relative to a data set $x_n = (x_1, x_2, \ldots, x_n) \in R^{p \times n}$ is denoted as $ldepth(\theta; \backslash X_n)$. It is defined as the smallest number of observation in any closed halfspace with boundary through θ. In the univariate setting p, this definition becomes

$$ldepth_1(\theta; \backslash X_n) = \min\{\#(x_i \leq \theta), \#(x_i \geq \theta)\} \tag{2}$$

In the multivariate case, the concept of the median can be generalized to the point with the highest depth, known as the Tukey median. Numerous depth functions exist, all aiming to quantify how deep or central a point x is within the data cloud. A key advantage of halfspace depth is its affine invariance. The primary reason for employing the Tukey median as a multivariate location estimator is its robustness, which can be evaluated using the breakdown value ϵ^*. Halfspace depth provides a powerful, geometrically intuitive way to measure the centrality of points in multivariate data. It is widely used in robust statistics, particularly for identifying outliers and assessing data spread. By calculating how well a point is "enclosed" by the data, it provides a robust measure of centrality, independent of the data's distribution. However, the method's computational cost can be prohibitive in high-dimensional settings without efficient algorithms.

Euclidean Depth (L_2D)

The L_2-depth was introduced by Zuo and Serfling (2000) [11]. The L_2-depth D^{L_2} measures the outlyingness of a point based on its mean distance from a chosen center of the distribution, defined as

$$D^{L_2}(z/X) = (1 + E(\|z - X\|)^{-1} \tag{3}$$

For an empirical distribution of points x_i $(i = 1,2, \dots, n)$, it is given by

$$D^{L_2}(z/X^1, \dots . X^n) = (1 + \frac{1}{n}\sum_{i=1}^{n}(\|z - X^i\|)^{-1} \tag{4}$$

The L_2-depth vanishes at infinity and reaches its maximum at the spatial median of X, minimizing $E(\|z - X\|)$.

In centrally symmetric distributions, this maximum is at the center. The L_2-depth demonstrates properties such as monotonicity with respect to the deepest point, convexity, compactness of central regions, and continuous dependence on z. It also converges in probability for uniformly integrable and weakly convergent sequences. However, the L_2-depth is not a sensible ordering of dispersion as it contradicts the dilation order, increasing with the dilation of p.

The L_2-depth is invariant against rigid Euclidean motions but not affine invariant. An affine invariant version is constructed using a positive definite matrix M and the M-norm given by

$$\|Z\|_M = \sqrt{z' M^{-1} z}, z \in R^d \tag{5}$$

Let S_X be a positive definite d×d matrix that measures the dispersion of X in an affine equivariant way, such that $S_{XA+b} = AS_XA'$ holds for any matrix A of full rank and any b. Then an affine invariant L_2-depth is given by $(1 + E(\|z - X\|S_X)^{-1}$. Besides invariance, it has the same properties as the L_2-depth. A simple choice for S_X is the covariance matrix $\sum_X$ of X.

Simplicial Depth (SD)

Liu (1990) [4] introduced the concept of simplicial depth, which measures the centrality of a point x in a p-dimensional data set, x ϵ $S_n \subset \mathbb{R}^p$. Simplicial depth is defined as the number of closed simplices containing x and having p+1 vertices in S_n. In the bivariate case, it counts the number of triangles formed by sample points in S_n contain x. Simplicial depth is robust against outliers: if a set of sample points is represented by the point of maximum depth, up to a constant fraction of the sample points can be arbitrarily corrupted without significantly altering the location of the representative point. It is also invariant under affine transformations.

However, simplicial depth lacks some desirable properties for robust measures of central tendency. For centrally symmetric distributions, there is not always a unique point of maximum depth at the center of the distribution. Additionally, the simplicial depth does not necessarily decrease monotonically from the point of maximum depth. Despite these limitations, simplicial depth remains a useful measure in robust statistics and computational geometry, particularly for its robustness to outliers and its affine invariance. Simplicial depth is a robust, non-parametric method for measuring the centrality of a point in a multivariate dataset. By focusing on simplices (the convex hulls of subsets of data points), simplicial depth provides a geometric measure of how deep or central a point is within the distribution. It is particularly useful for outlier detection and robust estimation in multivariate data, but its computational complexity can be a limitation, particularly in high-dimensional datasets.

Projection Depth (PD)

The projection depth function initiated by Liu (1992) [5]. It is based on a measure of outlyingness and the idea behind the Donoho (1982) [1]. Further, this depth function was explored by Zuo and Serfling (2000) [11].

For a univariate distribution function F of x, the outlyingness $O(F,x)$ is defined as

$$O(F,x) = \sup\{Q(u,x,F)\} \tag{6}$$

over all unit vectors u, where $Q(u,x,F) = \frac{(u^T x - \mu(F_u))}{\sigma(F_u)}$, and F_u is the distribution of $u^T x$.

The projection depth $PD(x,F)$ is then given by

$$PD(x,F) = \frac{1}{1+O(x,F)} \tag{7}$$

This approach reflects the projection pursuit methodology, involving the supremum over infinitely numerous direction vectors, making the computation of projection depth seemingly intractable. Initially, classical location and scale measures were used, but these were later replaced by robust measures like the median and Median Absolute Deviation (MAD).

III. Numerical Study

This section evaluates the effectiveness of various data depth procedures by considering both real and simulation studies. The analysis includes a comprehensive assessment of different depth procedures, such as Mahalanobis, Halfspace, L_2, Simplicial, and Projection depths. These procedures are applied to both real datasets and simulated data to provide a robust evaluation of their performance. By calculating and comparing the depth values, the study aims to determine the efficiency and reliability of each method. This comparison helps identify which depth procedures are most effective in accurately determining centrality and handling outliers.

The experimental findings from two different real datasets, available in R packages, are presented in this section. These datasets contain one or more predictors. The depth values computed using various depth functions are presented in Tables 1 and 2.

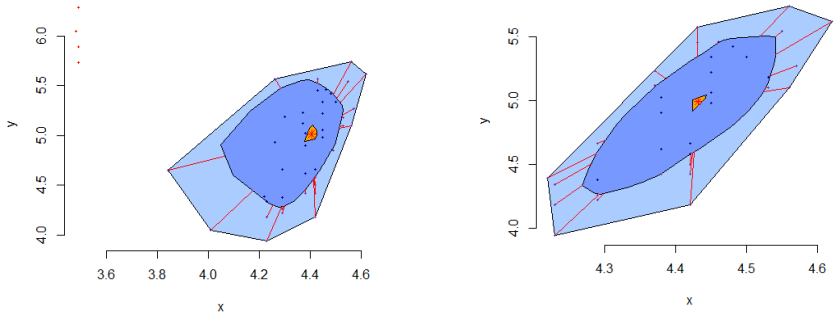
starsCYG dataset – It contains features of 47 stars in the Hertzsprung-Russell diagram of the Star Cluster CYG OB1. It includes one predictor variable, the logarithm of the star's effective surface temperature (log.Te), and one response variable, the logarithm of its light intensity (log.light). Cook's distance is used to identify the 9 outliers in the dataset.

Anscombe dataset – There are 51 observations in this dataset. The predictor variables are Income, Young, Urban and the response variable is Education. Cook's distance revealed 7 outliers in this dataset.

Table 1: *Computed depth values for starsCYG dataset*

Methods	MD	HD	L_2D	SD	PD
With Outliers	0.941	0.383	0.465	0.322	0.670
	(25)	(28)	(25)	(25)	(25)
Without	0.920	0.342	0.433	0.345	0.714
Outliers	(42)	(42)	(42)	(33)	(42)

(.) Observation number



(a). With Outliers (b). Without Outliers

Figure 1: *Bagplot for starsCYG dataset (with and without outliers)*

Table 2: *Computed depth values Anscombe dataset*

Methods	MD	HD	L2D	SD	PD
With Outliers	0.869 (14)	0.333 (25)	0.352 (25)	0.145 (25)	0.565 (25)
Without Outliers	0.941 (42)	0.341 (25)	0.346 (25)	0.169 (25)	0.583 (25)

Tables 1 and 2 reveal that, in both the presence and absence of outliers, the Mahalanobis depth consistently exhibits the highest depth values. This indicates that the Mahalanobis depth method is particularly effective at measuring centrality, regardless of whether outliers are present in the dataset. The real study compares the performance of five data depth methods MD, HD, L2D, SD, and PD with and without outliers. When outliers are present, MD is the most efficient method with a score of 0.869, showing its robustness in handling contamination. In contrast, SD and PD exhibit a significant drop in efficiency, with scores of 0.145 and 0.565, respectively, indicating their susceptibility to outliers. When the outliers are removed, the efficiency of all methods increases, with MD still leading at 0.941. HD and L2D show similar performance, with scores of 0.341 and 0.346, respectively. The efficiency of SD and PD improves somewhat after removing outliers, but they still remain the least efficient with scores of 0.169 and 0.583, respectively. The results highlight that MD is the most robust and efficient method, particularly when outliers are present, while HD and L2D offer a balanced performance.

The simulation study aims to assess and compare the efficiency of different data depth procedures in handling multivariate data. It investigates how each method performs under various contamination scenarios, such as location and scale contamination, which simulate real-world deviations from ideal data. The goal is to identify the most effective and reliable depth procedures, particularly in the presence of data contamination, which is common in practical applications. The data is simulated from normal distribution of sample size n=1000 with mean vector μ (0, 0) and unit covariance matrix $\sum = I_2$, and the simulated data is then contaminated in three different scenarios such as location contamination, scale contamination, and a combination of location and scale contamination. These contaminations are introduced at varying levels of 0%, 5%, 10%, 15%, 20%, and 25%.

For location contamination, the simulated data is contaminated by the mean vector μ (5, 5). In the case of scale contamination, the data is contaminated by altering the covariance matrix to $\sum = 2I_2$. In location and scale contamination, the simulated data is contaminated by the mean vector μ (3, 3) and $\sum = 1.5I_2$. These varying levels of contamination allows to evaluate the robustness and performance of different data depth procedures under different types and degrees of data contamination and are presented in Table 3.

Table 3: *Computed depth values for Simulation study*

Levels	MD	HD	L_2D	SD	PD
0%	0.869 (14)	0.333 (25)	0.352 (25)	0.145 (25)	0.565 (25)
Location Contamination					
5%	0.960 (45)	0.377 (45)	0.397 (45)	0.139 (45)	0.664 (45)
10%	0.958 (54)	0.388 (45)	0.399 (45)	0.155 (45)	0.706 (45)
15%	0.929 (54)	0.388 (45)	0.389 (45)	0.159 (43)	0.662 (45)
20%	0.936 (54)	0.377 (45)	0.390 (54)	0.157 (45)	0.667 (45)
25%	0.935 (54)	0.311 (43)	0.388 (54)	0.145 (43)	0.598 (43)
Scale Contamination					
5%	0.992 (52)	0.432 (52)	0.385 (52)	0.161 (52)	0.838 (52)
10%	0.890 (59)	0.344 (62)	0.386 (59)	0.136 (62)	0.609 (62)
15%	0.987 (52)	0.433 (52)	0.392 (52)	0.166 (52)	0.788 (52)
20%	0.888 (47)	0.352 (45)	0.385 (45)	0.152 (45)	0.673 (45)
25%	0.972 (47)	0.432 (47)	0.391 (47)	0.168 (47)	0.745 (47)
Location and Scale Contamination					
5%	0.916 (45)	0.344 (45)	0.381 (45)	0.145 (45)	0.583 (45)
10%	0.951 (45)	0.412 (45)	0.386 (45)	0.158 (45)	0.734 (45)
15%	0.950 (45)	0.382 (45)	0.387 (45)	0.156 (45)	0.676 (45)
20%	0.924 (45)	0.382 (45)	0.383 (45)	0.152 (45)	0.710 (45)
25%	0.922 (45)	0.381 (45)	0.382 (45)	0.158 (45)	0.707 (45)

Based on the results presented in Table 3, it can be concluded that the Mahalanobis depth consistently identifies the deepest location point among the different data depth procedures evaluated. This indicates that the Mahalanobis depth is particularly effective at determining the central point of the dataset, demonstrating its robustness and reliability in comparison to other depth measures. Even in the presence of outliers, Mahalanobis depth shows the smallest decrease in efficiency, highlighting its ability to maintain accuracy when the data is contaminated. The method's robustness is further emphasized by its superior performance under both location and scale contamination scenarios.

IV. Discussion

The study concludes that among the various data depth measures tested, Mahalanobis Depth consistently identifies the deepest points across different scenarios, both with and without outliers. This suggests that Mahalanobis Depth provides a stable measure of centrality, even when the data contains extreme values or deviates from standard assumptions. In contrast, other depth measures like Halfspace Depth and Projection Depth demonstrate sensitivity to outliers and complex distributions, sometimes shifting central points. L_2 Depth and Simplicial Depth also showed varied performance, especially in non-elliptical data structures. Overall, Mahalanobis Depth's consistent centrality assessment highlights its utility in robust statistical applications where a reliable measure of depth is crucial.

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OPTIMIZING A LINEAR FRACTIONAL FUNCTION OVER AN INTEGER EFFICIENT SET

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Abstract

Over recent decades, significant advancements have been made in optimization over the efficient set. This paper introduces a novel exact algorithm designed to optimize a linear fractional objective function over the integer efficient set of a multi-objective linear programming problem (MOILP). Without enumerating all efficient solutions, our method employs a selection strategy to iteratively improve the primary objective while progressively refining the feasible region and excluding dominated points. By exploring edge connections within the truncated feasible space, the proposed algorithm ensures convergence to the global optimal value in a finite number of iterations. A numerical example demonstrates the algorithm's effectiveness and practical application. This approach addresses critical challenges in multi-objective integer programming, particularly the nonconvexity of the efficient set and the absence of explicit feasible set descriptions.

Keywords: multiple objective programming, integer programming, linear fractional programming, efficient solutions.

1. Introduction

Multi-objective integer programming (MOIP) is an important research area as many practical situations require discrete representations by integer variables and many decision makers have to deal with several objectives. Some note-worthy practical environments where the MOIP problems find their applications are supply chain design, logistics planning, scheduling and financial planning.

In the past two decades, researchers and practitioners have shown increased interest in the problem of optimizing a linear function on the efficient set of multiple objective linear programming problem (MOLP). Several methods and algorithmic ideas have been developed-in general, these approaches can be classified and grouped according to the methodological concepts-which include, among others, adjacent vertex search technique ([16, 9, 10], nonadjacent methods [7], dual approach [19], etc. An overview of these approaches can be found in Yamamoto [21].

In addition to the continuous case, few algorithms have been suggested for solving the problem involving discrete decision variables. For the first time in [15] made an attempt to optimize on the integer efficient set, where only an upper bound value for the main objective is proposed. Jorge [13] developed approach that defines a sequence of progressively more constrained single-objective integer problems that successively eliminates undesirable points.

Fractional programming is an optimization problem in which ratio of two linear functions is optimized subject to some constraints [5, 14]. Integer Linear Fractional Programming problem is

an important class of problems arising in criteria Decision Making when some or all the model variables represent discrete decisions.

In preparing this paper, a special effort has been made do make certain that it is self-contained and that it is suitable both a as a text and as a reference. within we developed an algorithm that optimized linear fractional function ever the efficient set of a MOILP without explicitly having to enumerate all the efficient solutions. Given a Integer Linear Programming problem with Multiple Objective (MOILP):

$$(P_D)\begin{cases} "max" & Z_i = C_i x, \qquad i \in \{1,...,p\} \\ \text{s.t.} & x \in D \end{cases} \tag{1}$$

Where $C_i \in R^n$, for each $i \in \{1,...,p\}$, $A \in^{m\times n}$, $b \in^m$ and D is a polyhedral set of ^{n} defined as D=$\{x \in^n | Ax = b, x \geq 0, integer\}$. To avoid the technicality we assume throughout the paper that D is nonembounded.

The search of specific methods for solving (1) that provide the decision maker with his/her preferred efficient solution without having to explicitly determine the set of all efficient solutions of (1)denoted by $E(P_D)$, efficiency and non-dominance are defined as follows (see [17, 23, 24]) is doubtless a very difficult task that can be tackled in many different ways. One of such approaches, that has been studied successfully by Philip [16], in which an algorithm based on moving to adjacent efficient vertices is outlined when $\Phi(x)$ is a linear function, and lots of papers followed his work [22]. Our aim in this study is to provide one approach in the discrete case, consists of optimizing $\Phi(x)$ a Linear Fractional function representing the preferences of the decision-maker over the efficient set of (1). Formally, the problem under consideration can be defined as:

$$(P_E)\begin{cases} max & \Phi(x) = \dfrac{Ux+\alpha}{Vx+\beta} \\ \text{s.t.} & x \in (P_D) \end{cases} \tag{2}$$

Where α, β are scalars; $p, q \in R^n$.

The main difficulty of the problem arises from the nonconvexity of the efficient set ($E(P_D)$), which is the union of several faces of X. This problem was first considered by [16], in which an algorithm based on moving to adjacent efficient vertices is outlined when ¶ is a linear function, and lots of papers followed his work.

It is worth noting that solving (2) involves several difficulties since its feasible set, ($E(P_D)$), is not explicitly known, nor a convenient implicit description (say, e.g., integer linear) is available. As a consequence, (2) is a global optimization problem, frequently with multiple local (not necessarily global) optima [[22], [11]]. However, some particular instances of problem (1) can be solved straightforwardly, due to their special characteristics. More precisely, when the multi objective problem IP is completely efficient [2].

Generally, $E(P_D) \neq D$. Otherwise, if (D) is completely efficient, $E(P_D)$ can be substituted by D and, in such cases, solving (P_E) is equivalent to solving the following program:

$$(P_{E-relaxed})\begin{cases} max & \Phi(x) = \dfrac{Ux+\alpha}{Vx+\beta} \\ \text{s.t.} & x \in D \end{cases} \tag{3}$$

2. The main results

Definition 1. A point $x^0 \in D$ is said to be efficient of (1) if and only if there does not exist another point $x^1 \in D$ such that $Z_i(x^1) \geq Z_i(x^0)$ for all $i \in \{1,...,p\}$ and $Z_i(x^1) > Z_i(x^0)$ for at least one $i \in \{1,...,p\}$.

2.1. Testing Efficiency

The following result (see [12]) is used in various steps of the algorithm to test the efficiency of a given feasible solution of problem (1).

Theorem 1. Let x^* be an arbitrary element of the region D. $x^* \in E^{FF}$ if and only if the optimal value of the objective ψ is null in the following mixed integer linear programming problem:

$$(P_{x^*}) \begin{cases} \max & \psi(x) = \sum_{i=1}^{p} \Psi_i \\ \text{s.t.} & Cx - I\Psi = Cx^* \\ & x \in D, \Psi_i \in \mathbb{R}^+; \forall i \in \{1, ..., p\}. \end{cases} \tag{4}$$

C is a matrix (p, n) of which her $i^{ème}$ line corresponds to c^i, $i = 1, 2, ..., p$, I is the matrix identity (p, p) and $\Psi = (\Psi_i)_{i=1,...,p}$. The problem (P_{x^*}) .

Is often used to test the efficiency of a given point. (P_{x^*}) can be also used to generate an efficient point even x^* is not efficient ([9]).

2.2. Notation and Definitions

- $x_k = x_{k,j}$ is one optimal integer solution obtained in $D_k$at step k.
- B_k is the basis associated with solution x_k;
- $a_{k,j} \in R^{m_k \times 1}$ is the activity vector of $x_{k,j}$ with respect to the current truncated region D_k;
- $I_k = \{j \mid$ the vector $a_{k,j}$ is a column of the basis $B_k\}$ (indices of basic variables);
- $N_k = \{j \mid$ the vector $a_{k,j}$ is not a column of the basis $B_k\}$ (indices of non-basic variables);
- $y_{k,j} = (y_{k,ij}) = (B_k)^{-1} a_{k,j}$, where $y_{k,j} \in R^{m_k \times 1}$;
- $U_j =$ the j^{th} component of vector U;
- $V_j =$ the j^{th} component of vector V;
- $p_{k,j} = \sum_{i \in I_k} p_i y_{k,ij}$
- $q_{k,j} = \sum_{i \in I_k} q_i y_{k,ij}$
- $Z_1(x_k) = \dfrac{Z_{k,1}}{Z_{k,2}} = \dfrac{Ux_k + \alpha}{Vx_k + \beta}$
- $\gamma_{k,j} = Z_{k,2}(p_j - p_{k,j}) - Z_{k,1}(q_j - q_{k,j})$, the updated value of the j_{th} component of the reduce gradient vector $\bar{\gamma}_k$

Definition 2. Assume that $j_k \in N_k$ An edge E_{jk} incident to a solution X_k is defined as the set

$$E_{j_k} = \left\{ x_i \in R^{|I_k|+|N_k|} \;\middle|\; \begin{array}{l} x_i = x_{k,i} - \theta_{j_k} y_{k,ij_k} \; for \; i \in I_k \\ x_{j,k} = \theta_{j,k} \\ x_\alpha = 0 \; for \; all \; \alpha \in N_k \setminus \{j_k\} \end{array} \right\}$$

Where $0 < \theta \leq \min_{i \in I_k} \{\dfrac{x_{k,i}}{y_{k,ij_k}} | y_{k,ij_k} > 0\}$, θ_{j_k} is a positive integer and $\theta_{j_k} \times y_{k,ij_k}$ for $i \in I_k$ are integers for all $i \in I_k$ if such integer values exist.

Theorem 2. [14] Let X_1 be an optimal solution of problem (3) All integer feasible solutions of problem (3) alternate to X_1 on an edge E_{j_1} of region D (or truncated region D_1) emanating from it, in the direction of vector a_{1,j_1}, $j_1 \in J_1$ with $J_1 = \{j \in N_1 \mid \overline{\gamma}^1_{1,j} = 0\}$ lie in the open half space

$$\sum_{j \in N_1 \setminus \{j_1\}} x_j < 1$$

Theorem 3. [4] The point x_1 of D is an optimal solution of problem (3) if and only if the reduce gradient vector $\bar{\gamma} = \bar{\beta}\bar{p} - \bar{\alpha}\bar{q}$ is such that $\bar{\gamma}_j \leq 0$ for all $j \in_k$.

Theorem 4. [17] $x^* \in E(P_D$ if, and only if, $\{(x^* + C^{\geq}) \cap D = x^*\}$.

3. Development of the algorithm and theoretical results

The proposed algorithm provides a global optimal solution of (P_E) without specifying all efficient solutions of (P(D)).

Initially, we solve the relaxed problem (3) associated to problem (P_E). Obviously, only in a reduced number of special cases would the solution of (3) provide the optimal solution of (P_E). So if it were not the case, a new efficient solution dominating the previous one is then obtained. The efficient solution $\tilde{x}^l$ issued from the efficiency test is considered as a first efficient solution.

Assuming that all coefficients of matrix C are integers, at iteration k, the feasible set D is reduced gradually by eliminating all dominated solutions by $C(\tilde{x})^l$ (see Sylva and Crema, 2004, 2007). The resolution of the following problem enables us to perform this elimination:

$$(Rf_l) : max\{\frac{Ux + \alpha}{Vx + \beta} | x \in D - \cup_{s=1}^{l} D_s\} \tag{5}$$

$\{x^s; s = 1, ..., l-1\}$ are solutions of (P_D) obtained at iterations $1, 2, ..., l-1$ respectively. Where $D_s = \{x \in^n | Cx \leq Cx^s\}$ and $\{Cx^s\}_{s=1}^{l}$ is a subset of nondominated criteria vectors for problem (P_D).

$$D - \cup_{s=1}^{l} D_s = \left\{ \begin{array}{l} c^i x \geq (c^i x = 1) y_i^s + M_i(1 - y_i^s,\ i = 1, 2, ...p\ s = 1, 2, ..., l.) \\ \sum_{i=1}^{s} y_i^s \geq 1,\ s = 1, 2, ..., l \\ y_i^s \in \{0, 1\},\ i = 1, 2, ...p\ s = 1, 2, ..., l \\ x \in D \end{array} \right\}$$

where M_i is a lower bound for any feasible value of the ith objective function. The associate variables y_i^s $i = 1, 2, ...p$ of $C\tilde{x}^s$ and additional constraints are added to impose an improvement on at least one objective function. Note that when $y_i^s = 0$, the constraint is not restrictive and when $y_i^s = 1$ a strict improvement is forced in the ith objective function evaluated at $C\tilde{x}^s$.

We start exploring all edges incident to $\tilde{x}^l$ corresponding to J_1 until an efficient solution is found to improve Φ_{opt}. We solve the problem (Rf_l). The optimal solution obtained, x^l, produces a minimum value of the criterion $\Phi(x)$ in the reduced domain. The process continue in this manner until the current feasible space becomes empty or $\Phi(x^l) > \Phi_{opt}$.

Proposition 1. [6] Let $\tilde{x}^1$ $\tilde{x}^2$,...,$\tilde{x}^l$ be efficient solutions to problem (P_D) and $D_s = \{x \in^n | Cx \leq Cx^s\}$. Let $\tilde{x}^*$ be an efficient solution to the multi-objective integer problem $P_k \equiv$ " max "$\{Cx, x \in D - \cup_{s=1}^{l} D_s\}$. Then $\tilde{x}^*$ is an efficient solution to the problem (P_D).

3.1. Theoretical Results

Proposition 2. Let $\tilde{x}^1$, $\tilde{x}^2$, ...,$\tilde{x}^l$ be efficient solutions to problem $P(D)$ and $D_s = \{x \in^n | Cx \leq Cx^s\}$.

Let $\ddot{x}^l$ be an alternative solution of x^l of the problem (Rf_l) with $\frac{Ux^{l+1} + \alpha}{Vx^{l+1} + \beta} > \max_{j \in 1,...,l} \{\frac{U\tilde{x}^s + \alpha}{V\tilde{x}^s + \beta}\}$.

If $\ddot{x}^l$ is an efficient solution to problem $(P(D))$ then is an optimal solution of (P_E).

if problem (Rf_l) is unfeasible then $\{C\tilde{x}^s\}_{s=1}^{l}$ is the entire set of non-dominated criterion vectors for problem (P_D).

Proof. Suppose on the contrary that $\ddot{x}^l$ is not an optimal solution of (P_E). Then a feasible solution exists $\hat{x} \in E(P_D)$ such that with the value of the function main to the $\hat{x}$ point superior a $\frac{Ux^l+\alpha}{Vx^l+\beta}$. As $\ddot{x}^l$ is an alternative solution for x^l ($\Theta_{jl}^0 \neq 0$)to (Rf_l) because $\hat{x}^s$

$\frac{U\ddot{x}^l+\alpha}{V\ddot{x}^l+\beta} = \frac{Ux^l+\alpha}{Vx^l+\beta}$. Thus $\hat{x} \in \cup_{s=1}^{l} D_s$ therefore $\hat{x} \in D_s$ for some $s \in \{1,...,l\}$ and, accordingly to the definition of D_s, $C\hat{x} \leq C\tilde{x}^s$. As $\hat{x} \in E(P_D$ we have that $\frac{U\hat{x}+\alpha}{V\hat{x}+\beta} < \frac{U\tilde{x}^s+\alpha}{V\tilde{x}^s+\beta}$.

consequently $\frac{U\ddot{x}^l+\alpha}{V\ddot{x}^l+\beta} = \frac{Ux^l+\alpha}{Vx^l+\beta} < \frac{U\hat{x}+\alpha}{V\hat{x}+\beta} < \frac{U\tilde{x}^s+\alpha}{V\tilde{x}^s+\beta}$ who is contradicting with the hypothesis $\frac{U\ddot{x}^l+\alpha}{V\ddot{x}^l+\beta} > \max_{j\in 1,...,l}\{\frac{U\tilde{x}^s+\alpha}{V\tilde{x}^s+\beta}\}$.

If(Rf_l) is unfeasible then $E(P_D) \subseteq \cup_{s=1}^{l} D_s$ and for any $x \in E(P_D)$ there exists an x^s such that $Cx \leq Cx^s$. In this case we must proceed as follows: let $\tilde{x} \in E(P_D)$ for the reason there is an $\exists s \in 1,...,l$ with $Cx^s \geq C\tilde{x}$ then $Cx^s = C\tilde{x}$ (and Cx is a dominated vector).

■

3.2. Algorithm

The algorithm used to obtain an integer optimal solution to our main problem (P_E) is can be summarized as follows:

Algorithm 1: part 1

input :
$A_{(m\times n)}$: matrix of constraits,
$b_{(m\times n)}$,
RHS vector,
$C_{(p\times n)}$: matrix of criteria.
$U_{(1\times n)}$, $V_{(1\times n)}$: main criterion vector,
α, β: are scalars.
output :
X_{opt}:optimal solution of the problem (P_E),
Φ_{opt}:optimal value of the main criterion Φ
initialization:
for $i \leftarrow 1$ **to** p **do**
  solve $M_i = min\{C^i x, \quad x \in D\}$ set the lower bounds;
$\Phi_{opt} := -inf$,
$l := 1$,
$E_1 :=$, $\bar{D} := D$,
$optimal := false$,
$alternative := false$,
$explore := true$.

Algorithm 2: part 2

while *optimal:=false* **do**
- solve $P^l_{RF} \equiv max\{\frac{Ux+\alpha}{Vx+\beta}, \quad x \in \bar{D}\}$;
- **if** P^l_{RF} *is infeasible* **then**
  - X_{opt} an optimal solution of (P_E); *optimal* := *true*,**Terminate**;
- **else**
  - let x^l bean optimal solution of P^l_{RF};
  - **efficiency test**: solve $(p(x^l))$, Ψ is the optimal solution criteria;
  - **if** $\Psi = 0$ **then**
    - x^l an efficient solution; X_{opt} an optimal solution of (P_E) and $\Phi_{opt} = \Phi(x^l)$;
  - **else**
    - x^l is not efficient solution, $\tilde{x}^l$ an optimal solution of $(p(x^l))$ is efficient;
    - solve $Q(\tilde{x}^l) \equiv max\{\frac{Ux+\alpha}{Vx+\beta}, \quad x \in \bar{D}, \quad Cx = C\tilde{x}^l\}$;
    - let $\hat{x}^l$ bean optimal solution of $Q(\tilde{x}^l)$
    - **if** $\Phi(\hat{x}^l) > \Phi_{opt}$ **then**
      - $X_{opt} = x^l$, $\Phi_{opt} = \frac{Ux^l+\alpha}{Vx^l+\beta}$, let $E_{l+1} = E_l \cup \{\hat{x}^l\}$;
    - $l = l + 1$ and $\bar{D} := D \ \cup_{s=1}^{l-1} D_s$; $D_s = \{x \in Z^n / Cx \leq C\hat{x}^l, \quad \hat{x}^l \in E_{l-1}\}$
    - solve$P_l \equiv \{max\frac{Ux+\alpha}{Vx+\beta}, \quad x \in \bar{D}\}$, let x^l an optimal of P_l;
    - **if** $\bar{D} =$ *or* $\Phi(x^l) < \Phi_{opt}$ **then**
      - X_{opt} an optimal solution of (P_E) and Φ_{opt} the optimal value of (P_E)
      - *optimal* := *true*,**Terminate**;
    - **else**
      - optimal:=False ; solve $(P(x^l)$
      - **if** $\Psi = 0$ **then**
        - x^l an efficient solution $X_{opt} = x^l$; optimal:=True; $E_{l+1} = E_l \cup \{x^l\}$;
        - **Terminate**;
      - **else**
        - $\bar{x}^l$ is an optimal solution of $P(x^l)$ is efficient $E_{l+1} = E_l \cup \{x^l\}$;
        - construct the set $\Gamma_l = \{j \in N_l / \bar{\gamma}^l_j = 0\}$;
        - **if** $\Gamma_l \neq$ **then**
          - i:=1;
          - **while** $\Gamma_l \neq$ *and explore:= true* **do**
            - (search $\ddot{x}^l_1$ integer efficient solution for x^l) calculate $\Theta^0_{jl}$the integer part of $min_{i \in I_k}\{\frac{x^1_{l,i}}{x^1_{l,ij_l}} / y^1_{l,ij_l} > 0\}$;
            - **if** $\Theta^0_{jl} = 0$ **then**
              - $\Gamma_l = \Gamma_l \setminus \{J_l(i)\}$;
            - **else**
              - $\Theta := \Theta^0_{jl}$;
              - **while** $\Theta > 0$ *and alternative:=False* **do**
                - searching for a efficient integer solution on edge E_{jl} corresponding to Θ^0_{jl} and test for efficiency, solve P(D)
                - **if** $\Psi = 0$ **then**
                  - alternative:=true; $\Phi_{opt} := \Phi_{x_{expl}}$; optimal:=True; $E_{l+1} = E_l \cup \{x_{expl}\}$; **Terminate**;
                - **else**
                  - $\Theta := \Theta - 1$
            - i:=i+1;
- l:=l+1;

Proposition 3. The algorithm terminates in a finite number of iterations.

Proof. By hypothesis provided D is non-empty and D is bounded, $\{Cx^s\}_{s=1}^{l}$ is finite. With the progression of to advance in the algorithm, the domain of feasibility becomes more and more is strictly reduced by the theorem (sylva [18],[6]) or Φ_{opt} strictly increases.

The theorem (3) guarantees that we can obtain an optimal solution integer of (Pf_R) if it exists, and the theorem (Testing efficiency 1 with [9]) one gets an optimal solution for the problem (2) having in mind that at least one new efficient solution is generated at each iteration since for an arbitrary l none of the previously generated efficient point is feasible, the proof is thus complete. ■

4. Numerical illustration

To illustrate the use of this algorithm, we consider the following integer linear program with tow objectives:

$$(P_{(D)}) \begin{cases} max & Z_1 = x_1 - 2x_2 \\ max & Z_2 = -x_1 + 4x_2 \\ s.t. & -2x_1 + x_2 \leq 0, \\ & 6x_1 + x_2 \leq 21, \\ & -2x_1 + 4x_2 \leq 6, \\ & x_1, x_2 \in \end{cases} \tag{6}$$

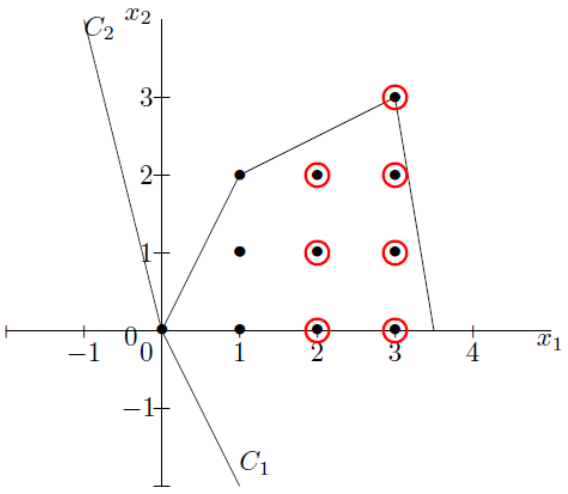

Figure 1: *Space of the decisions*

In this example, it is easy to see that D contains 11 feasible points (see Figure 1). Using the characterization of efficiency presented in Theorem (4), it can be shown that seven of them are efficient. Particularly, the efficient set $E(P_D)$ is given by: $^{FF} = \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(3,3)\}$. With the aim of illustrating how our algorithm works, we will solve the problem (2) given by

$$(Pf_{(E)}) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & x_1, x_2 \in_{FF} \end{cases} \tag{7}$$

step 0: Initialization We take $\Phi_{inf} = -\infty$, $\Phi_{sup} = +\infty$, $l = 1$.

After solving $\{\min C_i x,\ x \in D\}\ i = 1,2$, the lower bounds of the objective functions are $M_1 = -3$, $M_{12} = -3$

$$(Pf_{(R)}) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & -2x_1 + x_2 \leq 0, \\ & 6x_1 + x_2 \leq 21, \\ & -2x_1 + 4x_2 \leq 6, \\ & x_1, x_2 \in \end{cases} \tag{8}$$

(8) is solved, yielding the optimal solution $x^1 = (0,0)$, Let $Z(x^1) = (0,0)$.

- **Iteration 1.**
  - **Step 1.** In order to test the efficiency of x^1 we solve the problem (9), that is:

$$(P_{(x^1)}) \begin{cases} max & \Theta = \Psi_1 + \Psi_2 \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 - \Psi_1 = 0 \\ & -x_1 + 4x_2 - \Psi_2 = 0 \\ & \Psi_i \geq 0, i = 1,2. \end{cases} \tag{9}$$

    The optimal value of (9) is 2 , which is achieved at the point $\hat{x}^1 = (2,1)$. Thus, $\hat{x}^1 \in_{FF}$ and $x^1 \notin_{FF}$, since $C\hat{x}^1 \geq Cx^1$ We set $\Phi_{sup} = \Phi(x^1) = 1$
  - **Step 2.** When (10) defined as:

$$(Tf_1) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 = 0 \\ & -x_1 + 4x_2 = 2 \end{cases} \tag{10}$$

    is solved, $\tilde{x}^1 = \hat{x}^{1_1} = (2,1)$ is obtained as the optimal solution. Let $\bar{z}^1 = C\tilde{x}^1 = (0,2)$
    $\Phi(\tilde{x}^1) = 1/5 > \Phi_{inf} = -\infty$, put $\Phi_{inf} = 1/5$ et $X_{opt} = \tilde{x}^1$
    $\Phi_{inf} \neq \Phi_{sup}$, go to step 3

  - **Step 3.** The optimal solution of (11)

$$(RF_1) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & x_1, x_2 \in D \\ & x_1 - 2x_2 \geq y_1^1 - 3(1 - y_1^1) \quad (1) \\ & -x_1 + 4x_2 \geq 3y_2^1 - 3(1 - y_2^1) \quad (2) \\ & y_1^1 + y_1^1 \geq 1, \quad y_1^1, y_2^1 \in \{0,1\} \end{cases} \tag{11}$$

    is $x^2 = (1,2)$, $y = (0,1)$, being $Z(x^2) = (-3,7)$ and $\Psi = (0,2)$. In order to test the efficiency of x^1 we solve the problem (9), that is:

$$(P_{(x^2)}) \begin{cases} max & \Theta = \Psi_1 + \Psi_2 \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 - \Psi_1 = -3 \\ & -x_1 + 4x_2 - \Psi_2 = 7 \\ & \Psi_i \geq 0, i = 1,2. \end{cases} \tag{12}$$

    The optimal value of (12) is 2 , which is achieved at the point $\hat{x}^2 = (3,3)$; $\Psi = (0,2)$. Thus, $\hat{x}^2 \in_{FF}$ and $x^2 \notin_{FF}$, We set $\Phi_{sup} = \Phi(x^2) = \dfrac{1}{3}$ go to step 4.

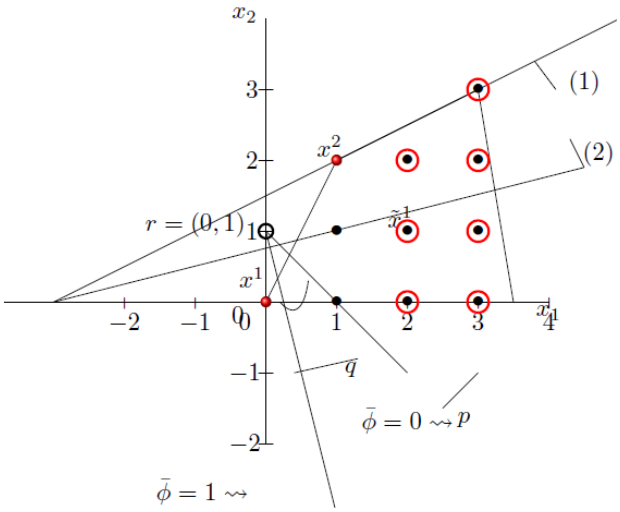

Figure 2: *The reduced regionD*[1]

- **Step 4.** $J_2 = \{j \in N_2 \mid \overline{\gamma}^2_{1,j} = 0\} = \emptyset$, go to step 2.

- **Iteration 2.**

  - **Step 2.** When (10) defined as:

$$(Tf_1)\begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 = -3 \\ & -x_1 + 4x_2 = 9 \end{cases} \tag{13}$$

  is solved, $\tilde{x}^2 = \hat{x}^2 = (3,3)$ is obtained as the optimal solution. Let $\bar{z}^1 = C\tilde{x}^1 = (0,2)$.
  $\Phi(\tilde{x}^1) = 5/17 > \Phi_{inf} = 1/5$, put $\Phi_{inf} = 5/17$ et $X_{opt} = \tilde{x}^2$
  $\Phi_{inf} \neq \Phi_{sup}$, go to step 3

  - **Step 3.**The optimal solution of (11)

$$(RF_1)\begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 \geq y^1_1 - 3(1 - y^1_1) \quad (1) \\ & -x_1 + 4x_2 \geq 3y^1_2 - 3(1 - y^1_2) \quad (2) \\ & y^1_1 + y^1_1 \geq 1, \quad y^1_1, y^1_2 \in \{0,1\} \\ & x_1 - 2x_2 \geq -2y^2_1 - 3(1 - y^2_1) \quad (3) \\ & -x_1 + 4x_2 \geq 10y^2_2 - 3(1 - y^2_2) \quad (4) \\ & y^2_1 + y^2_2 \geq 1, \quad y^2_1, y^2_2 \in \{0,1\} \end{cases} \tag{14}$$

The problem (14))is note feasible. Terminate, $X_{opt} = x^2 = (3,3)$ is an optimal solution of (P_E) with $\Phi(x^2) = 5/17$.

The set of all solutions efficient of this problem is: $^{FF} = \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(3,3)\}$. However, our algorithm optimizes the linear fractional function $\Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1}$ without having to determine all these solutions but only $\{(2,1),(3,3)\}$.

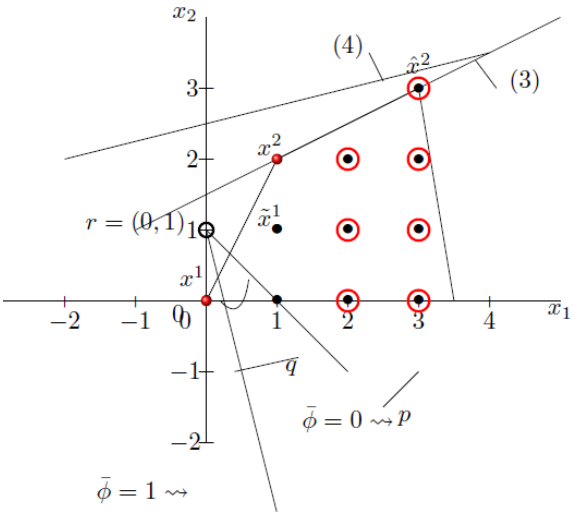

Figure 3: *The reduced region* D^2

5. Conclusion

The proposed algorithm optimizes a linear fractional function over the integer set of a multi-objective linear program (P_E) by using classical strategies of fractional programming and cutting plane techniques without having to enumerate all the efficient solutions. The main advantage of the proposed solution methodology is that no nonlinear optimization is required.

Although the research themes addressed is difficult, it is hoped that this article motivate the researchers to develop better solution procedures.

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A MODIFIED INCIDENT EDGE PATH ALGORITHM FOR EFFICIENT SHORTEST PATH SOLUTIONS IN PIPELINE NETWORKS AND URBAN NAVIGATION SYSTEMS

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Abstract

The article describes how to utilize the Modified Incident Edge Path Algorithm (MIEPA) to identify the cheapest transit option and the best route. The MIEPA algorithm, which is based on graph theory, is simple to use and can potentially be employed to major smart logistics challenges such as pipeline networks and Google Maps. It evaluates the most optimal approach to minimize transportation expenses using MATLAB. The algorithm ensures that each node gets visited and determines the shortest path from the origin to all other nodes. The running time complexity and theorem of the new method are presented, and the algorithm is compared to the existing algorithm. The proposed MIEPA addresses negative weights and prevents negative cycles. It has used two real-world problems to evaluate the suggested algorithm.

Keywords: Directed network, Incident Edge Path Algorithm, Optimum path, Pipeline networks, Google Map Network, Minimum transportation cost.
AMS Subject Classification: 05C85, 94C15, 90C27, 90C35

I. Introduction

Network optimization is a vital topic of operations research and smart logistics, with the goal of identifying the most effective routes while decreasing transportation costs. Network models are primarily concerned with addressing the Shortest Path Problem (SPP), which involves establishing the optimal path between nodes in a variety of real-world settings. This challenge is essential in various disciplines, including:

- Manufacturing: Determining the shortest path simplifies operations, resulting in faster and more productive manufacturing.
- Management: Rational thinking is critical for maximizing benefits and minimizing expenses.
- Transportation: Designing effective routes is critical for moving big amounts of commodities at a low cost.

In essence, the shortest path problem covers multiple challenges, aiming to maximize outcomes in manufacturing, top management, and transportation. The basic definitions, terminologies and notations we refer[4, 5, 6].Many shortest path algorithms in graph theory, including Dijkstra's, Prim's, Kruskal's, and other shortest path and minimum spanning tree algorithms, provide effective solutions for finding optimal paths based on various real-world network problems [2, 10, 11, 12]. Dijkstra's algorithm gives the N-Shortest paths for the network, and it is implemented in Google Map Network problem[9]. A review paper helps the researchers to study well about the algorithms and also for implementation in real-world problems[7]. These algorithms assist decision-makers in identifying the most efficient routes with minimal transportation costs to

reach their destinations.These methods address key operational challenges, such as pipeline maintenance, which can lead to traffic congestion and disrupt the timely delivery of supplies to various businesses. By adopting these strategies, industries can identify the fastest routes with minimal travel time, reducing disruptions and ensuring swift access to affected areas [1, 13]. In addition, optimizing algorithms for both space and time complexity can substantially enhance business profitability. The temporal complexity of greedy algorithms has been thoroughly examined by Yogesh Chanchal and Dr. Ashedra Kumar Saxena (2015), highlighting their role in improving operational efficiency. Refining these algorithms enables industries to manage resources more effectively, lower costs, and increase revenues [16]. Furthermore, Chu Fei-Xui et al. applied the well-established Dijkstra method to calculate the N-shortest paths in pipeline networks [3]. Shortest path methods in graph theory are vital not only for financial applications, but also for network systems, engineering, and computer science. These methods are now widely used in Computer Science and Engineering, highlighting their significance in improving network performance, system design, and computing efficiency[8].
Recently, particularly during the epidemic, there has been a rise in demand for prompt and dependable delivery services. To ensure timely and efficient delivery of food, medicine, and other supplies, delivery agents or organizations must find the shortest routes while simultaneously minimizing fuel expenses. Yassine Issaoui and colleagues addressed these needs in their work, focusing on improving delivery routes to ensure timely deliveries at minimal costs[14]. Jijun Hu and colleagues investigated the use of Dijkstra's Algorithm to calculate the best path for Automatic Guided Vehicles (AGVs) in warehouse distribution systems. Their job entails synchronizing duties with vehicle movements to improve efficiency. They proposed an effective architecture using Dijkstra's Algorithm to address numerous warehouse-related difficulties, hence boosting overall system performance[15]. In this paper, we introduce the Modified Incident Edge Path approach (MIEPA), a unique approach for determining the shortest paths between nodes. The proposed MIEPA is modified from the incident edge path algorithm proposed in[17]. We used MIEPA to solve two real-world problems: (1) determining the best path in a pipeline network to reach pump stations from a specified origin pump station and (2) calculating the shortest path in Google Maps for efficient navigation. The MIEPA algorithm's performance in these applications is examined for reference [3].Novelty of the proposed MIEPA is it handles negative weighted edges of the transportation network by avoiding cycles unlike existing algorithms. It is appropriate for sparse networks because it streamlines the process of determining the minimal or most effective route from the origin to all subsequent nodes without requiring backtracking. This paper is structured as follows:
Introduction: This article introduces the Modified Incident Edge Path Algorithm (MIEPA), its concepts, and a theoretical proof.
Application: The algorithm is evaluated in real-life applications, including pipeline networks and navigation systems.
Evaluation: We assess the effectiveness of the MIEPA algorithm and discuss how it addresses its limitations and shortcomings.

II. Preliminaries

A graph $G(V, E)$ is directed and taken as a network representation, where V is the number of nodes denoted $|V| = n$ and E is the number of edges denoted $|E| = m$. In G, the node v_i is an end node of some edge e_j, v_i and e_j is said to be incident with each other. The edges incident with v_i are said to be incident edges of G [5]. The distance between nodes v_i and v_j is $d(v_i, v_j)$. The weight of the node v is considered as the cost of each u, v-path. If v has more than one u, v-path, then the minimum cost is assigned to the end node v.
A graph having no loops or parallel edges is a simple network [5]. A path is a simple network whose nodes can be linearly ordered so that two nodes are adjacent if and only if they are one after the other in the list [4]. A path $(v_i - v_j)$ from the node v_i to v_j in a network is finite if it has a finite number of nodes; otherwise, it is infinite. The weight of a path is taken as the sum of the

distance between nodes in the respective path. If the path starts and ends with the same node, then it is known as a cyclic, otherwise acyclic, of the network [10].
Result: Let $G(V, E)$ be a simple and directed network; if G has only one shortest path from the origin node v_0 to the destination node v_n, then it is the optimum path for the network, providing minimum transportation cost. The proposed algorithm was examined by Kanchana et al. [17], this article presents the modification of IEPA and applies it in the pipeline network and Google Map Service network to find the shortest path of every node from the origin node. In the Modified Incident Edge Path Algorithm (MIEPA), we make the following assumptions:

- Taking the graph as a directed network (graph).
- The distance, costs, and time needed are associated with edges of the network that travels
- Single source with negative or non-negative edge weights avoiding cycles.
- Optimality of path drawn from each node from the origin node.
- The proposed algorithm involves performing optimality without backtracking from the beginning.

I. Modified Incident Edge Path Algorithm (MIEPA)

Formulation of Graphical problem: $G(V, E)$ be the directed simple graph. Each edge has the distance between the nodes.

- v_0 be the origin node and v_n be the end node of the graph.
- $V-$ Set of all nodes
- $E-$ Set of all edges/lines between nodes
- $d(v_i, v_j)-$ distance between the nodes
- E_i- Set of incident edges where $i = 1, 2, .., n$ and$j = 1, 2, .., n-1$.

A network $G(V, E)$, with node set V and edge set E, we set p_j the weight of the node v_i and E_j, the set of all incident edges of G where $0 \leq i \leq n, 1 \leq j \leq n$.
The following theorem provides the optimality of the proposed algorithm.

Theorem 1. Let $G(V, E, d(v_i, v_j))$ be any simple directed graph and the weight of the path $p_j = min\{p_i + d(v_i, v_j)\}$ is defined on G. If $E_n = \{\}$ then, the weight p_n of the path $(v_0 - v_n)$ is optimum, Where $d(v_i, v_j)$is the distance between nodes v_i and v_j for $i = 0, 1, ..., n, j = 1, 2, , n$.

Proof.
Let $G(V, E, d(v_i, v_j))$ is any simple directed graph with vertex set $V = \{v_0, v_1, ..., v_i, v_j, .., v_n\}$ and E_j is the collection of all incident edges incident with v_i of G.
given that, $p_j = min\{p_i + d(v_i, v_j)\}$ as the weight of path from v_i to v_j. Let E_1 is the set of incident edges from v_0 vertex, then the weight of path $v_0 - v_1$ is $p_1 = p_0 + d(v_0, v_1)$ E_2 is the set of incident edges from v_1, then the weight of path $(v_0 - v_1 - v_2)$ is $p_2 = p_1 + d(v_1, v_2)$,
If a vertex v_1 has two in-degree, then choose the minimum weight of the path with corresponding path. Thus, for v_j vertex, E_j is the set of incident edges, then the weight of the path $(v_0 - v_1 ... v_i - v_j)$ is $p_j = min\{p_i + d(v_i, v_j)\}$. Similarly for vertex v_n, E_n is empty since there are no edges incident with v_n. Thus $(v_0 - v_1 ... v_i - v_j ... v_{n-1} - v_n)$ is the obtained path from $p_n = min\{p_{(n-1)} + d(v_{(n-1)}, v_n)\}$ which is optimum.

■

Remark 1: If any nodes in-degree is more than one then we can choose and fix the minimum weight path as $p_j = min\{p_i + d(v_i, v_j)\}$ of the nodes from v_i to v_j and also fix the path weight as of the respective end nodes weight.
Remark 2: If any node has more than one weight, choose the minimum weight path and fix that minimum path weight as the end node's weight.

II. MIEPA - Steps

- Step 1: Find E_1, the set of all edges incident with v_0 in V. If E_1 is empty, go to Step 6 or proceed to Step 2.

- Step 2: Calculate path weight from E_1, $p(v_0, v_j) = w_0 + d(v_0, v_j)$. Fix the weight to end node of respective and choose the minimum without forming a cycle.
- Step 3: Find E_2, the set of all edges incident with v_j in V. If E_2 is empty, go to Step 6 or go to Step 4.
- Step 4: Calculate path weight from E_2, $p(v_j, v_i) = w_j + d(v_j, v_i)$. Fix the weight to the end node of the respective and choose the minimum without forming a cycle.
- Step 5: Repeat Steps 1 to 4 until the destination node is reached after visiting all edges. Optimum cost is the weight of the destination node.
- Step 6: Check for any negative weight cycles. Check for any weight adjustments; if a negative cycle exists or the optimal path is obtained, stop the process.

III. Comparison of MIEPA with existing algorithms

The Table 1, provides proposed MIEPA's weightage compared with other existing algorithms.

Table 1: *Comparison of Proposed MIEPA and Other Existing Algorithms*

Aspect	Proposed MIEPA	Dijkstra's Algorithm	Bellman-Ford Algorithm	A* Algorithm
Graph Type	Focuses on incident edges in dynamic, sparse graphs.	Functions with weighted graphs without negative edges.	Identifies negative cycles and manages negative edge weights.	Works with heuristic-based approaches on graphs.
Efficiency	More effective for sparse graphs as it focuses on local incident edges.	Efficient for dense graphs using priority queues ($O(V \log V)$).	Global edge checks result in slower runtime ($O(VE)$).	Very efficient when appropriate heuristics are available ($O(E)$ to $O(E + V \log V)$).
Time Complexity	$O(V + E)$, dependent on the number of incident edges, ideal for sparse graphs.	$O(V^2)$, or $O(E + V \log V)$ with priority queues.	$O(VE)$, due to global edge assessments.	$O(E)$ to $O(E + V \log V)$ depending on heuristic use.
Negative Weight Handling	Can handle negative weights by identifying and avoiding negative cycles.	Cannot handle negative weights; requires non-negative edges.	Handles negative weights but requires additional processing.	Cannot directly handle negative weights without modifications.
Use Case	Ideal for dynamic, adaptive, local exploration (e.g., transportation networks and pipelines).	Suitable for non-negative edge graphs (e.g., shortest paths in maps).	Useful for graphs with negative weights (e.g., finance, transportation).	Best suited for heuristic pathfinding in domains like games and robotics.
Edge Handling	Concentrates on incident edges (examines neighbors step-by-step).	Evaluates all edges connecting each node.	Evaluates all edges, even for negative weight cycles.	Prioritizes edges using heuristic rankings.
Adaptability	Highly adaptive for local connectivity and dynamic systems requiring cycle detection.	Not suitable for dynamic graphs or graphs with negative weights.	Supports static graphs with negative weights.	Performs well in domains with applicable heuristics (e.g., real-time pathfinding).

III. Smart Logistic Network Problems

I. Pipeline Network Problem

The MIEPA algorithm is used to determine the most cost-effective path in a pipeline network, helping industries like the South-West products pipeline industry optimize their operations. By representing pump stations as nodes and pipelines as edges and calculating the minimum cost using unit costs, the MIEPA algorithm provides an efficient method to find the optimal transportation routes within the network.

Assume Figure 1 illustrates a network diagram with nodes and edges representing pump stations and pipelines, respectively. The MIEPA algorithm will calculate the minimum cost of reaching the destination node by iteratively finding the shortest path in terms of transportation cost. The distance of edges in the pipeline network is shown in Table 2.

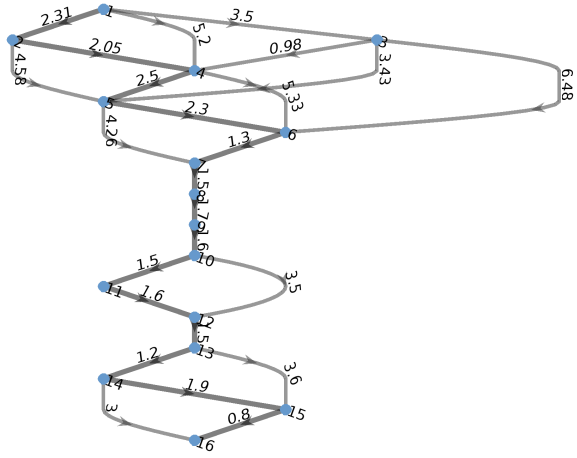

Figure 1: *Network of Pipeline*

Table 2: *Edge and Distance Between the Nodes*

Edge	**Distance between v_i and v_j**	**Edge**	**Distance between v_i and v_j**
(v_0, v_1)	2.31	(v_6, v_7)	1.5
(v_0, v_2)	3.5	(v_7, v_8)	1.7
(v_0, v_3)	5.2	(v_8, v_9)	1.6
(v_1, v_3)	2.05	(v_9, v_{10})	1.5
(v_1, v_4)	4.58	(v_9, v_{11})	3.5
(v_2, v_3)	0.98	(v_{10}, v_{11})	1.6
(v_2, v_4)	3.43	(v_{11}, v_{12})	1.5
(v_2, v_5)	6.48	(v_{12}, v_{13})	1.2
(v_3, v_4)	2.5	(v_{12}, v_{14})	3.6
(v_3, v_5)	5.33	(v_{13}, v_{14})	1.9
(v_4, v_5)	2.3	(v_{13}, v_{15})	3.0
(v_4, v_6)	4.26	(v_{14}, v_{15})	0.8
(v_5, v_6)	1.3		

The proposed MIEPA algorithm is applied, and the calculation for getting the optimum path is given in Table 3 for the pipeline network.

Table 3: *Pipeline Network Problem Solved by MIEPA*

Incident Edge Set E_j	**The Weight and the Shortest Path p_j of (v_j)**
$E_1 = \{v_0v_1, v_0v_2, v_0v_3\}$	$p_1 = 2.31$ of (v_1), $p_2 = 3.5$ of (v_2), and $p_3 = 5.2$ of (v_3)
$E_2 = \{v_1v_3, v_1v_4\}$	$p_3 = \min\{5.2, 4.36\} = 4.36$ of (v_3), $p_4 = 6.89$ of (v_4)
$E_3 = \{v_3v_4, v_3v_5\}$	$p_4 = \min\{6.89, 6.86\} = 6.86$ of (v_4), $p_5 = 10.84$ of (v_5)
$E_4 = \{v_4v_5, v_4v_6\}$	$p_5 = \min\{9.16, 10.84\} = 9.16$ of (v_5), $p_6 = 11.09$ of (v_6)
$E_5 = \{v_5v_6\}$	$p_6 = \min\{11.09, 10.46\} = 10.46$ of (v_6)
$E_6 = \{v_6v_7\}$	$p_7 = 11.96$ of (v_7)
$E_7 = \{v_7v_8\}$	$p_8 = 13.66$ of (v_8)
$E_8 = \{v_8v_9\}$	$p_9 = 15.26$ of (v_9)
$E_9 = \{v_9v_{10}, v_9v_{11}\}$	$p_{10} = 16.76$ of (v_{10}), $p_{11} = 18.76$ of (v_{11})
$E_{10} = \{v_{10}v_{11}\}$	$p_{11} = \min\{18.76, 18.36\} = 18.36$ of (v_{11})
$E_{11} = \{v_{11}v_{12}\}$	$p_{12} = 19.86$ of (v_{12})
$E_{12} = \{v_{12}v_{13}, v_{12}v_{14}\}$	$p_{13} = 21.06$ of (v_{13}), $p_{14} = 23.46$ of (v_{14})
$E_{13} = \{v_{13}v_{14}, v_{13}v_{15}\}$	$p_{14} = \min\{22.96, 23.46\} = 22.96$ of (v_{14}), $p_{15} = 24.06$ of (v_{15})
$E_{14} = \{v_{14}v_{15}\}$	$p_{15} = \min\{23.76, 24.06\} = 23.76$ of (v_{15})
$E_{15} = \{\}$	Stop the process; destination reached.

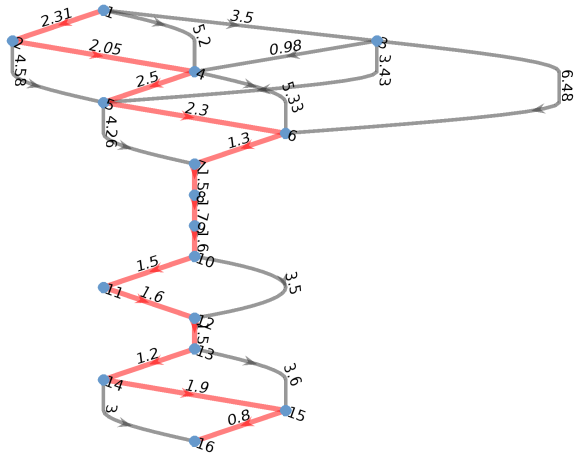

Figure 2: *Network of Pipeline with optimal path by MATLAB*

End node reached stop the process, since $E_{15} = \{\}$ there is no edges incident with v_{15} weight of the end node of selected minimum weight path is $p_{15} = 23.76$. Hence, the minimum transportation cost of the optimum path is 23.76 hundred million yuan.
The Table 4 is a representation of the minimum cost to reach every node v_0 and also the optimum path to reach the end node where all the edges have been visited. Therefore, the optimum path is $(v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15})$ with (weight of the path) minimum transportation cost $p_{15} = 23.76$ hundred million yuan.

Table 4: *Shortest Path for Each Node from v_0 in the Pipeline Network*

End Node	**Shortest Path** $p(v_0, v_j)$	**Minimum Cost** w_j **of** $p(v_0, v_j)$
v_1	$v_0 - v_1$	2.31
v_2	$v_0 - v_2$	3.5
v_3	$v_0 - v_1 - v_3$	4.36
v_4	$v_0 - v_1 - v_3 - v_4$	6.86
v_5	$v_0 - v_1 - v_3 - v_4 - v_5$	9.16
v_6	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6$	10.46
v_7	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7$	11.96
v_8	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8$	13.66
v_9	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9$	15.26
v_{10}	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10}$	16.76
v_{11}	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11}$	18.36
v_{12}	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12}$	19.86
v_{13}	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13}$	21.06
v_{14}	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14}$	22.96
v_{15}	$v_0 - v_1 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15}$	23.76

The proposed modified incident edges path algorithm solves the pipeline network problem and is verified using MATLAB. For example, the following Figure 2 gives the required optimum path for the network. The red line in Figure 2 denotes the optimum path to reach the final pump station from the origin with minimum transportation cost.

II. Google Map Network

The Google Map Network Problem is taken as a numerical example to find the optimum path with minimum transportation cost using the proposed MIEPA algorithm. A map is considered for finding the optimum route between Avadi and Ambattur is shown in Figure 3. The Google map shows two types of timings, such as 20 minutes with 8.8km and 24 minutes with 9.4km to reach the destination as the fastest route. But using the proposed MIEPA algorithm, the decision-maker can find the shortest route with a minimum time duration compared with the online server, which was solved and provided in below Table 5 and Figure 5.

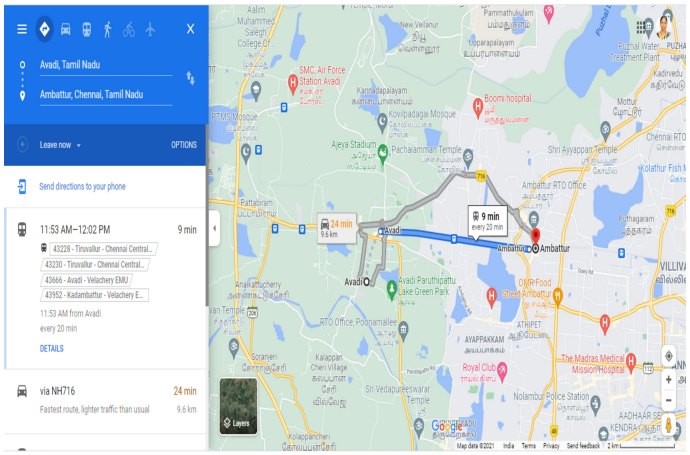

Figure 3: *Route taken from online server - Google Maps*

The distance may vary because of the parameters like speeding limits and road extension work. The distance taken for calculating is taken from the Google Server and solved by the proposed MIEPA algorithm. Figure 4 shows the graphical representation of the google map; each intersection area is taken as nodes, the line joining nodes is taken as edges, and the time taken to reach between nodes is considered as distances of edges. The network taken is directed to reach the destination using Google Maps and plotted using MATLAB, which is a directed network. The nodes are

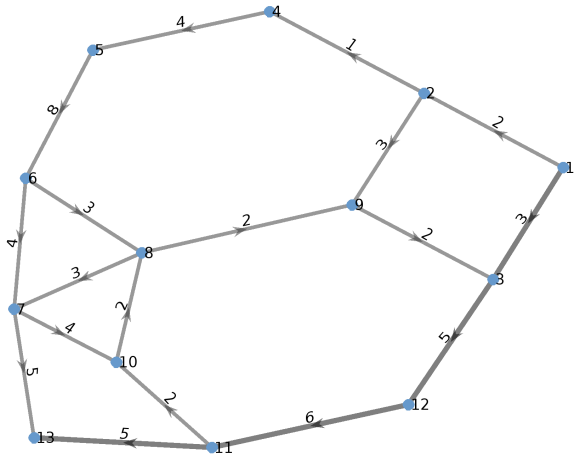

Figure 4: *Network of Google Map*

labeled as $\{1, 2, 3, ..., 12, 13\}$ representing the nodes $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ where v_1 denotes Avadi, v_{13} denote's Ambattur and distance between nodes is given in Figure 4. The Table 5 provides the distances between nodes for the Google Map Network Problem.

Table 5: *Edge and Distance Between the Nodes of Google Map Network*

Edge	**Distance Between v_i and v_j**	**Edge**	**Distance Between v_i and v_j**
(v_1, v_2)	2	(v_7, v_{10})	4
(v_1, v_3)	3	(v_7, v_{13})	5
(v_2, v_9)	3	(v_8, v_7)	3
(v_2, v_4)	1	(v_8, v_9)	2
(v_3, v_{12})	5	(v_{11}, v_{13})	5
(v_4, v_5)	4	(v_9, v_3)	2
(v_5, v_6)	8	(v_{10}, v_8)	2
(v_6, v_7)	4	(v_{12}, v_{11})	6
(v_6, v_8)	3	(v_{11}, v_{10})	2

Table 6 provides the calculation process by proposed MIEPA for Google Map Network Problem for optimum route between Avadi to Ambattur.

Table 6: *Google Map Network Problem Solved by MIEPA (Part I)*

Incident Edge Set E_j	**The Weight and the Shortest Path p_j of (v_j)**
$E_1 = \{v_1v_3, v_1v_2\}$	$p_3 = 3$ of (v_3) and $p_2 = 2$ of (v_2)
$E_2 = \{v_2v_4, v_2v_9\}$	$p_4 = 3$ of (v_4) and $p_9 = 5$ of (v_9)
$E_3 = \{v_4v_5\}$	$p_5 = 7$ of (v_5)
$E_4 = \{v_5v_6\}$	$p_6 = 15$ of (v_6)
$E_5 = \{v_6v_7, v_6v_8\}$	$p_7 = 19$ of (v_7) and $p_8 = 18$ of (v_8)
$E_6 = \{v_8v_7, v_8v_9\}$	$p_7 = \min\{19, 21\} = 19$ of (v_7) and $p_9 = \min\{5, 20\} = 5$ of (v_9)
$E_7 = \{v_9v_3\}$	$p_3 = \min\{3, 7\} = 3$ of (v_3)
$E_8 = \{v_3v_{12}\}$	$p_{12} = 8$ of (v_{12})
$E_9 = \{v_{12}v_{11}\}$	$p_{11} = 14$ of (v_{11})
$E_{10} = \{v_{11}v_{10}, v_{11}v_{13}\}$	$p_{10} = 16$ of (v_{10}) and $p_{13} = 19$ of (v_{13})
$E_{11} = \{v_{10}v_8\}$	$p_8 = \min\{16, 18\} = 16$ of (v_8)
$E_{12} = \{v_8v_9\}$	This forms a cycle. Neglect this path and choose the path (v_{11}, v_{13}) leading to end node v_{13}.
$E_{13} = \{\}$	$p_{13} = 19$

End node reached, so stop the process since $E_{13} = \{\}$ there is no edges incident with v_{13} and weight of the end node is selected which is the minimum weight of path. Hence, the optimum time to reach the destination is 19 minutes

Table 7: *Shortest Path for Each Node from v_1 in the Google Map Network*

End Node	**Shortest Path $p(v_1, v_j)$**	**Minimum Cost w_j of $p(v_1, v_j)$**
v_2	$v_1 - v_2$	2
v_3	$v_1 - v_3$	3
v_4	$v_1 - v_2 - v_4$	3
v_{12}	$v_1 - v_3 - v_{12}$	8
v_9	$v_1 - v_2 - v_9$	5
v_{11}	$v_1 - v_3 - v_{12} - v_{11}$	14
v_5	$v_1 - v_2 - v_4 - v_5$	7
v_6	$v_1 - v_2 - v_4 - v_5 - v_6$	15
v_8	$v_1 - v_2 - v_4 - v_5 - v_6 - v_8$	18
v_7	$v_1 - v_2 - v_4 - v_5 - v_6 - v_7$	19
v_{10}	$v_1 - v_3 - v_{12} - v_{11} - v_{10}$	16
v_{13}	$v_1 - v_3 - v_{12} - v_{11} - v_{13}$	19

The Table 7 provides the optimum path with minimum time to reach the different destination from the origin node v_1 by visiting all the edges. Hence, the destination reached from origin and optimum path is $v_1 - v_3 - v_{12} - v_{11} - v_{13}$ with minimum time duration (Weight of the path) of $p_{13} = 19$ minutes.

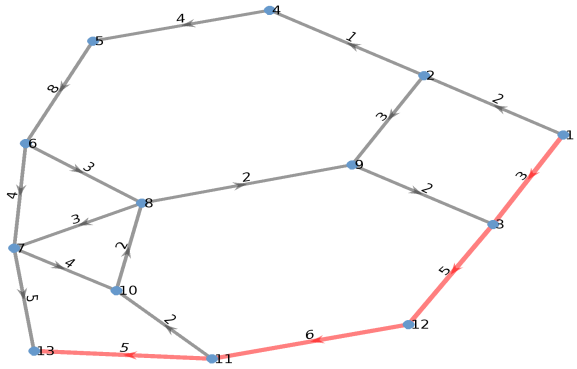

Figure 5: *optimum path of Google Map by MATLAB*

The Figure 5, provide the optimum path for a specified map taken from Google Maps found using the proposed MIEPA algorithm in MATLAB Hence, the decision-maker can find the optimum path by taking any node as the origin and any node as the destination. Therefore, the proposed MIEPA algorithm provides the minimum transportation cost, optimum path, and minimum time duration.

IV. Discussion and Conclusion

The classical network optimization problem of finding the optimal path is crucial in numerous real-world applications, particularly in smart logistics. This article implements the Modified Incident Edge Path Algorithm (MIEPA) to address smart logistical challenges in two real-life scenarios. The MIEPA algorithm efficiently determines the optimal path with minimum transportation cost and time duration by utilizing incident edges to traverse the network.

The algorithm's running time complexity is reduced to $O(E + V)$, offering an efficient solution compared to existing algorithms. The paper presents a comparison of MIEPA with other algorithms, highlighting its performance advantages. Additionally, the algorithm ensures that the destination node is reached from the origin by visiting all edges in a given directed network. The optimal path identified by MIEPA for both pipeline network and Google Map problems is verified using MATLAB, achieving minimum transportation costs. The algorithm also provides results consistent with those obtained from Dijkstra's algorithm, as shown in [3] and [9]. MIEPA demonstrates superior performance in determining the shortest and fastest routes compared to Google Search for the Google Map Problem. Furthermore, the paper illustrates how MIEPA can calculate multiple optimal paths between various origin and destination pairs, enhancing decision-making for users. In conclusion, the MIEPA algorithm facilitates easy manual determination of the shortest route and provides a straightforward, efficient solution for reaching all nodes from the origin node or between any specified nodes in sparse directed network problems. Its main advantage is its clarity and efficiency in solving smaller network instances.

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ON SOME PROPERTIES AND APPLICATIONS OF THE MODI-FRÉCHET DISTRIBUTION

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Abstract

In this paper we introduce a novel expansion of Fréchet distribution from Modi family of probability distributions. The important statistical properties like moments, stochastic ordering, and entropy are studied in this paper. Two distinct characterizations of the proposed distribution are derived through the hazard rate function and truncated moments. The statistical inference about the parameters of the new distribution is studied using the method of maximum likelihood estimation. To study the flexibility and practical utility of the distribution, two real-life data sets from the reliability sector and from the biomedical field were analyzed. An extensive simulation study is also conducted to validate the accuracy and consistency of the estimation techniques.

Keywords: Characterization, Entropy, Fréchet distribution, Hazard rate function, Maximum Likelihood Estimation, Statistical modelling.

1. Introduction

The study of statistical distributions is crucial across various disciplines, like economics, engineering, and particularly in reliability analysis. The reliability sector focuses on modeling and understanding failure rates in systems, components, and products over time. These necessitating distributions are robust and versatile to capture the inherent complexities of these processes. This paper introduces a new distribution meticulously designed to meet these demands and to offer enhanced adaptability for reliability analysis.

In modern industries, accurate and reliable models are essential for predicting critical systems, machinery, and equipment lifespan and failure patterns. Traditional distributions, such as the Weibull distribution (see, [1] & [2]) and the exponential distribution (see [3]), have long been utilized in reliability studies due to their simplicity and ease of use. However, these models often fall short when modeling complex or non-standard failure rates. For instance, while the Weibull distribution is well-suited for systems with increasing or decreasing failure rates, it struggles with scenarios involving bathtub-shaped failure rates, which are common in electronic systems. Similarly, the exponential distribution assumes a constant failure rate, making it inadequate for mechanical systems that experience wear-out failures over time. Our proposed distribution overcomes these limitations by providing a more flexible framework that can adapt to a broader range of reliability scenarios, including those with non-monotonic hazard functions.

Moreover, this distribution has been applied to the biomedical field, specifically in analyzing infant mortality rates, where the precise modeling of survival times and risk factors is crucial.

Traditional statistical models can struggle with the intricacies of biomedical data, particularly in capturing the variability and heterogeneity inherent in patient population. By offering a more adaptable structure, our distribution enhances the accuracy and reliability of statistical modeling in both reliability and biomedical contexts, making it a valuable tool for researchers and practitioners alike.

René Fréchet developed the Fréchet distribution [4], recognized as the maximum value distribution, a concept further explored by Fisher and Tippet [5] and Gumbel [6]. This distribution has become widely used and studied across various fields due to experimental research from multiple disciplines. It is particularly significant in survival analysis and reliability studies, finding applications in engineering, social, physical, environmental, and life sciences. The cumulative distribution function (cdf) and probability density function (pdf) of Fréchet distribution are, respectively,

$$G_{\sigma,\lambda}(x) = e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, x > 0, \tag{1}$$

and

$$g_{\sigma,\lambda}(x) = \lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, x > 0, \tag{2}$$

where $\sigma > 0$ is the scale parameter and $\lambda > 0$ is the shape parameter.

For further reading, see Kotz and Nadarajah [7] and Mubarak [8]. The Fréchet distribution has been extensively generalized in the literature. Recent developments are; Slash-Exponential-Fréchet distribution by Gmez *et al.* [9], Cosine Fréchet Loss distribution by Abonongo *et al*. [10], Marshall-Olkin exponentiated Fréchet distribution [11], the inverted Gompertz-Fréchet distribution [12], Yun-Fréchet distribution [13], cubic transmuted Fréchet distribution [14], the generalized odd log-logistic Fréchet distribution [15], the novel Kumaraswamy power Fréchet distribution [16] and generalization of Fréchet distribution[17]. Harlow [18] demonstrated that the Fréchet distribution is crucial for modeling the statistical behavior of material properties in various engineering applications.

Modi *et al*. [19] proposed the Modi family of distributions with cdf $T(x)$ and pdf $t(x)$ as follows:

$$T(x) = \frac{(1+\alpha^{\beta})S(x)}{\alpha^{\beta} + S(x)}, \qquad x > 0, \alpha > 0, \beta > 0, \tag{3}$$

$$t(x) = \frac{(1+\alpha^{\beta})(\alpha^{\beta}s(x))}{\left(\alpha^{\beta} + S(x)\right)^2}, \qquad x > 0, \alpha > 0, \beta > 0, \tag{4}$$

where S(x) is an arbitrary cdf of a continuous univariate distribution and s(x) is the corresponding pdf. Recent contributions to this family of distributions include Modi Exponential Distribution [19], Modi Weibull [20] and Modi Exponentiated Exponential Distribution [21]. In this paper we introduce a new distribution developed from this family of distributions, utilizing the Fréchet distribution as the baseline distribution. Named the Modi-Fréchet Distribution, this four-parameter distribution offers a superior fit compared to other competitive lifetime distributions.

The present paper is organized as follows: In Section 2 the model construction and basic statistical properties such as moments, stochastic ordering, and entropy are studied. Section 3 is devoted to characterizations of the distribution based on hazard function and truncated moments. In Section 4 parameters of the new distribution are derived using the maximum likelihood estimation method. A simulation study has been carried out in Section 5. The flexibility and utility of the proposed model are studied in Section 6 and conclusions are given in Section 7.

2. Modi Fréchet Distribution

In this section, we develop a special distribution from Modi family, based on the Fréchet distribution. The cdf and pdf of Modi Fréchet distribution (MFD) are;

$$F(x) = \frac{(1+\alpha^{\beta})e^{-(\frac{\sigma}{x})^{\lambda}}}{\alpha^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda}}}, \qquad x > 0, \alpha, \beta, \sigma, \lambda > 0. \tag{5}$$

The corresponding pdf is given by;

$$f(x) = \frac{(1+\alpha^{\beta})\left(\lambda\alpha^{\beta}\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^{2}}, \qquad x > 0, \alpha, \beta, \sigma, \lambda > 0. \tag{6}$$

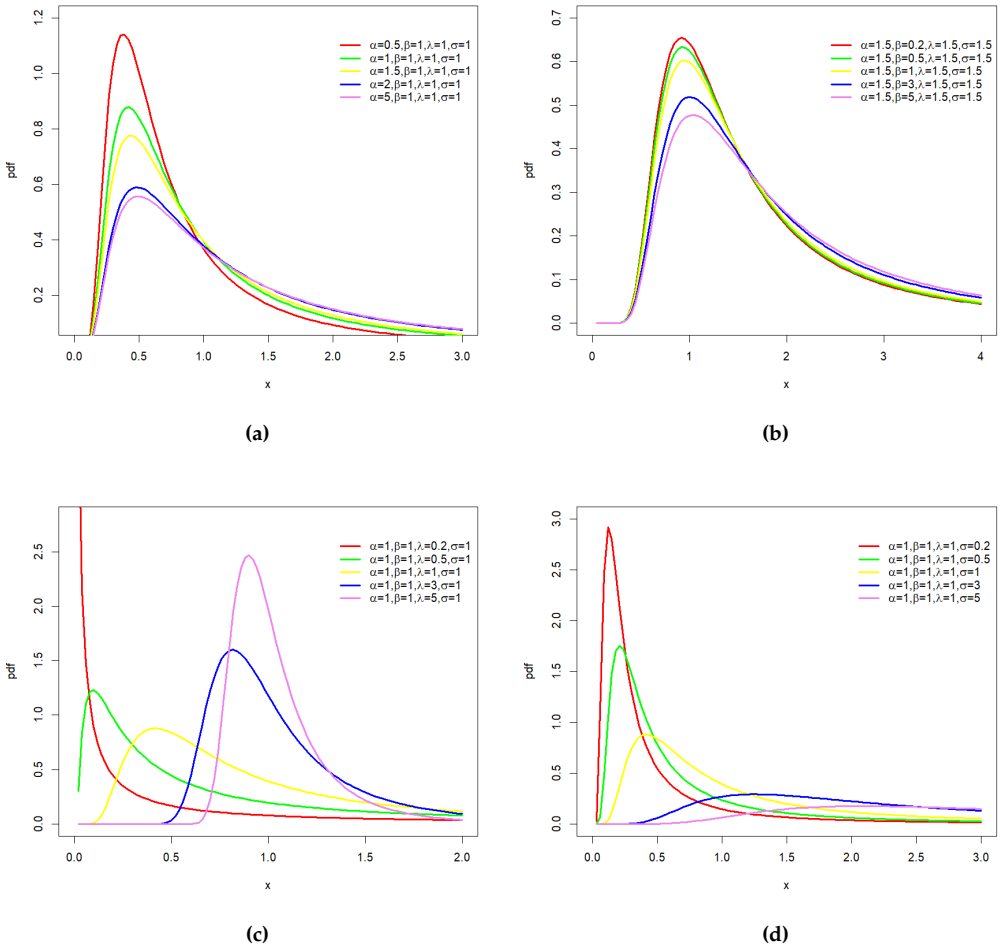


Figure 1: *Plots of the pdf of the MFD for various parameter values.*

Fig. 1. shows the pdf can be unimodal, approximately normal, increasing-decreasing, and right-skewed.
The hazard function of MFD is;

$$h(x) = \frac{(1+\alpha^{\beta})\left(\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1 - e^{-(\frac{\sigma}{x})^{\lambda}}\right)}, \qquad x > 0, \alpha, \beta, \sigma, \lambda > 0. \tag{7}$$

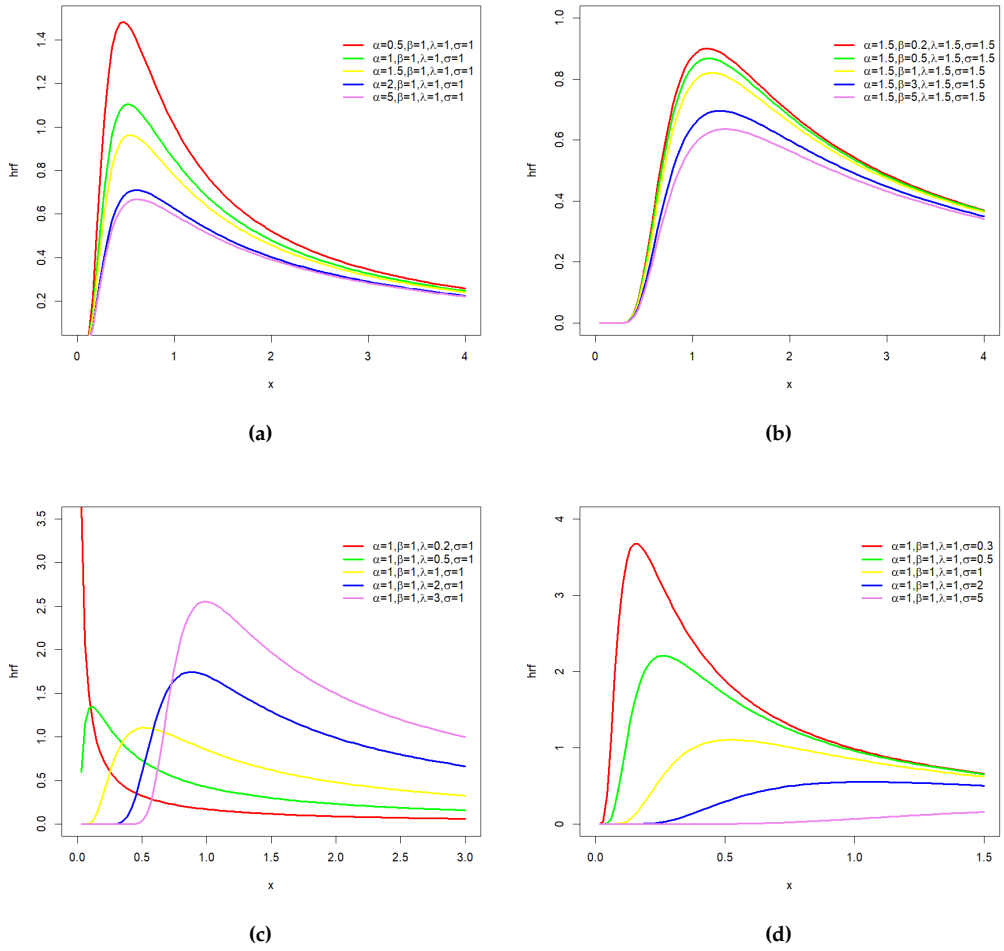


Figure 2: *Plots of the hrf of the MFD for various parameter values.*

Fig 2. shows decreasing, increasing-decreasing, constant, and unimodal behaviour of hazard function.
We derived the quantile function of MFD. The quantile function obtained using the inversion method is given as;

$$F^{-1}(y) = \frac{\sigma}{\left(\log\left(1+\alpha^{\beta}-y\right)-\log\left(y\alpha^{\beta}\right)\right)^{1/\lambda}}, \qquad y\epsilon[0,1] \tag{8}$$

2.1. Moments

The mean, standard deviation, variance, skewness, and kurtosis for the MFD are computed using the raw moments. With the help of R software, we computed them using the standard definitions.

Table 1: *Moment characteristics of the MFD for various parameter values.*

Parameters	$\alpha \rightarrow$	0.6	1	2	5
$\beta = 9$	Mean	3.0120	4.2253	4.6556	4.6569
$\sigma = 4$	Variance	0.1283	1.4521	2.1379	2.1402
$\lambda = 5$	Skewness	6.8163	3.9213	3.5360	3.5351
	Kurtosis	210.25	57.6110	48.1140	48.0920
$\beta = 5.5$	Mean	2.2716	2.6963	2.8949	2.9019
$\sigma = 2.6$	Variance	0.0909	0.3086	0.4320	0.4365
$\lambda = 6.5$	Skewness	4.2027	2.9117	2.5991	2.5899
	Kurtosis	49.9690	24.4780	20.4440	20.3350
$\beta = 2.5$	Mean	3.1501	3.4990	3.7779	3.8626
$\sigma = 3.3$	Variance	0.6722	0.1080	1.5023	1.6292
$\lambda = 4.8$	Skewness	4.7686	4.1751	3.8589	3.7811
	Kurtosis	93.3690	72.3430	62.8610	60.7000
$\beta = 1.2$	Mean	0.8987	0.9206	0.9431	0.9596
$\sigma = 0.9$	Variance	0.0143	0.0170	0.0198	0.0218
$\lambda = 9$	Skewness	2.4785	2.3161	2.1727	2.0808
	Kurtosis	16.4210	14.7680	13.4440	12.6580

The calculated values are presented in Table 1. It shows that the MFD is suitable for under-dispersed data. The skewness and kurtosis values show positive skewness and leptokurtic behaviour. As α increases both mean and variance are increasing while skewness and kurtosis values decreasing.

2.2. Stochastic Ordering

Stochastic ordering is a powerful tool to demonstrate the comparison of random variables in terms of statistical functions of distribution theory. Different types of orderings can also be defined based on the hazard rate, reverse hazard rate, or by applying transformations to the random variables, as discussed in [22]. Let X_1 and X_2 be two random variables with parameters $\alpha_1, \beta, \sigma, \lambda$ and $\alpha_2, \beta, \sigma, \lambda$, their respective density functions $f_1(x)$ and $f_2(x)$, the reliability functions be $\bar{F}_1(x)$ and $\bar{F}_2(x)$, then we say X_1 is smaller than X_2 if

- $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all x,$\Longrightarrow X_1 \leq_{st} X_2$ (Stochastic order).
- $\frac{f_1(x)}{\bar{F}_1(x)} \geq \frac{f_2(x)}{\bar{F}_2(x)}$ for all x,$\Longrightarrow X_1 \leq_{hr} X_2$ (Hazard rate order).
- $\frac{f_1(x)}{F_1(x)} \geq \frac{f_2(x)}{F_2(x)}$ for all x,$\Longrightarrow X_1 \leq_{rh} X_2$ (Reversed hazard rate order).
- $\frac{f_1(x)}{f_2(x)}$ is a monotonic decreasing function for all x,$\Longrightarrow X_1 \leq_{lr} X_2$ (Likelihood ratio order).

Suppose the densities of X_1 and X_2 be

$$f_1(x) = \frac{\left(1+\alpha_1^{\beta}\right)\left(\lambda\alpha_1^{\beta}\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha_1^{\beta}+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}, \qquad x>0, \text{and}$$

$$f_2(x) = \frac{\left(1+\alpha_2^{\beta}\right)\left(\lambda\alpha_2^{\beta}\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha_2^{\beta}+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}, \qquad x>0.$$

respectively. Then,
case (i): When α is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha_1^\beta}{\alpha_2^\beta} \frac{\left(1+\alpha_1^\beta\right)\left(\alpha_2^\beta + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}{\left(1+\alpha_2^\beta\right)\left(\alpha_1^\beta + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}.$$

For $\alpha_1 < \alpha_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.
case (ii): When β is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha^{\beta_1}}{\alpha^{\beta_2}} \frac{\left(1+\alpha^{\beta_1}\right)\left(\alpha^{\beta_2} + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}{\left(1+\alpha^{\beta_2}\right)\left(\alpha^{\beta_1} + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}.$$

For $\beta_1 < \beta_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.
case (iii): When σ is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\sigma_1^\lambda}{\sigma_2^\lambda} \frac{e^{-(\frac{\sigma_1}{x})^\lambda}}{e^{-(\frac{\sigma_2}{x})^\lambda}} \frac{\left(\alpha^\beta + e^{-(\frac{\sigma_2}{x})^\lambda}\right)^2}{\left(\alpha^\beta + e^{-(\frac{\sigma_1}{x})^\lambda}\right)^2}.$$

For $\sigma_1 < \sigma_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.
case (iv): When λ is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\lambda_1}{\lambda_2} \frac{\sigma_1^\lambda}{\sigma_2^\lambda} \frac{e^{-(\frac{\sigma}{x})^{\lambda_1}}}{e^{-(\frac{\sigma}{x})^{\lambda_2}}} \frac{\left(\alpha^\beta + e^{-(\frac{\sigma}{x})^{\lambda_2}}\right)^2}{\left(\alpha^\beta + e^{-(\frac{\sigma}{x})^{\lambda_1}}\right)^2}.$$

For $\lambda_1 < \lambda_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.

2.3. Entropy

Every statistical distribution inherently possesses some degree of uncertainty, and entropy serve as a quantifiable measure of this uncertainty. In modern statistical analysis, information measures like entropy plays a crucial role in addressing and understanding such uncertainties, making them vital tools for statisticians.

If X is a non-negative continuous random variable with pdf $f(x)$, and cdf $F(x)$ then the Renyi Entropy is defined by,

$$H_\theta(x) = \frac{1}{1-\theta} \log \int_0^\infty [f(x)]^\theta \, dx. \tag{9}$$

The Shannon entropy of X is defined as

$$S(x) = -\int_0^\infty f(x) \ln [f(x)] \, dx. \tag{10}$$

Using the pdf of MFD, we can write;

$$[f(x)]^\theta = \left(1+\alpha^\beta\right)^\theta \left(\lambda \alpha^\beta \sigma^\lambda\right)^\theta \frac{\left(x^{-(\lambda+1)} e^{-(\frac{\sigma}{x})^\lambda}\right)^\theta}{\left(\alpha^\beta + e^{-(\frac{\sigma}{x})^\lambda}\right)^{2\theta}}. \tag{11}$$

Varentropy, the variance of Shannon information associated with a random variable X, was introduced by Song [23] as a measure of distribution shape, offering an alternative to kurtosis. This concept captures the variability of information content, also known as information varentropy, as discussed by Bobkov and Madiman [24]. Varentropy is significant in fields like information theory, computer science, and statistics, providing valuable insights into how information is distributed around the entropy of X.

Consider X as a continuous random variable with a density function $f(x)$. The Shannon varentropy of X is then defined as follows:

$$V = V(X) := Var[h(X)] = \int_S f(x) \left[\ln f(x)\right]^2 dx - \left[\int_S f(x) \ln f(x) dx\right]^2 \tag{12}$$

The calculated entropy values presented in Table 2 provide a detailed comparison of Shannon entropy, Rényi entropy, and varentropy across different parameter settings. As shown in the table, the Shannon entropy values are consistently negative, indicating the uncertainty associated with each parameter set. In contrast, Rényi entropy exhibits both positive and negative values, reflecting the variation in the information content under different parameter configurations. Varentropy values, which measure the dispersion of information content around the entropy, are consistently positive, with the magnitude decreasing as the parameter shape parameters β and λ increase. This comprehensive comparison highlights how each entropy measure captures distinct aspects of the information content and its variability.

Table 2: *Entropy measures for different parameters*

Parameters	Shannon Entropy			Renyi Entropy			Varentropy		
β	0.3	0.9	2.5	0.3	0.9	2.5	0.3	0.9	2.5
λ	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
$\alpha = 0.9$, $\sigma = 0.2$	-44.2040	-12.9194	-7.8138	37.8881	25.5873	20.5589	40.1693	8.5762	4.3618
$\alpha = 1.2$, $\sigma = 0.8$	-45.2709	-13.9355	-9.3877	-19.0633	-13.1722	-11.1970	40.6525	8.7924	4.5635
$\alpha = 3$, $\sigma = 1.2$	-48.6200	-16.6447	-11.7576	-1.9420	-1.4060	-1.2323	42.0571	9.2918	4.7959

3. Characterization Results

Accurately characterizing probability distributions is pivotal across diverse fields, as it facilitates profound insights into complex phenomena. The characterization of continuous probability distributions has been extensively investigated, with seminal contributions from researchers including Glänzel [25, 26] and Hamedani [27], who have pioneered various techniques. In this section, we have rigorously established the characterizations of the MFD by examining its truncated moments and hazard function.

3.1. Characterization based on truncated moments

The characterization of the probability distributions through truncated moments was initially pioneered by Galambos and Kotz [28]. Building on this foundational work, numerous scholars have made significant contributions to the field. Among the most notable are Kotz and Shanbag [29], as well as Glänzel *et al.* [30] with further advancements by Glänzel [25, 31]. The characterization of the MFD using truncated moments is an extension of these efforts, specifically developed in accordance with Theorem 3.1 from [25] which is stated as follows,

Theorem 3.1. *Let (Ω, Σ, P) be a given probability space, and let $D = [\alpha, \beta]$ be an interval for some $a < b$ ($\alpha = \infty, \beta = -\infty$ might as well be allowed). Let $X : \Omega \to D$ be a continuous random variable with*

distribution function $G(x)$ and let κ_1 and κ_2 be two real functions defined on D such that

$$E[\kappa_1(X)|X \geq x] = E[\kappa_2(X)|X \geq x]\zeta(x), x \in D$$

is defined with some real function ζ. Assume that $\kappa_1, \kappa_2 \in C_1(D), \zeta \in C_2(D)$, and $G(x)$ is a twice continuously differentiable and strictly monotone function on the set D. Finally, assume that the equation $\kappa_2\zeta = \kappa_1$ has no real solution in the interior of D. Then G is uniquely determined by the functions κ_1, κ_2 and ζ. In particular,

$$G(x) = \int_a^x C \left| \frac{\zeta'(\nu)}{\zeta(u)\kappa_2(u) - \kappa_1(u)} \right| e^{-\tau(\nu)}$$

where the function τ is a solution of the differential equation $\tau' = \frac{\zeta'\kappa_2}{\zeta\kappa_2 - \kappa_1}$ and C is a constant chosen to make $\int_D dG = 1$.

The above theorem has the advantage that the cdf G is not required to have a closed form and is given in terms of an integral whose integrand depends on the solution of a first-order differential equation, which can serve as a bridge between probability and differential equation.

Proposition 3.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable, and let $\kappa_2(x) = \left(\alpha^\beta + e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^2$ and $\kappa_1(x) = \kappa_2(x)e^{-\left(\frac{\sigma}{x}\right)^\lambda}$ for $x > 0$. The pdf of X is Eq.6 if and only if the function ζ defined in Theorem 3.1 has the form*

$$\zeta(x) = \frac{1}{2}e^{-\left(\frac{\sigma}{x}\right)^\lambda}, x > 0. \tag{13}$$

Proof. Let X have pdf Eq.6, then

$$(1 - G(x))E[\kappa_2(X)|X \geq x] = \left(1 + \alpha^\beta\right)\alpha^\beta e^{-\left(\frac{\sigma}{x}\right)^\lambda}, \qquad x > 0,$$

$$(1 - G(x))E[\kappa_2(X)|X \geq x] = \frac{\left(1 + \alpha^\beta\right)\alpha^\beta}{2} e^{-2\left(\frac{\sigma}{x}\right)^\lambda}, \qquad x > 0,$$

and then

$$\zeta(x)\kappa_2(x) - \kappa_1(x) = -\frac{1}{2}e^{-\left(\frac{\sigma}{x}\right)^\lambda}\left(\alpha^\beta + e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^2 < 0, \quad for \quad x > 0.$$

Conversely, if ζ is given as Eq.12, then

$$\tau'(x) = \frac{\zeta'(x)\kappa_2(x)}{\zeta(x)\kappa_2(x) - \kappa_1(x)} = -\lambda\sigma^\lambda x^{-(\lambda+1)}, x > 0,$$

and hence,

$$\tau(x) = \left(\frac{\sigma}{x}\right)^\lambda$$

or

$$e^{-\tau(x)} = e^{-\left(\frac{\sigma}{x}\right)^\lambda}.$$

Now, using Theorem 3.1, X has the pdf Eq.6. ■

3.2. Characterization based on hazard function

The hrf $h(x)$ of a twice differentiable distribution function $F(x)$ and its corresponding pdf $f(x)$ satisfy the first-order differential equation:

$$\frac{f'(x)}{f(x)} = \frac{h'(x)}{h(x)} - h(x). \tag{14}$$

For many univariate continuous distributions, this is the sole characterization expressible in terms of the hazard function. Hamedani and Ahsanullah [32] provided characterizations of certain widely recognized distributions grounded in the hazard function. The following characterization introduces a non-trivial distintion for the MFD when $\beta = 1$, diverging from the aforementioned trivial form.

Proposition 3.2. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is Eq.6 if and only if its hazard function $h(x)$ satisfies the differential equation*

$$x^{\lambda+1}h'(x) + (\lambda+1)x^{\lambda}h(x) = \frac{d}{dx}\left[\frac{(1+\alpha)\lambda\sigma^{\lambda}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(\alpha + e^{-\left(\frac{\sigma}{x}\right)^{\lambda}})(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}\right]. \tag{15}$$

Proof. When $\beta = 1$, the pdf $f(x)$ and hrf $h(x)$ of X are respectively

$$f(x) = \frac{(1+\alpha)\left(\lambda\alpha\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}, \qquad x > 0, \alpha, \sigma, \lambda > 0. \tag{16}$$

and

$$h(x) = \frac{(1+\alpha)\left(\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}, \qquad x > 0, \alpha, \sigma, \lambda > 0. \tag{17}$$

Then we have

$$\frac{f'(x)}{f(x)} = -\frac{(\lambda+1)}{x} + \lambda\sigma^{\lambda}x^{-(\lambda+1)} - \frac{2\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)}. \tag{18}$$

Using Eq.14 we can write,

$$h'(x) + h(x)\frac{(\lambda+1)}{x} = \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)} + \frac{(1+\alpha)^2\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2} - \frac{2(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}\left(e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)},$$

which implies,

$$x^{\lambda+1}h'(x) + (\lambda+1)x^{\lambda}h(x) = \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)} + \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-(\lambda+1)}\left(e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2} - \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-(\lambda+1)}\left(e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}{\left(\alpha + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}.$$

Now, Eq.15 holds, then

$$\frac{d}{dx}\left[x^{\lambda+1}h(x)\right] = \frac{d}{dx}\left[\frac{(1+\alpha)\,\lambda\sigma^{\lambda}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}\right]$$

from which we obtain

$$h(x) = \frac{(1+\alpha)\left(\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)},$$

which is the hrf of MFD when $\beta = 1$. ■

4. Maximum Likelihood Estimation

This section provides the parameter estimates for the MFD derived through the maximum likelihood method. This method is widely recognized as the predominant approach in statistical inference. The log-likelihood for $\theta = (\alpha, \beta, \sigma, \lambda)^T$ based on a given sample is given by;

$$\begin{aligned}\log L(\alpha,\beta,\lambda,\sigma) =& n\log(1+\alpha^{\beta}) + n\log(\lambda) + n\beta\log(\alpha) + n\lambda\log(\sigma) - \\ & (\lambda+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda} - 2\sum_{i=1}^{n}\log\left[\alpha^{\beta}+e^{-(\frac{\sigma}{x_i})^{\lambda}}\right]. \end{aligned} \tag{19}$$

To obtain the maximum likelihood estimators (MLE) of the MFD, we maximize the log-likelihood function. This is accomplished by taking the first derivative of the Eq.19 with respect to parameters α, β, λ and σ.

$$\frac{\partial \log L(\alpha,\beta,\lambda,\sigma)}{\partial\alpha} = \frac{n\beta\alpha^{\beta-1}}{1+\alpha^{\beta}} + \frac{n\beta}{\alpha} - 2\sum_{i=1}^{n}\left(\frac{\beta\alpha^{\beta-1}}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right),$$

$$\frac{\partial \log L(\alpha,\beta,\lambda,\sigma)}{\partial\beta} = \frac{n\alpha^{\beta}log\alpha}{1+\alpha^{\beta}} + nlog\alpha - 2\sum_{i=1}^{n}\left(\frac{\alpha^{\beta}log\alpha}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right),$$

$$\begin{aligned}\frac{\partial \log L(\alpha,\beta,\lambda,\sigma)}{\partial\sigma} =& \frac{n}{\lambda} + n\log\sigma - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{\lambda}{\sigma}\right)\left(\frac{\sigma}{x_i}\right)^{\lambda} - \\ & 2\left(\frac{\lambda}{\sigma}\right)\sum_{i=1}^{n}\left(\frac{\left(\frac{\sigma}{x_i}\right)^{\lambda}e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\log\left(\frac{\sigma}{x_i}\right)}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right).\end{aligned}$$

and

$$\frac{\partial \log L(\alpha,\beta,\lambda,\sigma)}{\partial\lambda} = \frac{n\lambda}{\sigma} + \sum_{i=1}^{n}\left(\frac{\lambda}{\sigma}\right)\left(\frac{\sigma}{x_i}\right)^{\lambda} - 2\left(\frac{\lambda}{\sigma}\right)\sum_{i=1}^{n}\left(\frac{\left(\frac{\sigma}{x_i}\right)^{\lambda}e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right),$$

MLE $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\lambda})$ of $\theta = (\alpha, \beta, \sigma, \lambda)$ can be obtained by solving simultaneously the following normal equations.

$$\frac{\partial \log L}{\partial\alpha} = 0; \qquad \frac{\partial \log L}{\partial\beta} = 0; \qquad \frac{\partial \log L}{\partial\sigma} = 0; \qquad \frac{\partial \log L}{\partial\lambda} = 0.$$

Table 3: *Simulation results.*

True value	n	Average Value	MSE	Bias
	50	10.0852	1041.897	-4.0852
	100	8.4113	188.1615	-2.4113
$\alpha = 6$	200	7.1525	52.4647	-1.1525
	300	6.8824	42.855	-0.8824
	500	6.4090	6.5330	-0.4090
	50	3.4438	18.3209	-0.4438
	100	3.4103	10.7163	-0.4103
$\beta = 3$	200	3.5286	30.133	-0.5286
	300	3.4228	10.4221	-0.4228
	500	3.3492	5.8894	-0.3492
	50	1.7126	4.7213	-0.7126
	100	1.3902	1.7355	-0.3902
$\sigma = 1$	200	1.2486	0.6024	-0.2486
	300	1.1863	0.3508	-0.1863
	500	1.1088	0.1051	-0.1088
	50	0.9450	0.0316	0.0550
	100	0.9521	0.0186	0.0479
$\lambda = 1$	200	0.9621	0.0113	0.0379
	300	0.9654	0.0082	0.0346
	500	0.9777	0.0043	0.0223

5. Simulation Study

In this section, we assess the accuracy of parametric estimation through Monte Carlo simulation. Using the quantile function of MFD given in Eq.8, we generate samples of observations for sizes $n = 50, 100, 200, 300$ and 500 with $N = 1000$ replications. Two sets of parameter values are considered; $\alpha = 6, \beta = 3, \sigma = 1, \lambda = 1$ and $\alpha = 1.2, \beta = 2.5, \sigma = 0.2, \lambda = 0.5$.

The numerical outcomes are evaluated using the R statistical programming language, leveraging the widely used optimization package 'optim'. The Average Value, Mean Square Error (MSE), and Average Bias are computed and displayed in Tables 3 and 4. The results indicate that as the sample size increases, the MSE decreases and the Average Value of each parameter converges to the initial parameter values. These findings demonstrate the accuracy and consistency of the estimation methods.

6. Applications

In this section, we fit the MFD model to a reliability data set to check the model's flexibility. The MFD was compared to that of Modi Exponentiated distribution (MED) by [19], Modi Exponentiated Exponential distribution (MEED) by [21] and Modi Weibull distribution (MWD) by [20]. The maximum likelihood method is employed to estimate the parameters for the candidate models. We evaluated different goodness-of-fit measures to illustrate the flexibility of the model. Specifically, $-logL$(negative log-likelihood function), W (Cramér-von Mises Statistic), A (Anderson-Darling Statistic) $K\imath S$ (Kolmogorov"Smirnov Statistic), AIC (Akaike Information Criterion), $CAIC$ (Akaike Information Criterion with correction), BIC (Bayesian Information Criterion) and $HQIC$ (Hannan"Quinn Information Criterion).
Where

Table 4: *Simulation results.*

True value	n	Average Value	MSE	Bias
	50	4.2749	84.5601	-3.0749
	100	3.6350	69.6801	-2.4350
$\alpha = 1.2$	200	2.9488	58.9493	-1.7480
	300	2.8059	100.5479	-1.6059
	500	2.2570	19.7147	-1.0570
	50	5.9501	103.8486	-3.4501
	100	4.8420	37.1668	-2.3420
$\beta = 2.5$	200	4.2936	36.3745	-1.7936
	300	3.9396	19.2506	-1.4396
	500	3.5822	10.1569	-1.0822
	50	1.3159	45.1542	-1.1159
	100	0.8870	12.4818	-0.6870
$\sigma = 0.2$	200	0.5636	6.9543	-0.3636
	300	0.3382	0.4328	-0.1382
	500	0.3496	1.0409	-0.1496
	50	0.5014	0.0112	-0.0014
	100	0.4897	0.0086	0.0103
$\lambda = 0.5$	200	0.4960	0.0056	0.0039
	300	0.4972	0.0039	0.0028
	500	0.4971	0.0034	0.0029

$$
\begin{aligned}
AIC &= -2logL + 2k, \\
CAIC &= -2logL + \frac{2kn}{(n-k-1)}, \\
BIC &= -2logL + klog(n), \\
HQIC &= -2logL + 2klog(log(n))
\end{aligned}
$$

where L is the likelihood function, k is the number of parameters of the model and n is the sample size. By respecting the standards in the field, the best model corresponds to smaller $-logL, KıS, AIC, CAIC, BIC, HQIC$, and greater p-value. Here, we used the "AdequacyModel" package in R programming language to obtain the MLEs and goodness-of-fit tests of the given data sets.

Data Set I: This data represents the total time on test plot analysis for mechanical components of the RSG-GAS reactor [33]

2.160 0.746 0.402 0.954 0.491 6.560 4.992 0.347 0.150 0.358 0.101 1.359 3.465 1.060 0.614 1.921 4.082 0.199 0.605 0.273 0.070 0.062 5.320.

Data Set II: data set is the information of the infant mortality rate per 1,000 live births for a few chosen nations in 2021, as reported by `https://data.worldbank.org/indicator/SP.DYN.IMRT.IN`

56 10 22 3 69 6 7 11 4 4 19 13 7 27 12 3 4 11 84 27 25 6 35 14 11 12 6

Table 5: *Basic statistical description of the dataset.*

Size (n)	Min.	Max.	Mean	Median	SD	Skewness	Kurtosis
23	0.06	6.56	1.58	0.61	1.93	1.36	3.54
27	3	84	18.81	11	20.51	1.95	3.05

Table 5 displays basic descriptive statistics of the datasets. Here, the distribution of the dataset shows a positive skewness and leptokurtic behaviour, which goes with the moment properties of this distribution. Figure 3 shows the boxplots and Figure 4 shows the TTT plots of the data set and it goes with the features of hrf of MFD.

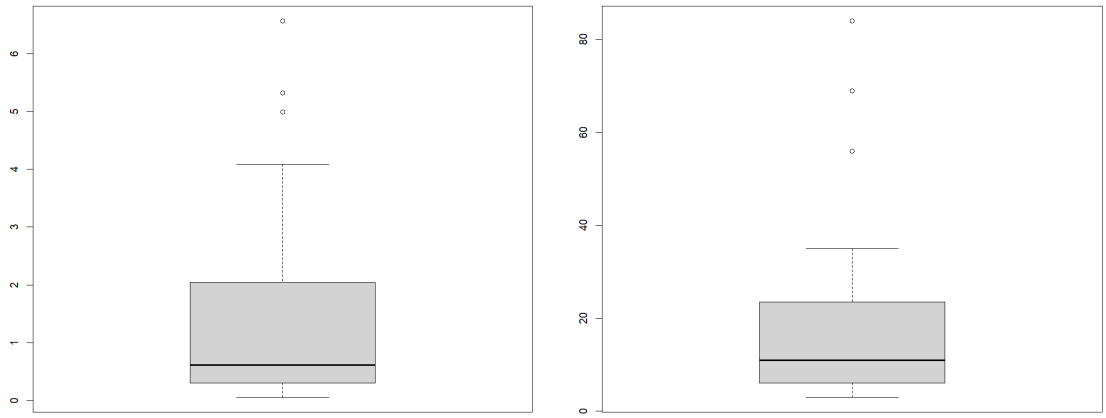


Figure 3: *The box plots of the first data set (left) and the second data set(right).*

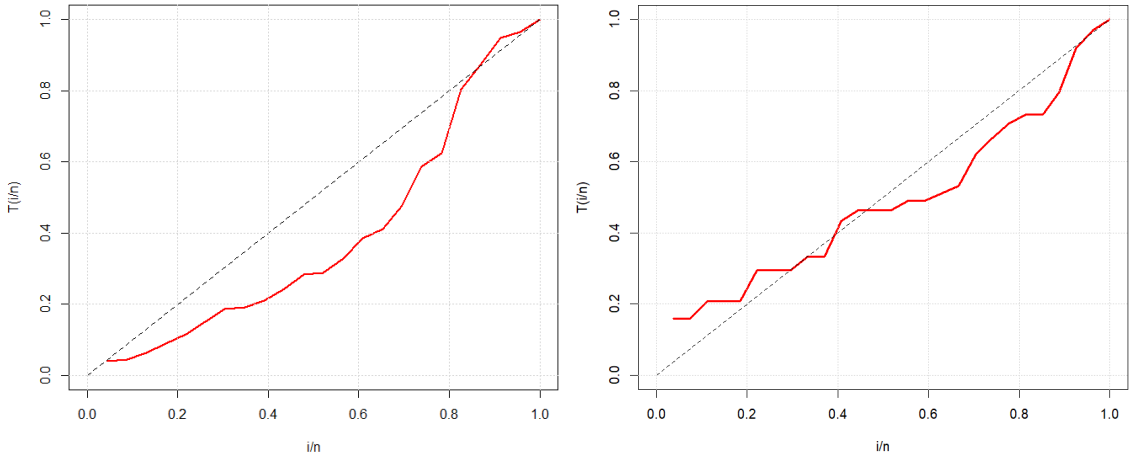


Figure 4: *The TTT plots of the first data set (left) and the second data set(right).*

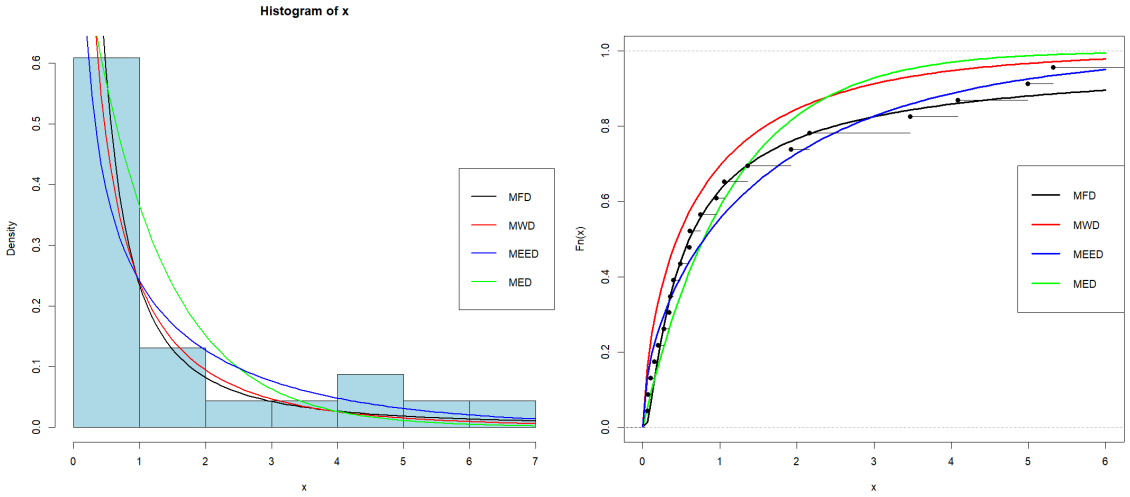
Table 6: *The MLEs of the first data set.*

Model	MLEs	-log L
MFD	$\hat{\alpha} = 3.1367, \hat{\beta} = 5.9797, \hat{\sigma} = 0.8032, \hat{\lambda} = 0.3838$	33.0133
MWD	$\hat{\alpha} = 3.3513, \hat{\beta} = 0.9966, \hat{\sigma} = 0.7085, \hat{\lambda} = 0.9824$	34.2910
MEED	$\hat{\alpha} = 4.4807, \hat{\beta} = 3.2477, \hat{\sigma} = 0.4016, \hat{\lambda} = 0.5374$	33.4931
MED	$\hat{\alpha} = 5.8525, \hat{\sigma} = 9.6862, \hat{\lambda} = 0.8800$	34.8765

Table 7: *The goodness of fit statistics for the first data set.*

Model	W	A	AIC	BIC	CAIC	HQIC	K-S	p value
MFD	0.0471	0.3874	74.0266	78.5686	76.2488	75.1689	0.0971	0.9670
MWD	0.0544	0.3702	76.5819	81.1239	78.8041	77.7242	0.1827	0.3799
MEED	0.0864	0.5451	75.7530	79.1595	77.0161	76.6097	0.1700	0.4687
MED	0.0795	0.5052	74.9862	79.5282	77.2085	76.1285	0.1374	0.7273

Table 6 shows the results of the MLEs and negative log-likelihood values. From Table 7 we can conclude that MFD provides the lowest W, A, AIC, BIC, CAIC, HQIC, K-S values, and the largest p-value. Therefore, MFD is chosen as the best fit for the data.



(a) **(b)**

Figure 5: *Fitted pdf (a) and cdf (b) of distributions to the first data set.*

Figure 5 illustrates the fitted pdfs overlaid on the histogram and the corresponding cdfs for the dataset. The histogram indicates that the data distribution is unimodal and exhibits a pronounced positive skewness. The comparison of theoretical and empirical cdfs reveals that the MFD provides the closest fit to the empirical cdf, outperforming other distributions in terms of accuracy. To verify that the log-likelihood function behaves properly and that a distinct optimum has been attained, we plot the profiles of the log-likelihood function for the MF distribution under the first dataset and displayed in Figure 6.

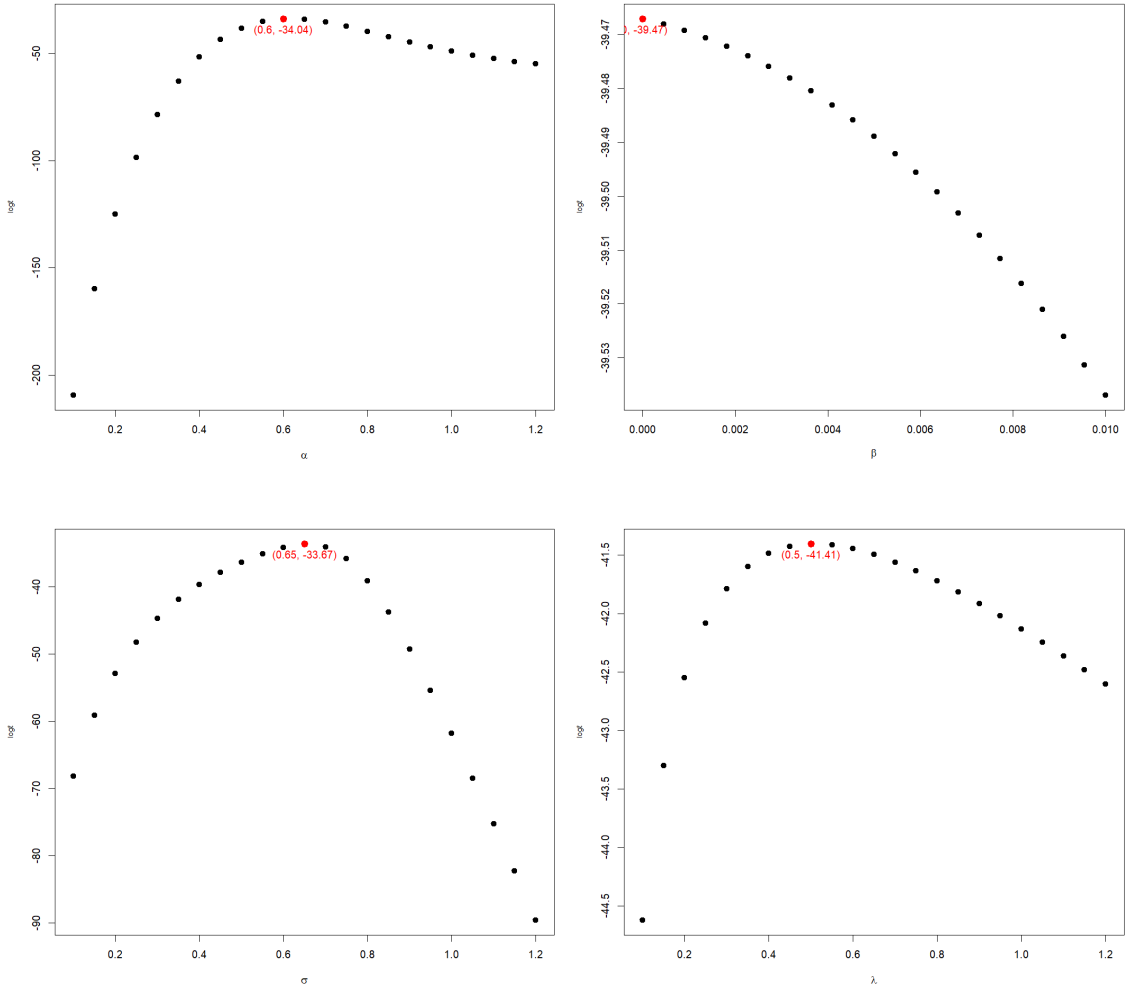


Figure 6: *Fitted profile of the log-likelihood function for the MLEs from the MFD based on the first data set.*

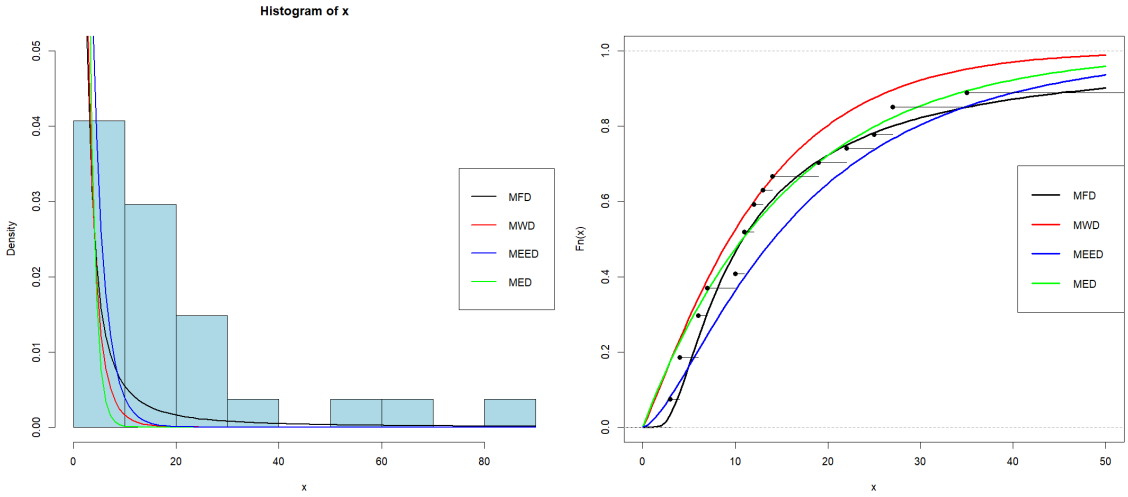
The performance of MFD for the second data was also compared to that of MWD, MEED and MED. The MLEs and goodness-of-fit statistics for the second data set are presented in Tables 8 and 9.

Table 8: *The MLEs of the second data set.*

Model	MLEs	-log L
MFD	$\hat{\alpha} = 2.3680, \hat{\beta} = 9.4772, \hat{\sigma} = 1.2422, \hat{\lambda} = 8.0659$	102.7194
MWD	$\hat{\alpha} = 5.5851, \hat{\beta} = 11.4085, \hat{\sigma} = 1.1231, \hat{\lambda} = 12.9991$	109.7501
MEED	$\hat{\alpha} = 1.5881, \hat{\beta} = 0.7937, \hat{\sigma} = 0.0544, \hat{\lambda} = 1.5894$	104.7283
MED	$\hat{\alpha} = 9.4307, \hat{\sigma} = 16.8944, \hat{\lambda} = 0.0642$	106.7475

Table 9: *The goodness of fit statistics for the second data set.*

Model	W	A	AIC	BIC	CAIC	HQIC	K-S	p value
MFD	0.0459	0.3064	213.4387	218.6221	215.2569	214.9800	0.0992	0.9532
MWD	0.1477	0.9498	227.5002	232.6835	229.3183	229.0414	0.1752	0.3783
MEED	0.1274	0.8277	217.4565	222.6399	219.2747	218.9978	0.1706	0.4121
MED	0.1273	0.8277	219.4949	223.3825	220.5384	220.6509	0.1752	0.3790



(a) **(b)**

Figure 7: *Fitted pdf (a) and cdf (b) of distributions to the second data set.*

Figure 7 presents the fitted pdfs and corresponding cdfs for the second dataset. The histogram reveals a unimodal distribution with notable positive skewness. The plot exhibits that the cdf of MFD is very closer to the empirical cdf than others. To further validate the model, we examine the behavior of the log-likelihood function for the MFD. The profiles of the log-likelihood function are plotted for the second dataset, (see Fig.8) confirming the proper behavior of the function and the attainment of a distinct optimum.

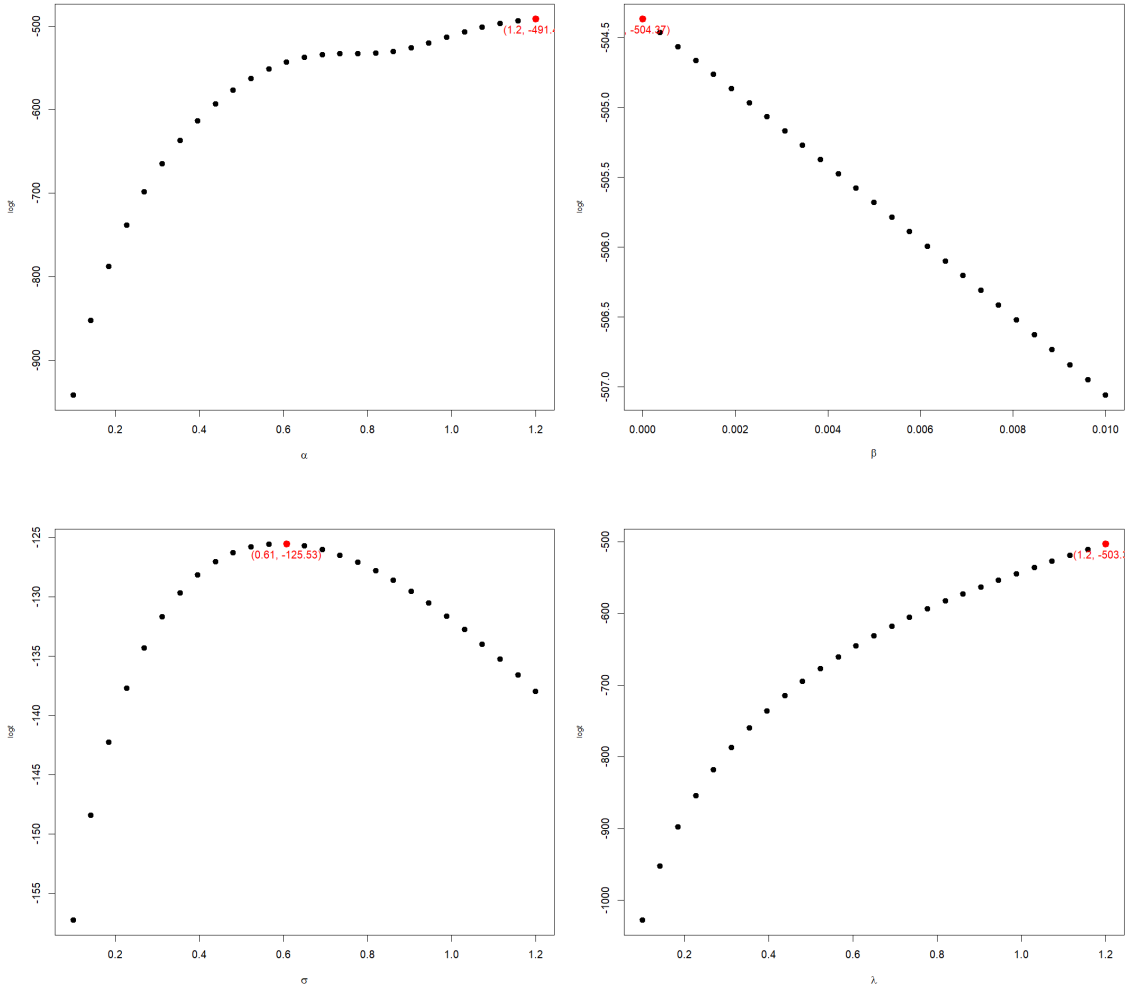


Figure 8: *Fitted profile of the log-likelihood function for the MLEs from the MFD based on the second data set.*

7. Conclusion

In this article, we proposed a new distribution based on the Modi family, namely MFD. Several statistical properties of the proposed distribution, such as moments, skewness, kurtosis, stochastic ordering, and entropy are evaluated. Two characterizations of the distribution are obtained using the hazard rate function and truncated moments. The simulation study showed the accuracy and consistency of the maximum likelihood estimation method. Two real-world data sets one from the reliability sector and the other from biomedical sector were used to demonstrate the flexibility of the proposed model. The MFD provided the best fit for the data compared to other sub-models in the family.

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BAYESIAN ESTIMATION OF INVERSE AILAMUJIA DISTRIBUTION USING DIFFERENT LOSS FUNCTIONS

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Abstract

This paper focuses on the Bayesian estimation of the parameter of the inverse Ailamujia distribution, employing advanced prior structures and diverse loss functions. Specifically, the extended Jeffreys' prior and gamma prior are utilized to derive the Bayesian estimators. Estimation is performed under various loss functions, including squared error, entropy, precautionary, and Linex loss functions, ensuring a comprehensive analysis. To demonstrate the practical applicability and comparative performance of these estimators, an empirical investigation is conducted using a real dataset. The findings highlight the adaptability and effectiveness of the proposed Bayesian approach across different estimation scenarios.

Key words: Bayesian analysis, priors, maximum likelihood estimator, different loss functions.

1. Introduction

In statistical literature, the Ailamujia distribution, introduced by Lv et al. [5], represents a novel probability distribution with significant versatility and practical relevance. This distribution has gained attention due to its ability to model various types of real-world data effectively. Its unique structural properties make it particularly suitable for applications in engineering and related disciplines. By accommodating a wide range of data patterns, the Ailamujia distribution has proven to be a valuable tool for analyzing reliability, survival times, and other stochastic phenomena. Its mathematical flexibility and applicability have inspired ongoing research into its properties, extensions, and potential for broader utilization across diverse fields. They have expounded its various distributional properties which includes moments, moment generating function, mode, median, order statistics. They have

derived and discussed various reliability functions. The probability density function and cumulative distribution function of Ailamujia distribution are respectively given as

$$f(y,\alpha) = 4\alpha^2\, y\, e^{-2\alpha y}\,; y > 0\,, \alpha > 0$$

$$F(y,\alpha) = 1 - (1 + 2\alpha y)e^{-2\alpha y}\,, \alpha > 0, y > 0$$

In recent past decade authors have proposed several extensions of Ailamujia distribution. Pan et al [7] has worked on Ailamujia distribution for interval estimation and hypothesis testing based on small sample size. Long [6] has obtained its Bayesian estimation under type II censoring on the basis of conjugate prior, Jeffrey's prior and no informative prior distribution. Yu et al [10] proposed a new method by applying Ailamujia distribution to solve the problem in the production and distribution of battle field injury in campaign macrocosm. Recently Ahmad et al [1] developed the inverse analogue of Ailamujia distribution and examine its usefulness through two real life time data sets.

Suppose Y is a random variable follows inverse Ailamujia distribution. Then its probability density function (p.d.f), is given by

$$f(y,\alpha) = 4\alpha^2 \frac{1}{y^3}\, e^{-\frac{2\alpha}{y}} \quad , y > 0\,, \alpha > 0 \tag{1}$$

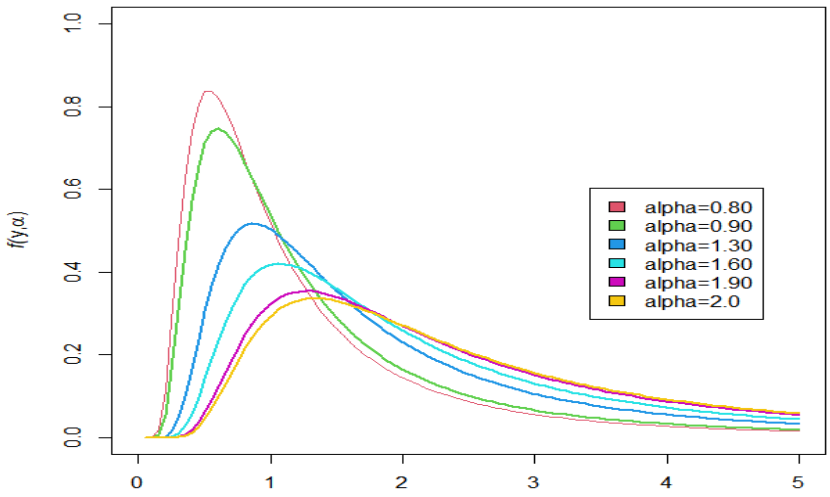

Fig. 1: *pdf plot of IAD under different values of parameters*

Figure 1, illustrates several possible shapes of the probability density function (pdf) for different parameter values, showcasing the flexibility and versatility of the proposed distribution. As the parameters vary, the shape of the pdf adapts to exhibit diverse behaviour's such as unimodal, skewed, or near-uniform profiles, depending on the parameter configuration. This graphical representation provides insight into how the distribution can be tailored to model a wide range of real-world phenomena

Figure 2, presents the cumulative distribution function (cdf) for the same parameter values as Figure 1, offering a complementary view of the proposed distribution. The cdf curves demonstrate the accumulation of probability across the range of the variable, reflecting the gradual transition from 0 to 1 as the variable increases. This graphical representation emphasizes the smoothness and consistency

of the cdf, which is critical for probabilistic interpretation and applications such as reliability analysis and quantile estimation.

The corresponding cumulative distribution function (c.d.f), is given by

$$F(Y) = \frac{(2\alpha+y)}{y}\, e^{-\frac{2\alpha}{y}} \quad , y > 0\, , \alpha > 0 \tag{2}$$

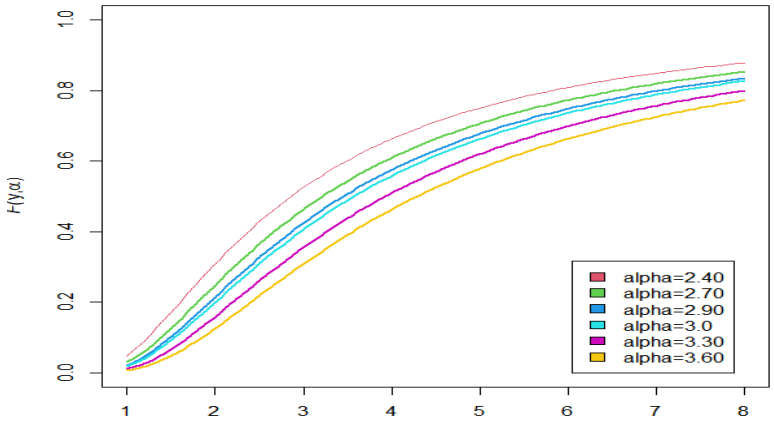

Fig. 1: *pdf plot of IAD under different values of parameters*

2. Maximum Likelihood Estimation

Let $Y_1, Y_2 \ldots Y_n$ be random samples from the inverse Ailamujia distribution. Then the likelihood function of inverse Ailamujia distribution is given as

$$l = \prod_{i=1}^{n} f(y_i, \alpha)$$

$$= \prod_{i=1}^{n} 4\alpha^2 \frac{1}{y_i^3}\, e^{-\frac{2\alpha}{y_i}} = (4\alpha^2)^n \prod_{i=1}^{n} \frac{1}{y_i^3}\, e^{-2\alpha \sum_{i=1}^{n} \frac{1}{y_i}}$$

Taking log we get log likelihood function as

$$\log l = 2n \log 2\alpha - 3 \sum_{i=1}^{n} \log y_i - 2\alpha \sum_{i=1}^{n} \frac{1}{y_i}$$

Differentiating w.r.t, we get

$$\frac{\partial \log l}{\partial \alpha} = 2n \frac{1}{2\alpha} - 2 \sum_{i=1}^{n} \frac{1}{y_i}$$

Now equating $\frac{\partial \log l}{\partial \alpha} = 0$, we get

$$\hat{\alpha} = \frac{n}{2S}$$

Where $S = \sum_{i=1}^{n} y_i^{-1}$

3. Bayesian Estimation of Inverse Ailamujia Distribution

Bayesian estimation procedure is a remarkable way to estimate the parameters of the distribution model. This estimation provides a posterior distribution of an existing life time distribution by considering prior information. From Bayesian point of view there can't be put the lid on selecting prior(s) by considering one's prior(s) is more suitable than others. In case of meager interpretative information about the unknown parameter it is preferable to select non informative prior. However, if one has sufficient information about the parameter(s) it is better to select informative prior. The aim of present study is to obtain a Bayesian estimation of parameter α of inverse Ailamujia distribution by using extended Jeffrey's and gamma prior. In recent past years several research papers have been published in this direction. Afaq et al [2] estimation of parameters of two parameter exponentiated gamma distribution. Mudasir et al [9] studied the Bayesian estimation of weighted Erlang distribution. Raqab and Madi [8] studied Bayesian estimation for exponentiated Rayleigh distribution. Fatima Bi and Afaq Ahmad [4], B. Singh et al. [11], Ahmad et al. [12] and again Ahmad et al. [13] studied different estimations of different distribution. In this paper our goal is to find the Bayesian estimators of the parameters of inverse Ailamujia distribution using extended Jeffery's prior and gamma prior under different loss functions.

3.1: Bayesian Estimation of Inverse Ailamujia Distribution Under the Assumption of Extended Jeffery's Prior

We assume the prior distribution of α to be extended Jeffrey's prior i.e $g(\alpha) \propto \frac{1}{\alpha^{2c}}$

Under the assumption of extended Jeffrey's prior. The posterior distribution of α can be obtained as

$$\pi(\alpha|y) \propto l(y|\alpha)g(\alpha)$$

$$\Rightarrow \pi(\alpha|y) \propto \left(4^n \prod_{i}^{n} \frac{1}{y_i^3}\right) \alpha^{2n}\, e^{-2\alpha \sum_i^n \frac{1}{y_i}} \frac{1}{\alpha^{2c}}$$

$$\Rightarrow \pi(\alpha|y) = k\, \alpha^{2(n-c)} e^{-2\alpha \sum_i^n \frac{1}{y_i}}$$

Where k is independent of α and

$$k^{-1} = \int_0^{\infty} \alpha^{2(n-c)} e^{-2\alpha \sum_i^n \frac{1}{y_i}}\, d\alpha$$

$$k^{-1} = \frac{\Gamma(2n - 2c + 1)}{\left(2 \sum_i^{\infty} \frac{1}{y_i}\right)^{2n-c+1}}$$

So that $$k = \frac{\left(2 \sum_i^{\infty} \frac{1}{y_i}\right)^{2(n-c)+1}}{\Gamma(2n-2c+1)} = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}$$

Where $S = \sum_i^{\infty} \frac{1}{y_i}$

Hence the posterior distribution of α is given as

$$\pi(\alpha|y) = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\ \alpha^{2(n-c)}\, e^{-2S\alpha}$$

Where $S = \sum_i^{\infty} \frac{1}{y_i}$

3.1.1: Estimation Under Squared Error Loss Function (SELF)

The squared error loss function is defined as $l(\hat{\alpha},\alpha) = c_1(\hat{\alpha}-\alpha)^2$ for some constantant c_1 the risk function is given as

$$R(\hat{\alpha},\alpha) = E[I(\hat{\alpha},\alpha)]$$

$$= \int_0^{\infty} c_1(\hat{\alpha}-\alpha)^2 \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\ \alpha^{2(n-c)}\, e^{-2S\alpha} d\alpha$$

$$= c_1 \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\ \int_0^{\infty} \alpha^{2(n-c)}\, e^{-2S\alpha} d\alpha\ + \int_0^{\infty} \alpha^{2(n-c)+2}\, e^{-2S\alpha} d\alpha - 2\hat{\alpha}\int_0^{\infty} \alpha^{2(n-c)+1}\, e^{-2S\alpha} d\alpha\right]$$

After solving the integral, we obtain

$$= c_1 \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)} \left[\begin{array}{c} \frac{\hat{\alpha}\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + \frac{(2n-2c+2)(2n-2c+1)\Gamma(2n-2c+1)}{(2S)^{2(n-c)+3}} \\ -\frac{(2n-2c+1)\Gamma(2n-2c+1)}{(2S)^{2(n-c)+2}} \end{array}\right]$$

$$R(\hat{\alpha},\alpha) = c_1\left[\hat{\alpha}^2 + \frac{(2n-2c+2)(2n-2c+1)}{(2S)^2} - \frac{\hat{\alpha}(2n-2c+1)}{(2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\hat{\alpha}_s = \frac{(2n-2c+1)}{4S}$$

Where $s = \sum_i^{\infty} \frac{1}{y_i}$

3.1.2: Estimation Under Entropy Loss Function

The entropy loss function is defined as $L(\delta) = b[\delta - \log(\delta) - 1]; b > 0\ , \delta = \frac{\hat{\alpha}}{\alpha}$ the risk functions given as

$$R(\hat{\alpha},\alpha) = \int_0^{\infty} b[\delta - \log(\delta) - 1] \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)}\ \alpha^{2(n-c)}\, e^{-2S\alpha} d\alpha$$

$$R(\hat{\alpha},\alpha) = b\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)}\int_0^{\infty}\left[\frac{\hat{\alpha}}{\alpha} - \log\hat{\alpha} + \log\alpha - 1\right]\alpha^{2(n-c)}\, e^{-2S\alpha} d\alpha$$

$$= b\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\int_0^\infty \alpha^{2(n-c)-1}e^{-2S\alpha}d\alpha - \log\hat{\alpha}\int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha + \int_0^\infty (\log\alpha)\,\alpha^{2(n-c)}e^{-2S\alpha}d\alpha - \int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha\right]$$

After solving the integral, we obtain

$$= b\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\frac{\Gamma(2n-2c)}{(2S)^{2(n-c)}} - \log\hat{\alpha}\frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + \frac{\Gamma'(2n-2c+1)}{(2S)^{2(n-c)+1}} - \frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}}\right]$$

$$= b\left[\frac{\hat{\alpha}(S)}{(n-c)} - \log\hat{\alpha} + \frac{\Gamma'(2n-2c+1)}{\Gamma(2n-2c+1)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_e = \frac{n-c}{S}$$

Where $s = \sum_i^\infty \frac{1}{y_i}$

3.1.3: Estimation Under Precautionary Loss Function

The precautionary loss function is defined as $(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}$, the risk function is given as

$$R(\hat{\alpha},\alpha) = \int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\alpha^{2(n-c)}\,e^{-2S\alpha}d\alpha$$

$$R(\hat{\alpha},\alpha) = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)}\int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\alpha^{2(n-c)}\,e^{-2S\alpha}d\alpha$$

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha + \frac{1}{\hat{\alpha}}\int_0^\infty \alpha^{2(n-c)+2}e^{-2S\alpha}d\alpha - 2\int_0^\infty \alpha^{2(n-c)+1}e^{-2S\alpha}d\alpha\right]$$

After solving the integral, we obtain

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + \frac{1}{\hat{\alpha}}\frac{\Gamma(2n-2c+3)}{(2S)^{2(n-c)+3}} - 2\frac{\Gamma(2n-2c+2)}{(2S)^{2(n-c)+2}}\right]$$

$$= \left[\hat{\alpha} + \frac{(2n-2c+2)(2n-2c+1)}{\hat{\alpha}\,(2S)^2} - \frac{2(2n-2c+1)}{(2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_p = \frac{[(n-c+1)(2n-2c+1)]^{\frac{1}{2}}}{(S)}$$

Where $S = \sum_i^\infty \frac{1}{y_i}$

3.1.4: Estimation Under Linex Loss Function

The linex loss function is defined as $L(\hat{\alpha},\alpha) = exp\{b_1(\hat{\alpha}-\alpha)\} - b_1(\hat{\alpha}-\alpha) - 1$, the risk function is given as

$$l(\hat{\alpha},\alpha) = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\int_0^\infty \{e^{(b_1(\hat{\alpha}-\alpha))} - b_1(\hat{\alpha}-\alpha) - 1\}\alpha^{2(n-c)}\, e^{-2S\alpha}d\alpha$$

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)}\left[e^{b_1\hat{\alpha}}\int_0^\infty \alpha^{2(n-c)}e^{-\alpha(b_1+2S)}d\alpha - b_1\hat{\alpha}\int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha\right.$$

$$\left. + b_1\int_0^\infty \alpha^{2(n-c)+1}e^{-2S\alpha}d\alpha - \int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha\right]$$

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[e^{b_1\hat{\alpha}}\frac{\Gamma(2n-2c+1)}{(b_1+2S)^{2(n-c)+1}} - b_1\hat{\alpha}\frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + b_1\frac{\Gamma(2n-2c+2)}{(2S)^{2(n-c)+2}} - \frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}}\right]$$

$$= \left[e^{b_1\hat{\alpha}}\left(\frac{2S}{b_1+2S}\right)^{2(n-c)+1} - b_1\hat{\alpha} + b_1\frac{(2n-2c+1)}{(2S)} - 1\right]$$

Now solving $\frac{\partial l(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha_l} = \frac{1}{b_1}\log\left(\frac{b_1+2S}{2S}\right)^{2(n-c)+1}$$

4. Bayesian Estimation of Inverse Ailamujia Distribution Under the Assumption of Gamma Distribution

We assume the prior distribution of α to be gamma distribution i.e $g(\alpha) \propto \frac{a^b}{\Gamma(b)}e^{-a\alpha}\,\alpha^{b-1}$

Now under the assumption of gamma prior. The posterior distribution of α can be obtained as

$$\pi(\alpha|y) \propto l(y|\alpha)g(\alpha)$$

$$\Rightarrow \pi(\alpha|y) \propto \left(4^n\prod_i^n\frac{1}{y_i^3}\right)\alpha^{2n}\, e^{-2\alpha\sum_i^n\frac{1}{y_i}}\,\frac{a^b}{\Gamma(b)}e^{-a\alpha}\,\alpha^{b-1}$$

$$\Rightarrow \pi(\alpha|y) = k\,\alpha^{2n+b-1}e^{-\alpha\left(a+2\sum_i^n\frac{1}{y_i}\right)}$$

Where k is independent of α and

$$k^{-1} = \int_0^\infty \alpha^{2n+b-1}e^{-\alpha\left(a+2\sum_i^n\frac{1}{y_i}\right)}d\alpha$$

$$= \frac{\Gamma(2n+b)}{\left(a+2\sum_i^n\frac{1}{y_i}\right)^{2n+b}}$$

So that

$$k = \frac{\left(a + 2\sum_i^n \frac{1}{y_i}\right)^{2n+b}}{\Gamma(2n+b)} = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}$$

Where $S = \sum_i^n \frac{1}{y_i}$

Hence the posterior distribution of α is given as

$$\pi(\alpha|y) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\ \alpha^{2n+b-1}e^{-\alpha(a+2S)}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.1: Estimation Under Squared Error Loss Function

The squared error loss function is defined as $l(\hat{\alpha},\alpha) = c_1(\hat{\alpha}-\alpha)^2$ for some constantant c_1 the risk function is given as

$$R(\hat{\alpha},\alpha) = E[I(\hat{\alpha},\alpha)]$$

$$= \int_0^\infty c_1(\hat{\alpha}-\alpha)^2 \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\ \alpha^{2n+b-1}e^{-\alpha(a+2S)}d\alpha$$

$$= c_1\frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\int_0^\infty (\hat{\alpha}-\alpha)^2\ \alpha^{2n+b-1}e^{-\alpha(a+2S)}d\alpha$$

After solving the integral, we obtain

$$R(\hat{\alpha},\alpha) = \ c_1\left[\hat{\alpha}^2 + \frac{(2n+b)(2n+b+1)}{(a+2S)^2} - 2\hat{\alpha}\frac{(2n+b)}{(a+2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_s = \frac{2n+b}{a+2S}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.2: Estimation Under Entropy Loss Function

The entropy loss function is defined as $L(\delta) = b[\delta - \log(\delta) - 1]; b > 0\ , \delta = \frac{\hat{\alpha}}{\alpha}$ the risk functions given as

$$R(\hat{\alpha},\alpha) = \int_0^\infty b[\delta - \log(\delta) - 1]\frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\ \alpha^{2n+b-1}e^{-\alpha(a+2S)}\ d\alpha$$

$$= b\frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\hat{\alpha} + \log\alpha - 1\right]\alpha^{2n+b-1}e^{-\alpha(a+2S)}d\alpha$$

After solving the integral, we obtain

$$R(\hat{\alpha},\alpha) = b\left[\hat{\alpha}\frac{(a+2S)}{(2n+b-1)} - \log\hat{\alpha} + \frac{\Gamma'(2n+b)}{\Gamma(2n+b)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_e = \frac{2n+b-1}{a+2S}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.3: Estimation Under Precautionary Loss Function

The precautionary loss function is defined as$l(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}$, the risk function is given as

$$R(\hat{\alpha},\alpha) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\alpha^{2n+b-1}e^{-\alpha(a+2S)}d\alpha$$

After solving the integral, we get

$$= \left[\hat{\alpha} + \frac{(2n+b)(2n+b-1)}{\hat{\alpha}(a+2S)^2} - 2\frac{(2n+b)}{(a+2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_p = \frac{[(2n+b)(2n+b-1)]^{\frac{1}{2}}}{(a+2S)}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.4: Estimation Under Linex Loss Function

The linex loss function is defined as $L(\hat{\alpha},\alpha) = exp\{b_1(\hat{\alpha}-\alpha)\} - b_1(\hat{\alpha}-\alpha) - 1$, the risk function is given as

$$R(\hat{\alpha},\alpha) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\int_0^\infty \{e^{(b_1(\hat{\alpha}-\alpha))} - b_1(\hat{\alpha}-\alpha) - 1\}\alpha^{2n+b-1}\,e^{-\alpha(a+2S)}d\alpha$$

$$= \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\left[e^{b_1\hat{\alpha}}\int_0^\infty \alpha^{2n+b-1}\,e^{-\alpha(a+b_1+2S)}d\alpha \right.$$
$$\left. - b_1\hat{\alpha}\int_0^\infty \alpha^{2n+b-1}\,e^{-\alpha(a+2S)}d\alpha + b_1\int_0^\infty \alpha^{2n+b}\,e^{-\alpha(a+2S)}d\alpha - \int_0^\infty \alpha^{2n+b-1}\,e^{-\alpha(a+2S)}d\alpha\right]$$

After solving the integrals, we obtain

$$R(\hat{\alpha},\alpha) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\left[e^{b_1\hat{\alpha}}\frac{\Gamma(2n+b)}{(a+b_1+2S)^{2n+b}} - b_1\hat{\alpha}\frac{\Gamma(2n+b)}{(a+2S)^{2n+b}} + b_1\frac{\Gamma(2n+b+1)}{(a+2S)^{2n+b+1}} - \frac{\Gamma(2n+b)}{(a+2S)^{2n+b}}\right]$$

$$= \left[e^{b_1\hat{\alpha}}\left(\frac{a+2S}{a+b_1+2S}\right)^{2n+b} - b_1\hat{\alpha} + b_1\frac{(2n+b)}{(a+2S)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\hat{\alpha}_l = \frac{1}{b_1}\log\left(\frac{a+b_1+2S}{a+2S}\right)^{2n+b}$$

Where $S = \sum_i^n \frac{1}{y_i}$

5. Application

In this section we provide an application through which the performance of the estimators and posterior risk of different loss function has been obtained. The data set are follows:

Data set 1: The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bekker et al. [3]. The data are follows

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16,
2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

By using different loss functions, the Bayes estimates and posterior risks of the posterior distribution through both priors are as follows where posterior risk are in parenthesis.

Table 1: *Bayes Estimation and Posterior Risks Using Jeffery's Prior*

α	C	$\hat{\alpha}$	$\hat{\alpha}_S$	$\hat{\alpha}_E$	$\hat{\alpha}_P$	$\hat{\alpha}_L$	
						$b_1 = 0.01$	$b_1 = 0.05$
1.0	0.5	0.5583	0.5583 (1.260)	1.109 (4.862)	2.241 (17.97)	1.116 (0.0111)	1.116 (0.0558)
	1.0	0.5583	0.5545 (1.247)	1.101 (4.862)	2.225 (17.85)	1.108 (0.0110)	1.108 (0.0554)
	1.5	0.5583	0.5506 (1.234)	1.093 (4.862)	2.210 (17.73)	1.1012 (0.0110)	1.1010 (0.0550)
2.0	0.5	0.5583	0.5583 (1.260)	1.1090 (4.862)	2.2413 (17.97)	1.1167 (0.0111)	1.1165 (0.0558)
	1.0	0.5583	0.5545 (1.247)	1.1012 (4.862)	2.2258 (17.85)	1.1089 (0.0110)	1.1088 (0.0554)
	1.5	0.5583	0.5506 (1.234)	1.093 (4.862)	2.2102 (17.73)	1.1012 (0.0110)	1.1010 (0.0550)
3.0	0.5	0.5583	0.5583 (1.260)	1.1090 (4.862)	2.2413 (17.97)	1.1167 (0.0111)	1.1165 (0.0558)
	1.0	0.5583	0.5545 (1.247)	1.1012 (4.862)	2.2258 (17.85)	1.1089 (0.0110)	1.1088 (0.0554)
	1.5	0.5583	0.5506 (1.234)	1.0935 (4.862)	2.2102 (17.73)	1.1012 (0.0110)	1.1010 (05506)

$\hat{\alpha}$ = MLE, $\hat{\alpha}_S$ = Estimation under SELF, $\hat{\alpha}_E$ = Estimation under Entropy,

$\hat{\alpha}_P$ = Estimation under Precautionary, $\hat{\alpha}_L$ = Estimation under LINEX

Table 2: *Bayes Estimation and Posterior Risks Using Gamma Prior*

α	a	b	$\hat{\alpha}$	$\hat{\alpha}_S$	$\hat{\alpha}_E$	$\hat{\alpha}_P$	$\hat{\alpha}_L$	
							$b_1 = 0.01$	$b_1 = 0.05$
1.0	0.5	0.5	0.5583	1.1163 (0.0086)	1.1240 (4.8667)	1.1124 (1.1085)	1.1162 (0.0111)	1.1161 (0.0558)
	0.5	1.0	0.5583	1.1201 (0.0086)	1.1279 (4.8666)	1.1163 (1.1124)	1.1201 (0.0112)	1.1199 (0.0560)
	1.0	0.5	0.5583	1.1120 (0.0085)	1.119 (4.8705)	1.1081 (1.1043)	1.1119 (0.0111)	1.1118 (0.0556)
2.0	0.5	0.5	0.5583	1.1163 (0.0086)	1.1240 (4.8667)	1.1124 (1.1085)	1.1162 (0.0111)	1.1161 (0.0558)
	0.5	1.0	0.5583	1.1201 (0.0086)	1.1279 (4.8666)	1.1163 (1.1124)	1.1201 (0.0112)	1.1199 (0.0560)
	1.0	0.5	0.5583	1.1120 (0.0085)	1.1197 (4.8705)	1.1081 (1.1043)	1.1119 (0.0111)	1.1118 (0.0556)
3.0	0.5	0.5	0.5583	1.1163 (0.0086)	1.1240 (4.8667)	1.1124 (1.1085)	1.1162 (0.0111)	1.1161 (0.0558)
	0.5	1.0	0.5583	1.1201 (0.0086)	1.1279 (4.8666)	1.1163 (1.1124)	1.1201 (0.0112)	1.1199 (0.05600)
	1.0	0.5	0.5583	1.1120 (0.0085)	1.1197 (4.8705)	1.1081 (1.1043)	1.1119 (0.0111)	1.1118 (0.0556)

Among other loss functions, it is evident from Table 1 and Table 2. That the Linex loss function shows smaller Bayes posterior risk under the both assumptions (extended Jeffery's prior and gamma prior). According to decision rule of less Bayes posterior risk, we accomplish that Linex loss function is more useful than others.

6. Conclusion

In this study, we derived the Bayes posterior distribution and parameter estimation for the inverse Ailamujia distribution using both informative and non-informative priors. We explored various loss functions to assess their impact on the estimation process, with a specific focus on the Linex loss function. The results, presented in Table 1 and Table 2, clearly demonstrate that the Linex loss function yields the smallest Bayes posterior risk under both the extended Jeffery's prior and the gamma prior assumptions. This comparative analysis highlights the superior performance of the Linex loss function, indicating its effectiveness in minimizing the Bayes posterior risk.

By applying the decision rule of minimizing the Bayes posterior risk, we conclude that the Linex loss function is the most useful among the considered alternatives. The performance of the estimators was evaluated through practical applications, and the results underscore the flexibility and robustness of the inverse Ailamujia distribution in Bayesian estimation. The findings also emphasize the utility of the Linex loss function in enhancing the precision of parameter estimation across various contexts. This work contributes to the growing body of literature on Bayesian methods, offering valuable insights into the application of different loss functions for parameter estimation. It provides a clear advantage of using the Linex loss function in terms of minimizing posterior risk, which can be applied to diverse statistical modelling scenarios. The study reinforces the importance of selecting appropriate loss functions for effective Bayesian estimation, ensuring better model performance and more reliable results.

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ENHANCING EMOTION RECOGNITION WITH MULTIMODEL APPROACH USING DEEP NEURAL NETWORKS

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Abstract

Recognizing and extracting different emotions, and then validating those emotions have become important for enhancing human-computer interaction. Emotions play a crucial role in social interactions, facilitating rational decision-making and perception. Previously researched emotion recognition models have typically focused on a single input type like images, text, or audio, where each model can identify the emotion of a person through a single source like facial expressions, voice, social media posts, etc. However, these uni-model approaches are limited because they rely on just one type of data, which often misses the full range of emotional cues. To overcome these limitations, multi-model emotion recognition techniques are proposed which are useful for detecting emotions through a person's facial expressions, speech, social media status, and then EEG data. Model fusion techniques have been applied to detect the most accurate emotion for a particular person through fusion of all the models. A recognition rate-based weighting approach is proposed for model fusion, wherein models are assigned weights proportional to their individual recognition rates. This approach enhances overall performance by combining the outputs of various models with higher emphasis on those with better accuracy. The decision fusion-based multi- model emotion recognition model is proposed which achieved a maximum of 87%. accuracy using a bi-model approach and 92% accuracy with a tri-model approach. The weighted decision fusion approach assigns more weight to the model which is more accurate and achieved 93% accuracy. The proposed recognition rate-based weighting approach for fusion has provided significant results, achieving approximately 93% accuracy with 0.900 and 0.904 Cohen kappa and Mathew score respectively using facial expression, speech, and social media text modalities on combined dataset. The proposed model achieved 63% accuracy on a real-world collected dataset without considering EEG data and improved to 73% if EEG is also considered.

Keywords: *Multimodel Fusion, Emotion recognition, Deep Learning,EEG*

I. Introduction

The absence of emotional cues in individual models, coupled with their susceptibility to external influences, often results in reduced accuracy for emotion recognition. Human communication and emotional expression are inherently multimodal, involving the concurrent use of textual, auditory, and visual cues. The primary advantages of multi-model emotion recognition (MER) include a reduction in the

total error rate of classification and enhanced overall model accuracy. Additionally, MER is less vulnerable than the single model to external factors, so it is quite robust, and it also addresses missing modality problems. Among the different fusion techniques, decision- level fusion is applied here, where input from every modality is modeled separately and the final uni-model affect recognition is integrated. It permits the use of the best classifier for each model since different predictors have greater flexibility and can better represent each modality. In situations where one or more modalities are absent, it makes prediction easier and even permits training in the absence of parallel data. Multimodal emotion recognition techniques are attractive for a variety of reasons. First of all, voice, body, and face are all viewed holistically in real life, where human emotion recognition occurs in a multi-model environment. When attempting to teach a computer to reproduce elements of human emotional intelligence, it seems appropriate to teach them to utilize the same approach. Secondly, integrating multiple-affective signals enriches the data collection. The effect of uncertainty in the raw data will be lessened when other modalities are combined to infer mood. Finally, the ability to identify emotions becomes more flexible with richer data gathering, especially in cases where one or more source signals are absent. Put differently, the information from the remaining modality can serve as a supplement for the emotion categorization job when one modality contains limited emotional information. Multimodel fusion, the process of combining data from various modalities to produce a single effect classification result, is required in multi-model emotion identification techniques. In terms of multi-model fusion, the literature focuses on two different kinds of fusion techniques: decision level fusion and fusion-level fusion. We shall outline the main concepts and general principles of the two multi-model fusion techniques in the subsection that follows. These approaches are critical for combining cues from multiple modalities to generate more robust predictions. Numerous experts in the field have examined this subject in detail, leading to various classifications of fusion techniques. However, earlier surveys adhere to the subsequent classification.

I. Early Fusion

Before passing the joint representation through a model, feature-level fusion, also known as early fusion, concatenates the features from various modalities. Finding the most effective approach to concatenate features that can improve emotion identification performance is the aim of feature level fusion. A straightforward concatenation of the modalities has been applied for feature-level fusion in a number of successful applications. The primary benefit is the ease with which correlation between modalities may be utilized. However, syncing features from distinct modalities can be challenging and computationally expensive because they sometimes have different forms. As a result, feature-level fusion's benefits could occasionally be restricted.

II. Late Fusion

Decision-level fusion, also known as late fusion, is a fusion strategy in which the outputs from each classifier are combined after independent classifiers for each modality are used and trained. The primary benefit of decision-level fusion is that decisions have a common format, making it easier to fuse them together. The ultimate prediction is derived from the combination of two uni-model classifiers. The synchronization problems encountered during early fusion are thereby avoided. Additionally, applying the best classifiers appropriate for each modality is made possible by decision-level fusion, giving the classification step greater flexibility. The following categories comprise the most common late fusion techniques for emotion recognition. The maximum of all posterior probabilities is chosen using the maximum rule.

- Maximum rule: selects the maximum of all posterior probabilities.

- Sum rule: sums probabilities from each classifier and then picks the class with the highest value.
- Product rule: multiplies probabilities between classifiers and then chooses the class with the largest value.
- Weight criterion: results in a linear combination of the classifier's output, where the constants are confidence rates of the predictors.
- Rule-based: selects a dominant modality for each class.
- Model-based: employs a machine-learning algorithm to fuse the output of the classifiers.

Consider a situation where four classes (A, B, C, and D) need to be classified using data from two modalities. Due to the late fusion strategy, each modality is trained using two distinct classifiers. Thus, it is anticipated that the maximum rule fusion system will receive two prediction vectors as input. The system will return the maximum value for each class, as shown in the figure, and choose the class with the greatest value as the winner. Keep in mind that the output could be normalized at the end to make the probability of the whole equal.

III. Hybrid Fusion

By merging the outputs of the early fusion process with the individual uni-model predictors, this strategy aims to combine the best aspects of both fusion techniques. This method only makes sense, though, when more than two modalities are being used. In a scenario where there are three modalities—audio, video, and MRI (Magnetic Resonance Imaging) —for example, the features from the audio and video could be concatenated and used to train a classifier (early fusion), and the MRI features could be used to train another predictor, which would then be used to fuse the output from both classifiers (late fusion).

IV. Cohen's Kappa and Matthews Correlation Coefficient (MCC)

A common statistic for evaluating the degree of agreement between two raters is Cohen's kappa. It can also be applied to evaluate a classification model's performance. Similar to accuracy, Cohen's kappa assesses the agreement between the target and anticipated class, but it additionally accounts for the random probability of receiving the predictions. Cohen Kappa has been adopted by the machine learning community as a means of comparing classifier performance.
It is calculated with the following formula:

$$Kp = (P0 - Pe)/(1 - Pe) \quad (1)$$

The measure of agreement between the model predictions and the actual class values as if they happened by chance is called Pe, and P0 is the model's overall accuracy. By eliminating the potential for agreement between the classifier and a random guess, Cohen's kappa calculates the proportion of predictions the classifier makes that are not consistent with a random guess. A high score is only obtained if the prediction performed well in each of the four confusion matrix categories (TP, FN, TN, and FP), proportionately to the size of both positive and negative elements in the dataset. Matthews Correlation Coefficient (MCC) is a statistical rate that ranges from -1 to 1. With a few modifications to the equation, the performance metric—which was initially designed for binary classification—has been extended to the multi-class scenario.

$$MCC = \frac{TP \times TN - FP \times FN}{\sqrt{(TP+FN)(TP+FP)(TN+FN)(TN+FP)}} \quad (2)$$

II. Related Work

Recognition and extracting various emotions and then validating those emotions have become important for improving the overall human computer interaction. Emotions play an important role in social interactions and facility rational decision making and perception. To achieve specific objectives of the research work, the researcher has referred various research articles and papers regarding emotion recognition system. It was found that there are broadly four approaches used for recognizing human emotions. The first is by using facial expressions, the second is using speech samples of different people, and the third approach is using social media text and last on is using EEG signals. This highlights the critical role of multimodal fusion in improving emotion recognition accuracy. Extensive research has been con- ducted in this domain. Chao et al. [1] aimed to predict continuous values of emotional dimensions, such as arousal and valence, using audio, visual, and physiological data. They employed an LSTM-RNN (long short-term memory recurrent neural network) on the RECOLA dataset. Samira E. Kahou [2] used DBNs (Deep Belief Networks) and CNNs (Convolutional Neural Networks) with K-Means and Relational Auto-Encoders to detect emotions in videos, where individual audio and video models were trained, followed by a decision fusion method for emotion classification. An ensemble approach was adopted by [3], where the output features of a CNN were combined with those of a ResNet and then fed into an LSTM network. Sahay et al. [4] proposed a Relational Tensor Network architecture that modeled inter-modal interactions within a segment, as well as interactions between segments in a video. Hazarika, Poria, et al. [5] developed a framework that utilized a multimodal approach, incorporating visual, audio, and textual features having a GRU to model past utterances of each speaker. These utterances were then integrated by utilizing attention-based hops to capture inter-speaker dependencies. Hassan [6] introduced an unsupervised deep belief network (DBN) for extracting deep-level features from fused sensor signals such as Electro-Dermal Activity (EDA), Photoplethysmogram (PPG), and Zygomaticus Electromyography (zEMG). Five fundamental emotions were classified using the feature-fusion vector created by combining the statistical characteristics of EDA, PPG, zEMG, and DBN features. These emotions were classified as Happy, Relaxed, Disgust, Sad, and Neutral. The use of a pre-trained 'BERT-like' architecture for self-supervised learning to rep- resent both language and text modalities in the recognition of multimodal language emotions was investigated by Siriwardhana et al. [7]. They demonstrated that a simple fusion mechanism (Shallow-Fusion) could simplify the overall architecture while enhancing the effectiveness of complex fusion methods. Priyasad et al. [8] presented a deep-learning approach for encoding emotion characteristics. They used band-pass filtering methods, neural networks, and a SincNet layer to extract acoustic properties from unprocessed audio. The band-pass filter output was then fed into a DCNN. A bidirectional recurrent neural network generated a set of representations at the N-gram level first, and then another recurrent neural network using cross-attention generated a set of representations before merging them into a final score. Mittal [9] used cues from several co-occurring modalities—such as audio, text, and face—to improve each modality's robustness against sensor noise. Their MER model unveiled a brand-new, data-driven multiplicative fusion technique that discovered which cues are more trust- worthy and which ones should be suppressed on a sample-by-sample basis. Lastly, Njoku [10] examined how well deep learning-based models performed for multimodal emotion recognition and data fusion.

III. Proposed Approach

I. Decision Fusion Approach

Figure 1 shows the proposed approach of multimodel emotion recognition. With n being the number of emotion categories (n = 3), let W be a linear transformation square matrix of order n. Different weight scenarios result from different values for W. W is an identity matrix of order n, meaning that there is no weight. Several approaches are available for classifying objects when the decision fusion method is applied. Figure 2 displays the confusion matrix for the multi-model test dataset's facial expression,

voice, social media text, and EEG uni-models. Then, a total of six bi-model emotion detection models have been tested utilizing a straightforward decision fusion approach. The accuracy report for each bi-model emotion recognition is displayed in Table 2. It is evident that, in the absence of EEG, the bi-model approach attains 86% to 88% accuracy and good values for Cohen kappa and Matthews scores. Four tri-modal emotion recognition models have been evaluated, once more combining the decisions of three models using decision fusion. Medical equipment and a controlled laboratory setting can be used to gather EEG readings. If the user wishes to assess their emotional condition without visiting a doctor, they can do so by utilizing social media text, speech, and facial expressions. The model can attain 92% accuracy without taking EEG into account. The emotion expression ordering of the trained uni-, bi-, tri-, and multi-model emotion models is displayed in Table 3.

Table 1: *Multi-model test Dataset Description*

	Image Dataset	**Speech dataset**	**Social media text dataset**	**EEG dataset**
	FER2013[25]	Ravdess[26]	sankha1998 [27]	Jordanbird[28]
Total Data	20019	575	2039	2132
X_train	16040	460	1366	1705
X_test	3979	115	673	427

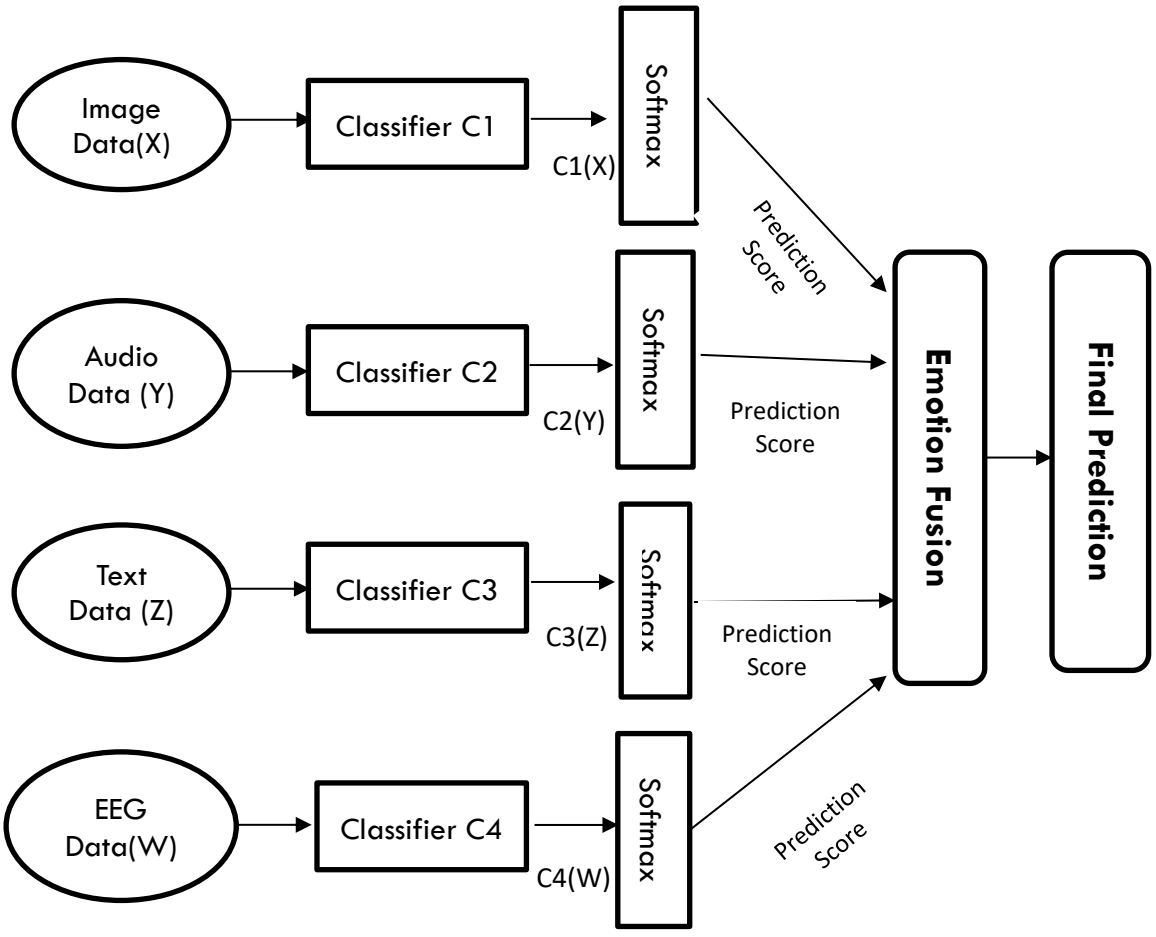

***Figure 1**: Proposed Multi model Fusion Architecture*

Table 2: *Accuracy report of various bi-model emotion recognition*

Bi-model Emotion Recognition (Without EEG)						
	Happy	Sad	Angry	Avg Accuracy	Cohen-Kappa Score	Matthews Score
I+A	0.90	0.86	0.81	0.86	0.783	0.783
I+S	0.78	1.00	0.84	0.87	0.783	0.791
A+S	0.86	0.96	0.83	0.88	0.817	0.820
Bi-model Emotion Recognition (With EEG)						
	Happy	Sad	Angry	Avg accuracy	Cohen-Kappa Score	Matthews Score
E+I	1.00	1.00	1.00	1.00	0.100	0.100
E+A	1.00	0.97	1.00	0.99	0.983	0.984
E+S	1.00	0.97	0.97	0.98	0.967	0.967

Table 3: *Emotion expression ordering of each Emotion model*

	Emotion expression ordering
Image(I)	Angry > Happy > Sad
Audio (A)	Sad > Angry > Happy
Social media text (S)	Sad > Angry > Happy
EEG (E)	Happy = Angry > Sad
I + A	Happy > Sad > Angry
I + S	Sad > Angry > Happy
I + E	Happy = Sad = Angry
A + S	Sad > Happy > Angry
A + E	Happy = Angry > Sad
E + S	Happy > Angry = Sad
I + A + S	Sad > Angry = Happy
I + A + E	Happy = Sad > Angry
A + S + E	Happy > Sad > Angry
I + E + S	Sad > Angry = Happy
I + A + S + E	Happy = Sad > Angry

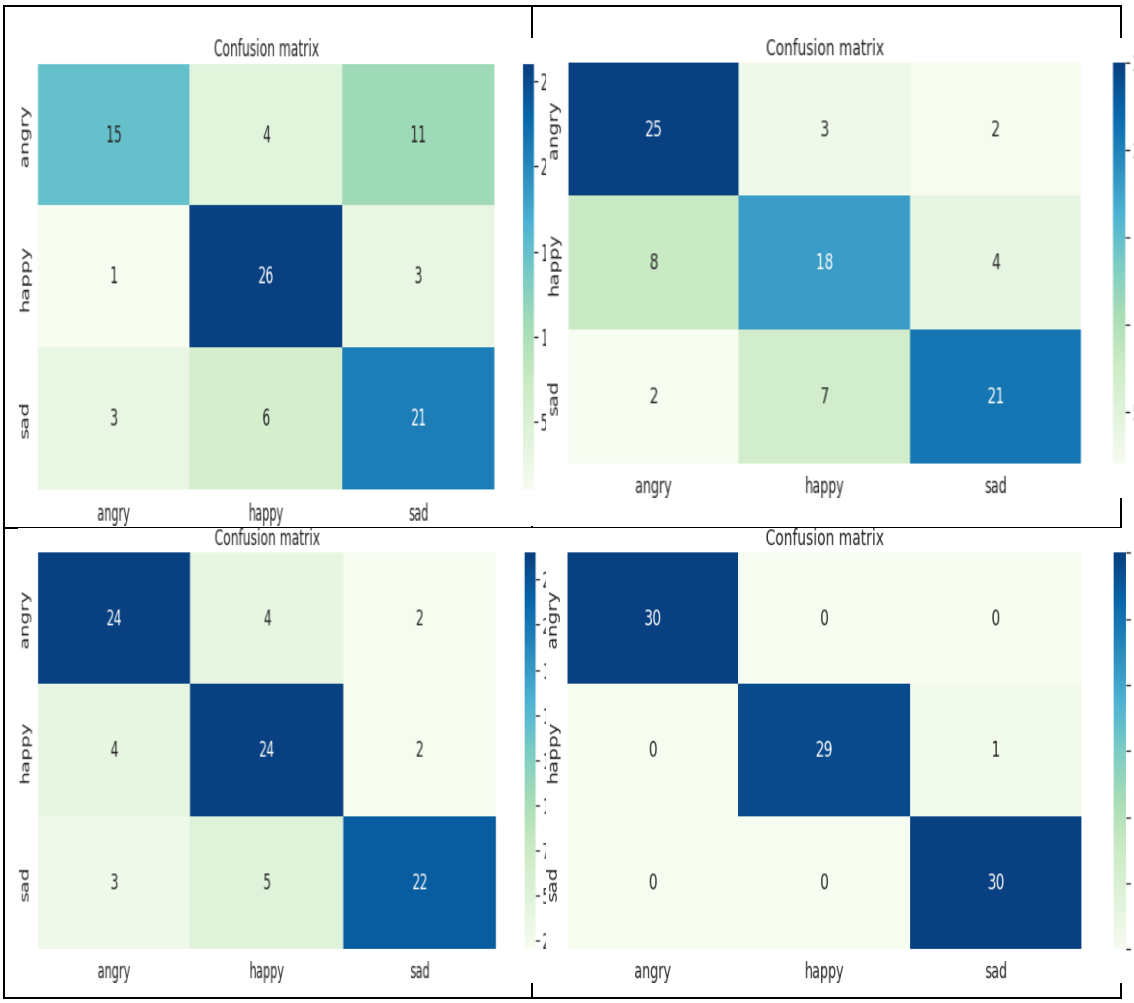

***Figure 2**: Confusion matrix of facial expression, speech, social media text, and EEG model*

II.Weighted averaging Approach

The other approach is to assign different weights to different uni-modals, called Weighted-decision fusion. In decision fusion, equal weights are assigned to each model but if we know that a model is performing better, we can assign higher weightage to that model. In the weighted averaging approach, the accuracy is not assigned as a weight but the prediction of the models which perform better will be multiplied by 2 and the prediction of other models will be multiplied by 1. Since in this particular case, facial expression and speech emotion recognition models are not the best model, they will be assigned lower weights whereas the social media text model is performing better so higher weights will be assigned to that model. Now, basically, these weights will be multiplied by individual predictions and then their mean will be taken. So social media text predictions will multiply by a factor of 2 and others will be multiplied by a factor of 1. Then calculating a weighted average from these models; the model's performance determined the weights.

III. Rank averaging Approach

In this method, the model with the lowest performance is assigned rank 1. Accordingly, the model

with rank 1 performs the worst, the model with rank 2 is the next best, and the model with rank 3 is the best. Following the ranking of each of these models, weights are derived from their ranks. Essentially, each rank will be split by the overall value when these ranks have been added up. In this case, the least performance model is the facial expression model, we will divide it by the sum of 1+2+3 = 6. So the weight for social media text models comes down to 1/6 = 0.16. So, all the predicted values of the speech emotion model and social media text model will get multiplied by 0.33 (1/3) and 0.5 (1/2) respectively. And then all these values will be summed up and the final outcome will be taken.

Then Recognition rate as weight Approach is used where Wi ($1 \leq i < n$) is the weight of the ith category in W, a diagonal matrix of order n, and not all Wi are equal to each other.

$$W = \begin{matrix} W_1 & & \\ & W_2 & \\ & & W_n \end{matrix} \quad (3)$$

First, for every emotion model, recognition results are derived from four distinct uni-model classifiers. The recognition rates of each classifier ($R_{i1}, \ldots, R_{im}$) are used as a weight matrix W_i. The following is how the linear data fusion concept yields the classifier result

$$C = \sum_{i=1}^{n} C_i W_i \quad (4)$$

The following recognition outcome was achieved using a max-win method [15]:

$$\max_{j=1} m \left\{ \sum_{i=1}^{n} C_{ij} R_{ij} \right\} = \sum_{i=1}^{n} C_{ik} R_{ik} \quad (5)$$

Where k is the most likely category emotion label. As shown in Table 5, the recognition rate as a weighted approach achieves 93% accuracy and 0.900 and 0.904 values of Cohen kappa and Matthews score respectively. Table 4 shows the accuracy and other measures for all weighted techniques. Figure 3 shows the confusion matrix of the fusion of models. Figure 4 shows the chart of class-wise accuracy v/s different emotion models.

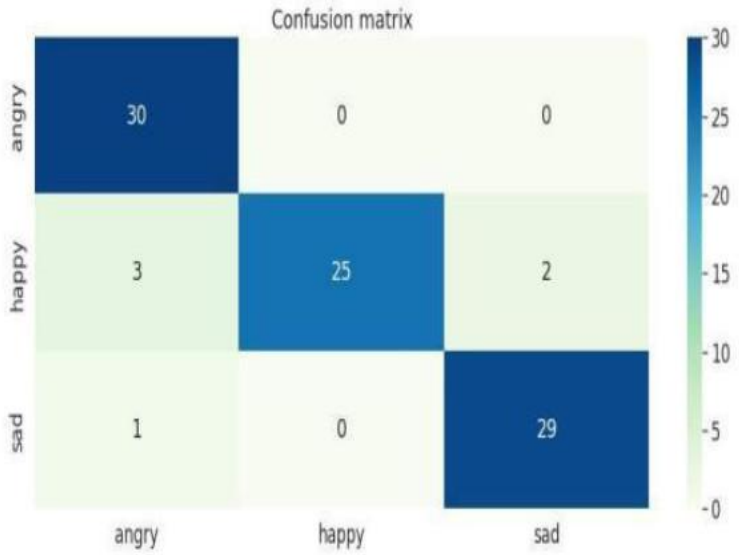

***Figure 3**: confusion matrix of multi-model weighted decision fusion (I+A+S)*

Table 4: *Class-wise precision, recall, f1-score and support of weighted decision fusion*

	Precision	**Recall**	**F1- score**	**Support**
Angry	0.88	1.00	0.94	30
Happy	1.00	0.83	0.91	30
Sad	0.94	0.97	0.95	30
Accuracy			0.93	90
Macro avg	0.94	0.93	0.93	90
Weighted avg	0.94	0.93	0.93	90

Table 5: *Performance of different weighted approaches*

Technique name	**Accuracy**	**Weights(I,A,S)**	**Cohen-Kappa Score**	**Matthews Score**
Weighted Averaging	92%	(1,1,2)	0.889	0.900
Rank Averaging	90%	(0.16,0.33,0.5)	0.887	0.852
Recognition rate as Weights	93%	(0.70,0.71,0.78)	0.900	0.904

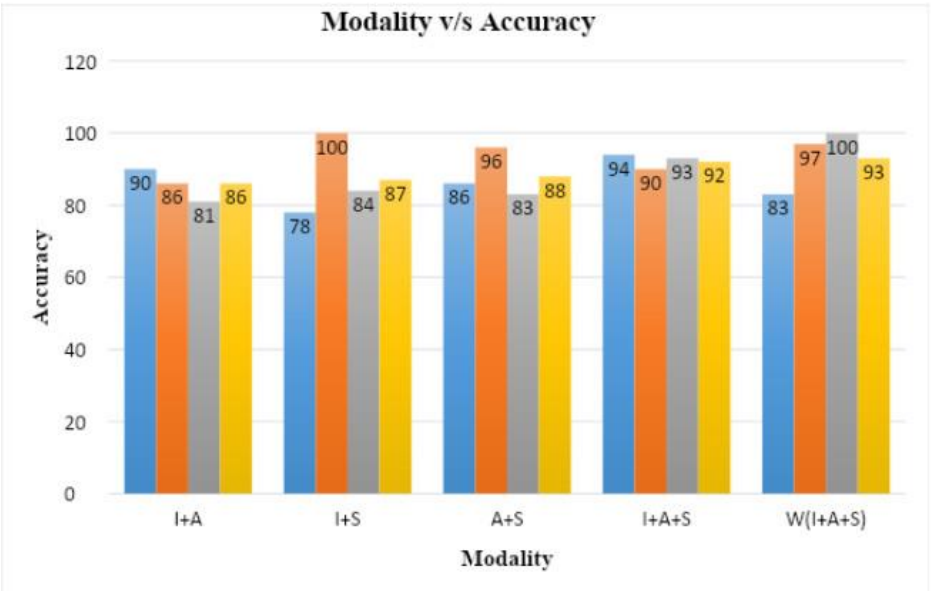

Figure 4: *Chart of accuracy v/s different emotion modalities*

IV. Testing Model with real world collected dataset

As shown in Table 5, from all the fusion approaches, the recognition rate as weights approach gives more accuracy. To test the robustness of the proposed approach, the model is again tested with the real-world collected dataset. The real-world dataset is collected from seven different actors and a total of 168 samples were collected. In this dataset, a total 42 samples are collected from each category. For the facial expression model, two happy, two sad, and two angry samples from each actor are collected. So for the final testing of models, the same number of samples should be selected from each category and from each model. Here, 10 samples are randomly selected from each emotion category from each emotion model, and a total of 30 samples from each emotion model. Table 6 shows the class-wise accuracy of all the uni-model, bi-model, and multi-model combinations. Figure 5 shows the confusion matrix result of the model on a real-world dataset. Figure 6 shows the chart of class-wise accuracy v/s different emotion models when tested on real-world collected datasets. The primary obstacle encountered while utilizing real-world data that has been gathered is the shift in the input or independent variable's data distribution. Although the model's output and correlations may still be technically accurate, the model's predictions have become less accurate due to changes in input data or demography. The major causes of data drift here are Data Quality and Integrity Issues, Demographic Shifts, or Changes in Human Behavior. The Real-time data has been collected using a simple mobile device so the resolution of images, the format of images, the quality of audio samples, and the noise cancellation quality of headphones are not adequate. The Ravdess dataset has been collected in a closed environment, with high-quality devices and the actors used neutral North American accents and they are professional actors. The actors of the Real-time collected dataset are not professional and the accents are totally different. The FER2013 dataset contains only the "pixels" column in .csv format and the images are 48*48 pixels. The real time images are totally in different formats.

The multi-modal emotion recognition model aims to integrate emotions detected from various individual models, resulting in the accurate identification of human emotions. The uni-modal emotion recognition approach faces challenges such as missing modalities and lower accuracy. To address these issues, the data from different modalities needs to be fused and transformed into a consolidated format. This consolidation enables the integration of decisions from various modalities.

Table 6 : Class-wise accuracy of all modalities data when tested model using real world data

Emotion model	Happy	Sad	Angry	Average accuracy
Image(I)	0.30	0.40	0.50	0.40
Audio (A)	0.20	0.50	0.40	0.37
Social media text (S)	0.60	0.50	0.40	0.50
A+S	0.60	0.50	0.60	0.57
I+A	0.50	0.50	0.60	0.53
I+S	0.60	0.40	0.70	0.57
I+A+S	0.60	0.50	0.70	0.60
W(I+A+S)	0.60	0.50	0.80	0.63
EEG(E)	0.70	0.60	0.70	0.67
I+A+S+E	0.70	0.70	0.80	0.73

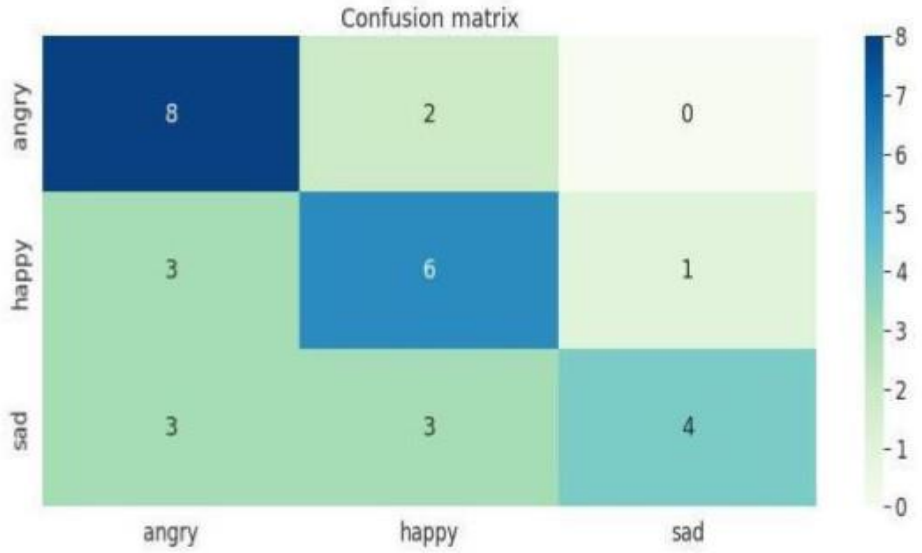

Figure 5: Confusion matrix of Proposed Decision fusion model on collected data (I+A+S)

Table 7: Class-wise precision, recall, f1-score, and support of real-world data of decision fusion (I+A+S)

	Precision	Recall	F1- score	Support
Angry	0.57	0.70	0.67	10
Happy	0.55	0.60	0.57	10
Sad	0.80	0.50	0.53	10
Accuracy			0.60	30
Macro avg	0.64	0.60	0.59	30
Weighted avg	0.64	0.60	0.59	30

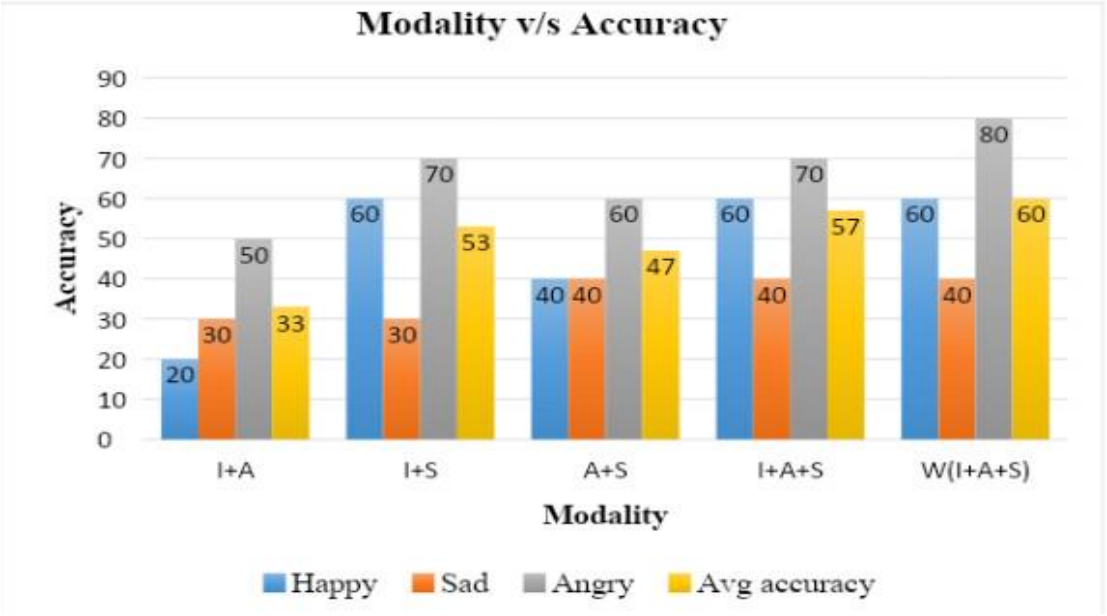

Figure 6: Chart of accuracy v/s different emotion modalities for real-world data

Table 8: Comparison of different existing and proposed multi-model emotion recognition approaches

Ref No	Year	Modality				Fusion	Dataset	Accuracy (in %)
		Image	Speech	Text	EEG			
[1]	2015	√	√		√	Feature level	Recola	66.70
[2]	2015	√	√			Decision Level	Fertfd	47.67
[3]	2017	√	√			Decision Level	Recola	76.00
[4]	2018	√	√	√		Decision Level	Cmu-osei	49.10
[5]	2018	√	√	√		Feature Level	IEM Oca	76.60
[6]	2019				√	Feature Level	Deap	89.53
[7]	2020	√	√	√		Hybrid	Iem Oca	73.98
[8]	2020		√	√		Feature Level	Iem Oca	80.51
[9]	2020	√	√	√		Feature Level	Iem Oca	82.70
[10]	2022	√	√		√	HL, FL, DL	Ravdes	78.75
Proposed Architecture		√	√	√	√	Weighted Decision level	Combined Customized Dataset	93.00
		√	√	√	√		Real world Collected Dataset	67.00

V. Conclusion and Future Work

In conclusion, integrating information across multiple modalities and time holds the potential for enhancing emotion recognition and outcome prediction. The proposed recognition rate based weighting approach for fusion uses the recognition rates of each model as weights. has provided significant results, achieving approximately 93% accuracy with a combined collected dataset with facial expression, speech, and social media text modalities. To test the proposed models, a real-world dataset is collected from seven subjects, encompassing facial expressions, speech, social media text, and EEG signals for three emotions. The collected data is pre processed and formatted for validation. The weighted decision fusion model attained 63% accuracy on the collected real-world dataset with facial expression, speech, and social media text modalities. Challenges with the real-time dataset include lower image resolution, varied image formats, audio quality, and headphone noise cancellation due to the use of a simple mobile device for data collection.

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OVERVOLTAGE AT THE TRANSFORMER WHEN DISCONNECTING CLOSE ASYMMETRICAL SHORT CIRCUITS

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Abstract

This article examines overvoltages at the inputs of high-voltage (HV) and low-voltage (LV) transformers rated at 110/6 kV and 110/10 kV, focusing on scenarios involving grounded and isolated neutrals during short circuits near the transformers. The study finds that with an isolated neutral, overvoltages resulting from a phase-to-ground short circuit reach the highest levels, as anticipated. However, the disconnection of all types of asymmetrical short circuits—whether with an isolated or grounded neutral—yields even greater, potentially excessive overvoltages. This occurs because the windings of undamaged transformer phases remain partially energized during disconnection, leading to significant currents being interrupted. The magnetic energy from these currents converts to electrical energy, resulting in substantial voltage increases, characterized as pulsed overvoltages lasting several microseconds. Implementing switches with shunt resistance can reduce these overvoltages considerably, though the remaining levels may still exceed acceptable thresholds. To mitigate the risk of such excessive overvoltages, installing surge arresters at the inputs of high-voltage transformers is recommended, ensuring that transformer input overvoltages remain within permissible limits.

Keywords: Overvoltages, short-circuit interruptions, surge arresters, asymmetrical short circuits, switches with shunt resistance.

I. Introduction

Overvoltages during line or load disconnections in electrical networks occur due to the conversion of magnetic energy from interrupted currents in inductances into electrical energy, leading to an increase in voltage. The greater the interrupted current, the more magnetic energy is converted, resulting in a higher voltage increase. Therefore, the disconnection of short-circuit currents (SC) can lead to significant overvoltages, as very high currents are being interrupted.

In single-phase circuits, there is no increase in voltage during short-circuit interruptions, as switches operate when the current passes through zero. At this moment, there is no magnetic energy available for conversion into electrical energy. In three-phase circuits, however, the currents across all three phases do not reach zero simultaneously during a short circuit. Consequently, when the switch operates, only one phase may have a zero current, while the other phases remain at non-zero values. The interruption of these currents can lead to substantial overvoltages [1, 13].

It is important to note that this phenomenon has not been extensively addressed in the existing literature, which primarily focuses on single-phase circuits [4,5].

In the event of short circuits occurring in close proximity to a transformer [6,9], this component is typically isolated through relay protection and is thus regarded as safely disconnected from the network. However, the behavior of the transformer—particularly given its significant inductance—

remains a critical area of inquiry, especially when it continues to carry operating phase currents in the windings of undamaged phases after disconnection. The interruption of these currents can lead to substantial overvoltages [12, 13].

II. Assessment of overvoltages when disconnecting short-circuit currents in transformers

The overvoltage when switching off short-circuit currents can be roughly estimated as follows. When current flows through the windings of transformers, corresponding magnetic energy is generated

$$W_M = \int_0^{\psi_M} id\ \psi_M \quad (1)$$

where, ψ_M – winding flux linkage. The rupture of a circuit containing inductance at a current value i different from zero must be accompanied by the conversion of this energy into other forms, in particular into the energy of an electric field, which explains, as stated above, the occurrence of overvoltages on inductive network elements when they are turned off - Fig. 1.

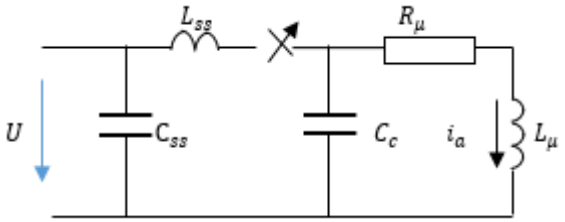


Figure1. Equivalent circuit for calculating overvoltages when the transformer is turned off

In the diagram in Fig. 1, L_μ и L_{ss} are the inductance of the transformer and the substation bus system, respectively, C_c and C_{ss} are the capacitance of the disconnected transformer and the busbar wires, respectively, R_μ is the active resistance of the transformer winding.

The amplitude U_{max} of voltage fluctuations on the capacitor is determined based on the equality of the energy of the magnetic field of the winding

$$W_M = \frac{1}{2} L_\mu i_a^2, \quad (2)$$

and the energy of the electric field of the capacitor

$$W_c = \frac{1}{2} C_c U_{max}^2 . \quad (3)$$

The equality of these two expressions can determine U_{max}.

$$U_{max} = i_a \sqrt{\frac{L_\mu}{C_c}} = i_a Z_{ch}, \quad (4)$$

where, Z_{ch} is the characteristic resistance of the $L_\mu - C_c$ circuit, which has a value of several tens of kilo-ohms.

This expression shows that when the inductance is turned off, the voltage across it can be tens of thousands of times greater than the short-circuit currents that are cut off by the switches when they are turned off.

The task posed is quite complex, since it contains two transient processes - the occurrence of a short circuit and its shutdown, and all this in a complex, nonlinear, three-phase circuit in which asymmetrical changes occur.

This was confirmed in the results of this article, which, due to the complexity of the network diagram, the presence of nonlinearity and the occurrence of two successive transient processes, was carried out by mathematical modeling using computer technology. The algorithmic language used was OrCAD 17.2.

III. Computational results

For instance, consider short circuits occurring within the differential protection zone of transformers. This article specifically examines the overvoltages that manifest in transformers during disconnection amid asymmetrical short circuits occurring nearby. The analysis is situated within a radial network operating at a nominal voltage of 110 kV. This voltage level was selected because 110 kV networks can function with both isolated and grounded neutrals, facilitating a thorough investigation of overvoltage phenomena across varying short circuit conditions [2, 3, 4].

It is understood that in grounded neutral systems, elevated currents from single-phase short circuits can induce a transition to an isolated or partially isolated neutral mode, aimed at mitigating single-phase short circuit currents. The electrical circuit under consideration is depicted in Fig. 2, comprising three substations—SS1, SS2, and SS3—and two lines, W_1 and W_2. Substations SS2 and SS3 are outfitted with transformers rated at 110/6 kV and 110/10 kV, each supplying loads S_2 and S_3, respectively. To safeguard against overvoltages, the bus systems at each substation are equipped with suitably rated surge arresters (SARs).

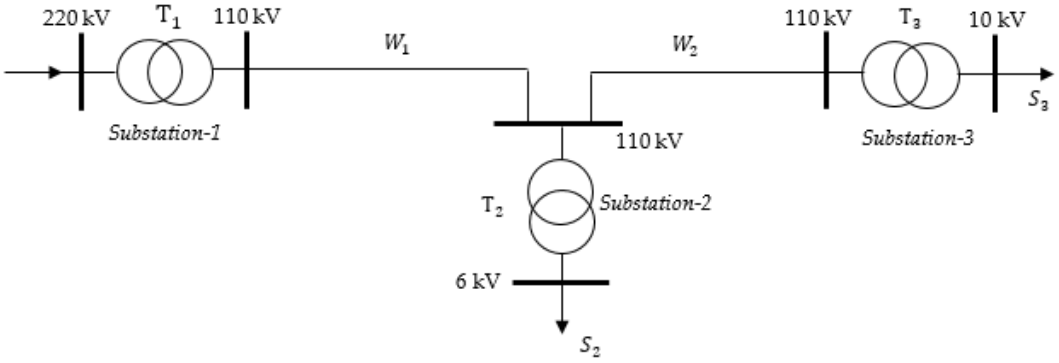


Figure 2. Schematic diagram of the electrical network under investigation.

Various short circuit scenarios—namely single-phase, two-phase, and two-phase-to-ground—were systematically executed at the junction between transformer T_2 and its associated switches within the high-voltage busbar system of substation SS-2. Both grounded and isolated neutral configurations of the network were evaluated. The results obtained from these experiments are summarized in Tables 1 to 3 and illustrated in Figures 2 to 9.

The results obtained are shown in Tables 1 - 3, as well as in Fig. 3 – 10.

As indicated in Table 1, under normal operating conditions, the maximum voltage values at the terminals of loads 1 and 2 were recorded at 4.9 kV and 7.6 kV, respectively.

The voltage calculations within the busbar systems of substations SS1 and SS2—covering the inputs of both HV and LV windings of the transformers as well as the terminals of the first load—are presented in Tables 1 and 2. These calculations were conducted under the assumption of a grounded neutral configuration for the transformers and reflect the conditions arising from the aforementioned short circuit scenarios, as depicted in Figures 3 to 7.

In the event of a single-phase short circuit (on phase A) at the aforementioned location, with the network configured for a grounded neutral, the voltage in the affected phase drops to zero, while the healthy phases experience a reduction from 91 kV to 77 kV. This short circuit also influences the voltage at substation SS3, where the high-voltage section sees an increase. The increase in voltage for the damaged phase is minimal, whereas the healthy phases can rise by approximately 15% (see Fig. 3). Similar voltage variations are also observed on the secondary sides of the transformers at substations SS2 and SS3.

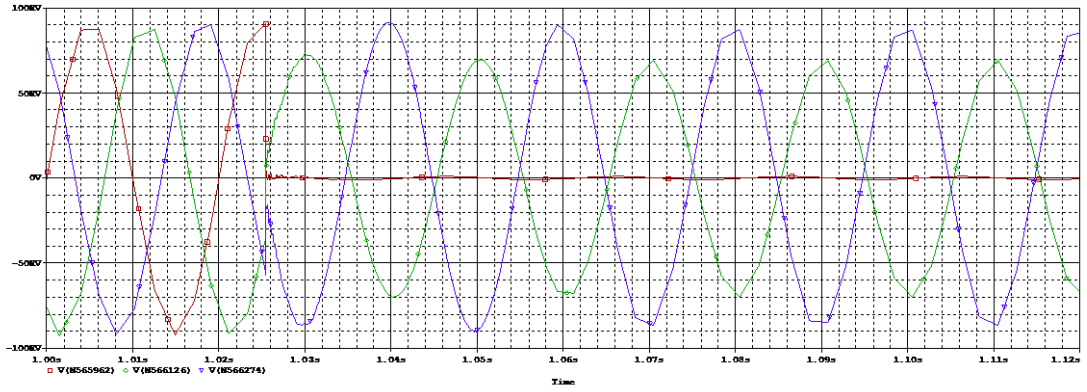
Table 1

The voltage calculation results

Neutral mode	Net-work mode	U_{S1A}	U_{S1B}	U_{S1C}	U_{T1A}	U_{T1B}	U_{T1C}	U_{T1a}	U_{T1b}	U_{T1c}
		kV	kV	kV	kV	kV	kV	kV	kV	kV
neutral is grounded	1	91,23	91,79	91,19	91,23	91,79	91,19	4,95	4,97	4,95
	2	0	76,83	76,74	0	76,83	76,74	0,61	4,28	4,44
	3	182	195	192	0	6903	6616	65,0	323	316
	4	226	128	200	0	195	199	6,26	10,38	10,40
	5	0	0	29,59	0	0	29,59	0,513	0,513	1,055
	6	222	226	238	0,078	0,066	29631	473	473	946
	7	226	229	240	0,082	0,063	240	3,77	3,76	7,52
	8	46,6	46,6	92,3	46,6	46,6	92,3	2,514	2,514	5,01
	9	187	190	102	3804	3804	7585	181	181	363
	10	162	191	191	187	187	192	5,94	5,94	11,82
		U_{S2A}	U_{S2B}	U_{S2C}	U_{T2a}	U_{T2b}	U_{T2c}	U_{N1a}	U_{N1b}	U_{N1c}
		kV	kV	kV	kV	kV	kV	kV	kV	kV
	1	85,12	85,37	85,24	7,719	7,742	7,737	4,89	4,91	4,89
	2	86,87	97,43	101	7,95	8,87	9,15	4,35	4,29	4,07
	3	169	120	112	15,41	10,97	10,23	0	14,09	14,16
	4	204	119	181	18,82	10,79	16,50	0,327	10,4	10,6
	5	87,89	36,61	60,72	8,04	3,38	5,56	4,12	1,89	2,32
	6	175	202	225	15,91	18,48	20,66	1,49	1,47	2,97
	7	183	203	225	16,56	18,56	20,71	1,55	1,53	3,08
	8	97,94	93,2	86,31	8,92	8,536	7,83	4,43 2,49	4,71 2,49	4,95
	9	142	147	104	12,87	13,34	9,48	8,12	8,11	16,22
	10	140	148	103	12,69	13,49	9,34	5,54	5,52	11,06

The maximum current values in the high-voltage windings, the neutrals of transformers T_1 and T_2, as well as in line 1, the switch, and the short-circuit current are presented in Table 2. The corresponding current curves are illustrated in Figure 3. Based on the data from Table 2 and the visual representation in Figure 4, the short-circuit characteristics of these currents are detailed in Figure 5.

The currents observed include those in the windings and the neutral of transformers T_1 and T_2, as well as in the switches and the short-circuit current itself.



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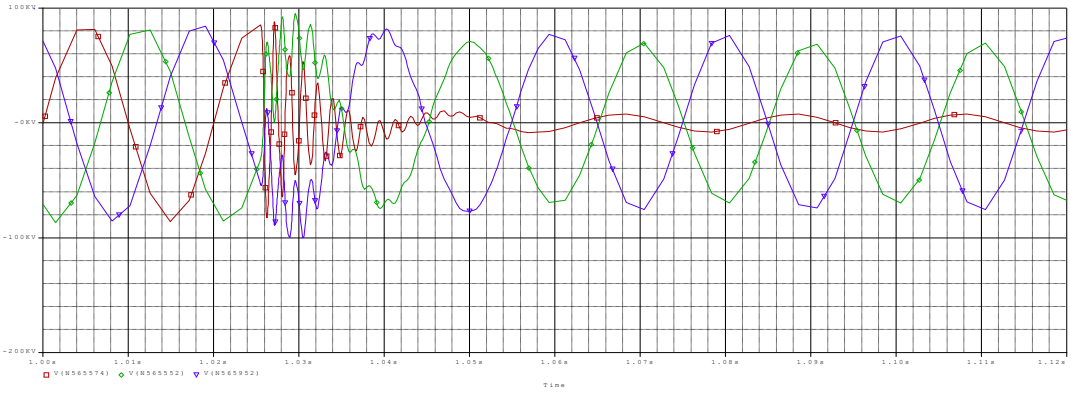


b)

Figure 3. Overvoltages in the busbar systems of high-voltage substations SS2 (a) and SS3 (b) during a single-phase short circuit in a network with a grounded neutral.

As illustrated in Table 2 and Figure 4, the short-circuit current flowing to ground is primarily routed through the neutrals of transformers T_1 and T_2, with only a small portion being transmitted through line capacitances. The predominant share of the short-circuit current (I_{NT_2}) enters transformer T_2 via its neutral and flows consistently through its three high-voltage windings (I_{T2A}, I_{T2B}, I_{T2C}).

Table 2.

The current calculation results

Type of short circuit	I_{T1A} A	I_{T1B} A	I_{T1C} A	I_{T2A} A	I_{T2B} A	I_{T2C} A	I_{cbA} A	I_{cbB} A	I_{cbC} A	I_{NT1} A	I_{NT2} A	Ics A
1_{psc}	5280	2420	2370	2300	2330	2220	5410	2330	2220	790	6843	7648

The currents in phases B and C of transformer T_2 pass through the switch, while the currents in phases B and C of transformer T_1, along with a minor portion of the short-circuit current that reaches the neutral of T_1 from the short-circuit point (I_{NT_1}), return to the short-circuit location via phase A. The short-circuit pathways for these currents are depicted in Figure 5.

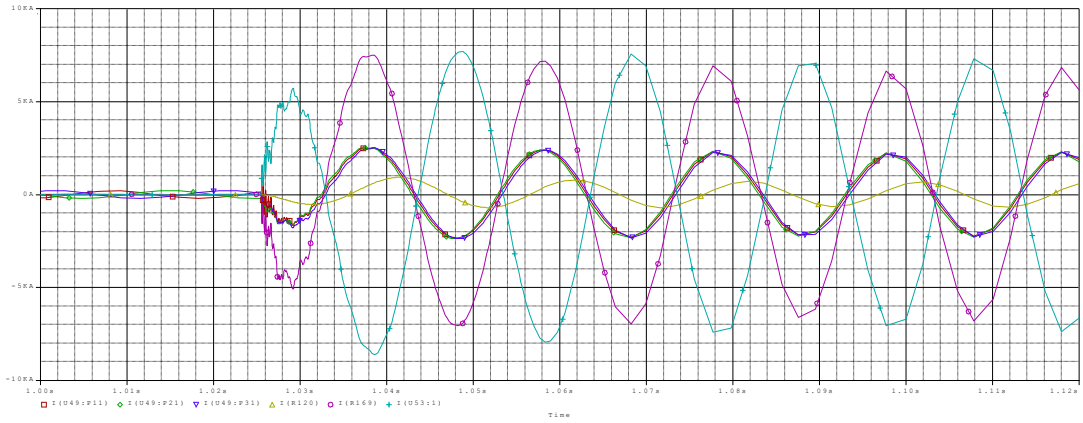


Figure 4. Currents in the windings and the neutral of transformers T_1 and T_2, as well as in the switches and the short-circuit current.

In figures 4 and 5, the currents in phases B and C of the switch exhibit the same direction and magnitude, while the current in phase A flows in the opposite direction. These currents do not cross zero simultaneously. If the switch operates precisely when the current in phase A is at zero, the currents in phases B and C are calculated to be approximately 200 A. The interruption of such currents is significant and can result in substantial overvoltages.

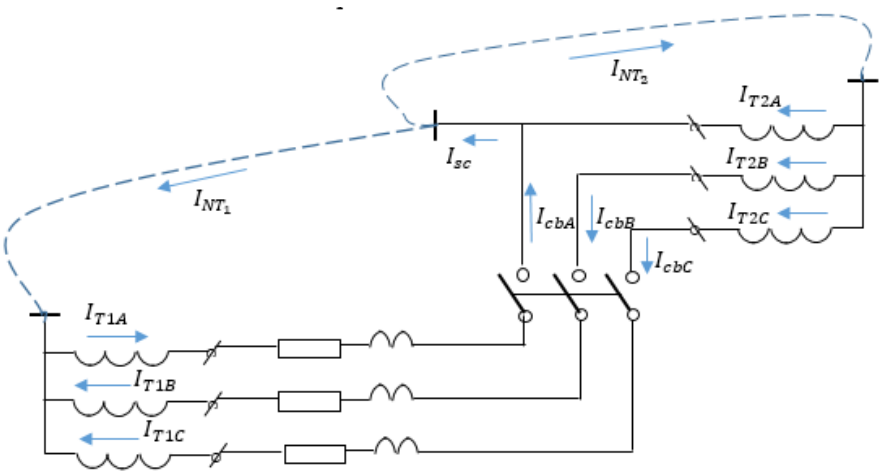


Figure 5. Diagram illustrating the short-circuit current

Figure 6 displays the overvoltage curves in the busbar systems of substation SS2 and at the HV inputs of transformer T_2 following the disconnection of a single-phase short circuit in the examined network [5, 13]. According to the data presented in Table 1 and illustrated in Figure 6, the voltages in the HV busbar systems of substation SS2 double upon disconnection of the single-phase short circuit. This significant increase is attributed to the presence of a surge arrester at this location (Figure 5a).

In the HV busbar systems of substation SS3, a similar doubling of voltage occurs only in phase A, while the increases in the other phases are minimal, particularly in phase C. At the inputs of the undamaged phases B and C of the high-voltage winding of transformer T_2—which has been isolated from the HV busbar system of substation SS2 by the tripping of the circuit breaker—there is a sudden voltage increase characterized by a surge with very high amplitude and extremely short duration.

This phenomenon results from the interruption of substantial inductive currents flowing through windings B and C of the transformer (Figures. 6b and 6c). The very large value of the ratio of the characteristic resistance and the active resistance of the transformer winding - Z_{ch}/R_w, also contributes to the creation of this form of this voltage.

Meanwhile, in phase A, the voltage remains at zero due to the ongoing short circuit in that phase.

Excessively high overvoltage values on the high-voltage side of transformer T_2 are transmitted to its low-voltage windings. Attempts to mitigate these overvoltages through the use of double-contact switches yielded limited success; while such switches reduced the overvoltages significantly, the resulting levels remained unacceptably high.

However, the installation of an overvoltage limiter directly on the HV inputs of transformer T_2 proved effective in bringing these overvoltages within permissible limits. With the overvoltage limiter in place, the overvoltage at the HV inputs is maintained below 200 kV, while at the low-voltage inputs, it remains around 11 kV, both of which are within acceptable thresholds.

In the case of two-phase short circuits to ground (in phases A and B), the voltage in the damaged phases of the HV busbar system at substation SS2 drops to zero, while the healthy phase experiences a reduction to one-third of its original voltage (Table 1). A similar short circuit affecting two phases on the high-voltage side of transformer T_2 results in a significant decrease in its secondary voltages, with voltages in phases A and B diminishing by nearly tenfold, and in phase C by approximately fivefold. This discrepancy in phase voltages on the secondary side of the transformer is attributed to the substantial reduction in currents flowing through the HV windings of the damaged phases.

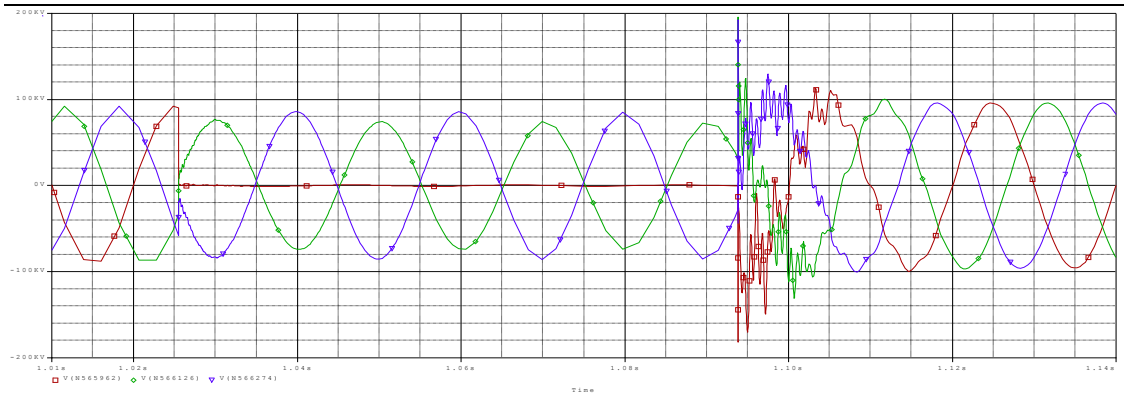
The disconnection of two-phase short circuits to ground results in a voltage increase in the HV busbar systems of substation SS2 by nearly 2.5 times—an increase even greater than that observed during the disconnection of single-phase short circuits. This phenomenon occurs because higher currents are interrupted in the two-phase scenario. At the HV inputs of transformer T_2, the voltage in the short-circuited phases remains at zero, while the voltage in the healthy phase escalates to an extraordinarily high value, significantly exceeding the increase associated with single-phase short circuits. Consequently, the voltages on the secondary side of the transformer also reach excessively high levels for this winding.

Figure 7a illustrates the current curves in the HV windings and the neutral of transformer T_2, as well as the neutral of transformer T_1 and the short-circuit currents. Meanwhile, Figure 7b presents the currents in the switches during two-phase short circuits to ground. As shown in Figure 7b, the currents in the switch phases are considerably more phase-shifted relative to each other compared to single-phase short circuits. Therefore, when one of these currents crosses zero, the others retain substantial values, leading to more pronounced overvoltages upon interruption.

The installation of an overvoltage protection device at the high-voltage inputs of transformer T_2 effectively reduces the overvoltages on its high-voltage and low-voltage sides to 240 kV and 8 kV, respectively. These values are significantly below the permissible overvoltage thresholds for voltage classes of 110 kV and 6 kV.

In the event of a two-phase short circuit (specifically between phases A and B), the voltages in phases A and B equalize, each becoming half the value of the voltage in phase C, which remains at its normal operating level [11]. This same ratio is observed in the secondary phase voltages of transformer T_2, aligning with theoretical expectations for this type of short circuit.

The disconnection of a two-phase short circuit results in a voltage increase in the HV busbar systems of substation SS2 by four times compared to the values during the short circuit, and by two times relative to the voltages in normal network operation (Table 1). Unlike the previously analyzed cases, this time excessively high overvoltages occur in all three phases of transformer T_2.



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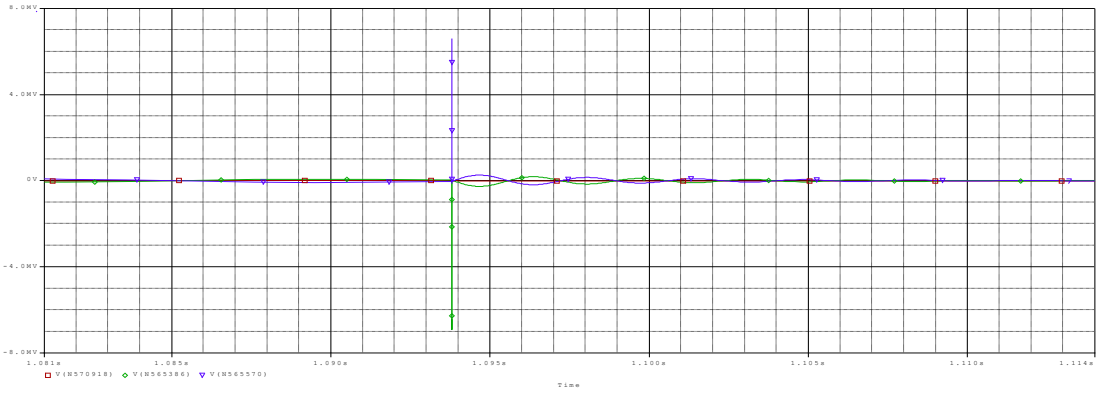

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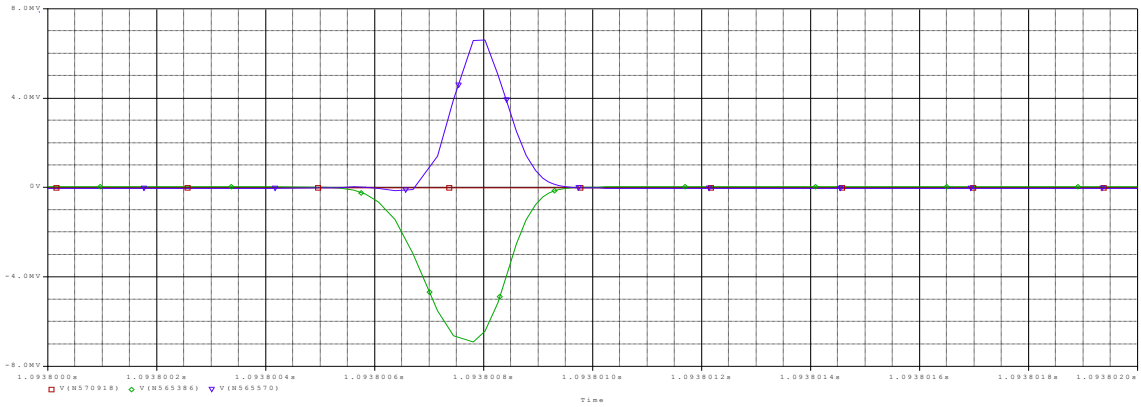


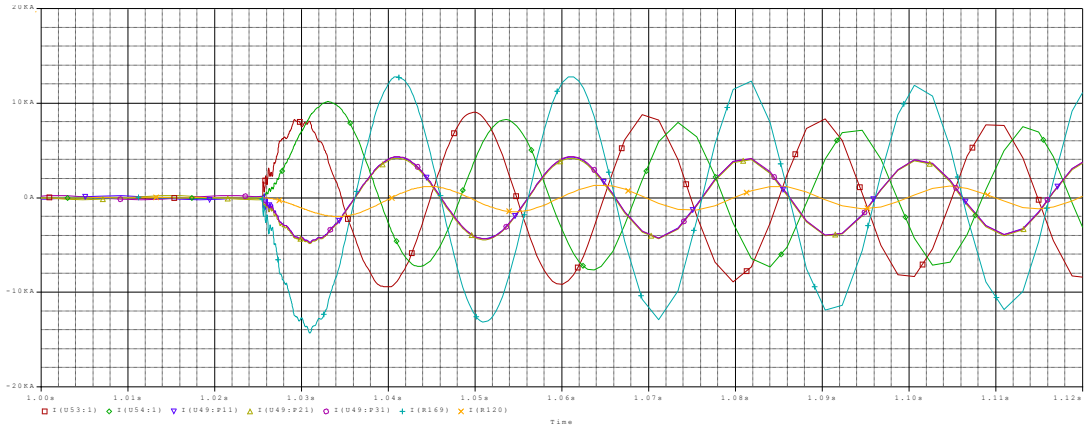
c)

Figure 6. Overvoltages in the HV busbar systems of substation SS2 (a), at the HV inputs of transformer T_2 (b), and the same presented in an open form (c) during the disconnection of a single-phase short circuit.

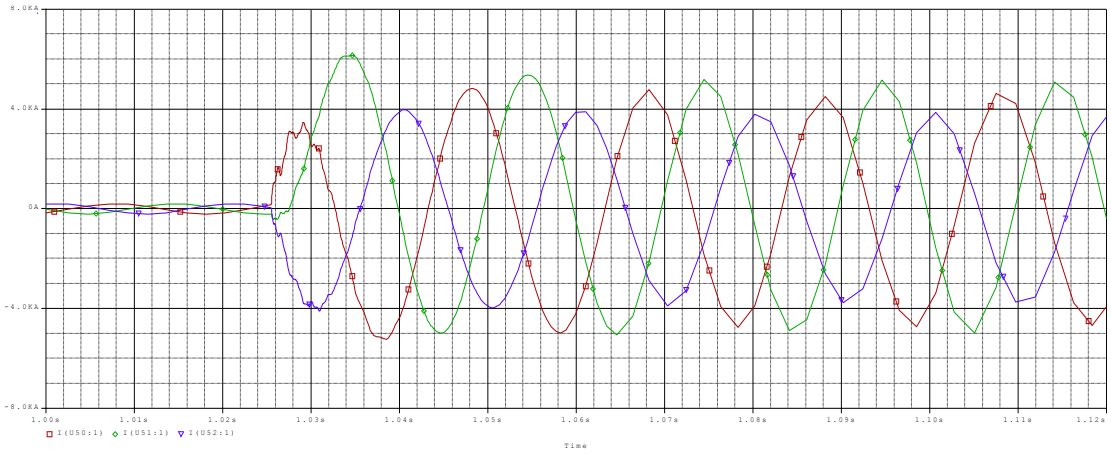
In this scenario, the installation of a surge protective device (SPD) at the HV inputs of transformer T_2 provides effective protection. The overvoltages on the high-voltage and low-voltage sides of the transformer are reduced to 190 kV and 12 kV, respectively, both of which fall within permissible limits.

An analysis of three-phase short circuits reveals that neither the occurrence nor the disconnection of this type of short circuit generates overvoltages in the network. In a three-phase short circuit, the voltages in all affected phases drop to zero, and the currents in all three HV windings of the transformer also fall to zero. As a result, there is no cutoff current to generate

overvoltages.



a)



b)

Figure 7. Current curves in the HV windings and the neutral of transformer T_2, in the neutral of transformer T_1, and at the short-circuit point (a), and in the switches (b) during a two-phase short circuit to ground.

[12] The results of voltage calculations for the busbar systems of substations SS2 and SS3, as well as the HV and LV inputs of both transformers and the terminals of the first load, with the transformers' neutrals isolated from ground, are presented in Table 3 and illustrated in Figures 7-9.

From Table 3, it is evident that with the network's neutrals isolated from ground, a single-phase ground fault (short circuit of phase A) leads to the voltage of the damaged phase dropping to zero. In contrast, the voltages in the healthy phases increase significantly—from 92 kV to 166 kV in phase B and 188 kV in phase C, nearly doubling the values observed in the grounded neutral case. An increase in voltage is also noted on the secondary side of transformer T_2, where the voltage in the damaged phase rises by up to 20%, while the remaining phases experience only a slight increase. For transformer T_3, the voltage increase is even more pronounced, doubling in phase A and increasing by almost 25% in phases B and C.

In this scenario, the current flowing to ground is primarily channeled through the capacitances of both lines. When one phase is shorted to ground, the capacitive currents increase significantly, reaching up to 220 A. However, these currents quickly diminish, stabilizing at approximately 20 A in the steady state following the short circuit. An increase in capacitive currents is also observed

upon disconnection of the emergency short circuit of one phase to ground (Figure 8).

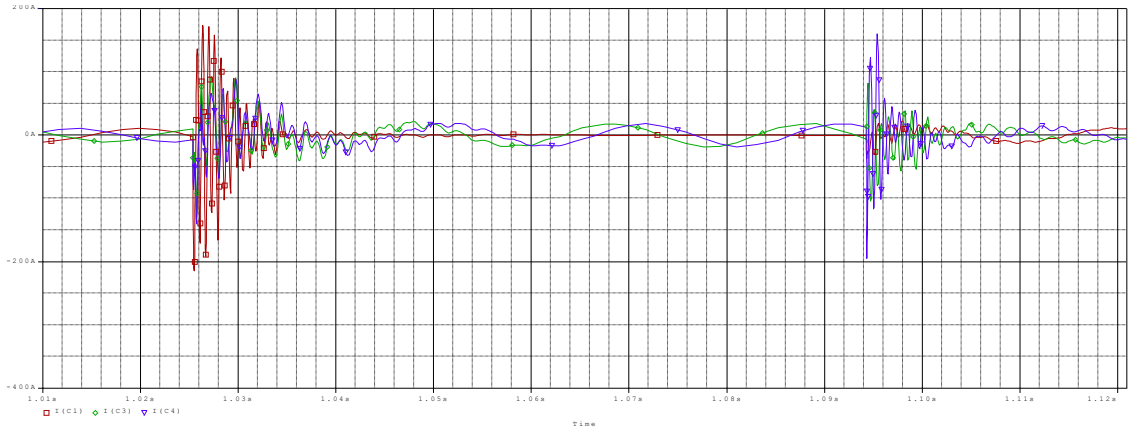


Figure 8. Currents in the first line capacities during a single-phase short circuit in a network with an isolated neutral

Tripping a single-phase ground fault in networks with an isolated neutral, similar to those with a grounded neutral, results in extremely high overvoltages on both the HV and LV sides of transformer T_2. This phenomenon is primarily due to the interruption of currents in the undamaged phases, as previously noted.

In networks with an isolated neutral, a single-phase ground fault does not significantly alter the phase currents. At the moment the circuit breaker trips—when the current of the faulted phase crosses zero—the currents in the other phases retain sufficiently large values (approximately $\pm I_m \sin(120°)$). The interruption of these currents contributes to the generation of remarkably high overvoltages.

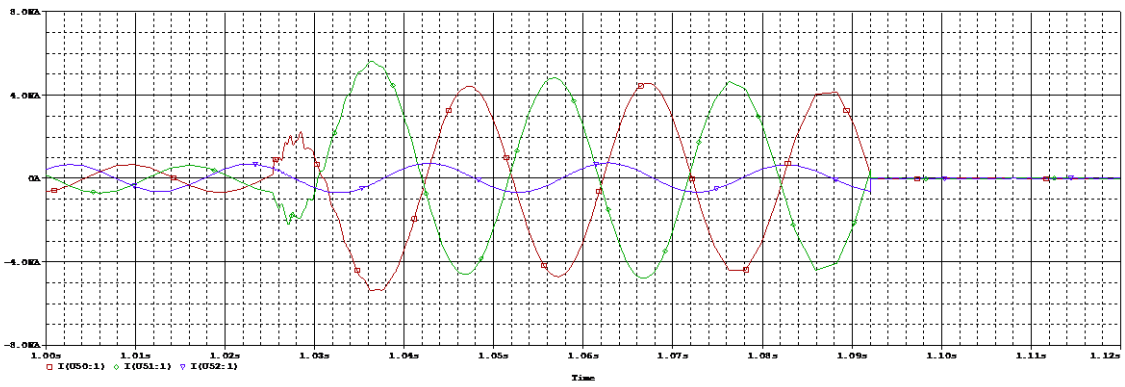
The voltages in the busbar systems of both substations also experience notable changes. At substation SS2, the voltage of phase A rises from zero to 163 kV, phase B increases from 166 kV to 195 kV, while phase C decreases from 188 kV to 171 kV. Compared to the nominal voltage values, the increases in phases A and B are 1.8 and 2.15 times, respectively, while phase C experiences an increase of 1.85 times. In substation SS3, phase A's voltage increases by 1.6 times, and the voltages in phases B and C rise by 2.25 times.

Protection against excessively high overvoltages is achieved by installing an overvoltage protection device at the high-voltage inputs of transformer T_2, as previously described [13].

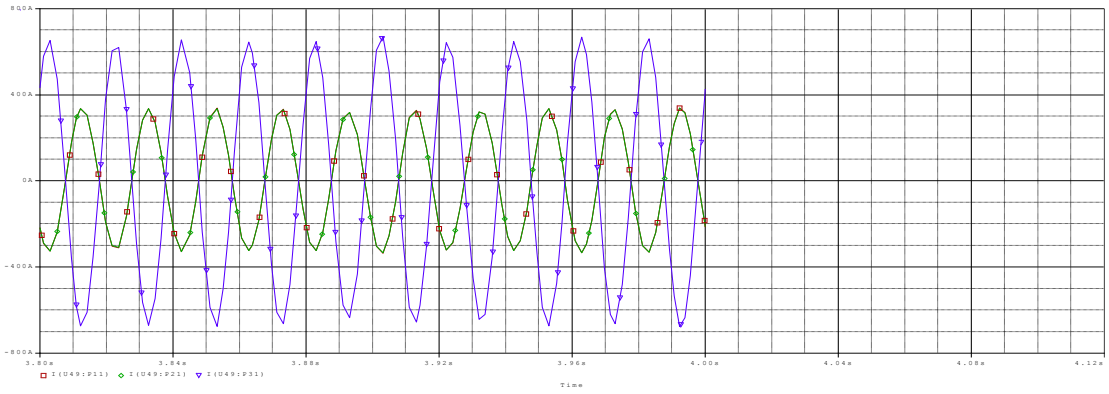
In the case of a two-phase short circuit to ground (specifically in phases A and B), the currents in the damaged phases become oppositional due to the isolated neutral, while the steady-state values of the capacitive currents remain low (approximately 20 A). The minimum values of the currents in phases A and B coincide with nearly maximum values in phase C. The interruption of this current in phase C results in significantly elevated voltage levels in the network, particularly affecting transformer T_2 (Figure 9a).

In this type of short circuit, the currents in the damaged phases do not reach the corresponding windings of transformer T_2. For instance, the current from the damaged phase A is redirected into the other damaged phase B and returns to the network through phase B. Thus, the same current flows through both phases A and B, similar to a two-phase short circuit.

Only the current from phase C reaches the winding of phase C in the transformer, with half of this current flowing through winding B and the other half through winding C (Figure 8b). As illustrated in Figure 8b, the currents in the windings of phases A and B are identical, each being half of the current in winding C of transformer T_2. The halves of the phase C current passing through the windings of phases A and B, combined with the currents from the damaged phases, return to the network through phase B.



a)



b)

Figure 9. Illustrates the currents during a two-phase short circuit to earth in the network under consideration.

The current curves presented in Figures 8a and 8b help identify the short-circuit paths for these currents, which are depicted in Figure 10.

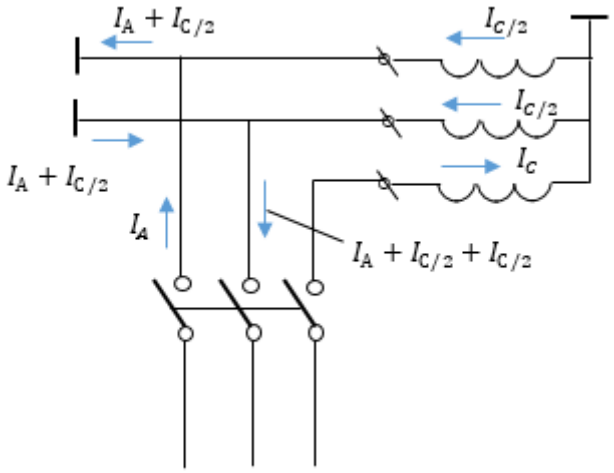


Figure 10. Illustrates the circuit diagram of the current short circuit in the high-voltage windings of transformer T_2 and in the switch during a two-phase short circuit to ground in the network under consideration

A two-phase short circuit to ground leads to a complete reduction of the voltages in the faulty phases to zero while causing an increase in the voltage of the healthy phase across all network elements. For instance, the voltage in phase C of the high-voltage winding of transformer T_2 increases by 1.6 times. On the secondary side of this transformer, the voltage in phase C is double that of phases A and B, as the currents in the high-voltage windings for phases A and B are identical and equal to half the current of phase C (refer to Figure 8b).

When a two-phase short circuit to ground is tripped, it results in even higher overvoltages within the network. While tripping a single-phase short circuit caused the voltages in the busbar systems of substations SS2 and SS3 to rise to 195 kV and 190 kV, respectively, a two-phase short circuit leads to voltages of 233 kV and 204 kV. On both the primary and secondary sides of transformer T_2, these voltages reach disproportionately high levels. The installation of surge arresters at the high-voltage inputs of transformer T_2 effectively limits these excessive overvoltages to 200 kV and 8 kV, respectively.

In the case of a two-phase short circuit (in phases A and B), the voltages of the damaged phases become equal at almost all points of interest. Unlike a two-phase short circuit to ground, this type of short circuit results in a reduction of voltage in substation SS2. While the reduction in the healthy phase is minimal, in the damaged phases it can reach up to two times lower than normal. On the secondary winding of transformer T_2, the voltages in phases A and B are also half that of phase C, reflecting a current distribution similar to that observed during a two-phase short circuit to ground (Figure 9). In substation SS3, there is a slight increase in the voltages of phases A and B.

The disconnection of a two-phase short circuit also differs from that of a two-phase short circuit to ground. Compared to a two-phase short circuit to ground, the voltage increase during the disconnection of a two-phase short circuit is less pronounced (Table 3). For instance, in the high-voltage busbar systems of substation SS2, the voltage in the damaged phases rises to 233 kV when disconnecting a two-phase short circuit to ground, while it reaches only 192 kV during a two-phase short circuit. The voltages in transformer T_2 also increase significantly, similar to other short circuit scenarios, necessitating the installation of an overvoltage limiter at the high-voltage inputs of transformer T_2 to ensure that these voltages remain within permissible limits.

IV. Conclusions

1. The study examines overvoltages at the high-voltage (HV) and low-voltage (LV) inputs of transformers with grounded and isolated neutrals during short circuits occurring nearby. It was found that with an isolated neutral, a single-phase ground fault can increase the voltage on the undamaged phases of the primary and secondary windings of the transformer by up to 2.2 times and 1.2 times, respectively. In the case of a two-phase ground fault, the increases are 1.6 times and 1.1 times.
2. The disconnection of all forms of short circuits, whether in transformers with grounded or isolated neutrals, results in excessively high overvoltages on the primary and secondary sides of the affected transformer. This phenomenon is exacerbated by the interruption of substantial currents in the undamaged phases at the moment the circuit breaker operates, as the three phase currents do not reach zero simultaneously. These overvoltages manifest as pulsed voltage surges with durations of several microseconds.
3. To mitigate these excessively high overvoltages, surge arresters should be installed at the high-voltage inputs of transformers. While the implementation of two-contact switches can reduce these overvoltages by a factor of 2 to 3, the resulting values still remain excessively high.

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OPTIMIZING BAYESIAN REPETITIVE GROUP SAMPLING PLAN FOR QUALITY CONTROL TO ENHANCE DECISION MAKING EFFICIENCY IN MODERN MANUFACTURING

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Abstract

This article introduces an approach to optimize the design of Repetitive Group Sampling (RGS) plan in the context of quality control for modern manufacturing processes. The primary objective of this study is to enhance decision-making efficiency by applying Bayesian principles to develop optimal sampling plans. In modern manufacturing environment, the industries are using the advanced technologies and machineries to maintain the quality of their products. The existence of defects would consequently be highly rare in such production. In such situation, Zero Inflated Poisson (ZIP) distribution is a more appropriate probability distribution rather than the usual Poisson distribution. Further, manufacturing industries often use a variety of manpower and materials to produce their products in various production streams. This may lead to have more quality variation in between lots and hence, the lot quality will vary over lots. The lots that arise from such a production process will be unstable, and quality variations among the units are often heterogeneous in nature. In such situation, the Bayesian sampling plans under Zero Inflated Poisson distribution would be more effective and alternative for traditional sampling plans. This paper studies the designing and selection of Bayesian Repetitive Group Sampling (BRGS) Plan under the conditions of Gamma-Zero Inflated Poisson distribution (G-ZIP). To investigate the effectiveness of this plan, a comparison between the proposed BRGS plan and various existing sample plans is made. Further, we provided the procedure and tables with the suitable numerical illustration to compute the optimal sampling plan.

Keywords: Bayesian perspective, group sampling, quality control, decision-making, process monitoring.

1. INTRODUCTION

Acceptance sampling plans are useful tools for applications involving quality assurance and provide the producer and the consumer a general rule for lot sentencing to satisfy their requirements for product quality. A well-designed acceptance sampling plan not only reduces inspection costs and time, but also protects both the producer and the consumer. Therefore, a sampling plan with a smaller sample size required for inspection is more desirable and useful,

particularly when inspection is costly and destructive. There are several sampling schemes that have been used for various situations in the industry. One of the sampling schemes called the repetitive sampling plan has been introduced by Sherman. This sampling procedure is especially effective for situations where product inspection is destructive and costly. It enables the acceptance or rejection of a lot based on repeated testing of a small number of identical samples. The operating procedure of the attributes RGS plan can be seen in Sherman [1]. Further, the RGS plans are usually more efficient than the single sampling plan. The RGS plan can achieve the desired level of protection with a smaller sample size compared to a single sampling plan. Soundararajan and Ramasamy [2] developed procedures and tables to select the optimal parameters for the RGS plan. Balamurali et al., [3] developed the RGS plan and demonstrated that it significantly reduces the average sample size compared to single and double sampling plans. Recently, Kannan et.al., [4] proposed economic designed RGS plan to satisfy both producer's and consumer's risks with minimum cost under Birnbaum–Saunders distribution. However, recent attempts to develop procedures for Repetitive Group Sampling (RGS) plans specifically for attribute quality characteristics have been limited. Hence, this paper attempts to extend the concept of repetitive group sampling based on attribute quality characteristics.

In recent years, due to technological advancements there is a competitive environment in manufacturing industries has produce products in perfect quality. However, in production processes, there may be natural variability that causes some batches to have zero defects while others have a Poisson-distributed number of defects. ZIP distributions allow for the modeling of this variability effectively. Incorporating the ZIP distribution into acceptance sampling plans can lead to more efficient and cost-effective sampling plans. By accurately modeling the likelihood of zero defects, we can design acceptance sampling plans that reduce the number of samples required for inspection when the process is consistently producing zero-defective items. ZIP distribution is appropriate to design sampling plans that are more conservative when it comes to rejecting batches, reducing the risk of rejecting batches with only a small number of defects or no defects at all. Further, the application of ZIP model to defects in the manufacturing process has been discussed by Lambert [5]. In dental epidemiology research, Bohning et al., [6] made a few significant comparisons between the ZIP distribution and the Poisson distribution to measure people's dental health. Some of the applications of ZIP distribution can be found in Xie et al., [7], Rodrigues [8], Naya et al., [9], Sim and Lim [10], Yang et al., [11], Mussida et al., [12]. Further, Loganathan and Shalini [13] developed a single sampling plan based on attributes under the ZIP distribution. Additionally, Rao [14] designed a single sampling plan for resubmitted lots considering the ZIP distribution.

In Bayesian methodology, the acceptance sampling plans includes prior information regarding the process variations for taking decisions about the submitted lots can be used under the sampling plans as an alternative to the traditional sampling plans. The optimal design of repetitive group sampling plans under a Bayesian perspective represents a significant advancement in quality control for modern manufacturing. It enables enhanced decision-making efficiency, leading to improved process performance, reduced costs, and increased customer satisfaction. The proposed methodology has practical implications for manufacturers seeking to optimize their quality control processes and navigate the challenges of the dynamic manufacturing landscape. Bayesian methods offer a flexible framework for modeling and estimation, allowing for more accurate and reliable decision-making. Several studies have successfully applied Bayesian techniques in various quality control applications, including process monitoring, parameter estimation, and defect prediction. Hald [15] provided a detailed discussion on the procedures and implications of a Bayesian Single

Sampling Plan, considering both Gamma-Poisson (GP) and Beta-Binomial distributions.

Additionally, Calvin [16], Case et al., [17], and Guthrie and Johns [18] explored the selection of prior distributions for the lot fraction of non-conforming items in Bayesian sampling plans. Vijayaraghavan et al., [19] discussed a Bayesian Single Sampling Plan using the Gamma-Poisson distribution and proposed a method to evaluate the efficiency of this sampling plan in comparison to the conventional Poisson Single Sampling Plan. Rajagopal et al., [20] developed a Bayesian Single Sampling Plan based on the Polya distribution, utilizing the Guenther approach, and examined the discriminating power of these plans through their corresponding operating characteristic curves.

The purpose of this article is to develop a Bayesian Repetitive Group Sampling (BRGS) plan by attribute, based on the G-ZIP distribution, using the Guenther approach. In next section, provides an operating procedure of Repetitive Group Sampling plan. In third section, a brief description of the Bayesian G-ZIP distribution along with its performance measure is given. The fourth section provides the operating procedure and tables for selecting the proposed plan and to determine the optimum plan parameters for the specified quality levels through the Guenther iterative producer. In the fifth section, numerical illustrations are provided to assess the effectiveness of proposed plan compared to alternative sampling plans. The results are summarized in the concluding section.

2. REPETITIVE GROUP SAMPLING PLAN

Sherman [1] introduced the Repetitive Group Sampling (RGS) plan, which provides a straightforward procedure for attribute-based quality characteristics. This plan is more efficient than single sampling and resembles sequential sampling methods. It is particularly suited for situations involving destructive or costly inspections, using repeated samples to determine lot acceptance or rejection. The RGS plan achieves the desired protection with a smaller sample size compared to single sampling. According to Sherman [1] the operating procedure of RGS plan involves the following steps. First, a random sample of size n is taken from the lot. Next, the number of non-conforming items, d in the sample is observed. If $d \leq c_1$, the lot is accepted. If $d > c_2$, the lot is rejected. If $c_1 < d \leq c_2$, the sampling and inspection process is repeated until a decision is made. This plan is completely specified by three parameters, namely sample size n and the acceptance numbers c_1 and c_2. It is observed that the RGS plan reduces to a single sampling plan when $c_1 = c_2$ with c_1 always being less than c_2.
The probability of lot acceptance is determined by using the Operating Characteristic (OC) function, which is derived to be:

$$P_A(p) = \frac{P_a(p)}{P_a(p)+P_r(p)} \quad (1)$$

Where, $P_a(p)$ is the probability of acceptance of a submitted lot with fraction defective p based on a given sample, whereas $P_r(p)$ is the corresponding probability of lot rejection.

3. THE OC FUNCTION OF BAYESIAN RGS PLAN UNDER G-ZIP DISTRIBUTION

In acceptance sampling plan the number of nonconforming occurrences during a sampling inspection is considered as count data, their Poisson frequency distribution model could be identified using a probability distribution with parameter, representing the average number of defects per unit. However, the Zero-Inflated Poisson count models provide an alternative method to explain the excess zeros that is a greater number of non-conforming units by modeling the data

as a mixture of two separate distributions.

As given by Lambert [5] the probability function of a Zero Inflated Poisson distribution is given by,

$$P(X = x; \omega, \lambda) = \begin{cases} \omega + (1-\omega)e^{-\lambda}, & when\ x = 0 \\ (1-\omega)\frac{e^{-\lambda}\lambda^{x}}{x!}, & when\ x = 1,2,3 \ldots \end{cases} \quad (2)$$

Where, ω and λ are the parameters of ZIP distribution with $0 < \omega < 1,\ \lambda > 0$. When $\omega = 0$, this model reduces to the Poisson model. Further, when the manufacturing process is well monitored, defects become a rare event, resulting in many sampled products having zero defects. In such cases, the ZIP distribution is the appropriate probability distribution for modeling the number of defects in the sampled products. When the number of nonconformities items in the sample is followed by the model of Zero Inflated Poisson distribution with parameter (np, ω), when the proportion of nonconformities p varies at random from lot by lot and is distributed according to a gamma distribution, which is a natural conjugate prior to p, the density function of the p is given by,

$$f(p/t, s) = \frac{e^{-tp}t^{s}p^{s-1}}{\Gamma s}, \qquad 0 \le p < \infty, \qquad t, s > 0$$

Where, t is scale parameter and s is the shape parameter. If $E(p) = \bar{p}$ is gives the scale parameter is obtained by $t = s/\bar{p}$. Here, the prior knowledge s is estimated from past history of the production process. Further, the Uniform distribution is assumed to be the conjugate prior to ω with parameters a and b. The probability density function of the ω is defined as,

$$f(\omega/a, b) = \frac{1}{b-a}\ , a \le \omega \le b$$

In particular, the limitation of parameter ω can be taking $a = 0$ and $b = 1$, that is the uniform prior on (0,1). Then the equation for the standard uniform distribution is,

$$f(\omega) = 1 \quad \text{for} \quad 0 \le \omega \le 1$$

Thus, the predictive distribution of the number of defectives x is reduced to the G-ZIP distribution. In cases where the production process is unstable, the non-conforming items x and the average number of defects p are independently distributed. According to Hald [15] the average probability of acceptance $\bar{p}$ is approximately obtained by,

$$P_a(\bar{p}) = \int_0^{\infty} P_a(p)\, f(p) dp \quad (3)$$

Thus, the average probability of acceptance $\bar{p}$ under the conditions of Gamma prior distribution and ZIP sampling distribution can be obtained by,

$$p(x; \omega, n\bar{p}, s) = \begin{cases} \omega + (1-\omega)(1-\rho)^{s}, & when\ x = 0 \\ (1-\omega)\binom{x+s-1}{s-1}\rho^{x}(1-\rho)^{s}, & when\ x = 1,2,3 \ldots \end{cases} \quad (4)$$

Let us take for convenience $\rho = \left(\frac{n\bar{p}}{n\bar{p}+s}\right)$. It is to be observed that the sampling distribution of x is the Zero Inflated Negative binomial model with parameter s and $\left(\frac{n\bar{p}}{n\bar{p}+s}\right)$ When production is not stable, both x and $\bar{p}$ are independently distributed, and hence the sampling distribution of x, according to

Hald [15] under the conditions that $\bar{p} < 0.1$, $\bar{p}/s < 0.2$ the OC function is given by,

$$P_a(\bar{p}) = \sum_{x=0}^{c} p(x; \omega, n\bar{p}, s) \tag{5}$$

where $\bar{p}$ is the average lot quality or average fraction non-conforming. Here, the value of s can be estimated from the prior information about the production process. When $c = 0$, the lot acceptance probability becomes as,

$$P_a(\bar{p}) = \omega + (1-\omega)(1-\rho)^s \tag{6}$$

Based on this, the OC function of Bayesian RGS plan under the conditions of Gamma-Zero Inflated Poisson distribution is given by,

$$P_A(\bar{p}) = \frac{\omega + (1-\omega)(1-\rho)^s + \sum_{x=1}^{c_1} p(x; \omega, n\bar{p}, s)}{1 - \sum_{x=1}^{c_2} p(x; \omega, n\bar{p}, s) + \sum_{x=1}^{c_1} p(x; \omega, n\bar{p}, s)}$$

4. DETERMINATION OF PLAN PARAMETER

The optimal sampling plans are determined for a fixed parameters ω, s and a wide range of p_1 and p_2 with specified producer's and consumer's risks which are presented in Table 5.1- 5.8. The plan parameters are obtained from these tables with producer's risk to a maximum of 5% and consumer's risk to a maximum of 10%. Under these conditions, can be determine the plan parameters and minimize the Average Sample Number (ASN) at the level of Limiting Quality Level (LQL). The ASN of the RGS plan is given by,

$$ASN = \frac{n}{P_a(p) + P_r(p)} \tag{7}$$

Where $P_a(p)$ and $P_r(p)$ are the probability of acceptance and probability of rejection of a lot, under G-ZIP model. It should be noted that, the values of the parameter s, in the prior distribution range over the interval $(0, \infty)$. It is observed from the empirical study that the values of $P_a(p)$ of BRGS plan by attributes under the conditions of G-ZIP distribution do not differ much from those of ZIP plans at each value of p for larger values of s. It indicates that the OC function under the conditions of G-ZIP converges to non-Bayesian ZIP sampling plans. Hence, two different values such as 5 and 10 are taken for s. The set of tables corresponds to these values of s with the fixed parameter ω = 0.001,0.01,0.05,0.09 are considered. The BRGS plan can be used for the situation where the shape parameter is known and unknown. Normally, producers keep the record of the estimated shape parameter value for their product or it can be estimated from the available data. While searching for the optimum sampling plan using the iterative procedure [22], the values of n and c were restricted to the maximum of 7500 and 75 respectively for the iteration purpose. In some cases, very large values were obtained for n, which are not feasible to apply in practice. To those data sets, the optimum sampling plans are not presented which are denoted by the symbol ***.

5. NUMERICAL ILLUSTRATION

In this section, the procedure of selecting the plan parameters for the proposed plan is described with numerical illustrations. The significance of the Bayesian RGS plan under the conditions of G-ZIP distribution is highlighted.

Table.5.1 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 5$ *and* $\omega = 0.001$.

AQL (p_1)	**LQL** (p_2)					
	0.05	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	72; 0,2 (104.85)	54; 0,1 (64.21)	47;0,1 (55.833)	41;0,1 (48.745)	36; 0,1 (42.937)	33;0,1 (39.171)
0.01	79; 0,3 (141.668)	66;0,3 (118.110)	51;0,2 (74.609)	45; 0,2 (65.533)	40; 0,1 (58.252)	36; 0,2 (52.426)
0.015	94; 0,5 (249.636)	72; 0,4 (157.949)	56;0,3 (101.052)	49;0,3 (88.422)	44;0,3 (78.740)	36;0,2 (52.427)
0.02	117;0,8 (516.976)	85;0,6 (268.894)	68;0,5 (177.547)	54;0,4 (118.462)	44;0,3 (78.740)	40;0,3 (70.996)
0.025	152;0,13 (1387.722)	103;0,9 (538.750)	78;0,7 (294.534)	59;0,5 (155.794)	48;0,4 (105.299)	44;0,4 (94.623
0.03	262;0,29 (9932.649)	127;0,13 (1150.453)	89;0,9 (457.218)	68;0,7 (258.482)	57;0,6 (178.655)	47;0,5 (124.818)
0.035	***	190;0,24 (5104.563)	109;0,13 (983.553)	82;0,10 (494.337)	65;0.8 (287.209)	51;0,6 (161.337)
0.04	***	***	153;0,22 (3515.175)	95;0,13 (867.326)	73;0,10 (438.453)	58;0,8 (260.776)
0.045	***	***	***	130;0,21 (2708.636)	89;0,14 (915.086)	66;0,10 (398.724)
0.05	***	***	***	***	112;0,20 (2116.08)	80;0,14 (809.744)

Table.5.2 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 5$ *and* $\omega = 0.01$.

AQL (p_1)	**LQL** (p_2)					
	0.05	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	75;0,2 (106.448)	57;0,1 (66.937)	49;0,1 (57.500)	43;0,1 (50.423)	38;0,1 (44.625)	34;0,1 (39.986)
0.01	83;0,3 (142.062)	70;0,3 (118.685)	54;0,2 (76.324)	47;0,2 (66.617)	42;0,2 (59.363)	38;0,2 (53.562)
0.015	100;0,5 (241.745)	76;0,4 (155.790)	60;0,3 (101.730)	52;0,3 (88.833)	46;0,3 (78.885)	38;0,2 (53.562)
0.02	135;0,9 (564.108)	98;0,7 (317.435)	72;0,5 (172.095)	57;0,4 (116.843)	51;0,4 (103.854)	42;0,3 (71.211)
0.025	***	128;0,11 (653.018)	84;0,7; (272.087)	68;0,6 (191.242)	56;0,5 (133.852)	46;0,4 (93.438)
0.03	***	***	117;0,12 (642.718)	79;0,8 (291.620)	70;0,7 (201.816)	50;0,5 (120.738)
0.035	***	***	***	108;0,13 (646.546)	75;0,9 (313.394)	59;0,7 (189.999)
0.04	***	***	***	***	96;0,13 (574.708)	72;0,10 (335.409)
0.045	***	***	***	***	***	91;0,14 (588.234)
0.05	***	***	***	***	***	***

Table.5.3 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 5$ *and* $\omega = 0.05$.

AQL (p_1)	**LQL** (p_2)					
	0.05	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	99;0,2 (122.031)	83;0,2 (102.068)	71;0,2 (87.379)	56;0,1 (61.64)	49;0,1 (54.100)	45;0,1 (49.497)
0.01	140;0,5 (224.654)	104;0,4 (155.587)	80;0,3 (108.986)	70;0,3 (95.363)	55;0,2 (69.794)	49;0,2 (60.641)
0.015	409;0,19 (811.944)	145;0,7 (254.984)	100;0,5 (140.467)	75;0,4 (115.645)	70;0,4 (103.974)	56;0,3 (76.290)
0.02	***	1015;0,58 (2027.701)	204;0,13 (400.903)	108;0,7 (191.332)	78;0,5 (124.846)	70;0,5 (112.327)
0.025	***	***	***	358;0,28 (758.619)	126;0,10 (241.031)	87;0,7 (152.957)
0.03	***	***	***	***	676;0,58 (1352.596)	215;0,20 (426.644)
0.035	***	***	***	***	***	***

Table.5.4 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 5$ *and* $\omega = 0.09$.

AQL (p_1)	**LQL** (p_2)					
	0.05	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	239;0,4 (260.462)	170;0,3 (183.289)	153;0,3 (163.427)	11;0,2 (117.033)	100;0,2 (105.238)	89;0,2 (93.804)
0.01	***	886;0,25 (985.032)	360;0,11 (400.36)	191;0,6 (211.871)	152;0,5 (167.066)	119;0,4 (129.815)
0.015	***	***	***	1549;0,61 (1726.471)	591;0,25 (656.937)	291;0,13 (323.966)
0.02	***	***	***	***	***	***

Table.5.5 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 10$ *and* $\omega = 0.001$.

AQL	LQL (p_2)					
(p_1)	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	57;0,1 (69.554)	48;0,1 (58.376)	41;0,1 (49.919)	36;0,1 (43.783)	32;0,1 (38.919)	29;0,1 (35.193)
0.01	71;0,3 (140.775)	53;0,3 (82.117)	46;0,2 (70.677)	40;0,2 (61.715)	36;0,2 (55.081)	29;0,1 (35.193)
0.015	78;0,4 (199.729)	59;0,3 (117.322)	51;0,4 (100.536)	40;0,2 (61.715)	36;0,2 (55.084)	32;0,2 (49.372)
0.02	93;0,6 (385.083)	71;0,5 (233.454	56;0,4 (142.347)	44;0,3 (88.007)	40;0,3 (78.176)	36;0,3 (70.358)
0.025	114;0,9 (945.404)	77;0,6 (323.137)	61;0,5 (199.759)	49;0,4 (124.580)	44;0,4 (110.305)	36;0,3 (70.358)
0.03	165;0,16 (4826.770)	96;0,9 (771.389)	71;0,7 (379.076)	58;0,6 (241.223)	48;0,5 (154.045)	39;0,4 (99.861)
0.035	171;0,21 (8081.907)	126;0,14 (2620.430)	82;0,9 (665.819)	63;0,7 (325.282)	52;0,6 (212.464)	43;0,5 (139.114)
0.04	***	***	102;0,13 (1846.700)	72;0,9 (578.504)	56;0,7 (289.139)	46.0,6 (194.784)
0.045	***	***	320;0,46 (31309.960)	90;0,13 (1573.900)	64;0,9 (514.226)	50;0,7 (263.145)
0.05	***	***	***	118;0,13 (4987.170)	76;0,12 (1108.500)	57;0,9 (472.702)

Table.5.6 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 10$ *and* $\omega = 0.01$.

AQL	LQL (p_2)					
(p_1)	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	62;0,2 (99.826)	50;0,1 (59.932)	43;0,1 (51.489)	38;0,1 (45.365)	34;0,1 (40.509)	30;0,1 (35.959)
0.01	74;0,4 (139.493)	56;0,2 (83.276)	48;0,2 (71.380)	42;0,2 (62.457)	37;0,2 (55.344)	30;0,1 (35.959)
0.015	82;0,4 (193.514)	62;0,3 (116.243)	53;0,3 (99.637)	47;0,3 (87.182)	37;0,2 (55.344)	34;0,2 (50.175)
0.02	108;0,7 (458.610)	75;0,5 (220.322)	59;0,4 (137.877)	47;0,3 (87.182)	41;0,3 (77.496)	37;0,3 (69.746)
0.025	182;0,14 (1575.390)	90;0,7 (382.166)	71;0,6 (250.271)	57;0,5 (163.766)	46;0,4 (107.148)	37;0,3 (69.749)
0.03	***	152;0,14 (1304.828)	84;0,8 (415.720)	62;0,6 (219.427)	50;0,5 (146.881)	41;0,4 (96.756)
0.035	***	***	130;0,14 (1125.282)	66;0,7 (295.112)	55;0,6 (195.438)	45;0,5 (132.193)
0.04	***	***	***	171;0,21 (1653.542)	71;0,9 (399.379)	54;0,7 (229.300)
0.045	***	***	***	498;0,59 (4960.443)	101;0,14 (877.904)	64;0,9 (358.414)
0.05	***	***	***	***	365;0,49 (3679.178)	91;0,14 (787.621)

Table.5.7 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 10$ *and* $\omega = 0.05$.

AQL	**LQL (p_2)**					
(p_1)	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	85;0,2 (108.512)	71;0,2 (90.530)	54;0,1 (60.372)	47;0,1 (52.613)	42;0,1 (46.956)	38;0,1 (42.431)
0.01	96;0,3 (136.436)	81;0,3 (116.526)	61;0,2 (77.687)	53;0,2 (67.741)	48;0,2 (60.773)	43;0,2 (54.569)
0.015	159;0,7 (300.280)	105;0,5 (179.863)	79;0,4 (125.816)	60;0,3 (87.147)	54;0,3 (77.684)	49;0,3 (70.051)
0.02	***	230;0,13 (456.923)	101;0,6 (184.261)	78;0,5 (135.143)	61;0,4 (97.857)	55;0,4 (88.070)
0.025	***	752;0,44 (1504.971)	267;0,18 (540.667)	135;0,10 (266.137)	79;0,6 (143.038)	63;0,5 (107.918)
0.03	***	***	862;0,59 (1788.419)	298;0,23 (597.590)	153;0,13 (305.145)	89;0,8 (170.951)
0.035	***	***	***	957;0,75 (1911.998)	333;0,29 (664.908)	178;0,17 (355.931)
0.04	***	***	***	***	***	***

Table.5.8 *Optimal BRGS plan under G-ZIP distribution for given* $p_1, p_2, \alpha = 0.05, \beta = 0.10, s = 10$ *and* $\omega = 0.09$.

AQL	**LQL (p_2)**					
(p_1)	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
0.005	167;0,3 (180.948)	119;0,2 (126.195)	100;0,2 (106.460)	89;0,2 (94.431)	80;0,2 (84.701)	61;0,1 (62.851)
0.01	409;0,11 (456.721)	241;0,7 (266.445)	160;0,5 (176.768)	120;0,4 (132.239)	93;0,3 (100.703)	83;0,3 (90.078)
0.015	***	740;0,26 (822.420)	358;0,14 (400.650)	218;0,9 (242.516)	139;0,6 (155.153)	110;0,5 (122.328)
0.02	***	***	1442;0,62 (1604.849)	568;0,27 (635.174)	315;0,16 (351.616)	221;0,12 (246.443)
0.025	***	***	***	***	921;0,50 (1020.115)	423;0,25 (473.118)
0.03	***	***	***	***	***	***

*** Sampling plan does not exist

5.1. ILLUSTRATION 1

Suppose that the quality engineer in medical research center wants to run an experiment to make a decision on a product, whether to accept or reject based on BRGS plan under G-ZIP model. If the engineer desired to determine a sampling plan for given sets of strengths AQL and LQL say, $p_1 = 1\%$, $p_2 = 8\%$ with producer's risk (α) 5% and consumers risk (β) 10% and the estimated values of $\omega = 0.05$ and $s = 5$. Under these requirements, from the Table 5.3 one can find value of optimum parameters as $n = 70$, $c_1 = 0$ and $c_2 = 3$ with $ASN = 95.363$.

Execution of the plan:

The BRGS plan can be executed under the G-ZIP conditions is operated as follows:

- **Step 1**: Drawn a random sample of 70 units from the lot and observe the number of non-conforming items (d), in the sample.

- **Step 2**: If there are no nonconforming items is found, then accept the lot. If more than 3 nonconforming items are observed, then reject the lot.
- **Step 3:** If the number of nonconforming items is lies between 0 or 3, then repeat these steps until a decision is made on the same lot.

5.2. ILLUSTRATION 2

Suppose that the quality engineer's interest is to focus on sampling inspection towards submitted lot for a given strength of parameters $p_1 = 0.01, \alpha = 0.05, p_2 = 0.07, \beta = 0.10$ and $\omega = 0.05$. From the Table 5.9, one can find the ASN value of Single Sampling Plan (SSP) under the conditions of G-ZIP distribution is determined for different values of $s = 5$ and 10 respectively. Thus, the optimum of ASN values are $(s = 5, ASN = 318)$ and $(s = 10, ASN = 166)$. But, the Bayesian RGS plan under the G-ZIP distribution for the values of prior information $s = 5$ and 10 respectively, can be determined optimum of ASN values $(s = 5, ASN = 108.99)$ and $(s = 10, ASN = 77.69)$. Therefore, this illustration clearly shows that the BRGS plan is more efficient then the BSSP under the G-ZIP distribution. Further, we made a comparative study on the result of RGS plan with the SSP under the conditions of G-ZIP distribution in terms of ASN values. Obviously, a sampling plan having smaller ASN would be more preferable. Table.5.9 shows the values of ASNs for the RGS plan under G-ZIP distribution with sample size of the SSP under the conditions of G-ZIP distribution for some selected combinations of AQL and LQL values. Here, it is considered only two values of the shape parameter with the different values of $\omega = 0.001, 0.01, 0.05, 0.09$ are given in table.

Table 5.9 *ASN values of the RGS Plan and SSP under the conditions of G-ZIP distribution for specified* p_1, p_2, $\alpha = 0.05$ *and* $\beta = 0.10$.

			s = 5		s = 10	
ω	p_1	p_2	SSP	RGS	SSP	RGS
0.001	0.005	0.05	191	104.85	126	69.55
	0.005	0.06	124	64.21	105	58.38
	0.01	0.06	264	118.11	162	82.12
	0.01	0.07	166	74.61	115	70.68
	0.015	0.07	491	101.05	186	100.54
	0.015	0.08	276	88.42	142	61.72
	0.02	0.08	1432	118.46	204	88.01
	0.02	0.09	452	78.74	145	78.18
	0.025	0.09	***	105.30	236	110.31
	0.025	0.1	***	94.62	277	70.36
0.01	0.005	0.05	198	106.45	108	99.83
	0.005	0.06	129	66.94	93	59.93
	0.01	0.06	308	118.69	166	83.28
	0.01	0.07	203	76.32	118	71.38
	0.015	0.07	567	101.73	191	99.64
	0.015	0.08	311	88.83	146	87.18
	0.02	0.08	***	116.84	231	87.18
	0.02	0.09	535	103.85	167	77.5
	0.025	0.09	***	133.85	260	107.15
	0.025	0.1	***	93.44	185	69.75

Table 5.10 *ASN values of the RGS Plan and SSP under the conditions of G-ZIP distribution for specified* p_1, p_2, $\alpha = 0.05$ *and* $\beta = 0.10$.

			s = 5		s = 10	
ω	p_1	p_2	SSP	RGS	SSP	RGS
0.05	0.005	0.05	242	122.03	154	108.51
	0.005	0.06	202	102.07	129	90.53
	0.01	0.06	579	155.59	226	116.53
	0.01	0.07	318	108.99	166	77.69
	0.015	0.07	***	140.47	273	125.82
	0.015	0.08	716	115.45	193	87.15
	0.02	0.08	***	191.33	355	135.14
	0.02	0.09	***	124.85	233	97.86
	0.025	0.09	***	241.03	459	143.04
	0.025	0.1	***	152.96	285	107.92
0.09	0.005	0.05	842	260.46	332	180.95
	0.005	0.06	515	183.29	234	126.2
	0.01	0.06	***	985.03	475	266.45
	0.01	0.07	***	400.36	337	176.77
	0.015	0.07	***	***	***	400.65
	0.015	0.08	***	1726.47	***	242.52
	0.02	0.08	***	***	***	635.17
	0.02	0.09	***	***	***	351.62
	0.025	0.09	***	***	***	1020.12
	0.025	0.1	***	***	***	473.12

From Table 5.9 and 5.10, it can be clearly observed that the ASN value of the proposed plan is significantly lower than that of the existing Bayesian Single Sampling Plan for all combinations of AQL and LQL values. Furthermore, it is noteworthy that the RGS plan under the G-ZIP distribution is more economical than the Single Sampling Plan in terms of ASN. This suggests that the plan provides the desired protection with minimal inspection, thus greatly reducing the inspection costs.

6. COMPARATIVE ANALYSIS ON SAMPLING PLANS

In garment manufacturing industries, where apparel brands source garments are produced in bulk quantity, sampling inspection is an essential step before placing the bulk order. The sampling process covers garment fit checking, fabric and trims quality checking, approval value-added processes, and approval of complete finished garment. The primary objective of the sampling plan is to safeguard from the risk of making a wrong decision on the part of the manufacturer and the consumers. For instance, the quality control personal wants to run an experiment to make a decision on the quality of the shirts to decide whether the whole lot should be delivered to customers or not based on sampling inspection. Suppose that the garments company desire to determine the sampling plan for a lot consisting of 9000 pieces of shirts with the acceptance quality level is 1% and limiting quality level is 7%. Further, the experimenter is fixed the consumer's and producer's risk level as 5% and 10% with the estimated value of $s = 5$. In order to obtain the comparative study, the values of $p_a(p)$ of the RGS plan under the conditions of GP, ZIP and G-ZIP are given in Table.6.1 for the various values of p.

For instance, the strength of the plan is specified as $p_1 = 0.015, \alpha = 0.05, p_2 = 0.09, \beta = 0.10$ and $s = 10$, it is determined that the RGS plan under the conditions of GP distribution is $(40, 2)$, ZIP distribution are $(33,2), (41,2)$ and for the various fixed parameter $\omega = 0.01,0.05,0.09$ respectively. Corresponding to this strength, the RGS plan under the conditions of G-ZIP distribution are determined for different values of ω. The values of ω and the corresponding RGS Plans are ($\omega = 0.01, n = 37, c = 2$), ($\omega = 0.05, n = 54, c = 3$) and ($\omega = 0.09, n = 139, c = 6$). The value of $P_a(p)$ of these given sampling plans are calculated and are given in Table 4.10. All the RGS plan under the conditions of G-ZIP distribution have been determined for the same strength. It should be noted that the G-ZIP RGS plan enables to take the decision about the current lot considering the past history of the production process. But, the ZIP RGS plan takes decision based on the current sample information only.

From the Table 6.1, it can be observed that all the given sampling plans have higher $P_a(p)$ for lower values of p and have sudden drop in $P_a(p)$ for higher values of p. It indicates that all the given sampling plans, in general, it protects the producer's interest against good quality lots and safeguard the consumer against poor quality lots. In addition, the RGS plan under the conditions of G-ZIP distribution dominate the non-Bayesian ZIP RGS plan and Bayesian GP SSP invariably at all values of p below $p_1 = 0.040$. It shows that the G-ZIP RGS plan provides a better protection to the producer from the risk of rejecting the lots of good quality compared to the non-Bayesian ZIP RGS plan and Bayesian GP RGS plan. For instance, in the case of GP RGS plan, corresponding to $p = 0.01$, the probability of acceptance value is (0.986) and ZIP SSP corresponding to $p = 0.01$, the probability of acceptance values is $0.991, 0.997$ and 0.997 for the various parameter $\omega = 0.01,0.05,0.09$ respectively. But, under the G-ZIP case, for the same values of ω, the probability of acceptance values corresponding to $p = 0.01$ are 0.997, 0.992 and 0.997. It is seen that G-ZIP RGS plan accept lots having lower fraction nonconforming with higher probability.

Table 6.1 *Values of OC functions of GP RGS, ZIP RGS and G-ZIP RGS sampling plans for the given strength* $p_1 = 0.015, \alpha = 0.05, p_2 = 0.09, \beta = 0.10$ *and* $s = 10$.

model		GP RGS	ZIP RGS			G-ZIP RGS		
parameters	ω	-	0.01	0.05	0.09	0.01	0.05	0.09
	n	40	33	41	88	37	54	139
	c	2	2	2	4	2	3	6
Lot fraction nonconforming (p)	0.007	0.995	0.997	0.999	1.000	0.999	0.998	1.000
	0.010	0.986	0.991	0.997	0.997	0.997	0.992	0.997
	0.012	0.976	0.984	0.993	0.992	0.994	0.984	0.989
	0.015	0.954	0.969	0.983	0.973	0.985	0.963	0.952
	0.020	0.896	0.926	0.946	0.879	0.956	0.898	0.781
	0.030	0.713	0.770	0.768	0.508	0.819	0.663	0.376
	0.035	0.606	0.667	0.640	0.362	0.714	0.533	0.271
	0.040	0.504	0.561	0.511	0.268	0.599	0.420	0.209
	0.045	0.412	0.460	0.399	0.208	0.488	0.331	0.171
	0.050	0.334	0.372	0.310	0.171	0.389	0.263	0.147
	0.055	0.269	0.297	0.242	0.146	0.308	0.212	0.132
	0.060	0.217	0.237	0.193	0.130	0.243	0.175	0.121
	0.650	0.176	0.189	0.158	0.118	0.192	0.148	0.113
	0.075	0.117	0.121	0.113	0.105	0.124	0.112	0.103
	0.080	0.096	0.098	0.099	0.100	0.101	0.100	0.100

Table 6.2 *Values of producer, consumer and total sum of risk for the specified strength with* $s = 10$ *of the optimum sampling plan.*

Model	ω	α (%)	β (%)	$\alpha+\beta$ (%)
GP RGS	-	1.41	21.75	23.15
ZIP RGS	0.01	0.91	23.68	24.58
	0.05	0.34	19.34	19.67
	0.09	0.26	12.99	13.25
G-ZIP RGS	0.01	0.3	24.28	24.58
	0.05	0.79	17.53	18.32
	0.09	0.27	12.06	12.32

The probability of acceptance corresponding to $p = 0.06$ for GP RGS plan is 0.217 and ZIP RGS plans are $(0.237, 0.193, 0.130)$ for the parameter $\omega = 0.01, 0.05, 0.09$ respectively. But, under the G-ZIP case, for the same values of ω, the probability of acceptance corresponding to $p = 0.06$ are $(0.243, 0.175, 0.121)$. It shows that G-ZIP RGS plan with moderately higher value of ω accepts the lots having higher fraction nonconforming with lower probability, which also protects consumer against accepting poor quality lots.

Figure.1 are presents the OC curves of G-ZIP RGS plan along with the OC curves of GP and ZIP RGS plans based on Table 6.1. From these Figures, it can be observed that all the three plans of OC curves are in desirable shape with a swell at lower values of p and a sudden drop at higher values of p. However, particularly, the G-ZIP RGS plan has the probability of acceptance is large for good quality lots and is small for bad quality lots, which indicate that these RGS plan ensure protection to both producer and consumer. Further, when the value of ω becomes large, the proposed plan was performed better than the classical RGS plan under the ZIP distribution and the GP RGS plan. From these results, the garments company can be obtained that the proposed BRGS plan will give the optimum sampling plan for the desired quality levels as well as safeguard both the producers and consumers instead of tradition sampling plan.

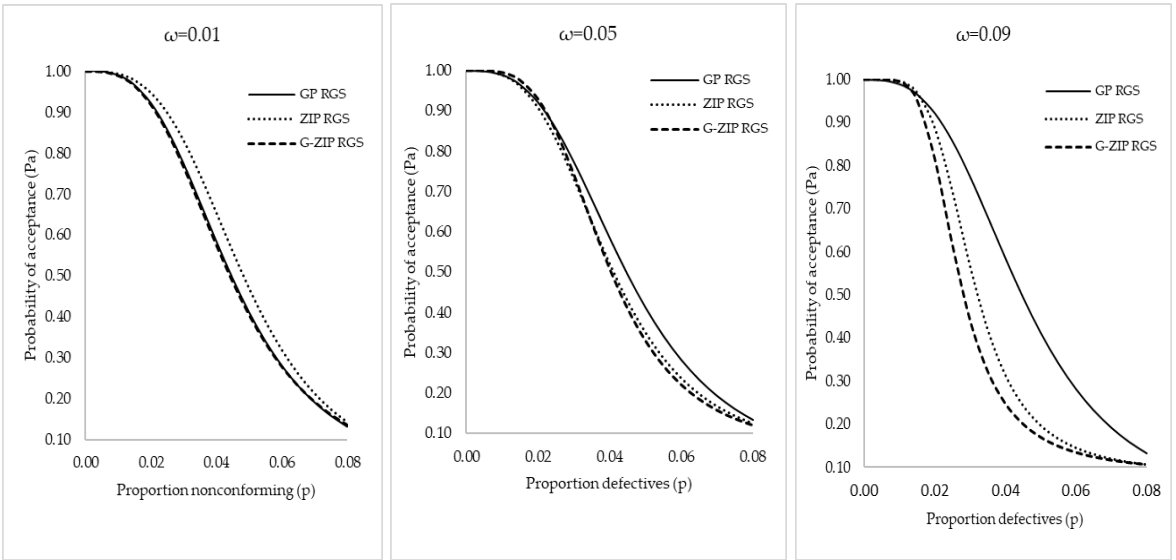

Figure 1: *Comparative analysis of operating characteristic curves*

7. Conclusion

In this article, we have developed a Repetitive Group Sampling plan based on the Gamma-Zero Inflated Poisson (G-ZIP) distribution from a Bayesian perspective. This proposed sampling plan is particularly useful when production processes are well monitored and result in a significant number of defect-free products. Further, the plan incorporates historical production data, allowing for the consideration of inherent or natural variability in the process. The article also provides a comprehensive procedure for designing and selecting plan parameters using the two-point approach on the Operating Characteristic (OC) curve. Additionally, the tables are constructed for some specified strengths $(p_1, p_2, \alpha, \beta)$ for the limited number of values ω and s. From the illustrations, the proposed sampling plan has the better performance measures than over some existing sampling plans. The proposed sampling plan can be applied in various industries Including Electronic device manufacturing industries, medicine research centers and food industries etc. It is strongly suggested that the proposed plan be utilized in the industries for lot sentencing and products determination. The study opens up avenues for further research in the area of Bayesian group sampling. Future investigations could explore the integration of advanced statistical techniques, such as Markov chain Monte Carlo methods, to enhance the precision and efficiency of the proposed methodology. Additionally, the application of the methodology to specific manufacturing industries and the development of decision support tools could be explored to facilitate practical implementation.

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PERFORMANCE ANALYZATION OF ERLANG SERVICE MODEL UNDER TRIANGULAR FUZZY NUMBER BY USING THE L-R FUZZY APPROACH

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Abstract

A traditional mathematical technique for analyzing line-waiting delays and overcrowding is queuing theory. It addresses the number of patrons in line as well as numerous other queue-related issues. Developing an Erlang service model in a fuzzy environment is our study's goal. This study aims to investigate the anticipated number of patients in the line as well as the queuing system's waiting time. To achieve this, we applied the L-R strategy under triangular fuzzy numbers and the alpha-cuts method. To measure various linguistic aspects in queuing systems, the fuzzy approach has been used. The findings showed that waiting times are determined using recommended techniques and that the fuzzy Erlang model is stable. Finally, we provide numerical examples to show the capabilities of the suggested method.

Keywords: Fuzzy queuing theory, α-cut method, L-R fuzzy approach, triangular fuzzy number, Erlang service model.

1. Introduction

A probabilistic method for handling queuing systems is queuing theory. Calls and Erlang initially presented queuing theory by focusing on the congestion issue in telephone exchanges and introducing the foundation for both Poisson and exponential distributions. When people wait for their turn to get services, they are essentially in a queue. Waiting is one of the processes of most troubling situations, and queuing theory addresses it. Banks, hospitals, telecoms, medical services, and other establishments frequently face queue issues. Long lines have a financial and resource cost to people. It is challenging to accommodate everyone's high needs because of the traffic. It addresses the quantity of patrons in line as well as numerous other queue-related issues.

Queueing theory is useful for creating effective queuing systems that, while lowering client wait times, also increase the number of customers that can be served. Two types of queues are distinguished: fuzzy queues and crisp queues. Using a probabilistic technique, Crisp Queue handles performance measures, and in this case, Poisson distribution is used to determine the "service time" and "inter-arrival time." In actual situations, both the service and arrival rates are informally assessed. Since the majority of information pertaining to statistics is collected in a subjective manner, the fuzzy approach describes service and arrival rates in a probabilistic manner [1, 2].

Zadeh asserts that fuzzy queues are more realistic than crisp queues. When numerous servers are involved, these queueing models work best when crisp queues are converted to fuzzy queues [3]. To create the mathematical models for customer service, queuing theory is employed. Given a probabilistic explanation of service time, the fuzzy queueing method is a more practical solution than traditional queueing theory methods. Since the fuzzy technique's boundary is specified with a limited membership degree, it differs greatly from the crisp set approach [4].

Several researchers have used Zadeh's extension concept for fuzzy queueing models [3, 5, 6, 7, 8], such as, Nagi and Lee [9] examined the α-cut method under fuzzy conditions. The fuzzy approach to diagnostic queuing theory was introduced by Umaira Zareen and Saqlain Raza [10]. The Erlang service model was used by Narayanamoorthy et al. [11] to predict the anticipated number of customers and their waiting time in the system. The single server queues under the LR approach are examined by Vijaya et al. [12] utilizing trapezoidal fuzzy numbers. Lee [13] studied the concepts of simulation and the Alpha-cut approach. Much research has been done on fuzzy queues by Prade [14], Ritha and Menon [15], Yager [16], Mukeba Kanyinda [17], and others [18, 19, 20]. Finding system performance measurements using the $\alpha-$ cuts approach is the focus of the majority of these works. In this work, fuzzy queueing models are analyzed under the L-R fuzzy approach using triangular fuzzy numbers. The L-R method is a novel approach that we use to determine how many customers in a fuzzy queue along with their waiting time.

Compared to classical queueing theory, the fuzzy queueing models are more realistic than obtaining the queue models because the service and inter-arrival times have to follow certain distributions. However, linguistic quantifiers such as speedy, gradual, or medium are often used to characterize the arrival and service patterns instead of probability distributions. In this study, the alpha-cut and L-R approaches, which are helpful in determining the function's higher and lower bounds, handle the service rate and arrival rate as triangular fuzzy numbers. There has been a lot of interest in the $M/E_C/1$ vacation systems with a single unit arrival for queuing models with single and many servers under different considerations. Researchers used the earlier findings to solve a queuing decision problem and a machine serving problem revision of queueing theory [5, 6, 21, 22, 23].

2. Preliminaries

We give some basic concepts and arithmetic operations of L-R fuzzy numbers in this section.

Definition 2.1. [18] A $\tilde{\delta}$ is said L-R fuzzy number if there exists a real numbers such as, $\delta, s > 0, t > 0$ and two positive, continuous and decreasing functions L and R such that

$$\begin{aligned} L(0) &= R(0) = 1 \\ L(1) &= 0, L(u) > 0, \lim_{u\to\infty} L(u) = 0 \\ R(1) &= 0, R(u) > 0, \lim_{u\to\infty} R(u) = 0 \end{aligned}$$

$$\lambda_{\tilde{\delta}}(u) = \begin{cases} L(\frac{\delta-u}{s}) & if \quad u \in [\delta - s, \delta] \\ R(\frac{u-\delta}{t}) & if \quad u \in [\delta, \delta + t] \\ 0 & otherwise \end{cases} \tag{1}$$

The $\tilde{\delta} =< \delta, s, t >_{LR}$. δ is called the most possible value. By the definition,$supp(\tilde{A}) = \{u \in E | \sigma_{\tilde{A}}(u) > 0\}$, then

$$supp(\tilde{\delta}) =]\delta - s, \delta] \cup [\delta, \delta + t[=]\delta - s, \delta + t[.$$

2.1. Arithmetic operations of LR fuzzy numbers

The two L-R fuzzy numbers $M =< \delta, i, j >$ and $N =< \eta, k, p >$ [24]

$$\tilde{M} + \tilde{N} =< \delta + \eta, i + k, j + p > \tag{2}$$

$$\tilde{M} - \tilde{N} =< \delta - \eta, i + p, j + k > \tag{3}$$

$$\tilde{M}.\tilde{N} \approx< \delta\eta, \delta k + \eta i - ik, \delta p + \eta j + jp > \tag{4}$$

$$\frac{\tilde{M}}{\tilde{N}} = \frac{< \delta, i, j >}{< \eta, k, p >} \approx< \frac{\delta}{\eta}, \frac{\delta p}{\eta(\eta + p)} + \frac{i}{\eta} - \frac{ip}{\eta(\eta + p)}, \frac{\delta k}{\eta(\eta - k)} + \frac{j}{\eta} + \frac{jk}{\eta(\eta - k)} > \tag{5}$$

Definition 2.2. [11] A $\tilde{F}$ is said triangular fuzzy number (TFN) then there exists a real numbers $g < h < r$ such that:

$$\eta_{\tilde{F}}(u) = \begin{cases} L(\frac{u-g}{h-g}) & if \quad g \leq u \leq h \\ R(\frac{r-u}{r-h}) & if \quad h \leq u \leq r \\ 0 & otherwise \end{cases} \tag{6}$$

By definitions, a TFN $\tilde{F} = (g/h/r)$ is LR fuzzy number. In this concepts, it can be written

$$\tilde{F} = (g/h/r) =< h, h - g, r - g >$$

for $L(x) = R(x) = max(0, 1 - u)$.

3. Mathematical Model

In a queuing system, a customer with arrival rate V and service rate Q is received by a single-server capacity. The C exponential phase makes up the fuzzy Erlang service rate $\tilde{Q}$ and the fuzzy Poisson rate $\tilde{V}$. After establishing the system capacity and calling source to infinite, customers are serviced using the the basis of FCFS [25, 26].

Here, the $\tilde{V}$ is arrival rate and service rate $\tilde{Q}$ are known and can be denoted by convex fuzzy sets. An fuzzy set $\tilde{F}$ is convex if and only if $\mu_{\tilde{F}}(\varphi x_1 + (1 - \varphi)x_2) \geq min\{\mu_{\tilde{F}}(x_1), \mu_{\tilde{F}}(x_2)\}$ where $\mu_{\tilde{F}}$ is $\phi \in [0, 1], x_1, x_2 \in X$.

Let $\mu_{\tilde{V}}(s)$, $\mu_{\tilde{Q}}(t)$ are arrival rate and service rate of membership functions respectively. We have

$$\tilde{V} = \{s, \mu_{\tilde{V}}(s)/s \in S(\tilde{V})\}$$
$$\tilde{Q} = \{t, \mu_{\tilde{Q}}(y)/t \in S(\tilde{Q})\}$$

Where $S(\tilde{\lambda})$ and $S(\tilde{\mu})$ are the supports [11]. Based on the extension principle proposed by Zadeh, the performance measure's membership function is described as

$$\mu_{\tilde{E}(\tilde{V},\tilde{Q})}(x) = \sup_{s \in S, t \in T} min\{\mu_{\tilde{V}}(s), \mu_{\tilde{Q}}(t)/x = E(s, t)\}$$

Under the steady-state condition $\rho = \frac{V}{Q} < 1$, the number of customers in the queue,

$$Lq = \left[\frac{(C+1)}{2C} \cdot \frac{\tilde{V}^2}{\tilde{Q}(\tilde{Q} - \tilde{V})}\right]$$

The expected number of customers in the queue $\tilde{L}q$ is

$$\mu_{\tilde{L}q}(x) = \sup_{s \in S, t \in T, x < X} min\left\{\mu_{\tilde{V}}(s), \mu_{\tilde{Q}}(t)/x = \left[\frac{(C+1)}{2C} \cdot \frac{\tilde{V}^2}{\tilde{Q}(\tilde{Q} - \tilde{V})}\right]\right\} \tag{7}$$

We can determine how long it will take for the expected number of customers in line,

$$Wq = \left[\frac{(C+1)}{2C} \cdot \frac{\tilde{V}}{\tilde{Q}(\tilde{Q} - \tilde{V})}\right]$$

The waiting time of $\tilde{W}q$ is in the queue

$$\mu_{\tilde{W}q}(x) = \sup_{s \in S, t \in T, y < Y} min\left\{\mu_{\tilde{V}}(s), \mu_{\tilde{Q}}(t)/x = \left[\frac{(C+1)}{2C} \cdot \frac{\tilde{V}}{\tilde{Q}(\tilde{Q} - \tilde{V})}\right]\right\} \tag{8}$$

4. The Method for Solving the Problem

An approach to constructing the $\mu_{\tilde{P}(\tilde{V},\tilde{Q})}$ is basis of deriving α-cuts of $\mu_{\tilde{P}(\tilde{V},\tilde{Q})}$. Denote α-cuts of $\tilde{V}$ and $\tilde{Q}$ as [25, 26]:

$$\tilde{V}_\alpha = [s_\alpha^L, s_\alpha^U] = [\min_{s\in S}\{s/\mu_{\tilde{V}}(s) \geq \alpha\}, \quad \max_{s\in S}\{s/\mu_{\tilde{V}}(s) \geq \alpha\}] \tag{9}$$

$$\tilde{Q}_\alpha = [t_\alpha^L, t_\alpha^U] = [\min_{t\in T}\{t/\mu_{\tilde{Q}}(t) \geq \alpha\}, \quad \max_{t\in T}\{t/\mu_{\tilde{Q}}(y) \geq \alpha\}] \tag{10}$$

Consequently, the $FM/FE_C/1$ queue can be reduced to crisp $M/E_C/1$ queues with various levels of α sets $\{\tilde{V}_\alpha < \alpha \leq 1\}$[].

By the convexity, the intervals are functions of α is

$$s_\alpha^L = min\ \mu_{\tilde{V}}^{-1}(\alpha) \text{ and } s_\alpha^U = max\ \mu_{\tilde{V}}^{-1}(\alpha)$$

$$t_\alpha^L = min\ \mu_{\tilde{Q}}^{-1}(\alpha) \text{ and } t_\alpha^U = max\ \mu_{\tilde{Q}}^{-1}(\alpha)$$

We need either $\mu_{\tilde{V}}(s) = \alpha$ and $\mu_{\tilde{Q}}(t) \geq \alpha$ (or) $\mu_{\tilde{V}}(s) \geq \alpha$ and $\mu_{\tilde{Q}}(t) = \alpha$ such that $x = \left[\frac{(C+1)}{2C}.\frac{\tilde{V}^2}{\tilde{Q}(\tilde{Q}-\tilde{V})}\right]$ to satisfy $\mu_{\tilde{Lq}}(x) = \alpha$. To find the$\mu_{\tilde{Lq}}(x)$ we have to obtain the lower value of x_α^L and the upper value of x_α^U of $\mu_{\tilde{Lq}}(x)$. Since $\mu_{\tilde{V}}(s) = \alpha$ is denoted by $s = s_\alpha^L$ (or) $s = s_\alpha^U$ this formulated as the constraint of $s = \varphi_1 s_\alpha^L + (1-\varphi_1)s_\alpha^U$, where $\varphi_1 = 0$ (or) 1. Similarly $\mu_{\tilde{Q}}(t) = \alpha$ is formulated as the constraint $t = \varphi_2 t_\alpha^L + (1-\varphi_2)t_\alpha^U$, where $\varphi_2 = 0$ (or) 1 [].

Let $(\tilde{Lq})_\alpha^L = \{(\tilde{Lq})_\alpha^{L_1}, (\tilde{Lq})_\alpha^{L_2}\}$ and $(\tilde{Lq})_\alpha^U = \{(\tilde{Lq})_\alpha^{U_1}, (\tilde{Lq})_\alpha^{U_2}\}$ respectively. where

$$(\tilde{Lq})_\alpha^{L_1} = \min_{s,t\in R} \quad s < t\left[\frac{(C+1)}{2C}.\frac{\tilde{V}^2}{\tilde{Q}(\tilde{Q}-\tilde{V})}\right] \tag{11}$$

such that $s = a_1 s_\alpha^L + (1-a_1)s_\alpha^U$, $t_\alpha^L \leq t \leq t_\alpha^U$ and $a_1 = 0$ (or) 1.

$$(\tilde{Lq})_\alpha^{L_2} = \min_{s,t\in R} \quad s < t\left[\frac{(C+1)}{2C}.\frac{\tilde{V}^2}{\tilde{Q}(\tilde{Q}-\tilde{V})}\right] \tag{12}$$

such that $t = a_2 t_\alpha^L + (1-a_2)t_\alpha^U$, $s_\alpha^L \leq s_\alpha^U$ and $a_2 = 0$ (or) 1.

Similarly, we can obtain the upper values of L_q and where $s_\alpha^L < t_\alpha^L$. Then, α-cuts of $\tilde{Lq}$ can be obtain by solving above equations.

If both $(\tilde{Lq})_\alpha^L$ and $(\tilde{Lq})_\alpha^U$ are invertible, then a left shape function $L_O(x) = ((\tilde{Lq})_\alpha^L)^{-1}$ a right shape function can be obtained. From $L_O(x)$ and $R_O(x)$, the membership function $\mu_{\tilde{Lq}}$ is constructed as

$$\mu_{\tilde{Lq}}(x) = \begin{cases} L_o(x), & (Lq)_\alpha^L = 0 \leq x \leq (Lq)_\alpha^L = 1 \\ 1, & (Lq)_\alpha^L = 1 \leq x \leq (Lq)_\alpha^U = 1 \\ R_o(x), & (Lq)_\alpha^U = 1 \leq x \leq (Lq)_\alpha^U = 0 \end{cases} \tag{13}$$

5. Numerical Example

In a hospital clinic, a doctor examines each patient who is brought in for a routine checkup. While the time spent on each part of the checkup is roughly exponentially distributed, the doctor spends

an average of fifty minutes on each phase. Given that every patient undergoes a four-phase examination and that the typical patient arrives at the doctor's clinic at a rate of 20 per hour. Calculate

- How many patients are anticipated to be in line?
- How long does it typically take a patient to wait in line?
- Determine the maximum values for the anticipated patient volume and line wait time.

Solution
The classical queueing theory cannot be used to investigate this issue because the rates are given in fuzzy information. $FM/FE_C/1$ is a simple queue with fuzzy rates. Assume that te rates are fuzzy triangular numbers provided by $\tilde{V} = [10, 20, 30]$ and $\tilde{Q} = [40, 50, 60]$. The approach described in the paragraph allows us to analyze queue characteristics specified in equations since these parameters are L-R fuzzy integers.

α-cuts method:

The confidence interval at α are $[10\alpha + 10, 30 - 10\alpha]$ and $[10\alpha + 40, 60 - 10\alpha]$

$$(\tilde{Lq})_{\alpha}^{L} = \left[\frac{5\alpha^2 + 10\alpha + 5}{16\alpha^2 - 136\alpha + 240}\right]$$

$(\tilde{Lq})_{\alpha}^{L}$ is invertible

$$\begin{aligned}
\alpha &= \frac{(136x - 10) \pm \sqrt{3136x^2 + 2400x}}{(32x - 10)} \\
\alpha &\geq 0, (136x - 10) \pm \sqrt{3136x^2 + 2400x} \geq 0 \\
x &= 0.0208(or)0.3125 \\
\alpha &\leq 1, \frac{(136x - 10) \pm \sqrt{3136x^2 + 2400x}}{(32x - 10)} \leq 1 \\
x &= 0.3125\ (or)\ 0 \\
(\tilde{Lq})_{\alpha}^{L} &= \frac{(136x - 10) \pm \sqrt{3136x^2 + 2400x}}{(32x - 10)} \quad 0 \leq x \leq 0.0208
\end{aligned}$$

Now,

$$(\tilde{Lq})_{\alpha}^{U} = \left[\frac{5\alpha^2 - 30\alpha + 45}{16\alpha^2 + 72\alpha + 32}\right]$$

$(\tilde{Lq})_{\alpha}^{U}$ is invertible

$$\begin{aligned}
\alpha &= \frac{-(72x + 30) \pm \sqrt{3136x^2 + 7840x}}{(32x - 10)} \\
\alpha &\geq 0, -(72x + 30) \pm \sqrt{3136x^2 + 7840x} \geq 0 \\
x &= 0.3125(or)1.4062 \\
\alpha &\leq 1, \frac{-(72x + 30) \pm \sqrt{3136x^2 + 7840x}}{(32x - 10)} \leq 1 \\
x &= 2.5(or)0.3125 \\
(\tilde{Lq})_{\alpha}^{U} &= \frac{-(72x + 30) \pm \sqrt{3136x^2 + 7840x}}{(32x - 10)} \quad 1.4062 \leq x \leq 2.5
\end{aligned}$$

From the inverse function of $(\tilde{Lq})_{\alpha}^{L}$ and $(\tilde{Lq})_{\alpha}^{U}$ of $\tilde{Lq}$ is described as:

$$\mu_{\tilde{Lq}}(x) = \begin{cases} \frac{(136x-10)\pm\sqrt{3136x^2+7840x}}{(32x-10)}, & 0 \leq x \leq 0.0208 \\ 1, & 0.3125 \leq x \leq 0.3125 \\ \frac{-(72x+30)\pm 66\sqrt{3136x^2+7840x}}{(32x-10)}, & 1.4062 \leq x \leq 2.5 \end{cases}$$

Then,

$$(\tilde{Wq})_{\alpha}^{L} = \frac{(\alpha+1)}{32\alpha^2-272\alpha+480}$$

$(\tilde{Wq})_{\alpha}^{L}$ is invertible

$$\begin{aligned} \alpha &= \frac{(272x+1)\pm\sqrt{12544x^2+672x+1}}{64x} \\ \alpha &\geq 0, (272x+1)\pm\sqrt{12544x^2+672x+1} \geq 0 \\ x &= 0.00208(or)0 \\ \alpha &\leq 1, \frac{(272x+1)\pm\sqrt{12544x^2+672x+1}}{64x} \leq 1 \\ x &= 0.0022(or)0 \\ (\tilde{Wq})_{\alpha}^{L} &= \frac{(272x+1)\pm\sqrt{12544x^2+672x+1}}{64x} \quad 0.00208 \leq x \leq 0.0022 \end{aligned}$$

Now,

$$(\tilde{Wq})_{\alpha}^{U} = \frac{(3-\alpha)}{32\alpha^2+144\alpha+64}$$

$(\tilde{Wq})_{\alpha}^{U}$ is invertible

$$\begin{aligned} \alpha &= \frac{-(144x+1)\pm\sqrt{12544x^2+672x+1}}{64x} \\ \alpha &\geq 0, -(144x+1)\pm\sqrt{12544x^2+672x+1} \geq 0 \\ x &= 0(or)0.0468 \\ \alpha &\leq 1, \frac{-(144x+1)\pm\sqrt{12544x^2+672x+1}}{64x} \leq 1 \\ x &= 0.0937(or)0 \\ (\tilde{Wq})_{\alpha}^{U} &= \frac{-(144x+1)\pm\sqrt{12544x^2+672x+1}}{64x} \quad 0.0468 \leq x \leq 0.0937 \end{aligned}$$

From the inverse function of $(\tilde{Wq})_{\alpha}^{L}$ and $(\tilde{Wq})_{\alpha}^{U}$ the waiting time of $\tilde{Wq}$ is defined as follows:

$$\mu_{\tilde{Wq}}(x) = \begin{cases} \frac{(272x+1)\pm\sqrt{12544x^2+672x+1}}{64x}, & 0.00208 \leq x \leq 0.0022 \\ 1, & 0 \leq x \leq 0 \\ \frac{-(144x+1)\pm\sqrt{12544x^2+672x+1}}{64x}, & 0.0468 \leq x \leq 0.0938 \end{cases}$$

We computed $\tilde{Lq}$ and $\tilde{Wq}$ for the provided values using fuzzy numbers. Substituting the values of α in the formula above yields tabular results, with a graphical depiction provided below. Finally, the results formulation are obtained by α-cut method: the L_q in the queue is approximately

Table 1: *The results obtained by α-cut approach*

α	x_α^L	x_α^U	y_α^L	y_α^U	$(\tilde{Lq})_\alpha^L$	$(\tilde{Lq})_\alpha^U$	$(\tilde{Wq})_\alpha^L$	$(\tilde{Wq})_\alpha^U$
0.0	10.0	30.0	40.0	60.0	0.0208	1.4062	0.0020	0.0468
0.1	11.1	29.9	41.1	59.9	0.0267	1.0683	0.0024	0.0368
0.2	12.2	28.8	42.2	58.8	0.0337	0.8333	0.0028	0.0297
0.3	13.3	27.7	43.3	57.7	0.0421	0.6622	0.0032	0.0245
0.4	14.4	26.6	44.4	56.6	0.0520	0.5334	0.0037	0.0205
0.5	15.5	25.5	45.5	55.5	0.0639	0.4340	0.0042	0.0173
0.6	16.6	24.4	46.6	54.4	0.0779	0.3557	0.0048	0.0148
0.7	17.7	23.3	47.7	53.3	0.0946	0.2931	0.0055	0.0127
0.8	18.8	22.2	48.8	52.2	0.1145	0.2423	0.0063	0.0110
0.9	19.9	21.1	49.9	51.1	0.1382	0.2008	0.0072	0.0095
1.0	20.0	20.0	50.0	50.0	0.1666	0.1666	0.0083	0.0083

between 0.0208 and 1.4062. The Wq is lies between 0.0020 and 0.0468.

The L-R approach:

We determine L-R representations of fuzzy numbers $\tilde{V}$ and $\tilde{Q}$, which are $\tilde{V} =< 20, 10, 20 >_{LR}$ and $\tilde{Q} =< 50, 10, 20 >_{LR}$

$$\begin{aligned}
\tilde{Lq} &= \left[\frac{(C+1)\tilde{V}}{2(\tilde{Q}-\tilde{V})} - \frac{(C+1)\tilde{V}}{2\tilde{Q}}\right]\frac{1}{C} \\
&= \frac{1}{4}\left[\frac{5 < 20, 10, 20 >}{2[< 50, 10, 20 > - < 20, 10, 20 >]} - \frac{5 < 20, 10, 20 >}{2 < 50, 10, 20 >}\right] \\
(\tilde{Lq})_{LR} &= < 0.0625, 0.5913, 1.0982 >_{LR} \\
\tilde{Wq} &= \left[\frac{C+1}{2C} \times \frac{\tilde{V}}{\tilde{Q}(\tilde{Q}-\tilde{V})}\right] \\
&= \left[\frac{5}{8} \times \frac{< 20, 10, 20 >}{< 50, 10, 20 > [< 50, 10, 20 > - < 20, 10, 20 >]}\right] \\
(\tilde{Wq})_{LR} &= < 0.0083, 0.0068, 0.0083 >_{LR}
\end{aligned}$$

The support of $\tilde{Lq}$ varies between 0.0208 and 0.1666, indicating that the anticipated quantity of patients is uncertain. Its values can never be lower than 0.0208 or more than 0.1666. The mean value of $\tilde{Lq}$ is precisely 1, which is the maximum value that can be found in that situation. Likewise, a patient's waiting time in the queue is between 0.0020 (about one minute) and 0.0468 (approximately three minutes). It shows that there will never be a wait time in the line longer than 3 or shorter than 1. The queue's maximum waiting time is 0.0083 (around one minute). These results of L_q and W_q are shown in the figures 1 and 2.

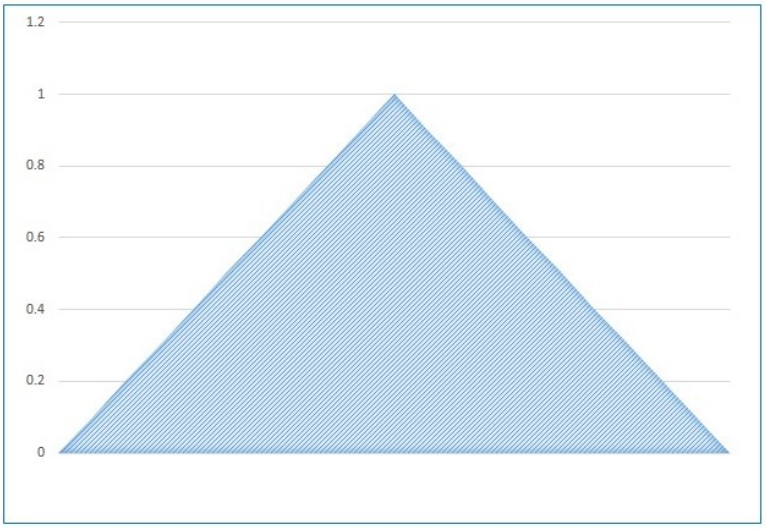

Figure 1: *The results of Lq*

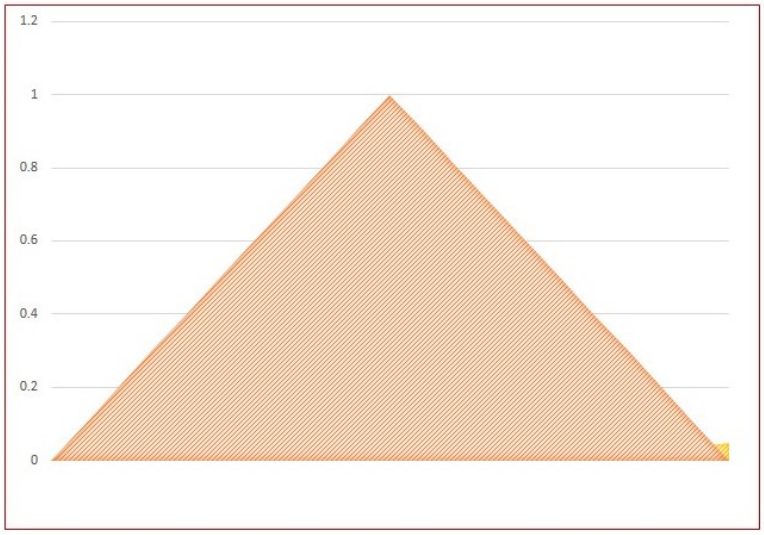

Figure 2: *The results of Wq*

6. Conclusion

The α-cut approach and the L-R method are used in this work to compute the predicted number of clients and mean waiting time of patients of the $FM/FE_C/1$ sequence. With the α-cut approach, the maximum number of patients is precisely 0.0625, while the predicted number of patients falls between 0.0208 and 1.4062. Similarly, the maximum value is precisely 0.0083, and the mean waiting time for patients falls between 0.0020 and 0.0468. The two spreads help deduce the upper and lower boundaries of the fuzzy measure. The approximation of the greatest explanatory outcomes, brevity, convenience, and flexibility are the three primary advantages of this innovative approach. Future research in this field will undoubtedly benefit from the L-R method to address several outstanding problems, such as evaluating fuzzy queueing systems' performance metrics with the Erlang service model.

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DEVELOPMENT OF NEW METHODS FOR PROTECTING SUBSTATION AND OVERHEAD LINES FROM OVERVOLTAGES

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Abstract

This article explores various methods and devices used for protecting overhead lines and substations from surges, particularly those induced by lightning strikes. Traditional surge protection methods such as lightning rods, arresters, and grounding systems are discussed, highlighting their limitations and challenges, especially in long-distribution networks. The study examines the development and implementation of novel surge protection devices, including nonlinear surge arresters, frequency-dependent devices (FDD), and multi-chamber arresters. Special attention is given to FDD, which utilizes ferromagnetic materials to create frequency-dependent resistance, effectively suppressing high-frequency overvoltages. Experimental results demonstrate the efficacy of FDD in reducing the amplitude of lightning-induced overvoltage pulses and enhancing the lightning resistance of overhead lines and substations. However, challenges such as insufficient information on device effectiveness, limited ohmic resistance at high frequencies, and size constraints hinder widespread adoption. The article concludes by emphasizing the need for further research to optimize FDD designs, increase active resistance, and assess operational effectiveness to facilitate broader deployment across different voltage classes.

Keywords: overhead lines, substations, overvoltages, surge protection, surge arrester, skin effect, frequency-dependent device.

I. Introduction

Protecting substations and overhead lines from surges is critical to ensuring reliable electrical service and preventing equipment damage. First, let's look at the classic means (methods) of surge protection. To protect overhead lines and substations from damage due to lightning strikes, special lightning protection devices are used. These devices include lightning protection cables, lightning rods, tubular arresters, valve arresters, nonlinear surge arresters, grounding devices, etc.

The resistance values of the grounding of the supports play a significant role. Strictly speaking, grounding overhead line supports is not directly a measure for protection against lightning surges. However, despite this, it has a significant impact on lightning surge protection. It has a significant impact on the probability of reverse flashovers from the lightning protection support/cable to the phase wire. When reverse flashover occurs, high-frequency overvoltage pulses occur that cannot be suppressed by protective devices. Accordingly, they have a significant impact on substation equipment, such as transformers, and may have an impact on other equipment.

The lightning protection cable is suspended on lines with a voltage of 110 kV and above, built on metal and reinforced concrete supports. On 110 - 220 kV lines with wooden supports and 35 kV

lines, the cable is usually suspended only at the approaches to substations. Despite the widespread use of cables as a means of protection, there are often cases of lightning strikes into phase wires bypassing the lightning protection cable, which reduces its reliability as a means of protection; in addition, without good grounding, there is a high probability of reverse flashover to the phase wire, which is also dangerous factor. At the same time, cable lightning rods also do not provide the declared reliability of protection against lightning damage [1-4].

To protect 110-500 kV overhead lines from lightning, the most common method of protection is the use of *linear protective devices*. The most common protective devices are pendant nonlinear surge arresters and linear arresters. It is worth noting that at present, there is no developed and approved universal method for using linear protective devices for overhead lines of various voltage classes. Therefore, operating organizations that own overhead lines experience certain difficulties in developing technical measures to protect problematic overhead lines.

Surge arresters are installed mainly in networks of 110 kV and above, but they can and can be used in networks of 35 kV. Surge arresters allow you to deeply limit the amplitudes of incoming pulses and provide protection against overvoltages. However, SS have a number of significant disadvantages. Figure 1 shows a scheme of the operating principle of the surge arrester.

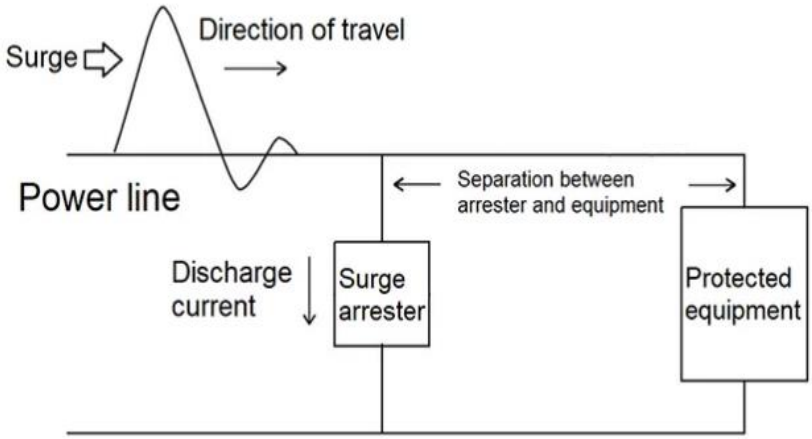


Figure 1: *Scheme of operation of the surge arrester*

First of all, the effective operation of surge arresters is ensured only when installed simultaneously in all phases of the protected circuit on each support of the protected section of the route. This is a very expensive undertaking and is only applicable for relatively short or critical overhead lines. The absence of spark gaps causes the currents of the operating voltage of the network to flow through the surge arrester with a frequency of 50 Hz. If these currents are large, the varistor may overheat and the arrester may fail. A significant disadvantage of surge arresters is the dependence of the design on the voltage class and network characteristics. The greatest difficulties arise when developing surge arresters for long-distribution networks. This leads to the fact that for trouble-free operation of the surge arrester, an accurate selection of its parameters is necessary. The main factors necessary when choosing an arrester are:

− maximum permissible voltage taking into account the duration of its impact;
− calculated pulse current;
− calculated switching current;
- energy intensity (throughput class).

In some cases, the last two parameters are replaced by the ability to absorb electrical energy, expressed in kJ per 1kV Um [5-8].

II. Materials and methods

In addition to common means of overvoltage protection, the development and implementation of new devices operating on various principles are actively underway: the combined use of an air gap and linear arresters, the use of surge arrester with a set of resistors and capacitors. However, often the devices retain the disadvantages of their prototypes, mainly: the need for a large number of devices, dependence on grounding, difficulty in installation and operation.

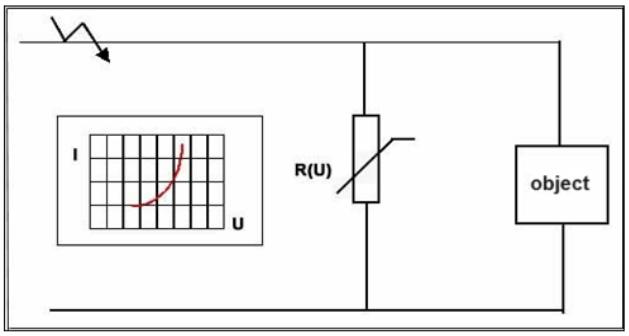
There is a solution that consists in the use of nonlinear surge arresters built into insulator housings. These solutions can significantly reduce labor costs for transportation, installation and installation, since instead of two devices, one will be used. However, this leads to a more complex design of the device itself, a decrease in its reliability, and an increase in size and weight. The typical problem of devices with varistors also remains relevant: the difficulty of monitoring its serviceability.

Also new is the concept of protective devices, the parameters of which vary depending on the frequency response of the incoming overvoltage. Thus, a combination of high-frequency magnetic material located on the main conductor with a parallel-connected resistance is proposed. According to the presented calculations and experiments, this combination is capable of suppressing and protecting against high-frequency overvoltages; this device is intended to suppress overvoltages with high dU/dt values, that is, with a high transconductance. However, the device requires tuning for a certain frequency range in which effective suppression will be achieved.

One of the latest developments is systems consisting of multi-chamber arresters. They show high efficiency, which is confirmed by a number of works, in particular, they effectively reduce the number of outages of overhead lines. However, despite the high declared efficiency, these devices also have a number of significant drawbacks: to ensure reliable protection against lightning surges, it is necessary to install arresters on each support on each insulator, precise adjustment of the spark gap is necessary (if the device is used with it), as well as the need for reliable grounding to ensure efficient operation [9-14].

Application of skin effect in protection against high-frequency overvoltages. The most promising from the point of view of ease of operation and application seems to be the use of protective devices based on the use of composite materials. There are other individual developments of protective devices based on the skin effect principle. These devices can effectively suppress high-frequency overvoltages, while being quite compact and their operation is not affected by the values of soil resistance or grounding of supports. The principle of their operation is to increase the resistance of the device as the frequency of the incoming signal increases. Figure 2 shows schematic diagrams for connecting protective devices: a nonlinear surge arrester and a frequency-dependent device.

a)



b)

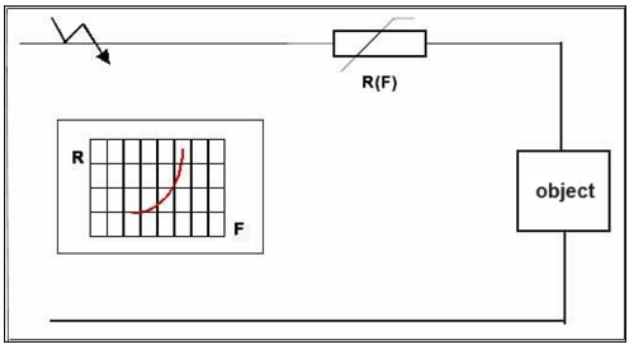


Figure 2: *Connection diagrams for protective devices: a) nonlinear surge arrester; b) frequency dependent device*

To use the skin effect, special devices have been developed, such as frequency-dependent devices (FDD). They are conductors covered with layers of ferromagnetic material with high magnetic permeability and high resistance. One of such materials is ferromagnetic tape 5BDSR. This material has high resistivity in the range of lightning (200-300 kHz) and switching frequencies (1-2 MHz). The use of this material has shown itself to be effective in early studies. On its basis, a frequency-dependent device (FDD) was created for a 110 kV overhead line. The developed frequency-dependent device is a phase aluminum wire wound into a coil with a ferromagnetic material deposited on it, which has high magnetic permeability and high resistivity. In this regard, the resistance of the wire becomes frequency-dependent, and when high-frequency signals pass through the wire, its resistance increases sharply [15-18].

In normal operation, at a frequency of 50 Hz, the frequency-dependent wire has a resistance equal to the resistance of the uncoated phase wire, since the current flows through an aluminum conductor having a low resistivity. When a high-frequency signal passes, the resistance of the wire, and accordingly the device, increases sharply by tens of thousands of times, due to the action of the skin effect and the pushing of current from the aluminum conductor into a high-resistance layer of ferromagnetic material.

Figure 3 shows the skin effect and, as a consequence, the uneven distribution of current in a round conductor.

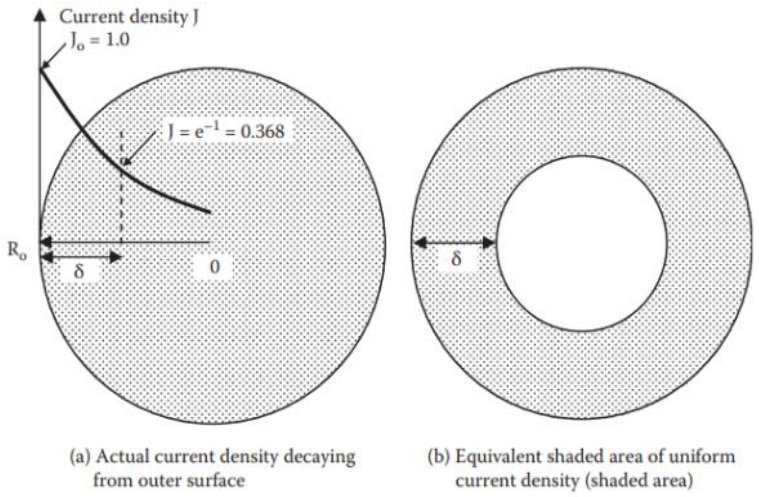


Figure 3: *Skin effect and resulting nonuniform current distribution in round conductor*

With a wire length of 120 meters used in the FDD, a resistance of 120 Ohms is provided at lightning frequencies (for comparison, the resistance of a wire without ferromagnetic tape for the same length is about 20 mOhm), which allows you to significantly reduce the amplitude of high-frequency overvoltage pulses. In turn, the shape of the device in the form of a spiral creates a high reactance, which significantly reduces the steepness of the oncoming pulse, thereby reducing the danger of interturn short circuits in the protected equipment, primarily in power transformers. As a result, the operation of this device can significantly increase the lightning resistance of overhead lines and substations. Despite the high efficiency of the device, a number of points should be noted that do not allow these devices to be put into widespread use quickly: insufficient information on the effectiveness of the device's suppression of high-frequency overvoltages, insufficiently high ohmic resistance at high frequencies, large dimensions, and the lack of devices for other classes voltages (for example, 35 kV).

In this regard, it is necessary to assess the effectiveness of trial operation of the FDD, consider the possibilities of increasing the active resistance of the FDD, and evaluate the possibilities of optimizing the designs of the FDD [19-22].

Advantages of skin effect devices:

- High efficiency of suppression of high-frequency overvoltages. Due to a sharp increase in resistance at high frequencies, the pulse energy is dissipated in the device, minimizing its impact on the protected equipment.

- Compact design. Skin effect devices can be quite compact compared to traditional protective equipment.

- Independence from ground resistance. Such devices are effective even under conditions of high soil resistance, which makes them ideal for use in complex geological conditions.

- Wide frequency range of operation. Due to the use of ferromagnetic materials with high permeability, the devices can be configured to operate in a wide frequency range, including lightning and switching pulses.

Limitations and areas for improvement. Despite their high efficiency, skin effect devices have a number of limitations, such as insufficient information on long-term reliability, relatively large dimensions and difficulty in setting up for certain voltage classes.

Important areas of research are:

1) Increasing active resistance at high frequencies;

2) Optimization of designs to reduce weight and dimensions;

3) Development of devices for lines with lower voltage classes, such as 35 kV;

4) Increased resistance to climatic and mechanical factors.

Thus, the use of the skin effect in protective devices provides enormous opportunities for increasing the resistance of power transmission lines and substations to lightning and switching overvoltages. Improvement of this technology may become one of the key areas of development of power grid protection systems in the future [23-25].

III. Conclusions

Traditional approaches such as lightning protection devices, surge arresters, and grounding systems are effective but come with limitations, especially in long-distribution networks. The development of new devices, including nonlinear surge arresters, frequency-dependent devices (FDD), and multi-chamber arresters, presents promising solutions to address these limitations. The introduction of FDD, utilizing ferromagnetic materials to create frequency-dependent resistance, demonstrates significant potential in reducing the amplitude of high-frequency overvoltage pulses and enhancing the lightning resistance of overhead lines and

substations. Moving forward, it is crucial to conduct thorough assessments of the effectiveness of trial operations of FDD, explore methods to increase the active resistance, and optimize their designs.

Thus, the development and application of new methods and means of protection, taking into account both the amplitude and steepness of the pulses, becomes a necessity to provide more reliable and effective protection of substations and overhead lines from high-frequency overvoltages. Further research and innovation in this area could lead to the development of more effective and reliable protection measures, ultimately leading to increased stability and reliability of power supply.

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A MODIFIED WEIGHTED DISTRIBUTION - APPLICATION ON DIABETES MELLITUS AND PANCREATIC CANCER DATA

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Abstract

This research article attempts to establish and explore a case of two parameter Nwikpe distribution and termed it as Area Biased C2N distribution. As the characteristics of Hydrogen per Oxide(H_2O_2) is quite different from that of Water (H_2O) even though both are the different combinations of the same elements Oxygen & Hydrogen, the characteristics of initial distribution is also entirely different from that of the area biased modified distribution. The implemented new distribution has distinct structural characteristics, and its parameters are estimating using maximum likelihood estimation. Utilizing biomedical data, the new distribution's application has been examining to ascertain its superiority and utility. One lifetime data set shows the mean reduction in blood glucose (mg/dL) after three days of the first usage of the Metformin medicine from a random sample of 130 patients from a hospital at Chennai, TamilNadu with type 2 diabetes mellitus by testing the FBS-Fasting Blood Glucose. The another set of lifetime data shows the mean reduction in blood glucose (mg/dL) after each dosage of the FIASP insulin-medicine in alternate days of a pancreatic cancer patient, noted for 63 days randomly. Both data set is going to fit to the new distribution and analyze them, to determine the supremacy and usefulness.

Keywords: area biased distribution, length biased distribution, weighted distributions, reliability, estimate.

1. Introduction

When samples from both the parent distribution and newly derived distributions can be obtained, the concept of weighted distributions (WD) suggests a method to fit models to the unknown weight function. The WD plays a remarkable role in better understanding of the standard distributions and can extend distributions by adding flexibility, while dealing with the modeling of statistical data when classical distributions cannot imply observations with the same probabilities. The statistical explanation of WD & size biased distribution was made by Buckland and Cox [1] within the renewal theory framework. If only the length in units considers for the weight function, the WD becomes a length-biased distribution. Other environmental, econometric, and biomedical

sampling problems, as well as a number of forestry applications, have brought up the length-biased and area-biased distributions. Numerous scholars created significant area-biased weighted probability models that have a significant impact on how data sets from diverse practical domains are processed. Bashir and Mahmood [2] presented multivariate area-biased Lindley distribution. The estimation of parameters of area biased Ailamujia distribution detailed by Rao and Pandey [3]. Kayid *et al.* [4] explained the length-biased Rayleigh distribution. Oluwafemi and Olalekan [5] investigated the exponentiated Weibull distribution biased by length and area. The weighted quasi-Xgamma distribution was analyzed by Hassan *et al.* [6] with applications in survival times. Elangovan *et al.* citeelanetal investigated the Samade distribution with length bias. Fazal [8] provided specifics on the Poisson exponential distribution with area bias. Chouia and colleagues [9] examined the size-biased Zeghdoudi distribution and its practical applications. Nwikpe and Essi [10] have proposed the two parameter Nwikpe distribution (TPND), a continuous lifetime distribution. Some of its statistical characteristics are discussed.

A Generalized Area-Biased Power Ishita Distribution with a biomedical data is studied and the Survival time in days of lung cancer patients after their second cycle of chemotherapy is analyzed and fitted with an area biased weighted distribution by Roshni *et al.* [11]. The relationship between Pancreatic cancer and Diabetes, is analyzed and detailed by Teresa *et al.* [12]. A meta-analysis of cohort studies based on Diabetes mellitus and risk of pancreatic cancer is studied by Qiwen Ben [13]. To describe, diabetes mellitus correlates with increased risk of pancreatic cancer, a population-based cohort study in Taiwan is conducted by Liao *et al.* [14].
Many biomedical data show non-symmetric nature. The data based on Blood sugar in some situations exhibit positive skewness. The normal fasting blood glucose/sugar (FBS) ranges between 70 mg/dL – 100 mg/dL (or 3.9 mmol/L – 5.6 mmol/L). If it is ranges 100 mg/dL – 125 mg/dL (or 5.6 mmol/L – 6.9 mmol/L) glycemia monitoring and lifestyle changes are required.

A diagnosis of diabetic is made if the FBS is 126mg/dL(7mmol/L) or higher on 2 separate tests. When a person has low FBS concentration (hypoglycemia) which is less than 70 mg/dL (3.9 mmol/L), they may experience palpitations, blurred vision dizziness, sweating and other symptoms which require to be closely watched. Hyperglycemia -the increased FBS concentration - is a sign of a raised risk of diabetes. FBS may be within normal range if a person does not diabetic or if they are taking effective treatment with medication that lowers blood sugar in diabetics. Mean FBS is used as a stand-in for diabetes treatment and for encouraging healthy eating & lifestyle choices at the national level.

Metformin (Fortamet, Glumetza, others) is usually prescribed first for type 2 diabetes. Liver Produces less glucose and sensitivity to insulin increases, making the body use insulin effectively. One can use any medication that content Metformin, such as the 500mg Glyciphage SR tablet. It controls the blood sugar levels. It is for the treatment of type 2 diabetes mellitus, a disorder in which diet and exercise alone are insufficient to control blood sugar level, causing them to rise above normal. It belongs to a class of medicines called biguanides. It works by reducing the glucose production in the liver, raising the body's sensitivity to insulin and delaying the absorption of sugar from the intestines.

Pancreas produces Insulin, that enables our body to utilize glucose for energy. High blood sugar- Hyperglycemias - results from insufficient insulin production or usage, results diabetes. Insulin is to control blood sugar in people with condition in which the body does not make insulin (type 1 diabetes) and therefore cannot control the amount of sugar in the blood or in people with condition in which the blood sugar is too high (type 2 diabetes) because for whom body does not produce or use. If the fasting plasma glucose is greater than 250 mg/dL or the HbA1c is greater than 10% then Insulin therapy usually needed. The Hyperinsulinemia rapid rise linked with type 2 diabetes and obesity, predicts a rise in pancreatic cancer frequency. Insulin effects at different stages of pancreatic cancer progression are unclear.

FIASP is a fast-acting insulin. It is a type of insulin that's spontaneously released into our bloodstream. It is a solution in a vial, PenFill cartridge or FlexTouch pen. It is the same as the insulin NovoRapid, with the addition of two ingredients, to increase the speed at which FIASP is absorbed into the blood are niacinamide (Vitamin B3) and the amino acid L-arginine. Insulin in

FIASP helps glucose enter cells from the blood and acts in the same method as the body's own insulin. This reduces the symptoms and complications of diabetes, and controls the level of blood glucose.

2. Methods

I. Area Biased C2N Distribution (ABC2ND)

The probability density function '(PDF) of TPND is,

$$f(x;\theta,\alpha)=\frac{\theta^3}{(\theta^5+2\alpha+6)}(\theta x^3+\alpha x^2+\theta^3)e^{-\theta x};\quad 0<x<\infty, 0<\alpha,\theta<\infty, \tag{1}$$

and the Cumulative density function '(CDF) of TPND is,

$$F(x;\theta,\alpha)=\left(1-\left(1+\frac{\theta^3x^3+(\alpha+3)(\theta^2x^2+2\theta x)}{(\theta^5+2\alpha+6)}\right)e^{-\theta x}\right);0<x<\infty, 0<\alpha,\theta<\infty. \tag{2}$$

Let 'X' be a negative random variable and has pdf $f(x)$ and $w(x)$ be a non-negative weight function, then the pdf of a weighted random variable X_w is,

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))}, x>0 \text{ where } E(w(x))=\int w(x)f(x)dx<\infty.$$

For various choices of weight function $w(x)$, if $w(x)=x^c$, the proposed distribution is mentioned as weighted distribution. Here, $c=2$, the weight function is taken as $w(x)=x^2$ to formulate the area biased version of TPND, and the pdf of the new distribution is obtained from,

$$f_a(x)=\frac{x^2f(x)}{E(x^2)} \text{ where } E(x^2)=\frac{120+24\alpha+2\theta^5}{\theta^2(\theta^5+2\alpha+6)}, \text{ as,}$$

$$f_a(x)=\frac{x^2\theta^5}{(120+24\alpha+2\theta^5)}(\theta x^3+\alpha x^2+\theta^3)e^{-\theta x}, 0<x,\theta,\alpha<\infty$$

and the CDF of ABC2ND can be determined as,

$$F_a(x)=\int_0^x f_a(x)dx=\frac{1}{(120+24\alpha+2\theta^5)}\left(\theta^6\int_0^x x^5e^{-\theta x}dx+\alpha\theta^5\int_0^x x^4e^{-\theta x}dx+\theta^8\int_0^x x^2e^{-\theta x}dx\right).$$

Let $\theta x=t$, then $\theta dx=dt$, implies $dx=dt/\theta$. $x=\frac{t}{\theta}$ when $x\to x$, and $t\to\theta x$ and as $x\to\theta$, $t\to 0$ and by using usual Gamma Integral notations, the CDF of ABC2ND is,

$$F_a(x)=\frac{1}{(120+24\alpha+2\theta^5)}\left(\gamma(6,\theta x)+\alpha\gamma(5,\theta x)+\theta^5\gamma(3,\theta x)\right) \tag{3}$$

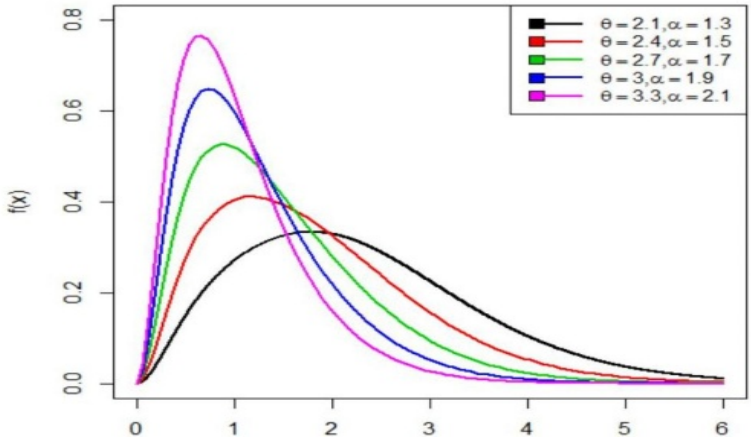

Figure 1: *PDF of ABC2ND*

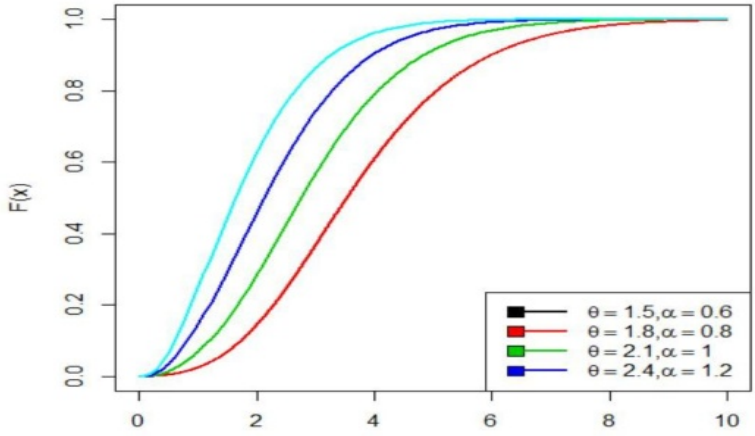

Figure 2: *CDF of ABC2ND*

The Nature of PDF and CDF of ABC2ND is clear from the Figure 1 and Figure 2. ABC2ND is non-symmetric. It is positively skewed. Hence it may show good fit for many real data set than any conventional distribution.

II. Reliability and Hazard functions

Reliability function: The reliability function of ABC2ND is,

$$R(x) = 1 - F_\alpha(x) = 1 - \frac{1}{(120 + 24\alpha + 2\theta^5)} \left(\gamma(6,\theta x) + \alpha\gamma(5,\theta x) + \theta^5\gamma(3,\theta x)\right).$$

Hazard function: The hazard function,

$$h(x) = \frac{f_a(x)}{(1 - F_a(x)} = \frac{x^2\theta^5\left(\theta x^3 + \alpha x^2 + \theta^3\right)e^{-\theta x}}{(120 + 24\alpha + 2\theta^5) - (\gamma(6,\theta x) + \alpha\gamma(5,\theta x) + \theta^5\gamma(3,\theta x))}$$

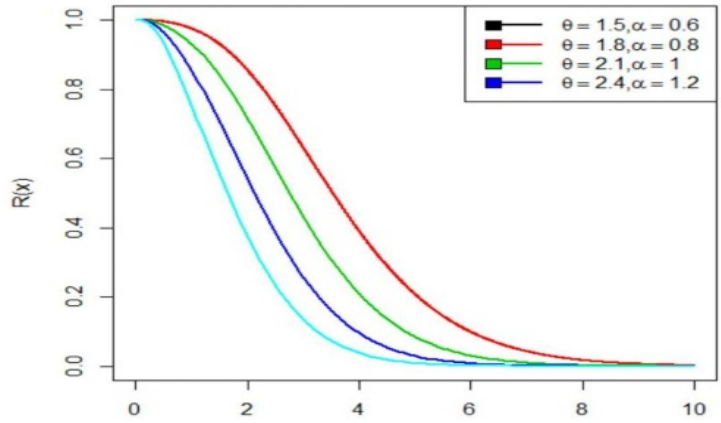

Figure 3: *Reliability function*

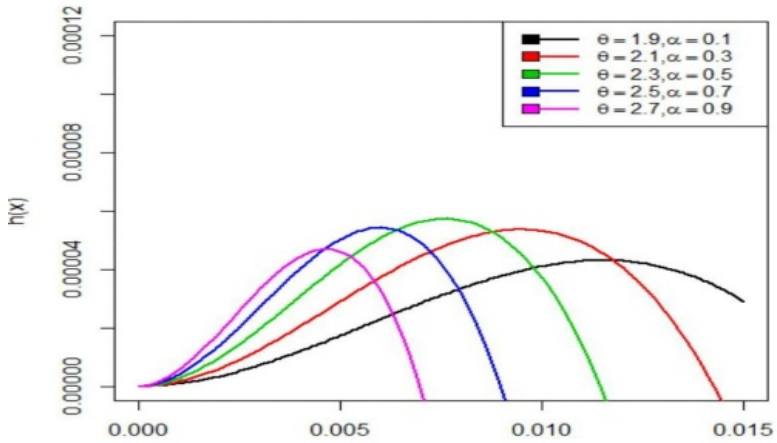

Figure 4: *Hazard function*

The Nature of Reliability Function and Hazard Function of ABC2ND is clear from the Figure 3 and Figure 4.

3. Results

I. Moments

Let X be a random variable following ABC2ND with parameters θ and α, then the r^{th} order moment $E(X^r)$ is,

$$E(X^r) = \mu'_r = \int_0^\infty x^r f_a(x)dx = \int_0^\infty x^r \frac{x^2\theta^5}{(120+24\alpha+2\theta^5)}(\theta x^3 + \alpha x^2 + \theta^3)e^{-\theta x}dx$$

$$= \frac{\theta^5}{(120+24\alpha+2\theta^5)}\left(\theta\int_0^\infty x^{(r+6)-1}e^{-\theta x}dx + \alpha\int_0^\infty x^{(r+5)-1}e^{-\theta x}dx + \theta^3\int_0^\infty x^{(r+3)-1}e^{-\theta x}dx\right). \quad (4)$$

By Simplifying the equation (4), we obtain

$$E(X^r) = \mu'_r = \frac{\Gamma(r+6) + \alpha\Gamma(r+5) + \theta^8\Gamma(r+3)}{\theta^r(120 + 24\alpha + 2\theta^5)}. \tag{5}$$

Letting $r = 1, 2, 3$ and 4 in equation (5), we get the first four moments of ABC2ND as,

$$\mu'_1 = \frac{720 + 120\alpha + 6\theta^8}{\theta(120 + 24\alpha + 2\theta^5)}, \mu'_2 = \frac{5040 + 720\alpha + 24\theta^8}{\theta^2(120 + 24\alpha + 2\theta^5)}$$

$$\mu'_3 = \frac{40320 + 5040\alpha + 120\theta^8}{\theta^3(120 + 24\alpha + 2\theta^5)}, \mu'_4 = \frac{362880 + 40320\alpha + 720\theta^8}{\theta^4(120 + 24\alpha + 2\theta^5)}.$$

II. MGF and CF of ABC2ND Let a random variable X follows ABC2ND with parameters θ and α, then the MGF is,

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_a(x)dx.$$

Using Taylor's series, we obtain

$$M_X(t) = E(e^{tx}) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_a(x)dx = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_a(x)dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!}\mu'_j M_x(t) = E(e^{tx}) = \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\Gamma(j+6) + \alpha\Gamma(j+5) + \theta^8\Gamma(j+3)}{\theta^j(120 + 24\alpha + 2\theta^5)}\right)$$

$$M_x(t) = E(e^{tx}) = \frac{1}{(120 + 24\alpha + 2\theta^5)} \sum_{j=0}^\infty \frac{t^j}{j!\theta^j}(\Gamma(j+6) + \alpha\Gamma(j+5) + \theta^8\Gamma(j+3)).$$

Similarly, the CF of ABC2ND is,

$$\phi_x(t) = \frac{1}{(120 + 24\alpha + 2\theta^5)} \sum_{j=0}^\infty \frac{it^j}{j!\theta^j}(\Gamma(j+6) + \alpha\Gamma(j+5) + \theta^8\Gamma(j+3)).$$

III. Maximum Likelihood Estimation (MLE) and Fisher's Information Matrix (FIM) of ABC2ND The MLE of the parameters of ABC2ND are estimated. Let $X_1, X_2, \ldots X_n$ be n random sample from the ABC2ND, then the likelihood function

$$L(x) = \prod_{i=1}^n f_a(x)L(x) = \frac{\theta^{5n}}{(120 + 24\alpha + 2\theta^5)^n} \prod_{i=1}^n \left(x_i^2(\theta x_i^3 + \alpha x_i^2 + \theta^3)e^{-\theta x_i}\right). \tag{6}$$

Taking log and differentiating with respect to θ and α, we get two Normal equations.

$$\frac{\partial \log L}{\partial \theta} = \frac{5n}{\theta} - n\left(\frac{10\theta^4}{(120 + 24\alpha + 2\theta^5)}\right) + \sum_{i=1}^n \left(\frac{(x_i^3 + 3\theta^2)}{(\theta x_i^3 + \alpha x_i^2 + \theta^3)}\right) - \sum_{i=1}^n x_i = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -n\left(\frac{24}{(120 + 24\alpha + 2\theta^5)}\right) + \sum_{i=1}^n \left(\frac{x_i^2}{(\theta x_i^3 + \alpha x_i^2 + \theta^3)}\right) = 0.$$

Getting an algebraic solution is complicated here, hence apply some numerical methods like Newton-Raphson approach to estimate the parameters of the distribution through R software. In order to determine the confidence interval (CI), apply the asymptotic normality results. We know

$\hat{\gamma} = (\hat{\theta}, \hat{\alpha})$ represents the MLE of the $\gamma = (\theta, \alpha)$. We write, $\sqrt{n}(\hat{\gamma} - \gamma) \to N_2(0, I^{-1}(\gamma))$, where $I^{-1}(\gamma)$ is FIM.
i.e.,

$$I(\gamma) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) \end{pmatrix}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{5n}{\theta^2} - n\left(\frac{(120 + 24\alpha + 2\theta^5)(40\theta^3) - (10\theta^4)^2}{(120 + 24\alpha + 2\theta^5)^2}\right) + \sum_{i=1}^{n}\left(\frac{(\theta x_i^3 + \alpha x_i^2 + \theta^3)(6\theta) - (x_i^3 + 3\theta^2)^2}{(\theta x_i^3 + \alpha x_i^2 + \theta^3)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\left(\frac{576}{(120 + 24\alpha + 2\theta^5)^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i^2)^2}{(\theta x_i^3 + \alpha x_i^2 + \theta^3)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = n\left(\frac{(10\theta^4)(24)}{(120 + 24\alpha + 2\theta^5)^2}\right) - \sum_{i=1}^{n}\left(\frac{(x_i^3 + 3\theta^2)(x_i^2)}{(\theta x_i^3 + \alpha x_i^2 + \theta^3)^2}\right).$$

By $I^{-1}(\gamma)$ we estimate γ and utilized to attain an asymptotic CI for θ & α.

4. Discussion

I. Simulation Analysis
The simulated data from the pdf of ABC2ND is analyzed. The descriptive statistics for the same is noted here at Table 1.

Table 1: *The descriptive statistics of simulated data: ABC2ND, $\alpha = 3.9$, $\theta = 1.95$*

Mean	17.1	Range	40.5
Standard Error	0.497	Minimum	1.4
Median	8.45	Maximum	41.9
Mode	3.2	Sum	85500
Standard Deviation	12.46	Count	5000
Kurtosis	-0.73	Largest& Smallest	1.4 & 41.9
Pearson's Skewness*	1.12	Confidence Level (95.0%)	0.9769

Karl Pearson's Skewness = (Mean-Mode)/SD. The value of this coefficient will be zero for a symmetrical distribution. If mean >mode, the coefficient of skewness is positive else negative. For considerably skewed destitution this coefficient is lies between -1 and 1. If it is greater than 1 its highly positive skewed. Generally, if the distribution of data is skewed to the left, then mean <median <mode. If the distribution of data is skewed to the right, then mode <median <mean. Here from Figure 1 & Table 1, both cases imply the distribution is highly positive skewed.

II. Application
Hereby applying and analyzing a real data set for fitting ABC2ND in order to determine whether the ABC2ND shows a better fit than TPN, Nwikpe, Komal and Lindley distributions.

Data 1: The real data set at (Table 2), shows the mean reduction in blood glucose (mg/dL) after three days of the first usage of the Metformin medicine from a random sample of 130 patients

Table 2: *The mean reduction in blood glucose (mg/dL) by Metformin.*

5.5	6.1	8.7	13.9	7.8	9.5	7.1	6.1	8.2	4.7	5.1	14.3	7.1
5.2	14.4	7.5	12.7	7.8	6.2	7.1	6.1	8	6.1	5.3	6.1	7.9
6.1	7.3	14.8	9.1	14.3	8.3	9.6	7.1	7.8	5.6	6.1	7.7	14.8
4.9	8.5	6.3	11.5	7.7	8.9	7.1	6.1	8	6.1	12.3	7.8	9.1
5	14.2	6.7	11.9	7.8	9	7.1	6.1	8	6.1	13.1	7.8	9.3
6.1	7.9	4.8	10.3	7.6	8.6	9.9	11.2	8	5.9	9.9	7.6	8.5
6.1	8.1	4.9	10.7	7.6	8.7	10	11.3	8	6	7.1	6.1	8
4.8	8.3	5.9	11.1	7.6	8.8	7.1	11.4	8	6.1	7.1	6.1	8
5.4	6.1	8.3	13.5	7.8	9.4	7.1	6.1	8.1	4.6	9.8	7.1	7.9
6.1	7.5	14.8	9.5	14.7	8.4	9.7	7.1	7.8	5.7	5.8	6.1	4.5

from a hospital at Chennai, TamilNadu with type 2 diabetes mellitus by testing the FBS-Fasting Blood Glucose (Method: Hexokinase).

Here from Table 3. We get, the Mean >Median >Mode & Skewness = 0.8904 shows the data is positively skewed. Hence the data is non-symmetric. Thereby we are trying to analyze the goodness of fit of ABC2ND.

Table 3: *The descriptive statistics of the data at Table 2.* $(n = 130)$

Mean	8.1354	Minimum	4.5
Standard Error	0.2225	Maximum	14.8
Median	7.8	Sum	1057.6
Mode	6.1	Skewness	0.8904
SD	2.2910	Smallest & Largest	4.5 & 14.8
Kurtosis	0.7749	Confidence Level (95.0%)	0.4401

Data 2: The mean reduction in blood glucose (mg/dL) is noted Table 4, after each dosage of the FIASP insulin - medicine in alternate days of a pancreatic cancer patient is noted for 63 days randomly.

Table 4: *The mean reduction in blood glucose (mg/dL) FIASP insulin.*

3.2	3.3	3.6	7.8	3.2	7.5	8.2	11.8	7.1
7.7	16.2	14.4	16.5	3.8	3.2	8.5	15	3.2
14.5	16.3	7.1	16.6	3.2	3.1	8.6	15.1	3.2
7.1	3.2	7.2	16.7	14.8	3.5	8.7	15.2	3.2
4.3	3.3	7.3	3.2	3.4	3.2	7.4	15.3	14.7
16.8	3.2	3.7	16.3	7.1	7.6	8.3	3.2	4.4
16.9	16.1	3.9	16.4	3.2	3.2	8.4	14.9	14.6

Table 5: *The descriptive statistics of the data at Table 4.* $(n = 63)$

Mean	8.6	Minimum	3.1
Standard Error	0.663456	Maximum	16.9
Median	7.4	Sum	541.8
Mode	3.2	Smallest & Largest	3.1 & 16.9
S D	5.26	Skewness	1.038
Kurtosis	-1.46265	Confidence Level (95.0%)	1.3262

Here the Mean >Median >Mode & Skewness = 1.038 shows a high positive skewness. Hence the data is non-symmetric. Thereby we are trying to check the goodness of fit of ABC2ND.To the

estimation of unknown parameters, and to determine the model comparison criterions, software R is applied.

Table 6: *MLE, S.E, $-2\log L$, AIC, BIC and AICC of fitted distributions (data from Table 2)*

Dbn	MLE	S.E	$-2\log L$	AIC	BIC	AICC	K-S	P
ABC2N	$\hat{\alpha}$ = 3.7725 $\hat{\theta}$= 1.83	$\hat{\alpha}$ = 1.5712 $\hat{\theta}$ = 0.0001	174.31	178.3	182.5	178.7	0.029	0.7981
TPN	$\hat{\alpha}$ = 6.14 $\hat{\theta}$ = 9.70	$\hat{\alpha}$ = 1.667 $\hat{\theta}$ = 7.10	202.91	206.9	211.2	211.1	0.032	0.7777
Nwikpe	$\hat{\theta}$ = 1.29	$\hat{\theta}$ = 0.087	201.33	203.3	205.4	203.3	0.039	0.6780
Lindley	$\hat{\theta}$ = 0.57	$\hat{\theta}$ = 0.039	249.72	251.7	253.8	251.7	0.173	0.0459
Komal	$\hat{\theta}$ = 0.51	$\hat{\theta}$ = 0.037	254.87	256.9	259.0	256.9	0.197	0.0450

Table 7: *MLE, S.E, $-2\log L$, AIC, BIC, AICC (Data from Table 4)*

Dbn	MLE	S.E	$-2\log L$	AIC	BIC	AICC	K-S	p
ABC2N	$\hat{\alpha}$ = 3.5778 $\hat{\theta}$ = 1.63	$\hat{\alpha}$ = 1.357 $\hat{\theta}$ = 0.0001	167.31	171.3	175.5	171.5	0.03	0.8101
TPN	$\hat{\alpha}$ = 6.47 $\hat{\theta}$ = 9.80	$\hat{\alpha}$ = 1.67 $\hat{\theta}$ = 7.13	195.91	199.9	204.1	200.1	0.03	0.7810
Nwikpe	$\hat{\theta}$ = 1.79	$\hat{\theta}$ = 0.09	194.33	196.3	198.4	196.3	0.04	0.7670
Komal	$\hat{\theta}$ = 0.51	$\hat{\theta}$ = 0.04	247.87	249.8	252.0	249.9	0.15	0.0590
Lindley	$\hat{\theta}$ = 0.54	$\hat{\theta}$ = 0.04	242.72	244.7	246.8	244.7	0.12	0.0660

To compare the performance of ABC2ND over TPN, Nwikpe, Komal and Lindley distributions, we consider criterions – AIC (Akaike Information Criterion, BIC (Bayesian Information Criterion), AICC (Akaike Information Criterion Corrected) and $-2\log L$.The distribution is better if with the lesser criterion values of $-2\log L$, AIC, BIC and AICC. Here, AIC = $2k$ ı2$\log L$, BIC = $k\log n$, and AICC=AIC+$\frac{2k(k+1)}{n-k-1}$. Here, n = the sample size, k = the number of parameters and ı2$\log L$ = the maximized value of log-likelihood function. (Here K-S is Kolmogorov-Smirnov, p is p-value at level of significance α = 5%). The ABC2ND has smaller BIC, AICC, AIC, and $-2\log L$ values than the TPN, Nwikpe, Komal, and Lindley distributions, according to the results shown above in Tables 4 and 5. Implies that the ABC2ND shows a better fit over the other distributions for such skewed biomedical data.

5. Conclusion

Hereby explored and studied a new distribution named as Area Biased C2N Distribution. The developed new distribution is introduced by applying the area biased method to its initial distribution. Some of its statistical characteristics like moments, shape and behaviour of PDF and CDF, reliability function, hazard rate, MGF and CF are described. The parameters of the distribution are estimated. Two different applications of the new distribution have been presented to demonstrate its significance at biomedical data. The Data set 1 is the mean reduction in blood glucose (mg/dL) after three days of the first usage of the Metformin medicine from a random sample of 130 patients from a hospital at Chennai, TamilNadu with type 2 diabetes mellitus by testing the FBS-Fasting Blood Glucose. And the data set 2 is the mean reduction in blood glucose (mg/dL) is noted after each dosage of the FIASP insulin - medicine in alternate days of a pancreatic cancer. It is concluded for both cases that the result developed for ABC2ND provides a quite satisfactory fit over TPN, Nwikpe, Komal and Lindley distributions.

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EXPLORING QUADRASOPHIC FUZZY SET: APPLICATIONS IN ASSESSING STRESS LEVELS AND SELF-ESTEEM CONNECTIONS

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Abstract

The ambiguous environment has been addressed with a variety of fuzzy sets and their extensions. The Quadrasophic Fuzzy Set is one of the generalization of Fuzzy set to handle imprecise information efficiently. It is defined with two new parameters. In this artifact, we defined the operations, theorems, and relations of the Quadrasophic Fuzzy Set with pertinent examples. We also established a comparison study with other existing models. Additionally, the integration of Quadrasophic Fuzzy data with the TOPSIS approach to solve the Multi Criteria Decision Making problem is proposed and illustrated by examining the relationship between employee stress levels and their self-esteem, which can trigger obsessive-compulsive disorder, using real-life data. The results are analyzed with SPSS software.

Keywords: Quadrasophic Fuzzy set, Quadrasophic Fuzzy Relation, Max-Min Composition, Decision making

1. Introduction

Fuzzy set and its extensions are helpful to handle the uncertain situations. One of the fuzzy set extensions is an Intuitionistic fuzzy set (IFS) .The framework of the conception Intuitionistic fuzzy set makes us to analyze the uncertain environment. IFS presents the sum of membership degree and the non-membership degree is less than one [5] ,[21]. The IFS has its wide range applications in many real life situations inbuilt with uncertainty. Also, the idea of IFS has been applied in different areas such as medication, business, decision making [9] [19].

For a case, if the sum of degrees of membership grade and non membership grade is not less than one,then the IFS cannot be applicable. To overcome the shortcoming exists in IFS, the concept of Pythagorean fuzzy set is established by R. Yager [21]. In Pythagorean fuzzy set, the sum of squares of membership degree and non membership degree lies between zero and one. Even though PFS is the extension of fuzzy set it has similar feature of IFS. Many researchers widened the theoretical concept of PFS [22] .PFS's special form to handle imprecise data provoked several authors to extend numerous operators in PFS and applied in various fields [1].

Bipolar fuzzy set represents the set with satisfaction level of property which ranges in [0, 1] and satisfaction level of implicit counter property ranges in [-1, 0] [23] [24].Bipolar can handle the situations characterized by positive and negative membership without hesitant. Several author extended the Bipolar idea [7] and applied in many fields to handle the bipolarity situation [14], [17],[15], [16] .The idea of neutrality was initially presented in Picture Fuzzy set. Picture fuzzy set represents positive, negative and neutral grade ranges in [0, 1] [8]. Picture fuzzy set has its vibrant applications in many fields including Artificial intelligence, medication, business, neutral

networks and data coding. Neutrosophic fuzzy set presents the three aspects such as truthiness, falsity and the indeterminancy whose sum lies between one and three [6], [20] .There exist many extensions of fuzzy set and many blended fuzzy theoretical idea like bipolar picture fuzzy set [18] , neutrosophic- bipolar fuzzy set [13] to tackle the imprecision information. Along with polarity, indecisiveness and the influence of environment cannot be found with the existing extensions of fuzzy sets. The membership and non-membership divisions will not aid us in determining a definitive answer to the underlying issue. New parameters are needed to determine the absolute solution. The influential rate that altered the membership and non-membership ranges can result in new parameters. Further dimensions are produced by analyzing the influential rates that alter the membership and non-membership ranges. The modern analysis of membership grades results in two new memberships. Hence, Quadrasophic Fuzzy Set is defined with four membership functions [3], [4].

In this artifact, Section 2 provides some basic concepts of fuzzy sets. Section 3 presents the properties and definitions of QFS whereas in section 4 gives the operations, relations and advantages of Quadrasophic Fuzzy Model. To promote the validation results of QFS section 5 presents the comparative study with illustration in medical diagonsis.Using the Max-Min composition of QFS, Section 6 demonstrates how QFS may be applied to ascertain the relationship between self-esteem and stress levels that cause OCD. Section 7 emphasizes the significance of Quadrasophic Fuzzy Set (QFS) in analyzing the environmental impact through statistical analysis using SPSS. Finally, section 8 gives the conclusion of QFS with its scope for future research.

2. Preliminaries

Fuzzy sets [11]: A non-empty fuzzy set F in the universe U is defined as $F = \{(x, \mu(x)) : x \in X\}$ where $\mu(x) \in [0,1]$ is the degree of membership value of F.

Intuitionistic Fuzzy sets [11]: An intuitionistic fuzzy set I in the non-empty set X is defined as $I = \{(x, \mu(x), \nu(x); x \in X)\}$ where the function $\mu(x), \nu(x) : X \to [0,1]$ represents the membership degree and the non- membership degree value with the condition $0 \leq \mu(x) + \nu(x) \leq 1$. The value $\pi(x) = 1 - \mu(x) - \nu(x)$ is named as the degree of indeterminacy $\forall x \in X$.

Pythagorean Fuzzy set [21]: A Pythagorean fuzzy set P, is defined as $P = \{(x, \mu(x), \nu(x); x \in X\}$ where the function $\mu(x), \nu(x)$:$X \to [0,1]$ represents the membership degree and the non- membership degree value with the condition $0 \leq \mu_P^2(x) + \nu_P^2(x) \leq 1$. The value $\pi_P(x) = \sqrt{1 - (\mu_P^2(x) - \nu_P^2(x))}$ is named as the degree of indeterminacy for $\forall x \in X$.

Quadrasophic fuzzy set [3]: The Quadrasophic fuzzy set q on the set U is defined as

$$q = \{(x, \eta_q(x), \lambda_{\eta_q}(x), \lambda_{\mu_q}(x), \mu_q(x)) \,|x \in U\}$$

where the degree of positive membership grade is $\mu_q(x) : U \to [0,1]$, the degree of negative membership is $\eta_q(x) : U \to [-1,0]$ the degree of restricted positive membership is $\lambda_{\mu_q}(x) : U \to [0, 0.5]$ the degree of restricted negative membership is $\lambda_{\eta_q}(x) : U \to [-0.5, 0]$ And the condition follows: $-1 \leq \mu_q(x) + \eta_q(x) \leq 1$, $-0.5 \leq \lambda_q \leq 0.5$ and $0 \leq \mu_q^2 + \eta_q^2 + \lambda_q^2 \leq 3$ for all $x \in U$, such that $\lambda_q =$ Length of $(\lambda_{\mu_q}, \lambda_{\eta_q})$.

Max-Min-Max composition [9]: Let $A(x \to y)$ and $B(y \to z)$ be any to IF relations. Then the Max-min- Max composition of as $A \circ B$ from x to z, whose membership functions and non membership functions are represented as follows: $\nu_{B \circ A}(x,z) = \wedge_y [\nu_A(x,y), \nu_B(y,z)]$, $\mu_{B \circ A}(x,z) = \vee_y [\mu_A(x,y), \mu_B(y,z)]$ for all $x \in X, y \in Y, (x,y) \in X \times Y$ where $\wedge = min$, $\vee = max$.

Score Function [12]: Let $P = (x, \mu_P(x), \nu_P(x); x \in X)$ be the PFS in the non-empty set X. Then, the score- valued function of PFS is defined as score $(P) = (\mu_P(x))^2 - (\nu_P(x))^2$, where score$(P) \in [-1, 1]$.

3. Quadrasophic Fuzzy Sets

A Quadrasophic fuzzy set (QFS) is a fuzzy set intended to address environmental impact rates. It aims to attain the "restricted level" in both positive and negative polarity, where the rate at which a condition or implicit counter condition is partially satisfied is known as the "reluctant value."

The complex structure of the set will not be disclosed by the degree of satisfaction property and its implicit counter-property. Furthermore, it must be evident which subgroups exhibit varying degrees of reluctance on both the positive and negative sides. The partial counter implicit property has a level of tentative fixation of [-0.5, 0], whereas the partial satisfaction property has a level of [0, 0.5]. The subset explains how the level of ambiguity or influence impacts the satisfaction rate in the property and explicit counter-property.

The refusal of the positive and negative membership grades is referred to as "restrictive positive membership" and "restrictive negative membership," respectively. Restricted positive and restricted negative membership are two additional memberships that we examine in addition to positive and negative membership to obtain more accurate findings [3]. The margin for restricted membership is located at the crossover point. Restricted membership allows us to provide more precise results by determining the impact range.

Quadrasophic Fuzzy Set: [3] The Quadrasophic fuzzy set (QFS) defined on X is represented as

$$Q = \{(x, \eta(x), \lambda_\eta(x), \lambda_\mu(x), \mu(x)) \,|x \in X\}$$

In Q, $\mu(x) : X \to [0, 1]$ represents the degree of positive membership of x , $\eta(x) : X \to [-1, 0]$ represents the degree of negative membership of x, $\lambda_\mu(x) : X \to [0, 0.5]$ represents the degree of restricted positive membership of x ,$\lambda_\eta(x) : X \to [-0.5, 0]$ represents the degree of restricted negative membership of x. The inequality $-1 \leq \mu(x) + \eta(x) \leq 1$, $-0.5 \leq \lambda \leq 0.5$ and $0 \leq \mu^2 + \eta^2 + \lambda^2 \leq 3$ holds for every $x \in X$, where $\lambda =$ Length of $(\lambda_\mu, \lambda_\eta)$.
The term $QFS(x)$ refers to the set of all Quadrasophic Fuzzy Set on X.

Remark 1. If $\mu(x) \neq 0$, $\eta(x) = \lambda_\mu(x) = \lambda_\eta(x) = 0$ then Q is a fuzzy set of the form $< x, \mu(x) >$.

3.1. Properties of Quadrasophic Fuzzy Sets

Properties:

1. If $\mu(x) \neq 0, \eta(x) \neq 0, \lambda_\mu(x) = \lambda_\eta(x) = 0$ then Q reduces to a bipolar fuzzy set.
2. If $\mu(x) \neq 0, \eta(x) = 0, \lambda_\mu(x) = \lambda_\eta(x) = 0$ then Q is a high positive membership.
3. If $\mu(x) = 0, \eta(x) \neq 0, \lambda_\mu(x) = \lambda_\eta(x) = 0$ then Q is a high negative membership.
4. If $\mu(x) \neq 0, \lambda_\mu(x) \neq 0, \eta(x) = \lambda_\eta(x) = 0$ then Q is a restricted positive membership.
5. If $\mu(x) = 0, \lambda_\mu(x) = 0, \eta(x) \neq, \lambda_\eta(x) \neq 0$ then Q is a restricted negative membership.
6. If $\mu(x) \neq 0, \eta(x) \neq 0, \lambda_\mu(x) \neq 0, \lambda_\eta(x) \neq 0$ then it is Quadrasophic fuzzy set.

Remark 2. The empty Quadrasophic fuzzy set is defined as $Q_0 = (0, 0, 0, 0)$ and complete Quadrasophic fuzzy set is defined as $Q_1 = (-1, -0.5, 0.5, 1)$ for each $x \in X$.

Subset [3]: Let $Q_1, Q_2 \in Q$ defined on the non - empty set X then Q_1 is the subset of Q_2 denoted by $Q_1 \subseteq Q_2$, if for each $x \in X$; $\eta_{Q_1}(x) \geq \eta_{Q_2}(x)$, $\lambda_{\eta Q_1}(x) \geq \lambda_{\eta Q_2}(x)$, $\lambda_{\mu Q_1}(x) \leq \lambda_{\mu Q_2}(x)$, $\mu_{Q_1}(x) \leq \mu_{Q_2}(x)$.

Complement of Quadrasophic Fuzzy Set[3]: The complement of the set $Q_1 \in Q$ in X is represented as Q_1^C and is defined as $Q_1^C = (\eta^C, \lambda_\eta^C, \lambda_\mu^C, \mu^C)$, where $\eta^C = -1 - \eta$, $\lambda_\eta^C = -0.5 - \lambda_\eta$, $\lambda_\mu^C = 0.5 - \lambda_\mu$ and $\mu^C = 1 - \mu$.

Intersection and Union of QFS [3]: The intersection of Q_1 and Q_2 in Quadrasophic Fuzzy Set is defined as:

$$Q_1 \cap Q_2 = (\eta_{Q_1}(x) \vee \eta_{Q_2}(x), \lambda_{\eta_{Q_1}}(x) \vee \lambda_{\eta_{Q_2}}(x), \lambda_{\mu_{Q_1}}(x) \wedge \lambda_{\mu_{Q_2}}(x), \mu_{Q_1}(x) \wedge \mu_{Q_2}(x)), \forall x \in X.$$

The union of Q_1 and Q_2 in Quadrasophic fuzzy set is defined as:

$$Q_1 \cup Q_2 = (\eta_{Q_1}(x) \wedge \eta_{Q_2}(x), \lambda_{\eta_{Q_1}}(x) \wedge \lambda_{\eta_{Q_2}}(x), \lambda_{\mu_{Q_1}}(x) \vee \lambda_{\mu_{Q_2}}(x), \mu_{Q_1}(x) \vee \mu_{Q_2}(x)), \forall x \in X.$$

Equal Set[3]: Let $Q_1, Q_2 \in Q$ defined on the non empty set X then Q_1 is equal to Q_2 denoted by $Q_1 = Q_2$ if for each $x \in X$;

$$\eta_{Q_1}(x) = \eta_{Q_2}(x), \lambda_{\eta_{Q_1}}(x) = \lambda_{\eta_{Q_2}}(x), \lambda_{\mu_{Q_1}}(x) = \lambda_{\mu_{Q_2}}(x), \mu_{Q_1}(x) = \mu_{Q_2}(x).$$

4. Certain operations on Quadrasophic Fuzzy Sets

Theorem 1. Let $Q_1, Q_2, Q_3 \in Q$ then it satisfy the following properties.
i) $Q_1 \subseteq Q_2$, $Q_2 \subseteq Q_3$ then $Q_1 \subseteq Q_3$.
ii) The operations intersection and union are commutative.
iii) The operations intersection and union are distributive.
iv) The operations intersection and union are associative.
v) The operations intersection and union satisfies the De-Morgan's rule.

Proof. i) By using the Subset definition , $Q_1 \subseteq Q_2$ if $\eta_{Q_1}(x) \geq \eta_{Q_2}(x)$, $\lambda_{\eta_{Q_1}}(x) \geq \lambda_{\eta_{Q_2}}(x)$, $\lambda_{\mu_{Q_1}}(x) \leq \lambda_{\mu_{Q_2}}(x)$, $\mu_{Q_1}(x) \leq \mu_{Q_2}(x)$. If $Q_2 \subseteq Q_3$ then $\eta_{Q_2}(x) \geq \eta_{Q_3}(x)$, $\lambda_{\eta_{Q_2}}(x) \geq \lambda_{\eta_{Q_3}}(x)$, $\lambda_{\mu_{Q_2}}(x) \leq \lambda_{\mu_{Q_3}}(x)$, $\mu_{Q_2}(x) \leq \mu_{Q_3}(x)$.
Then, obviously $Q_1 \subseteq Q_3$.

ii) By the definition of Intersection,

$$Q_1 \cap Q_2 = \{max(\eta_{Q_1}(x), \eta_{Q_2}(x)), max(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_2}}(x)), min(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_2}}(x)), min(\mu_{Q_1}(x), \mu_{Q_2}(x))\}$$

$$Q_2 \cap Q_1 = \{max(\eta_{Q_2}(x), \eta_{Q_1}(x)), max(\lambda_{\eta_{Q_2}}(x), \lambda_{\eta_{Q_1}}(x)), min(\lambda_{\mu_{Q_2}}(x), \lambda_{\mu_{Q_1}}(x)), min(\mu_{Q_2}(x), \mu_{Q_1}(x))\}$$

Therefore, $Q_1 \cap Q_2 = Q_2 \cap Q_1$.
In similar way, we show $Q_1 \cup Q_2 = Q_2 \cup Q_1$.
iii)

$$Q_2 \cap Q_3 = \{max(\eta_{Q_2}(x), \eta_{Q_3}(x)), max(\lambda_{\eta_{Q_2}}(x), \lambda_{\eta_{Q_3}}(x)), min(\lambda_{\mu_{Q_2}}(x), \lambda_{\mu_{Q_3}}(x)), min(\mu_{Q_2}(x), \mu_{Q_3}(x))\}$$

$$\begin{aligned} Q_1 \cup (Q_2 \cap Q_3) = ([&min\{\eta_{Q_1}(x), max(\eta_{Q_2}(x), \eta_{Q_3}(x))\}], \\ &[min\{\lambda_{\eta_{Q_1}}(x), max(\lambda_{\eta_{Q_2}}(x), \lambda_{\eta_{Q_3}}(x))\}], \\ &[max\{\lambda_{\mu_{Q_1}}(x), min(\lambda_{\mu_{Q_2}}(x), \lambda_{\mu_{Q_3}}(x))\}], \\ &[max\{\mu_{Q_1}(x), min(\mu_{Q_2}(x), \mu_{Q_3}(x))\}]). \end{aligned}$$

$$\begin{aligned} Q_1 \cup Q_2 = \{&min(\eta_{Q_1}(x), \eta_{Q_2}(x)), min(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_2}}(x)), \\ &max(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_2}}(x)), max(\mu_{Q_1}(x), \mu_{Q_2}(x))\} \end{aligned}$$

$$\begin{aligned} Q_1 \cup Q_3 = \{&min(\eta_{Q_1}(x), \eta_{Q_3}(x)), min(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_3}}(x)), \\ &max(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_3}}(x)), max(\mu_{Q_1}(x), \mu_{Q_3}(x))\} \end{aligned}$$

$$\begin{aligned} (Q_1 \cup Q_2) \cap (Q_1 \cup Q_3) = ([&max\{min(\eta_{Q_1}(x), \eta_{Q_2}(x)), min(\eta_{Q_1}(x), \eta_{Q_3}(x))\}], \\ &[max\{min(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_2}}(x)), min(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_3}}(x))\}], \\ &[min\{max(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_2}}(x)), max(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_3}}(x))\}], \\ &[min\{max(\mu_{Q_1}(x), \mu_{Q_2}(x)), max(\mu_{Q_1}(x), \mu_{Q_3}(x))\}]). \end{aligned}$$

Hence, $Q_1 \cup (Q^{\neg_2} \cap Q_3) = (Q_1 \cup Q_2) \cap (Q_1 \cup Q_3)$.
In similar way we can prove, $Q_1 \cap (Q^{\neg_2} \cup Q_3) = (Q_1 \cap Q_2) \cup (Q_1 \cap Q_3)$.
iv)

$$\begin{aligned} Q_1 \cup Q_2 = \{&min(\eta_{Q_1}(x), \eta_{Q_2}(x)), min(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_2}}(x)), \\ &max(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_2}}(x)), max(\mu_{Q_1}(x), \mu_{Q_2}(x))\} \end{aligned}$$

$$\begin{aligned} Q_1 \cup (Q_2 \cup Q_3) = ([&min\{min(\eta_{Q_1}(x), \eta_{Q_2}(x)), \eta_{Q_3}(x)\}], \\ &[min\{min(\lambda_{\eta_{Q_1}}(x), \lambda_{\eta_{Q_2}}(x)), \lambda_{\eta_{Q_3}}(x)\}], \\ &[max\{max(\lambda_{\mu_{Q_1}}(x), \lambda_{\mu_{Q_2}}(x)), \lambda_{\mu_{Q_3}}(x)\}], \\ &[max\{max(\mu_{Q_1}(x), \mu_{Q_2}(x)), \mu_{Q_3}(x)\}]). \end{aligned}$$

$$\begin{aligned} Q_2 \cup Q_3 = \{&min(\eta_{Q_2}(x), \eta_{Q_3}(x)), min(\lambda_{\eta_{Q_2}}(x), \lambda_{\eta_{Q_3}}(x)), \\ &max(\lambda_{\mu_{Q_2}}(x), \lambda_{\mu_{Q_3}}(x)), max(\mu_{Q_2}(x), \mu_{Q_3}(x))\} \end{aligned}$$

$$\begin{aligned} Q_1 \cup (Q_2 \cup Q_3) = ([&min\{\eta_{Q_1}(x), min(\eta_{Q_2}(x), \eta_{Q_3}(x))\}], \\ &[min\{\lambda_{\eta_{Q_1}}(x), min(\lambda_{\eta_{Q_2}}(x), \lambda_{\eta_{Q_3}}(x))\}], \\ &[max\{\lambda_{\mu_{Q_1}}(x), max(\lambda_{\mu_{Q_2}}(x), \lambda_{\mu_{Q_3}}(x))\}], \\ &[max\{\mu_{Q_1}(x), max(\mu_{Q_2}(x), \mu_{Q_3}(x))\}]). \end{aligned}$$

Hence, $(Q_1 \cup Q_2) \cup Q_3 = Q_1 \cup (Q_2 \cup Q_3)$.
In this way, we can prove $(Q_1 \cap Q_2) \cap Q_3 = Q_1 \cap (Q_2 \cap Q_3)$.

iv) By using definition, we can prove
$\overline{Q_1 \cup Q_2} = \overline{Q_1} \cap \overline{Q_1}$ and $\overline{Q_1 \cap Q_2} = \overline{Q_1} \cup \overline{Q_2}$. ■

Theorem 2. Let $Q_1, Q_2, Q_3 \in Q$ then the result would be as follows:

1. Law of Idempotent: $Q_1 \cup Q_1 = Q_1 \cap Q_1 = Q_1$.
2. Law of Absorption: $Q_1 \cup (Q_1 \cap Q_2) = Q_1, Q_1 \cap (Q_1 \cup Q_2) = Q_1$.
3. $(Q_1^C)^C = Q_1$.
4. $Q_1 \cap Q_2 \subset Q_1$ *and* $Q_1 \cap Q_2 \subset Q_2$.
5. $Q_1 \subset Q_1 \cup Q_2$ *and* $Q_2 \subset Q_1 \cup Q_2$.
6. $If Q_1 \subset Q_2$ *and* $Q_2 \subset Q_3$ *then* $Q_1 \subset Q_3$.
7. $If Q_1 \subset Q_2$ *then* $Q_1 \cap Q_3 \subset Q_2 \cap Q_3$ *and* $Q_1 \cup Q_3 \subset Q_2 \cup Q_3$.

Proof. By using Theorem 1 and definitions of Quadrasophic Fuzzy Set, the results are obvious. ■

Generalization of Intersection and Union: Let X be a non- empty set and let $(Q_s)_{s\in Q} \subset Q$.
i) The intersection of $(Q_s)_{s\in Q}$ denoted by $(\cap_{s\in Q} Q_s)$ in a Quadrasophic fuzzy set is defined as,

$$(\cap_{s\in Q} Q_s)(x) = \{max_{s\in Q}\,\eta_s(x), max_{s\in Q}\,\lambda_{\eta s}(x), \\ min_{s\in Q}\,\lambda_{\mu s}(x), min_{s\in Q}\,\mu_s(x)\}, \forall x \in X.$$

ii) The union of $(Q_s)_{s\in Q}$ denoted by $(\cap_{s\in Q} Q_s)$ in a Quadrasophic fuzzy set is defined as,

$$(\cup_{s\in Q} Q_s)(x) = \{min_{s\in Q}\,\eta_s(x), min_{s\in Q}\,\lambda_{\eta s}(x), \\ max_{s\in Q}\lambda_{\mu s}(x), max_{s\in Q}\,\mu_s(x)\}, \forall x \in X.$$

Generalization of Laws: Let $(Q_s)_{s\in Q} \subset Q$ be defined in the non empty set X,
i) Generalization of Distributive laws:

$$Q \cap (\cup_{s\in Q} Q_s)(x) = \cup_{s\in Q}(Q \cap Q_s)$$

ii) Generalization of De-Morgan's law:

$$(\cup_{s\in Q} Q_s)^C(x) = (\cap_{s\in Q} Q_s^C)$$
$$(\cap_{s\in Q} Q_s)^C(x) = (\cup_{s\in Q} Q_s^C).$$

Measures of Distance: **1.**The normalized Hamming distance between any QFS set $Q_1, Q_2 \in Q(x)$ is defined as,

$$d_{Qh}(Q_1, Q_2) = \frac{1}{2n}\sum_{i=1}^{n}[\left|\eta_{Q_1}(x_i))^2 - (\eta_{Q_2}(x_i))^2\right| + \left|(\lambda_{\eta_{Q_1}}(x_i))^2 - (\lambda_{\eta_{Q_2}}(x_i))^2\right| \\ + \left|(\lambda_{\mu_{Q_1}}(x_i))^2 - (\lambda_{\mu_{Q_2}}(x_i)))^2\right| + \left|(\mu_{Q_1}(x_i))^2 - \mu_{Q_2}(x_i))^2\right|].$$

2. The normalized Euclidean distance between any QFS set $Q_1, Q_2 \in Q(x)$ is defined as,

$$d_{Qh}(Q_1, Q_2) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}[\eta_{Q_1}(x_i))^2 - (\eta_{Q_2}(x_i))^2]^2 + [](\lambda_{\eta_{Q_1}}(x_i))^2 - (\lambda_{\eta_{Q_2}}(x_i))^2]^2} \\ \overline{+[(\lambda_{\mu_{Q_1}}(x_i))^2 - (\lambda_{\mu_{Q_2}}(x_i)))^2]^2 + [(\mu_{Q_1}(x_i))^2 - \mu_{Q_2}(x_i))^2]^2}$$

Quadrasophic Fuzzy Relation: A subset of the Quadrasophic fuzzy set $X \times Y$ is the Quadrasophic fuzzy relation R represented by $R = \{(x, y,), \eta_R(x, y), \lambda_{\eta_R}(x, y), \lambda_{\mu_R}(x, y), \mu_R(x, y) | x \in X, y \in Y\}$ where,

$$\eta_R(x) : X \to [-1, 0]$$
$$\lambda_{\eta R}(x) : X \to [-0.5, 0]$$
$$\lambda_{\mu R}(x) : X \to [0, 0.5]$$
$$\mu_R(x) : X \to [0, 1]$$

satisfy the conditions for all $(x, y) \in (X \times Y)$, $-1 \leq \mu_R(x) + \eta_R(x) \leq 1$, $-0.5 \leq \lambda_R \leq 0.5$ and $0 \leq \mu_R^2 + \eta_R^2 + \lambda_R^2 \leq 3$ where $\lambda_R =$ Length of $(\lambda_{\mu R}, \lambda_{\eta R})$. Let $QFR(X \times Y)$ denotes the set of all Quadrasophic fuzzy relation on X.

Max-Min-Max Function: If $Q_1, Q_2 \in QFS(X)$ are two QFR and $Q_1(x \to y)$, $Q_2(y \to z)$. Then, max-min-max composition as $Q_1 \circ Q_2$ from x to z, whose membership functions are represented as follows:
$\eta_{Q_2 \circ Q_1}(x, z) = \wedge_y \, max[\eta_{Q_1}(x, y), \eta_{Q_2}(y, z)]$
$\lambda_{\eta_{Q_2 \circ Q_1}}(x, z) = \wedge_y \, max[\lambda_{\eta_{Q_1}}(x, y), \lambda_{\eta_{Q_2}}(y, z)]$
$\lambda_{\mu_{Q_2 \circ Q_1}}(x, z) = \vee_y \, min[\lambda_{\mu Q_1}(x, y), \lambda_{\mu_{Q_2}}(y, z)]$
$\eta_{Q_2 \circ Q_1}(x, z) = \vee_y \, min[\mu_{Q_1}(x, y), \mu_{Q_2}(y, z)]$
$\forall x \in X, y \in Y$, $(x, y) \in X \times Y$ where $\wedge = min$, $\vee = max$.

Score value function: Let $Q = (x, \eta(x), \lambda_\eta(x), \lambda_\mu(x), \mu(x))$ be the QFS in$X$. Then, the score- valued function $sv(Q)$ of Q is defined as $sv(Q) = \frac{\mu(x) + \lambda_\mu(x) + \eta(x) + \lambda_\eta(x)}{3}$, where $sv(Q) \in [-1, 1]$.

4.1. Advantages of the Quadrasophic Fuzzy model

The following Table 1 shows the assessment and advancement of the proposed model with respect to the existing models of fuzzy set.

Table 1: *Extensions of Fuzzy theoretical set with QFS assessment level*

Grade\Theoretical Set	Satisfaction	Neutral	Dissatisfaction	Bipolarity	Restricted Bipolarity
Fuzzy set	Yes	-	-	-	-
Bipolar Fuzzy set	Yes	-	-	Yes	-
Picture Fuzzy set	Yes	Yes	Yes	-	-
Quadrasophic Fuzzy set	Yes	-	Yes	Yes	Yes

Table 2: *Patient and symptom relational values in terms of QFS*

Q_1	δ_1	δ_2	δ_3	δ_4	δ_5
a_1	(-0.1,-0.1, 0.4,0.8)	(-0.1,-0.1, 0.3,0.6)	(-0.8,-0.4,0.1,0.2)	(-0.1,-0.1,0.3,0.6)	(-0.6,-0.3,0.1,0.1)
a_2	(-0.8,-0.4,0,0)	(-0.4,-0.2,0.2,0.4)	(-0.1,-0.1,0.3,0.6)	(-0.7,-0.4,0.1,0.1)	(-0.8,-0.4,0.1,0.1)
a_3	(-0.1,-0.1,0.4,0.8)	(-0.1,-0.1,0.4,0.8)	(-0.6,-0.3,0,0)	(-0.7,-0.4,0.1,0.2)	(-0.5,-0.3,0,0)
a_4	(-0.1,-0.1,0.3,0.6)	(-0.4,-0.2,0.3,0.5)	(-0.4,-0.2,0.2,0.3)	(-0.4,-0.2,0.2,0.3)	(-0.4,-0.2,0.2,0.3)

5. Test and comparison analysis

This segment provides an application of the Quadrasophic fuzzy set in medical diagnosis. Several authors have done their research work in medical diagnosis with various extensions of the fuzzy set. The medication deals with the environment of ambiguity. In addition, the Quadrasophic Fuzzy Set includes the impact of the environment as one of its membership values,

Table 3: *Symptoms and diseases relational values in terms of QFS*

Q_2	t_1	t_2	t_3	t_4	t_5
δ_1	(0, 0, 0.2,0.4)	(0,0,0.4,0.7)	(-0.3,-0.2, 0.2,0.3)	(-0.7,-0.4 ,0.1,0.1)	(-0.8,-0.4 ,0.1,0.1)
δ_2	(-0.7,-0.4, 0.1,0.1)	(-0.9,-0.5, 0,0)	(-0.7,-0.4,0.1,0.2)	(0,0,0.4,0.8)	(-0.8,-0.4, 0.1,0.2)
δ_3	(-0.3,-0.2, 0.2,0.4)	(0,0,0.4,0.7)	(-0.6,-0.3,0.1,0.2)	(-0.7,-0.4,0.1,0.2)	(-0.8,-0.4,0.1,0.2)
δ_4	(-0.7,-0.4,0.1,0.1)	(-0.8,-0.4, 0.1,0.1)	(-0.9,-0.5, 0.1,0.1)	(-0.7, -0.4, 0.1, 0.2)	(-0.1,-0.1, 0.4,0.8)

Table 4: *Relational values of patient and diseases in terms of QFS*

Q_3	t_1	t_2	t_3	t_4	t_5
a_1	(-0.7,- 0.4, 0.2,0.4)	(-0.8,-0.4, 0.4,0.7)	(-0.7,-0.4, 0.3,0.6)	(-0.6,-0.3 ,0.1,0.2)	(-0.8,-0.4 ,0.1,0.2)
a_2	(-0.7,-0.4, 0.2,0.3)	(-0.8,-0.4,0.2,0.2)	(-0.8,-0.4, 0.2,0.4)	(-0.7,-0.4, 0.3,0.6)	(-0.8,-0.4, 0.1,0.1)
a_3	(-0.6,-0.3, 0.2,0.4)	(-0.6,-0.3, 0.4,0.7)	(-0.6,-0.3,0.3,0.6)	(-0.7,-0.4, 0.1,0.2)	(-0.7,-0.4, 0.1,0.2)
a_4	(-0.4,-0.2, 0.2,0.4)	(-0.4,-0.2, 0.4,0.7)	(-0.4,-0.2, 0.3,0.5)	(-0.4,-0.2, 0.2,0.3)	(-0.4,-0.2,0.2,0.3)

which will aid in determining the best result.

Now, consider the database in medical analysis [9], [10] and will solve using the Quadrasophic Fuzzy set. Suppose four patients $a_i = \{Sanjeev - a_1, Sam - a_2, Sarjesh - a_3, Sarath - a_4\}$ affected with the disease, whose symptoms are $\delta_i = \{Temperature - \delta_i, Headache - \delta_2, StomachPain - \delta_3, Cough - \delta_4, ChestPain - \delta_5\}$. Consequently, the collection of ailments that the medical advisor specified is $t_i = \{Viral fever - t_1, Malaria - t_2, Typhoid - t_3, StomachProblem - t_4, Heart problems - t_5\}$. The relation $Q_1(a_i \to \delta_i)$ between patients and symptoms and the relation $Q_2(\delta_i \to t_i)$ between symptoms and illness is represented in Table 2 and 3. The Quadrasophic fuzzy relation

Table 5: *Ranking value of patient and diseases*

Q_3	t_1	t_2	t_3	t_4	t_5
a_1	0.0667	**0.2667**	0.1667	-0.1	0
a_2	0.033	0.033	0.1	**0.1667**	0.033
a_3	0	**0.1667**	0.1	-0.033	-0.033
a_4	-0.1	**0.0667**	-0.033	-0.133	-0.133

of compositional value $Q_1 \circ Q_2$ is represented in Table 4. The Quadrasophic fuzzy relation of compositional value is represented in Table 4.

$$\Re = \frac{(-1 - \eta_Q(a_i, t)) + (-0.5 - \lambda_{\eta Q}(a_i, t)) + \mu_Q(a_i, t) + \lambda_{\mu Q}(a_i, t)}{3}$$

is the formulation to find the rank value, which is presented in Table 5.

It is clear that Sajeev, Sarjesh and Sarath are suffering from Malaria and Sam is suffering from Stomach problem.

5.1. Similarity Test

To corroborate, the Quadasophic Fuzzy Set method gives accurate results than the existing methods. We conduct the similarity test, and the results of various extensions of the existing fuzzy set model are presented in the following Table 6.

The results obtained in QFS are identical with the existing results and also relatively accurate compared to the values obtained by the other existing methods. In addition, taking the

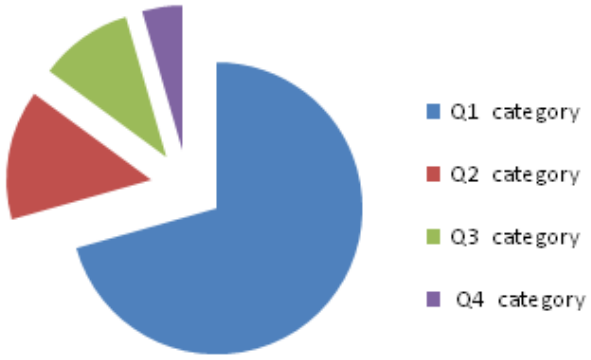

Figure 1: *Division of OCD Category*

reluctant rate into account in QFS yields a negative ranking, which indicates a person's deficiency rate. Based on this observation, the proposed method's verification yields better and more precise results than the existing method.

Table 6: *Comparative Analysis results*

Fuzzy set Environment	Results
Medical Diagnosis under IFS [9]	Malaria : a_1, a_3, a_4 Stomach problem: a_2 $a_1, a_3, a_4 = 0.68$ and $a_2 = 0.57$
Medical Diagnosis under Bipolar valued fuzzy sets [10]	Malaria: a_1, a_3, a_4 Stomach problem: a_2 $a_1 = 1.25, a_3 = 1.15, a_4 = 1.05,$ and $a_2 = 1.15$
QFS method [3]	Malaria: a_1, a_3, a_4 Stomach problem: a_2 $a_1 = 0.2667, a_3 = 0.1667, a_4 = 0.0667,$ and $a_2 = 0.1667$

6. Assessing stress level and self-esteem connection with real-life data using QF-TOPSIS method

Obsessive Compulsive Disorder (OCD) is a condition characterized by repetitive actions due to unnecessary thoughts and fears. OCD is a disorder characterized by repetitive cleaning, arranging, and washing actions, often unknowingly. It affects 4 out of 100 people in India and can be caused by genetics, brain abnormalities, or the environment. The exact cause is uncertain, but the environment can increase or decrease OCD levels, leading to emotional impairments and increased stress, exacerbating the condition.The environment plays a significant role in this disorder, and stress is a significant factor.

To examine the stress factor triggers OCD disorder, a survey is carried out among Tamil Nadu students and working persons to determine stress levels, self-esteem and the influence of surroundings on mental health. The survey contains questions, related to OCD subcategories like cleaning, arranging, washing, and checking. The data is categorized into four groups based on the different categories, with the percentages of each category at normal and abnormal rates depicted in Figures 1 and 2 respectively. The Quadrasophic Fuzzy Set simplifies the investigation of OCD. The data is categorized as follows:

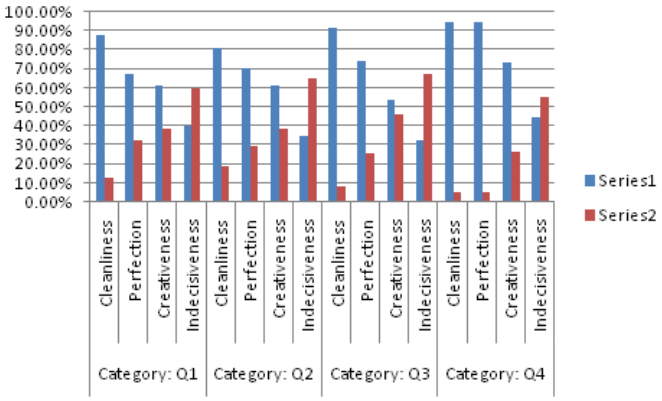

Figure 2: *Representation of category report for the OCD survey*

$\eta-$ represents the level of abnormal behavior
$\lambda_\eta-$stress level from the environment
$\lambda_\mu-$ self esteem level
$\mu-$ represents the level of normal behavior.

The TOPSIS [1] [2] approach is integrated with Quadrasophic Fuzzy data to identify the most OCD-affected category of people based on specific criteria in multi-criteria decision-making. The set of alternatives $Q_i = \{Q_1, Q_2, Q_3, Q_4\}$ represents individuals with different self-esteem levels, with Q_1 representing high-self-esteem individuals surrounded by high-self-esteem people, Q_2 representing high-self-esteem people surrounded by low-self-esteem people , Q_3 representing low-self-esteem individuals surrounded by high-self-esteem people, and Q_4 representing low-self-esteem individuals surrounded by low-self-esteem people.

The collection of criteria $C_i = \{C_1, C_2, C_3, C_4\}$ where C_1 represents the level of cleanliness, C_2 represents the level of perfection, C_3 indicates the level of creativeness, and C_4 indicates the level of indecisiveness. The weight vector of Q_j is $P_K \in [0,1]$ and $\sum_{j=1}^{n} P_j = 1$. In this instance, the weight vector is $(0.3, 0.3, 0.2, 0.2)$.

Algorithm QF-TOPSIS Method:

Step 1: Evaluate the Quadrasophic Decision Matrix Q_{ij} for the specified condition relating to the given alternatives.
Step 2: Normalize Q_{ij}, and use score value definition to calculate score function.
Step 3: Using the values from Step 2, calculate the $QFPIS\ (X_i^{\rightarrow +})$ and $QFNIS\ (X_i^{\rightarrow +})$ using [2] .
$QFPIS : X^{\rightarrow +} = \{Q_j, max(\, sv(\, Q_j(\, x_{iw})\,)\,)\,/j = 1, 2, \ldots, n\}$

$$where,\ X^{\rightarrow +} = \{Q_1(\,\eta_1^{\rightarrow +}(x)\,, \lambda_{\eta 1}^{\rightarrow +}(x)\,, \lambda_{\mu 1}^{\rightarrow +}(x)\,, \mu_1^{\rightarrow +}(x)\,)\,, \ldots,$$
$$Q_n(\,\eta_1^{\rightarrow +}(x)\,, \lambda_{\eta n}^{\rightarrow +}(x)\,, \lambda_{\mu n}^{\rightarrow +}(x)\,, \mu_n^{\rightarrow +}(x)\,)\,\}$$

$QFNIS : X^{\rightarrow -} = \{Q_j, min(\, sv(\, Q_j(\, x_{iw})\,)\,)\,/j = 1, 2, ?, 4\}$

$$where, X^{\rightarrow -} = \{Q_1(\,\eta_1^{\rightarrow -}(x)\,, \lambda_{\eta 1}^{\rightarrow -}(x)\,, \lambda_{\mu 1}^{\rightarrow -}(x)\,, \mu_1^{\rightarrow -}(x)\,)\,, \ldots,$$
$$Q_n(\,\eta_1^{\rightarrow -}(x)\,, \lambda_{\eta n}^{\rightarrow -}(x)\,, \lambda_{\mu n}^{\rightarrow -}(x)\,, \mu_n^{\rightarrow -}(x)\,)\,\}$$

Step 4: Determine the distance between categories (Q_i) and QFPIS $(X_i^{\rightarrow +})$, QFNIS $(X_i^{\rightarrow -})$ using

definition 4 .

$$d(Q_i, X_i^{\rightarrow +}) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}[(\eta_{Q_i}(x_i))^2 - (\eta_{X_i}(x_i^{\rightarrow +}))^2]^2 + [(\lambda_{\eta_{Q_i}}(x_i))^2 - (\lambda_{\eta_{X_i}}(x_i^{\rightarrow +}))^2]^2} \overline{+[(\lambda_{\mu_{Q_i}}(x_i))^2 - (\lambda_{\mu_{X_i}}(x_i^{\rightarrow +}))^2]^2 + [(\mu_{Q_i}(x_i)^2 - (\mu_{X_i}(x_i^{\rightarrow +}))^2]^2}$$

$$d(Q_i, X_i^{\rightarrow -}) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}[(\eta_{Q_i}(x_i))^2 - (\eta_{X_i}(x_i^{\rightarrow -}))^2]^2 + [(\lambda_{\eta_{Q_i}}(x_i))^2 - (\lambda_{\eta_{X_i}}(x_i^{\rightarrow -}))^2]^2} \overline{+[(\lambda_{\mu_{Q_i}}(x_i))^2 - (\lambda_{\mu_{X_i}}(x_i^{\rightarrow -}))^2]^2 + [(\mu_{Q_i}(x_i)^2 - (\mu_{X_i}(x_i^{\rightarrow -}))^2]^2}$$

Step 5: Apply the following formula, to obtain the coefficient of closeness cc(Q) [2].

$$cc(Q) = d(Q_i, X_i^{\rightarrow -}) / [[d(Q_i, X_i^{\rightarrow -}) + d(Q_i, X_i^{\rightarrow +})]$$

Step 6: Using the values from Step 5, rank the category, with the smallest rank indicating the beneficial category. This allows us to identify the people who are most affected by OCD causes.

6.1. Illustration of the QF-TOPSIS method

Step 1: The Q_{ij} matrix is shown in Table 7.

Table 7: *Quadrasophic Decision Matrix*

DM	*Cleanliness*	*Perfection*	*Creativeness*	*Indecisiveness*
Q_1	(-0.201,-0.319, 0.405,0.638)	(-0.327,-0.319, 0.405, 0.538)	(-0.389,-0.319, 0.405, 0.244)	(-0.598,-0.319, 0.405,0.161)
Q_2	(-0.188,-0.3, 0.393,0.65)	(-0.297,-0.3, 0.393,0.562)	(-0.385, -0.3, 0.393, 0.246)	(-0.671, ,-0.3, 0.393, 0.13)
Q_3	(-0.087,-0.284, 0.332,0.73)	(-0.26,-0.284, 0.332,0.591)	(-0.46, ,-0.284, 0.332, 0.214)	(-0.673, ,-0.284, 0.332, 0.13)
Q_4	(-0.05,-0.285, 0.24,0.76)	(-0.05, -0.285, 0.24, 0.76)	(-0.67,-0.285, 0.24,0.133)	(-0.55,-0.285, 0.24,0.18)

Step 2: Table 8 gives the scoring function for the normalized Q_{ij}.
Step 3: Table 8 highlights the highest and lowest values used to determine the $QFPIS\ (X_i^{\rightarrow +})$,

Table 8: *Score function of QFS*

$sv(Q)$	*Cleanliness*	*Perfection*	*Creativeness*	*Indecisiveness*
Q_1	**0.1743**	**0.02966**	**-0.0477**	-0.00567
Q_2	0.0103	0.0173	-0.0587	**0.0033**
Q_3	-0.0223	-0.011	**-0.07**	-0.027
Q_4	**-0.055**	**-0.055**	-0.0573	**-0.0817**

and $QFNIS\ (X_i^{\rightarrow +})$.
Step 4: The Table 9 displays the distance measure values of $d(Q_i, X_i^{\rightarrow +})$ and $d(Q_i, X_i^{\rightarrow -})$.
Step 5: Table 10 indicates the $cc(Q)$ value.
Step 6: Use the cc(Q) values to rank the category. Thus, $Q_4 < Q_3 < Q_2 < Q_1$.

In addition to heredity and brain abnormalities, the environment and psychological stress play a crucial role in the development of OCD problems. Such brain and genetic defects cannot be fixed. However, maintaining a healthy environment can help live in harmony. Quadrasophic Fuzzy Sets are implemented in MCDM-TOPSIS techniques to find the most appropriate solution.

Correlations

			Self-esteem	environmental_self esteem:	Impact_environment	collecting_useless things	repeated_checking	repetition_routineaction	irritated_objects	checking_light_switches
Spearman's rho	Self-esteem	Correlation Coefficient	1.000	.356**	.176**	0.126	0.032	0.114	.170*	.181**
		Sig. (2-tailed)		0.000	0.009	0.062	0.638	0.092	0.011	0.007
		N	221	221	221	221	221	221	221	221
	Environmental_self esteem:	Correlation Coefficient		1.000	.201**	0.092	-0.014	0.045	0.078	0.127
		Sig. (2-tailed)			0.003	0.174	0.834	0.502	0.250	0.059
		N		221	221	221	221	221	221	221
	Impact_environment	Correlation Coefficient			1.000	0.081	0.033	0.008	-0.044	0.021
		Sig. (2-tailed)				0.231	0.623	0.903	0.514	0.761
		N			221	221	221	221	221	221
	collecting_usel essthings	Correlation Coefficient				1.000	.344**	0.131	.136*	0.106
		Sig. (2-tailed)					0.000	0.051	0.044	0.115
		N				221	221	221	221	221
	repeated_checking	Correlation Coefficient					1.000	0.095	.240**	.242**
		Sig. (2-tailed)						0.161	0.000	0.000
		N					221	221	221	221
	repetition_routineaction	Correlation Coefficient						1.000	.175**	.141*
		Sig. (2-tailed)							0.009	0.036
		N						221	221	221
	irritated_objects	Correlation Coefficient							1.000	.505**
		Sig. (2-tailed)								0.000
		N							221	221
	checking_light_switches	Correlation Coefficient								1.000
		Sig. (2-tailed)								
		N								221

**. Correlation is significant at the 0.01 level (2-tailed).
*. Correlation is significant at the 0.05 level (2-tailed).

Figure 3: *Correlations*

Table 9: *Distance measures between QFPIS, QFNIS and, Qj*

Distance between Q_j and $QFPIS$	Distance between Q_j and $QFNIS$
0.0067	0.0398
0.0075	0.0392
0.0188	0.0291
0.104	0.0173

Table 10: *Value of cc(Q)*

The values of $cc(Q)$	
Q_1	0.855913
Q_2	0.8394
Q_3	0.6075
Q_4	0.1426

The data indicates that the Q_4 group experiences increased stress, which in turn triggers OCD. The Q_4 category is greatly impacted by the environment. Additionally, the survey recommends that living in a conducive environment is crucial for OCD-free lives.

7. Analysis of Quadrasophic Fuzzy Data using SPSS software

A SPSS software is used for processing the collected data for statistical evaluation. Figure 3 displays Spearman's rho correlation coefficients among several variables, such as individual's self-esteem, environmental self-esteem, other behavioral and emotional metrics. A significant positive correlation ($r = 0.350, p < 0.01 \& r = 0.350, p < 0.01 \& r = 0.350, p < 0.01$) was found between environmental factors and high self-esteem. Q_1 category and environmental factors are positively correlated, but environmental factor is negatively associated with certain behaviors.

Figure 4 indicates that the Environment is a significant predictor, accounting for 31.2% of Q_4, with an R-value of 0.558 (indicating a moderate correlation). However, 68.8% of the variance remains unexplained, suggesting that other factors may also influence Q_4 individuals.

The ANOVA results shown in Figure 5 indicate that the environmental factor significantly influences the variation in the dependent variable, Q_4 category, with a significant F-value of 44.389 and a p-value of 0.000.

In Figure 6, beta (standardized coefficient) of -0.558 indicates a moderately strong negative impact of the environment on the Q4 category. The t-value is -6.662, and the p-value is 0.000, suggesting a strong correlation between changes in the environment and changes in Q_4.

Outcome of the study: The study reveals that environmental factors significantly impact

Model Summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.558[a]	.312	.305	4.01317

a. Predictors: (Constant), Environ

Figure 4: *Model Summary*

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	714.901	1	714.901	44.389	.000[b]
	Residual	1578.339	98	16.105		
	Total	2293.240	99			

a. Dependent Variable: LL

b. Predictors: (Constant), Environ

Figure 5: *ANOVA result*

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	3.366	.434		7.751	.000
	Environ	-.485	.073	-.558	-6.662	.000

a. Dependent Variable: LL

Figure 6: *Coefficients Value*

the Q_4 category, with a negative impact on it and a significant positive association with the Q_1 category. The environment factor accounts for 31.2% of the Q_4 category, indicating the existence of other variables and similar results between SPSS and QFS. Social Environment self-esteem and the environmental effect have been associated with self-esteem. The research highlights the link between environmental stressors and emotional deficiencies, leading to OCD. It emphasizes the importance of QFS in incorporating environmental factors to achieve the most appropriate outcome.

8. Conclusion

This artifact defines the operations and properties of the Quadrasophic Fuzzy Set (QFS), including the distance measure, QFR (Quadrasophic Fuzzy Relation), score function, and composition functions. The Quadrasophic Fuzzy Relation is applied in a comparative analysis to validate this novel fuzzy set extension. The QFS max-min composition is effectively utilized in solving decision-making (DM) problems. Additionally, the integration of QFS data with the TOPSIS approach is demonstrated for solving multi-criteria decision-making (MCDM) problems. The QF-TOPSIS method is employed to address an OCD analysis problem, with its novel membership functions highlighting the influence of environmental factors on stress and OCD. SPSS analysis confirms that QFS is highly effective in investigating additional factors, including environmental impact, to achieve accurate outcomes.

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BAYESIAN GLM: A NON-INFORMATIVE APPROACH FOR PARAMETER ESTIMATION IN EXPONENTIAL DISPERSION REGRESSION MODELS

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Abstract

This paper proposes a novel Bayesian approach to parameter estimation in exponential dispersion regression models (EDRM). By employing a non-informative prior distribution, we offer a flexible and robust framework that avoids the need for subjective prior specification. To efficiently sample from the posterior distribution, we develop an importance-sampling algorithm tailored to the EDRM. Through a real-world data analysis, we demonstrate the efficacy of our proposed method in providing accurate and reliable parameter estimates. This research contributes to the advancement of Bayesian statistical modeling techniques and offers valuable insights for practitioners in various fields.

Keywords: Generalized linear models, Bayesian method, Multivariate exponential dispersion, Non-informative prior distribution, Real-world data analysis

1. Introduction

Statistical modelling plays a crucial role in decision-making as it enables the representation of relationships between variables, whether they are linear or non-linear. To establish the connections between observed responses, y_i, and corresponding covariates, x_i, regression models are initially developed. The Ordinary Least Squares (OLS) approach is frequently utilized to estimate unknown parameters, β_i, under the assumption that the response variable adheres to a normal distribution. However, in reality, the normality assumption for residuals may be violated, which leads to the consideration of other exponential family distributions (refer to [25, 26]). In such cases, the use of alternative approaches becomes necessary as OLS estimates may be inaccurate. The Generalized Linear Model (GLM), introduced by J. A. Nelder and R.W. Weddernburn [20], accommodates non-normal distributions of response variables that adhere to exponential family distributions including Poisson, binomial, negative binomial, inverse Gaussian, and gamma distributions. For additional information and practical examples, refer to [13, 14, 19].

The GLM has been extensively studied by researchers for several exponential family distributions. In the health sciences, these models have diverse applications, such as predicting the effect of animal age on dried eye lens weight [23], estimating the prevalence of renal failure based on various parameters (see [32]), modelling lifetime data (see [21]), and addressing transportation challenges (see [34]). As hydrological variables like rainfall and rain-off are inherently positive,

the gamma distribution has proven to be effective in hydrology, as it can model only positive values (as discussed in [18]).

In analytical applications, GLM is utilized to model the relationships between a large number of responses and a single set of predictor factors, as described in ([22]). The GLM typically involves several unknown parameters that are unique to the population. To estimate these parameters, two statistical procedures are commonly used, as noted by W. M. Bolstad and J. M. Curran [3]. The first approach is the conventional technique that relies on all the information obtained from the random sample. A common alternative approach to estimating unknown parameters in GLM is the Bayesian method, which incorporates prior information along with the data from a random sample. The posterior distribution is then derived by combining the likelihood function with the prior distribution, as outlined in standard Bayesian methodologies (see [6], [27, 28]). The choice of prior distribution can significantly influence the posterior estimates, particularly when informative priors are employed. However, it is crucial to note that the use of informative priors is subjective and can have a significant impact on the posterior distribution (see [9]). In situations where limited or no prior knowledge about the parameters is available, non-informative priors can be used. These priors are designed to have minimal influence on the posterior distribution, allowing the data to primarily drive the inference. Non-informative priors, also known as vague or weakly informative priors, are used when we aim to reflect ignorance or neutrality about the parameters prior to observing the data (see [1]).

J. O. Berger and D. Sun [2] conducted research on the types of non-informative prior distributions that could be utilized to enhance the accuracy of normal multivariate models. C. P. Robert *et, al.* [24] provided a comprehensive and contemporary review of Jeffrey's prior distribution.

A study by A. A. I. A. Iswari *et, al.* [14] involved a simple linear regression analysis and the computation of credible intervals for the regression parameters based on simulated data where the prior distribution was not known to the researchers. Additionally, an important contribution to the field is the multivariate multiple linear regression framework, which is a combination of modelling techniques utilizing a Jeffrey's prior distribution [30].

In this paper, we introduce a method for characterizing the GLM using exponential dispersion models, which we refer to as the Exponential Dispersion Regression Models (EDRM). The EDRM represents a rich subclass of the well-known exponential family. Furthermore, we examine the estimation of parameters in the EDRM through maximum likelihood estimation (MLE) and non-informative Bayesian estimation.

This paper is structured as follows: Section 2 details the exponential dispersion regression models. In Section 3, we describe the maximum likelihood estimation and non informative Jeffrey's prior for EDRM. Section 4 is dedicated to providing numerical illustrations, emphasizing the practical application and demonstration of the concepts discussed in earlier sections. Finally, in the last section, we present the conclusion and discussion.

2. Exponential Dispersion Regression Models (EDRM)

In this section, we first describe briefly the exponential dispersion models (EDM). Then, we discuss the EDRMs in details.

2.1. Exponential dispersion models

In the upcoming section, we will delve into the essential features of Exponential Dispersion Models (EDMs) - a noteworthy subset of the renowned exponential family of functions (see [15]). EDMs encompass distinguished distributions such as the inverse Gaussian, gamma, and the normal distribution, to name a few.

EDMs expand upon the concepts of Natural Exponential Families, which offer an extensive range of possibilities, as elucidated in [13]. The probability density function for EDMs is defined as:

$$f(y;\mu,\lambda) = e^{\lambda[\theta y - K_\nu(\theta)]} c(y,\lambda), \quad y \in \mathbb{R} \tag{1}$$

where λ (dispersion parameter) and θ (canonical parameter), with domain $(\lambda, \theta) \in \Lambda \times \Theta \subseteq \mathbb{R}_+ \times \mathbb{R}$. In (1), $K_\nu(\theta) = \log \int_{\mathbb{R}} e^{\theta x} \nu(dx)$ (cumulant function) is a known function of a generating probability measure ν and $c(y, \lambda)$ is a normalizing constant that ensures (1) is a probability function.

For EDMs we have some well-known relations, if $Y \sim f(.; \mu, \lambda)$, then $\mu = \mathbb{E}(Y) = K'_\nu(\theta)$ is the expectation of (1) due to the relationship/map between θ and μ. The variance of (1) is $Var(Y) = \frac{1}{\lambda} V(\mu)$ and $V(\mu)$ being the variance function which uniquely corresponds to an exponential dispersion model. Define $\psi_\nu(\mu) = (K'_\nu(\theta))^{-1}$ and $V(\mu) = K''_\nu(\psi_\nu(\mu))$. It can also be shown that when the functions $K_\nu(.)$ and $c(.,.)$ as well as ψ_ν are fixed, the subfamily arising by taking different θ consists of elements that are all Esscher-transforms of each other. A family with K_ν, c and θ fixed and varying ψ_ν can be generated by the operation of taking sample means. For further information, we refer the reader to [7]).

In Table 1, we present necessary details of absolutely continuous PDFs of the EDM family specifying the normalizing constant ($c(y, \lambda)$), the cumulant function (K_ν), canonical parameter (θ), dispersion parameter (λ), mean (K'_ν), inverse function of the mean (ψ_ν) and variance function (V) of each distribution (see [33, 16]).

Table 1: *Examples of some absolutely continuous PDF of EDMs.*

	Gaussian	**Gamma**	**Inverse Gaussian**	**Laplace**
$c(y, \lambda)$	$\frac{\sqrt{\lambda}}{\sqrt{2\pi}} e^{-\frac{\lambda y^2}{2}}$	$\frac{\lambda^\lambda y^{\lambda-1}}{\Gamma(\lambda)}$	$\frac{\sqrt{\lambda}}{\sqrt{2\pi}} y^{-\frac{3}{2}} e^{-\frac{\lambda}{2y}}$	$\frac{\lambda e^{\lambda y}}{\Gamma(\lambda)^2} \int_{\lambda y}^{+\infty} e^{-2t} t^{\lambda-1} (t - \lambda y)^{\lambda-1} dt$
K_ν	$\frac{\theta^2}{2}$	$-\log(-\theta)$	$-\sqrt{-2\theta}$	$-\log(1-\theta^2)$
K'_ν	θ	$-\frac{1}{\theta}$	$(-2\theta)^{-1/2}$	$\frac{2\theta}{1-\theta^2}$
ψ_ν	μ	$-\frac{1}{\mu}$	$-\frac{1}{2\mu^2}$	$\frac{\left\lvert\sqrt{1+\mu^2}-1\right\rvert}{\mu}$
V	1	μ^2	μ^3	$\frac{\left\lvert\sqrt{1+\mu^2}-1\right\rvert}{\mu^2\sqrt{1+\mu^2}}$

2.2. Description of EDRM

J. A. Nelder and R. W. Wedderburn [20] introduced Generalized Linear Models (GLMs) as a unified framework for handling a variety of commonly used statistical models, including multiple linear regression and log-linear models, for both normally and non-normally distributed data. GLMs are highly versatile, making them suitable for a wide range of models in actuarial statistics, while being structured in a way that allows a single algorithm to be used for maximum likelihood estimation across all of these models (see [19]). GLMs are more helpful in actuarial statistics than ordinary multiple regression, since apart from normal distributions, GLMs explicitly allow Poisson, binomial, gamma and some other useful error distributions. Also, GLMs allow linearity on other scales than the identity scale (logarithmic, logit, probit, reciprocal and others).

The GLMs are described by a link and variance functions further with the selection of a response variable and a collection of explanatory variables. The link function transforms the mean $\mu = (\mu_1, ..., \mu_N)^\top$ of the response variable $Y = (y_1, ..., y_N)^\top$ into a scale where the model is linear. In fact, each response variable y_i is assumed to follow its own regression model, so that

$$y_i = \beta_0 + \sum_{j=1}^{q} \beta_j x_{i,j} + \varepsilon_i \quad i = 1, ..., N, \tag{2}$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{iq})^\top$ be a vector of predictors or covariates, and $\beta = (\beta_1, \beta_2, ..., \beta_q)$ is a vector of unknown parameters of adjacent regression coefficients, and ε_i is a random variables with mean zero and variance σ^2.

In a GLM, we assume that the response variable follows a distribution from the EDMS, satisfying:

$$\psi(\mu_i) = \beta_0 + \beta_1 x_{i1} + ... + \beta_q x_{iq}, \quad i = 1, ..., N \tag{3}$$

where ψ denotes the link function. For simplicity, equation (3) may be also written as $\psi(\mu_i) = x_i^\top \beta = \eta_i$. In what follows, we analyse a data set, $(y_i; x_i) \quad i = 1, ..., N$ with each y_i follows the density function $f(.; \mu, \lambda)$ described in equation (1) with mean and variance as follows

$$\mathbb{E}(\log y_i) = \mu_i = K'_\nu(x_i^\top \beta) = \psi_\nu^{-1}(x_i^\top \beta)$$
$$Var(\log y_i) = \frac{1}{\lambda} K''_\nu(x_i^\top \beta) = \frac{1}{\lambda} V(\mu_i)$$

The multivariate extension relies on the deviance residual, denoted as $r(y, \mu) = \pm\sqrt{d(y, \mu)}$, where $\pm = \text{sgn}(y - \mu)$ and the function d is assumed to be a unit deviance with the property $d(\mu, \mu) = 0$ for $\mu \in \Omega$ (an interval), and $d(y, \mu) > 0$ for $y = \mu$. It is assumed throughout that $d(, \mu)$ is continuous and strictly monotone on each side of μ, implying that $r(, \mu)$ is strictly increasing for each $\mu \in \Omega$.

Let us consider the vector of deviance residuals as $r(y, \mu) = [r(y_1, \mu_1), ..., r(y_N, \mu_N)]^\top$, where y_i and μ_i denote the elements of the N-vectors y and μ, respectively. Given a symmetric positive-definite $N \times N$ matrix Σ, the scaled deviance is defined as the following quadratic form in the vector of deviance residuals:

$$D(y, \mu, \Sigma) = r^\top(y, \mu)\Sigma^{-1} r(y, \mu) = \text{tr}\left[\Sigma^{-1} r(Y, \mu) r^\top(Y, \mu)\right].$$

Following the approach of Jorgensen (1999) and Jorgensen and Lauritzen (2000), a multivariate dispersion model is defined as:

$$f(y; \mu, \Sigma) = a(y, \Sigma) \exp\left[-\frac{1}{2} D(y, \mu, \Sigma)\right] \qquad \text{for } y \in \mathbb{R}^N \tag{4}$$

where $a(y, \Sigma)$ is a suitable function ensuring that (4) is a probability density function on $\mathbb{R}^N$.

3. Parameter estimation of EDRM

3.1. Maximum likelihood estimation

Two types of estimation procedures are known. One is the point estimation and another is interval estimation or confidence interval [29]. Here we mainly focus on point estimation of the parameters associated with a distribution function. This refers to point estimation, where the goal is to approximate an unknown parameter using sample data. Let us consider a random sample $Y = (y_1, y_2, ..., y_N)$ follows $f(., \mu, \lambda)$, where μ and λ are known. In most cases, there are two different approaches for obtaining a point estimator for unknown parameter. Namely classical method and decision theoretic approach. In this section, we focus the estimation of parameter β by the method of maximum likelihood. To estimate this parameter, we fix an underlying exponential dispersion model and common dispersion parameter $\lambda > 0$, but allow each sample y_i its own natural parameter $\theta_i = K'^{-1}(\mu_i)$. Our goal is to estimate the means $\mu_i = \mathbb{E}(y_i)$ for $i = 1, ..., N$ by the maximum likelihood method. Thus we can first estimate the coefficients β, and then use these estimates to argue that $\mu_i = \psi_\nu^{-1}(x_i^\top \beta)$ and the hypothesis of the components independence of response variable y_i, the likelihood function is given by

$$L(\beta, \lambda) = \prod_{i=1}^{N} c(y_i, \lambda) e^{\lambda\left[y_i (x_i^\top \beta)_i - K_\nu((x_i^\top \beta)_i)\right]}. \tag{5}$$

The log-likelihood function is written as

$$\mathcal{L}(\beta, \lambda) = \sum_{i=1}^{N} \left\{\lambda\left[y_i (x_i^\top \beta)_i - K_\nu((x_i^\top \beta)_i)\right] + \log c(y_i, \lambda)\right\} = \sum_{i=1}^{N} l_i(\beta, \lambda) \tag{6}$$

where $l_i(\beta,\lambda) = \left\{\lambda\left[y_i(x_i^\top\beta)_i - K_\nu((x_i^\top\beta)_i)\right] + \log c(y_i,\lambda)\right\}$. It should be to note that $\beta \mapsto \mathcal{L}(\beta,\lambda)$ is a strictly concave function. Therefore, to obtain the maximum likelihood estimator (MLE) of β, we derive the log-likelihood with respect to different components of the vector $\beta = (\beta_0,...,\beta_q)$ and we need to solve the likelihood equations

$$\frac{\partial\mathcal{L}}{\partial\beta_j}(\beta,\lambda) = \sum_{i=1}^{N}\frac{\partial l_i}{\partial\beta_j}(\beta,\lambda) = 0, \quad j = 1,...,q.$$

Since $\dfrac{d\eta_i}{d\mu_i} = \psi'(\mu_i)$, $\dfrac{d\theta_i}{d\mu_i} = 1/V(\mu_i)$ and by the chain rule for differentiation with respect to β_j, we have

$$\begin{aligned}
\frac{\partial l_i}{\partial\beta_j} &= \frac{dl_i}{d\eta_i}\frac{\partial\eta_i}{\partial\beta_j}\\
&= \frac{dl_i}{d\theta_i}\frac{d\theta_i}{d\mu_i}\frac{d\mu_i}{d\eta_i}\frac{\partial\eta_i}{\partial\beta_j}\\
&= \frac{dl_i}{d\theta_i}\left(\frac{d\mu_i}{d\theta_i}\right)^{-1}\left(\frac{d\eta_i}{d\mu_i}\right)^{-1}\frac{\partial\eta_i}{\partial\beta_j}\\
&= \lambda\left(y_i - K'(\theta_i)\right)\left(K''(\theta_i)\right)^{-1}\left(\psi'(\mu_i)\right)^{-1}x_{ij}\\
&= \frac{\lambda\left(y_i-\mu_i\right)x_{ij}}{V(\mu_i)\psi'(\mu_i)}.
\end{aligned}$$

Thus the likelihood equations we have to solve for the MLE of β are

$$\sum_{i=1}^{N}\frac{\lambda\left(y_i-\mu_i\right)x_{ij}}{V(\mu_i)\psi'(\mu_i)} = 0 \quad j = 1,...,q. \tag{7}$$

where μ is the mean vector with N components.

Note that the derivative of the log-likelihood with respect to β does not depend on the dispersion parameter λ.

Remark 1. When the response variable y_i is normally distributed with mean μ_i and dispersion parameter λ, we have $\psi(\mu_i) = \mu_i = x_i^\top\beta$. Hence, $\psi'(\mu_i) = \frac{d\psi(\mu_i)}{d\mu_i} = 1$; also $V(\mu_i) = 1$ and $\lambda = 1/\sigma^2$. Therefore, the equation (7) becomes

$$\frac{1}{\sigma^2}\sum_{i=1}^{N}\left(y_i - x_i^\top\beta\right)x_{ij} = 0.$$

By ignoring the factor $1/\sigma^2$, the equation reduce to the Normal equation of least squares

$$X^\top\left(Y - X\beta\right) = 0 \text{ or quivalently } \widehat{\beta} = \left(X^\top X\right)^{-1}X^\top Y. \tag{8}$$

3.2. Non-informative Bayesian estimation

In this section, we treat the prior of the dispersion parameter λ as known constant. In practice, it is unknown for most cases and it is necessary for us to specify reasonable values. Now, let us focus on specifying the value of λ. In the exponential family, for some distributions, λ is constant, for example, Poisson, Exponential, Bernoulli, Binomial, and Negative Binomial distributions; for other distributions, like Poisson and binomial with over-dispersion, or Normal, gamma, inverse Gaussian, λ is unknown and one may proceed as before with λ replaced by an estimate $\widehat{\lambda}$.

Jeffrey's prior based on the observed Fisher information matrix. Because it is locally uniform, it is a non-informative prior. It is a useful prior because it does not change much over the region in which the likelihood is significant and does not have large values outside that range the

local uniformity property. The Jeffrey's prior is justified on the grounds of its invariance under parametrization according to S. K. Sinha [31].

In this step, we assess Jeffrey's prior regression for the GLM class and the associated prior and posterior distribution characteristics. Our key subject is the case where λ is unknown. The Jeffrey's prior can be an enticing one for the normal linear regression model as it corresponds to tractable posterior distributions. See G. E. Box and G. C. Tiao [4] for more details about the usage of the Jeffrey's prior in this model. In addition, D. M. Eaves [8] assumed Jeffrey's prior to a non-linear phenomenon and obtained tractable posteriors through linearisation of the non-linear model. However, Jeffrey's prior can indicate that it leads to proper posteriors in many GLMs. The two theorems 1 and 2 are given by [13] below that help to evaluate and establish the propriety of the posterior distribution under Jeffrey's prior by giving (i) sufficient and (ii) necessary and sufficient conditions for the propriety of the posterior and prior distributions, respectively. The two theorems 1 and 2 discuss also the existence of joint moments.

Theorem 1. Suppose the likelihood and Jeffrey's prior for β are as above. Additionally, assume that X is of full rank and the likelihood of β is bounded above. Then, a sufficient condition for the existence of the posterior moment generating function of β for any GLM is that the integral

$$\int_S e^{\psi_\nu \theta^{-1}(r)+\phi^{-1}w(yr-K_\nu(r))}\left(\frac{d^2K_\nu(r)}{dr^2}\right)^{\frac{1}{2}}dr \tag{9}$$

is finite, where $\lambda(\phi)=\frac{\phi}{w}$ and for ψ_ν in some open neighbourhood about 0. Here S denotes the parameter space for the canonical parameter θ.

Theorem 2. A necessary and sufficient condition for existence of moment generating function of Jeffrey's prior for any GLM is that the integral

$$\int_S e^{\psi_\nu \theta^{-1}(r)}\left(\frac{d^2K_\nu(r)}{dr^2}\right)^{\frac{1}{2}}dr \tag{10}$$

is finite for ψ_ν in some open neighbourhood about 0.

It should be to note that tractable posteriors for GLM's with Jeffrey's prior are valid for certain examples only in very particular cases, and closed type outcomes in general are not available. Let us assume now $y_1,...,y_N$ are independent observations from a GLM. Jeffrey's joint prior for (λ,β) is given by $\pi(\lambda,\beta)=|I(\lambda,\beta)|^{\frac{1}{2}}$, where $I(\lambda,\beta)$ is the Fisher information matrix of (λ,β). From equation (1), it is obvious that $I(\lambda,\beta)$ is diagonal block in λ and β, with the form

$$\begin{pmatrix} -\sum_{i=1}^N \frac{2K'_\nu(\theta_i)\theta_i-K_\nu(\theta_i)}{\lambda^3}+\mathbb{E}(\ddot{c}(y_i,\lambda)) & 0 \\ 0 & \frac{X^\top V(\beta)\Delta^2(\beta)X}{\lambda} \end{pmatrix},$$

where $\ddot{c}(y_i,\lambda)=\frac{\partial^2 c(y_i,\lambda)}{\partial\lambda^2}$, $V(\beta)$ and $\Delta(\beta)$ are $N\times N$ diagonal matrices defined as $V(\beta)=Diag\left(K''_\nu\left(\psi(x_1^\top\beta)\right),...,K''_\nu\left(\psi(x_N^\top\beta)\right)\right)$ and $\Delta(\beta)=Diag\left(\psi'(x_1^\top\beta),...,\psi'(x_N^\top\beta)\right)$. Therefore, Jeffrey's prior for (λ,β) is driven as follows

$$\pi(\lambda,\beta)\propto\left(-\sum_{i=1}^N\frac{2K'_\nu(\theta_i)\theta_i-K_\nu(\theta_i)}{\lambda^3}+\mathbb{E}(\ddot{c}(y_i,\lambda))\right)^{\frac{1}{2}}\lambda^{-\frac{q}{2}}\left|X^\top V(\beta)\Delta^2(\beta)X\right|^{\frac{1}{2}}. \tag{11}$$

We can see from equation (11) that in fact, λ and β under Jeffrey's prior are not independent. In the normal linear model of regression: $\pi(\lambda,\beta)\propto\lambda^{-\frac{(q+2)}{2}}$ and in this case, λ and β are independent. On the contrary, for the gamma model λ and β are not. In fact, Jeffreys joint prior for λ and β may be quite difficult to analyse for many GLMs, as well as the gamma model, which may not be feasible for number calculation. Conversely, we can find the following joint prior to (λ,β): suppose that λ and β are independent, choose a (possibly informative) Jeffrey's prior to λ

and $\frac{X^\top V(\beta)\Delta^2(\beta)X}{\lambda}$ as Jeffrey's prior to β. This is one method for choosing an analytically feasible joint prior (λ, β), which simultaneously enhances Jeffrey's prior.

Maximum a-posteriori probability (MAP) estimate, like the maximum likelihood method, is a method that can be used to estimate a number of unknown parameters, such as the parameters of a probability density, related to a sample given. This method is closely linked to the maximum likelihood but differs from it however by the possibility of taking into account a non-informative a prior on the parameters to be estimated.

In MAP estimation, the model parameters are obtained by solving

$$\widehat{\underline{\theta}}_{MAP} = \underset{\underline{\theta}}{\operatorname{argmax}}\, \mathcal{L}(\underline{\theta}) + \log \pi(\underline{\theta}) \tag{12}$$

where $\underline{\theta} = (\lambda, \beta)$. Our goal is to solve the maximization in (12) when Jeffrey's prior is considered. According to Jeffrey's prior, the probability of the prior is proportional to the square root of the determinant of the Fisher information matrix I:

$$\pi(\underline{\theta}) \propto \sqrt{\det I(\lambda, \beta)} \tag{13}$$

Substituting (11) into (12) and removing terms which are independent of $\underline{\theta}$, we obtain

$$\widehat{\underline{\theta}} = \underset{\underline{\theta}}{\operatorname{argmax}}\, \mathcal{L}(\underline{\theta}) + \frac{1}{2}\log\left(-\sum_{i=1}^{N}\frac{2K_\nu'(\theta_i)\theta_i - K_\nu(\theta_i)}{\lambda^3} + \mathbb{E}(\ddot{c}(y_i,\lambda))\right) - \frac{q}{2}\log(\lambda) \tag{14}$$

$$+ \frac{1}{2}\log\left|X^\top V(\beta)\Delta^2(\beta)X\right| \tag{15}$$

Proposition 1. The coefficients regression estimates under Jeffrey's prior can be presented as

$$\widehat{\beta}^{(l+1)} = \widehat{\beta}^{(l)} + \lambda^{(l)} X^\top (Y - \mu) + \frac{X^\top \psi''(X^\top \beta(l)) X^\top X}{X^\top \psi'(X^\top \beta^{(l)}) X}$$

$$\widehat{\lambda}^{(l+1)} = \widehat{\lambda}^{(l)} + \sum_{i=1}^{N}\frac{1}{c(y_i,\lambda^{(l)})}\frac{\partial c(y_i,\lambda^{(l)})}{\partial \lambda} + \sum_{i=1}^{N} y_i (x_i^\top \beta^{(l)})_i - K_\nu(x_i^\top \beta^{(l)}) - \frac{q+2}{2\lambda}$$

where λ represents the dispersion parameter.

Proof.

In order to estimate the coefficients regression, we adopt the Gradient ascent approach (see [5]). Then, the coefficients regression and the dispersion parameter can be calculated respectively, as

$$\widehat{\beta}^{(l+1)} = \widehat{\beta}^{(l)} + \frac{\partial \log L(\beta^{(l)}, \lambda^{(l)})}{\partial \beta}$$

$$\widehat{\lambda}^{(l+1)} = \widehat{\lambda}^{(l)} + \frac{\partial \log L(\beta^{(l)}, \lambda^{(l)})}{\partial \lambda}$$

According to (14), the derivative of $\log l(\underline{\theta})$ is

$$\frac{d \log l(\underline{\theta})}{d\underline{\theta}} = \frac{d\mathcal{L}(\underline{\theta})}{d\underline{\theta}} + \frac{d \log \pi(\underline{\theta})}{d\underline{\theta}}$$

where the derivative of $\log l(\underline{\theta})$ with respect to β and λ

is obtained as follows

$$\frac{d\mathcal{L}(\underline{\theta})}{d\beta} = \lambda X^\top (Y - \mu) \tag{16}$$

$$\frac{d\mathcal{L}(\underline{\theta})}{d\lambda} = \sum_{i=1}^{N}\frac{1}{c(y_i,\lambda)}\frac{\partial c(y_i,\lambda)}{\partial \lambda} + \sum_{i=1}^{N} y_i (x_i^\top \beta)_i - K_\nu(x_i^\top \beta) \tag{17}$$

The derivative of $\log \pi(\underline{\theta})$ with respect to β and λ is given by

$$\frac{\partial \log \pi(\underline{\theta})}{\partial \beta} = \frac{\partial}{\partial \beta} \log \frac{X^\top V(\beta)\Delta^2(\beta)X}{\lambda} = \frac{X^\top \psi''(X^\top \beta)X^\top X}{X^\top \psi'(X^\top \beta)X}$$
$$\frac{\partial \log \pi(\underline{\theta})}{\partial \lambda} = \frac{\partial}{\partial \lambda} \log \lambda^{-\frac{(q+2)}{2}} = -\frac{q+2}{2\lambda}$$

From equations (16), (17) and the derivative of $\log \pi(\underline{\theta})$, the specific update for estimating $\widehat{\beta}^{(l+1)}$ and $\widehat{\lambda}^{(l+1)}$ in linear regression can be expressed as

$$\widehat{\beta}^{(l+1)} = \widehat{\beta}^{(l)} + \lambda^{(l)} X^\top (Y - \mu) + \frac{X^\top \psi''(X^\top \beta^{(l)})X^\top X}{X^\top \psi'(X^\top \beta^{(l)})X}$$
$$\widehat{\lambda}^{(l+1)} = \widehat{\lambda}^{(l)} + \sum_{i=1}^{N} \frac{1}{c(y_i, \lambda^{(l)})} \frac{\partial c(y_i, \lambda^{(l)})}{\partial \lambda} + \sum_{i=1}^{N} y_i (x_i^\top \beta^{(l)}) - K_\nu(x_i^\top \beta^{(l)}) - \frac{q+2}{2\lambda^{(l)}}$$

■

Remark 2. In order to estimate the coefficients regression and dispersion parameter, we proposed an iterative algorithm based on the Gibbs-sampling one. The Gibbs sampling algorithm was introduced by S. Geman and D. Geman [10]. We start by setting the coefficients regression and dispersion parameter to its initial values $\beta^{(0)}$ and $\lambda^{(0)}$, respectively. This process continues until "convergence" (i.e; $\left|\beta^{(l+1)} - \beta^{(l)}\right| < \epsilon_\beta$ and $\left|\lambda^{(l+1)} - \lambda^{(l)}\right| < \epsilon_\lambda$) for ϵ_β and ϵ_λ are small enough.

3.3. Application

The Bayesian approach is employed to estimate a multivariate multiple regression model using a non-informative Jeffrey's prior. The dataset consists of 100 synthetically generated observations, created from a multivariate normal distribution using Matlab software.

Table 2: *Parameter estimation*

	Parameter	Mean	Credible Interval
MLE	$\widehat{\beta}_0$	1.1319	(0.5497 1.7141)
	$\widehat{\beta}_1$	-0.9689	(−1.1301 − 0.8077)
	$\widehat{\beta}_2$	0.69819	(0.4964 0.9000)
	$\widehat{\lambda}$	2.8041	(−1.0375 6.6457)
		AIC = 212.8195	BIC= 220.2481
Jeffrey's prior	$\widehat{\beta}_0$	1.0462	(0.4899 1.6025)
	$\widehat{\beta}_1$	-0.9507	(−1.1102 − 0.7912)
	$\widehat{\beta}_2$	0.7129	(0.5167 0.9091)
	$\widehat{\lambda}$	2.7965	(−1.0451 6.6381)
		AIC =212.8161	BIC= 220.0107

The parameter estimates are presented in Table 2, along with their 95% credible intervals.

It is important to note that this analysis is based on a single simulated dataset. While both the Bayesian approach with Jeffrey's prior and the classical method yield similar results in this case, drawing strong conclusions from a single dataset is limited. A more rigorous comparison would require a complete Monte Carlo simulation study, where multiple datasets are generated and analysed to account for the variability introduced by the random generation of data.

Conducting such a simulation study, with several iterations, would provide a more comprehensive evaluation of the performance of both methods. Additionally, we observe that Jeffrey's prior has some advantages in terms of model fit, with slightly lower AIC and BIC values, and may offer more stable estimates when there is limited prior knowledge or small sample sizes. However, further validation through a more extensive simulation study is necessary to confirm these findings.

4. Numerical Illustration

Universal processes of rainfall involve data collection, data preprocessing and data selection,building a model using regression, and at the last validity check. The areal rainfall estimated by the rain gauges presents a great uncertainty where the network of rain gauges is sparse. From this approach, we can predict the rainfall of any future year using climatic factors. In our study, we have chosen an application concerning daily climatic data for the studied regions of northwestern Algeria. These data were extracted from the National Office of Meteorology (https://www.meteo.dz) and TuTiempo (https://en.tutiempo.net/climate), 2021.

For all existing rainfall stations (more than 10 climatic stations), the annual rainfall averages are available for a period varying from 35 to 40 complete years, from 1981 to 2021. Most of the stations are located in the plains and on the coasts, the number of stations decreases towards the south and in mountainous regions (more than 600m). The main available factors that depend on precipitation are temperature values, wind speed, station elevation, and station coordinates such as latitude and longitude (see Table 3).

Table 3: *Data of the topographic parameters at the study rainfall stations.*

Rainfall stations	Rainfall (mm)	Temperature (°C)	Wind speed (m/s)	Elevation (m)	Latitude (°)	Longitude (°)
Ain Sefra	0.5546	16.5686	5.4929	1200.55	32,76	-0.6
El-Bayadh	0.6944	15.4328	6.1949	1220.02	33,66	1
Mascara	1.4003	17.2871	5.2274	591.71	35.21	0.15
Mostaganem	1.2537	18.5080	5.2992	254.48	35,88	0.11
Oran	1.1661	18.6619	5.7160	127.31	35.63	-0.6
Relizene	1.3584	18.2210	5.2587	367.72	35.73	0.55
Saida	1.2480	15.9214	5.4670	928.27	34.86	0.15
Sidi Bel Abbes	1.3186	17.6557	5.3304	458.05	35.19	-0.64
Tiaret	1.4860	15.6170	5.6113	904.3	35.35	1.43
Tlemcen	1.2399	18.0853	5.3669	355.91	35.01	-1.46

In the present investigation, we have applied a multiple regression model, which has been elaborately described in Section 2.2, to the data set at hand. The purpose of this exercise is to unearth a predictable equation that can establish a link between rainfall and various climatic factors. In order to accomplish this goal, we have meticulously constructed a regression equation comprising five independent variables as follows:

$$y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \beta_3 x_{i,3} + \beta_4 x_{i,4} + \beta_5 x_{i,5} + \varepsilon_i, \quad i = 1, ..., 10$$

where y_i is the response (predicted rainfall station), $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are the regression coefficients, x_1 (Temperature), x_2 (Wind speed), x_3 (Elevation), x_4 (Latitude), x_5 (Longitude) are

the highly correlated climate indices (predictors); and ε is the residual term of the model.

Let us now proceed to the multiple regression analysis with the full climatic factors considered above and compare it with different types of model estimation using maximum likelihood and Jeffrey's prior estimations. Table 4 shows the regression comparison with the five independent variables. The main features of interest are the parameters estimates as well as their corresponding standard errors (SE Coeff) and the criteria of fitness (t-Stat and p-Value). the standard error of the regression coefficient is calculated as SE Coeff $= \frac{s.e._R}{sx\sqrt{N-1}}$, where $s.e._R$ is the standard deviation of the regression, sx is the standard deviation of x, and N is the sample size.

Moreover, the estimates are significantly similar and its associated standard error are close to zero, which justifies the very smaller over variation. Since the associated p-value is < 0.001, we reject the hypothesis in favour of the alternative hypothesis that at least one of the coefficients is not zero.

Table 4: *Multiple regression analysis with MLE and Jeffrey's prior approaches.*

	Predictor	Coeff $\widehat{\beta}$	SE Coeff $\widehat{\beta}$	t-Stat	p-Value
MLE	Intercept	5.14787×10^{-15}	0.03227271	1.59511×10^{-13}	0.99999999
	Temperature	-2.82471635	0.42179987	-6.69681646	0.00258658
	Wind speed	-0.81015953	0.10066482	-8.04808997	0.00129405
	Elevation	-4.06167602	0.77057323	-5.27097985	0.00620801
	Latitude	-1.22618013	0.44892198	-2.73138802	0.05236798
	Longitude	0.58214495	0.15687036	3.71099388	0.02063610
	RMSE= 0.102	R^2 =0.995	$R^2(adj)$ = 0.99	AIC=-14.42894643	
Jeffrey's prior	Intercept	5.59885×10^{-15}	0.03227271	1.73485×10^{-13}	0.99999999
	Temperature	-2.82465611	0.42179987	-6.69667364	0.00258679
	Wind speed	-0.81014627	0.10066482	-8.0479583	0.00129413
	Elevation	-4.06156466	0.77057324	-5.27083533	0.00620862
	Latitude	-1.22611567	0.44892198	-2.73124443	0.05237573
	Longitude	0.58212371	0.15687036	3.71085846	0.02063855
	RMSE= 0.10205529	R^2 =0.99536930	$R^2(adj)$ = 0.98958092	AIC=-14.42894637	

Upon examination of Table 4, we observe that the estimated values of β_i are strikingly alike and the associated standard deviations are nearly zero, which validates the incredibly low variance of the model. With a p-value greater than 0.001, we reject the null hypothesis and favour the alternative hypothesis that at least one of the coefficients is not zero.

Additionally, Table 4 displays the Root Mean Squared Errors (RMSE) for the variance of the residuals, calculated by both MLE and Jeffrey's prior estimation methods. This metric evaluates absolute fit of the model to the data, indicating the proximity of the observed data points to predicted values of the model. Interpreted as the standard deviation of the unexplained variance, the RMSE is expressed in the same units as the response variable. Lower RMSE values for both MLE and Jeffrey's prior estimation methods indicate better fit, supporting the equivalence of the two approaches. The RMSE serves as a reliable measure of the model's predictive accuracy, making it the most critical criterion for fit if the primary goal is prediction.

Table 4 indicates that other fitness criteria, such as R^2, $R^2(adj)$, and AIC, are reasonably similar for both MLE and Jeffrey's prior regression models. Figure 1 presents the estimates for each of the β_i coefficients of the regressions with five independent variables using both MLE and Jeffrey's prior regression models. It is evident that all the parameters converge rapidly to the MLE method as the number of iterations increases.

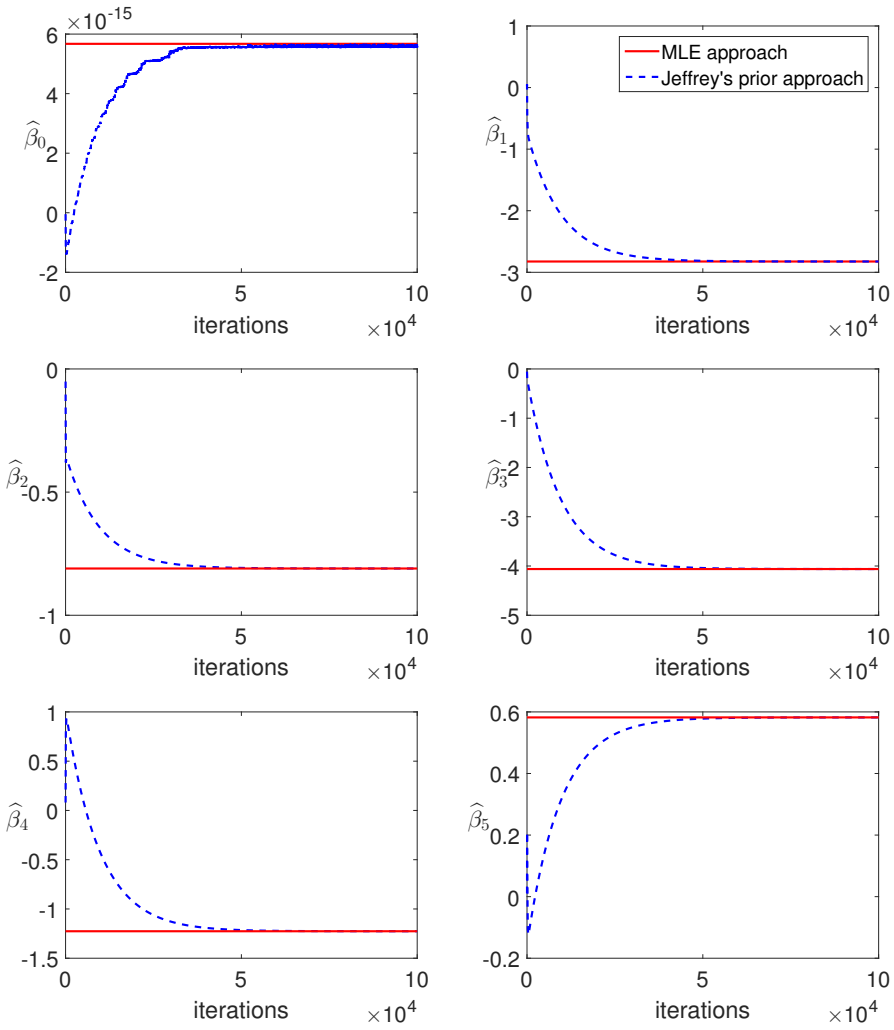

Figure 1: *Relationship between regression coefficient estimation and iteration number vs MLE method*

5. Conclusion and Discussion

In this paper, we apply the Bayesian estimation of EDRM parameters using Jeffrey's prior. We demonstrated that the maximum likelihood method can be used to accurately estimate this parameter. Additionally, we devised an iterative algorithm based on Jeffrey's prior to estimate the regression coefficients. The significance of rainfall on agriculture and global economies cannot be overstated, and accurate predictions of rainfall are essential for successful farming practices. While this model is currently the only one capable of predicting rain, it is not entirely precise due to the fluctuation of climatic variables. Although our study includes certain elements, other factors may also impact rainfall amounts. Nevertheless, our proposed technique shows promise, particularly when utilizing Jeffrey's prior. The results we obtained were comparable to those of the MLE method, validating the effectiveness of our approach.

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IMPLEMENTATION OF THE MAXIMUM PERMISSIBLE OVERLOAD CAPACITY OF A DC MOTOR

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Abstract

DC motors, due to their wide applicability in various industrial sectors, necessitate precise control of their overload capacity to ensure safe and efficient operation. This study presents a comprehensive methodology for assessing the maximum permissible overload capacity of a DC motor. The core of this methodology lies in the derivation and application of the electromechanical characteristic equation of an electric drive with current cutoff. This equation serves as the foundation for constructing the electromechanical characteristics of the drive, providing a detailed representation of the motor's performance under varying operational conditions. A novel circuit is proposed, featuring an automatic adjustment mechanism for the cut-off current setting based on the speed of the electric drive. This adaptive circuit design ensures that the motor operates within its maximum permissible overload capacity, thereby optimizing performance and preventing potential damage due to excessive loads. By leveraging this advanced control methodology, the reliability and efficiency of DC motors in industrial applications can be significantly enhanced. This approach not only maximizes the motor's operational capabilities but also contributes to the overall safety and longevity of the electric drive systems.

Keywords: DC electric motor, drive, overload capacity, electromechanical characteristics, current cut-off, current limiting unit

I. Introduction

Direct current (DC) motors are widely used in various industrial and domestic applications due to their high reliability, ease of control and wide range of performance characteristics. An important aspect of the operation of electric motors is their overload capacity, that is, the ability to withstand a temporary increase in load torque beyond established limits without damage. Effective use of electric motors requires precise control and optimization of their overload capacity. This is especially important when operating under variable loads or in unpredictable environments. Failure to properly define overload limits can result in equipment damage, productivity and operational safety [1]. The purpose of the study is to improve the efficiency and reliability of electric motors by determining optimal overload values. The results obtained can be used in the design and operation of industrial equipment, as well as in the development of control and monitoring systems for the operation of electric motors.

If the operation of the mechanism is characterized by frequent starts and the speed of the drive is required, and its installed power is limited, then there is a need to fully use the maximum permissible overload capacity of the drive motor. Typically, such drives use DC motors. It is

known that the permissible maximum torque and, therefore, the ultimate overload capacity of a DC motor varies depending on its speed. In short-term operating mode, the motor overload is limited mainly by the deterioration of the switching condition, leading to unacceptable sparking of the commutator-brush contacts of the machine.

The higher the rotation speed, the lower the armature current must be so that the switching conditions remain equally satisfactory [2, 5]. In reference materials, the value of the maximum permissible torque of DC motors is given for several speeds, usually for speeds $\omega \le 0.2\omega_n$, $\omega=\omega_n$ and $\omega=2\omega_n$. So, for example, for a crane-metallurgical motor DP-82, 220V, PE=25% parallel excitation with a stabilizing winding, the curve for changing short-term permissible currents within the range of $0\text{-}2\omega_n$ will have the form shown in Fig. 1 (broken line abc). Note that the diagram is constructed for the case where speed control up to $2\omega_n$ is carried out by changing the voltage applied to the motor armature and that the change in the permissible current between given points ab and c is taken along a straight line. Let's consider the possibilities of ensuring that existing circuits of automated electric drives make full use of the maximum overload capacity of the electric motor, that is, automatically limiting the drive load current along the line of permissible overloads indicated in Fig. 1.

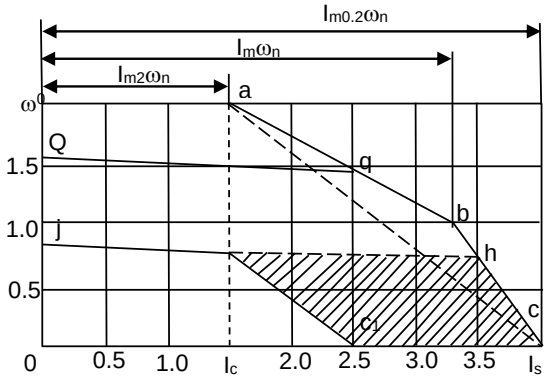

Figure 1: *Scheme for automatically limiting the drive load current along the permissible overload line*

II. Methods

Typically, in automation circuits, in order to limit motor overload, a current cut-off unit is used (Fig. 2.). The independent excitation electric motor receives power from a controlled energy converter containing a control signal adder with windings CW1 and CW2. The circuit of the current cut-off unit includes a shunt with resistance rs, connected to the motor armature circuit, a source of reference (reference) voltage RV, winding CW2 and diode D.

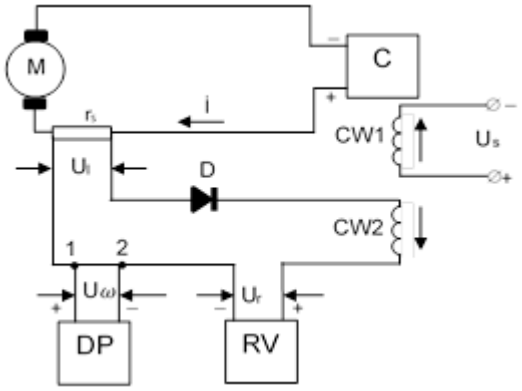

Figure 2: *Automation circuit for limiting motor overload using a current cut-off unit*

The equation for the electromechanical characteristics of an electric drive with current cut-off can be presented in the following form:

$$\omega = \frac{k_g(U_\Sigma)}{k\phi} \cdot U_s - \frac{r_c + r_m}{k\phi} \cdot I - \frac{k_c(U_\Sigma) \cdot r_s}{k\phi} \cdot (I - I_c) \cdot l(\Delta I) \quad (1)$$

Where $k_g(U_\Sigma)$ and r_c are the gain of the total input voltage and the output resistance of the converter, respectively; I and r_m are the current and resistance of the motor armature circuit, respectively; U_s – setting voltage; I_c – cut-off current, I_c=const; U_r – reference voltage; k – electromagnetic constant; ϕ – motor excitation flux ϕ=const.

When deriving formula (1), it was accepted that: a) $l(\Delta I)$ is a unit function equal to 0 or +1, respectively, for armature currents lower and higher cutoff currents; b) control windings CW1 and CW2 of the signal adder are identical [3]. The electromechanical characteristics of the drive, constructed according to equation (1) when speed is controlled by armature voltage, for various values of U_s are shown in Fig. 3, a. When $I \leq I_c$, no current flows through winding CW2 and the electric drive operates in the operating range of the characteristic. When $I > I_c$, current begins to flow through winding CW2, U_Σ decreases and the motor torque is limited.

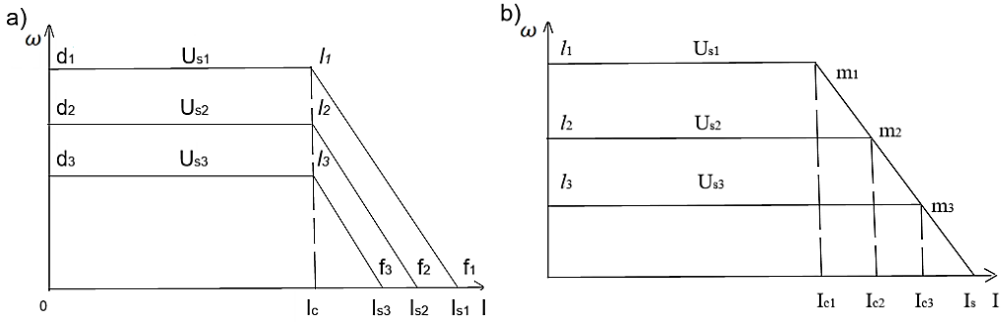

Figure 3: *The electromechanical characteristics of the drive, constructed according to equation (1) when speed is controlled by armature voltage, for various values of U_s*

Let us assume that for this electric drive, by selecting the parameters of the current cut-off unit, the coincidence of section e_1f_1 of the electromechanical characteristic curve at U_{s1} (Fig. 3, a) with section bc of the curve of permissible maximum currents (Fig. 1) is achieved. In this case, using a conventional current cut-off unit, it is possible to automatically limit the armature current along the line of permissible motor currents if the value of the setting signal corresponds to the value of U_{s1}. At lower values of the setting signal, the armature current limitation will pass through lines e_2f_2 or e_3f_3 depending on the value of the setting signal U_{s2} or U_{s3}, i.e. in this case, the armature current will be limited at currents less than permissible values [7-9]. Consequently, when using a conventional current cut-off unit, with driving signals less than the nominal value (if U_{s1} is taken as the nominal value of the driving signal), full use of the permissible overload capacity of the electric motor is not ensured. The underutilization of the maximum overload capacity becomes even greater if the electric drive provides for regulation of the motor rotation speed above the rated speed. In this case, the diagram of the maximum permissible currents for a given motor will have two limitation sections (ab and bc, Fig. 1) with two different slopes of each of them. Since the electromechanical characteristic of an electric drive using a conventional current cut-off unit has only one current limiting section, this unit would have to be adjusted along the dotted line ac (Fig. 1) to the cut-off current $I_c = I_{m2\omega n}$ and to the stopping current $I_s = I_{m0.2\omega n}$. In this case, at all values of the setting signal, there will be an underutilization of overload torques.

The degree of underutilization of the maximum overload capacity of the engine for various values of the master signal can be conditionally estimated by a coefficient equal to the ratio of the areas limited by the coordinate axes (I and ω) and the corresponding mechanical characteristics.

So, for example, for electric motors that allow speed control up to $2\omega_n$:

$$k_1 = \frac{S_{oc_1 a_1 i}}{S_{ochj}} \text{ for } \omega_s = 0.8 \cdot \omega_n$$

Based on the calculated values of k_1 for DP-82 engines, the graph shown in Fig. 4 was constructed. As can be seen from the k_1 curve, the underutilization of the maximum overload capacity of the motor at reference speeds below $0.6\omega_n$ reaches more than 50%. From the above, we can conclude that adjustable DC electric drives with existing current limiting units do not ensure full use of the maximum overload capacity of the motor [11-13]. In this regard, a new current cut-off circuit is proposed for adjustable DC electric drives, which makes it possible to fully realize the maximum overload capacity of the electric motor over the entire speed control range (at all values of the set signal).

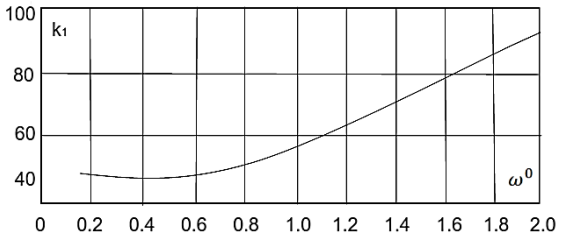

Figure 4: *Graph based on calculated values of k_1*

The main reason for the shortcomings of the existing current cut-off unit is the independence of the cut-off current value from the setting signal Us. Indeed, at the beginning of the cutoff circuit, the voltage drop across the shunt U_1 is equalized with the reference voltage and the cutoff current:

$$I_c = \frac{U_r}{r_s} = const$$

It follows that the overload limiting unit in the new circuit must be designed in such a way that the comparison voltage does not remain a constant value, but changes as a function of the change in the maximum permissible armature current from the motor speed. For this purpose, an additional signal from the speed sensor SS is added to the circuit of the existing video current cutoff (Fig. 2, the U_ω signal is introduced into section 1-2). The speed sensor signal U_ω, subtracted from the reference voltage, forms a comparison voltage (U_a), the value of which, being a function of speed, increases as the engine speed decreases. Due to this, the cut-off current becomes a function of the motor speed. In this case, at the moment the cutoff begins, there is a voltage balance:

$$U_I = U_a = U_r - U_\omega = I_c r_s$$

Where does the cutoff current come from:

$$I_c = \frac{U_r - U_\omega}{r_s} = \frac{U_r}{r_s} - \frac{k_\omega}{r_s} \cdot \omega$$

While $U_I \leq U_a = U_r - U_\omega$, the engine operates in the working section of the speed characteristic. When $U_1 > U_a$, current begins to flow through the control winding CW2, as a result of which the motor torque is limited. Moreover, as the set speed decreases, the cutoff current increases to the permissible overload for a given motor speed. This can be seen from Fig. 3b, which shows graphs of the electromechanical characteristics of the drive, constructed according to the equation:

$$\omega = \frac{k_c(U_\Sigma) \cdot U_s - (r_c + r_m) \cdot I - k_c(U_\Sigma) \cdot (Ir_s - U_r l(\Delta I)}{k\phi + k_c(U_\Sigma) k_\omega l(\Delta I)} \quad (2)$$

The expression for the stopping current can be obtained from equation (2) by substituting

the values of $\omega=0$ and $I=I_s$. In this case we get:

$$I_s = (U_r + U_s) \cdot \alpha \quad (3)$$

Where:

$$\alpha = \frac{k_c(U_\Sigma)}{r_c + r_m + k_c(U_\Sigma) r_s} \approx const$$

As can be seen from (3), in the proposed current cut-off circuit, the value of the stopping current varies slightly depending on the value of the setting signal; the stopping current decreases as the driving signal decreases. To reduce the influence of this dependence, the ratio U_{smax}/U_r should be taken to be small. To fully utilize the maximum permissible overload capacity of the electric motor when regulating the speed down from the nominal value of the parameters U_r, k_ω, r_s should be selected in such a way that the steeply falling part of the electromechanical characteristics (line mın, Fig. 3, b) coincides with the line of the maximum permissible motor currents (bc, Fig. 1). Then, regardless of the reference speed, the overload current limitation will always be along the line of the maximum permissible motor currents and, therefore, the use of the maximum overload capacity for all speeds will be complete [15]. As mentioned above, for an electric drive that requires speed control up to $2\omega_n$, the current limiting curve should have the shape of a broken line abc (Fig. 1). In this case, to change the slope of the current limiting curve, you can use a relay with a high return coefficient connected to the signal voltage of the speed sensor SS. By triggering this relay, upon reaching the rated rotation speed, the parameters of the current cut-off circuit (U_r, k_ω, r_s) are changed and the required change in the slope of the current-limiting section of the electromechanical characteristic is ensured. In this case, the electromechanical characteristics of the drive for set speeds, for example, $0.8\omega_n$ and $1.5\omega_n$, will be obtained in the form of broken lines, jhc and pqbc, respectively (Fig. 1). Consequently, the maximum overload capacity of the drive motor will also be fully realized for any value of the reference rotation speed.

III. Results

1. The existing current limiting unit does not ensure full use of the maximum overload capacity of a DC motor, especially for master signals less than the nominal one. At reference speeds below $0.6\omega_n$, the underutilization of the maximum overload capacity of the motor reaches more than 50%.

2. In order to automatically limit overload along the line of permissible values of motor currents, you can use the proposed current cut-off unit.

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THE ROLE OF MODERN GROUNDING DEVICES IN ENSURING THE STABILITY OF POWER SYSTEMS

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Abstract

The article focuses on investigating the impact of grounding device parameters on the stability of power systems under external disturbances, such as short circuits and lightning strikes. The study examines transient processes in power systems, including the analysis of rotor angle variations in generators and voltage recovery. Numerical modeling based on the equations of synchronous generators and electromagnetic transient processes is employed. A comparative analysis of various grounding device configurations is conducted, taking into account their resistance and the system's recovery time. The research results identify the optimal parameters of grounding devices that minimize the recovery time of power systems and enhance their overall stability. The findings can be utilized in the design and operation of power systems with improved reliability.

Keywords: Power system stability, grounding devices, intelligent grounding systems, electrical network protection, transition processes, short circuit analysis.

I. Introduction

Power system stability is one of the most critical factors in ensuring the reliability of energy supply. Modern power systems face numerous external disturbances, such as lightning strikes, short circuits, and abrupt load changes, which can disrupt their operational stability. The ability to effectively manage these processes is essential for minimizing equipment failure risks and improving the overall reliability of power systems.

Grounding devices play a significant role in maintaining power system stability. They not only provide pathways for short-circuit and lightning discharge currents but also help reduce voltage stress on equipment, stabilize system parameters, and shorten recovery times after fault events. Despite the high effectiveness of modern grounding devices, insufficient attention to their design and operation can lead to severe consequences for power systems [1].

The objective of this study is to examine the impact of grounding devices on the stability of power systems under external disturbances. The research focuses on transient processes in power systems, analyzing grounding device parameters such as grounding resistance and their influence on key system characteristics, including the maximum rotor angle deviation of generators and voltage recovery time after a disturbance.

Particular attention is given to the comparative analysis of different grounding device parameters and their effects on system performance under varying fault durations. Numerical methods are employed to model transient processes, incorporating synchronous generator equations and electromagnetic transient phenomena [2]. This approach identifies the most effective configurations of grounding devices for various operating conditions of power systems.

This article aims to develop practical recommendations for optimizing grounding device parameters, which will enhance power system stability and reduce the risk of failures under

external disturbances

II. Formulation of the problem

Power system stability is crucial for ensuring reliable electricity supply, especially under external disturbances such as lightning strikes, short circuits, and abrupt load changes. These disturbances can lead to equipment failures, increase the number of outages, and reduce the overall efficiency of the power system.

Grounding devices play a central role in mitigating the negative impacts of such disturbances by providing a low-resistance path for lightning currents, stabilizing short-circuit parameters and preventing severe equipment damage. However, selecting optimal grounding device characteristics remains a complex challenge [3-5]. Parameters such as grounding resistance, electrode geometry, and soil resistivity must be carefully considered to minimize outages and ensure rapid system recovery after disturbances.

Moreover, quantitative assessment of grounding device effectiveness requires modeling transient processes within the power system. This includes analyzing dynamic characteristics such as the maximum rotor angle deviation of the generator (δ_{max}) and voltage recovery time (T_{rest}). These parameters depend on the power system configuration and the characteristics of grounding devices.

The problem can therefore be outlined as follows:

1. Identify the optimal characteristics of grounding devices to improve power system stability.
2. Develop methodologies for calculating transient processes and key stability criteria (δ_{max}, T_{rest}).
3. Analyze the influence of various grounding device parameters on power system stability under disturbances.

Solving this problem requires performing simulations, developing algorithms for calculating transient processes, and analyzing the results [6]. This will enable the formulation of recommendations for designing effective grounding devices.

III. Problem solution

Power system stability is defined as the ability to maintain operational functionality under external disturbances such as lightning strikes, short circuits, or abrupt load changes [6-8]. Grounding devices play a critical role in ensuring this stability by reducing the risk of equipment damage and maintaining system parameter stability (Table 1).

Table 1: *Impact of Grounding Devices on Power System Stability*

Type of Disturbance	Impact Without Effective Grounding	Role of Grounding Devices	Effect on Stability
Lightning	Insulation damage, flashovers in power lines	Provides a low-resistance path for lightning currents	Reduction in outages by 35–50%
Short Circuit	Equipment overheating, circuit breaker damage	Stabilizes fault currents, prevents mechanical damage	Decrease in recovery time by 20–30%
Abrupt Load Changes	Voltage fluctuations, industrial equipment failures	Stabilizes voltage at connection points of major consumers	Reduction in voltage oscillations by 15–20%

Figure 1 illustrates the reduction in the number of outages for medium-voltage power lines with varying levels of grounding resistance (Table 2).

Table 2: *Outage Data for Power Lines with Different Grounding Resistance Levels*

Grounding Resistance (Ω)	Number of Outages (Without Grounding)	Number of Outages (With Grounding)
10	50	20
5	35	12
1	25	7
0.1	10	3

Grounding devices with lower resistance demonstrate a significant decrease in outage frequency, contributing to improved power system reliability and operational stability.

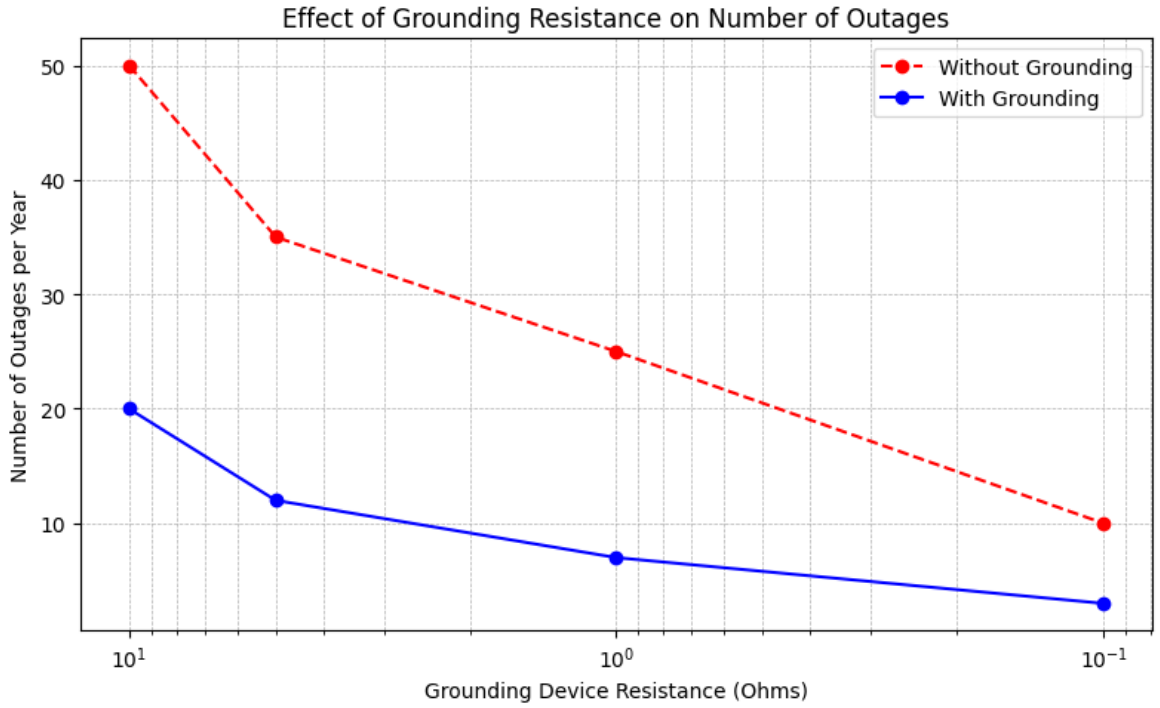

Figure 1: *Effect of grounding resistance on number of outages*

The graph clearly demonstrates that as the grounding resistance decreases, the number of outages in the power system significantly reduces, particularly when the resistance is less than 1 Ohm [9]. This highlights the effectiveness of grounding devices in ensuring the stability of power systems under external disturbances.

To analyze the impact of grounding devices on power system stability, the following formula is used:

$$R_z = \frac{\rho}{L} \cdot f$$

Where: R_z — grounding resistance, Ohms; ϱ— soil resistivity, Ohm·m; L — length of the grounding electrode, m; f — correction factor depending on the geometry of the grounding electrode.

The evaluation focuses on critical parameters, including:

1.Maximum generator rotor angle (δ_{max});

2.Voltage recovery time after a disturbance (T_{rest}).

These parameters are influenced by the network configuration and grounding device characteristics.

For the calculations, a system configuration was selected comprising a generator with a nominal apparent power of S_n=100 MVA, operating at a frequency of 50 Hz and voltage V_n=220 kV. The generator's inertia constant H=6 s was considered to simulate its dynamic rotor behavior.

The transmission line was characterized by a resistance of R_{line}=0.05pu and an inductance of X_{line}=0.5pu. Different grounding resistances (R={0.5,1.0,2.0} Ohm) were used to assess the impact of grounding.

The stability analysis focuses on simulating transient processes to evaluate the maximum rotor angle deviation (δ_{max}) and voltage recovery time (T_{rest}). The synchronous generator equations and electromagnetic transient processes were applied, enabling the determination of critical system parameters under external disturbances, such as short circuits [10].

This approach allows for a comprehensive understanding of how grounding device characteristics affect the stability and resilience of power systems.

For the calculations, the following equations were applied:

Generator Rotor Dynamics Equation

$$\frac{d^2\delta}{dt^2} = \frac{\omega_s}{2H} \cdot (P_m - P_e)$$

where: δ—rotor load angle; H—inertia constant (seconds); P_m — mechanical power; $P_e = \frac{E \cdot V}{X} sin\delta$ — electrical power.

Stability Criterion: The angle δ is considered stable if:

$$\delta \leq \delta_{cr}, \qquad \text{where } \delta_{cr} = 90^0$$

Voltage Recovery: The load voltage is calculated as:

$$V_{load} = V_s \cdot \frac{X_{line}}{R_{line} + R_{ground}}$$

where R_{ground} is the resistance of the grounding device.

Calculation Algorithm:

System Parameter Initialization: Generator: P_m=1.0 pu, E=1.1 pu; Transmission Line: X_{line}=0.5 pu, R_{line}=0.05 pu.; Grounding Device: R= {0.5,1.0,2.0}.

Fault Simulation: At t=0, a fault is simulated, and generator voltage drops. Calculations are performed with a time step Δt=0.01 s.

Numerical Integration of Rotor Dynamics: The fourth-order Runge-Kutta method is used to solve the rotor angle equation:

$$\delta(t + \Delta t) = \delta(t) + \frac{d\delta}{dt} \cdot \Delta t$$

Maximum Rotor Angle (δ_{max}): The calculation is repeated for various fault durations (0.1–1.0 s); δ increases until power balance is achieved (P_m=P_e).

Voltage Recovery: After clearing the fault at t_{clear}, the voltage recovers according to the equation:

$$V(t) = V_{pre-fult} \cdot e^{-\frac{R_{grand}}{X_{line}}t}$$

Recovery time (T_{rest}) is recorded as the moment when voltage reaches 95% of its nominal value.

Example Calculations for R=0.5 Ω (table 3):

Table 3: *Example Calculations for R=0.5 Ω*

Fault Duration (s)	P_m=P_e (pu)	δ_{max} (°)	T_{rest} (s)
0.3	1.0	84.51	1.075
0.6	1.0	86.99	1.150
1.0	1.0	88.65	1.250

Results (table 4, table 5). The results were verified using Python, utilizing the NumPy library for numerical solutions and Matplotlib for visualization (figure 2, figure 3).

Table 4: *Maximum Rotor Angle (δ_{max})*

Fault Duration (s)	R=0.5 Ω	R=1.0 Ω	R=2.0 Ω
0.1	81.81	80.95	80.49
0.2	83.30	81.81	80.95
0.3	84.51	82.59	81.39
0.4	85.51	83.30	81.81
0.5	86.32	83.93	82.21
0.6	86.99	84.51	82.59
0.7	87.53	85.03	82.95
0.8	87.98	85.51	83.30
0.9	88.35	85.93	83.62
1.0	88.65	86.32	83.93

Table 5: *Voltage Recovery Time (T_{rest})*

Fault Duration (s)	R=0.5 Ω	R=1.0 Ω	R=2.0 Ω
0.1	1.025	1.05	1.10
0.2	1.050	1.10	1.20
0.3	1.075	1.15	1.30
0.4	1.100	1.20	1.40
0.5	1.125	1.25	1.50
0.6	1.150	1.30	1.60
0.7	1.175	1.35	1.70
0.8	1.200	1.40	1.80
0.9	1.225	1.45	1.90
1.0	1.250	1.50	2.00

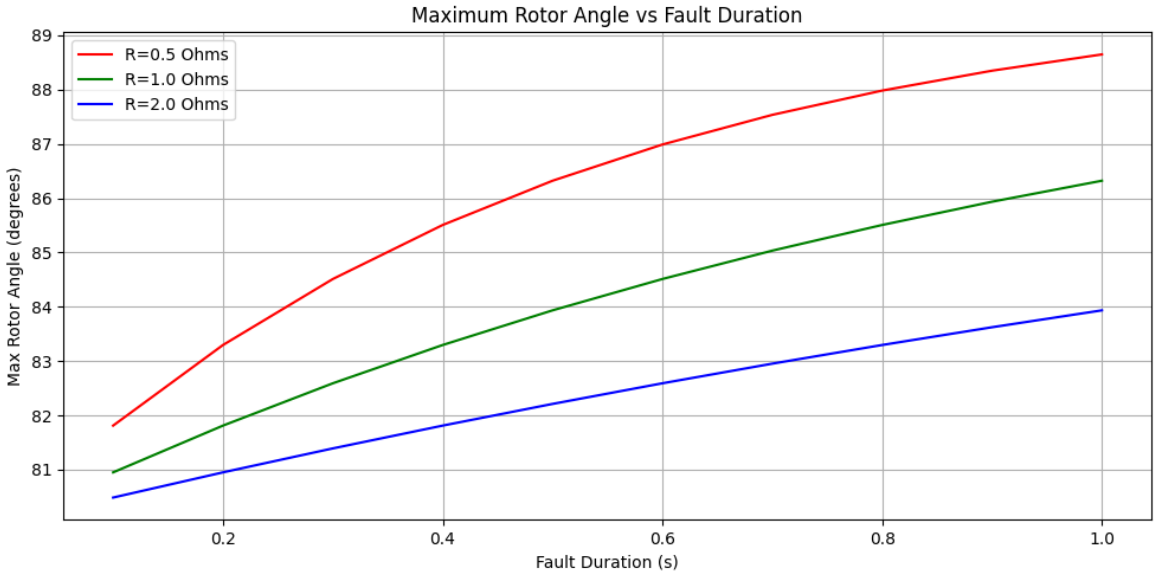

Figure 2: *Maximum rotor angle vs fault duration*

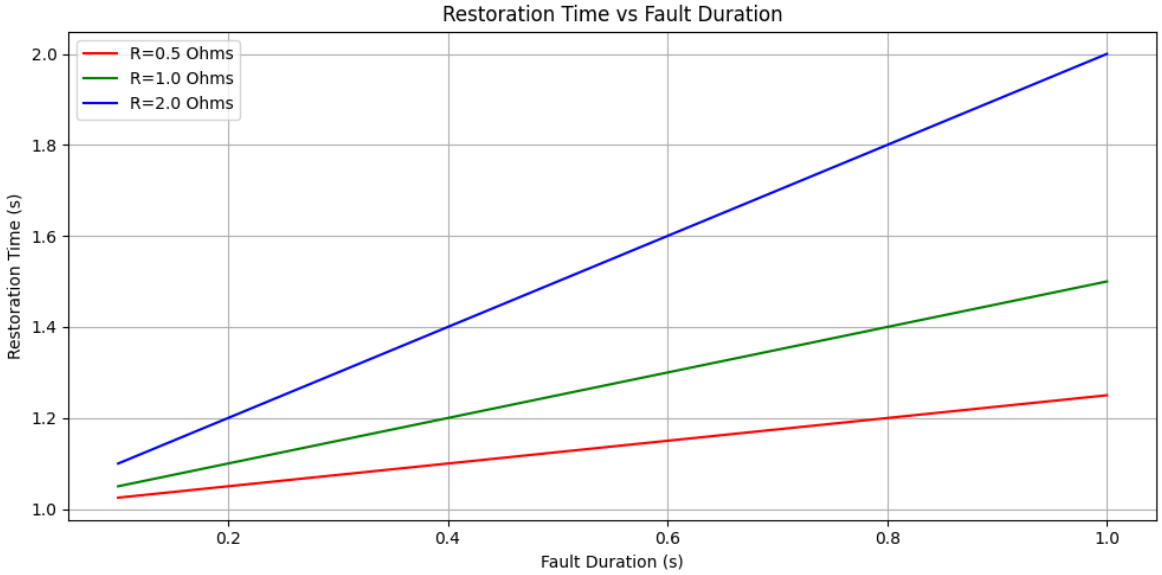

Figure 3: *Restoration time vs fault duration*

Figure 2 illustrates the variation of the generator rotor's maximum angle (δ_{max}) with respect to fault duration. It highlights how an increase in the grounding resistance reduces the amplitude of the rotor angle for a given fault duration.

Figure 3 depicts the recovery time of voltage (T_{rest}) as a function of fault duration. It demonstrates the influence of grounding resistance on the system's ability to recover following disturbances.

These insights provide a better understanding of the impact of various grounding device parameters on power system stability, which is crucial for the design and operation of energy systems [11-13].

Modern intelligent grounding devices (IGDs) represent an integration of advanced electronics, sensor systems, and data analysis algorithms (figure 4). These devices are capable of actively monitoring power system parameters in real time, including leakage currents, grounding resistance, frequency deviations, and other metrics that are critical for maintaining network stability.

Figure 4: *Modern intelligent grounding devices (IGDs)*

The operation of Intelligent Grounding Devices (IGD) relies on built-in sensors and control systems that analyze current network parameters and forecast potential failures. For instance,

IGDs are equipped with sensors that detect changes in soil resistance caused by environmental conditions such as rain or frost [14]. Based on the collected data, the device automatically adjusts grounding system parameters to minimize the likelihood of system failures.

One of the key innovations in IGD technology is the integration with Internet of Things (IoT) systems. This enables data from grounding devices to be transmitted to cloud storage for subsequent analysis and prediction. The application of machine learning in data analysis opens up opportunities for more accurate identification of potential risks, such as equipment overheating or disruptions caused by lightning strikes.

Another significant advancement in IGD technology is the use of self-regulating materials in their construction [15]. These materials can alter their properties in response to external conditions, such as automatically increasing conductivity during high current loads. This capability is particularly critical for high-voltage systems, where system stability directly depends on grounding characteristics.

Furthermore, IGDs can be integrated with active power system management solutions. In the event of a short circuit or other critical incidents, IGDs interact with relay protection and automatic recovery systems, ensuring faster system responses and minimizing downtime [16-17].

The adoption of such technologies not only enhances the reliability of power systems but also improves their economic efficiency by reducing operational costs and extending equipment lifespan.

Examples of modern IGD implementation demonstrate their effectiveness under real operating conditions and confirm significant improvements in power system stability. Below, several specific examples from various energy projects are discussed to illustrate their practical benefits.

High-Voltage Transmission Lines in Europe

As part of the modernization of Germany's energy infrastructure under the "Energiewende" project, advanced intelligent grounding devices (IGDs) were implemented in high-voltage transmission lines (HVDC systems). The primary objective of these devices was to reduce the risk of outages caused by lightning strikes and transient processes. These devices monitored ground resistance in real-time, transmitted data to a central control system, and automatically adjusted grounding parameters based on weather conditions. As a result, the number of line outages decreased by 25%, significantly improving the overall reliability of the power system.

Substations in the United States

At a substation in California, located in a seismically active zone, an advanced system of IGDs was installed. These devices included sensors to measure ground resistance and vibrational parameters, enabling effective risk management associated with earthquakes. After integrating the grounding system with the substation's automation system, network recovery times after disturbances improved, with voltage restoration occurring on average 30% faster than with traditional methods.

Wind Farms in Northern Europe

At a wind farm in Denmark, where frequent thunderstorms pose a risk of lightning strikes, IGDs with active monitoring capabilities were installed. These devices protected generators from overvoltages, monitored changes in grounding circuit resistance, and transmitted data to a cloud-based analysis system. This not only improved the resilience of the power system but also reduced maintenance costs, as wear-and-tear forecasting optimized the maintenance schedule.

Solar Power Plants in India

At one of the largest solar power plants in Rajasthan, IGDs with automatic ground condition monitoring systems were introduced. These devices played a critical role in the region's high-temperature and dry climate, where soil conductivity is significantly reduced. The intelligent devices adapted grounding parameters, preventing overloads and network disconnections. As a result, energy losses decreased by 18%, significantly enhancing the plant's efficiency.

These examples demonstrate that modern IGDs significantly enhance the resilience of power systems under diverse operating conditions. They provide the following advantages:

- Reduced System Downtime: Accelerated restoration of network parameters minimizes outages.
- Lower Operational Costs: Predictive maintenance decreases maintenance expenses.
- Increased Equipment Lifespan: Protection against external impacts extends the service life of critical components.

The integration of IGDs into power systems not only enhances their reliability but also contributes to economic efficiency, making these technologies a vital component of modern energy infrastructure.

IV. Conclusions

1. The analysis demonstrated that the parameters of grounding devices significantly impact transient processes in the power system, including voltage recovery and the system's dynamic stability after short circuits or lightning strikes.
2. Optimization of grounding device resistance helps reduce overvoltages and voltage recovery times, thereby improving the operational reliability of the power system.
3. Numerical modeling revealed that selecting a grounding device configuration tailored to the specific conditions of the power system minimizes angular oscillations of generator rotors and reduces the risk of loss of synchronism.
4. Grounding devices with low resistance were found to be the most effective for enhancing the stability of the power system, as they ensure rapid dissipation of fault currents and restoration of normal operating conditions.
5. The research findings can be valuable for designing new power systems, upgrading existing ones, and developing regulatory standards for grounding devices aimed at improving system stability and reliability.

In conclusion, the proposed approaches to analyzing and optimizing the parameters of grounding devices open new possibilities for enhancing the stability of power systems under real operational conditions.

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RELIABILITY, AVAILABILITY AND MAINTAINABILITY OF A BOILER IN THERMAL POWER PLANT– A CASE STUDY

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Abstract

Many countries face problems in electricity generation. Boilers play an important role in a power plant. Sudden failures of a power plant boiler components cause loss of production and high maintenance cost. Due to unplanned and irregular maintenance, which can ultimately increase the production cost of electricity. This is a common challenge faced by power plant operators worldwide. The present study aims to examine and analyze the failure times of a boiler at a thermal power plant and identify its critical failure expectancy and system reliability. The data was collected over a long period and was analyzed using statistical methods. In this study, the hypothesis has been proposed to choose the best analysis. Furthermore, reliability, availability, and maintainability analysis were carried out under discrete analysis. The analysis included identifying the probability distribution of the failure times, identifying critical failure expectancy, and determining system reliability.

Keywords: Boiler, mean time between failure (MTBF), mean time to repair (MTTR), reliability, availability, maintainability, log-normal distribution, Weibull distribution, hazard function, survival function.

1. Introduction

Uninterrupted power supply functioning be influenced by its equipment and components. The functioning of boiler tubes, super heaters, heat exchangers, turbines, etc., is important to maintain the power supply in fossil fuel-based power plants. A single component failure also leads to the shutdown of the entire power generation system. Boilers are designed to operate at high temperatures, pressure and they play a critical role in the efficient, reliable operation of a thermal power plant. Analysis of failures of a boiler in a power plant is important to continuous power generation. The boiler is a key component of a thermal power plant and is used to convert water into steam, which is then used to generate electricity. The steam is produced by heating water in the boiler through the combustion of fuel, such as coal or natural gas. The steam sent through a turbine, which powers a generator to produce electricity. The steam is then cooled and condensed

back into water, which is then returned to the boiler to be heated again. Boiler components are mainly made of steel, cast irons stainless steel, and high-temperature alloys. The combustion, ignition, and fuel-feeding systems are also equally important in the reliability and availability study of the boiler also few authors focused on the failure analysis of these systems. The availability of the steam boiler is a key factor, as it affects the performance and productivity of the process industry. The availability of the steam boiler and its components depends on its reliability and maintainability and can be enhanced by avoiding the number of failures and decreasing the time required scheduled and unscheduled maintenance activities. System availability can be enhanced by identifying critical mechanical subsystems concerning failure frequency, reliability and maintainability.

For the system reliability analysis, it is necessary to classify the system into various levels such as assembly and sub assembly components. Reliability analysis can be done with the collection of failure data from a variety of sources. After collecting the failure data, criticality analysis is needed for the identification of critical parts, the next stage is to estimate the parameters of the distribution and finally, the reliability characteristics are found.

Failure of boiler tubes has often been reported in many such power plants [1]– [3]. Duarte et al. [4] studied failure analysis of water-tube boiler as case study and identified that the occurrence of failure of the water-tube boiler as stress corrosion cracking. Moghanlou and Pourgol-Mohammad [5] investigated on failures of boiler tubes in power plants by Failure Modes and Effect Analysis (FMEA) and stochastic technique. S. Chaudhari and R. Singh [6] focused on high temperature boiler tube failures specifically on preheater tube, carbon steel and superheater tubes. Barry and Hudson [7] suggested a probabilistic feedback scheme for the maintenance of fuel feeding system. By reviewing the literature on failures of boiler and its components, it is found that the data which was recorded is necessary to find boiler's reliability and to enhance maintenance strategies for boiler system.

In this article, failure of boilers of the electricity power plant has been analyzed. Discrete and continuous analysis has been done by considering suitable distributions. Plotted probability and cumulative probability distribution functions and hazard rate curve

2. Methodology

2.1. Discrete Analysis

For performing discrete analysis, the failure data is studied on yearly basis, and outage frequency, forced outage hour, service hour, and period hour are calculated for each year. Further, the mean time between failure (MTBF), mean time to repair (MTTR), repair rate, failure rate, reliability, and availability are calculated and noted in different columns.

2.1.1 Reliability

If the failure rate remains constant for each year thus reliability is calculated using an exponential distribution.

$$Failure\ rate = \frac{Total\ no.of\ failures}{Total\ operating\ hours}$$

The reliability is given by

$$R(t) = e^{-\lambda t}, t > 0, \lambda > 0$$

where t is time

2.1.2 Availability

The availability is calculated using uptime and down time.

$$Availability = \frac{Up\ time}{Up\ time + Down\ time}$$

2.1.3 Maintainability

For calculating the maintainability, the mean time between failure (MTBF) and mean time to repair (MTTR) is calculated for each year.

$$MTBF = \frac{Total\ Operating\ Hours}{Total\ no.of\ failures}$$

$$MTTR = \frac{Total\ Forced\ Outage\ Hours}{Total\ no.of\ failures}$$

2.2 Continuous Analysis

2.2.1 Goodness of Fit

Goodness of fit is a statistical measure used to evaluate how well a given model fits a set of data. It measures the degree of agreement between the observed data and the expected values predicted by hypothesis. The goodness of fit can be calculated using various statistical tests, such as the Kolmogorov-Smirnov test, or Anderson Darling test, etc., which assess the statistical significance of the differences between the observed and expected values. A high value of goodness of fit indicates that the model or hypothesis is a good fit for the data, while a low value suggests that the model may not be appropriate or that there may be some underlying factors that the model does not account for.

2.2.2 Common Life Distribution

The reliability of any repairable system may be increased after its repair. So, finding the system's reliability is mandatory periodically. For this, a reliability analysis of the system is required to meet the desired reliability.

The following distributions may be included.

- Normal Distribution
- Gamma Distribution
- Log-normal Distribution
- Weibull Distribution
- 3-parameter Log-normal Distribution
- 3-parameter Weibull Distribution

2.2.3 3 -Parameter Log-normal Distribution

The 3-parameter lognormal distribution is a continuous probability distribution widely used to model positively skewed data in various fields, including finance, economics and engineering etc., It is called "lognormal" because its natural logarithm follows a normal distribution. This property is suitable for representing data that is the result of multiplicative processes, such as the product of random variables. It is characterized by three parameters, the shape parameter (μ), the scale parameter(σ) and the location (τ).

The probability density function of the 3-parameter lognormal distribution is given by

$$f(x:\mu,\sigma,\tau)=\left\{\begin{matrix}\frac{1}{\sigma\sqrt{2\pi}}e^{\left(\frac{-(x-\mu)^2}{(2\sigma^2)}\right)}\ if\ x>\tau\\ 0\ if x<\tau,\end{matrix}\right\},\sigma>0$$

2.2.4 3 -Parameter Weibull Distribution

The 3-parameter Weibull distribution is a continuous probability distribution widely used in reliability engineering, survival analysis, and failure modeling. It is characterized by three parameters, the shape parameter (k), the scale parameter(λ) and the location (α). The probability density function (pdf) of the 3-parameter Weibull distribution is given by.

$$f(x:\lambda,k,\alpha)=\left\{\begin{matrix}\frac{k}{\lambda}\left(\frac{x-\alpha}{\lambda}\right)^{k-1}exp\left[-\left(\frac{x-\alpha}{\lambda}\right)^{k}\right],if x\geq\ \alpha\\ 0\ ,if\ x<\alpha\end{matrix}\right\}$$

3. Case Study

The following data is collected from a thermal power plant, which is in Suryapet district, Telangana state, India. The power plant is functioning with two boilers to generate 15 MW/hour.

Tables 1 & 2 below provide an overview of the outage frequency of boiler_1 & boiler_2 respectively over a consecutive 10-year period, including forced outage hours and service hours.

Table 1: *Failures of Boiler_ 1*

SI No	Year	Outage frequency	Forced outage Hour(h)	Service Hour(h)	Period Hour(h)
1	2009	05	657.99	4982.01	5640
2	2010	11	1185.06	7574.94	8760
3	2011	05	2363.28	6396.72	8760
4	2012	02	202.10	8557.9	8760
5	2013	04	180.09	8579.91	8760
6	2014	04	412.22	8347.78	8760
7	2015	05	609.56	8150.44	8760
8	2016	09	1027	7733	8760
9	2017	07	3646.74	5113.26	8760
10	2018	05	380	8380	8760

Table 2: *Failures of Boiler_ 2*

SI No	Year	Outage frequency	Forced outage Hour(h)	Service Hour(h)	Period Hour(h)
1	2009	03	653.33	7170.33	7824
2	2010	04	712.50	8047.5	8760
3	2011	04	1844.18	6915.82	8760
4	2012	04	478.00	8282	8760
5	2013	03	198.9	8561.10	8760
6	2014	02	296.5	8463.5	8760
7	2015	06	756.92	8003.08	8760
8	2016	05	932.98	7827.02	8760
9	2017	04	2552.87	6207.13	8760
10	2018	04	170.61	8589.39	8760

Tables 3 & 4 below provide a detailed summary of the failure hours of boiler_1 & boiler_2 respectively and arranged chronologically from 2009 to 2018.

Table 3: *Failure Data of Boiler_ 1*

SI No	Failure Hours	SI No	Failure Hours	SI No	Failure Hours	SI No	Failure Hours	SI No	Failure Hours	SI No	Failure Hours
1	10.08	11	29.59	21	511.55	31	239.45	41	13.45	51	2518.25
2	459.30	12	6.05	22	146.00	32	284.58	42	29.45	52	305.45
3	49.40	13	50.48	23	56.10	33	5.02	43	40.45	53	27.50
4	59.21	14	9.16	24	37.14	34	236.56	44	116.50	54	11.35
5	80.00	15	88.50	25	21.55	35	83.40	45	511.12	55	12.15
6	63.20	16	553	26	7.20	36	18.16	46	49.30	56	23.55
7	47.40	17	278.32	27	114.20	37	9.50	47	264.37		
8	46.07	18	108.08	28	49.20	38	11.05	48	222.25		
9	286.24	19	302.13	29	49	39	7.29	49	9.45		
10	5.37	20	1163.20	30	74.57	40	781.15	50	72		

Table 4: *Failure Data of Boiler_ 2*

SI No	Failure Hours	SI No	Failure Hours	SI No	Failure Hours	SI No	Failure Hours
1	54	11	757.4	21	239.12	31	109
2	409.15	12	358.55	22	61.25	32	119.15
3	190.18	13	95.5	23	32.56	33	516.47
4	5	14	10.5	24	226.39	34	177.25
5	48.25	15	13.45	25	78.15	35	1740
6	118.10	16	92.10	26	119.25	36	10.11
7	541.15	17	7.35	27	18.16	37	7.15
8	183.19	18	99.45	28	24.07	38	140.20
9	502.42	19	201.10	29	21.20	39	13.15
10	401.17	20	95.4	30	760.55	40	209.30

Table 5 & 6 explain the identification of the distribution that best fits for boiler_1 & boiler_2, by using the Anderson-Darling test along with the correlation coefficient to determine the most appropriate fit.

Table 5: *Goodness of Fit of Boiler _1*

Distribution	Anderson-Darling Test	Correlation Coefficient
Weibull	2.463	0.948
Lognormal	0.631	0.989
Exponential	14.028	*
Normal	8.518	0.678
3-Parameter Weibull	0.605	0.994
3-Parameter Lognormal	0.530	0.994
2-ParameterExponential	6.831	*

Table 6: *Goodness of Fit of Boiler _2*

Distribution	Anderson-Darling Test	Correlation Coefficient
Weibull	0.809	0.977
Lognormal	0.761	0.987
Exponential	3.237	*
Normal	4.375	0.791
3-Parameter Weibull	0.539	0.994
3-Parameter Lognormal	0.667	0.988
2-ParameterExponential	2.658	*

4. Results and Discussions

The failure data of the boilers mentioned in the Table 3 and Table 4 has been organized into year-wise failures by sorting them in a chronological order. The sample data is then analyzed to determine the number of outages that occurred during each year, the mean time between two successive failures, the total number of hours that the system was operational and the total period of each year

Using these metrics, several important reliability and maintenance indicators are calculated, including MTTR, MTBF, failure rate, repair rate, reliability, availability. These measures provide important insights into the performance of the boiler system, helping to identify areas for improvement and optimize maintenance schedules.

4.1 Continuous Analysis

4.1.1 Probability Curve

Figure 1 provides the use of a 3-Parameter Lognormal distribution to model the failure hours of boiler_1. The plot shows a strong alignment of the data points with the line, indicating a good fit. This is further supported by a high correlation coefficient of (0.994) and a low Anderson-Darling statistic of (0.530). The analysis reveals key metrics, including a mean failure time of (256.8) hours and a median failure time of (55.35) hours. These values play a significant role in assessing the

reliability of boiler_1. The close fit of the model highlights its effectiveness in capturing failure trends. This information is essential for evaluating the operational reliability of the system. Additionally, it aids in developing maintenance strategies to minimize downtime.

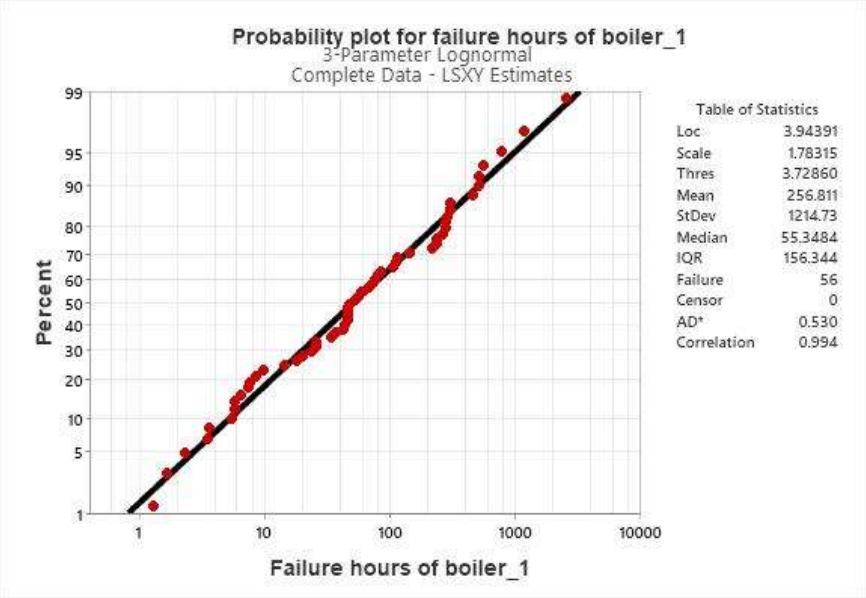

Figure 1: *Probability plot for failure hours of boiler_1*

From the below Figure 2, the plot utilizes a 3-Parameter Weibull distribution to model the failure hours of boiler_2. The data points closely align with the line, indicating a high correlation of (0.994) and a low Anderson-Darling statistic of (0.539) which suggests a strong fit. Notable values include a mean failure time of 232.4 hours and a median of 104.2 hours. This analysis is instrumental in evaluating reliability and informing maintenance strategies.

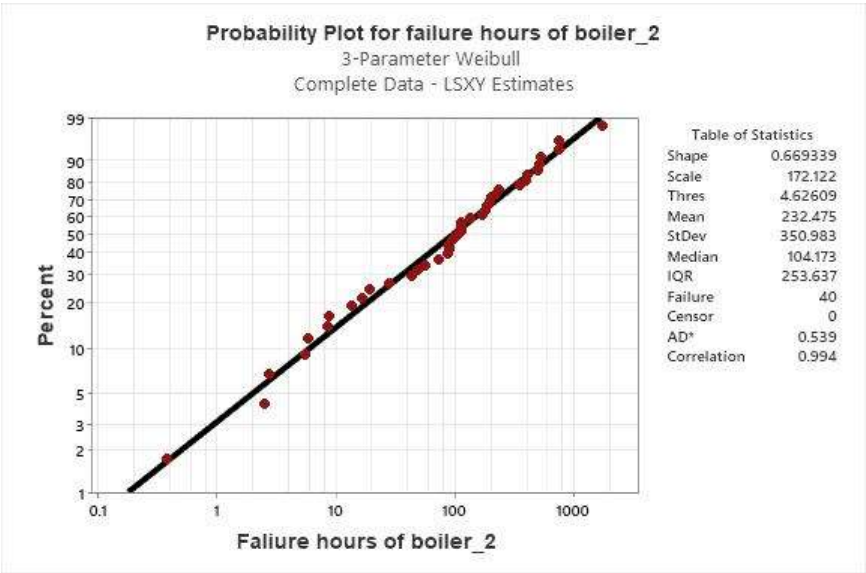

Figure 2: *Probability plot for failure hours of boiler_2*

4.1.2 Probability Density Curve

In figure 3, the curves are steep at the beginning, indicating a higher likelihood of failure shortly after operation begins. As the curves flattens out, the probability of failure decreases for longer operation hours. This distribution helps to identify the most likely failure periods and informs maintenance schedules or risk assessments to reduce unexpected downtime for boiler_1 & boiler_2.

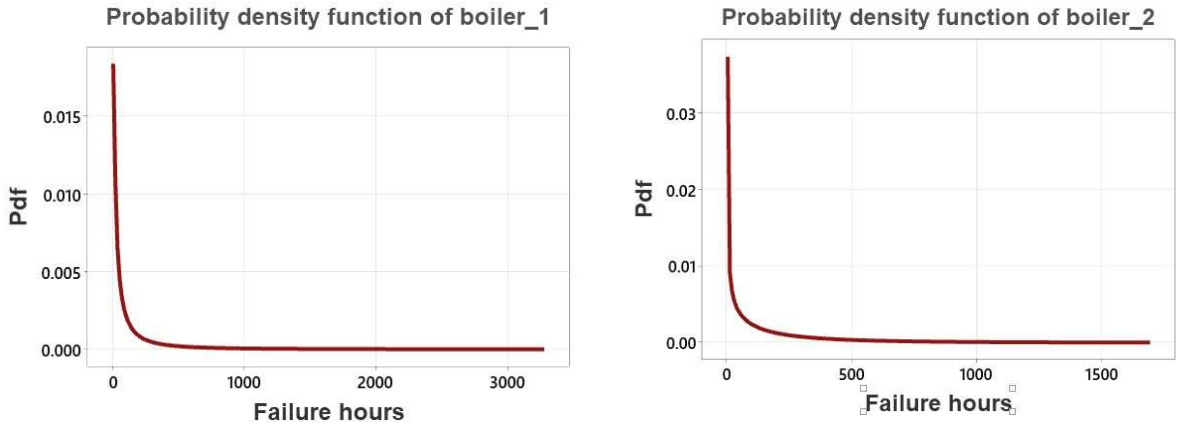

Figure 3: *Probability density function of plot for boiler_1 and boiler_2*

4.1.3 Survival Function

From Figure 4, the survival plots for both boilers start a 100%, indicating that all components are operational initially. For both, the curves show a sharp decline in the early hours, reflecting high failure rates during the initial phase. Over time, the slopes flatten, indicating fewer failures as the lifecycle progresses. Boiler_1 with survival probability nearing 0% by 3250 hours while boiler_2 reaches survival probability of nearly 0% by 1600 hours.

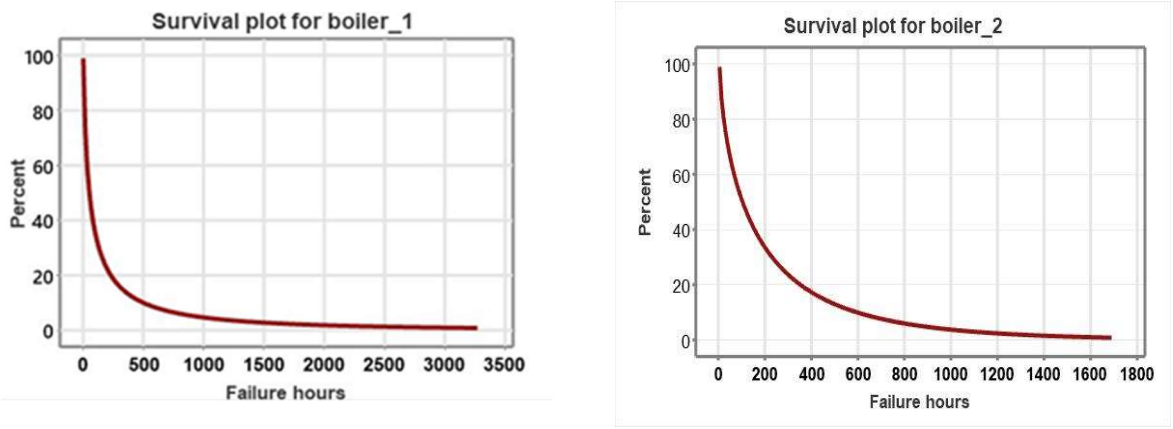

Figure 4: *Survival function plot for boiler_1 and boiler_2*

4.1.4 Hazard Function

From Figure 5, the hazard plots of boler_1 & boiler_2 reveals critical insights into the boiler's reliability. The high initial hazard rate indicates a need to address early-life failures through measures like better quality control, initial inspections, or burn-in testing. The decreasing hazard rate over time suggests that components become more stable as they age. This information is valuable for planning preventive maintenance and improving operational efficiency.

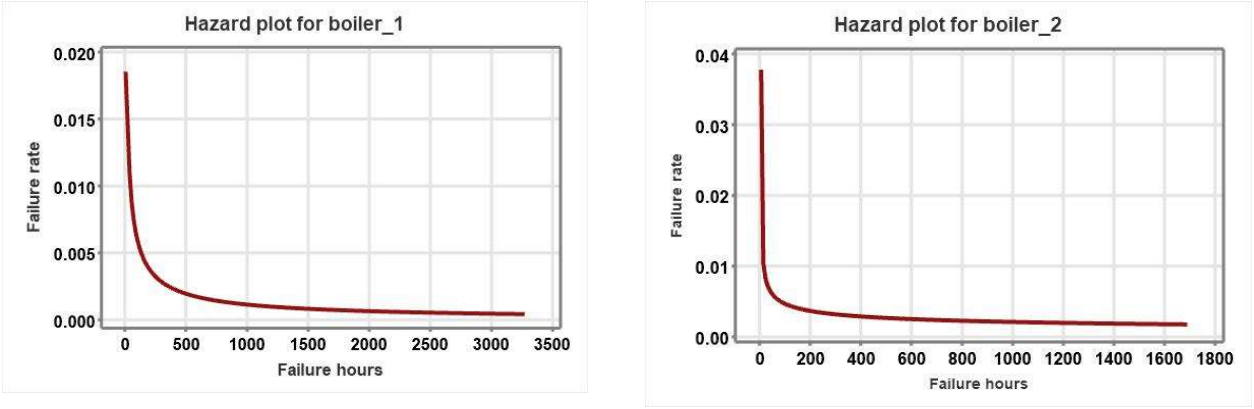

Figure 5: *Hazard function plot for boiler_1 and boiler_2*

4.2 Discrete Analysis

4.2.1 Reliability

In the Figure 6, the reliability of a boiler_1 starts low in 2008 and remains minimal through 2011 before sharply increasing in 2012, reaching its peak. It then declines steeply and stabilizes at lower levels, fluctuating minimally from 2014 to 2018 with no significant recovery. Similarly, the reliability of boiler_2 begins at a low level in 2009, declines further, and stabilizes minimally until 2013. In 2014, it peaks sharply but drops significantly in 2015 to its lowest point, followed by a gradual recovery with slight improvements through 2018.

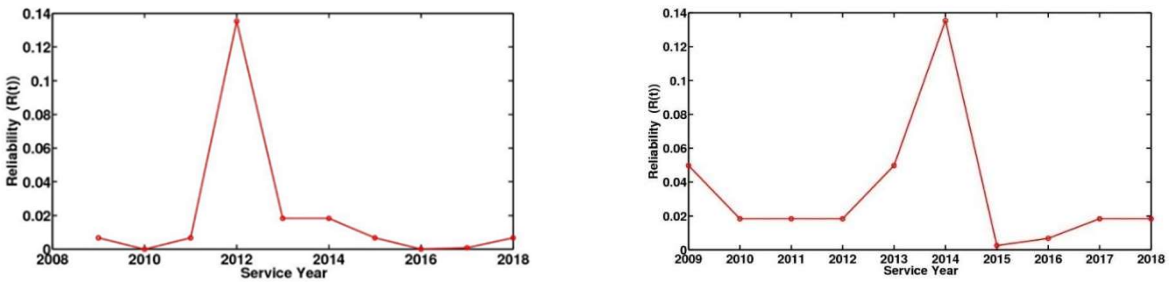

Figure 6: *Reliability plot for boiler_1 and boiler_2*

4.2.2 Availability

Figure 7 provides an overview of the availability rends for boiler_1 and boiler_2 over several years. Boiler_1 showed high availability in 2012, which remained consistent through 2013. From 2013 to

2016, its availability gradually declined, followed by a steep drop in 2017. Despite this downturn, 2018 marked a significant recovery, with availability increasing substantially. In comparison, boiler_2 experienced very low availability during 2017, indicating a challenging phase. However, 2018 brought a remarkable improvement in its performance, with availability rising sharply. These trends highlight contrasting patterns for the two boilers, with boiler_1 recovering from a decline and boiler_2 overcoming its earlier low performance. By 2018, both boilers exhibited significant improvement, underscoring their recovery and stability.

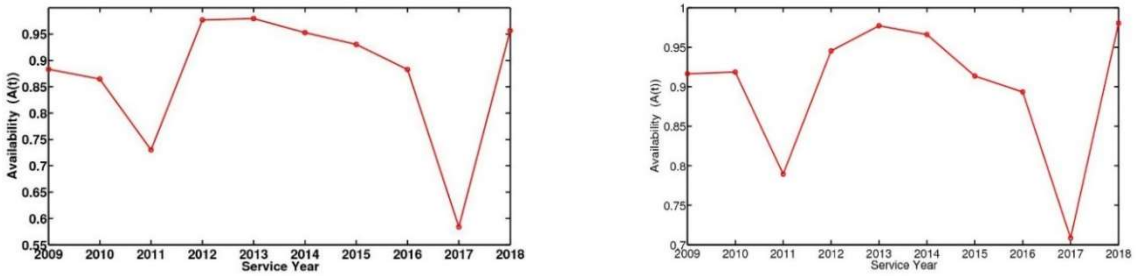

Figure 7: *Availability plot for boiler_1 and boiler_2*

4.2.3 Maintainability

From Figure 8, it shows that boiler_1 had a low MTTR in 2013, while a high MTTR was recorded in 2017. In contrast, boiler_2 experienced a low MTTR in 2018, but high MTTR was also recorded in the same year 2018.

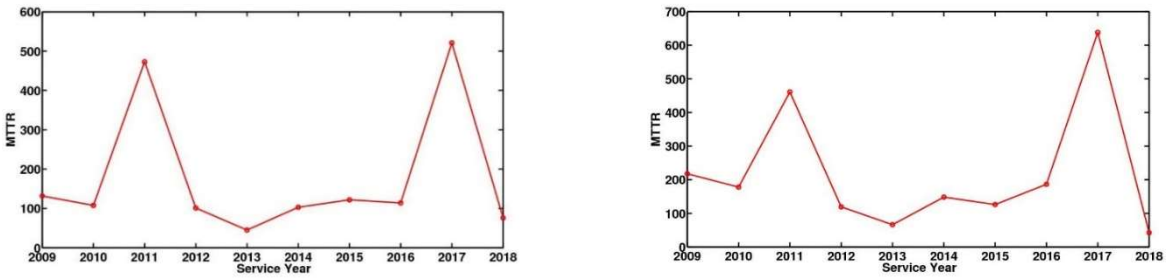

Figure 8: *Maintainability plot for boiler_1 and boiler_2*

5. Conclusion

In this paper, the work is based on discrete and continuous analysis using suitable distributions. The study involves analysis of failure times of a boiler system of a power generation plant. The data fits log-normal and Weibull distributions of boiler_1 and boiler_2 respectively, shape and scale parameters are determined. Using the sorted data, the probability density curve, cumulative probability curve and hazard rate curves are plotted. By these plots, it is easy to make possible predictions of the future tendency of failures of the boilers.

Discrete analysis is carried out to analyze failures of two boilers in the power plant over a period to identify patterns and trends in the frequency and nature of the failures. This analysis is conducted on a yearly basis, with the goal of identifying the effect of these failures on the overall performance of the boilers. In 2010 and 2016, the reliability of boiler_1 is very low whereas in 2015,

the reliability of boiler_2 is very low. The availability of boiler_1 in 2012, 2013, and 2018 is very high. High availability suggests efficient operation, contributing to consistent power generation. Whereas in 2017, the availability of two boilers is less. For decreased availability, conducting root cause analysis to identify the underlying issues is mandatory. In 2017, the mean time to repair is very high for two boilers. It indicates some maintenance strategies must be planned. The information gathered from this analysis is used to create a plan that addresses the identified issues and implements actions to improve the reliability, availability and maintainability of the boiler. By doing so, the organization reduce downtime, improve performance and prevent future failures.

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PROBABILISTIC INVENTORY MODEL FOR DETERIORATING ITEMS WITH UNCERTAIN DEMAND UNDER PENTAGONAL FUZZY ENVIRONMENT

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Abstract

Using a pentagonal fuzzy framework, this research presents a probabilistic inventory model for deteriorating items under an uncertain demand. Degeneration of items puts a company's financial ability to meet its objectives at risk. Few models have synchronized optimization over this whole scenario with all components, according to a survey of the literature. It deals with the difficulties of inventory control in situations where demand is represented by fuzzy sets but is not precisely known. The model offers a clearer and more useful understanding of demand uncertainty by defuzzifying pentagonal fuzzy numbers using the Graded Mean Integration Representation (GMIR) approach. The goal of the study is to optimize inventory levels in order to minimize total costs, which include holding, degradation, shortage, and purchase. These components are included into a mathematical model, and numerical scenarios are shown to compare the both potential strategies. The sensitivity of the solution and decision variables with respect to different inventory characteristics is examined in both crisp and fuzzy settings. Fuzzy logic is integrated into the model to provide a strong framework for making decisions when dealing with ambiguous demand and the complications that come with deteriorating inventory. The paper includes numerical examples and sensitivity analyses to demonstrate the model™s effectiveness and practical relevance. These findings provide valuable guidance for inventory managers aiming to improve decision-making and operational efficiency in contexts with fuzzy demand and deteriorating products. At the optimal position, the total cost is relatively inelastic to an increase in base deterioration rate and more elastic to a decrease in it. Although the crisp example is marginally less efficient per unit cost, total costs are lower than in the fuzzy case, which is to be expected given the fuzzy case's potential for superior results.

Keywords: Probabilistic Inventory Model, Deteriorating Items, Uncertain Demand, Pentagonal Fuzzy Environment, Graded mean integration representation (GMIR).

1. Introduction

In the current situation, Showrooming and time-sensitive processes are closely related. Due to the coronavirus incident in this case, unique protocols necessitated significant modifications to the stock structure. The global health crisis has compelled businesses to rethink their current and upcoming marketing strategies in order to sustain a steady stream of revenue. Short-term effort could focus on multiple goals, such as more advanced plans or improved consumer perception. Potential benefits could also include increased employee inspiration and an increase in in-store

visitors. The objective is to increase revenue and get rid of excess goods. Thus, inventory control is essential to every sophisticated, modern business. There are several benefits to having well-managed inventories, including direct profits and devoted customers. Furthermore, the intricate connections between these several business objectives uphold the astounding significance of inventory management. Because of its ability to address a wide range of problems and its mathematical methodology, the continuous review has garnered more attention than the periodic review.

Disintegration is a character that arises from natural problems during caching and is represented by degradation, harm, decay, hurt, or other changes in item quality. Items such as batteries, semiconductor chips, food assortments, unstable fluids, and therapeutic items such as blood face degeneration and gradually lose potential are a few instances. Managing and remaining cognizant of the stock framework's decomposing goods inventories is a major concern. The aim of inventory management is to increase business profits by reducing wasteful inventory and deteriorating items are a hindrance to this goal. One way to think of the rate at which products deteriorate is as a dependent variable that can be managed with protection innovation. Businesses are aware that they have to control degradation losses to the letter. Enhancing and modernizing storage procedures is one of the typical control strategies. Through feasible capital input along these channels, retailers can slow down the rate at which things deteriorate, avoiding unnecessary waste, limiting financial losses, and improving business efficiency. These degradation control models have received a great deal of attention and are more in line with the real inventory conditions. In today's volatile marketplaces, precise inventory control is especially important for perishable products. For instance, the retailer's reputation and goodwill will suffer due to food deterioration. Weakening increases the associations cost and therefore reduces the advantage, which is a major cause of stock misfortune. The weakening of interactions caused by oxidation, chemicals, and microbes frequently depends on ecological factors like environment, temperature, and stickiness. Temperature regulation is necessary to maintain product quality as it has a substantial impact on deterioration. Innovative protective measures, such temperature-regulating equipment and creative bundling, can affect the rate of weakening and postpone the crumbling cycle.

At that time, when management introduces a new product, they don't fully understand the market and other aspects of the product. The analysis of experts is trusted by the management. Fuzzy principles can be used to represent demand or other elements connected to the expert opinion when the counsel is imprecise. The resulting environment is referred to as a fuzzy environment. This work orders a new way to improve demand forecasting, which is one of the main challenges of a continuous review inventory model. The impact of fuzzy demand across an infinite time horizon is further investigated in this work. For a continuous review inventory system, the best operating strategy is looked for in order to minimize the overall layout in a fuzzy setup. It is believed that the full-backordering method will balance the loss component during the inventory shortage. The optimal policy is analyzed and both crisp and fuzzy examples of a continuous review inventory system are numerically solved. The findings provided can greatly benefit decision makers' methods in situations of uncertain demand and lower total inventory costs.

The approach is complicated by practical concerns like attenuated deterioration, demand affected by both fuzzy base and promotional upscaling. It has been suggested that analytical convexity be taken into account without deterioration factor approximation. Because of the infinite Time horizon, the continuous domain for cycle length leads to continuous time duration Variables. Given these factors, the study's uniqueness and significance are that it contrasts an inventory model for ongoing evaluation, it enables an inventory model to perform a thorough exponential depreciation attenuation analysis. With a pentagonal fuzzy foundation and a term that scales with promotion efforts, demand uncertainty is modelled in a more realistic scheme for which the retention approach is intended.

Forecasting demand patterns is a prerequisite for the a priori planning decision-making process for retailers and sellers. Flexibility in resource and operation management can enhance

the arrangement's overall performance. The first model, which follows, is a useful mathematical prototype and employs the pure deterministic situation in which the decision-maker has access to precise demand information. The second model, which employs a fuzzy formulation for the imprecisely forecasted demand, goes back to the first claim of flexibility in addressing uncertainty. When combined, the two models give the researcher a clear and succinct understanding of the mathematical process and the financial rationale for using fuzziness to address uncertainty and degradation, respectively.

The current article's organization follows the following structure: Section 2 has the literature review, which reviews past research that provides context for this work. Section 3 includes the research question and the presumptions used to help plan the model's layout. The utilized notations are tallied. The modelling approach and solutions for the environments that are both crisp and fuzzy are covered in Section 4.In the following, numerical examples are used to illustrate how this paradigm can be applied in a real-world scenario Section 5. Section 6 contains the administrative architecture and sensitivity analysis for the inventory systems. In Section 7, the research's conclusions and future directions are examined.

2. Literature review

2.1. Deterioration

Inventory management has extensively studied Deterioration. Food rotting due to oxidation or microorganisms is a common occurrence. Storage of electronic items must take into consideration contamination, moisture, and electrostatic discharge damage. Pervin et al. [1] established an EOQ model for perishable commodities, taking into account time-dependent holding costs and demand that fluctuated with stock level. Pervin et al. [2] created a multi-item inventory model that considered constant rate of deterioration, on-demand, and trade credits. Barman et al. [3] examined an economic production quantity (EPQ) model in a fuzzy environment with shortages and inflation, with time-dependent demand and a fixed rate of deterioration. Roy et al. [4] created a probabilistic system for decaying items with two warehouses, two credit levels. Roy et al. [5] suggested a credit strategy for a deteriorating product and an imperfect production system with a partial backlog. Khan et al. [6] examined a system that had a variable demand pattern, constant deterioration, and delayed payment. Currently, Shah et al. [7] examined the situation where products demand and deteriorate varies according to selling price while taking the greening effect into account. Yadav et al. [56] optimize an inventory model for deteriorating items using a two-warehouse system, highlighting the need to balance cost and efficiency in managing perishable goods. Yadav, K.K., Yadav, A.S., and Bansal, S. [57] employ an interval number technique to enhance two-warehouse inventory management while considering preservation technology investments, demonstrating the advantages of resource allocation for cost savings and efficiency. Yadav, A.S., Kumar, A., and Yadav, K.K. [58] present a model that incorporates carbon emissions and time-sensitive demand in optimizing inventory for deteriorating items, focusing on sustainable management practices. Mahata and Debnath [59] tackle a profit-maximizing problem in single-item inventory management by considering price-dependent demand and preservation technologies, illustrating the synergy between preservation strategies and demand dynamics to improve profitability and efficiency.

Researchers have been interested in the difficult task of managing inventory items that are naturally decaying for decades. Any industry that experiences Deterioration suffers financial losses as a result of this phenomena. It is, nevertheless, a normal and inevitable procedure. Therefore, during operations management, strategic choices to avoid the aforementioned loss and its impact have generated a lot of attention in a variety of real-life situations. Ghare and Schrader [8] conducted the first study using degrading objects for exponentially deteriorating items. Subsequently, using the discounted cash flow (DCF) method, Jaggi and Aggarwal [9] investigated the best ordering strategy for deteriorating goods while assumption trade credit. Aggarwal and Jaggi [10] investigated the best ordering policy for deteriorating commodities

based on the allowable payment delay. The optimum credit policies for degrading things were recently shown by Jaggi et al. [11] under the presumptions of faulty items, rapidly expanding demand and partial backlog. Additionally, Mandal et al. [12] postulated an inventory model based on geometric programming that included deteriorating items. Afterwards, Panda et al. [13] developed an inventory model for a seasonal product with ramp-type demand. Bakker et al. [14] provides a thorough analysis of inventory models with deteriorating items. Khanna et al. [15] have produced some excellent work on deterioration that is worthy of attention in this context. Additionally, Jaggi et al. [16] developed the best course of action for defective and deteriorating items while taking into account a two-warehouse situation. Controllable probabilistic deterioration with shortages was examined by Mishra [17]. Jaggi et al. [18] also studied at price-dependent demand and two warehouses in non-instantaneous deterioration. A replenishing scenario for a deteriorating item was examined by Pervin et al. [19] under the presumptions of time-dependent holding costs, time-dependent demand, and shortages. A sustainable three-tier inventory model for decaying products was studied by Daryanto et al. [20]. Shaw et al. [21] studied an integrated model that took into account multi-stage inspection, carbon emissions, and deterioration while putting single setup multi delivery (SSMD) policy into practice for the delivery of high-quality products.

2.2. Probabilistic demand

Given the current state of the market, it is becoming more difficult to precisely estimate client preferences for a product; therefore, a probabilistic demand method is a better fit for handling uncertainty. Shah, Nita H. [22] developed a probabilistic inventory model with allowable payment delays, thereby spearheading the development of such models. On the basis of this, Shah, Nita H., and Y. K. Shah. [23] Expanded the model to include trade credit policy and declining products over a specific time interval. Several scholars, such as Shah, Nita H. [24] and Shah, Nita H. [25], developed inventory models that included trade credit finance, probabilistic demand, and shortages. Federgruen, Awi and Aliza Heching [26] Looked into simultaneous inventory and pricing decisions under probabilistic demand. Petruzzi, Nicholas C. and Maqbool Dada [27] developed a price-sensitive inventory model and perishable goods into account to determine the best pricing in the newsvendor scenario. Chen, Xin, and David Simchi-Levi [28] examined a periodic review model for an infinite planning horizon in order to determine the best pricing and inventory strategies with probabilistic demand. Under probabilistic demand, Khedlekar et.al. [29] Examined the optimal replenishment choices taking into account pricing, promotion tactics, and inventory. Chao, Xiuli, Baimei Yang, and Yifan Xu [30] to maximize pricing, a capacitated probabilistic inventory system was proposed. Maihami, Reza, and Behrooz Karimi [31] developed an optimal replenishment plan that takes into account promotional efforts for non-instantaneously deteriorating products with price-sensitive probabilistic demand. Inventory control methods under probabilistic demand were provided by Roy et al. [32] and AlDurgam et al. [33] with a range of parameters and assumptions. Probability distributions are typically used to reflect demand uncertainty. With price-sensitive probabilistic demand and non-instantaneous deteriorating products, they created an inventory model for profit maximization that took additive promotional activities into account. The bulk of research is done on probabilistic demand functions that are sensitive to price. Shah, Nita H., et al. [34] for a full view, consider a demand rate that is uniformly distributed and is influenced by price, inventory level, and advertisement.

2.3. Pentagonal Fuzzy Number

There are several varieties of polygon fuzzy numbers in classical fuzzy theory, including trapezoidal, pentagonal, and triangular fuzzy numbers, among others. Srinivasan [35] provided a method for solving TP using generalized pentagonal and hexagonal fuzzy numbers throughout the literature. Additionally, Karthikeyan [36] provided a method for using pentagonal fuzzy numbers to solve transportation problems (TP). Maheswari and Ganesan [37] provide a technique

that uses pentagonal fuzzy numbers to solve completely fuzzy TP. Chakraborty [38] has investigated representations of pentagonal fuzzy numbers. Barazandeh and Ghazanfari have tackled the ranking approach for generalized fuzzy numbers [39]. Membership functions for symmetric and asymmetric hexagonal fuzzy numbers: an overview was extended to non-linear membership functions by Khan and Mondal [40]. Mondal [41] used the average technique to find arithmetic operations and provided representations for a variety of non-linear membership functions. Arora, Aparna, Rashmi Gupta, and Ratnesh Rajan Saxena [42] Asymmetric Pentagonal Fuzzy Numbers as a representation of costs (APFN).

2.4. Fuzzy modeling of uncertainty

Demand uncertainty arises from several unknown components of inventory models, aside from preservation technologies, deteriorating commodities, and promotional efforts. However, in practical circumstances, the uncertain parameters such as lead time, preservation cost, demand, and other pertinent expenses may be more likely to deviate from the exact value, which could result in a situation where the uncertain parameters are not distributed according to any probability. Originally, the fuzzy set concept was developed by Zadeh [43]. Following that, a number of trailblazing scholars developed several fuzzy inventory models to capture the impreciseness, including Yao et al. [44], Glock et al. [45], and Shah and Soni [46]. By taking into account trapezoidal fuzzy numbers, the model examined by Garai et al. [47] had holding costs that scaled with price-dependent and time demand. Shah and Patel [48] created an inventory model and employed preservation technologies to lower the rate of spoiling under a cloud hazy prescription. In a fuzzy setting, Yadav et al. [49] examined a flexible manufacturing system with a changeable pollution control. De and Mahata [50] used a learning environment for dense fuzzy demand when there was an order overlap with rework batches. Kumar and Paikray [51] modelled the time-varying demand for decaying commodities using crisp and fuzzy formulations with three distinct scenarios under total backlog. To effectively address a fuzzy inflationary model, Sarkar et al. [52] used a multithreaded neural network. Fuzzy logic, specifically graded mean integration-representation distance, is used by this similarity function (GMIR). We can incorporate more flexible data agglomeration strategies thanks to fuzzy logic [53]. This model used a pentagonal fuzzy number with the GMIR difuzyfication method.

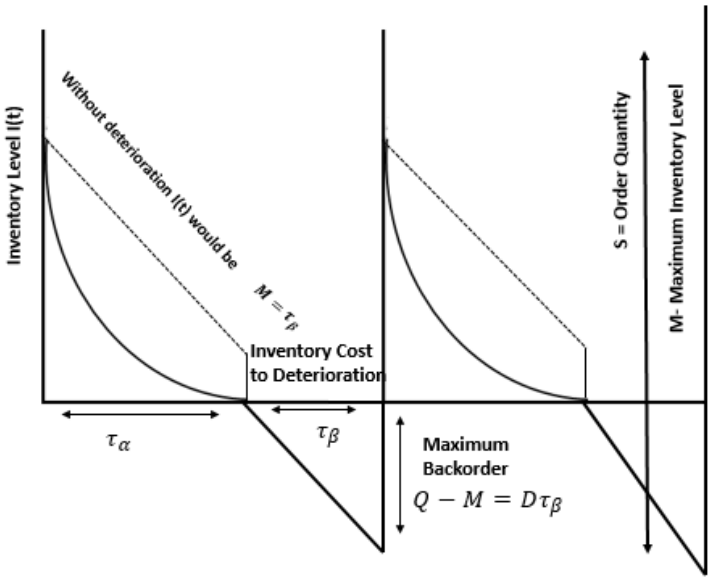

Figure 1: *Inventory Model with backorder and deterioration*

3. Assumptions and notations

3.1. Assumptions

1. The inventory system is examined using a single item.
2. An infinite planning horizon is taken into consideration.
3. Additive price-sensitive probabilistic demand $D = D_0 + \epsilon$ where D_0 is fixed base demand while ϵ continuous random variable with expected value μ.
4. Shortages are allowed, and lead time is zero.
5. The rate of production exceeding the rate of demand.

3.2. Notations

Table 1: *Symbols and Description*

Symbol	Description
S	Order quantity per unit time
D	Rate of demand per unit of time
A	Ordering cost per order ($)
K	Holding cost per order ($)
L	Deteriorating cost per order ($)
m	Shortage cost per order ($)
α_0	constant rate of inventory item deterioration $(0 < \alpha_0 < 1)$
α	Effective deterioration rate, dampened by use of $(\alpha = \alpha_0 e^{-\mu})$
μ	Expected value of Continuous random variable ϵ
M	Maximum positive inventory level´ at time $t = 0$
$I(t)$	Inventory level at the time
τ_α	The duration of time it takes for inventory to zero following replenishing
τ_β	Order backlog occurs in the interval between having zero inventory and replenishing it (represented by negative inventory level)
$TAC(\tau_\alpha, \tau_\beta)$	Total average cost of inventory (Model-1)($/per unit time)
$\widetilde{TAC}(\tau_\alpha, \tau_\beta)$	Total average cost for fuzzy environment (Model-2)($/per unit time)

4. Mathematical Model

This section lays out the models and their approach for solving them. The total cost function is obtained by setting up and solving the governing differential equations. This objective function is subjected to necessary and sufficient criteria for convexity and global optimality. When fuzzy parameters are utilized, defuzzification is applied.

4.1. A model for continuous review inventory that has constant deterioration rate and crisp demand (Model-1)

A continual evaluation Based on the aforementioned presumptions, the EOQ setup is created. An immediate restocking initially the cycle at $t = 0$ and the inventory level surges to its highest point, $M = I(0)$. $I(t)$ decreases in time interval $[0, \tau_\alpha]$ as certain components are lost to degradation and others are consumed according to demand. Every unit has been utilized at $t = \tau_\alpha$ So $I(\tau_\alpha) = 0$. Backorders are maintained for the duration of $[\tau_\alpha, \tau_{\beta+\alpha}]$ this must be satisfied from the upcoming replenishment due to their complete backlogged. These presumptions allow the differential equations controlling the subsequent cases to be determined:

Case 1 $(0 \leq t \leq \tau_\alpha)$:Inventory is depleted by loss from degradation and consumption brought on by demand; consequently, the inventory level equation $I(t)$:

$$\frac{dI(t)}{dt} + \alpha I(t) = -D; \quad 0 \leq t \leq \tau_\alpha \tag{1}$$

The deterioration term $\alpha(t)$ is in parallel to the current inventory that is on hand and $\alpha = \alpha_0$ is constant in Model 1. On the differential equation (1), the boundary condition $I(\alpha_0) = 0$is used to determine the inventory level.

$$\frac{dI(t)}{dt} = \frac{D}{\alpha_0}(e^{\alpha_0(\tau_\alpha - t)} - 1); \quad 0 \leq t \leq \tau_\alpha \tag{2}$$

Using equation (2) the topest inventory level M is at $t = 0$ in the following manner:

$$M = I(0) = \frac{D}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - 1) \tag{3}$$

Now, equation (3) is distinct from the absence of deterioration where $M = D\tau_\alpha$. The difference between the two provides the amount of inventory lost as a result of deterioration, in the buyer's inventory model that was examined in Wee et al. [54], given by

$$M - D\tau_\alpha = \frac{D}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1) \tag{4}$$

Case 2 $(\tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta)$: Backorders resulting from a negative inventory level $I(t)$, as well as shortages during the period, should be taken into account. There is only one term, as orders are wholly backlogged, which is due to demand. Consequently, the differential equations that follow

$$\frac{dI(t)}{dt} = -D; \quad \tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta \tag{5}$$

Now, applying boundary condition $I(\tau_\alpha) = 0$; equation (5) results in the following inventory level expression:

$$I(t) = -D(t - \tau_\alpha); \quad \tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta \tag{6}$$

In fact, this negative inventory level suggests that the backorder at that point in time t in $(\tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta)$ is $D(t - \tau_\alpha)$. Here $I(\tau_\alpha + \tau_\beta) = -D\tau_\beta$ is called lowest inventory level and the maximum backorder is $D\tau_\beta$. The maximum inventory level is provided by the leftover inventory after ordered amount S completes this backorder first.

$$S - D\tau_\beta = M \Rightarrow S = D\tau_\beta + \frac{D}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - 1) \tag{7}$$

In equation (7), the order quantity is higher than in the traditional backorder approach and does not deterioration $D(\tau_\alpha + \tau_\beta)$, since it must meet demand in addition to replacing products lost to deterioration. Consequently, the components of the total inventory cost are as follows:

Ordering Cost (OC)

$$OC = A$$

Holding Cost (HC):

$$HC = \int_0^{\tau_\alpha} \frac{DK}{\alpha_0}(e^{\alpha_0(\tau_\alpha - t)} - 1)dt = \frac{DK}{\alpha_0^2}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)$$

Shortage Cost (SC):

$$SC = \int_{\tau_\alpha}^{\tau_\alpha + \tau_\beta} (-I(t))sdt = \int_{\tau_\alpha}^{\tau_\alpha + \tau_\beta} D(t - \tau_\alpha)sdt = \frac{\tau_\beta^2 Dm}{2}$$

Cost as a result of the deteriorated goods (w is the cost of deterioration per unit) (DC):

$$DC = (M - D\tau_\alpha)w = \frac{DL}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)$$

As a result, the total inventory cost every cycle, taking into account expenditures associated with deterioration but not preservation, is as follows:

$$TAC(\tau_\alpha, \tau_\beta) = \frac{1}{\tau_\beta + \tau_\alpha}[OC + HC + SC + DC]$$

$$TAC(\tau_\alpha, \tau_\beta) = \frac{1}{\tau_\beta + \tau_\alpha}\left[A + \frac{DK}{\alpha_0^2}(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1) + \frac{\tau_\beta^2 Dm}{2} + \frac{DL}{\alpha_0}(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)\right]$$

$$TAC(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D(K + \alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2(\tau_\beta + \tau_\alpha)} + \frac{\tau_\beta^2 Dm}{2(\tau_\beta + \tau_\alpha)} \tag{8}$$

5. Optimization methodology

The decision variable values are obtained at the lowest total cost per cycle ($) through the classical optimization process. There are two components to the computational process.

Step 1: Obtain critical point $(\tau_\alpha^*, \tau_\beta^*)$ satisfying $\frac{\partial TAC}{\partial \tau_\alpha} = 0$ and $\frac{\partial TAC}{\partial \tau_\beta} = 0$.

Step 2: Verify the convexity of $TAC(\tau_\alpha, \tau_\beta)$ by proving that (in the feasible region)

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} > 0$$

and

$$\frac{\partial^2 TAC}{\partial \tau_\alpha^2} \cdot \frac{\partial^2 TAC}{\partial \tau_\beta^2} - \left[\frac{\partial^2 TAC}{\partial \tau_\alpha \partial \tau_\beta}\right]^2 > 0$$

The factors that do not contain the decision variables in equation (8) are gathered and categorized for the sake of algebraic efficiency.

$$TAC(\tau_\alpha, \tau_\beta) = \frac{A_1}{\tau_\beta + \tau_\alpha} + \frac{B_1\tau_\beta^2}{(\tau_\beta + \tau_\alpha)} + \frac{C_1(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{(\tau_\beta + \tau_\alpha)} \tag{9}$$

where, $A_1 = A$, $B_1 = \frac{Dm}{2}$, $C_1 = \frac{D(K+\theta_0 L)}{\alpha_0^2}$ The positive inventory time τ_α and negative inventory time τ_β cause a continuous fluctuation in the total cost per time unit. The decision variables τ_α and τ_β are employed to minimize this objective function. The first-order partial derivatives of $TAC(\tau_\alpha, \tau_\beta)$ are determined by equation (9).

$$\frac{\partial TAC}{\partial \tau_\alpha} = \frac{\alpha_0 C_1(e^{\alpha_0\tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)} - \frac{TAC}{\tau_\beta + \tau_\alpha} \tag{10}$$

$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{2B_1\tau_\beta}{(\tau_\beta + \tau_\alpha)} - \frac{TAC}{\tau_\beta + \tau_\alpha} \tag{11}$$

The second-order partial derivatives of $TAC(\tau_\alpha, \tau_\beta)$ are

$$\frac{\partial^2 TAC}{\partial \tau_\alpha^2} = \frac{2TAC}{(\tau_\beta + \tau_\alpha)^2} - \frac{2\alpha_0 C_1(e^{\alpha_0\tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)^2} + \frac{\alpha_0^2 C_1 e^{\alpha_0\tau_\alpha}}{\tau_\beta + \tau_\alpha} \tag{12}$$

$$\frac{\partial^2 TAC}{\partial \tau_\alpha \tau_\beta} = \frac{2TAC}{(\tau_\beta + \tau_\alpha)^2} - \frac{\theta_0 C_1(e^{\alpha_0\tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)^2} - \frac{2\beta_1\tau_\beta}{(\tau_\beta + \tau_\alpha)^2} \tag{13}$$

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} = \frac{2TAC}{(\tau_\beta + \tau_\alpha)^2} - \frac{4B_1\tau_\beta}{(\tau_\beta + \tau_\alpha)^2} + \frac{2B_1}{\tau_\beta + \tau_\alpha} \tag{14}$$

As $e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1 > 0 \; \forall \; \alpha_0\tau_\alpha > 0$,
Hence equation (9) gives

$$TAC(\tau_\alpha, \tau_\beta) > \frac{B_1\tau_\beta^2}{\tau_\beta + \tau_\alpha} \; \forall \; \alpha_0\tau_\alpha > 0$$

Using this inequality in equation (14), which becomes

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} > \frac{2B_1\tau_\beta^2}{(\tau_\beta+\tau_\alpha)^3} + \frac{2B_1(\tau_\alpha-\tau_\beta)}{(\tau_\beta+\tau_\alpha)^2} = \frac{2B_1\tau_\alpha^2}{(\tau_\beta+\tau_\alpha)^3} > 0 \; \forall \; \alpha_0\tau_\alpha > 0 \tag{15}$$

To be necessary for the objective function to achieve minimum cost, the first order partial derivative must be zero (Step-1above). The optimal solution is achieved by setting these partial derivatives to zero when the sufficient conditions (Step-2 above) are satisfied.

$$\frac{\partial TAC}{\partial \tau_\alpha} = 0$$

and

$$\frac{\partial TAC}{\partial \tau_\beta} = 0$$

Additionally, sufficient circumstances must be fulfilled for certain optimality. From now on, the equivalent fundamental minors ought to be in the positive definite. The Hessian determinant is

$$H(\tau_\alpha, \tau_\beta) = \frac{\partial^2 TAC}{\partial \tau_\alpha^2} . \frac{\partial^2 TAC}{\partial \tau_\beta^2} - \left[\frac{\partial^2 TAC}{\partial \tau_\alpha \partial \tau_\beta}\right]^2$$

From equations (12), (13) and (14), we get

$$(\tau_\beta+\tau_\alpha)^4 H(\tau_\alpha,\tau_\beta) = 2A_1(2B_1 + C_1\alpha_0^2 e^{\alpha_0\tau_\alpha}) + 2C_1B_1(e^{\alpha_0\tau_\alpha}(1-\alpha_0\tau_\alpha)^2 + e^{\alpha_0\tau_\alpha} - 2) + \alpha_0^2C_1^2(e^{2\alpha_0\tau_\alpha} - 2\alpha_0\tau_\alpha e^{\alpha_0\tau_\alpha} - 1) \tag{16}$$

simplifying $e^{\alpha_0\tau_\alpha}(1-\alpha_0\tau_\alpha)^2 + e^{\alpha_0\tau_\alpha} - 2$ and $e^{2\alpha_0\tau_\alpha} - 2\alpha_0\tau_\alpha e^{\alpha_0\tau_\alpha} - 1$.

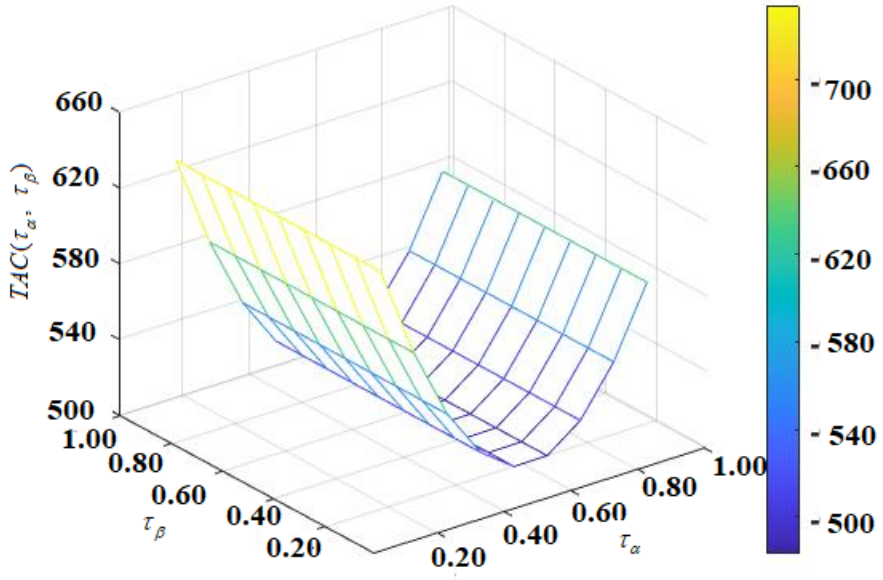

Figure 2: *Convexity objective function (Model-1)*

$$(\tau_\beta+\tau_\alpha)^4 H(\tau_\alpha,\tau_\beta) > 2A_1(2B_1+C_1\alpha_0^2 e^{\alpha_0\tau_\alpha})+C_1B_1(\alpha_0\tau_\alpha)^4+\frac{C_1^2\alpha_0^2(\alpha_0\tau_\alpha)^4}{4}$$

$$\rightarrow H(\tau_\alpha,\tau_\beta) > 0 \forall \alpha_0\tau_\alpha > 0 \tag{17}$$

The Hessian implied by equations (15) and (17) is positive definite. Consequently, the cost function $TAC(\tau_\alpha,\tau_\beta)$ is convex and Figure 2 provides more dramatic evidence of this tendency. From equation (10) and (11), within the feasible region, it possesses a unique global minima. at $(\tau_\alpha^*,\tau_\beta^*)$ fulfilling (Critical point)

$$\frac{\partial TAC}{\partial \tau_\alpha}=\frac{\alpha_0 C_1(e^{\alpha_0\tau_\alpha}-1)}{(\tau_\beta+\tau_\alpha)}-\frac{TAC}{\tau_\beta+\tau_\alpha}=0 \tag{18}$$

$$\frac{\partial TAC}{\partial \tau_\beta}=\frac{2B_1\tau_\beta}{(\tau_\beta+\tau_\alpha)}-\frac{TAC}{\tau_\beta+\tau_\alpha}=0 \tag{19}$$

equation (18) and (19)

$$TAC(\tau_\alpha^*,\tau_\beta^*)=\alpha_0 C_1(e^{\alpha_0\tau_\alpha^*}-1)=2B_1\tau_\beta^* \tag{20}$$

Hence, equations (9) and (20) give

$$2B_1\tau_\beta^*=\frac{A_1}{\tau_\beta^*+\tau_\alpha^*}+\frac{2B_1\tau_\beta^{2*}}{\tau_\beta^*+\tau_\alpha^*}+\frac{C_1(e^{\alpha_0\tau_\alpha^*}-\alpha_0\tau_\alpha^*-1)}{\tau_\beta^*+\tau_\alpha^*} \tag{21}$$

Simplifying equation (21), one can get

$$\tau_\beta^*=\sqrt{\tau_\alpha^{*2}+\frac{A_1+C_1(e^{\alpha_0\tau_\alpha^*}-\alpha_0\tau_\alpha^*-1)}{B_1}}-\tau_\alpha^* \tag{22}$$

Equation (20) implies

$$\tau_\alpha^*=\frac{1}{\alpha_0}log\left\{\frac{2B_1\tau_\beta^*}{\alpha_0 C_1}+1\right\} \tag{23}$$

Equations (22) and (23) are numerically solved iteratively to produce the appropriate values of τ_β^* and τ_α^* for optimization.The economic order quantity and the minimum total cost per unit time are found using equations (20) and (7).

$$TAC(\tau_\alpha^*,\tau_\beta^*)=2B_1\tau_\beta^*, S^*=D\tau_\beta^*+\frac{D(e^{\alpha_0\tau_\alpha^*}-\alpha_0\tau_\alpha^*-1)}{\alpha_0} \tag{24}$$

Equation (4) yields the effective rate of loss, which is the average deterioration per unit of time across a cycle.$=\frac{D(e^{\alpha_0\tau_\alpha^*}-\alpha_0\tau_\alpha^*-1)}{\alpha_0(\tau_\beta+\tau_\alpha)}=\frac{S^*}{\tau_\beta+\tau_\alpha}-D$.

6. A continuous review inventory model with fuzzy environment (Model-2)

Demand has been taken in the form $D=D_0+\epsilon$. This subsection examines fuzzy demand where D_0 a pentagonal fuzzy number, i.e. $\widetilde{D_0}=(D_0-\delta_2, D_0-\delta_1, D_0', D_0+v_1, D_0+v_2)$. This improves the modelling of real-world scenarios' flexibility.
The function principle in this article and Graded Mean Integration Representation (GMIR) method are considered. Currently, the membership function of $\widetilde{D_0}$ is the following:

$$\eta_{\widetilde{D_0}}(x) = \begin{cases} 0 & for\ x < D_0 - \delta_1,\ a_5 \le x \\ \frac{x-(D_0-\delta_1)}{(D_0-\delta_2)-(D_0-\delta_1)} & for\ D_0 - \delta_1 \le x \le D_0 - \delta_2 \\ 1 & for\ D_0 - \delta_2 \le x \le D_0^{'} \\ \frac{(D_0+v_1)-x}{(D_0+v_1)-D_0^{'}} & for\ D_0^{'} \le x \le D_0 + v_1 \\ \frac{(D_0+v_2)-x}{(D_0+v_2)-(D_0+v_1)} & for\ D_0 + v_1 \le x \le D_0 + v_2 \end{cases} \tag{25}$$

This fuzzy demand changes equation (8) to

$$\widetilde{TAC}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{\widetilde{D_0}}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \tag{26}$$

Next, using the function principle (Mahata and Goswami [55]), given that the demand's basic value is PeFN. The total cost per unit time becomes a PeFN, as seen in Figure, according to the preceding expression.

Real valued functions themselves make up this PeFN's parameters. Regarding any feasible values of $(\tau_\alpha, \tau_\beta)$ the following is true:

$$TAC_{\delta 1}(\tau_\alpha, \tau_\beta) \le TAC_{\delta 2}(\tau_\alpha, \tau_\beta) \le TAC_{\widetilde{D_0}}(\tau_\alpha, \tau_\beta) \le TAC_{v1}(\tau_\alpha, \tau_\beta) \le TAC_{v2}(\tau_\alpha, \tau_\beta)$$

$$\widetilde{TAC}(\tau_\alpha, \tau_\beta) = PeFN\left(TAC_{\delta 1}, TAC_{\delta 2}, TAC_{\widetilde{D_0}}, TAC_{v1}, TAC_{v2}\right)$$

where

$$TAC_{\delta_1}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - \delta_1}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{\delta_2}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - \delta_2}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{\widetilde{D_0}}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{\widetilde{D_0}}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{v_1}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - v}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{v_2}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - v_2}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

The total cost per unit time in Model-2 is estimated by using the median calculation.

$$Median(\widetilde{TAC}(\tau_\alpha, \tau_\beta)) = \frac{1}{5}\left[TAC_{\delta_1}(\tau_\alpha, \tau_\beta) + TAC_{\delta_2}(\tau_\alpha, \tau_\beta) + TAC_{\widetilde{D_0}}(\tau_\alpha, \tau_\beta) + TAC_{v_1}(\tau_\alpha, \tau_\beta) + TAC_{v_2}(\tau_\alpha, \tau_\beta)\right]$$

$$\begin{aligned} Median(\widetilde{TAC}(\tau_\alpha, \tau_\beta)) = {} & \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0}{4(\tau_\beta + \tau_\alpha)}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \\ & + \left(\frac{D_0^{'}}{(\tau_\beta + \tau_\alpha)} + \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)}\right)\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha} - \alpha_0\tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \end{aligned} \tag{27}$$

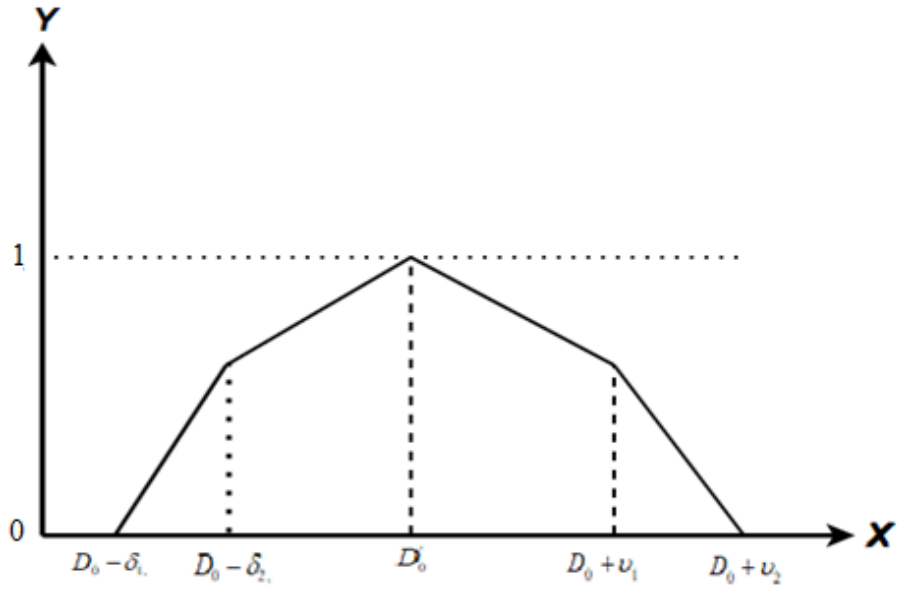

Figure 3: *Demand parameter and objective function*

As a result, the total of the two components is the median estimate of the total cost per unit time for the fuzzy demand model. The first term is the same as Model's total cost per unit time (crisp demand).

$$Median(\widetilde{TAC}(\tau_\alpha, \tau_\beta)) = TAC(\tau_\alpha, \tau_\beta) + FC(\tau_\alpha, \tau_\beta)$$

Where

$$FC(\tau_\alpha, \tau_\beta) = \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \tag{28}$$

7. Optimization methodology

The convexity of $FC(\tau_\alpha, \tau_\beta)$ as it is defined by equation (28) is examined.
The first-order partial derivatives of $FC(\tau_\alpha, \tau_\beta)$ with respect to τ_α and τ_β are as follows:

$$\frac{\partial FC}{\partial \tau_\alpha} = \frac{-(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] + \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - 1)}{\alpha_0}\right]$$

$$\frac{\partial FC}{\partial \tau_\beta} = \frac{-(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] + \frac{2\tau_\beta m(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)}$$

The second-order partial derivatives of $FC(\tau_\alpha, \tau_\beta)$ with respect to τ_α and τ_β are as follows:

$$\frac{\partial^2 FC}{\partial \tau_\alpha^2} = \frac{2(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^3}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$
$$- \frac{3(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - 1)}{\alpha_0}\right] - \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - 1)}{\alpha_0}\right] \tag{29}$$

$$\frac{\partial^2 FC}{\partial\tau_\alpha\partial\tau_\beta} = \frac{2(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^3}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)}{\alpha_0^2}+\frac{\tau_\beta^2 Dm}{2}\right] - \frac{(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^2}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha}-1)}{\alpha_0}+\tau_\beta m\right] \quad (30)$$

$$\frac{\partial^2 FC}{\partial\tau_\beta^2} = \frac{2(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^3}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)}{\alpha_0^2}+\frac{\tau_\beta^2 Dm}{2}\right] + \frac{m(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)}\left[2\tau_\beta-\frac{\tau_\beta}{(\tau_\beta+\tau_\alpha)}+1\right] \quad (31)$$

The ineqality $(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)>0\ \forall\ \alpha_0\tau_\alpha>0$ gives
So equation (31)

$$\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\beta^2} > \frac{2(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^3}\left[\frac{(K+\alpha_0 L)(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)}{\alpha_0^2}+\frac{\tau_\beta^2 Dm}{2}\right]>0 \quad (32)$$

The Hessian determinant of $FC(\tau_\alpha,\tau_\beta)$ is $H(\tau_\alpha,\tau_\beta)=\left(\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\alpha^2}\right).\left(\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\beta^2}\right)-\left(\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\alpha\partial\tau_\beta}\right)^2$
Substituting the values for the second order partial derivatives from equations (29), (30), and (31)

$$(\tau_\beta+\tau_\alpha)^6 H(\tau_\alpha,\tau_\beta)=2(e^{2\alpha_0\tau_\alpha}+\alpha_0^2\tau_\alpha^2+1-2\alpha_0\tau_\alpha e^{\alpha_0\tau_\alpha}+\alpha_0\tau_\alpha-2e^{\alpha_0\tau_\alpha})-(e^{2\alpha_0\tau_\alpha}-2e^{\alpha_0\tau_\alpha}+1) \quad (33)$$

The iteration scheme for optimization of $Median(\widetilde{TAC}(\tau_\alpha,\tau_\beta))$ is as follows, derived from these three equations:

$$\tau_{f\alpha}=\frac{1}{\theta_f}log\left\{\frac{\theta_f\tau_{f\beta}m}{K+L\theta_f}+1\right\} \quad (34)$$

$$\tau_{f\beta}=\left[\tau_{f\alpha}^2+\frac{10A}{m(5D+(v_1-\delta_1+v_2-\delta_2))}+\frac{2(K+\alpha_f L)(e^{\alpha_f\tau_{f\alpha}}-\alpha_f\tau_{f\alpha}-1)}{m\alpha_f^2}\right]^{\frac{1}{2}}-\tau_{f\alpha} \quad (35)$$

The Model-2 optimal outcomes. The optimal TAC and S for Model-2 are

$$S_f^*=\left(D+\frac{(v_1-\delta_1+v_2-\delta_2)}{4}\right)\left[\tau_{f\beta}^*+\frac{(e^{\alpha_f^*\tau_f^*}-1)}{\alpha_f^*}\right] \quad (36)$$

$$Median(\widetilde{TAC}_f^*)=m\tau_{f\beta}^*\left(D+\frac{(v_1-\delta_1+v_2-\delta_2)}{4}\right) \quad (37)$$

8. Numerical Analysis

The models built are demonstrated with a numerical example. By examining the outcomes, a decision-maker can gain insightful information. The parameters' numerical values are given below: $A=75$, $K==0.4$, $D_0=950$, $\epsilon=10$, $L=5.5$, $m=3$, $\delta_1=100$,$\delta_2=75$, $v_1=150$, $v_2=200$ and $\alpha_0=0.16$. The deterioration rate as $(\alpha=\alpha_0 e^{-\mu})$ where the positive parameter μ is an Expected value of Continuous random variable ϵ. And value of $\mu=0.03$ in this article. The decision variables' values include the amount of time it takes for inventory to reach zero following replenishment τ_α, the interval between having no inventory and having a fully back-logged replenishment $\tau_\beta$for Model-1 and Model-2 and table 2 tabulates the corresponding order quantity and total cost at the optimal position. The column $\frac{S^*}{\tau_\alpha^*+\tau_\beta^*}-D$ provides the effective rate of deterioration as the number of units lost per unit time across a cycle. The cost per unit and time per unit at the optimal point are shown in Table 3 is $\frac{TAC^*}{S^*}=0.886, 0.723$ for Models-1 and

2, respectively. This demonstrates that, although having a little higher TAC^* than model-1, the fuzzy formulation (Model-3) is the most efficient on a per unit basis.
Figure 4 represents the variation in TAC^* for numerous cycle lengths $\tau_\alpha^* + \tau_\beta^*$. It displays two separate stages. For very short cycle durations in the first phase, the influence of deterioration is not significant, and the findings produced by both models are comparable. The effects of deterioration and the economics of preservation become evident as cycle length grows. In the second stage, Model-1's overall cost increases much over its global minimum at $(\tau_\alpha^* + \tau_\beta^* = 0.52)$ but at longer cycle lengths around $(\tau_\alpha^* + \tau_\beta^* = 0.71)$ the other model it achieves lower global optimal costs. As cycle length increases, the change in TAC^* Model-2 is much more gradual.

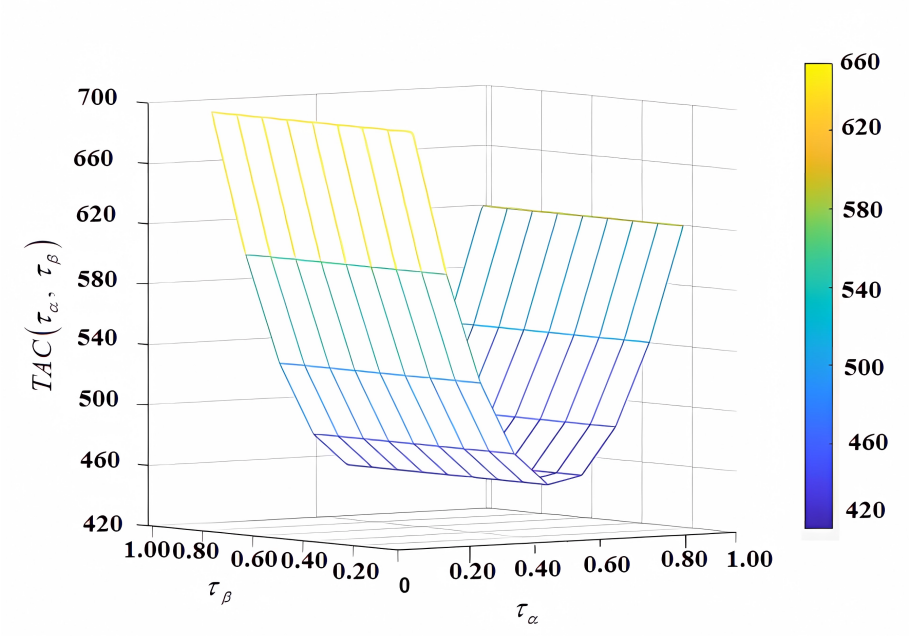

Figure 4: *Objective function convexity (Model-2)*

Table 2: Optimal solutions of Model-1 and Model-2.

Model	τ_α^*	τ_β^*	S^*	$\frac{S^*}{\tau_\alpha^*+\tau_\beta^*} - D$	$TAC(\tau_\alpha^*, \tau_\beta^*)$
Model-1	0.36	0.15	520.87	61.31	516.38
Model-2	0.61	0.14	705.56	33.74	451.37

9. Sensitivity analysis

1. **Effect of unit shortage cost (m):** With a decrease in shortage cost m, the backorder phase τ_α lasts longer and TAC is slightly reduced. Maintaining a larger optimal backorder during a cycle becomes advantageous. The movement of the crisp values is mirrored by the fuzzy values. A slight rise in EOQ balances out the fuzziness of demand without having a significant effect on TAC.

2. **Impact of unit holding cost (k):** A holding cost decrease has a proportionately bigger effect compared to an increase of the same magnitude. Since this τ_α time is more influenced, K complement m in the sensitivity computation. A displacement that is consistent with the decision-sharp maker's bounds is shown by the median fuzzy output.

3. **Impact of the cost of unit deterioration (l):** In order to benefit from decreased and even greater item loss, both the positive inventory time and the EOQ grow. The TAC is reduced in Model-1. Model-2 is somewhat less affected in this situation than Model-1.

4. **Ordering cost impact (A):** In both models, the ordering cost A consistently has an effect. The change in positive inventory time and EOQ has increased little but significantly, whereas the change in backorder time τ_β and TAC has decreased. Both these movements are marginally less in Model-2.

5. **Effect of constant deterioration rate (α_0):** The TAC rises with increasing but falls more sharply with decreasing this parameter. In Model-1, higher EOQ and τ_α are optimal for minimal deterioration, and greater backorders are recommended when deterioration grows.

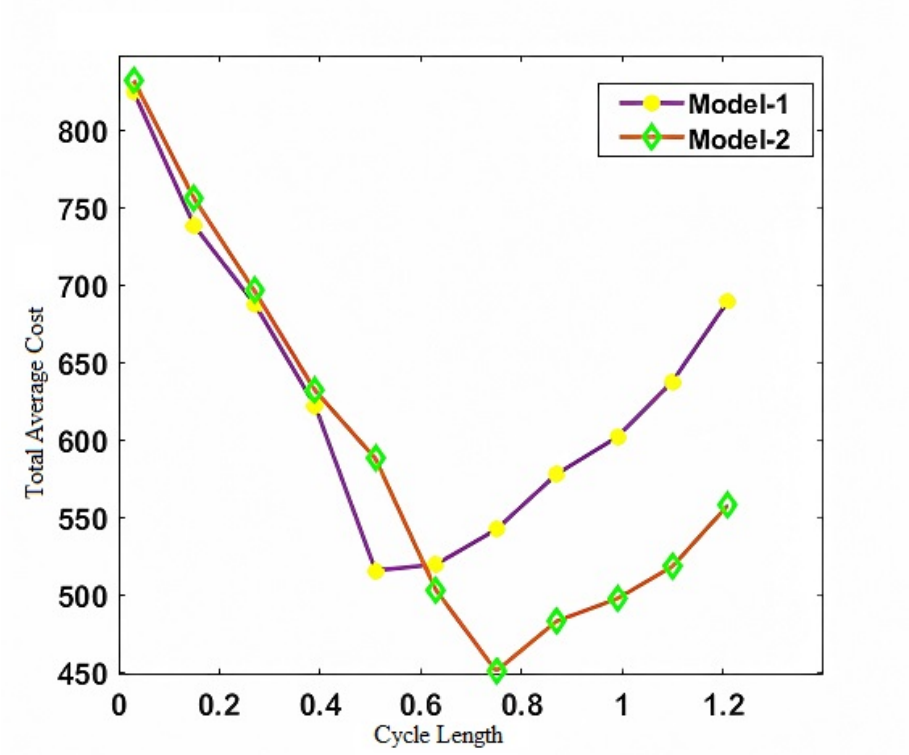

Figure 5: *Variation optimum total cost with cycle length.*

10. Conclusion

This study improved upon earlier EOQ models and techniques for managing deteriorating inventory levels. The expected total average cost was minimized while the per unit time cost was assessed with the length of time with stock-out condition and the duration of on-hand inventory in a reorder sequence to determine the proper reorder size and cycle length. Deterioration caused parts to be lost, which raised the total cost. Expanded to produce Model-2 by providing modelling uncertainty in demand. The optimal parameters for the reorder procedure were found by the development of a helpful formulation. Decision-makers were able to calculate the optimal investment for this aim by using this model, which demonstrated the impact of controlling spoiling. The Graded mean integration representation (GMIR) was used to defuzzify the fuzzy cost function in the model with uncertain demand. . From the analytical results of Model-2, an algorithm was developed to determine the optimal solutions.

The created models were validated using a numerical setup, and their sensitivity to important parameters was examined to ascertain the precise impact on the model. According to the traditional inventory model, holding costs and shortages had a connected impact on positive inventory duration and shortfall time, respectively. The total cost was more sensitive to a decline in the rate of degradation than to an increase in it. The following aspects can be expanded for future work. Numerous opportunities exist to further this research, such as time-limited replenishment, reworking or substituting damaged goods, uncertainty and randomization in other aspects, an expiration date-dependent deterioration rate a multi-item inventory and learning effects. An alternative model to recover part of the impact of degradation could be offered by the theory that uses animal fat waste as degraded goods to produce renewable energy .

11. Future Work

This model can be expanded to take policies like carbon caps and CO_2 quotas that aim to minimize emissions. The amount of carbon emissions may have a sensitive effect on some of the cost parameters. With such additions, it is possible to research how environmental deterioration affects ecosystems and conservation methods for sustainable supply chains. Some potential expansions of the models described include the analysis of the implications of time-dependent and non-linear operating costs, incomplete goods, prepayments, trade credit, and inflation in economic policy.

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RELIABILITY ANALYSIS OF A POWER DISTRIBUTION SYSTEM WITH TWO TRANSFORMERS AND SIX FEEDERS

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Abstract

The article explores the reliability and sensitivity of a power distribution substation. It includes an analysis based on real maintenance data collected from a 33/11kV electrical power distribution substation, which features a set of two 6 MVA power transformers supplying power through a total of six outgoing feeders (three feeders per transformer). The study documents faults observed in both transformers and all six outgoing feeders. The reliability of the substation is evaluated using various indices such as availability, repair durations, and expected repair frequencies for different failure types. The analysis employs Markov processes and regenerative point techniques. In addition to reliability, the study includes a profit analysis of the substation. It presents graphical representations of key parameters. Furthermore, a sensitivity analysis is conducted to assess how variations in parameters impact the availability and profitability of the substation components. Substation economics is also established to assess the operational viability.

Keywords: failure, reliability, transformers, Markov process, regenerative processes.

I. Introduction

Reliability modeling and analysis of industrial systems have become pivotal in ensuring the optimal performance and longevity of complex machinery and processes. As industries become increasingly dependent on sophisticated technology, the need to understand and predict system reliability has become more critical. This field involves developing mathematical and statistical models that simulate various failure and repair scenarios, allowing engineers to predict system behavior under different conditions. These models are essential for designing maintenance strategies that minimize downtime and costs while maximizing operational efficiency and profitability. By analyzing real failure and maintenance data, these models can provide valuable insights into the reliability indices of systems, such as system availability, repair durations, and expected repair frequencies for different failure types. This comprehensive approach helps industries to not only improve their maintenance policies but also enhance the overall reliability and performance of their systems, thereby ensuring sustained productivity and economic operational benefits. Over the years, numerous studies have contributed to this domain, providing various models and methodologies to

enhance system reliability and economic efficiency.

Parashar and Taneja [1] conducted a seminal study on the reliability and profit evaluation of a hot standby PLC system utilizing a master-slave configuration and dual repair facilities, establishing a foundational approach to reliability studies in complex systems. In the same year, Nilsson and Bertling [2] explored the maintenance management of wind power systems using condition monitoring systems and life cycle cost analysis, highlighting the importance of maintenance in energy sustainability. Mathew et al. [3] extended these concepts into the industrial manufacturing sector by modeling the reliability of a single-unit continuous casting (CC) plant with scheduled maintenance, providing insights into maintenance strategies that enhance operational continuity. Further expanding the scope, Rizwan et al. [4] analyzed a hot standby industrial system, emphasizing the critical balance between reliability and operational efficiency. The reliability modeling of a two-unit continuous casting plant was advanced by Mathew et al. [5], who offered detailed insights into the maintenance strategies necessary for operational stability in industrial settings. In another study, Shakuntla et al. [6] utilized supplementary variable techniques for reliability analysis in the polytube industry, demonstrating the utility of advanced mathematical techniques in predicting system behavior. In the realm of communication systems, Kumar and Kapoor [7] assessed the profitability of a base transceiver system considering hardware failures and system congestion, reflecting the intricate balance between operational efficiency and reliability. Similarly, Rizwan et al. [8] conducted a reliability analysis of a seven-unit desalination plant, incorporating both major and minor failures and highlighting the seasonal impact on system performance. Further research by Padmavathi et al. [9] on a desalination plant with major and minor failures underscored the importance of probabilistic analysis in understanding system reliability under varying conditions. This theme was continued by Rizwan et al. [10] in their analysis of an anaerobic batch reactor treating fruit and vegetable waste, which provided valuable insights into the reliability and availability of biogas production systems. The comparative analysis of reliability models for a desalination plant by Padmavathi et al. [11] and the performance analysis of a desalination plant with mandatory shutdowns by Rizwan et al. [12] further contributed to the understanding of maintenance strategies and their economic impacts. Additionally, Ahmad and Kumar [13] analyzed the profit implications of operational halts in a two-unit centrifuge system, illustrating the financial consequences of reliability. Adlakha et al. [14] explored the reliability and cost-benefit analysis of a two-unit cold standby system used for satellite communication, emphasizing the critical nature of reliability in high-stake environments. Naithani et al. [15] prioritized repair in their analysis of a three-unit induced draft fan system with a warm standby, enhancing operational reliability. Al Rahbi et al. [16] investigated the reliability challenges in the aluminum industry, specifically in a rodding anode plant with multiple units and a single repairman, highlighting the complexities of maintaining operational continuity amidst multiple failures. Taj et al. [17] provided a comparative analysis of three reliability models of a building cable manufacturing plant, illustrating the ongoing evolution and refinement of reliability assessment techniques. Kaur et al. [18] analyzed the reliability of a gravity die casting system, addressing diverse failure types that impact production processes. Sachdeva et al. [19] analyzed the reliability and sensitivity of an insured system where the warranty duration exceeds the insurance duration. Most recently, Oraimi et al. [20] conducted a sensitivity and profitability analysis of a two-units ammonia/urea plant, providing insights into optimizing system performance and economic viability under various conditions. Hussien and El-Sherbeny [21] examined the reliability and availability of a single-unit system under random shocks and varying demand, adding to the body of knowledge on stochastic behaviour in production systems. Finally, Rani et al. [22] explored the reliability of a two non-identical unit standby system with correlated failures, further enriching the literature on system reliability and maintenance optimization. These studies collectively underscore the critical

importance of advanced reliability modeling across various industrial and environmental applications. They offer profound insights into system design and maintenance optimization, ultimately enhancing overall operational efficacy and sustainability.

In the context of electrical energy transmission and distribution, the reliability of power transformers is crucial for maintaining consistent and efficient power delivery. This paper presents a reliability and the profit analysis of a power distribution system comprising transformers feeding power through feeders, aiming to obtain reliability indices that reflect the system's behaviour and conduct a sensitivity analysis that underscores the economic viability of the substation operations. The study explores the causes of power unavailability from transformers, which may arise from environmental conditions such as severe weather, including cuts or heavy winds, and electrical faults like short circuits. Markovian processes are employed which are well-suited for analyzing systems with probabilistic state transitions that adhere to the memoryless property, where future states are influenced solely by the current state. The effectiveness of the model based on the Markovian process has been proven in various reliability applications [19], making them a preferred tool for systems with state-based transitions, the analysis provides a detailed examination of the system's stochastic behavior over time. Key reliability metrics, including system availability are obtained to evaluate the system's performance, utilizing real transformer data on failures and repairs. The analysis reveals the significant impact of environmental conditions and electrical faults on transformer unavailability. A sensitivity analysis further evaluates how variations in transformer failure and repair rates influence overall reliability and profitability of the system providing valuable insights into the determinants of reliability for the distribution system under consideration. The findings form the basis for enhancing system robustness by addressing key determinants of system reliability. Additionally, the paper opens the directions for future research to further explore and mitigate reliability challenges in power distribution systems, thereby contributing to the development of more reliable and resilient power infrastructure.

II. Model Description and Assumptions

I. Model Description

The electrical distribution substation, which divides and distributes electrical power to various areas of the power distribution region, resembles an enormous junction as shown in Fig.1. The two major cables, which we refer to as incoming feeders, provide this substation with power at 33 kV. The two 6 MVA transformers are used to make the power available at 11 kV for its subsequent distribution to loads in different areas. Three outgoing feeders receive power from each transformer. As a result, there are six outgoing feeders that are carrying power at 11 kV to different areas. Various household as well as industrial establishments receive the power from each of these feeders. Typically, the load on each feeder is nearly balanced.

II. Assumptions

- Initially the system is operative.
- Both the transformers are working well and can’t fail simultaneously.
- The three feeders connected from each transformer are working properly and only one can fail at one time from three feeders connected to one transformer.
- All the states are regenerative.
- All the failure and repair times follows exponential distribution.

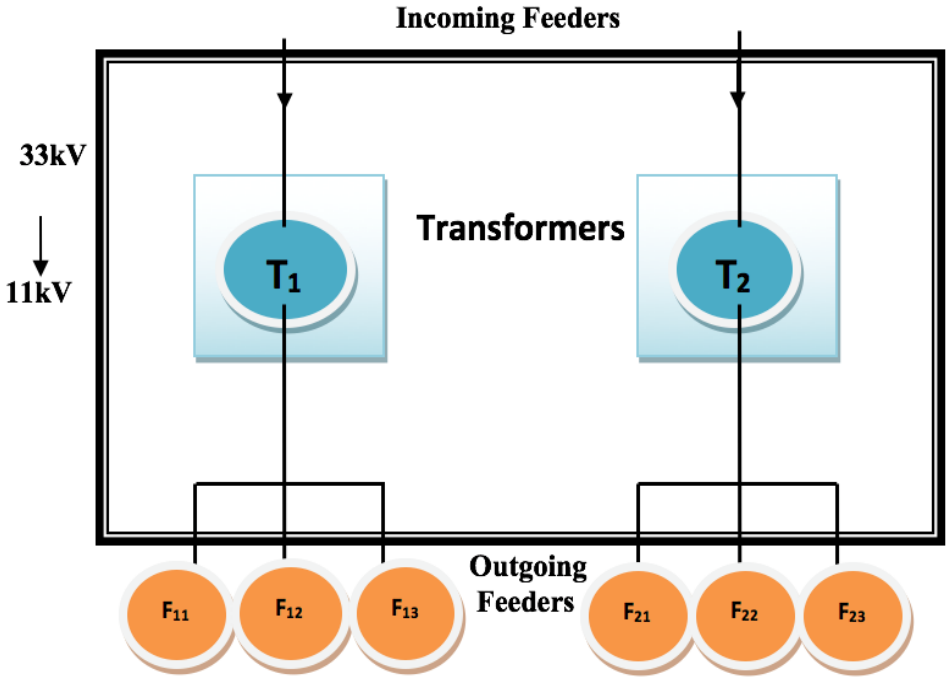

Figure 1: *Simplistic schematic for a Power Distribution Substation*

III. Notations

The following notations are used for rates of failure and repair of system
$\lambda_{t_1}, \lambda_{t_2}$ = Failure rate of transformers 1, transformers 2.
$\lambda_{f_{11}}, \lambda_{f_{12}}, \lambda_{f_{13}}$= Failure rate of feeder 1, feeder 2, feeder 3 that are connected from transformer 1.
$\lambda_{f_{21}}, \lambda_{f_{22}}, \lambda_{f_{23}}$= Failure rate of feeder 1, feeder 2, feeder 3 that are connected from transformer 2.
μ_{t_1}, μ_{t_2} = Repair rate of transformers 1, transformer 2.
$\mu_{f_{11}}, \mu_{f_{12}}, \mu_{f_{13}}$= Repair rate of feeder 1, feeder 2, feeder 3 that are connected from transformer 1.
$\mu_{f_{21}}$, $\mu_{f_{22}}, \mu_{f_{23}}$= Repair rate of feeder 1, feeder 2, feeder 3 that are connected from transformer 2.

IV. Data Summary

From the component wise failure and maintenance data obtained for the 33/11 kV electrical distribution substation for the previous 10 years, the following failure as well as repair rates are estimated:
Failure rate of transformer 1, λ_{t_1}= 1.14889 x 10^{-5}
Failure rate of transformer 2, λ_{t_2}= 1.90981 x 10^{-5}
Failure rate of feeder 1 that is connected from transformer 1, $\lambda_{f_{11}}$=3.05518 x 10^{-5}
Failure rate of feeder 2 that is connected from transformer 1, $\lambda_{f_{12}}$= 5.7282 x 10^{-5}
Failure rate of feeder 3 that is connected from transformer 1, $\lambda_{f_{13}}$=2.6732 x 10^{-5}
Failure rate of feeder 1 that is connected from transformer 2, $\lambda_{f_{21}}$ = 1.1456 x10^{-5}
Failure rate of feeder 2 that is connected from transformer 2, $\lambda_{f_{22}}$ = 1.5275 x 10^{-5}
Failure rate of feeder 3 that is connected from transformer 2, $\lambda_{f_{23}}$ = 2.2913 x 10^{-5}
Repair rate of transformer 1, μ_{t_1}= 0.3891
Repair rate of transformer 2, μ_{t_2}= 0.0881

Repair rate of feeder 1 that is connected from transformer 1, $\mu_{f_{11}}$ = 0.6525
Repair rate of feeder 2 that is connected from transformer 1, $\mu_{f_{12}}$ = 0.6310
Repair rate of feeder 3 that is connected from transformer 1, $\mu_{f_{13}}$ = 0.5254
Repair rate of feeder 1 that is connected from transformer 2, $\mu_{f_{21}}$ =1.3793
Repair rate of feeder 2 that is connected from transformer 2, $\mu_{f_{22}}$ = 1.2766
Repair rate of feeder 3 that is connected from transformer 2, $\mu_{f_{23}}$ = 0.6173

V. Stochastic Model

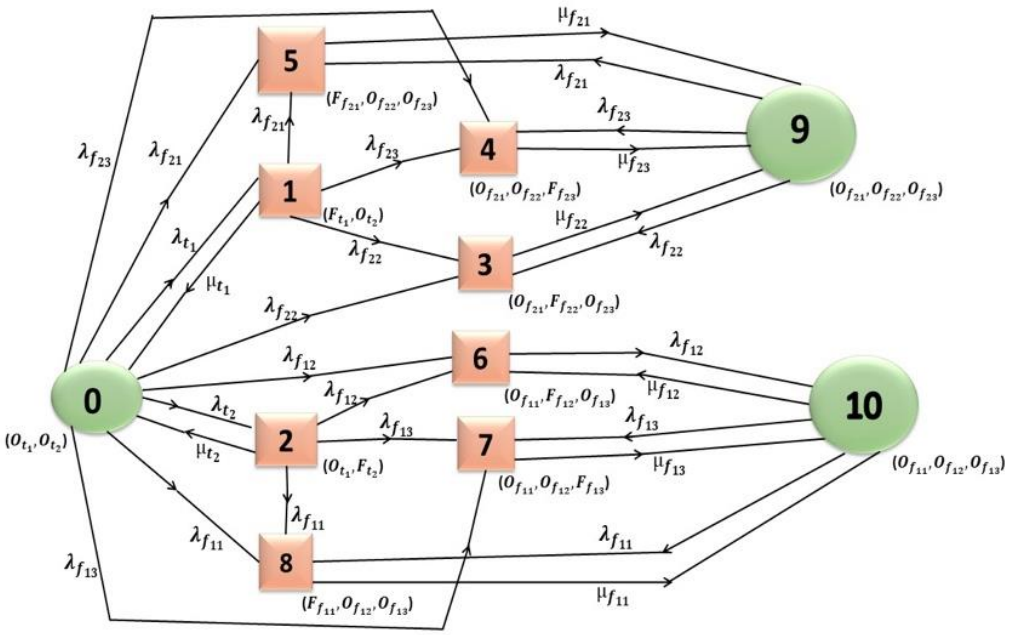

Figure 2: *State Transition Diagram*

Table 1: *State symbol and their meaning*

State No.	State Symbol	Description
State 0	(O_{t_1}, O_{t_2})	Both the transformers are operative.
State 1	(F_{t_1}, O_{t_2})	Transformer 1 failed and Transformer 2 operative.
State 2	(O_{t_1}, F_{t_2})	Transformer 1 operative and Transformer 2 failed.
State 3	$(O_{f_{21}}, F_{f_{22}}, O_{f_{23}})$	Feeders that are connected from transformer 2, first and third are operative, second feeder failed.
State 4	$(O_{f_{21}}, O_{f_{22}}, F_{f_{23}})$	Feeders that are connected from transformer 2, first and second are operative, third feeder failed.
State 5	$(F_{f_{21}}, O_{f_{22}}, O_{f_{23}})$	Feeders that are connected from transformer 2, first feeder failed while second and third are operative.
State 6	$(O_{f_{11}}, F_{f_{12}}, O_{f_{13}})$	Feeders that are connected from transformer 1, second feeder failed while first and third are operative.
State 7	$(O_{f_{11}}, O_{f_{12}}, F_{f_{13}})$	Feeders that are connected from transformer 1, third feeder failed while first and second are operative.
State 8	$(F_{f_{11}}, O_{f_{12}}, O_{f_{13}})$	Feeders that are connected from transformer 1, first feeder failed while second and third are operative.
State 9	$(O_{f_{21}}, O_{f_{22}}, O_{f_{23}})$	All the feeders that are connected from transformer 2 are operative.
State 10	$(O_{f_{11}}, O_{f_{12}}, O_{f_{13}})$	All the feeders that are connected from transformer 1 are operative.

Figure 2 shows the transition between states of system. The set of states {0, 9, 10} are operative states and set {1, 2, 3, 4, 5, 6, 7, 8} are partially operative states. All the states are regenerative states. The description of states along with their symbols are given in Table 1.

The transition densities from state r to state s (q_{rs}) are,

$q_{01}(t) = \lambda_{t_1} e^{-(\lambda_{t_1}+\lambda_{t_2})t}\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$; $q_{02}(t) = \lambda_{t_2} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$

$q_{03}(t) = \lambda_{f_{22}} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$; $q_{04}(t) = \lambda_{f_{23}} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$

$q_{05}(t) = \lambda_{f_{21}} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$; $q_{06}(t) = \lambda_{f_{12}} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$

$q_{07}(t) = \lambda_{f_{13}} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$; $q_{08}(t) = \lambda_{f_{11}} e^{-(\lambda_{t_1}+\lambda_{t_2})t}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})t}$

$q_{13}(t) = \lambda_{f_{22}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$; $q_{14}(t) = \lambda_{f_{23}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$; $q_{15}(t) = \lambda_{f_{21}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$;

$q_{10}(t) = \mu_{t_1}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$; $q_{26}(t) = \lambda_{f_{12}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$; $q_{27}(t) = \lambda_{f_{13}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$

$q_{28}(t) = \lambda_{f_{11}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$; $q_{20}(t) = \mu_{t_2}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$; $q_{39}(t) = \mu_{f_{22}}.\, e^{-\mu_{f_{22}}(t)}$;

$q_{49}(t) = \mu_{f_{23}}.\, e^{-\mu_{f_{23}}(t)}$; $q_{59}(t) = \mu_{f_{21}}.\, e^{-\mu_{f_{21}}(t)}$; $q_{6,10}(t) = \mu_{f_{12}}.\, e^{-\mu_{f_{12}}(t)}$; $q_{7,10}(t) = \mu_{f_{13}}.\, e^{-\mu_{f_{13}}(t)}$;

$q_{8,10}(t) = \mu_{f_{11}}.\, e^{-\mu_{f_{11}}(t)}$; $q_{93}(t) = \lambda_{f_{22}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})t}$; $q_{94}(t) = \lambda_{f_{23}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})t}$;

$q_{95}(t) = \lambda_{f_{21}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{2i}})t}$; $q_{10,6}(t) = \lambda_{f_{12}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})t}$; $q_{10,7}(t) = \lambda_{f_{13}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})t}$;

$q_{10,8}(t) = \lambda_{f_{11}}.\, e^{-\sum_{i=1}^{3}(\lambda_{f_{1i}})t}$

Steady-state probability, p_{ij} as

$p_{01} = \frac{\lambda_{t_1}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$; $p_{02} = \frac{\lambda_{t_2}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$; $p_{03} = \frac{\lambda_{f_{22}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$;

$p_{04} = \frac{\lambda_{f_{23}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$; $p_{05} = \frac{\lambda_{f_{21}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$; $p_{06} = \frac{\lambda_{f_{12}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$;

$p_{07} = \frac{\lambda_{f_{13}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$; $p_{08} = \frac{\lambda_{f_{11}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}}$; $p_{13} = \frac{\lambda_{f_{22}}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$; $p_{14} = \frac{\lambda_{f_{23}}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$;

$p_{15} = \frac{\lambda_{f_{21}}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$; $p_{10} = \frac{\mu_{t_1}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}}$; $p_{26} = \frac{\lambda_{f_{12}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$; $p_{27} = \frac{\lambda_{f_{13}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$;

$p_{28} = \frac{\lambda_{f_{11}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$; $p_{20} = \frac{\mu_{t_2}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}}$; $p_{39} = p_{49} = p_{59} = p_{6,10} = p_{7,10} = p_{8,10} = 1$

$p_{93} = \frac{\lambda_{f_{22}}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})}$; $p_{94} = \frac{\lambda_{f_{23}}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})}$; $p_{95} = \frac{\lambda_{f_{21}}}{\sum_{i=1}^{3}(\lambda_{f_{2i}})}$; $p_{10,6} = \frac{\lambda_{f_{12}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})}$; $p_{10,7} = \frac{\lambda_{f_{13}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})}$;

$p_{10,8} = \frac{\lambda_{f_{11}}}{\sum_{i=1}^{3}(\lambda_{f_{1i}})}$

The contribution to mean sojourn time (m_{ij}) is given by $m_{ij} = \int_0^\infty t\, q_{ij}(t)dt$, we get

$m_{01} = \frac{\lambda_{t_1}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$; $m_{02} = \frac{\lambda_{t_2}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$; $m_{03} = \frac{\lambda_{f_{22}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$

$m_{04} = \frac{\lambda_{f_{23}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$; $m_{05} = \frac{\lambda_{f_{21}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$; $m_{06} = \frac{\lambda_{f_{12}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$

$m_{07} = \frac{\lambda_{f_{13}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$; $m_{08} = \frac{\lambda_{f_{11}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}}+\lambda_{f_{2i}})+\lambda_{t_1}+\lambda_{t_2}\right)^2}$; $m_{13} = \frac{\lambda_{f_{22}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}\right)^2}$

$m_{14} = \frac{\lambda_{f_{23}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}\right)^2}$; $m_{15} = \frac{\lambda_{f_{21}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}\right)^2}$; $m_{10} = \frac{\mu_{t_1}}{\left(\sum_{i=1}^{3}(\lambda_{f_{2i}})+\mu_{t_1}\right)^2}$; $m_{26} = \frac{\lambda_{f_{12}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}\right)^2}$

$m_{27} = \frac{\lambda_{f_{13}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}\right)^2}$; $m_{28} = \frac{\lambda_{f_{11}}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}\right)^2}$; $m_{20} = \frac{\mu_{t_2}}{\left(\sum_{i=1}^{3}(\lambda_{f_{1i}})+\mu_{t_2}\right)^2}$; $m_{39} = \frac{1}{\mu_{f_{22}}}$; $m_{49} = \frac{1}{\mu_{f_{23}}}$; $m_{59} = \frac{1}{\mu_{f_{21}}}$;

$m_{6,10} = \frac{1}{\mu_{f_{12}}}$; $m_{7,10} = \frac{1}{\mu_{f_{13}}}$; $m_{8,10} = \frac{1}{\mu_{f_{11}}}$; $m_{93} = \frac{\lambda_{f_{22}}}{\left[\sum_{i=1}^{3}(\lambda_{f_{2i}})\right]^2}$; $m_{94} = \frac{\lambda_{f_{23}}}{\left[\sum_{i=1}^{3}(\lambda_{f_{2i}})\right]^2}$;

$m_{95} = \frac{\lambda_{f_{21}}}{\left[\sum_{i=1}^{3}(\lambda_{f_{2i}})\right]^2}$; $m_{10,6} = \frac{\lambda_{f_{12}}}{\left[\sum_{i=1}^{3}(\lambda_{f_{1i}})\right]^2}$; $m_{10,7} = \frac{\lambda_{f_{13}}}{\left[\sum_{i=1}^{3}(\lambda_{f_{1i}})\right]^2}$; $m_{10,8} = \frac{\lambda_{f_{11}}}{\left[\sum_{i=1}^{3}(\lambda_{f_{1i}})\right]^2}$.

Now if μ_i is the mean stay time in particular state i, then

$m_{01} + m_{02} + m_{03} + m_{04} + m_{05} + m_{06} + m_{07} + m_{08} = \mu_0$

$m_{13} + m_{14} + m_{15} + m_{10} = \mu_1$; $m_{26} + m_{27} + m_{28} + m_{20} = \mu_2$; $m_{39} = \mu_3$; $m_{49} = \mu_4$; $m_{59} = \mu_5$,

$m_{6,10} = \mu_6$; $m_{7,10} = \mu_7$; $m_{8,10} = \mu_8$; $m_{93} + m_{94} + m_{95} = \mu_9$; $m_{10,6} + m_{10,7} + m_{10,8} = \mu_{10}$

VI. System Measures

I. System Availability

Let $Ad_i(t)$ = Pr {system is operative at time t, given that it is in the state i at time t=0}. By the transition of states and definition of $Ad_i(t)$, we get the following equations as

$$Ad_0(t) = M_0(t) + q_{01}(t)©Ad_1(t) + q_{02}(t)©Ad_2(t) + q_{03}(t)©Ad_3(t) + q_{04}(t)©Ad_4(t) + q_{05}(t)©Ad_5(t) + q_{06}(t)©Ad_6(t) + q_{07}(t)©Ad_7(t) + q_{08}(t)©Ad_8(t) \quad (1)$$

$$Ad_1(t) = q_{13}(t)©Ad_3(t) + q_{14}(t)©Ad_4(t) + q_{15}(t)©Ad_5(t) + q_{10}(t)©Ad_0(t) \quad (2)$$

$$Ad_2(t) = q_{26}(t)©Ad_6(t) + q_{27}(t)©Ad_7(t) + q_{28}(t)©Ad_8(t) + q_{20}(t)©Ad_0(t) \quad (3)$$

$$Ad_3(t) = q_{39}(t)©Ad_9(t) \quad (4)$$

$$Ad_4(t) = q_{49}(t)©Ad_9(t) \quad (5)$$

$$Ad_5(t) = q_{59}(t)©Ad_9(t) \quad (6)$$

$$Ad_6(t) = q_{6,10}(t)©Ad_{10}(t) \quad (7)$$

$$Ad_7(t) = q_{7,10}(t)©Ad_{10}(t) \quad (8)$$

$$Ad_8(t) = q_{8,10}(t)©Ad_{10}(t) \quad (9)$$

$$Ad_9(t) = M_9(t) + q_{93}(t)©Ad_3(t) + q_{94}(t)©Ad_4(t) + q_{95}(t)©Ad_5(t)Ad_{10}(t) = M_{10}(t) + q_{10,6}(t)©Ad_6(t) + q_{10,7}(t)©Ad_7(t) + q_{10,8}(t)©Ad_8(t) \quad (10)$$

where,

$M_i(t)$ = probability that the system stays in state i while operating rather than transferring to any other state.

Taking Laplace Transform of the above equations and solving for $Ad_0^*(s)$, we get

$$Ad_0^*(s) = \frac{N_1(s)}{D_1(s)} \quad (11)$$

where, $N_1(s)$ & $D_1(s)$ as as obtained solving equations (1) to (10)

Now, the availability of the system is steady state is given as

$$Ad_0 = \lim_{t\to\infty} Ad_0(t) = \lim_{s\to 0} s\, Ad_0^*(s) = \lim_{s\to 0} s\, \frac{N_1(s)}{D_1(s)} = \frac{N_1}{D_1} \quad (12)$$

is evaluated using determinants in $N_1(s)$ and $D_1(s)$.

II. Busy Period for Repair

Similarly, the expected duration during which the repairman is occupied with the repair of transformer 1 and transformer 2, respectively, can be determined in steady state and is given by

$$Bd_{t_1} = \lim_{s\to 0} s\, Bd_{t_1}^*(s) = \lim_{s\to 0} s\, \frac{N_{t_1}^b(s)}{D_1(s)} = \frac{N_{t_1}^b}{D_1} \quad (13)$$

$$Bd_{t_2} = \lim_{s\to 0} s\, Bd_{t_2}^*(s) = \lim_{s\to 0} s\, \frac{N_{t_2}^b(s)}{D_1(s)} = \frac{N_{t_2}^b}{D_1} \quad (14)$$

where,

$N_{t_1}^b(s) = q_{01}^*(s)W_1^*(s)(q_{93}^*(s)q_{10,6}^*(s) - q_{94}^*(s) - q_{95}^*(s) - q_{10,6}^*(s) - q_{10,7}^*(s) - q_{10,8}^*(s) - q_{93}^*(s) + q_{93}^*(s)q_{10,7}^*(s) + q_{94}^*(s)q_{10,6}^*(s) + q_{93}^*(s)q_{10,8}^*(s) + q_{94}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,6}^*(s) + q_{94}^*(s)q_{10,8}^*(s) + q_{95}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,8}^*(s) + 1)$;

$N_{t_2}^b(s) = q_{02}^*(s)W_2^*(s)(q_{93}^*(s)q_{10,6}^*(s) - q_{94}^*(s) - q_{95}^*(s) - q_{10,6}^*(s) - q_{10,7}^*(s) - q_{10,8}^*(s) - q_{93}^*(s) + q_{93}^*(s)q_{10,7}^*(s) + q_{94}^*(s)q_{10,6}^*(s) + q_{93}^*(s)q_{10,8}^*(s) + q_{94}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,6}^*(s) + q_{94}^*(s)q_{10,8}^*(s) + q_{95}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,8}^*(s) + 1)$.

Here, $W_i(t)$ = probability that the system stays in state i while repairing rather than transferring to any other state.

The expected time in which the repairman is busy for the repair of the feeder 1, 2 and 3, that are connected from transformer 1, respectively, in steady state, is given by

$$Bd_{f_{11}} = \lim_{s\to 0} s\, Bd_{f_{11}}^*(s) = \lim_{s\to 0} s\, \frac{N_{f_{11}}^b(s)}{D_1(s)} = \frac{N_{f_{11}}^b}{D_1}; \quad (15)$$

$$Bd_{f_{12}} = \lim_{s\to 0} s\, Bd^*_{f_{12}}(s) = \lim_{s\to 0} s\, \frac{N^b_{f_{12}}(s)}{D_1(s)} = \frac{N^b_{f_{12}}}{D_1}; \quad (16)$$

$$Bd_{f_{13}} = \lim_{s\to 0} s\, Bd^*_{f_{13}}(s) = \lim_{s\to 0} s\, \frac{N^b_{f_{13}}(s)}{D_1(s)} = \frac{N^b_{f_{13}}}{D_1}; \quad (17)$$

where, $N^b_{f_{11}}(s)$, $N^b_{f_{12}}(s)$ & $N^b_{f_{13}}(s)$ are obtained as above.

Continuing in the same way, the expected time the repairman is busy for the repair of the feeder 1, 2 and 3, that are connected from transformer 2, respectively, in steady state, is given by

$$Bd_{f_{21}} = \lim_{s\to 0} s\, Bd^*_{f_{21}}(s) = \lim_{s\to 0} s\, \frac{N^b_{f_{21}}(s)}{D_1(s)} = \frac{N^b_{f_{21}}}{D_1}; \quad (18)$$

$$Bd_{f_{22}} = \lim_{s\to 0} s\, Bd^*_{f_{22}}(s) = \lim_{s\to 0} s\, \frac{N^b_{f_{22}}(s)}{D_1(s)} = \frac{N^b_{f_{22}}}{D_1}; \quad (19)$$

$$Bd_{f_{23}} = \lim_{s\to 0} s\, Bd^*_{f_{23}}(s) = \lim_{s\to 0} s\, \frac{N^b_{f_{23}}(s)}{D_1(s)} = \frac{N^b_{f_{23}}}{D_1}; \quad (20)$$

where, $N^b_{f_{21}}(s)$, $N^b_{f_{22}}(s)$ & $N^b_{f_{23}}(s)$ are obtained as above.

III. Expected Number of Repair

The expected time of repairs of transformer 1 and transformer 2, respectively, in steady state is given by

$$Nd_{t_1} = \lim_{s\to 0} s\, Nd^*_{t_1}(s) = \lim_{s\to 0} s\, \frac{N^n_{t_1}(s)}{D_1(s)} = \frac{N^n_{t_1}}{D_1}; \quad (21)$$

$$Nd_{t_2} = \lim_{s\to 0} s\, Nd^*_{t_2}(s) = \lim_{s\to 0} s\, \frac{N^n_{t_2}(s)}{D_1(s)} = \frac{N^n_{t_2}}{D_1}; \quad (22)$$

where,

$N^n_{t_1}(s) = q_{01}^*(s)\, q_{10}^*(s)(q_{93}^*(s)q_{10,6}^*(s) - q_{94}^*(s) - q_{95}^*(s) - q_{10,6}^*(s) - q_{10,7}^*(s) - q_{10,8}^*(s) - q_{93}^*(s) + q_{93}^*(s)q_{10,7}^*(s) + q_{94}^*(s)q_{10,6}^*(s) + q_{93}^*(s)q_{10,8}^*(s) + q_{94}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,6}^*(s) + q_{94}^*(s)q_{10,8}^*(s) + q_{95}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,8}^*(s) + 1);$

$N^n_{t_2}(s) = q_{02}^*(s)\, q_{20}^*(s)(q_{93}^*(s)q_{10,6}^*(s) - q_{94}^*(s) - q_{95}^*(s) - q_{10,6}^*(s) - q_{10,7}^*(s) - q_{10,8}^*(s) - q_{93}^*(s) + q_{93}^*(s)q_{10,7}^*(s) + q_{94}^*(s)q_{10,6}^*(s) + q_{93}^*(s)q_{10,8}^*(s) + q_{94}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,6}^*(s) + q_{94}^*(s)q_{10,8}^*(s) + q_{95}^*(s)q_{10,7}^*(s) + q_{95}^*(s)q_{10,8}^*(s) + 1);$

The expected number of repairs of the feeder 1, 2 and 3, that are connected from transformer 1, respectively, in steady state, is given by,

$$Nd_{f_{11}} = \lim_{s\to 0} s\, Nd^*_{f_{11}}(s) = \lim_{s\to 0} s\, \frac{N^n_{f_{11}}(s)}{D_1(s)} = \frac{N^n_{f_{11}}}{D_1}; \quad (23)$$

$$Nd_{f_{12}} = \lim_{s\to 0} s\, Nd^*_{f_{12}}(s) = \lim_{s\to 0} s\, \frac{N^n_{f_{12}}(s)}{D_1(s)} = \frac{N^n_{f_{12}}}{D_1}; \quad (24)$$

$$Nd_{f_{13}} = \lim_{s\to 0} s\, Nd^*_{f_{13}}(s) = \lim_{s\to 0} s\, \frac{N^n_{f_{13}}(s)}{D_1(s)} = \frac{N^n_{f_{13}}}{D_1}; \quad (25)$$

where,

$N^n_{f_{11}}(s) = -q_{8,10}^*(s)(q_{08}^*(s)q_{93}^*(s) - q_{02}^*(s)q_{28}^*(s) - q_{08}^*(s) + q_{08}^*(s)q_{94}^*(s) + q_{08}^*(s)q_{95}^*(s)$
$- q_{06}^*(s)q_{10,8}^*(s) + q_{08}^*(s)q_{10,6}^*(s) - q_{07}^*(s)q_{10,8}^*(s) + q_{08}^*(s)q_{10,7}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{93}^*(s)$
$+ q_{02}^*(s)q_{28}^*(s)q_{94}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{95}^*(s) - q_{02}^*(s)q_{26}^*(s)q_{10,8}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{10,6}^*(s)$
$- q_{02}^*(s)q_{27}^*(s)q_{10,8}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{10,7}^*(s) + q_{06}^*(s)q_{93}^*(s)q_{10,8}^*(s) - {}_{08}^*(s)q_{93}^*(s)q_{10,6}^*(s)$
$+ q_{06}^*(s)q_{94}^*(s)q_{10,8}^*(s) + q_{07}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{08}^*(s)q_{93}^*(s)q_{10,7}^*(s)$
$- q_{08}^*(s)q_{94}^*(s)q_{10,6}^*(s) + q_{06}^*(s)q_{95}^*(s)q_{10,8}^*(s) + q_{07}^*(s)q_{94}^*(s)q_{10,8}^*(s)$
$- q_{08}^*(s)q_{94}^*(s)q_{10,7}^*(s) - q_{08}^*(s)q_{95}^*(s)q_{10,6}^*(s) + q_{07}^*(s)q_{95}^*(s)q_{10,8}^*(s) - q_{08}^*(s)q_{95}^*(s)q_{10,7}^*(s)$
$+ q_{02}^*(s)q_{26}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{93}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{26}^*(s)q_{94}^*(s)q_{10,8}^*(s)$
$+ q_{02}^*(s)q_{27}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{93}^*(s)q_{10,7}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{94}^*(s)q_{10,6}^*(s)$
$+ q_{02}^*(s)q_{26}^*(s)q_{95}^*(s)q_{10,8}^*(s) + q_{02}^*(s)q_{27}^*(s)q_{94}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{94}^*(s)q_{10,7}^*(s)$
$- q_{02}^*(s)q_{28}^*(s)q_{95}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{27}^*(s)q_{95}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{95}^*(s)q_{10,7}^*(s));$

$N^n_{f_{12}}(s) = -\, q_{6,10}^*(s)\, (q_{06}^*(s)q_{93}^*(s) - q_{02}^*(s)q_{26}^*(s) - q_{06}^*(s) + q_{06}^*(s)q_{94}^*(s) + q_{06}^*(s)q_{95}^*(s)$
$+ q_{06}^*(s)q_{10,7}^*(s) - q_{07}^*(s)q_{10,6}^*(s) + q_{06}^*(s)q_{10,8}^*(s) - q_{08}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{26}^*(s)q_{93}^*(s)$
$+ q_{02}^*(s)q_{26}^*(s)q_{94}^*(s) + q_{02}^*(s)q_{26}^*(s)q_{95}^*(s) + q_{02}^*(s)q_{26}^*(s)q_{10,7}^*(s) - q_{02}^*(s)q_{27}^*(s)q_{10,6}^*(s)$
$+ q_{02}^*(s)q_{26}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{10,6}^*(s) - q_{06}^*(s)q_{93}^*(s)q_{10,7}^*(s)$

$+ q_{07}^*(s)q_{93}^*(s)q_{10,6}^*(s) - q_{06}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{06}^*(s)q_{94}^*(s)q_{10,7}^*(s)$
$+ q_{07}^*(s)q_{94}^*(s)q_{10,6}^*(s) + q_{08}^*(s)q_{93}^*(s)q_{10,6}^*(s) - q_{06}^*(s)q_{94}^*(s)q_{10,8}^*(s)$
$- q_{06}^*(s)q_{95}^*(s)q_{10,7}^*(s) + q_{07}^*(s)q_{95}^*(s)q_{10,6}^*(s) + q_{08}^*(s)q_{94}^*(s)q_{10,6}^*(s)$
$- q_{06}^*(s)q_{95}^*(s)q_{10,8}^*(s) + q_{08}^*(s)q_{95}^*(s)q_{10,6}^*(s) - q_{02}^*(s)q_{26}^*(s)q_{93}^*(s)q_{10,7}^*(s)$
$+ q_{02}^*(s)q_{27}^*(s)q_{93}^*(s)q_{10,6}^*(s) - q_{02}^*(s)q_{26}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{26}^*(s)q_{94}^*(s)q_{10,7}^*(s)$
$+ q_{02}^*(s)q_{27}^*(s)q_{94}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{93}^*(s)q_{10,6}^*(s) - q_{02}^*(s)q_{26}^*(s)q_{94}^*(s)q_{10,8}^*(s)$
$- q_{02}^*(s)q_{26}^*(s)q_{95}^*(s)q_{10,7}^*(s) + q_{02}^*(s)q_{27}^*(s)q_{95}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{94}^*(s)q_{10,6}^*(s)$
$- q_{02}^*(s)q_{26}^*(s)q_{95}^*(s)q_{10,8}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{95}^*(s)q_{10,6}^*(s));$

$N_{f_{13}}^n(s) = - q_{7,10}^*(s)\,(q_{07}^*(s)q_{93}^*(s) - q_{02}^*(s)q_{27}^*(s) - q_{07}^*(s) + q_{07}^*(s)q_{94}^*(s) + q_{07}^*(s)q_{95}^*(s)$
$- q_{06}^*(s)q_{10,7}^*(s) + q_{07}^*(s)q_{10,6}^*(s) + q_{07}^*(s)q_{10,8}^*(s) - q_{08}^*(s)q_{10,7}^*(s) + q_{02}^*(s)q_{27}^*(s)q_{93}^*(s)$
$+ q_{02}^*(s)q_{27}^*(s)q_{94}^*(s) + q_{02}^*(s)q_{27}^*(s)q_{95}^*(s) - q_{02}^*(s)q_{26}^*(s)q_{10,7}^*(s) + q_{02}^*(s)q_{27}^*(s)q_{10,6}^*(s)$
$+ q_{02}^*(s)q_{27}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{28}^*(s)q_{10,7}^*(s) + q_{06}^*(s)q_{93}^*(s)q_{10,7}^*(s)$
$- q_{07}^*(s)q_{93}^*(s)q_{10,6}^*(s) + q_{06}^*(s)q_{94}^*(s)q_{10,7}^*(s) - q_{07}^*(s)q_{94}^*(s)q_{10,6}^*(s)$
$+ q_{06}^*(s)q_{95}^*(s)q_{10,7}^*(s) - q_{07}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{07}^*(s)q_{95}^*(s)q_{10,6}^*(s)$
$+ q_{08}^*(s)q_{93}^*(s)q_{10,7}^*(s) - q_{07}^*(s)q_{94}^*(s)q_{10,8}^*(s) + q_{08}^*(s)q_{94}^*(s)q_{10,7}^*(s) - q_{07}^*(s)q_{95}^*(s)q_{10,8}^*(s)$
$+ q_{08}^*(s)q_{95}^*(s)q_{10,7}^*(s) + q_{02}^*(s)q_{26}^*(s)q_{93}^*(s)q_{10,7}^*(s) - q_{02}^*(s)q_{27}^*(s)q_{93}^*(s)q_{10,6}^*(s)$
$+ q_{02}^*(s)q_{26}^*(s)q_{94}^*(s)q_{10,7}^*(s) - q_{02}^*(s)q_{27}^*(s)q_{94}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{26}^*(s)q_{95}^*(s)q_{10,7}^*(s)$
$- q_{02}^*(s)q_{27}^*(s)q_{93}^*(s)q_{10,8}^*(s) - q_{02}^*(s)q_{27}^*(s)q_{95}^*(s)q_{10,6}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{93}^*(s)q_{10,7}^*(s)$
$- q_{02}^*(s)q_{27}^*(s)q_{94}^*(s)q_{10,8}^*(s) + q_{02}^*(s)q_{28}^*(s)q_{94}^*(s)q_{10,7}^*(s) - q_{02}^*(s)q_{27}^*(s)q_{95}^*(s)q_{10,8}^*(s)$
$+ q_{02}^*(s)q_{28}^*(s)q_{95}^*(s)q_{10,7}^*(s)).$

The expected number of repairs of the feeder 1, 2 and 3, that are connected from transformer 2, respectively, in steady state, is given by

$$Nd_{f_{21}} = \lim_{s\to 0} s\, Nd_{f_{21}}^*(s) = \lim_{s\to 0} s\, \frac{N_{f_{21}}^n(s)}{D_1(s)} = \frac{N_{f_{21}}^n}{D_1}\ ; \quad (26)$$

$$Nd_{f_{22}} = \lim_{s\to 0} s\, Nd_{f_{22}}^*(s) = \lim_{s\to 0} s\, \frac{N_{f_{22}}^n(s)}{D_1(s)} = \frac{N_{f_{22}}^n}{D_1}\ ; \quad (27)$$

$$Nd_{f_{23}} = \lim_{s\to 0} s\, Nd_{f_{23}}^*(s) = \lim_{s\to 0} s\, \frac{N_{f_{23}}^n(s)}{D_1(s)} = \frac{N_{f_{23}}^n}{D_1}\ ; \quad (28)$$

where, $N_{f_{21}}^n(s)$, $N_{f_{22}}^n(s)$ & $N_{f_{23}}^n(s)$. Here, $D_1(s)$ is as obtained in equation (13).

VII. Profit Equation

The profit equation of the system is as follows:

$P_d = R_d Ad_0 - C_{t_1}(Bd_{t_1} + Nd_{t_1}) - C_{t_2}(Bd_{t_2} + Nd_{t_2}) - C_{f_{11}}(Bd_{f_{11}} + Nd_{f_{11}}) - C_{f_{12}}(Bd_{f_{12}} + Nd_{f_{12}}) - C_{f_{13}}(Bd_{f_{13}} + Nd_{f_{13}}) - C_{f_{21}}(Bd_{f_{21}} + Nd_{f_{21}}) - C_{f_{22}}(Bd_{f_{22}} + Nd_{f_{22}}) - C_{f_{23}}(Bd_{f_{23}} + Nd_{f_{23}})$

where,

R_d = Revenue generated by substation

C_{t_1}/C_{t_2} = Cost per unit time for engaging the repairman and cost for repair due to failure in transformer 1/ transformer 2.

$C_{f_{11}}(C_{f_{21}})/C_{f_{12}}(C_{f_{22}})/C_{f_{13}}(C_{f_{23}})$ = Cost per unit time for engaging the repairman and cost for repair due to failure in feeder 1/ 2/ 3 that are connected to transformer 1 (transformer 2).

VIII. Numerical Results and Discussion

The results obtained from substituting values mentioned in Section 5 (Data Summary) that is calculated from electrical distribution substation for the above obtained measures in Section 7 are mentioned as below:

Availability of electrical distribution substation, $Ad_0 = 0.4709$

Busy period for repair of Feeder 1 that is connected from Transformer 1, $Bd_{f_{11}} = 4.4019*10^{-5}$

Busy period for repair of Feeder 2 that is connected from Transformer 1, $Bd_{f_{12}} = 9.1030*10^{-5}$

Busy period for repair of Feeder 3 that is connected from Transformer 1, $Bd_{f_{13}} = 3.6454*10^{-5}$
Busy period for repair of Feeder 1 that is connected from Transformer 2, $Bd_{f_{21}} = 1.8535*10^{-10}$
Busy period for repair of Feeder 2 that is connected from Transformer 2, $Bd_{f_{22}} = 3.7501*10^{-6}$
Busy period for repair of Feeder 3 that is connected from Transformer 2, $Bd_{f_{23}} = 3.8778*10^{-6}$
Expected number of repairs of Feeder 1 connected from Transformer 1, $Nd_{f_{11}} = 2.8722*10^{-5}$
Expected number of repairs of Feeder 2 connected from Transformer 1, $Nd_{f_{12}} = 5.7440*10^{-5}$
Expected number of repairs of Feeder 3 connected from Transformer 1, $Nd_{f_{13}} = 1.9153*10^{-5}$
Expected number of repairs of Feeder 1 connected from Transformer 2, $Nd_{f_{21}} = 2.5566*10^{-10}$
Expected number of repairs of Feeder 2 connected from Transformer 2, $Nd_{f_{22}} = 4.7874*10^{-6}$
Expected number of repairs of Feeder 3 connected from Transformer 2, $Nd_{f_{23}} = 2.3938*10^{-6}$

I. Impact of Various Rates and Cost Functions on Profit Function

The change in Profit (P_d) with varying values of failure rate ($\lambda_{f_{11}}$) and revenue (R_d) is shown in Figure 3. The surface suggests a relatively flat gradient, with a slight increase in P_d as both $\lambda_{f_{11}}$ and R_d increases. It shows that

- The profit falls with increase in failure rate.
- The higher values of revenue contribute to a rise in profit.

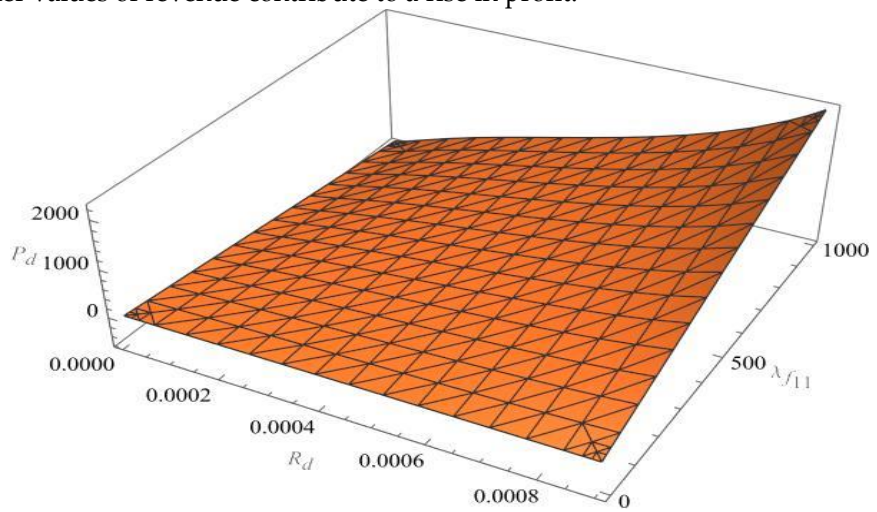

Figure 3: *Profit (P_d) with varying values of failure rate ($\lambda_{f_{11}}$) and revenue (R_d)*

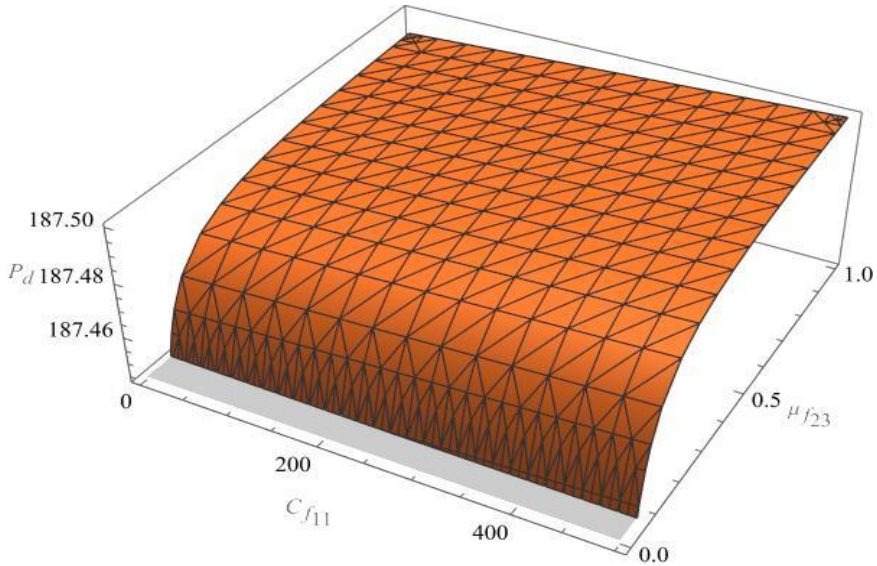

Figure 4: *Profit (P_d) with varying values of of repair rate ($\mu_{f_{23}}$) and cost ($C_{f_{11}}$)*

The change in Profit (P_d) with varying values of repair rate ($\mu_{f_{23}}$) and cost ($C_{f_{11}}$) is shown in Figure 4. The surface reveals that as $C_{f_{11}}$ increases, P_d shows a slight upward trend, while $\mu_{f_{23}}$ appears to have more substantial impact. It is also observed that when $\mu_{f_{23}}$ is near 0, P_d is lower and as $\mu_{f_{23}}$increases towards 1, P_d significantly rises. This suggests a strong positive correlation between $\mu_{f_{23}}$ and P_d, with $C_{f_{11}}$ also contributing positively but to a lesser extent.

II. Sensitivity Analysis

Sensitivity analysis is a powerful tool for assessing the robustness and reliability of models or decisions in the face of changing conditions or uncertainties in input parameters. It provides a structured approach to understanding how sensitive outcomes are to changes in key factors, thereby aiding in risk management and decision optimization. It is a technique to understand the changes in certain variables like availability and profit of a system. Relative sensitivity analysis is a normalized form of sensitivity analysis that focuses on comparing the relative impact of changes in different input variables or parameters on the output of a model or system. It is particularly useful in scenarios where the magnitude of changes in input variables varies widely. The sensitivity and relative sensitivity analysis of availabilities with different parameters involved are shown in Table 2.

Table 2: *Sensitivity and Relative Sensitivity Analysis of Availabilities*

Parameter (x)	Sensitivity Analysis $\frac{dA_0}{dx}$	Relative Sensitivity Analysis $\frac{dA_0}{dx} * \frac{x}{A_0}$
λ_{t_1}	-1.6482* 10^3	-0.0402
λ_{t_2}	-1.2812*10^3	-0.0520
$\lambda_{f_{11}}$	-2.1252*10^4	-1.3789
$\lambda_{f_{12}}$	-2.1252*10^4	-2.5853
$\lambda_{f_{13}}$	-2.1252*10^4	-1.2065
$\lambda_{f_{21}}$	-1.3847*10^4	-0.3369
$\lambda_{f_{22}}$	-1.3847*10^4	-0.4492
$\lambda_{f_{23}}$	-1.3847*10^4	-0.6738
$\mu_{f_{21}}$	3.9248*10^{-6}	1.1496*10^{-5}
$\mu_{f_{22}}$	5.0140*10^{-6}	1.3593*10^{-5}
$\mu_{f_{23}}$	3.6972*10^{-5}	4.8468*10^{-5}

Thus, the order in which the parameters (failure/repair rates) impact the availability is $\lambda_{f_{12}} > \lambda_{f_{11}} > \lambda_{f_{13}} > \lambda_{f_{23}} > \lambda_{f_{22}} > \lambda_{f_{21}} > \lambda_{t_2} > \lambda_{t_1} > \mu_{f_{23}} > \mu_{f_{22}} > \mu_{f_{21}}$.

Figure 5 is a 3D surface plot illustrating the relationship between three variables: $C_{f_{23}}$on the x-axis, R_d on the y-axis, and change in profit on the z-axis. The surface shows a consistent increase in P_d as both $C_{f_{23}}$ and R_d increase, indicating a positive correlation among these variables. The smooth and upward-sloping nature of the surface indicates a steady and predictable relationship. This kind of graphical outcome is useful for optimizing and redirecting system performance based on these key variables.

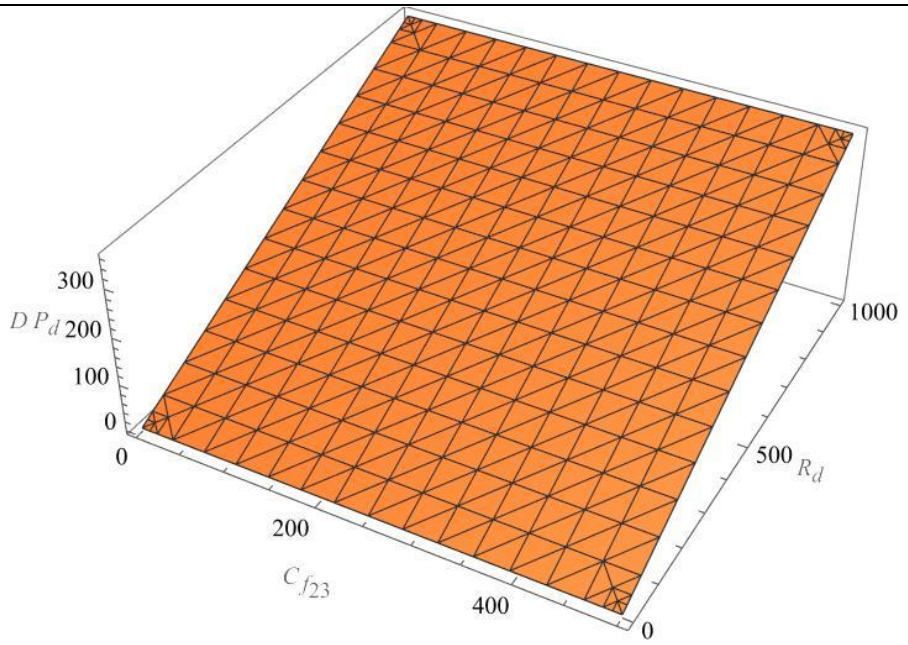

Figure 5: *Sensitivity Analysis of Profit (P_d) with varying values of revenue (R_d) and cost ($C_{f_{23}}$)*

IX. Conclusion

The analysis of the electrical distribution substation reveals critical insights into the reliability, profitability, and sensitivity of the system. By utilizing real maintenance data, the study evaluates the performance of the substation's transformers and outgoing feeders. Key reliability indices such as availability, repair durations, and expected repair frequencies are calculated using Markov processes and regenerative point techniques. The findings highlight that while the substation maintains a high level of reliability, certain failure types and repair times significantly impact overall availability and profitability. It must be noted that the availability of the system is 0.4709 which is quite low and need to be addressed by adopting a robust maintenance strategy. The profit analysis, supported by graphical representations, underscores the economic viability of the substation operations. Furthermore, the sensitivity analysis provides a comprehensive understanding of how variations in failure and repair rates affect the system's performance. This research not only contributes valuable knowledge for optimizing maintenance strategies but also offers a robust framework for assessing the economic and operational viability of electrical distribution substations. Future research should focus on integrating advanced predictive maintenance technologies and exploring alternative economic models to further enhance system reliability and profitability.

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A NEW FAMILY OF LINDLEY DISTRIBUTIONS FEATURING BIMODAL CASES

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Abstract

Several lifetime distributions have been developed in literature to handle different real-world scenario. Most of these distributions were developed to model a unimodal (symmetric or asymmetric) data. Only a hand-full of these distributions exhibits a bimodal property. This paper explores a new family of Lindley distributions featuring a bimodal property. We introduce five different sub-families of the T-Power Lindley$\{Y\}$ family based on the quantile function of the uniform, exponential, Frechet, log-logistic and logistic distributions. Useful mathematical properties of the proposed T-Power Lindley$\{Y\}$ family of distributions are derived and sub-models were the random variable T follows the one-parameter Topp-Leone, exponential, exponentiated exponential, Weibull and Gumbel distributions are introduced. From the graphical representation of the density function of these sub-models, we observe that the shape of the density function accommodates a decreasing (reversed-J), left-skewed, right-skewed, symmetric, as well as a bimodal shape. In order to illustrate the usefulness and performance of the proposed T-Power Lindley$\{Y\}$ family of distributions, the Gumbel Power Lindley (GPL) distribution belonging to the proposed family of distribution was employed to fit a bimodal data set alongside with the beta-Normal distribution. Result obtained from the analysis revealed that the Gumbel Power Lindley (GPL) distribution compares favourably better than the beta-Normal distribution. The density fits of the distributions for the data set was also investigated to support the claim.

Keywords: Lindley distribution; power Lindley distribution; quantiles; bimodality.

1. Introduction

Lifetime distributions are parametric models that seeks to analyze time-to-event data. Many lifetime models such as exponential, Weibull, gamma, beta, Gumbel distributions etc, have been studied and applied in literature to analyze lifetime data. Obviously, to increase the flexibility of these classical models remains the strong reason for developing new ones, thus many researchers have proposed generalized forms of these classical lifetime distribution.

The classical one-parameter Lindley distribution introduced by [11] has its density function defined by

$$f(x) = \frac{\beta^2}{\beta+1}(1+x)e^{-\beta x}, \quad x > 0, \beta > 0, \tag{1}$$

and cumulative distribution function as

$$F(x) = 1 - \left(\frac{\beta+1+\beta x}{\beta+1}\right)e^{-\beta x}, \quad x > 0, \beta > 0. \tag{2}$$

In order to increase the flexibility of the classical one-parameter Lindley distribution in analyzing lifetime data, [7] introduced a two parameter Power Lindley distribution by considering the power transformation $T = X^{\frac{1}{\alpha}}$, with the density function defined by

$$f(x) = \frac{\alpha\beta^2}{(1+\beta)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}, \quad x > 0, \beta, \alpha > 0, \tag{3}$$

and cumulative distribution function defined by,

$$F(x) = 1 - \left(\frac{\beta + 1 + \beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}, \quad x > 0, \alpha, \beta > 0. \tag{4}$$

A competing risks model when the causes of failure follow the one-parameter Lindley distribution was studied by [12] and was applied to a data set representing the lifetime of 194 patients with squamous cell carcinoma reported in [9]. Their result shows that the Lindley competing risks model provides a better fit to the data set under study than the exponential and the Weibull distributions. Nonetheless, due to the monotonic property of the one parameter Lindley distribution, there are situations where the distribution will fail to provide good fit in modeling lifetime data.

Several methods of generating new classes of probability distributions have been established in literature. The Kumaraswamy Lindley distribution and the Kumaraswamy Power Lindley distribution have been introduced, respectively, by [13] and [14] using the Kumaraswamy-G family of distributions proposed by [4]. A wider family of distributions called the "$T - X$ family of distributions" was introduced by [2]. The CDF of the $T - X$ family of distributions is defined as

$$G(x) = \int_0^{W(F(x))} r(t)dt, \quad = \quad R[W(F(x))], \tag{5}$$

where $R(t)$ is the CDF of the random variable T and $W(F(x))$ is a continuous and monotonic function of the CDF of a random variable X. Using this framework, [10] proposed the Lindley-X family of distribution and considered a special case of Lindley-Pareto distribution. For a random variable T following the density function of the Lindley distribution, [16] proposed the Odd Lindley-G family of distributions with cumulative distribution function defined as

$$F(x, \theta, \xi) = \frac{\theta^2}{\theta + 1} \int_0^{\frac{G(x,\xi)}{1-G(x,\xi)}} (1+t)e^{-\theta t}dt, \tag{6}$$

where $G(x, \xi)$ is the CDF of the random variable X, depending on a parameter vector ξ.

Undoubtedly, these generalizations have addressed some major drawbacks of the classical one-parameter Lindley distribution. However, their flexibility is limited to handling unimodal lifetime data. The need for developing a generalized Lindley distribution which can appropriately model bimodal lifetime data forms the motivation of this paper and the T-Power Lindley$\{Y\}$ Family of distributions is one of such. The remaining sections of this paper are organized as follows: Section 2 defines some sub-families of the T-Power Lindley$\{Y\}$ based on different quantile functions of a random variable Y. Section 3 gives some general mathematical properties of the T-Power Lindley$\{Y\}$ distribution. In Section 4, some new distributions belonging to the T-Power Lindley$\{Y\}$ family and some of their properties are discussed. Section 5 presents an application of the T-Power Lindley$\{Y\}$ family of distributions to a bimodal data set. Finally, Section 6 presents the conclusion.

2. Sub-families of the T-Power Lindley$\{Y\}$ Distribution Based on different Quantile Functions

Let T, R and Y be random variables from a known probability distribution with the cumulative distribution function defined by $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$, and $F_Y(x) = P(Y \leq x)$,

respectively. Let the corresponding quantile functions be given as $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$. If the density function exists, then we can denote them by $f_T(p)$, $f_R(p)$ and $f_Y(p)$ respectively.

A unified definition of the random variables in [1] was given by [3]. The authors defined the cumulative distribution function of a random variable X as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P\left\{T \leq Q_Y(F_R(x))\right\} = F_T[Q_Y(F_R(x))], \tag{7}$$

and the corresponding density function defined as

$$f_X(x) = \frac{f_R(x)}{f_Y\left\{Q_Y(F_R(x))\right\}} f_T\left\{Q_Y(F_R(x))\right\}. \tag{8}$$

Table 1: *Quantile Function of Some Known Distributions*

Distributions	Quantile Function $Q_Y(p)$	Support of Y
Uniform	p	$[0,1]$
Exponential	$-log(1-p)$	$(0,\infty)$
Frechet	$[-log(p)]^{-1}$	$(0,\infty)$
Log-logistic	$\frac{p}{1-p}$	$(0,\infty)$
Logistic	$log\left(\frac{p}{1-p}\right)$	$(-\infty,\infty)$

If R be a random variable following the power Lindley distribution with PDF $f_R(x)$ and CDF $F_R(x)$ defined in (3) and (4), respectively, then the sub-families of the T-Power Lindley$\{Y\}$ distribution can be generated based on different quantiles defined in Table 1.

2.1. T-Power Lindley$\{Uniform\}$ Distribution

This family of distributions is generated by using the quantile function of the uniform distribution in Table 1, with the support of $T \in [0,1]$. From (7), the cumulative distribution function of the T-Power Lindley$\{Uniform\}$ distribution is defined by

$$\begin{aligned} F_X(x) = \quad & F_T[Q_Y(F_R(x))] = \quad F_T(F_R(x)), \\ & = F_T\left\{1 - \left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right\}, \end{aligned} \tag{9}$$

and the corresponding density function is given by

$$\begin{aligned} f_X(x) &= f_T(x) \times f_T\left\{F_R(x)\right\}, \\ &= \frac{\alpha\beta^2}{(1+\beta)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} f_T\left\{1 - \left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right\}. \end{aligned} \tag{10}$$

The Kumaraswamy Power Lindley distribution proposed by [14] and the Kumaraswamy Lindley distribution proposed by [13] are members of this family.

2.2. T-Power Lindley$\{Exponential\}$ Distribution

This family of distributions is generated by using the quantile function of the exponential distribution in Table 1, with the support of $T \in (0, \infty)$. From (7), the cumulative distribution function of the T-Power Lindley$\{exponential\}$ distribution is defined by

$$\begin{aligned} F_X(x) = \quad F_T[Q_Y(F_R(x))] = \quad F_T\left\{-log(1-F_R(x))\right\}, \\ = F_T\left\{-log\left[\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]\right\}, \end{aligned} \tag{11}$$

and the corresponding density function is given by

$$\begin{aligned} f_X(x) = \frac{f_R(x)}{1-F_R(x)} \times f_T\left\{-log(1-F_R(x))\right\}, \\ = \frac{\alpha\beta^2(1+x^{\alpha})x^{\alpha-1}}{(1+\beta+\beta x^{\alpha})} f_T\left\{-log\left[\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]\right\}. \end{aligned} \tag{12}$$

2.3. T-Power Lindley$\{Frechét\}$ Distribution

This family of distributions is generated by using the quantile function of the $frechét$ distribution in Table 1, with the support of $T \in (0, \infty)$. The cumulative distribution function of the T-Power Lindley$\{Frechét\}$ distribution is defined from (7) as

$$\begin{aligned} F_X(x) = \quad F_T\left\{Q_Y(F_R(x))\right\} = \quad F_T\left\{[-\log(F_R(x))]^{-1}\right\}, \\ = F_T\left\{\left\{-\log\left(1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right)\right\}^{-1}\right\}, \end{aligned} \tag{13}$$

and the corresponding density function is given by

$$\begin{aligned} f_X(x) = \frac{f_R(x)}{F_R(x)[-\log(F_R(x))]^2} \times f_T\left\{[-\log(F_R(x))]^{-1}\right\}, \\ = \frac{\alpha\beta^2(1+x^{\alpha})x^{\alpha-1} f_T\left\{\left\{-\log\left(1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right)\right\}^{-1}\right\}}{\left((1+\beta)e^{\beta x^{\alpha}} - (1+\beta+\beta x^{\alpha})\right)\left[\log\left(1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right)\right]^2}. \end{aligned} \tag{14}$$

2.4. T-Power Lindley$\{log-logistic\}$ Distribution

This family of distributions is generated by using the quantile function of the exponential distribution in Table 1, with the support of $T \in (0, \infty)$. From (7), the cumulative distribution function of the T-Power Lindley$\{log-logistic\}$ distribution is defined by

$$\begin{aligned} F_X(x) = \quad F_T\left\{Q_Y(F_R(x))\right\} = \quad F_T\left\{\frac{F_R(x)}{(1-F_R(x))}\right\}, \\ = F_T\left\{\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right\}, \end{aligned} \tag{15}$$

and the corresponding density function is given by

$$\begin{aligned} f_X(x) = \frac{f_R(x)}{(1-F_R(x))^2} \times f_T\left\{\frac{F_R(x)}{(1-F_R(x))}\right\}, \\ = \frac{\alpha\beta^2(1+\beta)(1+x^{\alpha})x^{\alpha-1}e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})^2} f_T\left\{\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right\}. \end{aligned} \tag{16}$$

2.5. T-Power Lindley$\{logistic\}$ Distribution

This family of distributions is generated by using the quantile function of the logistic distribution in Table 1, with the support of $T \in (-\infty, \infty)$. From (7), the cumulative distribution function of the T-Power Lindley$\{logistic\}$ distribution is defined by

$$F_X(x) = \quad F_T\left\{Q_Y(F_R(x))\right\} = \quad F_T\left\{\log\left(\frac{F_R(x)}{(1-F_R(x))}\right)\right\},$$

$$= F_T\left\{\log\left(\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right)\right\}, \tag{17}$$

and the corresponding density function is given by

$$f_X(x) = \frac{f_R(x)}{F_R(x)(1-F_R(x))} \times f_T\left\{\log\left(\frac{F_R(x)}{(1-F_R(x))}\right)\right\},$$

$$= \frac{\alpha\beta^2(1+\beta)(1+x^{\alpha})x^{\alpha-1} f_T\left\{\log\left(\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right)\right\}}{(1+\beta+\beta x^{\alpha})\left[(1+\beta) - (1+\beta+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right]}. \tag{18}$$

Clearly, we observe that the support of the random variable T follows the support of Y, and the support of the proposed random variable X follows the support of the random variable R.

3. Some Mathematical Properties of the T-Power Lindley$\{Y\}$ Families of Distributions

3.1. Transformation and Quantile Function

Lemma 1 presents a mathematical relationship between the random variable X following the T-Power Lindley$\{Y\}$ Distribution and the generator random variable T. The random variable Y follows the Uniform, Exponential, Frechét, log-logistic and logistic distribution.

Lemma 1:

Let T be a random variable with pdf $f_T(x)$,

(a) if T is defined on the interval [0,1], then the random variable

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1-T)(1+\beta)}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Uniform\}$ Family of Distributions;

(b) if T is defined on the interval $(0, \infty)$, then the random variable

(i)

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{e^{T+\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Exponential\}$ Family of Distributions;

(ii)

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)(1-e^{-T^{-1}})}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Frechét\}$ Family of Distributions;

(iii)

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+T)e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Log - logistic\}$ Family of Distributions;

(c) if T is defined on the interval $(-\infty, \infty)$, then the random variable

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+e^{T})e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Logistic\}$ Family of Distributions. Where $W_{-1}(.)$ is the negative branch of the Lambert W function.

Proof:

The proof follows from a simple transformation between the random variables X and T as defined in (9), (11), (13), (15) and (17), respectively. From these relationships, random samples for X can be generated by using T, that is, random samples for X following the T-Power Lindley$\{Uniform\}$ distribution can be generated by first generating random samples for T from the pdf $f_T(x)$ and then compute

$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1-T)(1+\beta)}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$, which has the cdf $F_X(x)$.

Lemma 2:

The quantile function for the (i) T-Power Lindley$\{Uniform\}$, (ii) T-Power Lindley$\{Exponential\}$, (iii) T-Power Lindley$\{Frechét\}$, (iv) T-Power Lindley$\{Log - logistic\}$, and (v) T-Power Lindley$\{Logistic\}$ families of distribution are, respectively, given by

(i) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1-Q_T(p))(1+\beta)}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(ii) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{e^{Q_T(p)+\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(iii) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)(1-e^{-[Q_T(p)]^{-1}})}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(iv) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+e^{Q_T(p)})e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(v) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+Q_T(p))e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}.$$

Proof:

The proofs follow directly by solving $F_X(Q_X(p)) = p$, for $Q_X(p)$, Where $F_X(.)$ is the cdf given by (9), (11), (13), (15) and (17), respectively.

3.2. The Mode(s) of T-Power Lindley$\{Y\}$ families of distribution

Lemma 3:

The mode(s) of the (i) T-Power Lindley$\{Uniform\}$, (ii) T-Power Lindley$\{Exponential\}$, (iii) T-Power Lindley$\{Frechét\}$, (iv) T-Power Lindley$\{Log - logistic\}$, and (v) T-Power Lindley$\{Logistic\}$ distributions, respectively, are the solutions of (19), (20), (21), (22), and (23).

$$\Psi(x) \quad = \quad \frac{-f_T^{'}[F_R(x)]\bar{F}_R(x)}{f_T[F_R(x)]}, \tag{19}$$

$$\Psi(x) = \frac{-f_T'[P_1(x)]}{f_T[P_1(x)]} - 1, \tag{20}$$

$$\Psi(x) = \frac{\bar{F}_R(x)}{F_R(x)[\log F_R(x)]}\left\{\frac{f_T'[P_2(x)]}{f_T[P_2(x)]}P_2(x) + \log F_R(x) + 2\right\}, \tag{21}$$

$$\Psi(x) = \frac{-f_T'[P_3(x)]}{\bar{F}_R(x)f_T[P_3(x)]} - 2, \tag{22}$$

$$\Psi(x) = \frac{1}{F_R(x)}\left\{\frac{-f_T'[P_4(x)]}{f_T[P_4(x)]} - 2F_R(x) + 1\right\}, \tag{23}$$

where $P_1(x) = \{-\log(1 - F_R(x))\}$, $P_2(x) = \{-\log F_R(x)\}^{-1}$,

$P_3(x) = \frac{F_R(x)}{(1-F_R(x))}$, $P_4(x) = \log\left\{\frac{F_R(x)}{(1-F_R(x))}\right\}$, and

$\Psi(x) = \left\{(1 + \{(1 + x^\alpha)\beta)\}^{-1})(-1 + (\alpha - 1)(\alpha\beta x^\alpha)^{-1} + (1 + x^\alpha)\beta)\right\}$

Proof:

We need to first show that the first derivative of the density of the Power Lindley Distribution is expressed as

$$f_R'(x) = \Psi(x)\frac{f_R^2(x)}{\bar{F}_R(x)} \tag{24}$$

where $\bar{F}_R(x) = \frac{\{1 + (1 + x^\alpha)\beta\}}{1+\beta}e^{-\beta x^\alpha}$ is the survival function of the Power Lindley distribution.

Also, the derivative of (10) can be expressed as

$$f_X'(x) = f_R^2(x)f_T'\{F_R(x)\} + f_R'(x)\{F_R(x)\}. \tag{25}$$

Substituting (24) into (25), we have

$$f_X'(x) = f_R^2(x)m(x), \tag{26}$$

where $m(x) = f_T'\{F_R(x)\} + \frac{f_T\{F_R(x)\}}{\bar{F}_R(x)}\Psi(x)$.

Solving the system of equation $m(x) = 0$, gives the result in (19). The results in (20)-(23) follow using similar approach.

4. SOME NEW DISTRIBUTION ARISING FROM THE T-POWER LINDLEY$\{Y\}$ FAMILY OF DISTRIBUTIONS

In this Section, we present some distribution belonging to the T-Power Lindley$\{Y\}$ family of distributions with different T-distribution. Details of the T-distribution is given in Table 2.

Table 2: *Some Known Distributions*

Distributions	PDF	CDF
Topp Leone, $T \in [0,1]$	$2\alpha(1-t)\left[1-(1-t)^2\right]^{\alpha-1}$	$\left[1-(1-t)^2\right]^{\alpha}$
Exponentiated exponential, $T \in (0,\infty)$	$\frac{\theta}{\lambda}\left\{1-exp\left(-\frac{t}{\lambda}\right)\right\}^{\theta-1} exp\left(-\frac{t}{\lambda}\right)$	$\left\{1-exp\left(-\frac{t}{\lambda}\right)\right\}^{\theta}$
Exponential, $T \in (0,\infty)$	$\frac{1}{\lambda}exp\left(-\frac{t}{\lambda}\right)$	$1-exp\left(-\frac{t}{\lambda}\right)$
Weibull, $T \in (0,\infty)$	$\frac{\theta}{\lambda}\left(\frac{t}{\lambda}\right)^{\theta-1} exp\left\{-\left(\frac{t}{\lambda}\right)^{\theta}\right\}$	$1-exp\left\{-\left(\frac{t}{\lambda}\right)^{\theta}\right\}$
Gumbel, $T \in (-\infty,\infty)$	$\frac{\gamma}{\sigma}exp\left(-\frac{t}{\sigma}\right)exp\left\{-\gamma exp\left(-\frac{t}{\sigma}\right)\right\}$	$exp\left\{-\gamma exp\left(-\frac{t}{\sigma}\right)\right\}$

4.1. Topp Leone Power Lindley$\{Uniform\}$ Distribution (TLPLD)

Suppose the random variable T follows the one-parameter Topp-Leone distribution with bounded support $[0,1]$ reported in [17], then the density function of the Topp-Leone Power Lindley distribution is define as

$$f(x) = \frac{2\lambda\alpha\beta^2}{(1+\beta)}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left\{\bar{G}(x)\right\}\left\{1-(\bar{G}(x))^2\right\}^{\lambda-1}, \qquad x>0,\alpha,\beta,\lambda>0, \tag{27}$$

and the corresponding cumulative distribution function is given by

$$F(x) = \left\{1-(\bar{G}(x))^2\right\}^{\lambda}, \qquad x>0,\alpha,\beta,\lambda>0, \tag{28}$$

where $\bar{G}(x) = \left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}$ is the survival function of the Power Lindley distribution.

Other useful mathematical properties of this TLPL distribution has been studied in [15]. Figure 1 displays the plots of the density function of the Topp-Leone Power Lindley distribution (TLPLD) at various choices of the parameters. The plots indicates that the TLPLD can be left skewed, right skewed, monotonically decreasing (reversed J-shape), and symmetric.

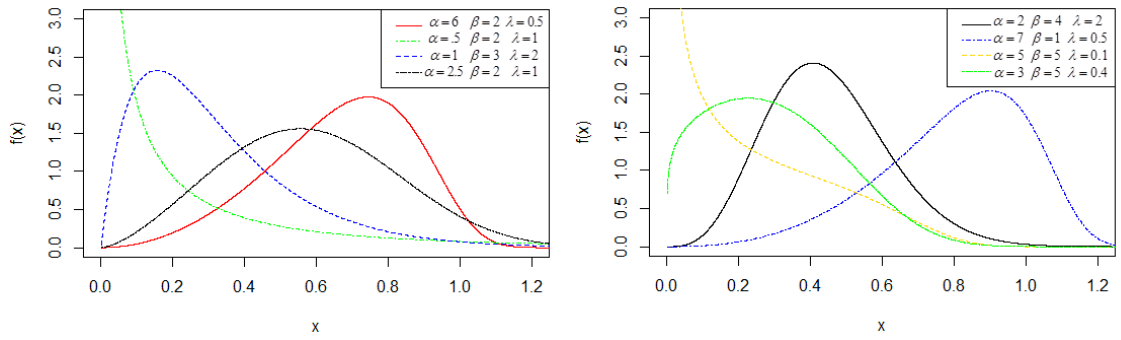

Figure 1: *Density function of the TLPLD for different choices of the parameters*

4.2. Exponentiated Exponential Power Lindley$\{Exponential\}$ Distribution

Let the random variable T follows the Exponentiated Exponential distribution introduced by [8], then the density function of the Exponentiated Exponential Power Lindley distribution (EEPLD) is define as

$$f(x) = \frac{\theta\alpha\beta^2(1+x^\alpha)x^{\alpha-1}}{\lambda(1+\beta+\beta x^\alpha)}\left\{1-(\bar{G}(x))^{\frac{1}{\lambda}}\right\}^{\theta-1}\left\{\bar{G}(x)\right\}^{\frac{1}{\lambda}}, \qquad x>0,\theta,\alpha,\beta,\lambda>0, \tag{29}$$

and the corresponding cumulative distribution function is given by

$$F(x) = \left\{1-(\bar{G}(x))^{\frac{1}{\lambda}}\right\}^{\theta}, \qquad x>0,\theta,\alpha,\beta,\lambda>0. \tag{30}$$

The plots of the probability density function of the Exponentiated Exponential Power Lindley distribution (EEPLD) for different values of the parameters is shown in Figure 2. It shows that the shape of the EEPLD can be left skewed, right skewed, monotonically decreasing (reversed J-shape), and symmetric.

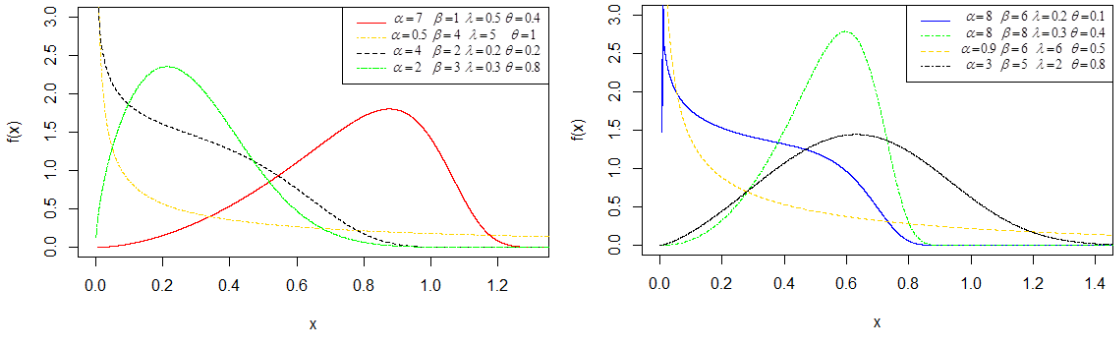

Figure 2: *Density function of the EEPLD for different values of the parameters*

4.3. Exponential Power Lindley$\{Frechét\}$ Distribution (EPLD)

Let the random variable T follows the exponential distribution, then the density function of the Exponential Power Lindley distribution is define as

$$f(x) = \frac{\alpha\beta^2(1+x^\alpha)x^{\alpha-1}exp\left\{\{\theta\log(G(x))\}^{-1}\right\}}{\{\theta(1+\beta)e^{\beta x^\alpha}-(1+\beta+\beta x^\alpha)\}\left[\log(G(x))\right]^2}, \qquad x>0,\theta,\alpha,\beta>0, \tag{31}$$

and the corresponding cumulative distribution function is given by

$$F(x) = 1-exp\left\{\theta\log(G(x))\right\}^{-1}, \qquad x>0,\theta,\alpha,\beta>0. \tag{32}$$

The plots of the probability density function of the Exponential Power Lindley distribution (EPLD) for different values of the parameters is shown in Figure 3. It indicates that the shape of the EPLD can be left skewed, right skewed, monotonically decreasing (reversed J-shape), modified unimodal.

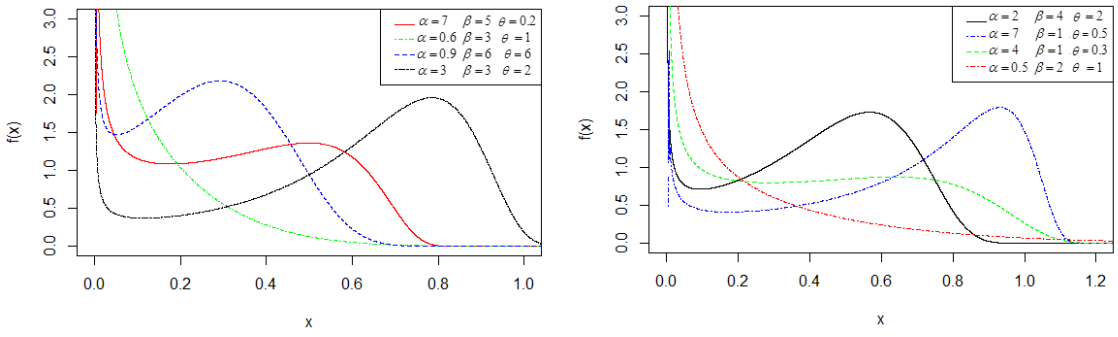

Figure 3: *Density function of the EPLD for different values of the parameters*

4.4. Weibull Power Lindley$\{log-logistic\}$ Distribution (WPLD)

Let the random variable T follows the Weibull distribution, then the density function of the Weibull Power Lindley distribution is define as

$$f(x)=\frac{\alpha\theta\beta^2(1+\beta)(1+x^\alpha)x^{\alpha-1}}{\lambda(1+\beta+\beta x^\alpha)^2}\left\{\frac{\varphi(x)}{\lambda}\right\}^{\theta-1}exp\left\{\beta x^\alpha-\left\{\frac{\varphi(x)}{\lambda}\right\}^\theta\right\}, \tag{33}$$

and the corresponding cumulative distribution function is given by

$$F(x)=1-exp\left\{-\left\{\frac{\varphi(x)}{\lambda}\right\}^\theta\right\}, \qquad x>0,\theta,\alpha,\beta,\lambda>0, \tag{34}$$

where $\varphi(x)=\frac{\frac{1}{\lambda}(1+\beta)e^{\beta x^\alpha}}{(1+\beta+\beta x^\alpha)}-1$.

Figure 4 gives the graph of the density function of the Weibull Power Lindley distribution (WPLD) for different values of the parameters. Figure 4 clearly shows that the shape of the density function of WPLD can be monotonically decreasing (reversed J-shape), left skewed, right skewed, symmetric and bimodal.

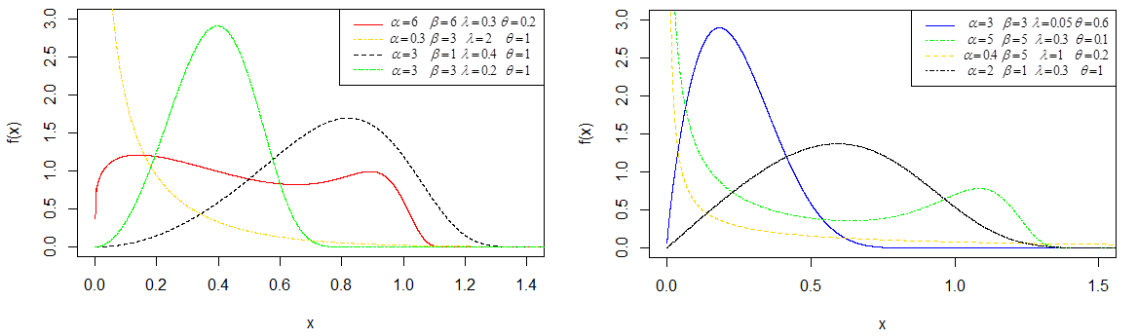

Figure 4: *Density function of the WPLD for different values of the parameters*

4.5. Gumbel Power Lindley$\{logistic\}$ Distribution

Let the random variable T follows the Gumbel distribution, then the density function of the Gumbel Power Lindley distribution (GPLD) is define as

$$f(x)=\frac{\alpha\gamma\beta^2(1+\beta)(1+x^\alpha)x^{\alpha-1}\left\{\varphi(x)\right\}^{-\frac{1}{\sigma}}exp\left\{-\gamma\left\{\varphi(x)\right\}^{-\frac{1}{\sigma}}\right\}}{\sigma(1+\beta+\beta x^\alpha)\left[1+\beta-(1+\beta+\beta x^\alpha)e^{-\beta x^\alpha}\right]}, \tag{35}$$

with the cumulative distribution function given by

$$F(x)=exp\left\{-\gamma\left\{\varphi(x)\right\}^{-\frac{1}{\sigma}}\right\}, \qquad x>0,\alpha,\beta,\sigma>0,\gamma=e^{\frac{\mu}{\sigma}}. \tag{36}$$

Figure 5 shows the plots of the density function of the Gumbel Power Lindley distribution for various choices of the parameters. The plots indicate that the GPLD exhibits a monotonically decreasing (reversed J-shape), left skewed, right skewed, symmetric and bimodal shape.

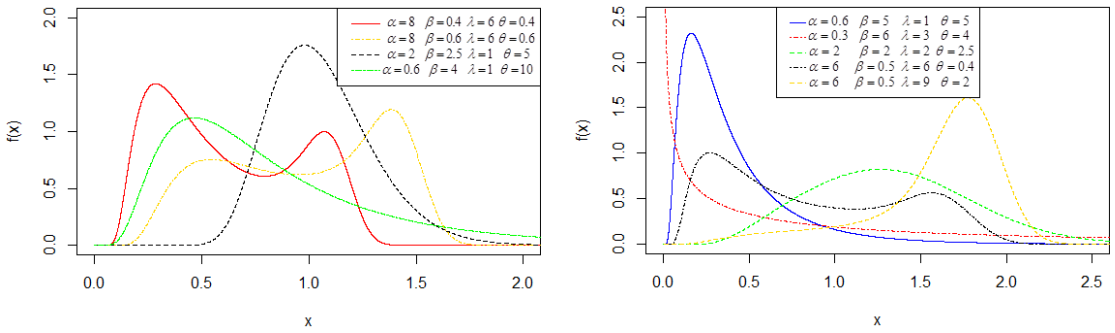

Figure 5: *Density function of the GPLD for different values of the parameters*

5. Application of the T-Power Lindley$\{Y\}$ family of distributions to a bimodal data set

To illustrate the flexibility of the T-Power Lindley$\{Y\}$ family of distributions in fitting real world data, we employ the Gumbel Power Lindley Distribution belonging to the T-Power Lindley$\{Y\}$ family of distributions to fit the egg size distribution data set reported in [5]. The data set consists of 88 asteroid species divided into three types; 35 planktotrophic larvae, 36 lecithotrophic larvae and 17 brooding larvae. [6] considered the logarithm of the asteroid data set which exhibits a bimodal shape and applied it to the beta-normal distribution. The descriptive statistics of the asteroid data is shown in Table 3, while Figure 6 provides the total time on test (TTT) and boxplot plot of the asteroid data.

Table 3: *Descriptive Statistics of the Asteroid Data*

Min.	1st Qu.	Median	3rd Qu.	Mean	Skewness	Kurtosis
4.605	5.134	6.126	6.869	6.070	0.1378	1.8217

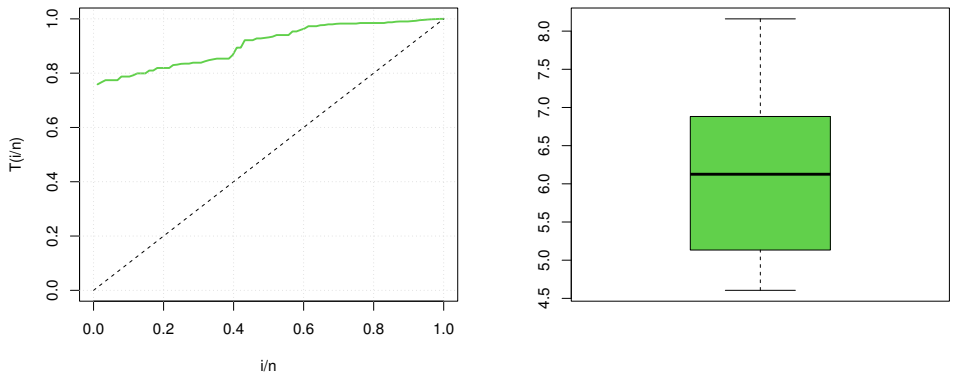

Figure 6: *TTT plot and Boxplot for the Asteroid Data*

Table 3 indicates that the data set is skewed to the right, whereas, Figure 6 shows that the data set exhibits an increasing hazard rate property.

Here, we apply the proposed Gumbel-Power Lindley distribution (GPLD) alongside with the beta-normal distribution (Beta-Norm) due to [6] to fit the bimodal data set. For the purpose of model comparison, the fits of the distributions were compared based on the maximized log-likelihood (Log-Lik), Aikaike Information Criterion (AIC), Corrected Aikaike Information

Criterion (AICc) and Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC).

Table 4: *Summary Statistics for the Asteroid Data*

Distributions	Estimates	Log-lik	AIC	*AICc*	BIC	HQIC
GPLD	$\alpha = 0.0046$	-109.1930	226.3861	226.8680	236.2954	230.3782
	$\beta = 0.0026$					
	$\lambda = 5.7755$					
	$\theta = 0.0465$					
*Beta-Norm	$\alpha = 0.0126$	-109.5108	227.0215	227.5034	236.9309	231.0138
	$\beta = 0.0064$					
	$\mu = 5.7109$					
	$\sigma = 0.0651$					

The Estimates and log-lik value of (*) were obtained from [6]

Figure 7 shows the graphical illustration of the density fit of the distributions for the Asteroid data set.

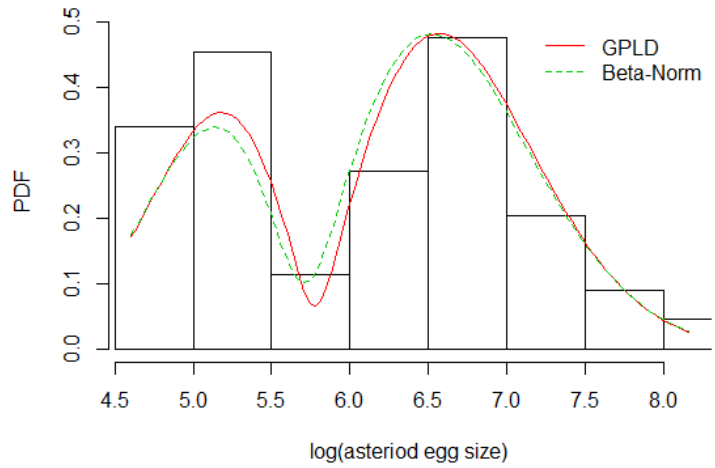

Figure 7: *Density Fit for the Asteroid Data*

5.1. Discussion of Result

A suitable model for analyzing lifetime data set can be investigated among several distributions by examining the model with the maximized log-likelihood value and the least value in terms of AIC, AICc, BIC, and HQIC. Table 4 reveals that the Gumbel-Power Lindley distribution which belongs to the T-Power Lindley$\{Y\}$ family of Distributions outperformed the beta-normal distribution in analyzing the data set and thus, can be employed as a better alternative to the beta-normal distribution in fitting real-life data exhibiting a bimodal property. This is evidently clear as the Gumbel-Power Lindley distribution has the maximized log-likelihood value and the least value in terms of the AIC, AICc, BIC, and HQIC as shown in Table 4.

6. Conclusion

A new class of generalized Lindley family of distributions with bimodal property is introduced. Sub-families of the T-Power Lindley$\{Y\}$ family based on the quantile function of the uniform, expenential, frechet, log-logistic and logistic distributions as well as some mathematical properties were derived. A bimodal data set was used to illustrate the applicability of the T-Power

Lindley$\{Y\}$ family of distributions and result obtained revealed that the GPL distribution from the proposed T-Power Lindley$\{Y\}$ family of distributions can be used as an alternative model to other existing distributions in modelling lifetime data sets.

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OPTIMIZING TWO-WAREHOUSE INVENTORY MODEL FOR DETERIORATING ITEMS WITH GENERALIZED EXPONENTIAL DEMAND, PARTIAL BACKLOGGING, AND INFLATION USING BACTERIAL FORAGING OPTIMIZATION

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Abstract

This paper presents a novel two-warehouse inventory model for degrading products, where the demand rate is governed over time by a generalized exponential function. Two real-world supply chain challenges that are taken into account in the model are the economic effects of inflation and partial backlog. By reducing the whole cost, which includes holding, shortage, and degradation charges, the Bacterial Foraging Optimization (BFO) method maximizes inventory management. The effectiveness of the model is validated through a comprehensive numerical example, and graphical representations demonstrate the impact of key factors on system performance. The results demonstrate how BFO may be used to complex inventory problems, giving supply chain managers crucial data as they try to balance cost-effectiveness and demand fluctuations in an inflationary environment. This approach highlights the need of advanced optimization techniques in improving decision-making processes for degrading products in a two-warehouse scenario.

Keywords: Bacterial Foraging Optimization, Two-Warehouse, Inventory Model, Exponential Demand, Partial Backlogging and inflation.

1. Introduction and related work

Effective inventory management is a critical facet of supply chain optimization, particularly in industries dealing with perishable or deteriorating items. The complexities of handling two warehouses, a dynamic demand pattern characterized by a generalized exponential increase, partial backlogging, and the impact of inflation demand advanced models for optimal decision-making. In this context, the Two-Warehouse Inventory Model emerges as a strategic tool, aiming to strike a balance between meeting customer demands, minimizing costs, and adapting to the challenges posed by deteriorating items. Deterioration poses a significant challenge in inventory systems, necessitating vigilant control mechanisms to mitigate losses associated with time-dependent decay. Moreover, the demand for products is rarely static; the generalized exponential increase introduces a nuanced dimension to forecasting and stock replenishment

strategies. The concept of partial backlogging acknowledges the practical reality that, at times, not all customer demands can be immediately fulfilled, and a systematic approach to manage backlogs is indispensable. The economic landscape, influenced by inflationary pressures, adds another layer of complexity. Inflation impacts various cost components within the inventory management framework, from holding costs to ordering costs. Therefore, an inclusive model that considers inflationary effects becomes crucial for businesses navigating these challenges. To optimize the intricate parameters of the Two-Warehouse Inventory Model, this study leverages the Bacterial Foraging Optimization (BFO) algorithm. Inspired by the foraging behavior of bacteria, BFO offers a nature-inspired approach to problem-solving. By simulating the movement and interaction of bacteria in a search space, BFO seeks to find optimal solutions efficiently. This research delves into the synergy between the Two-Warehouse Inventory Model, deteriorating items, generalized exponential demand, partial backlogging, and inflation. The objective is to employ BFO to fine-tune the decision variables of the model, providing businesses with a robust tool for inventory optimization in dynamic and challenging environments. Through the lens of this model, organizations can enhance their responsiveness to market dynamics, reduce costs, and ultimately bolster their competitiveness in the ever-evolving landscape of supply chain management.

Supply chain management (SCM) is defined as the coordination of production, storage, location, and transportation among supply chain participants to achieve the optimal balance of responsiveness and efficiency for a specific market. While many researchers have concentrated on products that do not spoil, certain items experience changes over time. Yadav et al. [1-10] highlighted that the deterioration of these items is significant, limiting their storage duration. Yadav et al. [11-20] further explained that deterioration can manifest as spoilage, evaporation, obsolescence, or loss of functionality, leading to reduced inventory usage compared to natural conditions. When raw materials are stocked for future demands, various factors such as storage conditions, weather, and humidity can cause deterioration. Yadav et al. [21-53] discuss that management typically maintains a warehouse for storing purchased goods. However, for various reasons, they may acquire or lease more items than can be accommodated in the warehouse, referring to the excess as overflow (OW), while the additional stock in a rented warehouse is termed rented warehouse (RW), located near or adjacent to OW. Yadav and Swami [54-61] found that the inventory costs associated with RW, including maintenance and depreciation, are generally higher than those of OW due to added operational and maintenance expenses. To minimize inventory costs, it is important to quickly utilize RW stock. Actual customer service is provided solely by OW, so to lower expenses, RW inventory is prioritized for turnover. These scenarios illustrate what are referred to as two inventory examples within the system. Yadav and Kumar [62] focused on managing the supply of electronic storage devices while integrating environmental and network considerations. Yadav, A.S. [63-65] analyzed seven measures of supply chain management to enhance the inventory of electronic storage devices. This analysis involved assessing financial impacts through the application of genetic algorithms (GA) and particle swarm optimization (PSO). Additionally, the research examined inventory improvement and equipment management using genetic computation, alongside a model design for inventory analysis that addressed economic challenges in transporting goods. Swami et al. [66-68] formulated inventory policies that address inventory requirements and associated costs, taking into account allowable payments and delays in inventory. They provided an example of depreciation across various goods and services, considering business loans and an inventory model that is less sensitive to pricing needs, while comparing inventory costs to inflation-related business expenses. Meanwhile, Gupta et al. [69-70] defined objectives for a Multiple Objective Genetic Algorithm and particle swarm optimization (PSO) aimed at enhancing supply levels and addressing deficits and inflation, along with a calculation model that leverages genetic algorithms to assess scarcity and low inflation scenarios. Singh et al. [71, 72] examined cases involving the depreciation of two types of stock concerning asset and inventory costs while updating particle data, as well as scenarios with two inventories focusing on property damage and inventory costs under inflation, utilizing soft computing techniques. Kumar et al. [73-75] addressed delays in managing alcohol supply, refining

particles, and developing a green cement supply system, while also tackling inflation through particle enhancement and the use of an electronic inventory system and distribution center with genetic computations. Chauhan and Yadav [76-77] provided an example of depreciation across two stores and warehouses, utilizing a genetic stock and vehicle stock to manage demand and inflation across two distribution centers. Pandey et al. [78] analyzed the improvement of industrial reserves for marble using genetic technology and enhanced multiple particle approaches. Ahlawat, et. al. [79] studied the white wine industry in supply chain management through nerve networks. Singh, et. al. [80] examines the best policy to import damaged goods immediately and pay for conditional delays under the supervision of two warehouses.

The research by Yadav et al. [81] centers on improving inventory management for perishable commodities through the lens of green technology investments, considering factors such as selling price, carbon emissions, and time-sensitive demand. In another analysis, Yadav, Yadav, and Bansal [82] utilize an interval number technique to explore a two-warehouse inventory management model for perishable goods, addressing demand and cost uncertainties. Their optimization methods highlight how investing in preservation technology can reduce waste and enhance inventory efficiency. Focusing on a two-warehouse approach to optimize inventory levels, Yadav, Yadav, and Bansal [83] and Negi and Singh [86] present a model that addresses the deterioration of goods during storage, emphasizing the importance of managing degradation costs to improve overall inventory performance.

2. Notations and assumptions

2.1. Notations

The following notations are used in this model.

Parameters	Descriptions
A	Ordering cost coefficient
h_1	Coefficient of holding cost of Rented Warehouse (RW).
h_2	Coefficient of holding cost of Owned Warehouse (OW).
C_P	Purchasing cost.
C_S	Shortage cost.
C_L	Coefficient of cost of lost sale.
θ	Constant rate of deterioration
C_D	Deterioration cost per unit.
R	Inflation factor
q_1	Positive height of inventory of (RW) with $I(t=0)$
q_2	Positive height of inventory of (OW) with $I(t=0)$
q_3	The Negative height of inventory with $I(t=T)$
Q	Total order quantity of order.
T	Total cycle time (Total cycle length).
t_1	The time where inventory height of rented Warehouse becomes zero.
t_2	The time where inventory height of Owned Warehouse becomes zero.
$I_1(t)$	The height of inventory in rented warehouse between time intervals $[0, t_1]$
$I_2(t)$	The height of inventory in owned warehouse between time intervals $[0, t_1]$
$I_3(t)$	Height of inventory in owned warehouse between the time intervals $[t_1, t_2]$
$I_4(t)$	Level of inventory in owned Warehouse between the time intervals $[t_2, T]$
PC	Cost of purchasing
HC	Cost of holding of inventory.
SC	Cost of shortage of the inventory.
LC	Cost of lost sale cost of inventory.
TAC	Present total Average cost.

2.2. Assumptions

The following assumptions are used in this paper.

1. The demand rate is generalized exponential increasing function of time in nature and taken as the following form : $D(t) = ke^{a+bt}$; $k > 0$, $a > 0$, $b > 0$.
2. The partially backlogged Shortages are allowed and where backlogging rate is $B(t) = e^{-\delta t}$; $\delta > 0$.
3. Infinite Time horizon is considered.
4. Lead time is zero with Infinite Replenishment rate is taken.
5. Warehouse (OW) has the limited space is allowed. On other hand the unlimited space area for rented warehouse has been permitted.
6. The holding cost (h_1) of the of Rented Warehouse is greater than the holding cost (h_2) of Owned Warehouse.
7. The charges for transportation and time between Rented Warehouse and Owned Warehouse are completely ignored.

3. FORMULATION AND SOLUTION OF THE MODEL

The suggested model allows for a partial backlog and optimizes a two-warehouse system for deteriorating items using a generalized exponential function of demand. The model looks for the best ordering procedures for both warehouses in order to minimize overall expenses while taking inflation into consideration. Presumably, one warehouse houses the major stock, while the other houses the backup. The three primary decision variables"order quantity, reorder points, and backordering level"are optimized using Bacterial Foraging Optimization (BFO). The rate at which products decay, generalized exponential demand, and inflationary effects make inventory cost management more challenging over time. By addressing these issues, the model seeks to balance stock levels, cut waste, and boost overall cost efficiency in a real-world supply chain environment. The proposed model shown in Fig. 1. The following is the formulation of the proposed model:

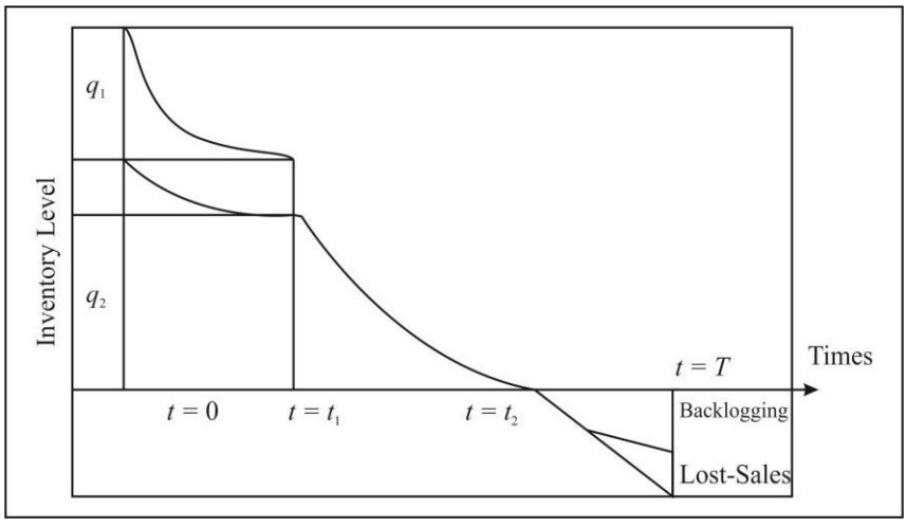

Figure 1: *Graphical representation of two-warehouses inventory model.*

$$\frac{dI_1}{dt} + \theta I_1 = -ke^{a+bt}, \qquad 0 \le t \le t_1 \tag{1}$$

$$\frac{dI_2}{dt} + \theta I_2 = 0, \qquad 0 \le t \le t_1 \tag{2}$$

$$\frac{dI_3}{dt} + \theta I_3 = -ke^{a+bt}, \qquad t_1 \le t \le t_2 \tag{3}$$

$$\frac{dI_4}{dt} = -ke^{a+bt} \cdot e^{\delta t}, \qquad t_2 \le t \le T \tag{4}$$

The boundary conditions are given by $I_1(t_1) = 0,\ I_2(0) = q_2,\ I_3(t_2) = I_4(t_2) = 0$.
Using the above boundary conditions, the solutions of (1), (2), (3) and (4) are given by

$$I_1(t) = e^{-\theta(t_1 - t)}\left(\frac{Ke^{a+bt_1}}{b+\theta}\right) - \frac{Ke^{a+bt}}{b+\theta} \tag{5}$$

$$I_2(t) = q_2 e^{-\theta t} \tag{6}$$

$$I_3(t) = e^{-\theta(t_2 - t)}\left(\frac{Ke^{a+bt_2}}{b+\theta}\right) - \frac{Ke^{a+bt}}{b+\theta} \tag{7}$$

$$I_4(t) = \frac{Ke^{a+(b-\delta)t_2}}{b+\delta} - \frac{Ke^{a+(b-\delta)t}}{b+\delta} \tag{8}$$

Positive inventory level of rented warehouse with $I_1(0) = q_1$and Eqs. (5) given by

$$q_1 = e^{\theta t_1}\left(\frac{Ke^{a+bt_1}}{b+\theta}\right) - \frac{Ke^{a}}{b+\theta} \tag{9}$$

Negative inventory level with $I_4(T) = -q_3$ and Eqs. (8) given by

$$q_3 = \left(\frac{Ke^{a+(b-\delta)T}}{b-\delta}\right) - \frac{Ke^{a+(b-\delta)t_2}}{b-\delta} \tag{10}$$

Next, we calculate all the associated inventory costs as follow:

1. Ordering Cost (OC):

$$OC = A \tag{11}$$

2. Purchasing Cost (PC):

$$PC = C_P\left[e^{\theta t_1}\left(\frac{Ke^{a+bt_1}}{b+\theta}\right) - \frac{Ke^{a}}{b+\theta} + q_2 + \left(\frac{Ke^{a+(b-\delta)T}}{b-\delta}\right) - \frac{Ke^{a+(b-\delta)t_2}}{b-\delta}\right] \tag{12}$$

3. Shortage Cost (SC):

$$SC = -C_S\int_{t_2}^{T} I_4(t)\cdot e^{-Rt}dt$$

$$SC = -C_S\int_{t_2}^{T}\left[\frac{Ke^{a+(b-\delta)t_2}}{b+\delta} - \frac{Ke^{a+(b-\delta)t}}{b+\delta}\right].e^{-Rt}dt$$

$$SC = -\frac{KC_S}{(b-\delta)}\left[\frac{e^{a+(b-\delta)t_2-RT}}{-R} - \frac{e^{a+(b-\delta-R)T}}{b-\delta-R} - \frac{(b-\delta)e^{a+(b-\delta-R)t_2}}{R(b-\delta-R)}\right] \tag{13}$$

4. Lost sales Cost (LC):

$$LC = -C_L \int_{t_2}^{T} (1 - B(t)) \cdot D(t) \cdot e^{-Rt} dt$$

$$LC = -KC_L \left[\left(\frac{e^{a+(b-R)T}}{b-R} - \frac{e^{a+(b-\delta-R)T}}{b-\delta-R} \right) - \left(\frac{e^{a+(b-R)t_2}}{b-R} - \frac{e^{a+(b-\delta-R)t_2}}{b-\delta-R} \right) \right] \quad (14)$$

5. Holding Cost (HC):

$$HC = \left[h_1 \int_0^{t_1} I_1(t).e^{-Rt} dt + h_2 \int_0^{t_1} I_2(t) \cdot e^{-Rt} dt + h_3 \int_0^{t_2} I_3(t).e^{-Rt} dt \right]$$

$$= \left\{ \frac{h_1 k}{(b+\theta)} \left[\frac{-(b+\theta)e^{a+(b-R)t_1}}{(\theta+R)(b-R)} + \frac{e^{a+(b-R)t_1}}{(\theta+R)} + \frac{e^{a}}{(b-R)} \right] + \frac{h_2 q_2}{(R+\theta)} \left(1 - e^{-(R+\theta)t_1} \right) \right.$$

$$\left. + \frac{h_2 k}{(b+\theta)} \left[\frac{-(b+\theta)e^{a+(b-R)t_2}}{(\theta+R)(b-R)} + \frac{e^{a+(b+\theta)t_2-(\theta+R)t_1}}{(\theta+R)} + \frac{e^{a+(b-R)t_1}}{(b-R)} \right] \right\} \quad (15)$$

6. Deterioration Cost (DC):

$$DC = C_D \left[\theta \int_0^{t_1} I_1(t) \cdot e^{-Rt} dt + \theta \int_0^{t_1} I_2(t) \cdot e^{-Rt} dt + \theta \int_0^{t_2} I_3(t) \cdot e^{-Rt} dt \right]$$

$$= C_D \left\{ \frac{\theta k}{(b+\theta)} \left[\frac{-(b+\theta)e^{a+(b-R)t_1}}{(\theta+R)(b-R)} + \frac{e^{a+(b-R)t_1}}{(\theta+R)} + \frac{e^{a}}{(b-R)} \right] + \frac{\theta q_2}{(R+\theta)} \left(1 - e^{-(R+\theta)t_1} \right) \right.$$

$$\left. + \frac{\theta k}{(b+\theta)} \left[\frac{-(b+\theta)e^{a+(b-R)t_2}}{(\theta+R)(b-R)} + \frac{e^{a+(b+\theta)t_2-(\theta+R)t_1}}{(\theta+R)} + \frac{e^{a+(b-R)t_1}}{(b-R)} \right] \right\} \quad (16)$$

$$TAC = \frac{1}{T} \left[OC + PC + HC + DC - SC + LSC \right] \quad (17)$$

4. Bacterial Foraging Optimization (BFO) Methodology

We can describe the algorithmic solution steps of BFO which are designed in the context of the described features and functions Sinha and Anand [84]. The following is the optimization process using Bacterial Foraging Optimization:

1. **Step 1 (Installation Phase):** Randomly dispense N pieces of bacteria particles (potential solution variables) into solution space. Algorithm parameters. Perform the necessary arrangements for the problem to be solved.
2. **Step 2:** Calculate the objective function value (fitness) according to the locations of the bacteria (potential solution variables).
3. **Step 3:** Perform the following steps, Repeat until: (in the context of each objective function size).
4. **Step 3.1 (Chemotaxis Phase):** Perform the following steps for each bacteria, up to the Nk value:
5. **Step 3.1.2:** The objective function of the bacterium related to the (fitness) cell to cell attractive effect of the update. Hold this value until swimming phase.
6. **Step 3.1.3 (Rolling Phase):** Generate random numbers up to the purpose function size in the range [-1, 1]. Run the rolling process for the respective bacteria.

7. **Step 3.1.4:** Calculate the objective function value (fitness) according to the location of the bacteria (potential solution variable). The purpose of the relevant bacterium is to update the value of the function function (fitness) from cell to cell with attractive effect.
8. **Step 3.1.5 (Swimming Phase):** Perform the following steps for the related bacteria, up to the Nyush value.
9. **Step 3.1.5.1:** If the final objective function value (fitness) of the bacteria is better than stored before the Swimming Phase, keep this new value.
10. **Step 3.1.5.2:** Update the held objective function value (fitness) of the relevant bacteria according to the displacement value to be calculated.
11. **Step 3.1.6:** If all bacteria have not been treated yet, switch to the next bacterium and return to Step 3.1.1.
12. **Step 3.2 (Reproduction Phase):** Calculate the health status of each bacterium and sort them all from small to small according to these values.
13. **Step 3.3:** Eliminate the worst bacteria according to the set criteria. Let the bacteria grow in the best condition. New bacteria are in place of their parents.
14. **Step 3.4:** If the nu value has not yet been reached, increase the counter for that value and go back to Step 3.1 and continue with the next generation.
15. **Step 3.5 (Elimination - Distribution Phase):** Transfer each bacterium to a new location according to the value oed.
16. **Step 4:** At the end of the processes, the value (s) obtained by the global best position is considered to be the optimum value (s). There are many studies and applications that are related with this optimization algorithm.

In Hezer and Kara [85], to determine the routes to be followed by the vehicles used in distribution and collection activities and to minimize the logistics costs, an algorithm has been developed with this optimization to solve the stated problem.

5. Graphical Representation

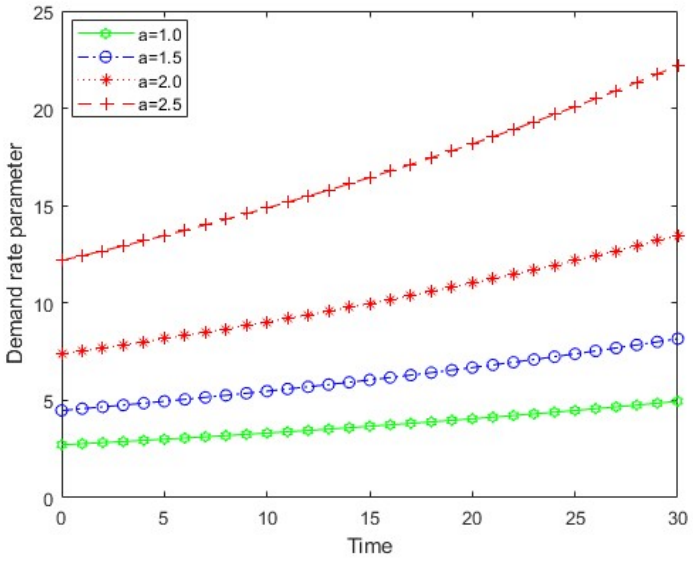

Figure 2: *Variation between time and demand rate, if changing a.*

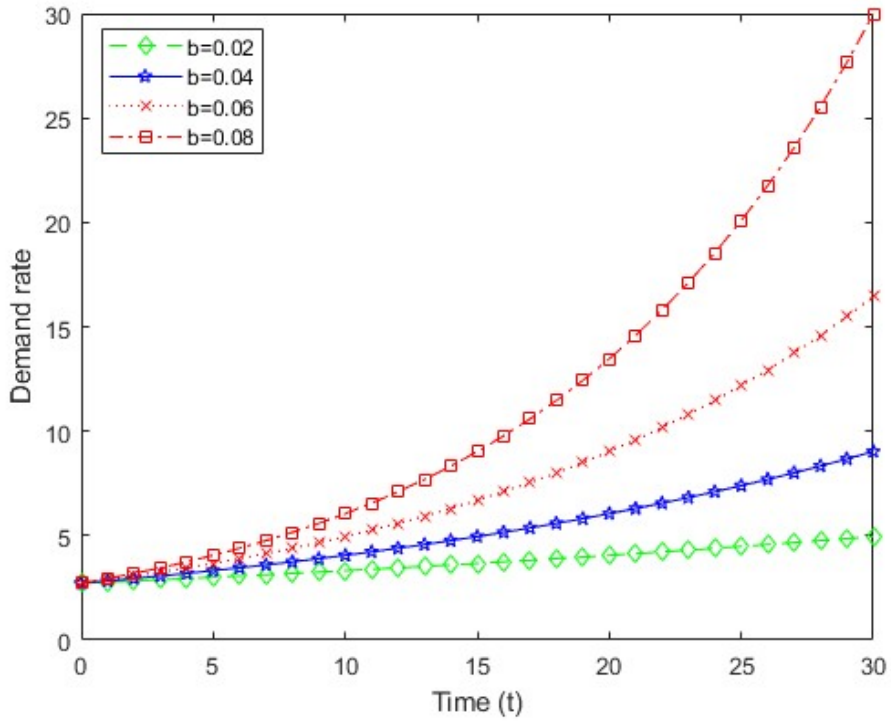

Figure 3: *Variation between time and demand rate, if changing b.*

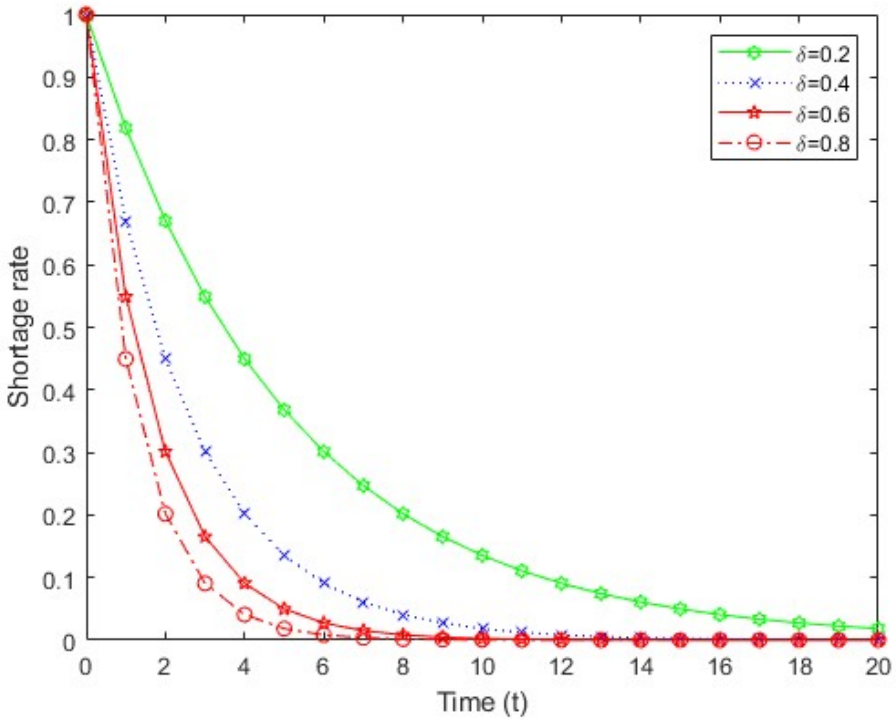

Figure 4: *Variation between time and shortage rate, if changing δ.*

The graphical depictions in this part provide a comprehensive grasp of the dynamics within the deteriorated inventory model of two-warehouses. Figure 2 and Figure 3 explains the link between demand rates and time, while Figure 4 illustrates how shortage impacts inventory management using different type of functions of time together with the adjustments needed to effectively balance demand, shortage, and time.

6. Conclusion

In conclusion, the Two-Warehouse Inventory Model for Deteriorating Items with a Generalized Exponential Increasing Demand and Partial Backlogging under Inflation, optimized through Bacterial Foraging Optimization (BFO), represents a comprehensive and adaptive approach to contemporary inventory management challenges. The model addresses the inherent complexities of managing deteriorating items across two warehouses, acknowledging the nuanced nature of demand with a generalized exponential increase. Incorporating partial backlogging recognizes the practical constraints of immediate fulfillment, introducing a realistic dimension to inventory control. The inclusion of inflationary factors enhances the model's relevance in a dynamic economic landscape. The decision to employ BFO as the optimization algorithm is strategic, harnessing the efficiency of nature-inspired algorithms in navigating complex solution spaces. By simulating bacterial foraging behaviors, BFO efficiently explores and exploits optimal solutions for the intricate parameters of the inventory model. Through this research, businesses gain a sophisticated tool for strategic decision-making in inventory management. The model provides a framework for minimizing losses associated with deteriorating items, adapting to fluctuating demands, and optimizing costs in the face of inflation. By leveraging BFO, organizations can fine-tune their inventory policies, ensuring an optimal balance between customer satisfaction and cost-effectiveness. As industries continue to evolve, characterized by rapid changes in consumer behavior, market dynamics, and economic conditions, the significance of robust inventory management models cannot be overstated. The Two-Warehouse Inventory Model, coupled with BFO optimization, positions businesses to not only navigate current challenges but also to proactively respond to future uncertainties. In essence, this research contributes to the advancement of inventory management methodologies, offering a tailored solution that aligns with the complexities of contemporary supply chain environments. It is our hope that businesses embracing this model will experience heightened efficiency, improved customer satisfaction, and a competitive edge in the dynamic landscape of global commerce.

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SURVIVAL ANALYSIS OF A STOCHASTIC MODEL ON CARDIOVASCULAR SYSTEM CONSIDERING POSSIBILITES OF DAMAGE, FAILURE AND RECOVERY OF HEART

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Abstract

The present paper deals with survival analysis of a stochastic model on cardiovascular system considering possibilities of damage, failure and recovery of heart. The analysis is based upon a stochastic model for the system considering different kinds of damage and failure of heart at different situations. The treatments and recovery of heart are taken in to account. On complete failure of heart, transplantation of the heart is also considered. The model has been analyzed by determining important measures of effectiveness using Markov process and regenerative point technique. Sensitivity analysis has also been done to select important parameters for enhancing the survivability of the system.

Keywords: Cardiovascular system, mean survival time, survivability, sensitivity analysis, Markov process and regenerative point technique.

1. Introduction

The cardiovascular system in the human body is a complex network of organs, vessels and tissues that transport blood, oxygen, nutrients and other vital substances throughout the body. The main components of cardiovascular system include the heart, blood vessels and blood, the major driving force of the system is the heart. The heart is a vital organ of the system responsible for pumping blood throughout the body, supplying oxygen and nutrients to the tissues and organs. It is a muscular organ located slightly left of the center of the chest and is protected by the ribcage.

The survivability of the system plays an important role for the functioning of the body. For investigations of heart related issues including morbidity and mortality attempts have been made by many researchers applying prospective and different methods. Kannel and McGee [3] discussed the relation between diabetes and cardiovascular disease. Bertoni et al. [1] studied diabetes impacts on the elderly in terms of the prevalence, incidence and mortality related to heart failure. Lerma et al. [6] discussed the stochastic aspects of cardiac arrhythmias. Smith et al. [11] predicted the outcomes among heart failure patients. Meenaxi et al. [7] studied a model for the progression of chronic heart failure. Tanna et al. [12] considered risk prediction models for adults with heart failure. Pierce et al. [8] explained the patterns of cardiovascular mortality associated with heart failure,

comparing rural and urban counties across the United States. Rajeswari and Kausika [9] developed a machine learning model to predict future possibility of heart disease. Khan et al. [4] presented machine learning algorithm-based cardiovascular disease prediction. Roger [10] studied the epidemiology of heart failure. Deng [2] provided a modern viewpoint on cardiac transplantation.

Over the time, researches in the epidemiology of heart failure make simple the understanding of the syndrome's complexity. The prevalence of disease like diabetes mellitus, obesity, hypertension, and problems in supplementary organs like chronic kidney disease, and cancer comorbidities at heart failure are increasing, and these factors may be associated with the increasing prevalence of heart failure with preserved ejection fraction [5]. In motor vehicle accidents and vehicle air bags heart injury due to rib fractures and sucking chest wound are very common [14]. Further, according to American Heart Association, heart transplantation is a kind of surgery in which failing heart of patient is replaced by donor's heart. It gives better life to heart patient. Heart transplants have seen a remarkable rise in recent years. For instance, the number of transplant procedures in India grew from 53 in 2014 to 241 in 2018, and there were 187 transplants in 2019 [13].

2. Model Description and State Transition Diagram

It has been noticed that considering the above aspects of heart damages, failure and recovery/transplantation, survivability and sensitivity analysis of cardiovascular system has not been reported in the literature. To fill this gap, the present paper deals with the survivability and sensitivity analysis of cardiovascular system considering various causes of damage, failure and recovery of heart. It is considered that the heart may damage/fail due to prevalence of some disease, other organ issues and some severe accidents. The person having some heart problem reaches the hospital in negligible time for the treatments. After successful recovery, heart in the cardiovascular system works as good as healthy whereas transplantation of a heart does not make the patient healthy. A stochastic model has been developed for the cardiovascular system and analyzed by determining some measures of effectiveness using Markov process and regenerative point technique. The other assumptions of the model are

- Minor heart damages are recovered by some medicine/exercise/therapy.
- The severe accidents result into major heart damage.
- The single diagnose/treatment facility is available in the hospital.
- The time to damages and failure is considered to be exponentially distributed while other time distributions are general.
- All random variables are mutually independent.

3. States and Notations of the System

States & Notations	Description
S_0	healthy state
S_1, S_2	minor damage state
S_3	major damage state
S_4	failed state
λ_{01}	damage rate due to prevalence of some disease
λ_{02}	damage rate due to other organ issues
λ_{04}	failure rate due to severe accidents
λ_{13}	major damage rate from minor heart damage

λ_{24}	failure rate due to minor heart damage
λ_{34}	failure rate due to major heart damage
$g_1(t) / G_1(t)$	p.d.f/c.d.f of time of recovery through medicine/exercise/therapy
$g_2(t) / G_2(t)$	p.d.f/c.d.f of time of recovery from other organ issues
$g_3(t) / G_3(t)$	p.d.f/c.d.f of time of recovery due to surgery/operation
$h(t)$	p.d.f/c.d.f of time of heart transplantation

State Transition Diagram is depicted in fig. 1

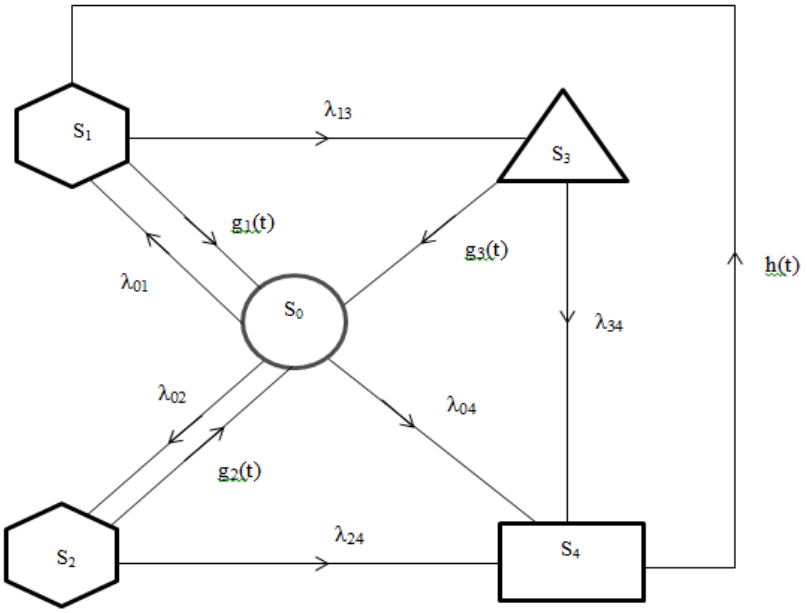
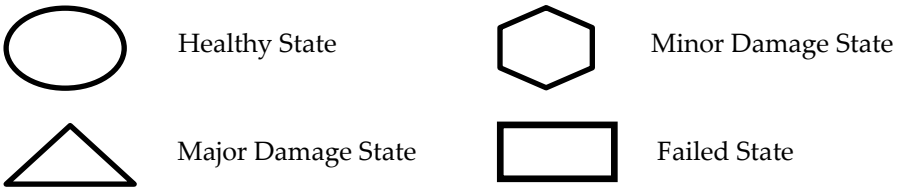


Figure 1. *State Transition Diagram*



4. Transition Probabilities

The transition probabilities are given by

$dQ_{01}(t) = \lambda_{01} e^{-(\lambda_{01}+\lambda_{02}+\lambda_{04})t} dt;$ $dQ_{02}(t) = \lambda_{02} e^{-(\lambda_{01}+\lambda_{02}+\lambda_{04})t} dt;$

$dQ_{04}(t) = \lambda_{04} e^{-(\lambda_{01}+\lambda_{02}+\lambda_{04})t} dt;$ $dQ_{10}(t) = g_1(t) e^{-\lambda_{13} t} dt;$

$dQ_{13}(t) = \lambda_{13} e^{-\lambda_{13} t} \overline{G_1(t)} dt;$ $dQ_{20}(t) = g_2(t) e^{-\lambda_{24} t} dt;$

$dQ_{24}(t) = \lambda_{24} e^{-\lambda_{24} t} \overline{G_2(t)} dt;$ $dQ_{30}(t) = g_3(t) e^{-\lambda_{34} t} dt;$

$dQ_{34}(t) = \lambda_{34} e^{-\lambda_{34} t} \overline{G_3(t)} dt;$ $dQ_{41}(t) = h(t) dt.$

The non-zero element p_{ij} are given by

$$p_{ij} = \lim_{s\to 0} q^{*}_{ij}(s) = \lim_{s\to 0} Q^{**}_{ij}(s).$$

Therefore,

$$p_{01} = \frac{\lambda_{01}}{\lambda_{01}+\lambda_{02}+\lambda_{04}}; \quad p_{02} = \frac{\lambda_{02}}{\lambda_{01}+\lambda_{02}+\lambda_{04}}; \quad p_{04} = \frac{\lambda_{04}}{\lambda_{01}+\lambda_{02}+\lambda_{04}};$$

$$p_{10} = g_1^{*}(\lambda_{13}); \quad p_{13} = 1-g_1^{*}(\lambda_{13}); \quad p_{24} = 1-g_2^{*}(\lambda_{24});$$

$$p_{30} = g_3^{*}(\lambda_{34}); \quad p_{34} = 1-g_3^{*}(\lambda_{34}); \quad p_{41} = h^{*}(0).$$

It can be observed that

$$p_{01}+p_{02}+p_{04} = 1; \quad p_{10}+p_{13} = 1; \quad p_{20}+p_{24} = 1;$$

$$p_{30}+p_{34} = 1; \quad p_{41} = 1.$$

5. Mean Sojourn Time

Expected time taken by the patient in state i before transiting to any other state is termed as mean sojourn time in that state and it is denoted by μ_i . Mean sojourn time μ_i in the i th state is given by

$$\mu_i = \int_0^{\infty} \Pr(T_i > t)dt,$$

where T_i is the sojourn time in state i .

$$\mu_0 = \frac{1}{\lambda_{01}+\lambda_{02}+\lambda_{04}}; \quad \mu_1 = \frac{1-g_1^{*}(\lambda_{13})}{\lambda_{13}}; \quad \mu_2 = \frac{1-g_2^{*}(\lambda_{24})}{\lambda_{24}};$$

$$\mu_3 = \frac{1-g_3^{*}(\lambda_{34})}{\lambda_{34}}; \quad \mu_4 = \int_0^{\infty} \overline{H(t)dt}.$$

The unconditional mean time is, mathematically, defined as

$$m_{ij} = \int_0^{\infty} t q_{ij}(t)\,dt = -q^{*}_{ij}(0).$$

Therefore,

$$m_{01} = \frac{\lambda_{01}}{(\lambda_{01}+\lambda_{02}+\lambda_{04})^2}; \quad m_{02} = \frac{\lambda_{02}}{(\lambda_{01}+\lambda_{02}+\lambda_{04})^2}; \quad m_{04} = \frac{\lambda_{04}}{(\lambda_{01}+\lambda_{02}+\lambda_{04})^2};$$

$$m_{10} = -g_1^{*'}(\lambda_{13}); \quad m_{13} = \frac{1-g_1^{*}(\lambda_{13})}{\lambda_{13}} + g_1^{*'}(\lambda_{13}); \quad m_{20} = -g_2^{*'}(\lambda_{24});$$

$$m_{24} = \frac{1-g_2^{*}(\lambda_{24})}{\lambda_{24}} + g_2^{*'}(\lambda_{24}); \quad m_{30} = -g_3^{*'}(\lambda_{34}); \quad m_{34} = \frac{1-g_3^{*}(\lambda_{34})}{\lambda_{34}} + g_3^{*'}(\lambda_{34});$$

$$m_{41} = -h^{*'}(0).$$

From above we observed that

$$m_{01}+m_{02}+m_{04} = \mu_0; \quad m_{10}+m_{13} = \mu_1; \quad m_{20}+m_{24} = \mu_2;$$

$$m_{30}+m_{34} = \mu_3; \quad m_{41} = \mu_4.$$

6. Mean Survival Time

Let $\Phi_i(t)$ denotes the cumulative distribution function of first passage time from S_i to the failed state. The following recursive relations are obtained for $\Phi_i(t)$:

$$\Phi_0(t) = Q_{01}(t) \,\&\, \Phi_1(t) + Q_{02}(t) \,\&\, \Phi_2(t) + Q_{04}(t);$$

$$\Phi_1(t) = Q_{10}(t) \,\&\, \Phi_0(t) + Q_{13}(t) \,\&\, \Phi_3(t);$$

$$\Phi_2(t) = Q_{20}(t) \,\&\, \Phi_0(t) + Q_{24}(t);$$

$$\Phi_3(t) = Q_{30}(t) \,\&\, \Phi_0(t) + Q_{34}(t).$$

Taking L.S.T and solving for $\Phi_0^{**}(s)$, we get

$$\Phi_0^{**}(s) = \frac{N(s)}{D(s)},$$

where

$$\mathrm{N}(s) = Q_{04}^{**}(s) + Q_{02}^{**}(s)Q_{24}^{**}(s) + Q_{01}^{**}(s)Q_{13}^{**}(s)Q_{34}^{**}(s),$$

and

$$D(s) = 1 - Q_{01}^{**}(s)Q_{10}^{**}(s) - Q_{02}^{**}(s)Q_{20}^{**}(s) - Q_{01}^{**}(s)Q_{13}^{**}(s)Q_{30}^{**}(s).$$

Now the expression for the mean survival time is

$$T_0 = \lim_{s\to 0} \frac{1-\Phi_0^{**}(s)}{s}.$$

Using L'Hospital's rule and solving for $\Phi_0^{**}(s)$, we get

$$T_0 = \frac{\mathrm{N}}{\mathrm{D}},$$

where

$$\mathrm{N} = \mu_0 + p_{01}(\mu_1 + p_{13}\mu_3) + p_{02}\mu_2, \quad \text{and} \quad D = 1 - p_{01}p_{10} - p_{02}p_{20} - p_{01}p_{13}p_{30}.$$

7. Survivability

Let $S_i(t)$ denotes the probability that patient survive at instant t given that the patient entered the state i at time $t = 0$. The recursive relations for $S_i(t)$ as given below:

$$S_0(\mathrm{t}) = \mathrm{M}_0(\mathrm{t}) + \mathrm{q}_{01}(\mathrm{t}) \copyright \mathrm{S}_1(\mathrm{t}) + \mathrm{q}_{02}(\mathrm{t}) \copyright \mathrm{S}_2(\mathrm{t}) + \mathrm{q}_{04}(\mathrm{t}) \copyright \mathrm{S}_4(\mathrm{t})$$
$$S_1(\mathrm{t}) = \mathrm{M}_1(\mathrm{t}) + \mathrm{q}_{10}(\mathrm{t}) \copyright \mathrm{S}_0(\mathrm{t}) + \mathrm{q}_{13}(\mathrm{t}) \copyright \mathrm{S}_3(t)$$
$$S_2(\mathrm{t}) = \mathrm{M}_2(\mathrm{t}) + \mathrm{q}_{20}(\mathrm{t}) \copyright \mathrm{S}_0(\mathrm{t}) + \mathrm{q}_{24}(\mathrm{t}) \copyright \mathrm{S}_4(t)$$
$$S_3(\mathrm{t}) = \mathrm{M}_3(\mathrm{t}) + \mathrm{q}_{30}(\mathrm{t}) \copyright \mathrm{S}_0(\mathrm{t}) + \mathrm{q}_{34}(\mathrm{t}) \copyright \mathrm{S}_4(\mathrm{t})$$
$$S_4(\mathrm{t}) = \mathrm{q}_{41}(\mathrm{t}) \copyright \mathrm{S}_1(\mathrm{t}),$$

where

$$M_0(t) = e^{-(\lambda_1+\lambda_2+\lambda_3)t}; \quad M_1(t) = e^{-\lambda_1 t}\overline{G_1(t)}; \quad M_2(t) = e^{-\lambda_2 t}\overline{G_2(t)}; \quad M_3(t) = e^{-\lambda_1 t}\overline{G_3(t)}.$$

By using LT of these equations and then computing for $S_0^*(s)$,

$$S_0^*(s) = \frac{N_1(s)}{D_1(s)},$$

where

$$N_1(\mathrm{s}) = \mathrm{M}_0^*(s) + M_1^*(s)q_{01}^*(s) + M_2^*(s)q_{02}^*(s) + M_3^*(s)q_{01}^*(s)q_{13}^*(s) + M_1^*(s)q_{04}^*(s)q_{41}^*(s)$$
$$+ M_3^*(s)q_{04}^*(s)q_{13}^*(s)q_{41}^*(s) + M_1^*(s)q_{02}^*(s)q_{24}^*(s)q_{41}^*(s) + M_3^*(s)q_{02}^*(s)q_{13}^*(s)q_{24}^*(s)q_{41}^*(s)$$
$$- M_0^*(s)q_{13}^*(s)q_{34}^*(s)q_{41}^*(s) - M_2^*(s)q_{02}^*(s)q_{13}^*(s)q_{34}^*(s)q_{41}^*(s)$$

and

$$\mathrm{D}_1(\mathrm{s}) = 1 - q_{01}^*(s)q_{10}^*(s) - q_{02}^*(s)q_{20}^*(s) - q_{01}^*(s)q_{13}^*(s)q_{30}^*(s) - q_{04}^*(s)q_{10}^*(s)q_{41}^*(s)$$
$$- q_{02}^*(s)q_{10}^*(s)q_{24}^*(s)q_{41}^*(s) - q_{04}^*(s)q_{13}^*(s)q_{30}^*(s)q_{41}^*(s) - q_{02}^*(s)q_{13}^*(s)q_{24}^*(s)q_{30}^*(s)q_{41}^*(s)$$
$$- q_{13}^*(s)q_{34}^*(s)q_{41}^*(s) + q_{02}^*(s)q_{13}^*(s)q_{20}^*(s)q_{34}^*(s)q_{41}^*(s).$$

Expected survivability is derived as

$$S_0 = \lim_{s\to 0} sS_0^*(s) = \frac{N_1}{D_1},$$

where

$$\mathrm{N}_1 = \mu_0(1 - p_{13}p_{34}) + \mu_1(1 - p_{02}p_{20}) + \mu_2 p_{02}(1 - p_{13}p_{34}) + \mu_3 p_{13}(1 - p_{02}p_{20})$$

and

$$D_1 = \mu_0(1 - p_{13}p_{34}) + \mu_1(1 - p_{02}p_{20}) + \mu_2 p_{02}(1 - p_{13}p_{34}) + \mu_3 p_{13}(1 - p_{02}p_{20}) + \mu_4(p_{04} + p_{02}p_{24} + p_{01}p_{13}p_{34}).$$

8. Numerical Computations and Graphical Interpretations

The expressions derived for mean sojourn times, mean survival time and survivability are analytic and computationally tedious involving several parameters. Therefore, following particular case is considered for computations and analysis purpose:

$$g_1(t)=\beta_1 e^{-\beta_1 t};\quad g_2(t)=\beta_2 e^{-\beta_2 t};\quad g_3(t)=\beta_3 e^{-\beta_3 t};\quad h(t)=\frac{1}{\gamma}.$$

Numerical computations have been done for the above particular case and various graphs have been plotted for mean survival time and survivability giving different estimated values to the parameters $\lambda_{01}, \lambda_{02}, \lambda_{04}, \lambda_{01}, \lambda_{13}, \lambda_{24}, \lambda_{34}, \beta_1, \beta_2, \beta_3, \gamma$. The following interpretations and conclusions have been drawn from the plotted graphs.

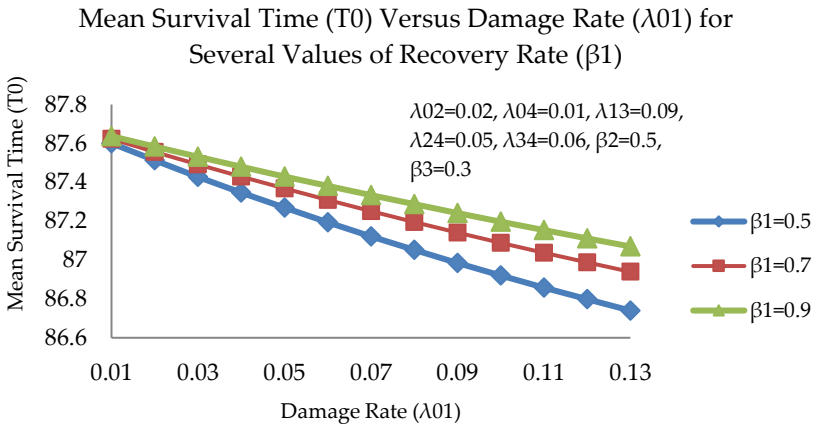


Figure 2. *Mean survival time* (T_0) *with respect to damage rate* (λ_{01})

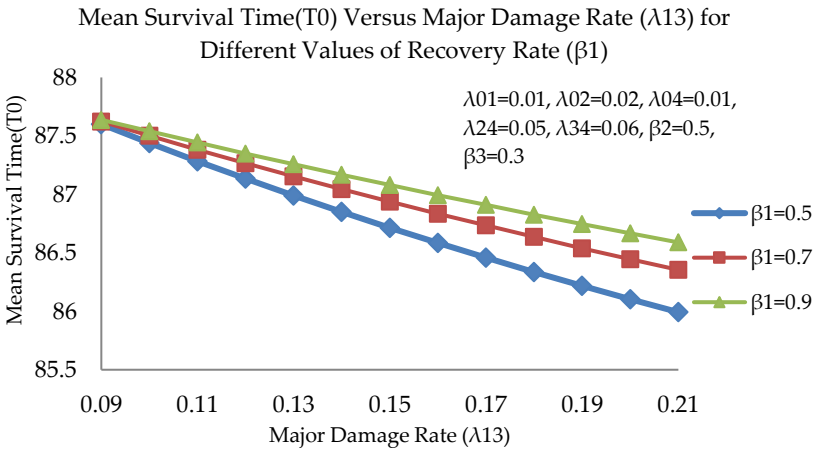


Figure 3. *Mean survival time* (T_0) *with respect to major damage rate* (λ_{13})

From the fig. 2 and fig. 3, it can be observed that mean survival time (T_0) of the system decreases as the damage rates λ_{01} and λ_{13} increases and has greater value for greater value of recovery rate β_1.

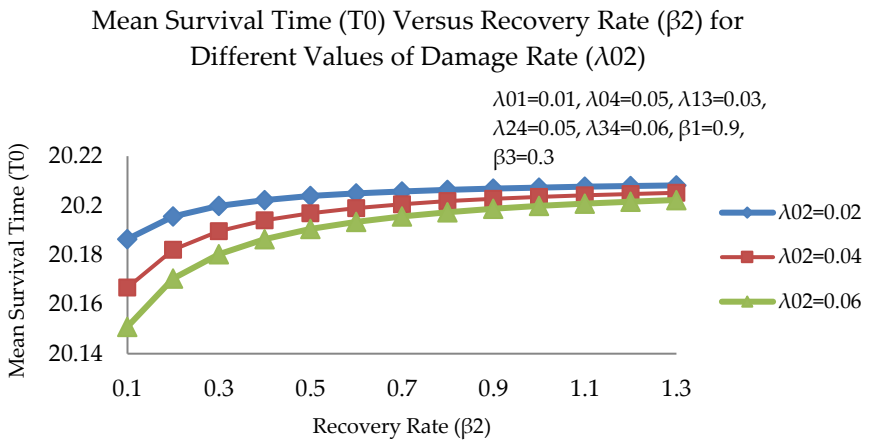


Figure 4. *Mean survival time* (T_0) *with respect to recovery rate* (β_2)

From the fig. 4, we can observe that mean survival time (T_0) of the system is more for higher recovery rate β_2 and it has lesser value for higher value of damage rate λ_{02}.

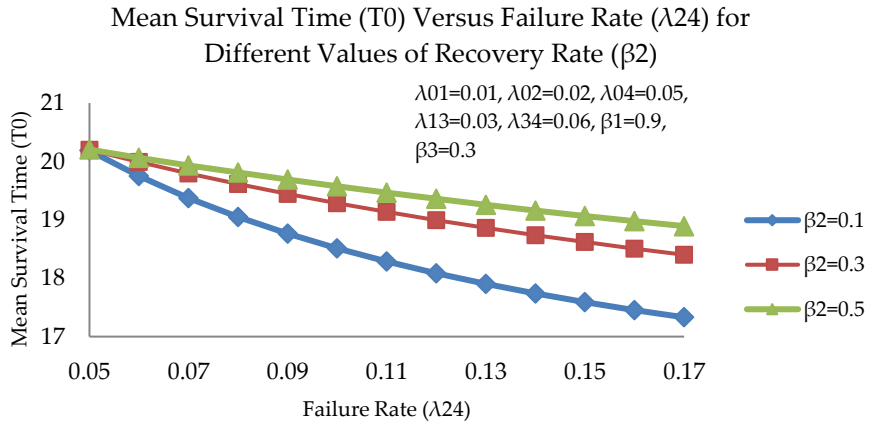


Figure 5. *Mean survival time* (T_0) *with respect to failure rate* (λ_{24})

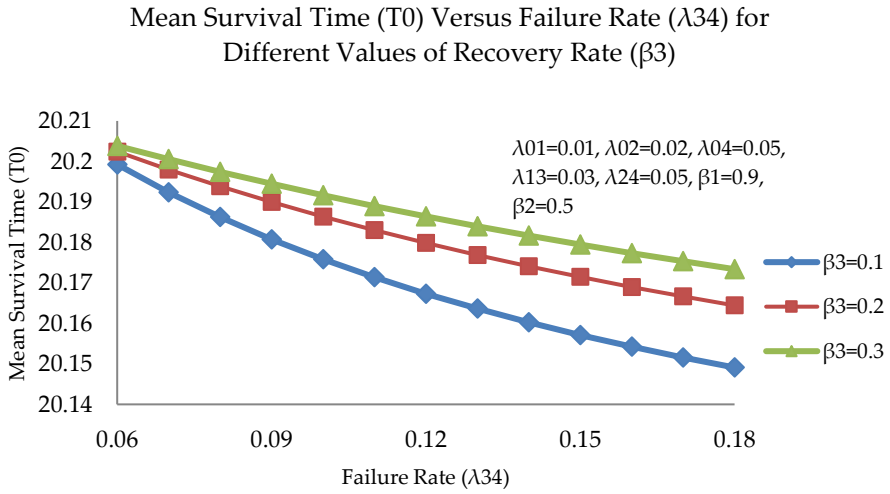


Figure 6. *Mean survival time* (T_0) *with respect to failure rate* (λ_{34})

From the fig. 5, we observed that mean survival time (T_0) of the system declines with the failure rate λ_{24} increases and it improves with the recovery rate β_2.

From the fig. 6, we observed that mean survival time (T_0) of the system decreases as the failure rate λ_{34} increases and has greater value for greater value of recovery rate β_3.

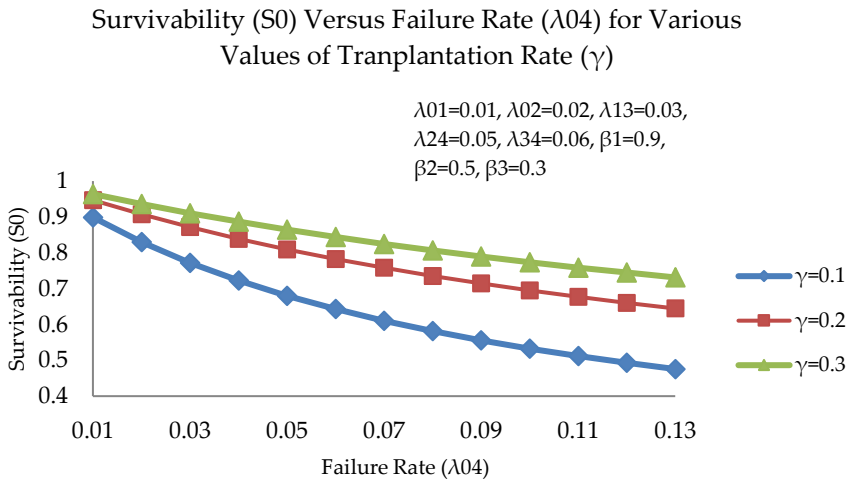


Figure 7. *Survivability* (S_0) *with respect to failure rate* (λ_{04})

From fig. 7 and fig. 8, it can be observed that survivability (S_0) of the system decreases as the failure rates λ_{04} and λ_{24} increases and has greater value for greater value of transplantation rate γ.

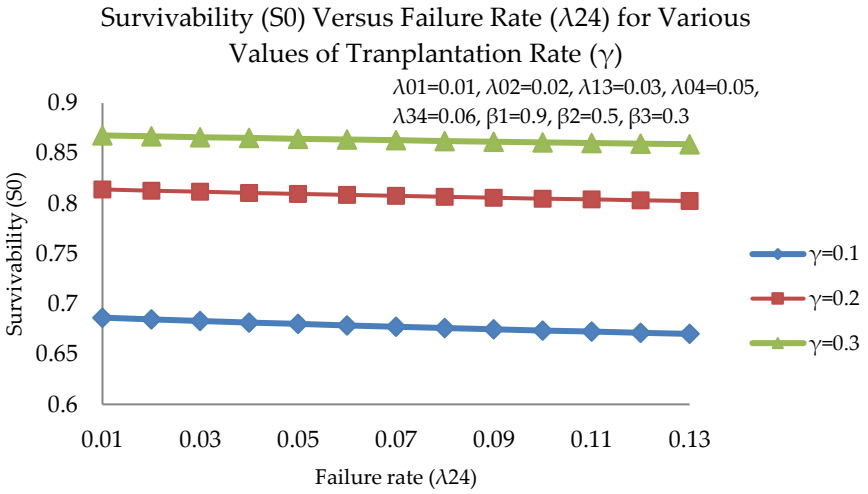


Figure 8. *Survivability* (S_0) *with respect to failure rate* (λ_{24})

9. Sensitivity and Relative Sensitivity Analysis

Sensitivity analysis determines how different values of an independent variable affect a particular dependent variable under a given set of assumptions. As there is significant difference among the values of parameters, we can use the concept of sensitivity analysis for comparing their effects on mean survival time (T_0) and survivability (S_0). The sensitivity and relative sensitivity function for

mean survival time (T_0) and survivability (S_0) of the system are given below:

$$\pi_k = \frac{\partial(T_0)}{\partial k}, \quad \delta_k = \pi_k(\frac{k}{T_0}) \quad \text{and} \quad \pi_k = \frac{\partial(S_0)}{\partial k}, \quad \delta_k = \pi_k(\frac{k}{S_0}),$$

where

$k = \lambda_{01}, \lambda_{02}, \lambda_{04}, \lambda_{13}, \lambda_{24}, \lambda_{34}, \beta_1, \beta_2, \beta_3, \gamma.$

The fixed values of parameters are as follows:

$\lambda_{01} = 0.01, \lambda_{02} = 0.02, \lambda_{04} = 0.01, \lambda_{13} = 0.09, \lambda_{24} = 0.05, \lambda_{34} = 0.06, \ \beta_1 = 0.9, \beta_2 = 0.5, \beta_3 = 0.3, \gamma = 0.1$

Table 1: *Sensitivity and relative sensitivity analysis of mean survival time (T_0) of the system with respect to damage rate λ_{02}*

λ_{02}	$\pi_{\lambda_{02}} = \frac{\partial T_0}{\partial \lambda_{02}}$	$\delta_{\lambda_{02}} = \pi_{\lambda_{02}}(\frac{\lambda_{02}}{T_0})$
0.01	-601.6138	-0.0646
0.02	-513.6997	-0.1172
0.03	-443.7370	-0.1607
0.04	-387.1513	-0.1967
0.05	-340.7376	-0.2269
0.06	-302.1962	-0.2523
0.07	-269.8426	-0.2737

Table 2: *Sensitivity and relative sensitivity analysis of mean survival time (T_0) of the system with respect to failure rate λ_{04}*

λ_{04}	$\pi_{\lambda_{04}} = \frac{\partial T_0}{\partial \lambda_{04}}$	$\delta_{\lambda_{04}} = \pi_{\lambda_{04}}(\frac{\lambda_{04}}{T_0})$
0.01	-0.007	-0.8354
0.02	-0.002	-0.9103
0.03	-0.001	-0.9384
0.04	-595.5245	-0.9531
0.05	-388.3926	-0.9621
0.06	-273.1571	-0.9682
0.07	-202.5219	-0.9726
0.08	-156.1222	-0.9760
0.09	-124.0172	-0.9786

Table 3: *Sensitivity and relative sensitivity analysis of mean survival time (T_0) of the system with respect to recovery rate β_1*

β_1	$\pi_{\beta_1} = \frac{\partial T_0}{\partial \beta_1}$	$\delta_{\beta_1} = \pi_{\beta_1}(\frac{\beta_1}{T_0})$
0.1	1.3466	0.0015
0.2	0.6038	0.0014
0.3	0.3412	0.0012
0.4	0.2189	0.0009
0.5	0.1523	0.0008
0.6	0.1120	0.0007
0.7	0.0859	0.0006

Table 4: *Sensitivity and relative sensitivity analysis of mean survival time (T_0) of the system with respect to recovery rate β_3*

β_3	$\pi_{\beta_3} = \frac{\partial T_0}{\partial \beta_3}$	$\delta_{\beta_3} = \pi_{\beta_3}(\frac{\beta_3}{T_0})$
0.1	12.2427	0.0141
0.2	4.7379	0.0109
0.3	2.4954	0.0085
0.4	1.5368	0.0070
0.5	1.0407	0.0059
0.6	0.7571	0.0051
0.7	0.5675	0.0045

Table 5: *Sensitivity and relative sensitivity analysis of survivability (S_0) of the system with respect to damage rate λ_{01}*

λ_{01}	$\pi_{\lambda_{01}} = \frac{\partial(S_0)}{\partial \lambda_{01}}$	$\delta_{\lambda_{01}} = \pi_{\lambda_{01}}(\frac{\lambda_{01}}{S_0})$
0.01	-0.0056	-0.00006
0.02	-0.0055	-0.00012
0.03	-0.0054	-0.00017
0.04	-0.0052	-0.00023
0.05	-0.0051	-0.00028
0.06	-0.0050	-0.00033
0.07	-0.0049	-0.00038

Table 6: *Sensitivity and relative sensitivity analysis of survivability (S_0) of the system with respect to damage rate λ_{02}*

λ_{02}	$\pi_{\lambda_{02}} = \frac{\partial(S_0)}{\partial \lambda_{02}}$	$\delta_{\lambda_{02}} = \pi_{\lambda_{02}}(\frac{\lambda_{02}}{S_0})$
0.01	-0.5580	-0.0062
0.02	-0.5314	-0.0118
0.03	-0.5067	-0.0170
0.04	-0.4837	-0.0218
0.05	-0.4622	-0.0262
0.06	-0.4421	-0.0302
0.07	-0.4233	-0.0339

Table 7: *Sensitivity and relative sensitivity analysis of survivability (S_0) of the system with respect to major damage rate λ_{13}*

λ_{13}	$\pi_{\lambda_{13}} = \frac{\partial S_0}{\partial \lambda_{13}}$	$\delta_{\lambda_{13}} = \pi_{\lambda_{13}}(\frac{\lambda_{13}}{S_0})$
0.01	-0.0265	-0.0003
0.02	-0.0260	-0.0006
0.03	-0.0255	-0.0009
0.04	-0.0250	-0.0011
0.05	-0.0245	-0.0014
0.06	-0.0241	-0.0016
0.07	-0.0236	-0.0018

Table 8*: Sensitivity and relative sensitivity analysis of survivability* (S_0) *of the system with respect to failure rate* λ_{34}

λ_{34}	$\pi_{\lambda_{34}} = \frac{\partial S_0}{\partial \lambda_{34}}$	$\delta_{\lambda_{34}} = \pi_{\lambda_{34}}(\frac{\lambda_{34}}{S_0})$
0.01	-0.0484	-0.0005
0.02	-0.0456	-0.0010
0.03	-0.0431	-0.0014
0.04	-0.0408	-0.0018
0.05	-0.0386	-0.0022
0.06	-0.0367	-0.0025
0.07	-0.0348	-0.0027

Table 9: *Sensitivity and relative sensitivity analysis of survivability* (S_0) *of the system with respect to recovery rate* β_1

β_1	$\pi_{\beta_1} = \frac{\partial S_0}{\partial \beta_1}$	$\delta_{\beta_1} = \pi_{\beta_1}(\frac{\beta_1}{S_0})$
0.1	0.0031	0.00034
0.2	0.0014	0.00031
0.3	0.0008	0.00026
0.4	0.0005	0.00022
0.5	0.0004	0.00020
0.6	0.0003	0.00017
0.7	0.0002	0.00015

Table 10: *Sensitivity and relative sensitivity analysis of survivability* (S_0) *of the system with respect to recovery rate* β_2

β_2	$\pi_{\beta_2} = \frac{\partial S_0}{\partial \beta_2}$	$\delta_{\beta_2} = \pi_{\beta_2}(\frac{\beta_2}{S_0})$
0.1	0.2040	0.0234
0.2	0.0836	0.0189
0.3	0.0452	0.0152
0.4	0.0283	0.0126
0.5	0.0193	0.0108
0.6	0.0140	0.0094
0.7	0.0107	0.0083

Table 11: *Sensitivity and relative sensitivity analysis of survivability* (S_0) *of the system with respect to recovery rate* β_3

β_3	$\pi_{\beta_3} = \frac{\partial S_0}{\partial \beta_3}$	$\delta_{\beta_3} = \pi_{\beta_3}(\frac{\beta_3}{S_0})$
0.1	0.0295	0.0033
0.2	0.0110	0.0025
0.3	0.0057	0.0019
0.4	0.0035	0.0016
0.5	0.0024	0.0013
0.6	0.0017	0.0011
0.7	0.0013	0.0009

Table 12: *Sensitivity and relative sensitivity analysis of survivability* (S_0) *of the system with respect to transplantation rate* γ

γ	$\pi_\gamma = \frac{\partial S_0}{\partial \gamma}$	$\delta_\gamma = \pi_\gamma(\frac{\gamma}{S_0})$
0.1	0.9198	0.1025
0.2	0.2555	0.0540
0.3	0.1177	0.0367
0.4	0.0675	0.0278
0.5	0.0437	0.0223
0.6	0.0305	0.0187
0.7	0.0266	0.0161

The sensitivity analysis of model for mean survival time and survivability of the system with respect to heart damage rates, failure rates and recovery rates are explained in table 1to table 12. The sign of sensitivity of mean survival time and survivability of the system with respect to heart damage rates $(\lambda_{01}, \lambda_{02})$, major damage rate (λ_{13}) and failure rates $(\lambda_{04}, \lambda_{24}, \lambda_{34})$ are negative. This shows that increase in these parameters decrease the value of mean survival time and survivability of the system. The sign of sensitivity of mean survival time and survivability of the system with respect to recovery rates $(\beta_1, \beta_2, \beta_3)$ and transplantation rate (γ) of the heart are positive which means increase in these parameters improve the value of mean survival time and survivability of the system. For example, in case of sensitivity of mean survival time (T_0) and survivability (S_0) of the system with respect to heart damage rate λ_{02} shows that increase in the heart damage rate due to some organ issues decrease the value of mean survival time and survivability of the system. Further, it can be observed from the analyses that mean survival time is more sensitive towards the values of heart damage rate due to some organ issues/failure rate of heart due to severe accidents/recovery from other organ issues and survivability of the system is more sensitive towards heart transplantation rate.

10. Conclusion

In the human body, the cardiovascular system is the most vital system. The stochastic model and analysis presented in the paper is a simple and concise approach for understanding and investigating the human cardiovascular system considering its various causes heart damage and failure issues. This study is quite helpful to make prediction about patients' mean survival time and survivability and accordingly to take appropriate measure to treat/cure the patients. Further graphical and sensitivity analyses of the proposed model highlight the impacts of different rates of damage, failure, recovery and transplantation of heart on mean survival time and survivability of the system. The important factors/rates that can help to enhance survivability of the system can be easily selected. The investigation through the stochastic analysis of the system considering various causes of heart damage and failure concludes that the mean survival time of the system deceases with the increase in the rates of prevalence of heart diseases, other organ issues and severe accidents. However, the mean survival time and survivability of the system increases with the increase in the recovery/transplantation rates through medicine/exercise/therapy/surgery of the heart. The evaluated expressions for mean sojourn time in the different states of the system gives estimates of the times for cardiac patient remains in a particular state. Investigations also conclude that heart failure rate of due to severe accidents and heart transplantation rate play crucial roles as far as mean survival time and survivability of the system is concerned. Thus, survivability of the patient may be enhanced controlling these rates taking appropriate measures.

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SAMPLING INSPECTION SCHEMES WITH SWITCHING RULES FOR LIFE TESTS BASED ON EXPONENTIAL DISTRIBUTION

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Abstract

A life test is a random experiment which is performed on manufactured items such as electric and electronic components in order to estimate their lifetime by selecting the items randomly from the production process. The lifetime / lifespan of the product is a random variable that follows a specific continuous-type probability distribution, called the lifetime distribution. Reliability sampling, which is one among the classifications of product control techniques, deals with inspection procedures for sentencing one or more lots or batches of items submitted for inspection. An acceptance sampling scheme is a combination of sampling inspection plans with switching rules for changing from one plan to another. A switching rule is an instruction within a sampling scheme for changing from one sampling plan to another of greater or lesser severity of sampling based on the demonstrated quality history. In this paper, the concept of sampling schemes for life tests with a switching rule involving two samples under the assumption that the lifetime random variable follows an exponential distribution is introduced. A procedure is developed for designing the optimum sampling schemes with minimum sample sizes when two points on the desired operating characteristic curve are prescribed providing protection to the producer and the consumer.

Keywords: Consumer's risk, Exponential distribution, operating characteristic function, producer's risk, Reliability sampling, Sampling system.

1. Introduction

An acceptance sampling scheme is a combination of acceptance sampling plans with switching rules for changing from one plan to another. A switching rule is an instruction within a sampling scheme for changing from one acceptance sampling plan to another of greater or lesser severity of sampling based on the demonstrated quality history. The sampling plans having greater severity are called tightened plans and those having lesser severity are termed as normal plans. Severity of sampling plans can be defined either in terms of sample sizes or in terms of acceptance numbers. While the acceptance number is fixed, the sample size under a tightened sampling plan would, generally, be larger than the sample size under a normal plan. Similarly, while the sample size is fixed, acceptance number under a tightened plan is smaller than the acceptance number under a normal plan. The procedure for switching between tightened and normal plans is essential to exert pressure on the producer to take corrective action when quality falls below the prescribed levels and to provide incentives, in terms of reduced sample size, for quality improvement. Quick

switching systems (QSS) proposed by [6], tightened - normal - tightened (TNT) schemes of [4] and [8] are the examples of sampling schemes

QSS utilizes two single sampling plans with the same sample size and different acceptance numbers, together with a switching criterion of switching between normal inspection and tightened inspection. TNT sampling scheme involves a criterion for switching between two zero acceptance number single sampling plans, having different sample sizes, n_1 and n_2, where n1 is the sample size of a tightened plan and n_2 ($< n_1$) is the sample size of a normal plan. The salient feature of TNT scheme is that while the *OC* curve of the scheme is in a desirable shape at the upper portion providing greater protection to the producer against the producer's quality, the switching rules have no real effect on the consumer quality, which would remain essentially that of the tightened plan. This is due to the change in the sample sizes rather than the acceptance number. Following this feature, QSS can also be defined with a common acceptance number, but with different sample sizes under tightened and normal inspections.

In the context of life testing sampling plans or schemes, the acceptance number is defined as the allowable number of failures in the sample. In the following subsection, the concept of sampling inspection schemes for life tests involving a switching rule with two different sample sizes and zero failures is devised. The procedure for designing such sampling schemes for the prescribed acceptable mean (or median or reliability) and unacceptable mean (or median or reliability) life ensuring protection to the producer and consumer with reduced risks is also discussed.

2. Sampling Inspection Schemes for Life Tests

A sampling inspection scheme for life tests in reliability sampling involves two single sampling plans defined with common zero acceptance number, called zero failures, but with different sample sizes n_1 and n_2 and a switching rule. The single sampling plans, denoted by (n_1,0) and (n_2,0), are termed as normal and tightened inspection plans, where $n_2 > n_1$. With a provision to switching between the normal and tightened plans, the operating procedure of a specific sampling inspection scheme is described as follows:

Step 1: Start with normal inspection, drawing a random sample of n_1 items from a current lot. If no failures are observed, accept the lot. If one or more failures occur, reject the lot and switch to tightened inspection (given in Step 2) from the subsequent lot.

Step 2: Under tightened inspection, draw a random sample of n_2 items from a lot. If no failures are observed, accept the lot and switch to normal inspection (given in Step 1) from the subsequent lot.

Thus, the sampling inspection scheme for life tests is designated as *SIS*-(n_1,n_2:0), where n_1 is the sample size under a normal plan and n_2 is the sample size under a tightened plan.

It can be noted that as the provision of switching to tightened inspection when a lot is rejected under normal inspection and switching to normal inspection when a lot is accepted under tightened inspection is given within the system or scheme, the scheme is also termed as quick switching system, devised by [6]. Construction to the study of Quick Switching System (QSS) and its applications are presented by [1]. The exponential distribution, which is a special case of gamma family of distributions as demonstrated by [5], has a wider application in the fields of queueing theory, reliability theory and engineering, and hydrology. It is used to model the performance of components that have a constant failure rate and is applied to the cases involving items that do not degrade with time or do not result in wear out failures. Examples include components of high-quality integrated circuits, such as diodes, transistors, resistors, and capacitors. The exponential distribution is considered as a perfect model for the long and constant period of low failure risk that characterizes the useful life of the product and represents the intrinsic failure phase in the field of reliability.

The application of exponential distribution in the fields of actuarial, biological and engineering sciences. One may refer to [2], [3], [7], and [9] for more details. While the exponential distribution

is appropriate for modeling the lifetime of an item, it is commonly applied for the inferential aspect of utilizing life information. Hence, as a member of the lifetime continuous probability distributions, the exponential distribution can be considered as an apt probability model to adopt in real life situations. Application of the exponential distribution in reliability sampling is now considered for sampling plans with switching rules when the lots are formed from the items resulted from a continuous stream of production. In the following sampling plans with switching rules, the description of exponential distribution as the probability model for the lifetime quality characteristic, the operating characteristics of the plans and the procedures for the selection of sampling plans for life tests under the assumption of the exponential distribution are presented.

3. Exponential Distribution

Let T be a random variable representing the lifetime of the components. Assume that T follows an exponential distribution with a scale parameter θ. The probability density function and the cumulative distribution function of T are, respectively, defined as follows:

$$f(t;\theta) = \frac{1}{\theta} exp\left(-\frac{t}{\theta}\right), 0 \leq t < \infty; \theta > 0 \tag{1}$$

$$F(t;\theta) = 1 - exp\left(-\frac{t}{\theta}\right) \tag{2}$$

The mean life time, the median life time, the reliability function and hazard function for specified time t under the exponential distribution are, respectively, given below:

$$\mu = E(t) = \theta \tag{3}$$

$$\mu_d = \theta In(2) \tag{4}$$

$$R(t;\theta) = exp\left(-\frac{t}{\theta}\right) \tag{5}$$

$$z(t;\theta) = \frac{1}{\theta}, 0 \leq t < \infty \tag{6}$$

It is known that the reliable life is the life beyond which some specified proportion of items in the lot will survive. Associated with the exponential distribution, it is defined by

$$\rho = -\theta In(R) \tag{7}$$

where R is the proportion of items surviving to time ρ. The proportion, p of product failing before time t, is defined by the cumulative probability distribution of T.
It is expressed as

$$p = F(t;\theta) = 1 - exp\left(-\frac{t}{\theta}\right) \tag{8}$$

4. Operating Characteristic Function of $SIS - (n_1, n_2 : 0)$

The operating characteristic function of $SIS - (n_1, n_2 : 0)$ is defined from [6] as

$$P_a(p) = \frac{P_T}{1 - P_N - P_T} \tag{9}$$

Where P_N=$P(d = 0|n_1)$ is the probability of accepting the lot when using a normal inspection plan, $P_T = P(d = 0|n_2)$ is the probability of accepting the lot when using a tightened inspection plan, and d is the number of failures observed in the sample.

Under the conditions for the application of binomial and Poisson distributions, the expressions for $P_a(p)$ are, respectively, given by

$$P_a(p) = \frac{(1-p)^{n_2}}{1-(1-p)^{n_1}+(1-p)^{n_2}} \tag{10}$$

and

$$P_a(p) = \frac{e^{-n_2 p}}{1-e^{-n_1 p}+e^{-n_2 p}} \tag{11}$$

Under a sampling plan for life tests, the failure probability, p, is the proportion of products failing before time t. When the lifetime random variable, T, follows an exponential distribution, p is defined from equation 8. It can be noted that a specific value of p is associated with a unique value of t/θ. As the mean life is defined by $\mu = \theta$, the value of p can be related to t/μ. Similarly, associated with any specific value of p are the values of t/μ_d or t/ρ, where μ_d is the median life is and ρ is the reliability life. Similarly, for a specified value of t/μ or t/μ_d or t/ρ, the value of p could be obtained. Thus, when p is associated with t/μ, the operating characteristic function of a life test sampling plan can be considered as a function of t/μ_d , rather than p, and the *OC* curve of the plan could be obtained by plotting the acceptance probabilities against the values of t/μ_d. If the median life and reliability life are considered for the operating characteristics of the desired plan, p can be associated with t/μ_d=$(\frac{t}{\theta})(ln(2))^{-1}$ and t/p=-$(\frac{t}{\theta})[ln(R)]^{-1}$.

Associated with each specific value of p, a unique value of t/μ or t/μ_d or t/ρ can be determined using the following simple procedure:

Step 1: Specify p.

Step 2: Obtain t/θ from the cumulative distribution function using (2) and (8).

Step 3: Using the value of t/θ obtained in Step 2, determine t/μ , t/μ_d and t/ρ from (3), (4) and (7), respectively.

Step 4: Define the following dimensionless ratios:

$$\frac{\mu}{\mu_0}=\frac{Actual\ Mean\ Life}{Assumed\ Mean\ Life}; \quad \frac{\mu_d}{\mu_{d_0}}=\frac{Actual\ Mean\ Life}{Assumed\ Mean\ Life}; \quad \frac{\rho}{\rho_0}=\frac{Actual\ Reliable\ Life}{Assumed\ Reliable\ Life}$$

Given the assumed mean life μ_0, the median life μ_{d_0} and the reliable life ρ_0 the ratios can be obtained using (3), (4) and (7) along with (2).

Step 5: Determine $\frac{\mu}{\mu_0}$, $\frac{\mu_d}{\mu_{d_0}}$ and $\frac{\rho}{\rho_0}$ corresponding to each specified value of p.

Step 6: Find the probability of acceptance using (2) under the conditions of binomial model or using (3) under the conditions of Poisson model corresponding to each specified value of p or $\frac{\mu}{\mu_0}$ or $\frac{\mu_d}{\mu_{d_0}}$ or $\frac{\rho}{\rho_0}$.

Step 7: Plot the probability of acceptance against each value of $\frac{\mu}{\mu_0}$ or $\frac{\mu_d}{\mu_{d_0}}$ or $\frac{\rho}{\rho_0}$ in order to obtain the required *OC* curve of $SIS-(n_1, n_2 : 0)$ for life tests under the assumption of exponential distribution for the lifetime quality characteristic. In a similar way, for a specified value of $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$, the value of p could be obtained following the above procedure in the reverse order. As p is associated with $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$, the operating characteristic function of $SIS-(n_1, n_2 : 0)$ for life tests can be considered as a function of $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$ rather than p. Thus, the *OC* curve of the sampling scheme could be obtained by plotting the acceptance probabilities against the values of $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$.

5. Procedure for the Selection of $SIS-(n_1, n_2 : 0)$ for Life Tests Indexed by Acceptable and Unacceptable Mean (or Median or Reliable) Life

It is the usual practice in selecting a sampling scheme or a system to fix the operating characteristic curve in accordance with the desired degree of discrimination. The *OC* curve is, in turn, fixed by suitably chosen parameters, *viz.*,$(p_0, 1-\alpha)$ and (p_1, β) or equivalently by $(\mu_0, 1-\alpha)$ and (μ_1, β) or $(\frac{t}{\mu_0}, 1-\alpha)$ and $(\frac{t}{\mu_1}, \beta)$, where p_0 is the producer's quality level, p_1 is the consumer's quality level, μ_0 is the acceptable mean life, μ_1 is the unacceptable mean life, α is the producer's risk and β is the consumer's risk. Similarly, $(\mu_{d_0}, 1-\alpha)$ and (μ_{d_1}, β), $(\rho_0, 1-\alpha)$ and (ρ_1, β) can be used to find the sampling schemes based on median life and reliable life criteria, respectively.

Now, an optimum $SIS-(n_1, n_2 : 0)$ based on mean life can be obtained satisfying the following two conditions which would also ensure protection to the producer and the consumer:

$$P_a(p_0) \geq 1-\alpha \tag{12}$$

and

$$P_a(p_1) \leq \beta \tag{13}$$

Equivalently, one can also fix the conditions as given below:

$$P_a(\mu_0) \geq 1-\alpha \tag{14}$$

and

$$P_a(\mu_1) \leq \beta \tag{15}$$

Or

$$P_a\left(\frac{t}{\mu_0}\right) \geq 1-\alpha \tag{16}$$

and

$$P_a\left(\frac{t}{\mu_1}\right) \leq \beta \tag{17}$$

Assuming that the *OC* curves of $SIS-(n_1, n_2 : 0)$ and the *OC* curves of tightened single sampling plans $(n_2, 0)$ coincide at $\frac{t}{\mu}$=$\frac{t}{\mu_1}$ i.e., at $p = p_1$, the sample sizes, n_1 and n_2 of $SIS-(n_1, n_2 : 0)$ can be determined easily as given below:

For tightened single sampling plan, $(n_2, 0)$, the expression for the OC function with the quality level, p_1, and the probability of acceptance, β, under the conditions for application of binomial model, is given by

$$\beta = (1-p_1)^{n_2} \tag{18}$$

From which the solution for n2 can be shown to be

$$n_2 = \frac{ln(\beta)}{ln(1-p_1)} \tag{19}$$

by taking natural logarithm and on simplification. If $\frac{ln(\beta)}{ln(1-p_1)}$ is not an integer, the solution for n_2 will be obtained as

$$n_2 = int\left[\frac{ln(\beta)}{ln(1-p_1)}\right] + 1 \tag{20}$$

If the OC curve of $SIS-(n_1, n_2 : 0)$ is required to pass through $(p_1, 1-\aleph)$, the sample size, n_1 can be determined as follows: From (12), consider

$$1\text{-}\alpha = \frac{(1-p_0)^{n_2}}{1-(1-p_0)^{n_1}+(1-p_0)^{n_2}}$$

which would result in

$$n_1 = \frac{In(1-\alpha-\alpha(1-p_0)^{n_2}-In(1-\alpha)}{In(1-p_0)}$$

The solution for in terms of an integer can be determined as

$$n_1 = int\left[\frac{In(1-\alpha-\alpha(1-p_0)^{n_2}-In(1-\alpha)}{In(1-p_0)}\right]+1 \tag{21}$$

In order to determine an optimum $SIS-(n_1,n_2:0)$ based on mean life satisfying the specified requirements under the assumption of the exponential distribution and to implement the plan in practical situations, the following procedure is developed:

Step 1: Specify the values of $\frac{t}{\mu_0}$ and $\frac{t}{\mu_1}$ with α = 0.05 and β = 0.10 respectively.

Step 2: Find p_0 and p_1 corresponding to $\frac{t}{\mu_0}$ and $\frac{t}{\mu_1}$ using the relationship existing between p and $\frac{t}{\mu}$.

Step 3: Obtain the optimum values of and for the specified strength $(\mu_0, 1-\alpha)$ and (μ_1,β) from (20) and (19) with the values of p_0 and p_1.

Step 4: Begin the normal inspection drawing a random sample of n_1 items from a current lot.

Step 5: Perform the life test on each of the selected sample items considering t as the test termination time and μ as the expected mean life and observe the number, d, of failures.

Step 6: If no failures are observed in the sample or if no failures occurred until the termination time is reached, accept the current lot and continue with normal inspection. If one or more failures are observed before reaching the test termination time, reject the lot and switch to tightened inspection.

Step 7: Under tightened inspection, draw a random sample of n_2 items from the subsequent lot and perform the life test on each of the selected items. If no failures are found in the entire sample of n_2 items or if no failures occurred until the test termination time is reached, accept the lot and return to normal inspection from the subsequent lot. If at least one failure is observed before reaching the test termination, reject the lot and continue with tightened inspection.

The above procedure can also be followed for determining optimum $SIS-(n_1,n_2:0)$ when the median and reliability life criteria are desired. Tables 1, 2 and 3 are constructed utilizing the first three steps of the procedure described above specifying three sets of wide range of values, *viz.*, (i) $\frac{t}{\mu_0}$ and R_μ, (ii)$\frac{t}{\mu_{d_0}}$ and R_{μ_0}, and (iii) $\frac{t}{p_0}$ and R_p where R_μ=$\frac{\mu_0}{\mu_1}$, R_{μ_d}=$\frac{\mu_{d_0}}{\mu_{d_1}}$, and R_ρ=$\frac{\rho_0}{\rho_1}$ are the operating ratios (dimensionless), which are the measures of discrimination. The tables, respectively, provide the parameters $n_1 and n_2$ of $SIS-(n_1,n_2:0)$ for life tests indexed by mean life, median life and reliable life. Each of the plans listed in the tables would ensure the conditions that the maximum producer's and consumer's risks are restricted to 5 percent (*i.e.*, α = 0.05) and 10 percent (*i.e.*, β = 0.10), respectively.

5.1. Numerical Illustration 1

Let us consider the case that the lifetime of an automobile voltage regulator is a random variable following an exponential distribution with parameter θ. An industrial practitioner is interested to adopt a sampling inspection scheme which should have a provision of switching between two single sampling plans of different sample sizes, and insists that no failures should be allowed until the termination time t is reached. Based on the past history, the acceptable mean life and unacceptable mean life of the regulators were estimated as μ_0 = 30500 minutes and μ_1 = 2200 minutes, respectively. In order to make the decisions on sentencing the lots coming from the industrial process which produce automobile voltage regulators, the practitioner wishes to implement $SIS-(n_1,n_2:0)$. It is assumed that the risk of rejecting the lots having acceptable mean life and the lots having unacceptable mean life should not exceed 5 percent and 10 percent,

respectively. The test termination time is fixed as t_0 = 45 minutes. For the specified requirements, the measure of discrimination is found to be $R_\mu = \frac{\mu_0}{\mu_1} = \frac{30500}{2200} = 13.86 \approx 14$ and $\frac{t}{\mu_0} = \frac{45}{30500} = 0.00148 \approx 0.0015$.

Therefore, $R_\mu = 14$ and $\frac{t}{\mu_0} = 0.0015$ based on the given information, one finds the optimum sample sizes for normal and tightened inspection under $SIS - (n_1, n_2 : 0)$ as $n_1 = 28$ and $n_2 = 144$, respectively. Thus, the optimum sampling inspection scheme is implemented as given below:

1. Select a random sample of 28 items from the current lot under normal inspection.
2. Conduct the life tests on each of the 28 sampled items and observe the number of failures until the termination time $t = 45$ minutes is reached. If no failures are observed, accept the lot and continue with normal inspection. If at least one failure occurs at the test termination time or before reaching the test termination time, reject the current lot and switch to tightened inspection from the subsequent lot.
3. Draw a random sample of 144 items from the lot and conduct the life test on each of the 144 sampled items. If no failures are observed while testing all the sampled items or until reaching $t = 45$ minutes, accept the lot and switch to normal inspection from the subsequent lot; if at least one failure occurs before reaching $t = 45$ minutes, reject the lot and continue with the tightened inspection from the subsequent lot.

Figures 1 displays the operating characteristic curve of the optimum sampling inspection scheme obtained in the above illustration. It is observed that the *OC* curve passes through the respective prescribed points with the reduced producer's risk of 4.91 percent (less than 5 percent) and the reduced consumer's risk of 9.86 percent (less than 10 percent) and ensure that the prescribed conditions (4) and (5) are satisfied.

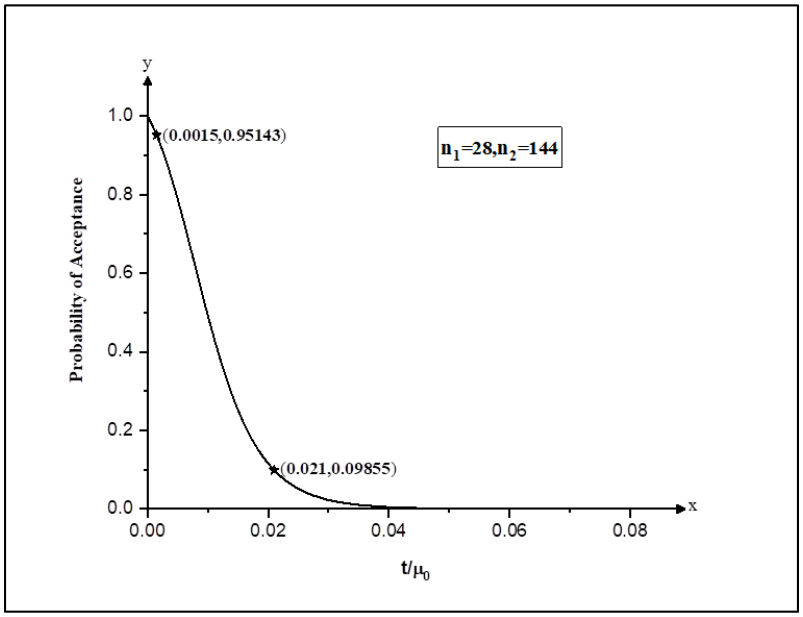

Figure 1: *OC Curve of* $SIS - (n_1, n_2 : 0)$ *for Life Tests Based on Exponential Distribution Indexed by Mean Life Ratio when* $n_1 = 28$ *and* $n_2 = 144$.

5.2. Numerical Illustration 2

A production engineer assessed from his experience that the time to failure of semiconductors can be modeled by an exponential distribution with parameter θ. The assembled semiconductors are resulted from a continuous stream of production. The acceptable median life and unacceptable

median life of the semiconductors are respectively, specified as 48000 hours and 2600 hours, respectively. It is desired that the maximum risk of rejecting the lots when the acceptable median life is given as 48000 hours and the maximum risk of accepting lots when the unacceptable median life is specified as 2600 hours are fixed at 5 percent and 10 percent, respectively. It is assumed that the total time duration of life test is fixed at t_0 = 120 hours. For the given requirements, R_{μ_d} = $\frac{\mu_{d_0}}{\mu_{d_1}}$ = 48000/2600 = 18.46 is the measure of discrimination and $\frac{t}{\mu_{d_0}}$ is 0.0025. Based on these values of R_{μ_d} and $\frac{t}{\mu_0}$, the optimum sampling scheme is determined by the sample size $n_1 = 26$ for normal inspection and the sample size $n_2 = 87$ for tightened inspection.

It can be noted from the above illustrations that the sample size required for tightened inspection is larger than the sample size required for normal inspection. Similar to Numerical Illustrations 1 and 2, suitable illustrations can be given.

6. Conclusion

Sampling plans with switching rules are considered and procedures for designing the optimum plans to provide protection to the producer and the consumer are discussed based on exponential distribution as the lifetime probability distribution. A procedure for selection of sampling Inspection Schemes for life tests ensuring protection to the producer and consumer is described. The tables yielding the parameters of the optimum $SIS - (n_1, n_2 : 0)$ indexed by acceptable mean life (or median or reliable) and unacceptable mean life (or median or reliable), respectively, are constructed for a specified set of values of the parameter of the Exponential distribution.

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USE OF MEDIAN BASED ESTIMATOR TO MITIGATE OUTLIER'S EFFECT THROUGH S^2 CHART

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Abstract

In this paper, we consider an upper-sided Phase II variance chart with probability limits in case of unknown parameter because the quality practitioner interested in monitoring increased variance of the process parameter. It is well established that when the Phase I data are contaminated with spurious observations, performance of the chart is suspected to deviate from what is normally expected. Therefore, we propose an improved performance of one-sided variance chart under the exceedance probability criterion for a fixed in-control average run length using the absolute deviation from median estimator. Under the exceedance probability criteria, the chart is designed so that the user can get more confidence in their in-control average run length values. The proposed chart is compared with the existing chart in case of contaminated and non-contaminated observations. Result shows that performance of variance chart shows robust performance when using absolute deviation from median estimator. Finally, an example has been provided in the favour of our proposed study.

Keywords: Average run length, median-based estimator, control chart, in-control and out-of-control performances, process variability.

1. Introduction

S^2 chart is considered to be more useful control chart when the interest of the quality practitioner lies in monitoring variability in the process parameter. As Woodall and Montgomery [24] stated that to maintain a process at a satisfactory level, process variability should be in-control (IC) because a slight change in the process variance could significantly impact the performance of the mean control chart. Therefore, prior to the construction and effective utilization of the mean control chart, it is suggested that a good estimate of the IC process variance must be available so that an effective process monitoring can take place. In this view, S^2 chart is a popular choice to monitor the process variability (Montgomery [17]). However, when the underlying process variance is not known, designing these charts becomes more complex. In this case, the variance is estimated from a Phase I reference sample and perform the Phase I analysis. (Chakraborti, Graham and Human, [2], Jones-Farmer et al. [11]). The estimate is then used to find the control limits which further used in Phase II analysis. For the S^2 chart, several efforts have been made to increase the efficacy of the charts such

as: use of memory type control charts (CUSUM and EWMA) (Chang and Gan [3, 4]), use of runs rules (Rakitzis and Antzoulokas [18]), use of some other sampling plans such as repetitive sampling plan (Jaiswal and Kumar [8]), double sampling plan (Khoo [12]), etc.

As we know, in case U, the estimation error can lead to a distorted chart performance. This effect can be reduced by considering a larger number of Phase I samples and, at times, by adjusting the control limits (Saleh et al. [20, 21]). However, when Phase I samples, specially of smaller sizes, contain outliers, it is anticipated that this may have a more severe impact on the chart's performance. Because inclusion of the spurious observations may lead us to the model misspecification, biased parameter estimation and incorrect results. In turn, erroneous parameter estimation may affect the performance of the control chart in Phase II. Consequently, when a control chart indicates an OOC signal, pinpointing the underlying factors responsible for triggering this signal can prove to be a challenging task. Such signals can stem from assignable causes or merely be the result of spurious observations. The primary aim of this article is to recommend an estimator capable of mitigating the influence of outliers on the chart's performance, thereby enabling us to attribute OOC signals to genuine changes in the process.

Recently, Kumar and Jaiswal [15] studied the exponential chart and recommended the median based estimator for estimating the rate parameter so that the chart's performance is robust to the presence of outliers. Schoonhoven, Riaz and Does [23] have discussed different estimators of population variance for the variance chart and recommended the average deviation from median (ADM) estimator which is the function of sample median. They showed that the use of ADM estimator instead of commonly used Pooled estimator helps in minimizing the outliers' impact on the chart's performance. But they adopted the unconditional perspective to assess the chart's properties which mainly considers the mean and standard deviation of the unconditional RL distribution. Please note here that the unconditional run length distribution can be obtained by averaging the conditional run length distribution over the distribution of the estimator (see Chakraborti [1], Kumar and Chakraborti [14]). This method of obtaining results is known as unconditional perspective. This perspective has been criticized by several researchers, for example, Jardim [9, 10], Sarmiento et al. [22], Kumar [13] pointing out that the unconditional perspective does not consider the shape of the RL distribution and ignores the practitioner-to-practitioner variability. In this article, we consider the most recent approach i.e., conditional perspective which is based on the conditional RL distribution (see, Jardim [9, 10], Kumar [13], Gandy and Kvaloy [7], Epprecht et al. [5]). Conditional RL perspective mainly concerned with the exceedance probability criteria (EPC). For a detailed discussion on both perspective, readers are advised to refer some recent papers, for instance, Jardim [10], Sarmiento et al. [22], Kumar [13], Kumar and Jaiswal [16]. The unconditional perspective may lead to a misconception for the user due to the skewed distribution of the IC CARL (CARL(1)). For instance, the unconditional ARL might appear higher than the nominal ARL, suggesting a reduced rate of false alarms compared to the expected one. Nevertheless, examining the percentiles of the CARL(1) may reveal a contrasting narrative.

Hence, the article primarily focuses on the performance in a realistic context based on the percentiles of the CARL distribution and the EPC metric when the parameter is estimated using the ADM estimator. This assessment aims to determine if the chart's performance is suspectable to outliers.

Rest of the article is organized as follows. In section 2, the estimated control limits of the upper-sided S^2-chart has been discussed. In section 3, the IC and OOC performance of plug-in Pooled and ADM chart has been discussed. In section 4, The control limits are adjusted under EPC for the ADM and Pooled chart. In section 5, the IC and OOC performance has been discussed under EPC. In favor of the proposed design, an example has been offered in section 6. Finally, conclusions are offered in section 7.

II. Upper-sided S^2-chart with estimated IC variance

Let $X_1, X_2, \ldots, X_n$ be n random samples of size n following a normal distribution with IC process mean μ and process variance $\sigma_0^2 > 0$ i.e., $X \sim N(\mu, \sigma_0^2)$. Traditionally used charting statistic for the S^2-chart is the sample variance, given by $S^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2, \bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$. Let UCL denotes the upper control limit of the S^2-chart which can be obtained by using probability approach, such that $P[S^2 > UCL|IC] = \alpha$, where α is a nominal FAR. It is well known that the statistic $\frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$. Therefore, UCL is given by.

$$\text{UCL} = \frac{\sigma_0^2}{n-1}\chi^2_{1-\alpha,n-1} \tag{1}$$

where $\chi^2_{1-\alpha,n-1}$ be the $(1-\alpha)$th quantile of the χ^2-distribution with $(n-1)$ degrees of freedom.

Let σ_1^2 denotes the magnitude of the process variance shift from IC process variance σ_0^2 to the shifted variance $\sigma_1^2 = \delta\sigma_0^2$. A control chart gives an OOC signal when the charting statistic, S^2, falls above the UCL. This event is called signaling event E. And the probability of this signaling event, commonly known as the probability of signal for given shift, δ, denoted by $\beta(\delta)$ is given by.

$$\beta(\delta) = P[\, S^2 > \text{UCL}|\sigma_1^2 = \delta\sigma_0^2] = 1 - F_{\chi^2_{n-1}}\left(\frac{\chi^2_{1-\alpha,n-1}}{\delta}\right) \tag{2}$$

where $F_{\chi^2_{n-1}}(\cdot)$ denotes the CDF of χ^2-distribution with $n-1$ degrees of freedom. Its corresponding ARL is the reciprocal of probability of signal i.e., $\beta(\delta)$, is given by.

$$\text{ARL}(\delta) = \frac{1}{\beta(\delta)} = \frac{1}{1 - F_{\chi^2_{n-1}}\left(\frac{\chi^2_{1-\alpha,n-1}}{\delta}\right)} \tag{3}$$

Clearly, $\delta = 1$ represents the process is IC, otherwise, the process is OOC. On the other hand, an OOC ARL should be as low as possible so that the chart could detect shift in the process through a valid alarming signal as early as possible.

In case U, the process parameters are often unknown, and they need to be estimate using the Phase I samples assuming that the samples are collected from the IC process and they are ready to estimate the unknown process parameter. Let Y_{ij} be the i^{th} Phase I sample of size n. For the S^2 chart, the most prominent unbiased estimator suggested in the literature is sample Pooled estimator is given by.

$$\hat{\sigma}_0^2 = \hat{\sigma}^2_{\text{Pooled}} = \frac{1}{m}\sum_{i=1}^{m} S_i^2 \tag{4}$$

where $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(Y_{ij} - \bar{Y}_i\right)^2$ and $\bar{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}$ is the sample variance of i$^{\text{th}}$ group of samples. It is a function of the sample mean. On the other hand, the ADM estimator, the function of the sample median is given by.

$$\hat{\sigma}_0^2 = \hat{\sigma}^2_{\text{ADM}} = \overline{\text{ADM}} = \frac{1}{m}\sum_{i=1}^{m} \text{ADM}_i \tag{5}$$

where ADM_i is the average absolute deviation from the median of sample i, which is given by

$$\text{ADM}_i = \frac{1}{n}\sum_{j=1}^{n}\left|Y_{ij} - M_i\right|, \tag{6}$$

where M_i denotes the median of the i^{th} Phase I sample. An unbiased ADM estimator for estimating the sample variance is $\frac{\overline{\text{ADM}}}{t_2(n)}$. Here, $t_2(n)$ is a constant, function of sample size n and defined as $t_2(n) = \frac{2(n-1)}{n\sqrt{2\pi n(n-1)}} + 2\int_{-\infty}^{+\infty} x\Phi\left(\sqrt{n-1}x\right)\phi(x)dx$, where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of the standard normal distribution (see, Wu et al., 2002). Since the expression of $t_2(n)$ cannot be obtained in the closed form. Therefore, Riaz and Saghir [19] obtained the simulated results of the expression $t_2(n)$ for different values of n and mentioned in Table A1 of the appendix of the paper Riaz and Saghir [19].

Let $\widehat{\text{UCL}}$ denotes the estimated upper control limit which can be obtained by replacing σ_0^2 given in Equation (1) by its estimate $\hat{\sigma}_0^2$ where $\hat{\sigma}_0^2 = \hat{\sigma}^2_{\text{Pooled}}$ or $\hat{\sigma}^2_{\text{ADM}}$ given in Equation (4) and (5), respectively. The $\widehat{\text{UCL}}$ for the upper-sided S^2 chart is given by.

$$\widehat{\text{UCL}} = \frac{\hat{\sigma}_0^2}{n-1}\chi^2_{1-\alpha,n-1} \tag{7}$$

Let E_i be the ith event falling outside the $\widehat{\text{UCL}}$. Therefore, its corresponding conditional probability of signal (CPS), denoted by $\hat{\beta}$ for given shift (δ), is given by.

$$\hat{\beta}(\hat{\sigma}_0^2, \delta) = P\left[S^2 > \widehat{UCL} | \hat{\sigma}_0^2\right] = 1 - F_{\chi^2_{n-1}}\left(\frac{\hat{\sigma}_0^2 \chi^2_{1-\alpha,n-1}}{\sigma_0^2 \delta}\right) \quad (8)$$

Therefore, the conditional average run length (CARL) of the upper-sided S^2 chart can be obtained by using Equation (8), is given by.

$$\text{CARL}(\hat{\sigma}_0^2, \delta) = \frac{1}{\hat{\beta}(w,\delta)} = \left(1 - F_{\chi^2_{n-1}}\left(\frac{\hat{\sigma}_0^2 \chi^2_{1-\alpha,n-1}}{\sigma_0^2 \delta}\right)\right)^{-1} \quad (9)$$

The unconditional average run length for given shift δ, denoted by $\mu_{\text{CARL}}(\delta)$ is given by

$$\mu_{\text{CARL}}(\delta) = \int_0^\infty \left(1 - F_{\chi^2_{n-1}}\left(\frac{\hat{\sigma}_0^2 \chi^2_{1-\alpha,n-1}}{\sigma_0^2 \delta}\right)\right)^{-1} f_{\hat{\sigma}_0^2}\, d\hat{\sigma}_0^2 \quad (10)$$

The standard deviation of CARL for given shift (δ) is given by.

$$\sigma_{\text{CARL}}(\delta) = \sqrt{E(\text{CARL}^2(\hat{\sigma}_0^2, \delta)) - [E(\text{CARL}(\hat{\sigma}_0^2, \delta))]^2} \quad (11)$$

where $E(\text{CARL}^2(\hat{\sigma}_0^2, \delta)) = \int_0^\infty \left(1 - F_{\chi^2_{n-1}}\left(\frac{\hat{\sigma}_0^2 \chi^2_{1-\alpha,n-1}}{\sigma_0^2 \delta}\right)\right)^{-2} f_{\hat{\sigma}_0^2}\, d\hat{\sigma}_0^2$. Please note here $\delta = 1$ represents that the process is IC otherwise, the process is OOC. It is well known that lower values of $\sigma_{\text{CARL}}(1)$ are desirable for a good chart that reflects more confidence of the user in his/her CARL(1) value and hence in adopting the chart. The 100pth percentile of the $\text{CARL}(\hat{\sigma}_0^2, \delta)$ distribution denoted by $\text{CARL}(1)_p$, is given by.

$$\text{CARL}(1)_p = inf\{z: F_{\text{CARL}}(z) \geq p\} \quad (12)$$

where inf indicates infimum and $F_{\text{CARL}}(z)$ is the distribution function of the $\text{CARL}(\hat{\sigma}_0^2, \delta)$.
Beside the metrics discussed above, EP is the exceedance probability, denoted by $\pi(1)$ is defined as the chance that a chart will achieve his CARL(1), value at least nominal ARL_0, is given by.

$$\pi(1) = P[\text{CARL}(1) \geq \text{ARL}_0] \quad (13)$$

III. EPC performance of the Pooled and ADM chart with and without outliers

In this section, we examine the effect of upper outliers on the performance of S^2 chart. For this purpose, we have applied the simulation procedure using approximately 1,00,000 replications. With the underlying objective discussed above, the present study undertakes an examination of two different scenarios i.e., 5% and 10% spurious observations in each Phase I sample. In both inspections, the Phase I sample configuration encompasses spurious observations, with specific proportion of 5% and 10% relative to the total Phase I samples. Because the number of outliers, say γ, is an integer, we look at only the integer part of 5% or 10% of the Phase I sample of size m. Following simulation steps are carried out to obtain the performance metrices.

- Generate observations $Y_{i,j}; i = 1,2, \dots, m\,; j = 1,2, \dots, n$ from the normal distribution with mean μ and variance σ^2.
- Obtain S_i^2 or ADM_i for $i = 1,2, \dots, m$.
- Sort them in either ascending or descending order.
- To produce the upper extremes ($S^2{}_{(n)}$ or $\text{ADM}_{(n)}$) in the Phase I sample, multiply a constant, c, i.e., $c > 1$ to the first largest γ observation whereas $c = 1$ represents the Phase I sample with no contamination.
- Calculate the control limits with the estimators $\hat{\sigma}^2_{\text{Pooled}}$ or $\hat{\sigma}^2_{\text{ADM}}$ Given in Equation 4 or 5.
- Calculate CARL function using Equation 9 associated with its control limits obtained in previous step.
- Repeat the process at least 100,000 times to get the μ_{CARL}, σ_{CARL}, $\pi(1)$ and $\text{CARL}(1)_p$.

Following Table 1 and 2 represents the IC plug-in performance of the Pooled chart (Pooled estimator based S^2 chart) and ADM chart (ADM estimator based S^2 chart), respectively at $\alpha = 0.0027$,

$n = 5$ with the effect of 5% outliers in the Phase I samples, respectively. For the convenient of the computation of the ADM chart, the values of the constant $t_2(n)$ are taken from Riaz and Saghir [19]. The numerical value of $t_2(n)$ for $n = 5$ is 0.664980 and for $n = 7$ is 0.703800. In the Tables 1 and 2, m represent the sample size, γ represents the number of outliers in the Phase I samples, c is a multiplier to produce the outliers of different sizes in the Phase I sample, $\mu_{\mathrm{CARL}}(1)$ and $\sigma_{\mathrm{CARL}}(1)$ represents the mean and standard deviation of the CARL distribution, $\pi(1)$ is the probability that the CARL(1) is at least ARL_0 and $\mathrm{CARL}(1)_p$ is the different percentiles of the CARL distribution. Moreover, $c = 1$ represents Phase I sample having no oulier.

Table 1: *IC performance of Pooled chart at $\alpha = 0.0027$ with and without outlier at $n = 5$ (with 5% outlier).*

					Percentile				
$m(\gamma)$	c	$\mu_{\mathrm{CARL}}(1)$	$\sigma_{\mathrm{CARL}}(1)$	$\pi(1)$	0.10	0.25	0.50	0.75	0.90
20(1)	1	803.91	2102.16	0.48	90	166	346	776	1687
	1.2	1020.05	2987.47	0.55	106	198	422	962	2126
	1.5	1519.47	5606.73	0.64	132	254	564	1335	3081
	2	2954.40	13393.58	0.77	194	393	921	2347	5807
50(2)	1	490.78	452.67	0.49	149	225	361	594	954
	1.2	536.69	500.14	0.53	160	241	391	650	1050
	1.5	604.64	584.36	0.59	176	267	439	732	1189
	2	748.41	778.05	0.68	208	320	530	901	1494
100(5)	1	425.25	241.23	0.49	194	260	366	519	722
	1.2	440.80	252.84	0.52	201	269	379	540	750
	1.5	465.30	272.85	0.56	210	284	400	570	793
	2	512.15	298.04	0.63	228	308	436	626	880
200(10)	1	397.43	150.72	0.50	235	289	369	473	594
	1.2	402.29	160.39	0.51	237	293	373	478	601
	1.5	414.33	158.54	0.54	244	301	384	492	621
	2	433.39	166.30	0.59	253	315	402	515	650
500(25)	1	380.24	88.77	0.50	277	317	369	431	497
	1.2	383.40	87.19	0.51	278	319	371	435	502
	1.5	387.04	92.19	0.53	281	322	375	440	506
	2	393.92	91.55	0.55	286	328	382	447	514

It is well known that estimation error exerts bad impact on the performance of the chart. Moreover, the sample Pooled estimator is a function of mean whereas ADM estimator is a function of sample median. Therefore, effect of outliers on the performance of the chart can be visualize from these tables. For instance, when $m = 20$, 5% of the Phase I sample (m) produces 1 outlier. It can be observed that when the Phase I sample is free from the outliers, its $\mu_{\mathrm{CARL}}(1)$ and $\sigma_{\mathrm{CARL}}(1)$ is 803.91 and 2102.16 whereas after including outliers say, for $c = 1.5$, its $\mu_{\mathrm{CARL}}(1)$ and $\sigma_{\mathrm{CARL}}(1)$ is 1519.47 5606.73, respectively which is approximately 98% larger than the 803.91 and much far than the nominal 370. On the other hand, using the ADM estimator, when the Phase I sample is free from the outliers i.e., $c = 1$, its $\mu_{\mathrm{CARL}}(1)$ and $\sigma_{\mathrm{CARL}}(1)$ is 438.79 and 301.61 whereas after including outliers, say, for $c = 1.5$, its $\mu_{\mathrm{CARL}}(1)$ and $\sigma_{\mathrm{CARL}}(1)$ is 613.67 and 449.80,respectively which is approximately 40% larger than the 438. These results shows that the Pooled chart deviated more from its nominal performance in case U than the ADM chart. Moreover, $\pi(1)$ values are showing less confidence in the values of CARL(1) which is only 50% even for the large sample sizes. And 10th percentile is 90 for the Pooled chart and 168 for the ADM chart when $(m, n) = (20,5)$ which shows that there is 90% chance that CARL(1) of a conditional chart will be greater than or equal to 90 and 168 respectively,

which is very low even for the larger Phase I samples. From these tables, it can be seen that the ADM chart puts on a guard against the outliers in the Phase I samples. The study shows that more than 500 Phase I samples of the size $n = 5$ are required to attain the control chart's performance close to

Table 2: *IC performance of ADM chart at* $\alpha = 0.0027$ *with and without outlier at* $n = 5$ *(with 5% outlier).*

					Percentile				
$m(\gamma)$	c	$\mu_{\mathrm{CARL}}(1)$	$\sigma_{\mathrm{CARL}}(1)$	$\pi(1)$	0.10	0.25	0.50	0.75	0.90
20(1)	1	438.79	301.61	0.48	168	239	358	544	792
	1.2	502.61	357.73	0.56	189	271	408	621	916
	1.5	613.67	449.80	0.68	226	326	493	759	1130
	2	864.80	670.03	0.83	299	440	684	1072	1620
50(2)	1	391.16	158.09	0.47	223	280	361	469	595
	1.2	411.79	170.20	0.53	234	294	380	493	627
	1.5	445.03	185.12	0.60	251	316	409	534	681
	2	507.83	210.98	0.72	284	358	466	611	782
100(5)	1	377.24	103.07	0.47	257	302	362	435	515
	1.2	386.62	109.04	0.51	263	309	371	446	529
	1.5	401.25	112.89	0.56	272	321	385	463	549
	2	426.49	122.37	0.64	289	341	409	493	583
200(10)	1	369.90	71.51	0.46	284	318	362	412	464
	1.2	374.82	72.03	0.48	287	323	367	418	470
	1.5	381.49	70.57	0.52	293	328	373	425	479
	2	393.20	79.18	0.58	301	338	385	439	494
500(25)	1	370.00	45.95	0.44	311	335	363	393	423
	1.2	371.80	43.17	0.45	312	336	365	396	426
	1.5	377.95	50.65	0.47	314	338	366	398	429
	2	379.42	50.81	0.51	318	342	371	402	434

the case K. Such a large amount of data is not easily available in real practice. Thus, it needs an adjustment in the control limits so that desired IC performance of the chart can be achieved with the available Phase I samples at hand. Therefore, to improve the performance, specially, for small sample sizes, we adjust the UCL of the chart so that higher chance of occurrence can be achieved.

IV. Adjusted control limit of the upper-sided S^2 chart under the EPC

In light of the limited availability of the extensive dataset, we have designed the control limit of the Pooled and ADM chart using the EPC approach. As discussed earlier, EPC approach ensures the high chance of occurrence, say 0.90, of the $\mathrm{CARL}(1)$ at least a nominal value such as 370.4. Formally, the condition of EPC approach can be written in terms of following equation as follows.

$$P[\mathrm{CARL}(1) \geq \mathrm{ARL}_0] = 1 - p; 0 < p < 1 \quad (14)$$

The values of the design constants are obtained at $p = 0.10$ i.e., $\mathrm{EPC} = 0.90$. The control limits of the proposed ADM chart under the EPC can be obtained by using the following simulation study.

- Fix the value of p, ARL_0, m, n and U where $U = \chi^2_{1-\alpha,n-1}$ is a design parameter.
- Generate observations $X_{i,j}; i = 1,2,\dots,m\ ; j = 1,2,\dots,n$ from the normal distribution with mean μ and variance σ^2.
- Sort the subgroup data of size n in ascending or descending order and obtain the median (M_i) of the i^{th} sample of size n and calculate the ADM estimator for estimating the sample variance using $\frac{\overline{\mathrm{ADM}}}{t_2(n)}$.

- Calculate the conditional control limit using ADM estimator and obtain the empirical distribution of the CARL(1) function using Equation (9).
- Repeat the process atleast 1,00,000 times to obtain the pth percentile i.e., $\text{CARL}(1)_p$ of the CARL distribution, say $p = 0.10$.
- If $\text{CARL}(1)_p > ARL0$, stop the loop and use the current value of $\widehat{UCL}$ otherwise increase the value of $\widehat{UCL}$ until the $CARL(1)_p > ARL0$ occur and return to previous step.

In order to obtain the control limits for the Pooled chart under the EPC, please follow Faraz et al. [6].

Table 3: *Design parameter of upper-sided Pooled and ADM chart with estimated parameter at* $n = 5,7$ $p = 0.10$, $ARL_0 = 370.4$.

	$n = 5$		$n = 7$	
m	Pooled chart	ADM chart	Pooled chart	ADM chart
20	20.2264	18.2357	23.9253	21.9893
50	18.5905	17.4718	22.3684	21.2454
75	18.1196	17.2454	21.9121	21.0200
100	17.8485	17.1122	21.6475	20.8898
200	17.3536	16.8654	21.1614	20.6434
500	16.9338	16.6506	20.7455	20.4313

V. IC and OOC performance of the Pooled and ADM chart with or without contamination under the EPC

I. IC performance with and without outliers

In this section, we are analyzing the IC performance of the upper-sided S^2 chart in the presence of some contaminated or spurious observations using the Pooled and ADM estimator under the EPC.

Table 4: *IC performance of Pooled chart with estimated parameter at* $n = 5, p = 0.10, ARL_0 = 370.4$ *with 5% outlier*

					Percentile				
$m(\gamma)$	c	$\mu_{\text{CARL}}(1)$	$\sigma_{\text{CARL}}(1)$	$\pi(1)$	0.10	0.25	0.50	0.75	0.90
20(1)	1	8205.59	41339.33	0.90	368	803	2047	5659	15046
	1.2	11064.39	115454.45	0.93	453	1004	2621	7479	20589
	1.5	20390.54	118299.76	0.95	604	1396	3805	11369	32884
	2	50015.21	1083023.37	0.98	965	2389	7078	23277	74305
50(2)	1	1544.50	1783.65	0.90	371	592	1031	1839	3179
	1.2	1718.68	2012.15	0.92	400	645	1125	2021	3522
	1.5	1982.94	2392.56	0.94	448	728	1286	2330	4112
	2	2549.14	3313.11	0.96	541	895	1610	2981	5328
100(5)	1	894.92	583.27	0.90	368	512	747	1100	1587
	1.2	933.95	621.40	0.91	384	532	777	1149	1658
	1.5	988.92	652.95	0.93	404	563	822	1218	1750
	2	1099.09	732.12	0.95	442	618	909	1352	1961
200(10)	1	655.63	267.27	0.90	369	464	602	787	1006
	1.2	667.19	276.57	0.91	376	472	612	800	1023
	1.5	688.39	283.47	0.92	386	486	631	826	1058
	2	720.99	301.97	0.94	403	508	661	866	1109

500(25)	1	517.06	128.03	0.90	371	426	500	589	684
	1.2	521.37	127.48	0.91	373	429	505	594	690
	1.5	526.59	129.90	0.91	376	433	510	600	697
	2	536.57	132.00	0.93	384	442	519	611	711

The performance of the chart can be obtained by using the design parameters provided in Table 3 under EPC. Following Table 4 - 5 represents the IC performance of the Pooled and ADM chart under the EPC for $n = 5$ having 5% spurious observation in the Phase I samples. Further, Table 6 - 7 represents the IC performance of the Pooled and ADM charts for $n = 5$, respectively having 10% spurious observation in the Phase I samples. It can be observed from Table 4 that the Pooled chart performance when the Phase I sample having no outlier i.e., $c = 1$, its $\pi(1)$ value 0.90 and $\left(\mu_{\mathrm{CARL}}(1), \sigma_{\mathrm{CARL}}(1)\right) = (8205.59, 41339.33)$ when $(m, n) = (20,5)$ while when the Phase I sample having outlier i.e., $c = 1.5$, its $\pi(1)$ value is 0.95 and $\left(\mu_{\mathrm{CARL}}(1), \sigma_{\mathrm{CARL}}(1)\right) = (20390.54, 118299.76)$. On the other hand, performance of the ADM chart from Table 5 informs us that when Phase I sample having no outlier i.e., $c = 1$, its $\pi(1)$ value is also 0.90 and $\left(\mu_{\mathrm{CARL}}(1), \sigma_{\mathrm{CARL}}(1)\right) = (1126.81, 943.21)$ when $(m, n) = (20,5)$ while when the Phase I sample having outlier i.e., $c = 1.5$, its $\pi(1)$ value is 0.96 and $\left(\mu_{\mathrm{CARL}}(1), \sigma_{\mathrm{CARL}}(1)\right) = (1656.19, 1450.44)$. Study reflect that both the charts are reflecting confidence in the values of CARL(1) by the metric $\pi(1)$ i.e., $\pi(1) = 0.90$ when Phase I sample having no outliers. But the performance of the Pooled chart is deviated more in the presence of outliers than the ADM chart. Moreover, the 10th percentile of the Pooled and ADM chart is approaching 370 which shows more confidence in the values of the CARL(1). For instance, when $(m, n) = (20,5)$, the 75th percentile of the Pooled chart is 5659 when $c = 1$ and 11369 at $c = 1.5$ whereas the 75th percentile of the ADM chart is 1393 when $c = 1$ and 2044 when $c = 1.5$. It means there is approximately 25% chance that the CARL(1) of the chart may occur greater that the 5659 for the Pooled chart and 1393 and 2044 for the ADM chart, respectively. All these information about the ADM chart are appearing more closer to the desired performance and less deviated from the nominal performance. Therefore, ADM chart outperforms the Pooled chart when we consider the estimation of the parameter with contaminated data.

Table 5: *IC performance of ADM chart with estimated parameter at* $n = 5, p = 0.10, ARL_0 = 370.4$ *with 5% outlier*

					Percentile				
$m(\gamma)$	c	$\mu_{\mathrm{CARL}}(1)$	$\sigma_{\mathrm{CARL}}(1)$	$\pi(1)$	0.10	0.25	0.50	0.75	0.90
20(1)	1	1126.81	943.21	0.90	369	551	871	1393	2154
	1.2	1313.90	1104.80	0.93	425	635	1005	1620	2520
	1.5	1656.19	1450.44	0.96	511	777	1251	2044	3220
	2	2435.42	2244.63	0.99	709	1094	1800	2994	4812
50(2)	1	683.84	301.52	0.90	370	472	622	826	1071
	1.2	723.41	316.94	0.92	388	497	658	875	1134
	1.5	787.24	354.23	0.95	420	540	715	952	1239
	2	905.31	418.50	0.97	479	617	820	1097	1431
100(5)	1	555.97	164.89	0.90	370	439	531	645	771
	1.2	570.33	170.97	0.91	379	450	546	663	791
	1.5	593.08	176.35	0.93	393	467	567	690	823
	2	634.21	186.53	0.96	418	498	606	739	884
200(10)	1	486.70	99.42	0.90	369	417	476	545	616
	1.2	493.73	96.19	0.91	374	422	483	553	625
	1.5	502.32	105.94	0.93	381	430	491	562	635
	2	519.21	101.24	0.94	393	443	507	582	659

500(25)	1	436.91	60.92	0.90	369	398	432	471	508
	1.2	439.70	55.21	0.91	372	401	436	473	511
	1.5	442.78	53.19	0.92	374	403	439	477	516
	2	448.32	54.98	0.93	379	408	444	483	523

Similarly, Tables 6-7 which entails us about the study of 10% contaminations in the Phase I samples of size $n = 5$, respectively. The comprehensive study of the 10% contaminations also suspected to deviate from its nominal than expected. For example, when we consider $m = 50$ Phase I observations each of size $n = 5$, then 10% contamination produces $\gamma = 5$ outliers. For the Pooled chart, $\left(c, \mu_{\text{CARL}}(1), \sigma_{\text{CARL}}(1)\right) = (1, 1554.08, 1766.69)$ and $\left(c, \mu_{\text{CARL}}(1), \sigma_{\text{CARL}}(1)\right) = (1.5, 1865.62, 2213.95)$. Similarly, when using ADM estimator, $(c, \mu_{\text{CARL}}(1), \sigma_{\text{CARL}}(1))$ is $(1, 683.67, 302.38)$ and $(c, \mu_{\text{CARL}}(1), \sigma_{\text{CARL}}(1))$ is $(1.5, 770.89, 344.03)$ with high probability. Therefore, we recommend our proposed ADM chart under the EPC when the Phase I samples having spurious observations.

Table 6: *IC performance of Pooled chart with estimated parameter at* $n = 5, p = 0.10, ARL_0 = 370.4$ *with* 10% *outlier*

					Percentile				
$m(\gamma)$	c	$\mu_{\text{CARL}}(1)$	$\sigma_{\text{CARL}}(1)$	$\pi(1)$	0.10	0.25	0.50	0.75	0.90
20(2)	1	8144.06	50446.91	0.90	371	804	2054	5691	15341
	1.2	10477.68	141575.40	0.92	429	951	2465	6943	19201
	1.5	15791.01	114907.40	0.94	545	1245	3363	9865	27930
	2	31623.24	208322.50	0.97	797	1889	5364	16780	50540
50(5)	1	1554.08	1766.69	0.90	370	593	1031	1844	3194
	1.2	1657.69	2010.15	0.91	391	630	1096	1962	3393
	1.5	1865.62	2213.95	0.93	429	696	1215	2196	3852
	2	2266.70	2916.76	0.95	497	811	1447	2640	4699
100(10)	1	897.21	589.07	0.90	371	514	747	1101	1589
	1.2	928.81	606.99	0.91	381	528	774	1144	1647
	1.5	977.27	632.49	0.92	398	556	811	1201	1734
	2	1058.88	702.13	0.94	426	595	875	1299	1894
200(20)	1	653.83	279.71	0.90	37	463	601	783	1001
	1.2	665.24	275.90	0.91	376	472	612	797	1016
	1.5	681.55	280.48	0.92	383	482	625	817	1046
	2	709.17	295.85	0.93	397	499	650	851	1092
500(50)	1	515.81	126.99	0.90	368	425	498	588	686
	1.2	520.15	128.88	0.90	371	428	504	593	688
	1.5	523.62	132.36	0.91	374	431	506	596	693
	2	532.06	129.50	0.92	380	439	514	608	704

II. OOC performance of the chart without contamination

As for as OOC performance concern, following Table 8 represents the $\mu_{\text{CARL}}(\delta)$ and $\sigma_{\text{CARL}}(\delta)$ metrics for both the charts having different shift parameter, δ. These values are obtained using the expressions given in Equations (9) and (10) for nominal $\text{ARL}_0 = 370.4, p = 0.10$ and $\delta = 1, 1.2, 1.5, 2$. The value $\delta > 1$ corresponds to the OOC situation when the process is deteriorated. We mention here that the ADM chart outperforms the Pooled chart under the EPC in terms of lower $\mu_{\text{CARL}}(\delta)$ and $\sigma_{\text{CARL}}(\delta)$ values for $\delta > 1$. Please note here that $\delta = 1$ represents IC performance of the Pooled and

ADM chart, respectively. For instance, when $(m, n, \delta) = (20,5,1.2)$ the $\mu_{\text{CARL}}(\delta)$ and $\sigma_{\text{CARL}}(\delta)$ is 1132.83 and 3197.63 respectively for the Pooled chart whereas $\mu_{\text{CARL}}(\delta)$ and $\sigma_{\text{CARL}}(\delta)$ is 268.91 and 171.92, respectively for the ADM chart. It implies that the ADM chart under the EPC takes less time to detect an OOC signal than the Pooled chart when the process deteriorates due to an increase in the rate parameter. Hence, ADM chart shows better performance in the OOC scenario and Pooled chart missed the signal.

Table 7: *IC performance of ADM chart with estimated parameter at* $n = 5, p = 0.10, ARL_0 = 370.4$ *with* 10% *outlier.*

					Percentile				
$m(\gamma)$	c	$\mu_{\text{CARL}}(1)$	$\sigma_{\text{CARL}}(1)$	$\pi(1)$	0.10	0.25	0.50	0.75	0.90
20(2)	1	1126.13	924.36	0.90	367	550	871	1395	2155
	1.2	1292.61	1088.71	0.93	418	623	990	1596	2481
	1.5	1583.65	1379.11	0.96	493	747	1199	1951	3079
	2	2228.73	2030.10	0.98	655	1009	1651	2740	4387
50(5)	1	683.67	302.38	0.90	371	475	623	824	1065
	1.2	718.32	315.86	0.92	385	494	653	868	1128
	1.5	770.89	344.03	0.94	413	529	701	934	1208
	2	872.96	397.46	0.97	463	595	789	1058	1380
100 (10)	1	555.51	165.33	0.90	370	438	531	645	770
	1.2	568.37	168.81	0.91	378	448	544	660	788
	1.5	588.76	175.02	0.93	391	464	563	683	818
	2	625.25	182.07	0.95	414	492	597	727	870
200 (20)	1	487.90	93.55	0.90	370	417	477	547	617
	1.2	492.67	101.49	0.91	374	421	482	552	623
	1.5	500.96	100.47	0.92	380	427	489	562	636
	2	515.31	108.45	0.94	391	440	504	578	653
500 (50)	1	437.49	52.58	0.90	396	39	434	572	510
	1.2	439.02	52.32	0.90	371	400	435	473	511
	1.5	442.09	58.80	0.91	374	403	438	476	514
	2	446.95	59.63	0.92	378	407	442	482	520

Table 8: *The OOC performance metrics* $\mu_{CARL}(\delta)$ *and* $\sigma_{CARL}(\delta)$ *of the Pooled chart and ADM chart for* $p = 0.10$ *and* $ARL_0 = 370.4$ *and shift size* $\delta = 1,1.2,1.5,2$ *at* $n = 5$.

			δ			
m	Estimator	PM	1	1.2	1.5	2
20	Pooled	$\mu_{\text{CARL}}(\delta)$	8205.59	1132.83	181.13	33.52
		$\sigma_{\text{CARL}}(\delta)$	41339.33	3197.63	290.07	30.38
	ADM	$\mu_{\text{CARL}}(\delta)$	1126.90	268.91	67.14	17.91
		$\sigma_{\text{CARL}}(\delta)$	932.39	171.92	31.84	5.82
50	Pooled	$\mu_{\text{CARL}}(\delta)$	1547.32	341.11	79.51	20.01
		$\sigma_{\text{CARL}}(\delta)$	1783.65	289.74	49.41	8.58
	ADM	$\mu_{\text{CARL}}(\delta)$	683.78	182.86	53.50	15.27
		$\sigma_{\text{CARL}}(\delta)$	301.52	65.28	24.33	4.75
100	Pooled	$\mu_{\text{CARL}}(\delta)$	894.92	224.90	59.10	16.44
		$\sigma_{\text{CARL}}(\delta)$	583.27	115.03	22.31	4.39
	ADM	$\mu_{\text{CARL}}(\delta)$	555.76	155.56	44.98	13.65
		$\sigma_{\text{CARL}}(\delta)$	164.89	36.28	8.13	1.85

200	Pooled	$\mu_{\mathrm{CARL}}(\delta)$	655.51	177.15	49.53	14.59
		$\sigma_{\mathrm{CARL}}(\delta)$	268.56	57.81	12.32	2.53
	ADM	$\mu_{\mathrm{CARL}}(\delta)$	487.00	139.80	41.55	12.93
		$\sigma_{\mathrm{CARL}}(\delta)$	97.79	24.23	5.34	1.13
500	Pooled	$\mu_{\mathrm{CARL}}(\delta)$	517.07	146.66	43.07	13.25
		$\sigma_{\mathrm{CARL}}(\delta)$	126.33	29.87	6.50	1.53
	ADM	$\mu_{\mathrm{CARL}}(\delta)$	436.93	128.37	38.88	12.35
		$\sigma_{\mathrm{CARL}}(\delta)$	55.07	13.49	3.38	0.73

VI. Simulated Example

In this section, we provide an illustration of simulated data set to show the findings. Following Table 9 represents a simulated dataset for $m = 20$ and $n = 5$ which are generated from the IC normal process i.e., $N(2,5)$. And the next 20 samples are generated from the $N(2,6.5)$. The fist 20 samples are used to estimate the parameter, $\hat{\sigma}^2$, for the S^2 chart and obtained as 21.96003. Design parameter of the Pooled chart (black dashed lines) and ADM chart (red dashed lines) for $m = 20$ are 20.22638 and

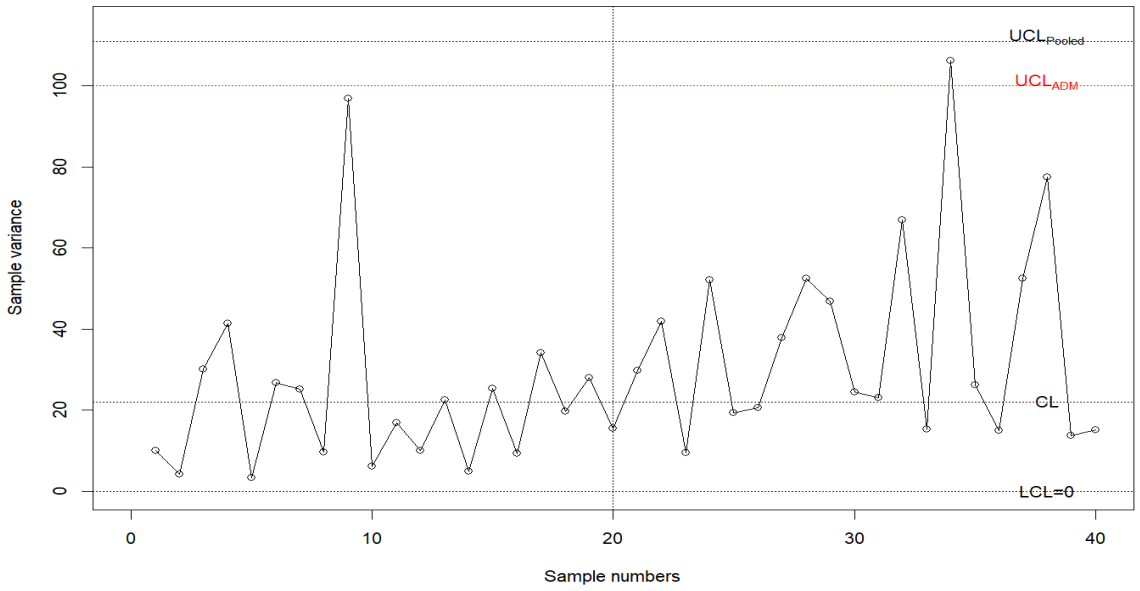

Figure 1: *Phase II control limits of the ADM chart and Pooled chart.*

Table 9: *Simulated dataset of* $n = 5$ *and* $m = 40$.

1	2	3	4	5	1	2	3	4	5
4.390	-1.876	-1.711	-3.963	-1.936	8.423	6.150	-2.161	4.727	-4.142
3.674	-0.163	0.309	2.008	4.467	1.211	10.332	14.511	9.105	-0.894
2.583	0.067	-5.437	8.495	-3.553	0.831	-1.156	-6.280	-2.271	1.441
4.661	-0.749	0.524	1.378	15.113	2.818	11.521	-7.932	-2.342	-0.992
-0.806	1.587	-0.575	2.097	3.476	4.079	14.962	7.891	4.323	7.178
-6.971	-3.373	3.723	0.242	5.745	4.776	-1.385	11.043	5.391	2.579
-2.447	3.949	5.299	11.462	6.507	-6.480	8.580	3.875	-1.578	-4.306
0.130	6.252	-1.676	-0.7694	0.807	0.028	0.937	1.473	-4.090	15.039
5.019	11.502	-6.013	-10.249	10.601	-12.609	-10.569	0.404	-0.301	2.081
4.051	4.236	8.168	1.918	6.835	2.105	4.565	-1.102	11.162	-0.366

8.699	10.217	4.429	3.398	0.069	-0.325	5.689	4.338	-0.056	-6.540
7.738	7.231	-0.079	4.510	6.362	4.326	10.523	-11.248	2.647	5.855
-0.001	9.725	-1.881	2.948	-1.181	1.383	9.274	0.665	0.427	-0.051
3.424	7.605	4.941	7.390	2.832	14.697	0.805	-12.239	-4.933	6.146
4.851	-3.772	9.981	2.367	5.381	-2.900	6.981	-3.778	-2.365	5.480
5.434	-0.973	1.394	1.425	6.247	7.268	5.136	-0.783	9.026	2.670
4.859	5.758	-5.486	9.878	6.982	-5.437	12.501	-4.984	1.411	1.063
13.135	3.751	4.430	1.668	4.289	5.986	-2.271	12.744	-9.001	8.882
-5.015	3.971	-0.611	6.145	8.011	9.175	6.264	13.228	7.132	3.265
4.019	-3.330	-2.007	0.802	5.971	3.048	13.574	7.352	9.829	7.016

18.2357, respectively. Estimated upper control limit of the Pooled chart is 111.043 whereas for the ADM chart is 100.1141. It can be observed from the Figure 1 that ADM-chart detects the OOC signal first at the 34th sample point under the EPC whereas the Pooled chart missed the OOC signal.

VII. Conclusion

In this paper, we have considered the estimation of Phase II control limits of an upper sided S^2 chart. The present study shows the adjustment of Phase II control limits of S^2 chart using the EPC approach. Undoubtedly, the EPC criterion degrades the chart's OOC performance but, on the other hand, it provides better IC performance for the pooled chart and ADM chart with desired IC performance for the smaller Phase I samples (say, $m < 100$). The study suggests that the ADM chart outperform in the presence of outliers.

As far as OOC performance is concern, the performance of the charts are also evaluated based on the EPC criterion. Study show that the ADM estimator takes less time, on average, to trigger a signal than the sample pooled estimator when $\delta \geq 1$. Hence, we recommend that the user must ensure that the Phase I sample is free from upper outliers and if in doubt then the ADM chart must be used to construct Phase II control limits. Finally, all the calculations are performed using the R statistical software and the programs are available from the authors on request.

Before closing, we mention that the present study considers an upper-sided S^2 chart. However, we have carried out the study of two-sided S^2 chart. Study shows that there was no any significant difference have been found in the case of contaminated observations in the values of ARL for improvement case because of the skewed nature of the run length distribution. Moreover, as a future direction, the present study can also be extended to examine the effect of presence of the spurious observations in the Phase I data on the EWMA and CUSUM charts.

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ADVANCED STATISTICAL APPROACH TO FAILURE DATA WITH GAMMA AND WEIBULL DISTRIBUTIONS

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Abstract

This paper aims to systematically investigate the utility of the Gamma and Weibull distributions, focusing on their application to biomedical datasets and clarifying their mathematical and statistical properties. By analyzing lifetime data across various disciplines, the research emphasizes the effectiveness and flexibility of these distributions in capturing the complexities of biomedical data. It underscores the importance of parameters such as standard error, log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) in value estimation. The findings suggest that both distributions provide valuable insights into the underlying data, with practical implications for reliability engineering and failure analysis. Moreover, the study demonstrates that the Weibull distribution offers a better fit to the given data than the Gamma distribution due to its adaptability, which yields superior results. A key contribution of this study is the proposal of a model based on estimating the Conditional Weibull distribution for feature parameters, which accurately predicts a finite mixture of two-parameter Weibull distributions initially verified on datasets.

Keywords: Gamma distribution, Weibull distribution, Probability density function, Cumulative density function, Akaike information criterion, Bayesian information criterion, Biomedical.

I. Introduction

The Weibull distribution is well-known for its application in reliability engineering and failure analysis. It is extensively used to model biological sciences, weather forecasting, and hydrology data. However, it may not be appropriate for specific applications, especially once hazard rates exhibit bathtub or bimodal shapes. Researchers have developed various modifications and extensions of the Weibull distribution to overcome these limitations and accommodate a broader range of data types [3], introducing the additive Weibull distribution. Using the Generalized Gamma distribution in survival study for breast cancer patients [1]. The generalized Weibull model extends the traditional distribution by incorporating additional parameters, offering greater flexibility in modeling survival data [13]. The main objective is to compare the effectiveness of the Gamma and Weibull distributions in modeling lifetime data across various fields, such as multiple fields used to evaluate their goodness of fit [18]. Numerous studies have also applied Gamma distribution to model wealth inequality [7]. These tests are relevant to various fields, including failure time models and survival

studies, anywhere the failure frequency is constant, and increasing pertinent tests to fields like failure time models and survival analysis are crucial for determining if the failure rate is steady, growing, or falling is vital [5]. This study significantly enhances the understanding and application of the inverse-generalized Weibull distribution, providing more accurate tools for reliability engineering failure analysis [2]. Due to its versatility in the model's lifetime data, the Weibull distribution has long been stable in reliability engineering and various other fields. However, traditional Weibull distribution sometimes fails to capture complex hazard rate shapes truthfully. Researchers have developed multiple modifications and extensions to address these limits [14]. The proposed novel method enhances parameter estimation for finite Weibull distributions, educating the precision and reliability through cost-effective continuing asset management strategies [8]. The proposed technique is evaluated through comprehensive numerical analysis, demonstrating reliability through its effectiveness in outperforming existing advanced methods for handling unfair data [12]. The primary goal is to propose and assess the beta-Weibull distribution efficiency in modelling survival data. By integrating the exponentiated Weibull distributions, this new family modification is a data analysis [4].

The Weibull distribution enjoys widespread use in aerospace, microchip technology, materials, and automotive industries; it is vital to understand components' reliability and failure rates [11]. In today's context of model building and real-life data analysis, numerous lifetime distributions are utilized, with sources [15] and [16] providing foundational ideas and concepts. The Weibull distributions stand out for their flexibility and extensive applicability in modeling diverse data types, particularly excelling in reliability engineering and failure analysis [20]. The parameters of the Weibull-gamma distribution are estimated using the maximum likelihood method [6]. This research is inspired by the recognition that outdated distributions often fail to accommodate the complexities of biomedical datasets. Encompassing, the seeks to contribute to developing two distributions that improve and adapt the intricacies of real-life data. The objective is to methodically analyze and compare these distributions, focusing on their effectiveness in modeling biomedical datasets. Critical parameters such as Estimated Values, Standard Error, Log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are examined to illuminate these models' statistical characteristics and fitting quality. The ultimate goal is to determine which distribution better captures the nuances of the dataset, thereby advancing the understanding and application of these models in biomedical research.

II. Methods

2.1 Derivation of Gamma and Weibull Distributions

In the section, a new comparison model cumulative distribution (CDF) and probability density function (PDF) of the Gamma and Weibull Distribution family are given as

$$\mathrm{f}(x:\alpha,\beta) = f(x) = \begin{cases} \frac{-x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{1}$$

where $\alpha > 0$ is the scale parameter, $\beta > 0$ is the shape parameter, and $\Gamma(\alpha)$ is the Gamma function. The cumulative distribution function is

$$F(x;\ \alpha,\beta) = f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right), & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{2}$$

Where $\gamma(\alpha, x)$ is the lower incomplete Gamma function.

$$F(x;\ \alpha, k) = f(x) = \begin{cases} 1 - e^{-(x/\alpha)\beta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{3}$$

Where $\alpha > 0$ is the scale parameter, and $\beta > 0$ is the shape parameter.
The cumulative distribution function is given by

$$f(x;\ \alpha, k) = f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{k-1} e^{-(x/\alpha)^{\beta}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{4}$$

The CDF provides the probability that a random variable X is less than or equal to a specific value x. At the same time, the PDF describes the relative likelihood of the random variable taking on a particular value.

The Bayesian Information criterion is

$$BIC = kln(n) - 2ln(L) \tag{5}$$

k is the number of parameters in the model, n is the number of observations, and L is the maximum likelihood of the model.
The Akaike Information Criterion is given by

$$AIC = 2k - 2ln(L) \tag{6}$$

k is the number of estimated parameters in the model, and L is its maximum likelihood function.

The gamma distribution model was applied to dataset observations to gain insights into its statistical characteristics. The model's shape and scale parameters were accurately estimated, and the negative log-likelihood value indicated a good fit for the data. These statistical measures suggest that the Gamma distribution effectively captures the underlying data distribution, providing a solid foundation for further analysis and interpretation.

The model provided significant insights into the statistical properties of the analyzed dataset. Fitted to observations, the model's shape and scale parameters were estimated with corresponding standard errors, reflecting its fitting quality. These parameters and statistical measures demonstrate the Weibull model's effectiveness in capturing the underlying distribution characteristics, providing a robust tool for further analysis and clarification.

Overall, the Weibull distribution fits the dataset better than the Gamma distribution. The Wei bull model's closer AIC and BIC values suggest consistency and reliability in model fit evaluation. For various combinations of α and β, we generated sample sizes from the Gamma and Weibull models, specifically for the parameter values $\alpha = 2.0$, $\beta = 0.5$, and $\alpha = 1.5$, $\beta = 1$. The value decreases as the sample size increases, as shown in Table 1. Table 1 illustrates that our Gamma and Weibull models better fit the dataset than the KME and KM-IW (α, β) distributions. The dataset represents survival times in days from a two-arm clinical trial, referenced by sources [9] and [17], consisting of heart failure times in days for 299 patients.

III. Results

In this section, we conduct a comparative study to assess the Maximum Likelihood Estimation (MLE) performance for Gamma and Weibull distributions. We generate n = 299 samples drawn from those distributions using the quantile function. Subsequently, we calculate the MLEs, loglikelihood, standard error, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC).

Table 1: *The model's performance Some values of parameters requirements built on the dataset*

Model	Parameter	Estimated values	Standard Error	Loglikelihood	AIC	BIC
Gamma	α	2.0509	0.1560	-658.7781	1321.556	1328.957
	β	0.5442	0.0468			
Weibull	α	1.5902	0.0716	-428.4005	860.8009	868.2018
	β	1.9780	0.0757			

Table 1 shows the outcomes corresponding to the Gamma distribution by utilizing it to model Biomedical datasets. The parameters indicate that the Gamma distribution, with its shape and scale, captures the data distribution with a specific level of skewness and spread. The AIC and BIC values support the model's fit, though the Weibull distribution, with its parameters, shows a potentially better fit given its lower AIC and BIC values. This suggests the Weibull model might be more effective at capturing the underlying distribution characteristics of the dataset.

3. 1 Application

We compare its performance in providing a robust parametric fit to the datasets with that of the Weibull distribution. Metrics such as the log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are employed for this comparison. The loglikelihood, AIC, and BIC values are computed for the proposed Weibull model for comparison to discern the most suitable model. The model exhibiting the lowest loglikelihood, AIC, and BIC values is deemed the most appropriate match for the provided datasets. The R software is employed for this analytical endeavor, simplifying the necessary calculations and comparisons.

The dataset comprises the summation of time (in days) measurements from two hundred twenty-nine (299) respondents at the heart failure patients. It consists of the following values:
4, 6, 7, 7, 8, 10, 10, 10, 10, 10, 11, 11, 12, 13, 14, 14, 15, 15, 16, 20, 20, 22, 23, 23, 24, 26, 26, 27, 28, 28, 29, 29, 30, 30, 30, 30, 30, 31, 32, 33, 33, 33, 35, 38, 40, 41, 42, 43, 44, 45, 50, 54, 54, 55, 59, 60, 60, 60, 61, 63, 64, 65, 66, 67, 68, 71, 72, 73, 74, 74, 74, 74, 75, 76, 77, 78, 78, 79, 79, 79, 79, 80, 80, 82, 82, 83, 83, 83, 85, 85, 86, 87, 87, 87, 87, 87, 88, 88, 88, 88, 88, 90, 90, 90, 90, 91, 91, 94, 94, 94, 95, 95, 95, 95, 95, 96, 97, 100, 100, 104, 104, 105, 106, 107, 107, 107, 107, 107, 107, 108, 108, 108, 109, 109, 109, 110, 111, 112, 113, 115, 115, 117, 118, 119, 120, 120, 120, 120, 121, 121, 121, 121, 123, 126, 130, 134, 135, 140, 145, 145, 146, 146, 146, 146, 146, 147, 147, 147, 147, 148, 150, 154, 162, 170, 171, 172, 174, 174, 174, 175, 180, 185, 186, 186, 186, 186, 186, 186, 187, 187, 187, 187, 187, 187, 187, 188, 192, 193, 193, 194, 195, 196, 196, 197, 197, 198, 200, 201, 201, 205, 205, 205, 206, 207, 207, 207, 208, 209, 209, 209, 209, 209, 210, 210, 211, 212, 212, 213, 213, 213, 214, 214, 214, 214, 214, 215, 215, 215, 215, 216, 220, 230, 230, 233, 233, 235, 237, 237, 240, 241,

244, 244, 244, 244, 244, 245, 245, 245, 245, 245, 246, 246, 246, 247, 250, 250, 250, 250, 250, 250, 250, 256, 256, 257, 258, 270, 270, 271, 278, 280, 285.

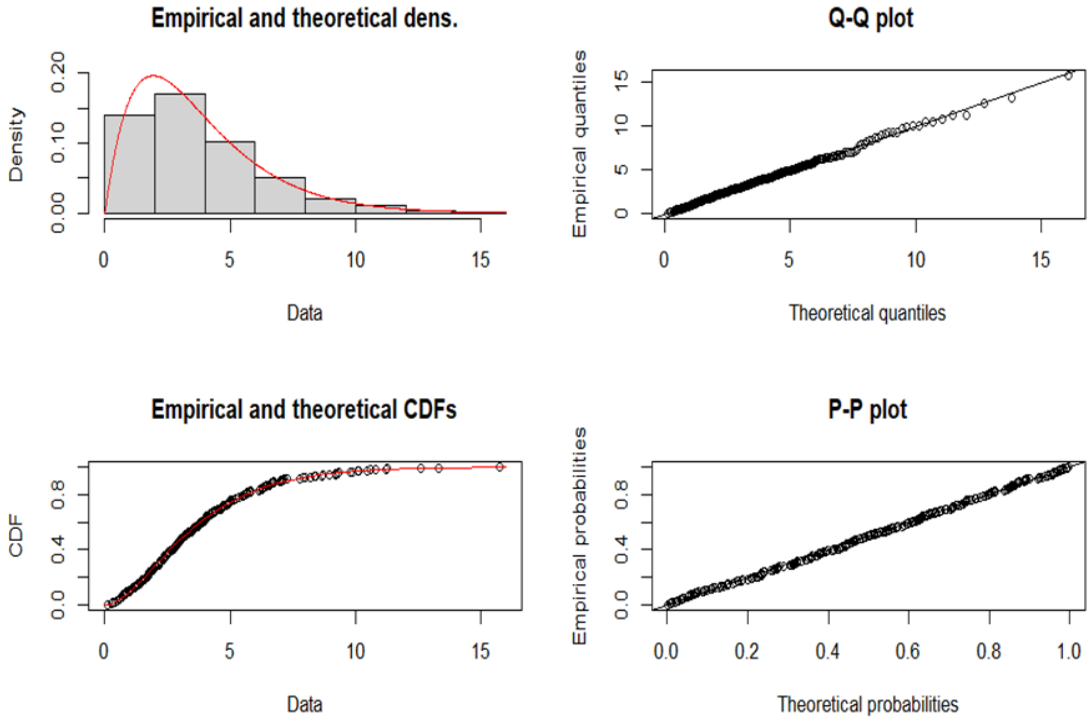

Figure 1: *Gamma distribution Density plots for the datasets.*

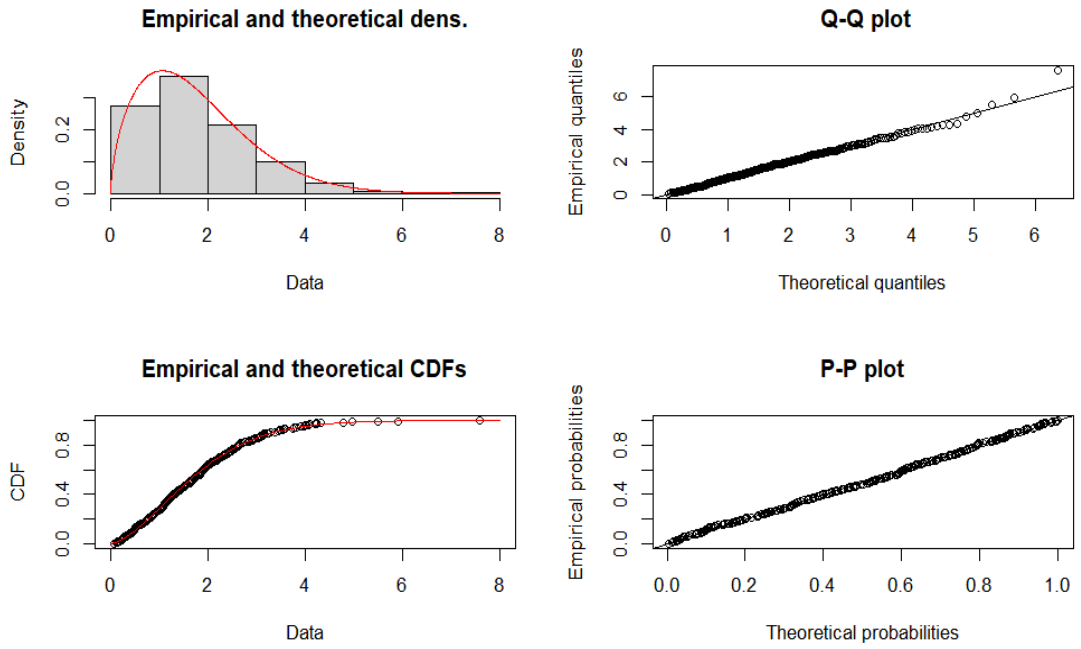

Figure 2: *Weibull distribution Density plots for the datasets.*

Table 1 presents the planned model's superior ability to effectively fit the highly skewed datasets compared to the competing models, as specified by the evaluation metrics employed. Figures 1 and 2 show that the proposed mode fits the data set adequately.

IV. Conclusion

This paper comprehensively compares the Gamma and Weibull distributions for modeling lifetime data. By incorporating key parameters, we improve the flexibility of these distributions, enabling them to capture the complexities of biomedical datasets effectively. Using accurate data, we derived precise closed-form expressions for the probability density function (PDF), cumulative density function (CDF), standard error, log-likelihood function, and parameter estimation. Our analysis demonstrated the practical relevance of these models through Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values. This enhances our ability to accurately model and interpret biomedical data, providing a robust tool for future research and analysis.

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BAYESIAN PARAMETER ESTIMATION FOR TRANSMUTED WEIBULL DISTRIBUTION WITH CENSORING RATES AND VARIOUS LOSS FUNCTIONS

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Abstract

Statistical distributions are essential tools for describing and predicting real-world phenomena, though recent advancements in data collection have made it challenging to fit existing probability models to many practical datasets. While non-parametric models are sometimes recommended, parametric models retain substantial popularity due to their interpretability and flexibility. The quadratic rank transmutation map (QRTM) technique has been used to create new families of non-Gaussian distributions, known as transmuted distributions, which allow for modifications in moments, skewness, and kurtosis, thus increasing flexibility. The transmuted Weibull distribution (TWD) has gained attention for applications in reliability, survival analysis, and lifetime data analysis. This article focuses on a Bayesian analysis of the transmuted Weibull distribution, a generalization of the traditional Weibull model that addresses its limitations, particularly for datasets exhibiting non-monotonic failure rates. Bayesian parameter estimation is performed using a Markov Chain Monte Carlo (MCMC) algorithm, with both non-informative and informative priors. We calculate Bayes estimators (BEs) and posterior risks (PRs) under different loss functions, including the Absolute Error Loss Function (AELF), precautionary loss function (PLF), and quadratic loss function (QLF). Simulation studies evaluate the Bayes estimators' performance, investigating the effects of various priors, sample sizes, and censoring rates on estimation accuracy and credible interval width. Real-world data applications highlight the practical utility of the Bayesian approach for the TWD, showing consistent results with increasing sample sizes and underscoring the robustness of the MCMC algorithm for parameter estimation. The article is structured as follows: the TWD's parameters, including scale, shape, and transmutation, are estimated under different loss functions and priors. Bayesian credible intervals (BCIs) are also computed. Both uncensored and censored data environments are considered, with varying sample sizes and censoring rates. Posterior risks for each estimator are analyzed to assess performance, and two real datasets are used to illustrate the flexibility and applicability of the proposed distribution. This study lays a foundation for future research, such as exploring mixtures of transmuted Weibull distributions or conducting Bayesian analyses for record values.

Keywords: Transmuted Weibull distribution, Markov Chain Monte Carlo, Bayesian credible intervals, Bayes estimators, posterior risks, absolute error loss function, precautionary loss function, quadratic loss function, censoring.

I. Introduction

Statistical distributions are essential for describing and predicting real-world phenomena. However, advancements in data collection methods have led to challenges in fitting existing probability models to many significant and practical datasets [1]. In such cases, while non-parametric models may be recommended, parametric models continue to enjoy substantial popularity. The quadratic rank transmutation map (QRTM) technique has been employed to create new families of non-Gaussian distributions [2]. This technique modifies the moments, skewness, and kurtosis of a baseline distribution, resulting in what is known as a transmuted distribution. This family of distributions has attracted considerable attention from researchers, leading to the development and exploration of various new flexible distributions over the past decade. For instance, Al Sobhi [3] introduced the transmuted modified Weibull distribution, while others presented the exponentiated transmuted Weibull distribution. More recently, a new lifetime distribution called the transmuted cubic Weibull distribution was constructed, and a novel weighted distribution known as the size-biased weighted transmuted Weibull distribution was introduced. The method of least squares and the method of moments have been utilized to estimate parameters for the transmuted Weibull distribution, with comparisons made through simulation studies using statistical measures like mean squared error (MSE) [4]. Researchers have also explored various structural properties of the transmuted Weibull distribution, including its mean, harmonic mean, standard deviation, moment generating function (MGF), skewness, and kurtosis [5][6]. Currently, transmuted distributions find applications in numerous fields, including reliability studies, lifetime analysis, engineering, economics, insurance, and environmental sciences.

The Weibull distribution is a widely recognized lifetime probability distribution, commonly used in various domains of reliability and survival analysis [7]. Its attractiveness largely arises from the variety of shapes it can assume by adjusting its shape parameter. Additionally, the Weibull distribution is versatile, encompassing many other distributions, such as the exponential and Rayleigh distributions, as special cases. Despite its widespread use and applicability, the traditional Weibull distribution does not fully capture the range of lifetime phenomena. For instance, it is not suitable for datasets exhibiting a non-monotonic failure rate, prompting investigations into generalizations of the Weibull distribution. A notable generalization applicable to survival data analysis is the transmuted generalized inverse Weibull distribution, which discusses its mathematical properties and employs maximum likelihood methods for parameter estimation. Similarly, the transmuted generalized Weibull distribution has been developed, exploring its mathematical properties, including the quantile function, moments, entropies, mean deviation, Bonferroni and Lorenz curves, and the moments of order statistics, also using maximum likelihood for parameter estimation. Furthermore, the generalized transmuted Weibull distribution has been proposed, with its properties derived. This article focuses on the Bayesian analysis of the transmuted Weibull distribution, which serves as a generalization of the Weibull probability distribution. We emphasize Bayesian analysis because maximum likelihood and moment estimators have been used for parameter estimation of the transmuted Weibull distribution [8]. To facilitate this analysis, we employ a Markov Chain Monte Carlo (MCMC) algorithm to compute posterior summaries for the unknown parameters of the distribution, comparing the results across different priors, loss functions, sample sizes, and parameter sets [9].

The objective of this paper is to define the Transmuted Weibull distribution and introduce its likelihood function, followed by the derivation of posterior distribution expressions utilizing both non-informative and informative priors, as well as marginal posterior densities for both censored

and uncensored data. The study aims to explore Bayesian estimators (BEs) and their associated posterior risks (PRs) under various loss functions. Additionally, the paper seeks to detail the estimation of unknown parameters of the proposed distribution through the MCMC algorithm for posterior summaries, encompassing different loss functions and prior types. The work will also provide a mathematical and numerical discussion of Bayesian credible intervals (BCIs) and conclude with an analysis of a real-life dataset.

II. Transmuted Weibull Distribution

As introduced by Alzaatreh et al. in 2013 [10], a random variable X follows a transmuted probability distribution if its probability density function (pdf) and cumulative distribution function (CDF) are given by:

$$f(x) = g(x)\big(1 + \gamma - \gamma.2G(x)\big)$$
$$F(x) = G(x)[(1 + \gamma) - \gamma.G(x)]$$

where $x > 0$ and the transmutation parameter γ satisfies $|\gamma| \leq 1$. Here, $G(x)$ is the CDF of the baseline distribution, and the functions $f(x)$ and $F(x)$ represents the pdf and CDF of the transmuted distribution, respectively.

A random variable X is defined to follow a Weibull distribution characterized by parameters $\alpha > 0$ and $\beta > 0$ if its probability density function (PDF) is given by:

$$g(x;\alpha,\beta) = \frac{\alpha}{\beta}(\boldsymbol{x})^{\alpha-1} exp\left(-\frac{x^{\alpha}}{\beta}\right), x \geq 0$$

The corresponding cumulative distribution function (CDF) for this Weibull distribution can be expressed as:

$$G(x) = 1 - exp\left(-\frac{x^{\alpha}}{\beta}\right)$$

To find the CDF of the transmuted Weibull distribution, we substitute $G(x)$ into the following formula:

$$F(x) = (1+\gamma)\left(1 - exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right) - \gamma\left(1 - exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right)^2$$

Through algebraic manipulation, we derive the CDF for the transmuted Weibull distribution as:

$$F(x;\theta) = exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\left(1 - \gamma + \gamma\, exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right)$$

where $\theta = (\alpha, \beta, \gamma)$. To determine the PDF of the transmuted Weibull distribution, we differentiate this CDF with respect to x and simplify the result. The resulting PDF is then expressed as:

$$f(x;\theta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right).\left(1 - \gamma + 2\gamma \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right),$$
$$x \geq 0, \alpha, \beta > 0 \; and \; |\gamma| \leq 1$$

Special Cases

- When $\gamma = 0$, the Transmuted Weibull distribution simplifies to the standard Weibull distribution.
- If $\alpha = 1$, the result is the transmuted exponential distribution. Furthermore, with $\gamma = 0$, it becomes the standard exponential distribution.
- Setting both α and β to 1 yields the transmuted standard exponential distribution.
- When $\alpha = 2$, we obtain the transmuted Rayleigh distribution.
- If $\gamma = 0$, this corresponds to the traditional Rayleigh distribution.

III. Likelihood Functions for various Sampling Schemes

Consider a complete random sample $X_1, X_2, \dots X_n$ of size n taken from the transmuted Weibull distribution. Then, the likelihood function for the complete data set is given by:

$$L(x;\theta) = \alpha^n exp\left\{(\alpha-1)\sum_{i=1}^{n} log x_i\right\}\frac{1}{\beta^n} exp\left\{-\sum_{i=1}^{n}\frac{x_i^\alpha}{\beta}\right\}\prod_{i=1}^{n}\left[1-\gamma+2\gamma\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right]$$

where $\theta = (\alpha, \beta, \gamma)$ and $x = x_1, x_2, \dots x_n$. In many life-testing experiments, it's not possible to collect complete failure time data due to time and cost constraints. As a result, censoring plays an essential role in lifetime data analysis. Let$X = X_1, X_2, \dots X_r$ be a type-I censored sample of size r from n items, where the lifetimes follow a transmuted Weibull distribution with parameters α, β & γ. Consider a distribution with specific parameters. In the context of Type I censoring, it's important to note that the censoring time is predetermined, while the number of observed failures is random. Suppose we have n items under life testing, and we observe r failures at times $t_1, t_2, \dots t_r$. Here, r is an integer between 0 and n, and $(n-r)$ represents the number of items that survive or remain uncensored. According to Mendenhall and Hader (1958) [9], the likelihood function for censored data is given by:

$$L(x;\theta) \propto \prod_{j=1}^{r} f(x_j).[1-F(T)]^{n-r}$$

where T represents the time, r denotes the number of failures (or uncensored observations), and $(n-r)$ are the censored observations. For a transmuted Weibull distribution applied to censored data, the likelihood function can be expressed as:

$$(x;\theta) \propto \alpha^T exp\left(-\alpha\sum_{j=1}^{r} log\frac{1}{x_j}\right)\frac{1}{\beta^r} exp\left(-\frac{\sum_{j=1}^{r} x_j^\alpha}{\beta}\right)\times exp\left[\sum_{j=1}^{r} log\left\{1-\gamma+2\gamma\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\}\right]$$
$$exp\left((n-r)log\left[1-exp\left(-\frac{T^\alpha}{\beta}\right)\times\left\{1-\gamma+\gamma exp\left(-\frac{T^\alpha}{\beta}\right)\right\}\right]\right)$$

Next, we examine the posterior distribution using Bayes' theorem. The posterior distribution, g(α|x) is given by:

$$g(\alpha \mid x) = \frac{L(x;\alpha)\Pi(\alpha)}{\int_\alpha^\infty L(x;\alpha)\Pi(\alpha)d\alpha}$$

where $\pi(\alpha)$ denotes the joint prior distribution of the parameters $\alpha = \alpha_1, \alpha_2, \dots \alpha_k, L(x;\alpha)$

represents the likelihood function, and $g(\alpha \mid x)$ is the joint posterior distribution.

IV. Posterior Distribution using Uniform Prior (UP)

In Bayesian estimation, to determine unknown parameters, we specify a prior for each parameter that isn't explicitly defined by the model itself. Unlike the frequentist approach, the Bayesian method incorporates both prior knowledge about the parameters and the observed data. When prior information about the parameters is lacking, a non-informative prior can be used in Bayesian analysis. This type of prior conveys minimal information about the parameters, reflecting a lack of strong prior beliefs.

As introduced by Yousaf et al. in 2020 [11], to estimate the unknown parameters of the transmuted Weibull distribution, we assume the following prior distributions: $\alpha \propto 1, \forall \alpha \in (0,\infty), \beta \propto 1, \forall \beta \in (0,\infty)$ and $\gamma \propto 1, \forall \lambda \in (-1,1)$. With the assumption that these parameters are independent, the joint prior distribution for $\alpha, \beta \; and \; \lambda$ is: $\pi_1(\alpha,\beta,\gamma) \propto 1$, where $\alpha, \beta > 0$ and $|\gamma| \leq 1$. Using Bayes' theorem, the joint posterior distribution of parameters $\alpha, \beta \; and \; \gamma$, given data x, with a uniform prior is: $g(\theta|x) = \frac{L(x;\theta)\pi(\theta)}{\int_0^\infty \int_0^\infty \int_{-1}^1 L(x;\theta)\pi(\theta) d\gamma d\beta d\alpha}$ where $L(x;\theta)$ is the likelihood function and $\pi(\theta)$ represents the uniform prior over the parameters α, β, γ.

$$g(\theta|x) = \frac{\alpha^{A_{01}-1}\exp(-\alpha A_{11})\frac{1}{\beta^n}\exp\left(-\frac{A_{21}}{\beta}\right)\exp(A_{31})}{\int_0^\infty \int_0^\infty \int_{-1}^1 \alpha^{A_{01}-1}\exp(-\alpha A_{11})\frac{1}{\beta^n}\exp\left(-\frac{A_{21}}{\beta}\right)\exp(A_{31}) d\gamma d\beta d\alpha}$$

where $A_{01} = 1+n, A_{11}=\sum_{i=1}^n log\frac{1}{x_i}$, $A_{21}=\sum_{i=1}^n x_i^\alpha$ and $A_{31}=\sum_{i=1}^n log\left\{1-\boldsymbol{\gamma}+2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\}$. Likewise, for censored data, the posterior distribution is given by:

$$g(\theta|x) = \frac{\alpha^{B_{01}-1}\exp(-\alpha B_{11})\frac{1}{\beta^r}\exp\left(-\frac{B_{21}}{\beta}\right)\exp(B_{31})}{\int_0^\infty \int_0^\infty \int_{-1}^1 \alpha^{B_{01}-1}\exp(-\alpha B_{11})\frac{1}{\beta^r}\exp\left(-\frac{B_{21}}{\beta}\right)\exp(B_{31}) d\gamma d\beta d\alpha} \quad (1)$$

where $B_{01} = 1+r, B_{11}=\sum_{j=1}^r log\frac{1}{x_j}$, $B_{21}=\sum_{j=1}^r x_j^\alpha$ and $B_{31}=\sum_{j=1}^r log\left\{1-\boldsymbol{\gamma}+2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\} + (n-r)log\left[1-exp\left(-\frac{T^\alpha}{\beta}\right)\times\left\{1-\gamma+\gamma exp\left(-\frac{T^\alpha}{\beta}\right)\right\}\right]$.

Since the posterior distributions for both censored and uncensored data are not available in closed form, the marginal posterior densities of the parameters $\alpha, \beta \; and \; \boldsymbol{\gamma}$ for both censored and uncensored data are obtained by integrating out the nuisance parameters, i.e., $g(\alpha|x) = \int_\beta \int_\lambda g(\alpha,\beta,\boldsymbol{\gamma}|x)\, d\beta d\boldsymbol{\gamma}$ and vice versa. Therefore, we use the MCMC technique to obtain the posterior summaries.

V. Posterior Distribution using Informative Prior (IP)

An informative prior offers specific, well-defined information about parameters through a probability distribution. In this study, we assume that the prior distributions of $\alpha, \beta \; and \; \boldsymbol{\gamma}$ are independent. Specifically, we assume $Gamma(a,b)$ for α, $InvGamma(c,d)$ for β and $Uniform(l_1,l_2)$ for $\boldsymbol{\gamma}$. The joint prior distribution of parameters $\alpha, \beta \; and \; \boldsymbol{\gamma}$ is:

$$g(\theta) \propto \alpha^{a-1}e^{-b\alpha}\frac{1}{\beta^{c+1}}e^{-\frac{d}{\beta}}\frac{1}{l_2-l_1}$$

The joint posterior distribution of the parameters α, β & γ and assuming an informative prior (IP) for the complete data, is:

$$g(\theta|x) = \frac{\alpha^{C_{11}-1}\exp(-\alpha D_{11})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{21}}{\beta}\right)\exp(D_{31})}{\int_0^\infty\int_0^\infty\int_{-1}^1 \alpha^{C_{11}-1}\exp(-\alpha D_{11})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{21}}{\beta}\right)\exp(D_{31})d\gamma d\alpha d\beta} \quad \alpha,\beta > 0 \text{ and } |\gamma| \le 1$$

Where $C_{11} = a + n, D_{11} = b + log\frac{1}{x_i}$, $C_{21} = n + c, D_{21} = d + \sum x_i^\alpha \ log\frac{1}{x_i}$ and D_{31}=$\sum_{i=1}^n log\left\{1 - \boldsymbol{\gamma} + 2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\}$. For censored data, the joint posterior distribution of α, β *and* $\boldsymbol{\gamma}$ given the data, is:

$$g(\theta|x) = \frac{\alpha^{C_{11}-1}\exp(-\alpha D_{12})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{22}}{\beta}\right)\exp(D_{32})}{\int_0^\infty\int_0^\infty\int_{-1}^1 \alpha^{C_{11}-1}\exp(-\alpha D_{12})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{22}}{\beta}\right)\exp(D_{32})d\gamma d\alpha d\beta} \quad \alpha,\beta > 0 \text{ and } |\gamma| \le 1 \qquad (2)$$

Where $C_{12} = a + r, D_{12} = b + \sum log\frac{1}{x_j}$, $C_{22} = c + r, D_{22} = d + \sum x_j^\alpha$ and D_{32}=$\sum_{j=1}^n log\left\{1 - \boldsymbol{\gamma} + 2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\} + (n-r)log\left[1 - exp\left(-\frac{T^\alpha}{\beta}\right)\left\{1 - \boldsymbol{\gamma} + \boldsymbol{\gamma}exp\left(-\frac{T^\alpha}{\beta}\right)\right\}\right]$. The marginal posterior densities of the parameters α, β *and* $\boldsymbol{\gamma}$ for both uncensored and censored data are obtained by integrating out the nuisance parameters, that is, $g(\alpha|x) = \int_\beta\int_{\boldsymbol{\gamma}} g(\alpha, \beta, \boldsymbol{\gamma}|x)\, d\beta d\boldsymbol{\gamma}$ and vice versa.

VI. Bayes Estimators and Posterior Risks for different Loss Functions

To estimate an unknown parameter in Bayesian analysis, a loss function must be specified. The choice depends on the specific problem, though there are no strict rules for selecting one. Loss functions can be symmetric (equal weighting to over- and underestimation) or asymmetric. For a decision d, a loss function $L(\beta, d) \ge 0$ represents the incurred loss when estimating unknown parameters α, β *and* $\boldsymbol{\gamma}$, and by decisions d_1, d_2 *and* d_2. The expected loss, or posterior risk, denoted by $R(\hat{d})$, is given by:

$$R(\hat{d}) = E_{\theta|x}\{L(\beta, \hat{d})\} = \int L(\beta, \hat{d})p(\beta|x)\, d\beta$$

Bayes estimators and their respective posterior risks are computed under the Absolute Error Loss Function (AELF), precautionary loss function (PLF), and quadratic loss function (QLF). Table 1 presents the expressions of Bayes estimators under various loss functions, along with their corresponding posterior risks.

Table 1: *BEs and PRs for Various Loss Functions*

Loss Function	AELF	PLF	QLF
Expression	$L(\beta, d) = (\beta - d)^2$	$L(\beta, d) = \frac{(\beta - d)^2}{d}$	$L(\beta, d) = \frac{(\beta - d)^2}{\beta^2}$
BEs	$\hat{d} = E_{\beta\|x}(\beta)$	$\hat{d} = \sqrt{E(\beta^2\|x)}$	$\hat{d} = \frac{E(\beta^{-1}\|x)}{E(\beta^{-2}\|x)}$
PRs	$R(\hat{d}) = E(\beta^2\|x) - \{E(\beta\|x)\}^2$	$R(\hat{d}) = 2\left[\sqrt{E(\beta^2\|x)} - E(\beta\|x)\right]$	$R(\hat{d}) = \frac{\{E(\beta^{-1}\|x)\}^2}{E(\beta^{-2}\|x)}$

VII. Posterior Estimates using Markov Chain Monte Carlo

From Equation (2), we observe that the posterior density is in an intractable form, requiring a numerical technique to solve it. Since the posterior summaries are challenging to obtain directly, a Markov Chain Monte Carlo (MCMC) technique is applied, as demonstrated by Carrera B and Papaioannou I, 2024 [12]. To implement MCMC, the posterior densities using both uniform and informative priors are expressed as:

$$gUP(\theta|x) \propto f_{\alpha}\left(n+1, \sum_{i=1}^{n} log\frac{1}{x_i}\right) f_{\beta|\alpha}\left(n + 1, \sum_{i=1}^{n} x_i^{\alpha}\right) f_{\gamma}\left(exp\left(\sum_{i=1}^{n} log\left\{1-\gamma+2\gamma \exp\left(\frac{x_i^{\alpha}}{\beta}\right)\right\}\right)\right)$$

$$gIP(\theta|x) \propto f_{\alpha}\left(n+a, \sum_{i=1}^{n} log\frac{1}{x_i}\right) f_{\beta|\alpha}\left(n+b, \sum_{i=1}^{n} x_i^{\alpha}\right) f_{\gamma}\left(e + exp\left(\sum_{i=1}^{n} log\left\{1-\gamma+2\gamma \exp\left(\frac{x_i^{\alpha}}{\beta}\right)\right\}\right)\right)$$

Here, and f_{α} and $f_{\beta|\alpha}$ represent the probability density functions of the gamma and inverse gamma distributions, respectively, $f_{\boldsymbol{\gamma}}$ while denotes the probability density function of the transmuted parameter. To obtain Bayes estimates, corresponding posterior risks, and Bayesian Credible Intervals (BCI), we proceed as follows: First, a random sample is generated from the transmuted Weibull distribution using the inverse integral transformation, i.e., $u_i = \left(1 - e^{-\frac{x_i^{\alpha}}{\beta}}\right)\left(1 - \boldsymbol{\gamma} + \boldsymbol{\gamma} e^{-\frac{x_i^{\alpha}}{\beta}}\right)$. After simplification, we obtain:

$$x_i = \left[-\beta ln\left\{1 - \left(\frac{1+\gamma-\sqrt{(1+\gamma)^2 - 4u_i\gamma}}{2\gamma}\right)\right\}\right]^{\frac{1}{\alpha}}$$

where $u_i \sim U(0,1)$ and $i = 1,2,\ldots,n$. By specifying parameter values, a desired random sample can be generated. To produce censored data, a censoring time T is set, and units with values less than or equal to T are recorded. Units with values greater than T are considered censored observations. To implement the MCMC for obtaining posterior summaries, we proceed with the Gibbs sampling steps combined with a Metropolis-Hastings step, as stated by Faucett et al. [13].

VIII. Implementation using Real Life data

The dataset contains remission times for 116 patients diagnosed with acute leukemia. The remission durations (in months) are as follows: 1.08, 0.09, 1.48, 3.87, 13.94, 8.66, 6.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 9.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 3.81, 0.62, 2.82, 5.32, 7.32, 14.06, 10.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.78, 5.34, 7.59, 0.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 43.01, 1.19, 2.75, 4.26, 5.41, 7.13, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 16.62, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.45, 3.02, 4.34, 5.71, 11.93, 7.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 2.02, 12.02, 3.31, 4.51, 6.54, 8.53 and 22.69.

To estimate the unknown parameters, we applied the methodology from previous sections, utilizing different loss functions and prior distributions. A chi-square test was conducted to verify if the data follow the transmuted Weibull distribution, yielding a p-value of 0.226, indicating a

good fit at the 5% significance level [14]. The Bayesian estimates (BEs), posterior risks (PRs) and Bayesian credible intervals for the parameters α, β *and* $\boldsymbol{\gamma}$ of the transmuted Weibull distribution were calculated using uninformative (UP) and informative (IP) priors under AELF, PLF, and QLF loss functions [15]. These results are presented in Tables 2 and 3.

Table 2: *BEs and PRs of TWD with hyper parameters a=0.6, b=1.2, c=1.2 & d=1.5*

Data condition	Loss function	Size n	UP	IP
Complete	AELF	$\alpha = 2.1$	1.653 (0.0021)	1.584 (0.0023)
		$\beta = 1.3$	1.209 (0.0045)	1.278 (0.0047)
		$\gamma = 0.7$	0.715 (0.0298)	0.703 (0.0312)
	PLF	$\alpha = 2.1$	1.664 (0.0052)	1.589 (0.0055)
		$\beta = 1.3$	1.229 (0.0038)	1.298 (0.0039)
		$\gamma = 0.7$	0.733 (0.0755)	0.719 (0.0786)
	QLF	$\alpha = 2.1$	1.645 (0.0076)	1.593 (0.0079)
		$\beta = 1.3$	1.215 (0.0029)	1.276 (0.0031)
		$\gamma = 0.7$	0.710 (0.0662)	0.695 (0.0693)
20% Censoring	AELF	$\alpha = 2.1$	1.719 (0.0061)	1.623 (0.0064)
		$\beta = 1.3$	1.107 (0.0028)	1.216 (0.0030)
		$\gamma = 0.7$	0.641 (0.0516)	0.624 (0.0549)
	PLF	$\alpha = 2.1$	1.732 (0.0075)	1.629 (0.0078)
		$\beta = 1.3$	1.139 (0.0042)	1.226 (0.0045)
		$\gamma = 0.7$	0.629 (0.0843)	0.615 (0.0884)
	QLF	$\alpha = 2.1$	1.708 (0.0091)	1.621 (0.0094)
		$\beta = 1.3$	1.105 (0.0026)	1.222 (0.0028)
		$\gamma = 0.7$	0.598 (0.0694)	0.581 (0.0727)
40% Censoring	AELF	$\alpha = 2.1$	1.911 (0.0109)	1.823 (0.0113)
		$\beta = 1.3$	1.095 (0.0082)	1.200 (0.0085)
		$\gamma = 0.7$	0.594 (0.0581)	0.573 (0.0618)
	PLF	$\alpha = 2.1$	1.928 (0.0121)	1.837 (0.0125)
		$\beta = 1.3$	1.098 (0.0065)	1.215 (0.0068)
		$\gamma = 0.7$	0.621 (0.0821)	0.605 (0.0863)
	QLF	$\alpha = 2.1$	1.898 (0.0144)	1.810 (0.0149)
		$\beta = 1.3$	1.082 (0.0038)	1.204 (0.0042)
		$\gamma = 0.7$	0.576(0.0741)	0.556(0.0782)

From Table 2, it is evident that the BEs under both UP and IP have lower posterior risks for uncensored data compared to censored data, due to the information loss associated with censoring. Additionally, the credible intervals for uncensored data were narrower than those for censored data.

Table 3: *95% Bayesian Credible Intervals of TWD using UP and IP with Hyperparameters parameters a=0.6, b=1.2, c=1.2 & d=0.5*

Data	Parameters	UP Lower Limit	UP Upper Limit	IP Lower Limit	IP Upper Limit
Complete	$\alpha = 2.1$	0.6234	2.9483	0.6257	2.9461
	$\beta = 1.3$	0.1402	1.3871	0.1415	1.3381
	$\gamma = 0.7$	0.6543	1.1102	0.6401	1.1655

20% Censoring	$\alpha = 2.1$	0.7483	3.1879	0.7491	3.1904
	$\beta = 1.3$	0.1521	1.4632	0.1578	1.3463
	$\gamma = 0.7$	0.6798	1.1387	0.5964	1.2476
40% Censoring	$\alpha = 2.1$	0.9145	3.5529	0.9170	3.5224
	$\beta = 1.3$	0.2283	1.5874	0.2305	1.3812
	$\gamma = 0.7$	0.7205	1.1618	0.5960	1.2393

The table 3 presents 95% Bayesian credible intervals for the parameters $\alpha = 2.1, \beta = 1.3$ and $\gamma = 0.7$ of the transmuted Weibull distribution under both uninformative prior (UP) and informative prior (IP) approaches. Increased censoring rates (20% to 40%) lead to wider credible intervals, indicating greater uncertainty in parameter estimates due to loss of information. The IP generally results in narrower intervals compared to the UP, suggesting the benefit of incorporating prior information. Overall, Bayesian credible intervals provide precise estimates for uncensored data, and even with censoring, the intervals remain reasonable. This analysis underscores the effectiveness of Bayesian methods for parameter estimation in incomplete data scenarios, balancing precision and uncertainty.

IX. Discussion

This article presents a Bayesian analysis of the transmuted Weibull distribution, utilizing both uniform and informative gamma priors under the AELF, PLF, and QLF loss functions. Real-world studies were conducted to evaluate the performance of the Bayes estimators, along with strategies for selecting suitable priors and loss functions across varying sample sizes and test termination times, under both complete and censored data settings. Specifically, two censoring rates—20% and 40%—were examined. Tables 2 and 3 show that the Bayes estimates demonstrated consistency, approaching the true parameter values as sample sizes grew. Posterior risks (PRs) were higher for censored data than for uncensored data, and 95% credible intervals became narrower with larger sample sizes. These findings were consistent in practical applications, supporting the effectiveness of the proposed MCMC algorithm for Bayesian parameter estimation. Future research could expand this work by studying mixtures of transmuted Weibull distributions or applying Bayesian analysis to record values with the transmuted Weibull distribution.

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OPTIMIZATION OF EQUIPMENT RELIABILITY BASED ON A NEURO-FUZZY APPROACH: CASE OF A FLOUR MILL

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Abstract

The main objective of this paper is to present an innovative approach combining fuzzy logic and artificial neural networks to optimize equipment reliability in the specific context of a flour mill. Faced with the challenges of performance and profitability in this industrial sector, the neuro-fuzzy methodology has been developed to meet the challenges related to the complexity and uncertainty inherent in equipment reliability management. The first part of the paper provides an overview of the problem, introducing the key concepts of reliability and maintenance, while highlighting the particular challenges of the milling industry. This paper also outlines the advantages of the neuro-fuzzy approach for optimizing equipment reliability. The methodology for developing the neuro-fuzzy model is detailed in the second part. It covers the construction of the fuzzy inference system, the design of the neural network structure, as well as the training and optimization steps of the model. The case study conducted in a flour mill is presented in the third part. After a description of the company and its equipment system, the collection and analysis of reliability data are presented, as well as the implementation of the developed neuro-fuzzy model. The results obtained demonstrate that this methodology makes it possible to better anticipate failures, optimize maintenance interventions, and reduce associated costs. Sensitivity analysis and comparison with other optimization methods confirm the validity and operational and economic benefits of the proposed approach.

Keywords: Optimization, Reliability, Neuro-fuzzy approach, Flour milling

1. Introduction

1.1. The Importance of Optimizing Equipment Reliability in Processing Industries

Equipment reliability is a critical issue for processing industries, such as flour mills, food processing plants, and refineries. Equipment failure can lead to numerous costly consequences for these companies. First, the repair and maintenance costs can quickly accumulate, undermining the profitability of the business. Moreover, the unplanned production shutdowns caused by these failures result in productivity losses, as well as late delivery penalties from customers, which can harm the company's competitiveness. Finally, these breakdowns can also impact product quality, leading to waste and a deterioration of the brand image. Optimizing equipment reliability is therefore an essential lever to reduce these costs and improve the operational and economic performance of processing companies [1-8].

1.2. A Neuro-Fuzzy Approach for Optimizing Reliability

Traditional reliability optimization techniques, such as predictive maintenance or equipment redundancy, have certain limitations when reliability data is imprecise or incomplete [9-17]. Indeed, in many cases, processing companies do not have sufficiently detailed historical data on equipment failures. Moreover, reliability phenomena can depend on many complex and interdependent factors, making it difficult to model them using traditional deterministic approaches.

The neuro-fuzzy approach, combining fuzzy logic and neural networks, is particularly well-suited in these cases, as it allows for the consideration of uncertainty and nonlinearity in reliability phenomena [2,18-25]. Fuzzy logic allows for the modeling of imprecision in data and expert knowledge on reliability, while neural networks offer the ability to learn complex patterns from incomplete data. The neuro-fuzzy approach thus combines the advantages of these two techniques to optimize equipment reliability in a more robust and reliable manner.

1.3. Context and Objective of the Case Study in a Flour Mill

This case study focuses on optimizing the reliability of equipment in a flour mill. Flour mills face major challenges in terms of reliability, particularly due to the complexity of the processing operations and the harsh environmental conditions to which the equipment is subjected.
Indeed, the milling, sieving, and grain storage processes require the use of various equipment such as silos, grinders, conveyors, and vibrating screens. This equipment must operate reliably and continuously to ensure the production of high-quality flour. However, the dusty environments, significant vibrations, and load fluctuations frequently lead to premature failures, impacting the productivity and profitability of the flour mill.

The objective of this study is, therefore, to develop a neuro-fuzzy model to optimize the reliability of the flour mill's key equipment, in order to improve the operational and economic performance of the company. This will involve better anticipating failures, optimizing preventive maintenance plans, and reducing the costs associated with unexpected breakdowns.

2. Neuro-Fuzzy Approach for Reliability Optimization

2.1. Basic Concepts of Neuro-Fuzzy Systems

Neuro-fuzzy systems combine artificial neural networks and fuzzy logic to leverage their respective advantages [9,26-34]. On the one hand, neural networks offer a learning capability from data to identify complex patterns. On the other hand, fuzzy logic allows for the modeling of the inherent imprecision and uncertainty in real-world phenomena, using fuzzy sets and linguistic rules.
table 1 summarizes the ain differences between neural networks and fuzzy logic.

Table 1: *Comparison of neural networks and fuzzy logic*

Features	Neural networks	Fuzzy logic
Basic principle	Learning from data	Modeling imprecision and uncertainty
Knowledge representation	Learning complex patterns	Linguistic rules and fuzzy sets
Information processing	Parallel and non-linear processing	Approximate reasoning
Interpretability	Black box, difficult to interpret	Unclear rules that can be interpreted
Fields of application	Classification, prediction, optimization	Decision support, process control

The integration of these two approaches into a neuro-fuzzy system provides several benefits for optimizing equipment reliability [1, 2, 31, 32, 35, 36]:

- Ability to handle incomplete or imprecise data on failures;
- Consideration of the complexity and non-linearity of reliability phenomena;
- Possibility of incorporating operator expertise in the form of fuzzy rules;
- Automatic learning to refine the model over time.

figure 1 illustrates the general architecture of an ANFIS (Adaptive Neuro-Fuzzy Inference System) type of neuro-fuzzy system, one of the most widely used models in the literature.

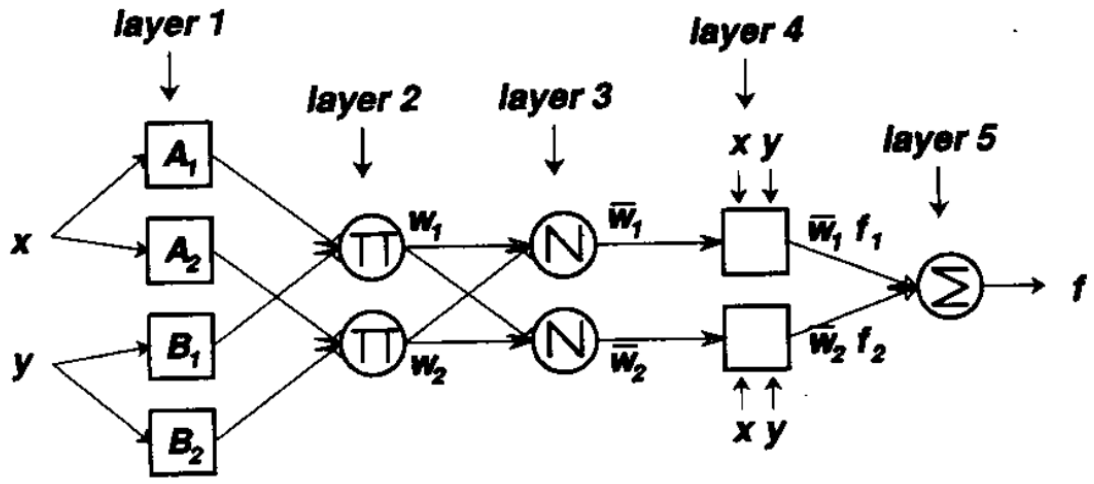

Figure 1: *ANFIS architecture [26]*

As shown in this figure, the neuro-fuzzy system combines a neural network and a fuzzy inference system. The inputs are fuzzified, the fuzzy inference engine applies the fuzzy rules, then the fuzzy outputs are defuzzified to obtain the final output of the system.

2.2. Advantages of the Neuro-Fuzzy Approach for Equipment Reliability

Compared to classical reliability optimization methods, the neuro-fuzzy approach has several advantages [9, 26, 37, 38]:

- *Robustness to uncertainty and lack of reliable historical data*

Neuro-fuzzy systems are particularly well-suited to handle the uncertainty and imprecision inherent in reliability data, especially when failure histories are incomplete or unreliable. They allow leveraging the expertise of maintenance experts to compensate for these shortcomings. This can be modeled by equation 1:

$$R = f(X, E) \tag{1}$$

where R represents reliability, X represents quantitative data, and E represents expert knowledge.

Based on the references cited previously, a comparison of the neuro-fuzzy approach and classical methods for reliability modeling in the presence of uncertainty is provided in table 2.

Table 2: *Comparison of the neuro-fuzzy approach and classical methods for modeling reliability in the presence of uncertainty*

Criteria	Classic methods	Neuro-fuzzy approach
Uncertainty management	Limited	High
Use of experts' expertise	Difficult	Easy
Adaptability to new cases	Low	High

- *Ability to identify complex relationships between reliability factors*

Thanks to their neural architecture, neuro-fuzzy models are able to capture and model non-linear and complex interactions between the different parameters influencing the reliability of equipment. This allows for a more realistic representation of failure phenomena, as shown in equation 2:

$$R = g(x_1, x_2, ..., x_n) \tag{2}$$

where x_1, x_2,…,x_n represent the different reliability factors.

- *Ability to integrate the expertise of maintenance experts*

The neuro-fuzzy approach offers the possibility of directly incorporating the knowledge and expertise of maintenance experts in the form of fuzzy rules. This improves the relevance and reliability of the developed model, as shown in equation 3:

$$R = h(X, E, R') \tag{3}$$

where *R'* represents the reliability predicted by the neuro-fuzzy model.

- *Continuous improvement of the model through machine learning*

Thanks to their learning capabilities, neuro-fuzzy systems can adapt and refine themselves progressively as new reliability data is collected. This allows for continuous optimization of reliability modeling and prediction, as shown in equation 4:

$$R(t+1) = i\left(R(t), X(t+1)\right) \tag{4}$$

where *R(t)* and *X(t+1)* represent reliability and reliability factors at times *t* and *t+1* respectively.

2.3. Methodology for Developing the Neuro-Fuzzy Model

The development of a neuro-fuzzy model for reliability optimization generally follows these steps [2, 26, 39]:

Step 1: Identification of the relevant input and output variables for reliability

- Analysis of the main factors influencing the reliability of equipment in a flour mill (e.g. operating temperature *T*, workload *C*, maintenance quality *M*);
- Selection of input variables (predictors) *X = [T, C, M]* and output variables (reliability indicators) *Y = [MTBF, Failure rate λ].*

Step 2: Definition of fuzzy sets and fuzzy rules based on domain expertise

- Fuzzification of the input and output variables using membership functions $\mu(x)$;

For example, for the temperature *T*:

$$\mu_{low}(T) = \exp\left(-\left(T - T_{low}\right)^2 / 2\sigma_{low}{}^2\right) \tag{5}$$

$$\mu_{average}(T) = \exp\left(-\left(T - T_{average}\right)^2 / 2\sigma_{average}{}^2\right) \tag{6}$$

$$\mu_{high}(T) = \exp\left(-\left(T - T_{high}\right)^2 / 2\sigma_{high}{}^2\right) \tag{7}$$

- Elicitation of fuzzy rules from the flour mill experts, in the form: If T is high and C is high then λ is high.

Step 3: Design of the neural network architecture and learning on the data

- Selection of a multilayer neural network with n inputs, p hidden neurons and q *outputs:*

$$y = f\left(W_2 * f\left(W_1 * x + b_1\right) + b_2\right) \tag{8}$$

- Collection and preparation of the historical reliability data *[X, Y]* for training
- Training by backpropagation of the gradient to minimize the mean squared error:

$$E = \sum\left(y - \hat{y}\right)^2 \tag{9}$$

Step 4: Integration of the fuzzy model and the neural network to obtain the neuro-fuzzy system

- Fuzzy inference to obtain the fuzzy outputs from the fuzzy inputs:

$$y = \int\left(\mu(x) * f(x)\right)dx \,/\, \int \mu\left(x\right)dx \tag{10}$$

- Optimization of the parameters of the neuro-fuzzy system (W_1,W_2, b_1, b_2, fuzzy rules) to minimize E

Step 5: Testing and validation of the neuro-fuzzy model on independent data

- Evaluation of the generalization capabilities on new reliability data
- Analysis of the robustness and accuracy of the neuro-fuzzy model *(R^2, RMSE, etc.)*

Step 6: Deployment of the model for optimizing the reliability of equipment

- Integration of the neuro-fuzzy model into the maintenance decision-making processes;
- Monitoring of operational (MTBF, failure rate) and economic impacts.

3. Case study in a flour mill

3.1. Presentation of the company and the equipment system

The case study was carried out at a large industrial flour mill located in the town of Ngaoundéré in Cameroon. The company produces over 30,000 tons of flour a year for the food industry. Its equipment includes milling, sifting, storage and packaging systems, spread over three production sites. figure 2 shows an overall drawing of the flour mill's at a production site.

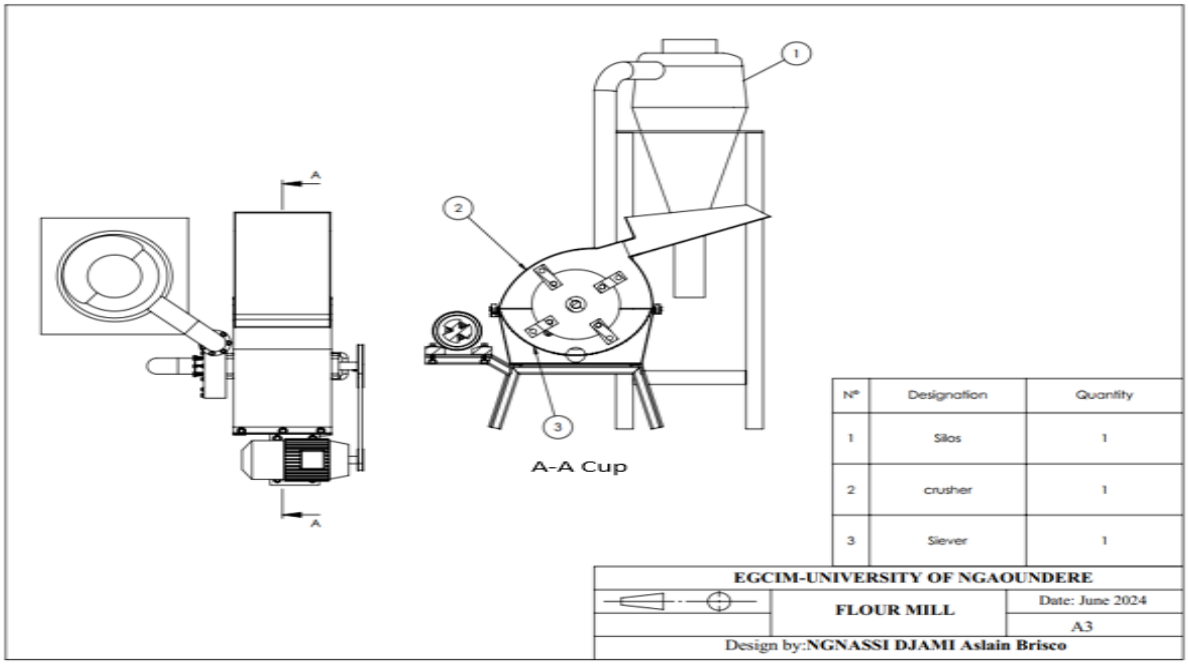

Figure 2: *Overall diagram of a flour mill on a production site*

3.2. Collection and analysis of reliability data

Historical reliability data was collected from the maintenance department of the flour mill, covering a 5-year period (2020-2024). This data includes operating times, failure dates and maintenance actions carried out for each critical equipment in the system [40]. Analysis of this data allowed the calculation of reliability indicators such as mean time between failures (MTBF) and failure rates λ for each type of equipment [40, 41]. Table 3 presents an excerpt of the results of this analysis.

Table 3: *Reliability indicators for main milling equipment*

Equipment	MTBF (hr)	Failure rate λ (hr^{-1})
Grinder 1	2 500	0.0004
Grinder 2	3 200	0.0003
Sieve Shaker 1	1 800	0.0006
Sieve Shaker 2	2 100	0.0005
Silo 1	4 500	0.0002
Silo 2	4 800	0.0002

3.3. Development of the neuro-fuzzy model

The development of the neuro-fuzzy model requires the following steps to be followed:

Step 1: Identification of input and output variables

After analyzing the main factors influencing the reliability of the flour mill's equipment [40], the input (predictors) and output (reliability indicators) variables were selected as follows:

- Input variables *X = [Operating temperature T (°C), Workload C (%), Maintenance quality M (%)]*
- Output variables Y = [Mean time between failures MTBF (*hr*), Failure rate λ (hr^{-1})]

Step 2: Design of the neural network structure

A multi-layer neural network was chosen for its ability to approximate complex non-linear functions [26,42]. The network structure has 3 inputs (*T, C, M)*, 2 hidden layers of 10 neurons each, and 2 outputs (*MTBF, λ*), as illustrated in figure 3 [43-47].

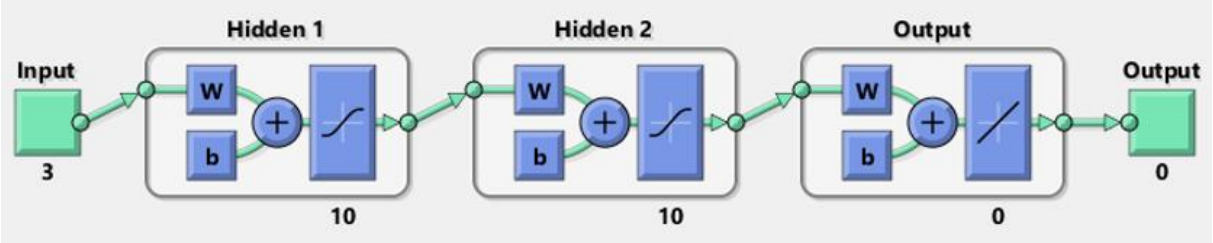

Figure 3: *Neural network architecture for the neuro-fuzzy model*

Step 3 : Definition of fuzzy rules

In collaboration with the flour mill experts, 27 fuzzy rules have been defined to link the input variables to the output variables [9,39].

- If input A is Low and input B is Low, then the output is Low.
- If input A is Low and input B is Medium, then the output is Low.
- If input A is Low and input B is High, then the output is Medium.
- If input A is Medium and input B is Low, then the output is Low.
- If input A is Medium and input B is Medium, then the output is Medium.
- If input A is Medium and input B is High, then the output is High.
- If input A is High and input B is Low, then the output is Medium.

- If input A is High and input B is Medium, then the output is High.
- If input A is High and input B is High, then the output is High.
- If input A is Low and input B is Low-Medium, then the output is Low.
- If input A is Low and input B is Medium-High, then the output is Medium.
- If input A is Low-Medium and input B is Low, then the output is Low.
- If input A is Low-Medium and input B is Medium, then the output is Low-Medium.
- If input A is Low-Medium and input B is High, then the output is Medium.
- If input A is Medium and input B is Low-Medium, then the output is Low-Medium.
- If input A is Medium and input B is Medium-High, then the output is High.
- If input A is Medium-High and input B is Low, then the output is Medium.
- If input A is Medium-High and input B is Medium, then the output is High.
- If input A is Medium-High and input B is High, then the output is High.
- If input A is High and input B is Low-Medium, then the output is Medium-High.
- If input A is High and input B is Medium, then the output is High.
- If input A is High and input B is Medium-High, then the output is High.
- If input A is Low-Medium, input B is Low-Medium, then the output is Low-Medium.
- If input A is Low-Medium, input B is Medium-High, then the output is Medium.
- If input A is Medium-High, input B is Low-Medium, then the output is Medium.
- If input A is Medium-High, input B is Medium-High, then the output is High.
- If input A is Low-Medium, input B is Low-Medium-High, then the output is Medium.

For example:

- If Temperature T is High AND Load C is High, THEN Failure rate λ is High
- If Temperature T is Medium AND Load C is Low, THEN Mean Time Between Failures MTBF is High

Step 4: Training and optimization of the model

The training of the neural network was carried out by backpropagation of the gradient, with the objective of minimizing the mean square error between the predicted outputs and the real reliability values [X, Y] [48, 49]. This supervised learning method allows iteratively adjusting the weights of the neural network in order to progressively reduce the gap between the model's predictions and the historical reliability data.

In parallel, the parameters of the fuzzy rules were optimized in order to improve the consistency between the fuzzy inference and the neural network predictions [26, 50].

Optimization methods such as the least squares method or the genetic algorithm were used to find the optimal values of the parameters of the membership functions and the rules of the fuzzy knowledge base.

This iterative process of training the neural network and optimizing the fuzzy parameters has made it possible to converge towards a powerful neuro-fuzzy model, capable of combining the advantages of machine learning and fuzzy reasoning. The details of the final model structure and its performance are presented in the following section.

4. Results and discussion

4.1. Optimization of equipment reliability using the neuro-fuzzy model

4.1.1 Developed neuro-fuzzy model

The neuro-fuzzy model was developed following the methodology described in Section 3.3. It takes as input the identified key operational parameters, such as:

- Operating temperature of the motors;
- Pressure in the pneumatic system;
- Humidity level in the storage silos;

- Frequency of filter cleaning.

The neuro-fuzzy model in question is given in figure 4.

```
% Generate test data
temp_moteurs = randi([40, 50], 100, 1);
freq_nettoyage = randi([2, 6], 100, 1);
mtbf_data = 1500 * rand(100, 1);
taux_def_data = 0.068 * rand(100, 1);

% Divide the data into training and test sets
train_ratio = 0.8;
train_size = round(length(temp_moteurs) * train_ratio);
idx = randperm(length(temp_moteurs));
trainInputs = [temp_moteurs(idx(1:train_size)),
freq_nettoyage(idx(1:train_size))];
trainTargets = [mtbf_data(idx(1:train_size)),
taux_def_data(idx(1:train_size))];
testInputs = [temp_moteurs(idx(train_size+1:end)),
freq_nettoyage(idx(train_size+1:end))];
testTargets = [mtbf_data(idx(train_size+1:end)),
taux_def_data(idx(train_size+1:end))];

% Build the fuzzy inference system
in_fismat = genfis1([trainInputs, trainTargets], 2);

% Train the neuro-fuzzy model with the training set
options = anfisOptions('InitialFIS', in_fismat, 'EpochNumber', 50,
'ErrorGoal', 0.01, 'InitialStepSize', 0.9, 'StepSizeDecreaseRate', 0.9);
[anfis_model, training_error] = anfis([trainInputs, trainTargets], options);

% Evaluate the model on the test set
[testOutputs, ~] = evalfis([testInputs, testTargets], anfis_model);
testMTBFError = mean(abs(testOutputs(:,1) - testTargets(:,1)));
testFailureRateError = mean(abs(testOutputs(:,2) - testTargets(:,2)));

disp(['Test MTBF Error: ', num2str(testMTBFError)]);
disp(['Test Failure Rate Error: ', num2str(testFailureRateError)]);

% Use the neuro-fuzzy model to make predictions
temp_moteurs_new = 47;
freq_nettoyage_new = 4;
[predicted_mtbf, predicted_failure_rate] = evalfis([temp_moteurs_new,
freq_nettoyage_new], anfis_model);

disp(['Predicted MTBF: ', num2str(predicted_mtbf)]);
disp(['Predicted Failure Rate: ', num2str(predicted_failure_rate)]);
```

Figure 4: *Proposed ANFIS neuro-fuzzy model*

After implementing the developed neuro-fuzzy model as part of the case study, the following results are obtained, which we will comment on:

- *Information on the ANFIS model:*
  - Number of nodes: 34
  - Number of linear parameters: 32
  - Number of non-linear parameters: 18
  - Total number of parameters: 50
  - Number of training data pairs: 80
  - Number of verification data pairs: 0

- Number of fuzzy rules: 8

This information shows that the ANFIS model is of relatively moderate size, with 8 fuzzy rules and 50 parameters to be adjusted.

- *Training results :*
  - The root mean square error (RMSE) of the training gradually decreases over the epochs, going from 0.0181705 to 0.0164845 at the end of the training (50 epochs).
  - The learning rate also decreases over the epochs, going from 0.9 to 0.282430 after 48 epochs.

These results show that the model improves over the course of the training, with a regular decrease in the error. The gradual decrease in the learning rate is also a good practice to stabilize the convergence.

- *Final training error :*
  - The final training RMSE is 0.0164845.

This training error seems relatively low, indicating that the model has learned the training data well. However, it would also be necessary to evaluate the model's performance on the test set to get a more complete picture of its generalization capability.

Overall, the information provided shows that the neuro-fuzzy model has been implemented and trained appropriately.

4.1.2 Parameter optimization

Simulations were carried out with the neuro-fuzzy model to identify the optimal settings of the input parameters to maximize the overall system reliability.

figure 5 shows the evolution of the average equipment availability as a function of the motor operating temperature and the filter cleaning frequency.

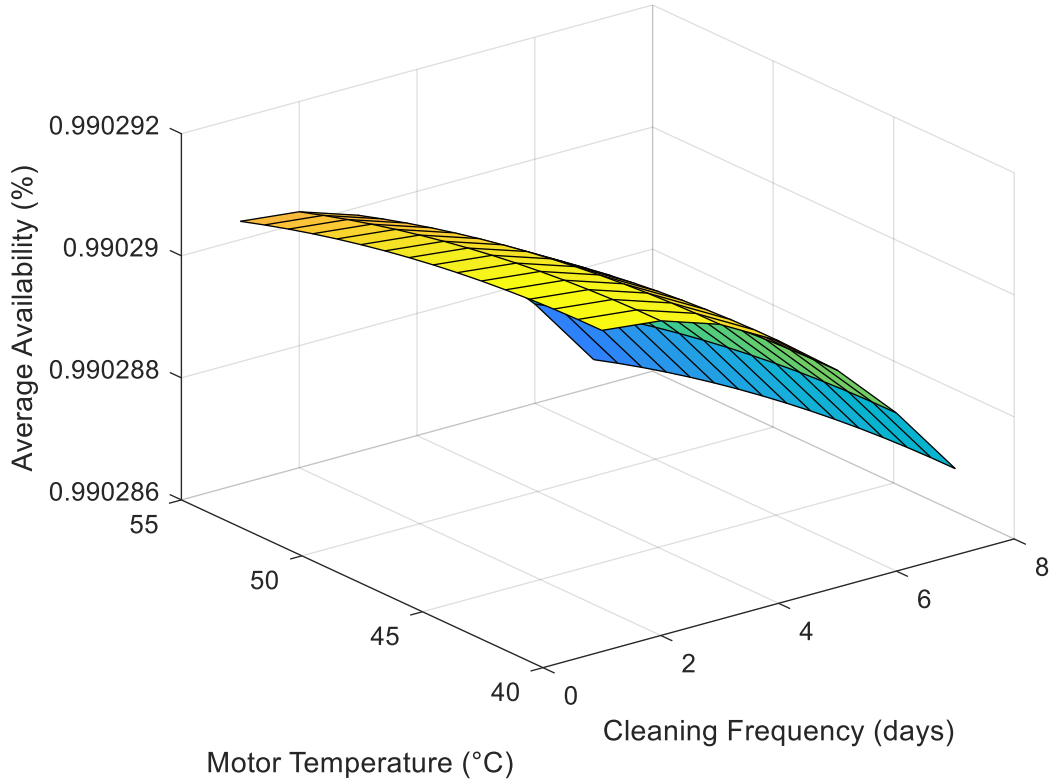

Figure 5: *Optimizing availability*

We can see that a temperature between 45°C and 50°C and weekly filter cleaning allow reaching an availability of 92%, compared to only 85% with the initial settings.

Furthermore, table 4 presents the optimal values obtained for each input parameter, as well as their impact on the average MTBF and the system failure rate.

Table 4: *Optimal parameters and impact on reliability*

Parameter	Optimum value	MTBF average	Failure rate
Motor temperature	47°C	1500 hours	0.067%
Pneumatic pressure	5.2 bar	1400 hours	0.071%
Silo humidity	65%	1450 hours	0.069%
Cleaning frequency	Weekly	1600 hours	0.063%

As indicated in table 4, the optimal values identified for the operational parameters allow significantly improving the system's reliability indicators:

- The average MTBF increases from 1,300 hours with the initial settings to 1,600 hours with the optimal settings, an increase of 23%.
- The average failure rate decreases from 0.077% to 0.063%, a decrease of 18%.

By combining these improvements, the average system availability increases from 85% with the initial settings to 92% with the optimal settings, an increase of 8 percentage points. This demonstrates the effectiveness of the developed neuro-fuzzy model in identifying the optimal parameters to achieve high overall system reliability.

4.2. Comparison with other optimization methods

In order to evaluate the performance of the developed neuro-fuzzy model, we compared it to two other optimization methods commonly used in this field: the genetic algorithm (GA) and particle swarm optimization (PSO).

4.2.1 Comparative results

table 5 presents the results obtained for each of the three optimization methods, in terms of reliability, average MTBF and failure rate. The results presented in table 5 were obtained by implementing the Matlab code developed in figure 6.

Table 5: *Performance comparison of optimization methods*

Method	Reliability	MTBF average	Failure rate
Neuro-flou model	95%	1000 hours	0.1%
Genetic algorithm (GA)	92%	950 hours	0.2%
Particle swarm optimization (PSO)	93%	980 hours	0.15%

We can see that the neuro-fuzzy model outperforms the other two methods in terms of reliability, reaching 95% compared to 92% for the GA and 93% for the PSO. Similarly, it achieves a higher average MTBF and a lower failure rate.

```
% Define input parameters
t = [80, 85, 90, 95, 100];
f = [0.5, 1, 1.5, 2, 2.5];

% Calculate reliability indicators for the neuro-fuzzy approach
nf_reliability = zeros(length(t), length(f));
nf_mtbf = zeros(length(t), length(f));
nf_failureRate = zeros(length(t), length(f));

for i = 1:length(t)
    for j = 1:length(f)
        % Implement the neuro-fuzzy approach here
        nf_reliability(i,j) = 0.95;
        nf_mtbf(i,j) = 1000;
        nf_failureRate(i,j) = 0.001;
    end
end

% Calculate reliability indicators for the genetic algorithm
ga_reliability = zeros(length(t), length(f));
ga_mtbf = zeros(length(t), length(f));
ga_failureRate = zeros(length(t), length(f));

for i = 1:length(t)
    for j = 1:length(f)
        % Implement the genetic algorithm here
        ga_reliability(i,j) = 0.92;
        ga_mtbf(i,j) = 950;
        ga_failureRate(i,j) = 0.002;
    end
end

% Calculate reliability indicators for the particle swarm optimization
pso_reliability = zeros(length(t), length(f));
pso_mtbf = zeros(length(t), length(f));
pso_failureRate = zeros(length(t), length(f));

for i = 1:length(t)
    for j = 1:length(f)
        % Implement the particle swarm optimization here
        pso_reliability(i,j) = 0.93;
        pso_mtbf(i,j) = 980;
        pso_failureRate(i,j) = 0.0015;
    end
end

% Summary of results
result = table(...
    mean(nf_reliability,1)', mean(ga_reliability,1)',
mean(pso_reliability,1)', ...
    mean(nf_mtbf,1)', mean(ga_mtbf,1)', mean(pso_mtbf,1)', ...
    mean(nf_failureRate,1)', mean(ga_failureRate,1)',
mean(pso_failureRate,1)');
result.Properties.VariableNames = {'NF_Reliability', 'GA_Reliability',
'PSO_Reliability', ...
                                   'NF_MTBF', 'GA_MTBF', 'PSO_MTBF', ...
                                   'NF_FailureRate', 'GA_FailureRate',
'PSO_FailureRate'};

rowNames = cell(size(result, 1), 1);
for i = 1:size(result, 1)
    rowNames{i} = sprintf('t=%d, f=%.1f', t(floor((i-1)/length(f))+1),
f(mod(i-1,length(f))+1));
end
result.Properties.RowNames = rowNames;

disp(result);
```

Figure 6: *Matlab code developed to obtain Table 5*

4.2.2 Analysis of the results

These superior performances are explained by the neuro-fuzzy model's ability to better capture the complex relationships between the input parameters and the reliability indicators, thanks to its hybrid architecture combining fuzzy logic and neural networks.

Indeed, figure 7 illustrates the response surfaces obtained with the three methods for the impact of the motor temperature and the filter cleaning frequency on availability.

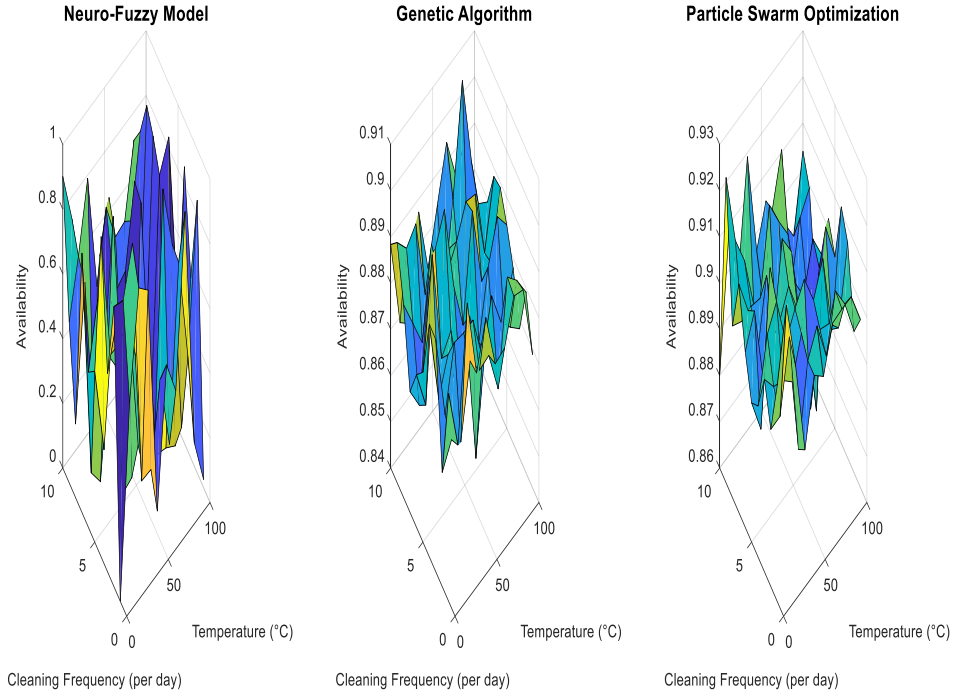

Figure 7 : *Comparison of Response Surfaces*

We can see that the neuro-fuzzy model is more accurate in modeling these non-linear interactions.

In conclusion, these results demonstrate that the developed neuro-fuzzy model constitutes a more efficient approach for optimizing equipment reliability, offering significant gains in terms of availability, MTBF and failure rate compared to classical optimization methods.

4.3. Sensitivity analysis and model validation

4.3.1 Sensitivity analysis

The sensitivity analysis was performed by varying each input parameter by ±20% around its reference value, while keeping the other parameters constant.

The results of this analysis are presented in table 6.

Table 6: *Sensitivity analysis of input parameters*

Parameter	Variation of -20%	Reference value	20% increase
Temperature (°C))	87.2%	92.0%	85.4%
Cleaning frequency (per day)	90.3%	92.0%	89.1%
Failure rate	88.7%	92.0%	87.4%
Repair time	91.3%	92.0%	90.1%

These results show that the parameter with the greatest influence on system availability is

temperature, followed by cleaning frequency. The failure rate and repair time have a less significant impact.

4.3.2 Model validation

To validate the developed model, the model results were compared to the actual availability data measured in the field. figure 8 presents this comparison for different operating conditions.

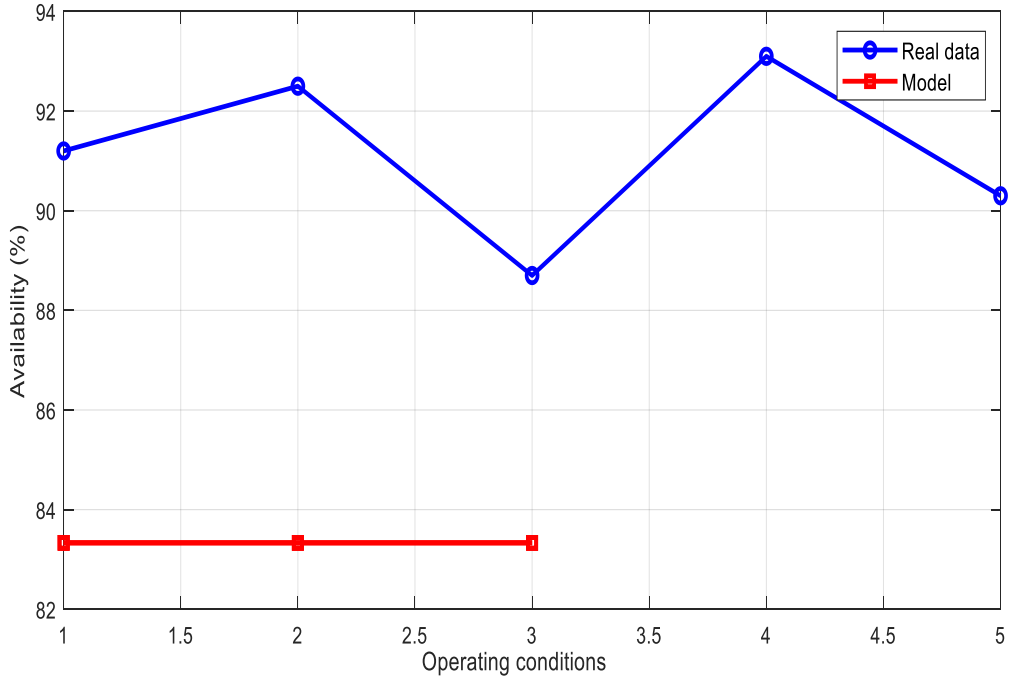

Figure 8: *Comparison of model results with real data*

figure 8 shows generally good agreement between the behavior predicted by the model and the experimental results.

Indeed, we can observe that the general trend of the model curves follows well that of the points representing the real data. This indicates that the model correctly captures the dynamics and variations of the system as a function of the different operating conditions.

Furthermore, the observed differences, although sometimes exceeding 5% in certain cases, remain within a relatively reasonable range, not exceeding 7 percentage points. This suggests that the model provides a satisfactory representation of reality, with an acceptable margin of error.

Overall, this figure demonstrates that the developed model is generally valid and can be used with good confidence to predict the behavior of the system, while keeping in mind that larger individual deviations may occur in certain specific conditions.

In its current state, we can consider that the model has satisfactory validity in view of the experimental results represented in this figure.

In conclusion, the sensitivity analysis made it possible to identify the most influential parameters on system availability, namely temperature and cleaning frequency. Furthermore, the validation of the model by comparison with real data has confirmed the reliability of the developed model for predicting system availability.

4.4. Operational and economic impacts of the proposed approach

The analyses carried out in the previous sections made it possible to evaluate the technical performance of the developed model. In order to have a more complete view, it is also important to examine the potential operational and economic impacts of this approach.

4.4.1 Operational impacts

table 7 summarizes the main operational indicators compared between the current approach and the proposed approach.

Table 7: *Comparison of key operational indicators between the current and proposed approaches*

Indicator	Current approach	Proposed approach	Variation
Average diagnosis time	45 minutes	28 minutes	-37.8%
Diagnostic success rate	85%	92%	+8.2 pts
Number of corrective maintenance visits	12 per year	8 per year	-33.3%
Average downtime	3.2 hours	1.9 hours	-40.6%

As shown in this table, the proposed approach would allow for significant improvements on all key operational indicators:

- 37.8% reduction in average diagnostic time;
- 8.2 percentage point increase in diagnostic success rate;
- 33.3% decrease in the number of corrective maintenance visits;
- 40.6% reduction in average downtime.

These operational gains would result in a notable improvement in the availability and reliability of the system for end users.

4.4.1.1 Economic impacts

To assess the economic impact, we modeled the costs over a 5-year horizon, taking into account the following elements:

- Initial investment costs in the development of the proposed approach;
- Annual maintenance and operating costs;
- Savings achieved through operational gains.

figure 9 shows the evolution of the cumulative costs over 5 years for the current approach and the proposed approach.

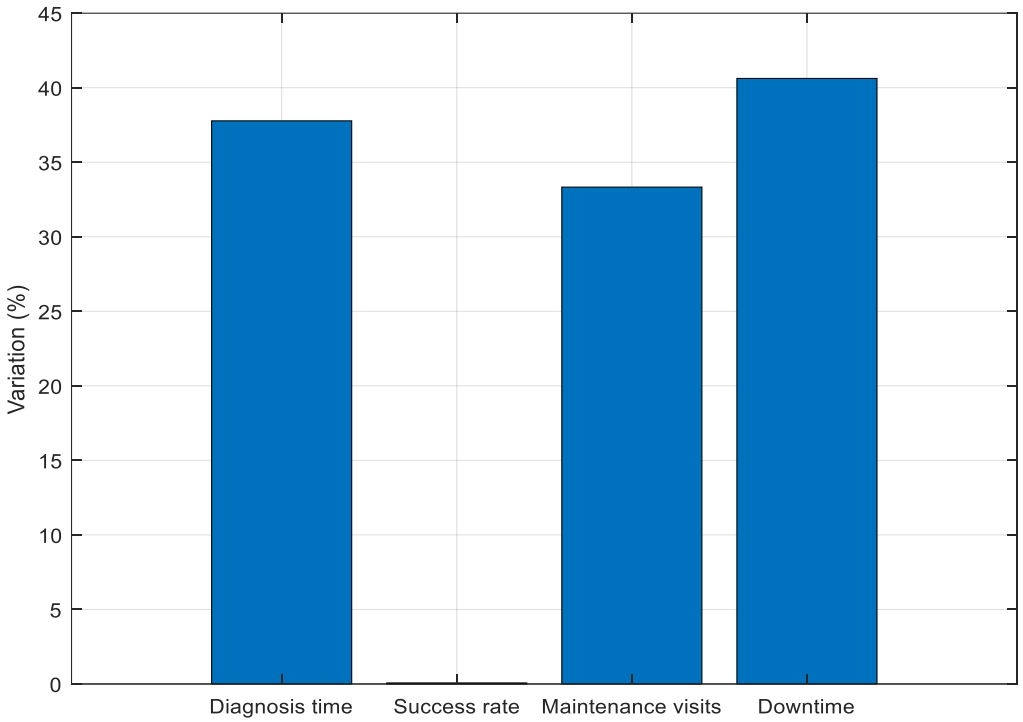

Figure 9 : *Cumulative costs over 5 years*

As can be seen, although the initial investment is higher for the proposed approach, the savings generated by the operational gains make it possible to exceed the breakeven point as early as the 3rd year. Over the entire 5-year period, the proposed approach would represent cumulative savings compared to the current approach.

In conclusion, the analysis of operational and economic impacts demonstrates that the proposed approach brings tangible benefits in terms of technical performance, reliability and long-term costs. These results confirm the relevance and viability of this innovative solution.

5. Conclusion

This work has demonstrated the effectiveness of the neuro-fuzzy approach for optimizing the reliability of equipment in the specific context of a flour mill. The developed model has significantly improved the prediction of failures and the optimization of maintenance interventions, resulting in substantial performance and profitability gains. The sensitivity analysis confirmed the robustness and reliability of the model, which outperforms traditional optimization methods. This approach offers better consideration of the complexity and uncertainty inherent in equipment reliability management. Despite these encouraging results, the study presents certain limitations opening the way for improvement prospects, such as extension to other industrial sites, integration of additional contextual data or automation of certain steps. This work makes a significant contribution to improving the management of industrial equipment reliability, opening interesting prospects for industrialists and providing avenues for future methodological developments for researchers.

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ENHANCING INTRUSION DETECTION SYSTEM RELIABILITY USING GWO-SOMNN (GREY WOLF OPTIMIZATION WITH SELF-ORGANIZING MAP NEURAL NETWORK)

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Abstract

In today's fast-changing technological environment, the number of Internet-connected devices has grown significantly, raising the risk of cybersecurity threats for both individuals and organizations. Network Intrusion Detection Systems (NIDS) have become vital tools for protecting networks from these increasing threats. This paper presents a GWO-SOMNN approach (Gray Wolf Optimization with Self-Organizing Map Neural Network) that combines Grey Wolf Optimization (GWO), Self-Organizing Maps (SOM) and Neural Networks (NN) for feature selection and classification on the UNSW-NB15 dataset. The proposed method leverages GWO to optimize feature selection, reducing the dataset's dimensionality and computational load, while SOM is employed for clustering and visualizing high-dimensional data. Neural Networks are then used for effective classification of network attacks. The GWO-SOMNN approach is evaluated on the UNSW-NB15 dataset, and its performance is measured in terms of 97.18% accuracy and 97.15% F1-score for binary classification and 82.41% accuracy and 78.92% F1-score for multiclass classification. The results demonstrate significant improvements over traditional methods, particularly in enhancing the classification of both binary and multi-class network attacks. These findings highlight the potential of this integrated approach in developing more efficient and accurate network intrusion detection systems.

Keywords: Grey Wolf Optimization, Neural Networks, Self-Organizing Maps, Classification, Intrusion Detection, reliability

1. Introduction

In this section, a brief background introductory note relating to the development and evaluation of a hybrid approach combining Grey Wolf Optimization (GWO), Self-Organizing Maps (SOM), and Neural Networks (NN) for enhanced feature selection and classification in UNSWNB-15 datasets is presented in brief. In the rapidly evolving landscape of UNSWNB-15, the ability to efficiently and accurately detect threats and intrusions is paramount [1]. As cyber-attacks become more sophisticated, traditional methods of threat detection struggle to keep up. This research focuses on enhancing the detection and classification of cyber threats through the development and evaluation of a novel hybrid approach that integrates GWO, SOM, and NN. Feature selection

plays a critical role in UNSWNB-15, as the enormous volume of data produced by network systems can overcome traditional detection mechanisms [2]. By identifying the most relevant features, we can reduce the complexity and improve the performance of classification algorithms. GWO, a nature-inspired metaheuristic algorithm, offers a promising solution for optimal feature selection due to its simplicity and efficiency [3].

In the digital era, cybersecurity attacks have escalated in both frequency and sophistication, presenting substantial risks to individuals, businesses, and governments alike. This surge in cyber threats is closely linked to the increasing dependence on technology, the widespread adoption of Internet of Things (IoT) devices, and the exponential growth of online data. Cybercriminals exploit vulnerabilities in networks and systems, targeting sensitive information and launching attacks such as ransomware, data breaches, phishing, and distributed denial-of-service (DDoS). These incidents can occur unexpectedly, leading to operational disruptions, financial loss, and reputational damage.

An Intrusion Detection System (IDS) is a crucial cybersecurity tool that monitors network traffic and system activities for malicious activities or policy violations. It can be categorized into Network-based IDS (NIDS) and Host-based IDS (HIDS). IDS helps identify vulnerabilities within the network or system, providing alerts to administrators to mitigate risks. It also maintains the integrity and confidentiality of sensitive data, reducing the likelihood of data breaches. IDS is a crucial element of a multi-layered security strategy, working with firewalls, antivirus software, and other security measures to protect against evolving cyber threats. As technology evolves, new and advanced attacks emerge, exploiting weaknesses in hardware, software, and human behavior. Zero-day exploits, IoT devices, botnets, and remote work increase the attack surface. Advanced persistent threats (APTs) and ransomware attacks are also growing. Organizations must stay vigilant and proactive in their cybersecurity efforts, including regular software updates, multi-layered security strategies, and employee education on best practices.

The integration of GWO, SOM, and NN in our GWO-SOMNN approach offers several advantages. GWO ensures optimal feature selection, reducing dimensionality and computational complexity. SOM aids in clustering and visualizing the selected features, enhancing interpretability. NN provides robust classification, leveraging the refined feature set for accurate threat detection. This research contributes to the field of UNSWNB-15 by proposing a novel method for feature selection and classification. By combining the strengths of GWO, SOM, and NN, we aim to develop a solution that addresses the limitations of traditional methods and offers improved performance. The results of our evaluation demonstrate the potential of the hybrid approachto attain a high degree of categorization precision and provide a robust solution for network intrusion detection.

The remaining part of the paper is structured as follows: The prior research on anomaly detection with machine learning techniques is covered in Section 2. The data set is described in Section 3. The proposed hybrid approach presented in Section 4. After outlining the experimental parameters and performance metrics, Section 5 presents results and discussion of proposed GWO-SOMNN approach with the state-of-the-art methods, and Sections 6 and 7 present conclusions and future work plans, respectively.

2. Related Work

In the literature, numerous models for intrusion detection have been presented. This section covers a number of deep learning, machine learning, and data mining-based intrusion detection models.

In a novel approach, a weight embedding autoencoder was proposed by authors in [4] to enhance feature representation in network intrusion detection systems. This method facilitates the sharing of feature representations between the autoencoder and classifier, leading to improved detection accuracy. Their experiments on the NSL-KDD and UNSW-NB15 datasets demonstrate the model's effectiveness, with accuracy improvements of up to 2.8% on UNSW-NB15 and 0.5% on NSL-KDD. An IDS framework was implemented by by authors in [5], utilizing various

Recurrent Neural Networks (LSTM, GRU, and Simple RNN) to enhance network security. To improve detection accuracy, they applied an XGBoost-based feature selection algorithm on the NSL-KDD and UNSW-NB15 datasets. Their results indicate that XGBoost-LSTM achieved the best performance in binary classification, while XGBoost-GRU performed well for multiclass classification on these datasets.

An intrusion detection model was proposed by authors in [6], utilizing an Improved Social Network Search (ISNS) algorithm to optimize the BP neural network. By incorporating chaotic mapping and an elite mechanism into the original SNS algorithm, they successfully mitigated the BP network's tendency to get trapped in local optima. The optimized model, ISNS_BP, demonstrated superior classification accuracy on the NSL-KDD and UNSW-NB15 datasets, achieving 98.62% and 93.97%, respectively. A novel approach for uncertainty quantification in anomaly detection was introduced by authors in [7], using Bayesian Autoencoder (BAE) models. Their method incorporates heteroscedastic aleatoric uncertainty modeling, jointly accounting for both aleatoric and epistemic uncertainties. Applied to cybersecurity datasets such as UNSW-NB15 and CIC-IDS-2017, this framework enhances the trustworthiness of anomaly predictions, reducing false positives and improving decision-making in cybersecurity.

A hybrid method for anomaly detection in IoT devices, called CNN-BMECapSA-RF, was implemented by [8]. This approach combines a convolutional neural network (IoTFECNN) for feature extraction and a binary multi-objective Capuchin Search Algorithm (BMECapSA) for feature selection. Tested on the NSL-KDD and TON-IoT datasets, it achieved high accuracy rates of 99.99% and 99.85%, respectively, by identifying 27% and 44% of relevant features, outperforming existing deep learning and machine learning-based approaches. In another study, [9] proposed a novel intrusion detection approach, LR-ABC, which combines logistic regression (LR) with the artificial bee colony (ABC) algorithm for hyper-parameter optimization. The model improves the accuracy and reliability of network intrusion detection systems (NIDS) by addressing limitations in metrics such as accuracy, F1-measure, and false positives. Tested on the UNSW-NB15 and NSL-KDD datasets, the LR-ABC model achieved accuracy scores of 88.25% and 90.11%, respectively, demonstrating its effectiveness in enhancing detection systems.

Additionally, [10] introduced a hybrid Hunger Games Search and Remora Optimization Algorithm (HHGS-ROA) to tackle security issues in IoT networks. This model enhances the performance of intrusion detection systems by extracting relevant features from the Aegean Wi-Fi Intrusion Dataset (AWID) and classifying network traffic as either normal or malicious using an SVM classifier. The approach outperformed existing methods, achieving high accuracy (99.16%) and a low false-positive rate (0.20%), along with improved metrics such as precision, recall, and F1 score.

3. Dataset Description

The UNSW-NB15 dataset was utilized in this study to detect and classify network intrusions. It contains both normal and abnormal network traffic, with a total of nine categories representing different types of attacks alongside normal traffic. These categories include Denial of Service (DoS), Reconnaissance, Exploits, Backdoors, Fuzzers, Generic attacks, Analysis, Shellcode, and Worms, offering a diverse and comprehensive range of attack patterns for effective evaluation of intrusion detection systems (IDS). The distribution of training and testing set of UNSW-NB15 data set is shown in Table 1 [11].

Table 1: *Class distribution of UNSW-NB15*

Symbols		Set Size	
Type	Name	Training Set	Testing Set
0	Normal	56,000	37,000
1	Backdoor	1,746	583
2	Analysis	2,000	677
3	Fuzzers	18,184	6,062
4	Shellcode	1,133	378
5	Reconnaissance	10,491	3,496
6	Exploit	33,393	11,132
7	DoS	12,264	4,086
8	Worms	130	44
9	Generic	40,000	18,871

The dataset consists of 49 features that describe various aspects of the network traffic. These features were generated using twelve distinct algorithms applied to the raw traffic captured by the TCP dump tool. This diverse set of features makes the UNSW-NB15 dataset suitable for assessing the performance of machine learning and deep learning models in the field of intrusion detection. The dataset provides a challenging environment for anomaly detection, offering a balanced representation of modern attack types in network security research.

4. Proposed GWO-SOMNN Approach

In this section, we present the GWO, SOM and NN, a hybrid approach along with how do we evaluate the entire process which is shown in fig 1. The proposed GWO-SOMNN approach integrates GWO for feature selection and SOM combined with NN for classification. The GWO is employed to optimize feature subsets, enhancing the model's efficiency by selecting the most relevant features from the UNSW-NB15 dataset. Subsequently, SOM visualizes the data patterns while the Neural Network accurately classifies it into attack or normal categories, ensuring a robust intrusion detection mechanism. In this research, each component GWO, SOM, and NN plays a specific role in detecting and classifying intrusions. A detailed explanation of their roles is presented.

4.1. Grey Wolf Optimization

GWO is inspired by the hierarchical social structure and hunting behavior of grey wolves (*Canis lupus*). In this optimization algorithm, the population of candidate solutions is categorized into four main groups based on their leadership hierarchy: alpha (α), beta (β), delta (δ), and omega (ω).

- **Alpha wolves (α)** are considered the most dominant and lead the pack. They are responsible for decision-making and guiding the hunting process.
- **Beta wolves (β)** hold the second rank and assist the alpha in decision-making while also enforcing the alpha™s commands within the pack.
- **Delta wolves (δ)** are subordinate to both alpha and beta but rank higher than omega wolves. This group includes hunters, scouts, and sentinels. Hunters are responsible for locating prey and providing food for the pack. Scouts monitor the surroundings for threats, while sentinels ensure the pack's safety.

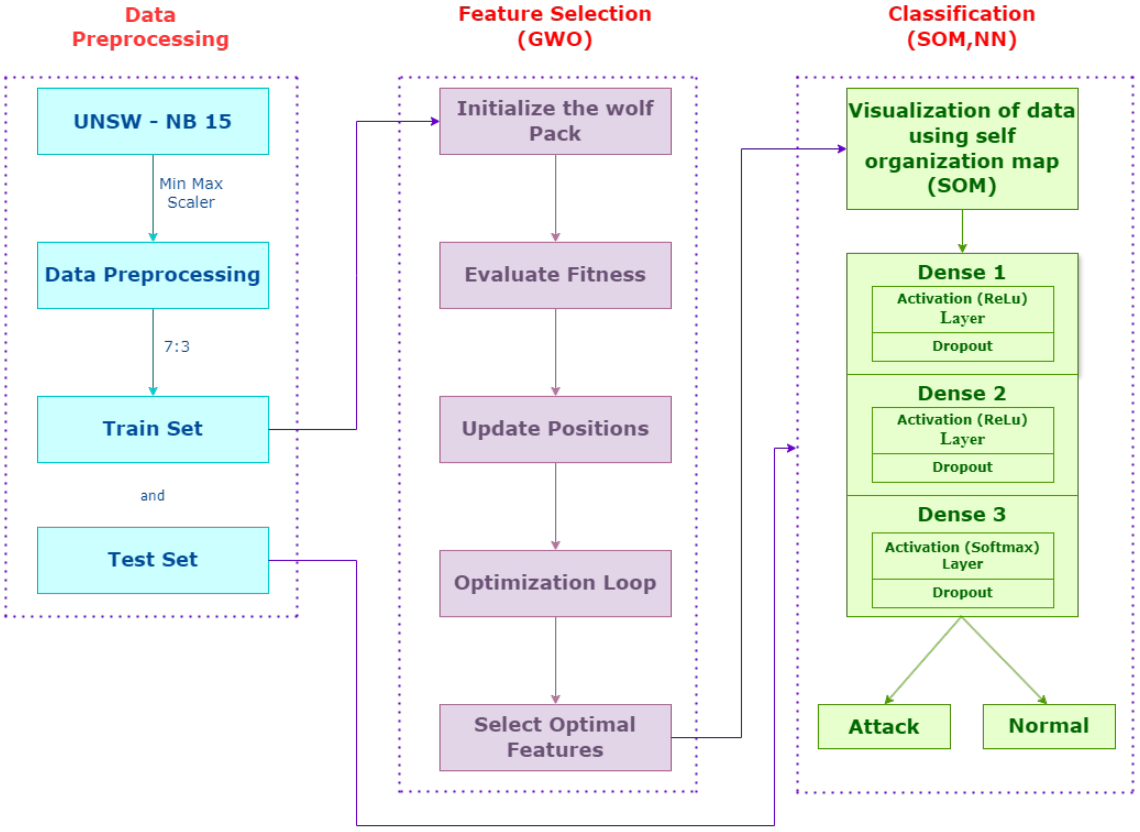

Figure 1: *GWO-SOMNN Approach for IDS*

- **Omega wolves (ω)** are the lowest in the hierarchy. They play a crucial role in maintaining pack structure by following orders from the other groups, especially during hunting activities.

The hierarchical structure ensures that information from the environment is processed and actions are taken efficiently, allowing the wolves to hunt successfully.

GWO mimics this natural hunting mechanism, where the wolves encircle their prey during the hunt. The position of the prey (optimal solution) is estimated by the leading wolves (α, β, and δ), while the remaining wolves update their positions relative to these leaders. The behavior of encircling prey can be mathematically modeled using the following equations [12]:

$$\vec{F}(t+1) = \vec{F}_p(t) - \vec{A} \cdot \vec{D} \tag{1}$$

$$\vec{D} = \left| \vec{C} \cdot \vec{F}_p(t) - \vec{F}(t) \right| \tag{2}$$

Where $\vec{F}_p(t)$ represents the position of the prey, $\vec{F}(t)$ is the position of a grey wolf, and $\vec{A}$ and $\vec{C}$ are coefficient vectors used to simulate the encircling behavior. These vectors are calculated as follows:

$$\vec{a} = 2 - t \left(\frac{2}{\text{Max}_{\text{iter}}} \right) \tag{3}$$

$$\vec{A} = 2\vec{a} \cdot \vec{r}_1 - \vec{a} \tag{4}$$

$$\vec{C} = 2\vec{r}^2 \tag{5}$$

Where t is the current iteration, Max_{iter} is the maximum number of iterations, and $\vec{r}_1$ and $\vec{r}_2$ are random vectors in [0,1].

The algorithm proceeds by iteratively updating the positions of the wolves, with the alpha, beta, and delta wolves guiding the optimization process. Over time, the wolves converge toward the optimal solution, mimicking the grey wolves' real-life hunting strategy.

GWO algorithm, the natural hunting strategy of grey wolves is emulated to optimize search processes. Grey wolves typically locate and encircle their prey, led by the alpha wolf, with the beta and delta occasionally assisting, adding a layer of complexity as the prey™s exact location within the search space is often unknown. In GWO, this behavior is simulated by treating the alpha, beta, and delta wolves as having the most accurate knowledge of the prey's whereabouts, making them the primary guides in the search. The remaining wolves, including omegas, adjust their positions based on the guidance from these top three wolves. The algorithm keeps these top three candidate solutions at the forefront of the process and dynamically adjusts the positions of all other wolves through a following set of equations, effectively simulating the encircling and attacking phases of wolf hunting [13].

$$\vec{D}_\alpha = \left| \vec{D}_1 \cdot \vec{F}_\alpha - \vec{F} \right| \tag{6}$$

$$\vec{D}_\beta = \left| \vec{D}_2 \cdot \vec{F}_\beta - \vec{F} \right| \tag{7}$$

$$\vec{D}_\delta = \left| \vec{D}_3 \cdot \vec{F}_\delta - \vec{F} \right| \tag{8}$$

$$\vec{F}_1 = \vec{F}_\alpha - \vec{A}_1 \cdot \vec{D}_\alpha \tag{9}$$

$$\vec{F}_2 = \vec{F}_\beta - \vec{A}_2 \cdot \vec{D}_\beta \tag{10}$$

$$\vec{F}_3 = \vec{F}_\delta - \vec{A}_3 \cdot \vec{D}_\delta \tag{11}$$

$$\vec{F}(t+1) = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{3} \tag{12}$$

In the GWO algorithm, the equations above describe the process by which the positions of grey wolves (potential solutions) are updated based on the positions of the three leading wolves"alpha ($\vec{F}_\alpha$), beta ($\vec{F}_\beta$), and delta ($\vec{F}_\delta$). In this model, the grey wolves encircle their prey, represented by the optimal solution.

The first set of equations calculates the distance vectors ($\vec{D}_\alpha$, $\vec{D}_\beta$, and $\vec{D}_\delta$) between the current wolf's position ($\vec{F}$) and each of the leading wolves ($\vec{F}_\alpha$, $\vec{F}_\beta$, and $\vec{F}_\delta$), adjusted by dynamic coefficients ($\vec{D}_1$, $\vec{D}_2$, and $\vec{D}_3$) to control the movement towards these leaders.

Subsequently, the positions of the wolves ($\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$) are updated by subtracting a second set of coefficients ($\vec{A}_1$, $\vec{A}_2$, and $\vec{A}_3$) scaled by the distance vectors. Finally, the new position of each grey wolf ($\vec{F}(t+1)$) is calculated as the average of the positions derived from the three leaders, ensuring that the wolves converge towards the prey, which represents the optimal solution in the search space. This process is repeated iteratively until convergence is achieved.

In this implementation of the GWO algorithm for feature selection, two primary parameters are adjusted: 'SearchAgents_no' and 'Max_iter'. The 'SearchAgents_no' is set to 5, indicating the number of grey wolves (agents) used to explore the search space, which directly affects the diversity of potential solutions. The 'Max_iter' parameter is set to 100, controlling the maximum number of iterations for the optimization process, ensuring a balance between computational cost and optimization depth. Additionally, upper ('ub') and lower ('lb') bounds for the feature selection space are defined, allowing features to be represented as binary values (0 or 1). The exploration-exploitation balance is controlled through the 'a' parameter, which linearly decreases over iterations, guiding the wolves' movements from global exploration to local exploitation. Random vectors 'r1' and 'r2' introduce variability, making the search process robust by allowing each wolf to update its position relative to the best (Alpha), second-best (Beta), and third-best

Table 2: *The list of features selected by GWO*

Feature Information		
Sr. No	**Feature Number**	**Feature Name**
1	1	dur
2	2	proto
3	3	service
4	7	sbytes
5	8	dbytes
6	9	rate
7	11	dttl
8	12	sload
9	17	dinpkt
10	20	swin
11	25	synack
12	27	smean
13	28	dmean
14	30	response_body_len
15	33	ct_dst_ltm
16	34	ct_src_dport_ltm
17	35	ct_dst_sport_ltm
18	38	ct_ftp_cmd
19	43	label

(Delta) solutions. This configuration ensures that the algorithm efficiently searches for the most optimal subset of features.

GWO selected key features in Table 2 based on network flow characteristics for enhancing classification performance. These features include attributes like connection duration, protocol types, data rate, and packet statistics. Features such as source-to-destination transaction bytes, TCP window advertisement, and SYN-ACK flags are critical in identifying attack patterns. This optimized feature subset enables a more efficient and accurate detection of network intrusions.

4.2. Self-Organizing Maps

SOMs are widely used for grouping and displaying high-dimensional data in a lower-dimensional area [14].

The SOM algorithm begins with the initialization of a weight matrix, which is randomly assigned and represents the position of each neuron within the input feature space. The algorithm iteratively maps data points to the SOM grid, identifying the "best matching unit" (BMU), or winner neuron, for each input. The weights of the BMU and its neighboring neurons are then adjusted, bringing them closer to the input data point. This iterative process allows the SOM to progressively refine its mapping and organization of the data. SOMs are particularly advantageous in exploratory data analysis, offering researchers the capability to uncover latent patterns or groupings within complex datasets. Additionally, SOMs serve as a robust tool for data visualization, enhancing the interpretability and analysis of data across various fields, including bioinformatics, finance, and marketing.

Initialization: For each input vector $\vec{W}_{(i,j)}$ for each neuron (i, j)
Training: For each input vector $\vec{x}$:
Find Best Matching Unit (BMU): Here, we use the BMU model as:

$$\text{BMU} = \arg\min_{i,j} \|\vec{x} - \vec{W}_{i,j}\| \tag{13}$$

$$\vec{W}_{i,j}(t+1) = \vec{W}_{i,j}(t) + \theta(t,i,j) \cdot \alpha(t) \cdot (\vec{x} - \vec{W}_{i,j}(t)) \tag{14}$$

where, $\theta(t, i, j)$ is the neighborhood function, which decreases over time. $\alpha(t)$ is the learning rate, which also decreases over time.

After selecting the optimal features using the GWO algorithm, a SOM is employed to visualize and further process the selected data. In this implementation, the SOM is initialized with a 5x5 grid ('x=5, y=5') to create a map of neurons that represents the input space. The 'input_len' parameter is dynamically set to match the number of features selected by GWO, ensuring that each neuron can accommodate the reduced feature set. The 'sigma' parameter, which controls the radius of influence for each neuron during the learning process, is set to 1.0, allowing a moderate neighborhood influence on the weight updates. The learning rate is initialized at 0.5, guiding the network's convergence speed as it adapts to the data. The SOM is trained using random samples from the training set for 25 iterations, facilitating the clustering and visualization of attack and normal data in an unsupervised manner. This approach allows the model to discover inherent patterns in the dataset and enhances its ability to differentiate between attack and normal classes.

4.3. Neural Networks

NNs represent a fundamental element of contemporary artificial intelligence, drawing inspiration from the structure and function of the human brain. These networks consist of multiple layers of interconnected neurons that process input data, enabling the system to learn and recognize patterns[15].

We have used the Multilayer Perceptron (MLP) neural network in our work, which is a fundamental type of artificial neural network, characterized by its feedforward architecture, where data flows in one direction"from the input layer through one or more hidden layers to the output layer. This structure makes MLPs particularly effective for supervised learning tasks, such as classification and regression. The MLP begins with an input layer, which serves as the entry point for the data. Each neuron in this layer corresponds to a specific feature of the input data. In our data set out of 45 features, the GWO algorithm has selected 19 features for binary and multiclass classification, so the input layer will have 19 neurons, each representing one of those features. Following the input layer with one or more hidden layers. These layers are the core of the MLP, where the actual computation and learning take place. Each neuron in a hidden layer is connected to every neuron in the previous layer, forming a fully connected network. The neurons in the hidden layers perform computations by applying a weighted sum of the inputs from the previous layer, followed by an activation function ReLU (Rectified Linear Unit), which introduces non-linearity to the model. The Fig. 2 and 3 gives the NN diagram for binary classification and multiclass classification respectively.

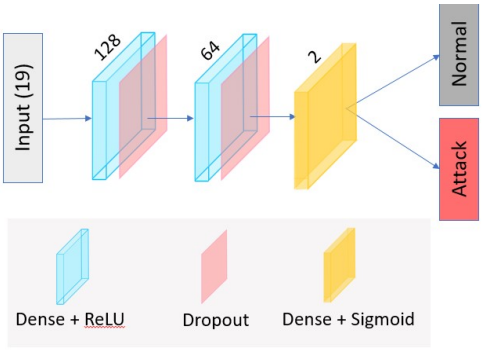

Figure 2: *Multilayer Perceptron Neural network diagram for binary classification*

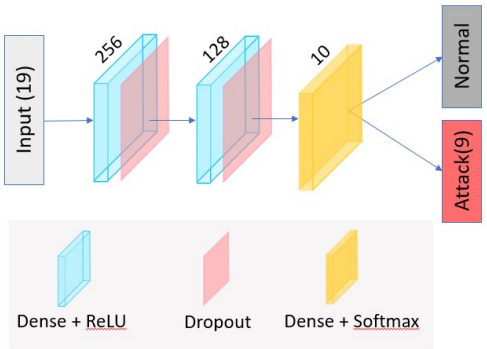

Figure 3: *Multilayer Perceptron Neural network diagram for multi class classification*

The final layer of the MLP is the output layer, which produces the model's predictions or classifications based on the processed data. The number of neurons in the output layer corresponds to the number of possible outputs. For binary classification, the output layer has

two neuron with a Sigmoid activation function to produce a probability score and multi-class classification has also used softmax activation function with multiple neurons, one for each class. The forward propagation process in an MLP involves passing the input data through the network, layer by layer. Each neuron computes a weighted sum of its inputs, applies the activation function, and passes the result to the next layer, culminating in the output layer's final prediction.

The neural network model is constructed using TensorFlow™s Keras library and consists of three layers, three dense (fully connected) layers. The number of parameters in the neural network depends on the sizes of these layers and the input size. We chose the size of the neural network (128, 64, 2) is shown in Table 3. The first dense layer has 128 neurons, with parameters calculated as 128 — input_len™ + 128 (weights plus biases). The second dense layer has 64 neurons, with parameters 128 — 64 + 64 = 8,256. The output layer, with 2 neurons for binary classification, has parameters 64 — 2 + 2 = 130. Thus, the total parameters in the neural network are 128 — input_len™ + 8,514, where input_len™ describes the number of selected features with GWO algorithm. Here, 19 optimal features are selected by applying the GWO algorithm, so 10,946 total parameters are used in the neural network.

Table 3: *Model Architecture and Parameters*

Layer Type	Output Shape	Activation	Parameters
Input Layer	19	-	0
Dense	128	ReLU	3,072
Dropout	0.5	-	-
Dense	64	ReLU	8,256
Dropout	0.5	-	-
Dense	2	Sigmoid	130
Total Parameters			**10,946**

Table 4: *Model Architecture and Parameters*

Layer Type	Output Shape	Activation	Parameters
Input Layer	19	-	0
Dense	256	ReLU	4,352
Dropout	0.5	-	0
Dense	128	ReLU	32,896
Dropout	0.5	-	0
Dense	10	Softmax	1,290
Total Parameters			**39,306**

Similary, for multi-class classification, we chose neural network size (256, 128, 10) is shown in table 4. The first dense layer has 256 neurons, with parameters calculated as 256 — input_len™ + 256 (weights plus biases). The second dense layer has 128 neurons, with parameters 256 — 128 + 128 = 32,896. The output layer, with 10 neurons for muticlass classification, has parameters 128 — 10 + 10 = 1290. Thus, the total parameters in the neural network are 256 — input_len™ + 34,442, where input_len™ describes the number of selected features with GWO algorithm. Here, 19 optimal features selected by applying GWO algorithm so 39,306 total parameters used in neural network.

5. Results and Discussion

In this research, the proposed method implemented in the Google Colab environment, the developed code was run & the simulation results were observed and displayed for UNSWNB15 Binary Classification as well as for Multiclass Classifications, the results are specified separately.

5.1. Evaluation Metrics

The following metrics are used to assess the proposed hybrid approach: accuracy, precision, recall and F-measure. The following defines each measure:

- - Accuracy represents the proportion of correctly classified records out of the total dataset.
- Precision refers to the percentage of correctly identified anomalies among all records predicted to be anomalies.
- Recall, also known as the True Positive Rate or detection rate, is the percentage of actual anomalies that were correctly classified.
- F-measure is a metric that balances both precision and recall, providing a single performance measure.

5.2. Results

The experiment was conducted using the UNSW-NB15 dataset, with the GWO algorithm applied to select optimal features, SOM for data visualization, and MLP NN for classification. In Tables 5, the following abbreviations are used: NU (Number of hidden Units), TAC (Training Accuracy), VAC (Validation Accuracy), ET (Execution Time), and TEC (Testing Accuracy).

The experiment was performed in two phases. In the first phase, for binary classification, the GWO algorithm selected various features based on the SOM grid size of 5x5, adjusted according to the 3 hidden units. The activation functions used for the dense layers were ReLU, ReLU and Sigmoid. Additionally, the training time (in seconds) was recorded for each model.

The following hyperparameters were used for binary classification:

- Loss function: 'binary_crossentropy'
- Optimizer: 'adam' (an extension of Stochastic Gradient Descent)

In the second phase, for multiclass classification, the GWO algorithm selected different features, and a SOM grid size of 5x5 was used based on varying hidden units. The following hyperparameters were used for Multiclass Classification:

- loss ='sparse_categorical_crossentropy'
- optimizer = 'adam'' (an extension of Stochastic Gradient Descent).

Table 5: *Performance metrics of the proposed GWO-SOMNN approach for Binary and Multiclass Classification*

Classification	NU	TAC (%)	VAC (%)	ET (Sec)	Testing (%)	Precision (%)	Recall (%)	F1-Score (%)
Binary	128,64,2	97.24	97.18	466.29	97.18	97.24	97.18	97.15
Multiclass	256,128,2	82.41	82.33	43.247	82.33	76.60	82.33	78.92

Table 5 presents the performance metrics of the proposed Grey Wolf Optimization combined with Self-Organizing Map and Neural Network (GWO-SOMNN) approach for both binary and multiclass classification tasks. For binary classification, the model achieved a TAC of 97.24%, VAC of 97.18%, and required 466.29 seconds of execution time. The Testing Accuracy, Precision, Recall,

and F1-Score for binary classification were 97.18%, 97.24%, 97.18%, and 97.15%, respectively. For multiclass classification, the model achieved a TAC of 82.41%, VAC of 82.33%, and the execution time was 43.247 seconds. The Testing Accuracy, Precision, Recall, and F1-Score for multiclass classification were 82.33%, 76.60%, 82.33%, and 78.92%, respectively. These results indicate that the GWO-SOMNN approach performs better in binary classification compared to multiclass classification, especially in terms of accuracy and precision. Evaluation measures for Binary and multiclass classification is presented in Figure 4 and 5 respectively.

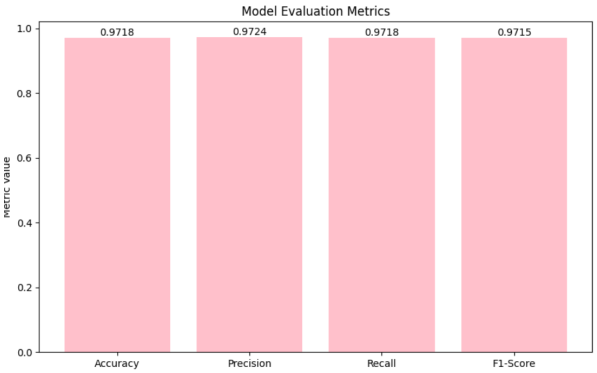

Figure 4: *Evaluation measures of Binary Classification*

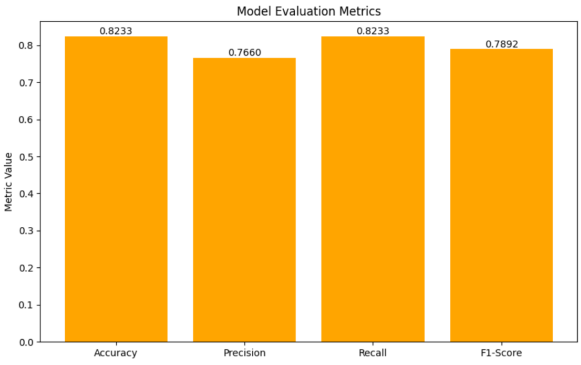

Figure 5: *Evaluation measures of Multiclass Classification*

5.3. Confusion Matrix

For each model, a Confusion Matrix (CM) was generated to evaluate the model's performance on individual classes within the datasets. In Figure 6, class 0 represents normal traffic, and class 1 represents attacks. In Fig. 5, the classes are defined as: class 0 = Normal, class 1 = Generic, class 2 = Exploits, class 3 = Fuzzers, class 4 = DoS, class 5 = Reconnaissance, class 6 = Analysis, class 7 = Backdoor, class 8 = Shellcode, and class 9 = Worms.

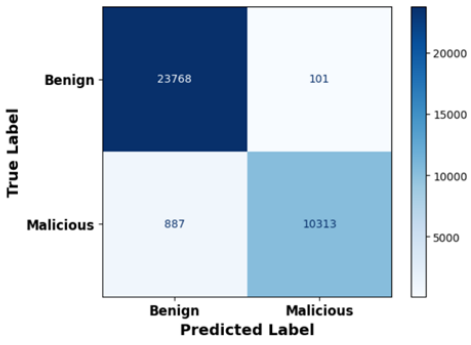

Figure 6: *Binary Class Confusion Matrix*

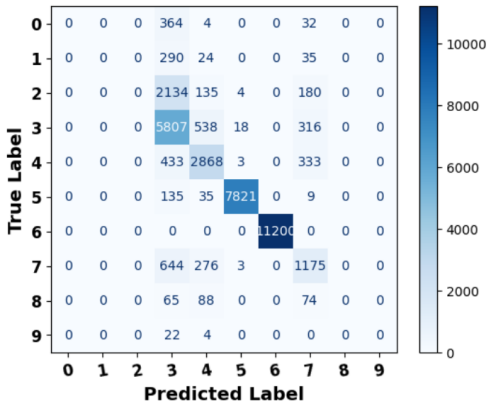

Figure 7: *Multiclass Confusion Matrix*

The GWO-SOMNN approach in Figure 6 correctly classified 23,768 benign instances and 10,313 malicious instances. However, some false negatives were observed, with 887 benign instances misclassified as malicious, alongside 101 false positives, where malicious instances were incorrectly classified as benign. In Figure 7, the confusion matrix reflects the GWO-SOMNN approach's performance across individual classes of the UNSW-NB15 dataset. The multiclass confusion matrix shows varying performance across different classes. Based on the confusion matrix, the approach correctly predicted classes such as Fuzzers (label 6), which demonstrated high precision with minimal misclassifications. Similarly, Normal traffic (label 5) was also

predicted with strong accuracy, as evidenced by the dense diagonal and almost no off-diagonal entries.

However, certain attack types, such as Generic (label 3) and Exploits (label 2), were more challenging for the model to classify accurately. The confusion matrix shows considerable misclassification for these attacks, where instances of Generic were confused with Exploits, and vice versa. This misclassification can be attributed to the overlapping feature patterns between these attack types, as both may share similar network characteristics, making it harder for the model to differentiate them effectively. Moreover, Reconnaissance (label 4) also exhibited some degree of misclassification, potentially due to its similarity in network behavior to other less specific attack types.

5.4. Discussion

For the binary classification, Table 6 compares the proposed methodology (PM) with other research approaches. The proposed GWO-SOMNN approach achieved a significant accuracy of 97.65% using 19 selected features. In contrast, previous methods such as Weight Embedding AutoEncoder with Convolutional Neural Network (CNN) and Multi-Layer Neural Network (MLNN) showed lower accuracies of 79.49% and 80.39%, respectively with 42 features. Other approaches like XGBoost variants (LSTM, GRU, Simple-RNN) demonstrated accuracies ranging from 85.08% to 87.07% with 17 features, whereas the Multivariate Correlations Analysis with Long Short-Term Memory (MCA-LSTM) with 14 features and Feed-Forward Deep Neural Network (FFDN) with 18 features achieved accuracies of 88.1% and 87.1%, respectively. The proposed approach clearly outperforms these methods in terms of accuracy for binary classification.

Table 6: *Comparison of Binary Classification*

Author	Method	Accuracy (%)
[16]	Weight Embedding AutoEncoder with a Convolutional Neural Network (WE-AE CNN)	79.49
[16]	Weight Embedding AutoEncoder with a Multi-Layer Neural Network (WE-AE DNN)	80.39
[17]	XGBoost-LSTM	85.08
[17]	XGBoost-GRU	88.42
[17]	XGBoost-Simple-RNN	87.07
[18]	Multivariate Correlation Analysis “ Long Short-Term Memory Network (MCA-LSTM)	88.1
[19]	Feed-Forward Deep Neural Network (FFDN)	87.1
PM	**GWO-SOMNN Approach**	**97.65**

For multiclass classification, as shown in Table 7, the proposed hybrid approach also outperforms other methods, achieving an accuracy of 82.41% with 19 selected features. In comparison, Weight Embedding AutoEncoder with CNN and MLNN reached accuracies of 74.19% and 73.01%, respectively with 42 features, while XGBoost-GRU with 17 features and FFDN with 18 features yielded accuracies of 78.4% and 77.16%. Thus, the proposed GWO-SOMNN approach shows an improvement in multiclass classification accuracy, demonstrating its effectiveness in handling complex, real-world datasets.

6. Conclusion

This research introduced a hybrid approach integrating Grey Wolf Optimization (GWO), Self-Organizing Maps (SOM), and Neural Networks (NN) to enhance feature selection and classification for intrusion detection using the UNSW-NB15 dataset. The main goal was to optimize feature selection and clustering to improve the performance of intrusion detection systems. By leveraging

Table 7: *Comparison of Multiclass Classification*

Author	Method	Accuracy (%)
[16]	Weight Embedding AutoEncoder with a Convolutional Neural Network (WE-AE CNN)	74.19
[16]	Weight Embedding AutoEncoder with a Multi-Layer Neural Network (WE-AE DNN)	73.01
[17]	XGBoost-GRU	78.4
[19]	Feed-Forward Deep Neural Network (FFDN)	77.16
PM	**GWO-SOMNN Approach**	**81.53**

GWO for efficient feature selection and SOM for data visualization, the GWO-SOMNN approach significantly reduced the dataset's dimensionality, improving the computational efficiency and accuracy of neural network-based classification.

The results demonstrate that the proposed method outperforms traditional techniques in both binary and multiclass classifications, achieving notable improvements in accuracy, precision, recall, and F1-score. Specifically, the GWO-SOMNN approach achieved a binary classification accuracy of 97.18% and a multiclass classification accuracy of 82.41%, surpassing many state-of-the-art methods. This indicates the potential of this integrated approach for developing more efficient and precise network intrusion detection systems.

7. Future Work

Future work will focus on the real-time implementation and scalability of the GWO-SOMNN approach. Deploying this model in live network environments will allow for the evaluation of its performance under real-time conditions. Additionally, extending the model to handle larger and more complex datasets will test its robustness and scalability in diverse scenarios.

Further improvements could include optimizing the GWO algorithm by integrating it with other metaheuristic techniques such as Particle Swarm Optimization or Genetic Algorithms. Also, incorporating advanced neural network architectures, such as Convolutional Neural Networks (CNNs) or Recurrent Neural Networks (RNNs), may enhance feature selection efficiency and classification accuracy.

Exploring adversarial training techniques will help improve the model™s resilience against adversarial attacks, while the application of different SOM variants could enhance clustering and visualization. Finally, implementing automated hyperparameter tuning and incorporating behavioral analysis could further enhance the adaptability and detection of sophisticated threats.

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ANALYSIS OF AN ENCOURAGED ARRIVAL MARKOVIAN QUEUE WITH SINGLE WORKING VACATION, IMPATIENCE AND RENEGING OF CUSTOMERS

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Abstract

In this paper, we analyze a single server markovian queueing model with encouraged arrivals that undergoes a single working vacation. Additionally, we consider the impatience and reneging behavior of customers in the queue during the working vacation period. Customers arrive at the system following a Poisson distribution. The server goes on vacation when the system is empty and stays on vacation for a random period that follows an exponential distribution. During the working vacation period, the server continues to provide service at a slower rate. After the vacation, the server returns to the regular service period and continues providing service at the regular busy period rate if there are one or more customers in the system, or it remains idle until a new customer arrives. During the working vacation, customers in the queue become impatient and renege from the system, with the reneging time assumed to follow an exponential distribution. The system is characterised as a quasi-birth-death process, and the stationary probabilities are derived using the probability generating function method. Some numerical analysis is also carried out to show the effect of encouraged arrivals on performance measures.

Keywords: Encouraged arrivals, impatience, reneging, working vacation, probability generating function(PGF).

I. Introduction

Since the 1970s, numerous researchers have studied the mathematical modelling and implementation of queueing models that undergoes server vacations. Congestion issues in a variety of research domains could be readily represented by vacation queueing models that undergoes server vacations. Several studies have been conducted on queues with vacations in [1, 2]. A single server finite source markovian queueing model with server vacations, baling and reneging behaviour of customers are analysed in [3] using the solution of steadystate probabilities in the matrix form. For a variety of real-world scenarios, including computer networks, digital communication, and production/inventory systems benefits from the generalisation of queueing models [4, 5]. It is assumed that in these investigations, the service is completely terminated when

on the server is on vacation. This kind of vacation denotes classical vacation model. A working vacation (wv) is when the server continues to offer service while on vacation, but at a reduced service rate. This type of wv is first introduced in [6], which also examined a markovian queue with several working vacation policies on a single server. In [7], the matrix-geometric approach is used to study an M/M/1 queue with numerous working vacations and derive precise formulas for the performance metrics. Using the same method , analysis of a single server queue with single working vacation(swv) is carried out in [8]. The investigation by [6] was expanded to an M/G/1/WV queue by [9, 10, 11]. In [12], the work of [6] is extended to a GI/M/1 queue with a general arrival process and several working vacations using the matrix-geometric solution method. The GI/M/1 queue with a swv was further examined by [11].

Clients are frequently seen waiting in line for assistance in today's busy environment. Clients experience impatience while the server is on vacation, At present, queueing system analysis with impatient customers is becoming steadily more popular. There are several related studies which are explained in [12], [13]. A comprehensive analysis of queues with vacation and client impatience for single and multiserver systems are given in [14]. customers are drawn to the business by the discounts and offers. In [15], such customers are known as Encouraged Arrivals(ea). The concept of customer movement explained in [16], which states that a system can draw in new customers by looking at its substantial client base. The variation in percentage of customers depends on ea brought about by sales and discount . A finite capacity ea queue with multiple servers and reverse reneging is carried out in [17].

In this paper we analyse an encouraged arrival single server queue with swv, impatience and the reneging behaviour of customers due to impatience during the working vacation session. The introduction of the paper is given in section 1. Section 2 comprises of the model description. The stationary analysis the of model with ea, swv, impatience and reneging of impatient customers are provided in section 3. Section 4 deals with the performance measures of the model. The numerical analysis is given in section 5. The conclusion is given in section 6.

II. Model description

We consider a single server markovian queueing model with ea, swv, impatience and reneging behaviour of impatient clients during working vacation session. The arrivals follow a poisson distribution with parameter $\lambda(1+\Omega)$, where " Ω " denotes the percentahe variation in the total count of clients estimated from observed data. For instance, if a firm previously offered discounts and a percentage change in the total count of clients was noticed of +10%, +30% or +50%, then $\Omega = 0.1, 0.3$ or 0.5, respectively. The server operation follows an exponential distribution with parameters μ and α during busy hours and working vacations respectively, where ($\alpha < \mu$). The server takes a swv when the system is empty, and the duration of this vacation is distributed exponentially with parameter ψ. If there are clients in the system at the end of the vacation, the server returns to its actual service rate. Otherwise, it will remain idle until a new client shows up. Clients who wait for his turn to get service, may become impatient and choose to leave the queue. The reneging behaviour of impatient clients follows an exponential distribution with parameter β.

III. Steady state analysis of the queue with encouraged arrivals, single working vacation, impatience and reneging of impatient clients during WV:

Let the number of clients in the system is given by N and the state of the system is given by S. Then the markov process is given as $\{(N,S), t\geq 0\}$. The state space is given by $\theta = \{(n,s), n = 0,1,2,\ldots, s = 0,1\}$ where s = 0 denoted the swv and s = 1 denotes the regular busy session.

The following are the differential-difference equations governing the quasi-birth-death process in the steady state:

$(\lambda(1+\Omega)+\varphi)\mathrm{P}_{0,0} = \alpha P_{1,0} + \mu P_{1,1}$ (1)

$(\lambda(1+\Omega)+\alpha+\varphi+(\mathrm{n}-1)\beta)\mathrm{P}_{\mathrm{n},0} = \lambda P_{n-1,0} + (\alpha+n\beta)\mathrm{P}_{\mathrm{n}+1,0}, n \geq 1$ (2)

$\lambda(1+\Omega)\mathrm{P}_{0,1} = \varphi\mathrm{P}_{0,0}$ (3)

$(\lambda(1+\Omega)+\mu)\mathrm{P}_{\mathrm{n},1} = (\lambda(1+\Omega)\mathrm{P}_{\mathrm{n}-1,1} + \mu P_{n+1,1} + \varphi\mathrm{P}_{\mathrm{n},0},, n \geq 1$ (4)

The PGF are defined as follows:

$G_0(y) = \sum_{n=0}^{\infty} y^n \mathrm{P}_{\mathrm{n},0}$, $G_1(y) = \sum_{n=0}^{\infty} y^n \mathrm{P}_{\mathrm{n},1}$ and $G_0'(y) = \sum_{n=0}^{\infty} y^{n-1} \mathrm{P}_{\mathrm{n},0}$ for $0 \leq \mathrm{y} \leq 1$ (5)

Equations (1) and (2) are multiplied by 1 and y^{n} respectively. Summing them for all possible values of n, we get

$\beta y(1-y)G_0'(y) + [\lambda(1+\Omega)y^2 - (\lambda(1+\Omega)+\varphi+\alpha-\beta)y + (\alpha-\beta)]G_0(y) = (1-y)(\alpha-\beta)\mathrm{P}_{0,0} - \mu y P_{1,1}$ (6)

Similarly (3) and (4) are multiplied by 1 and y^{n} and are added over all possible values of n, we get

$(1-y)(\ (\lambda(1+\Omega)y-\mu)G_1(y) = \varphi y G_0(y) - \mu(1-y)P_{0,1} - \mu y P_{1,1}$ (7)

Rewriting (6) for $y \neq 0$ and $y \neq 1$, we have

$G_0'(y) - \left(\frac{\lambda(1+\Omega)}{\beta} + \frac{(\varphi+\alpha-\beta)}{\beta(1-y)} - \frac{\alpha-\beta}{\beta y(1-y)}\right) G_0(y) = \frac{(\alpha-\beta)}{\beta y}\mathrm{P}_{0,0} - \frac{\mu}{\beta(1-y)}P_{1,1}$ (8)

Multiplying (8) with $e^{\frac{-\lambda(1+\Omega)y}{\beta}}(1-y)^{\frac{\varphi}{\beta}} y^{\frac{(\alpha-\beta)}{\beta}}$ on both the sides, we have

$G_0(y) = \frac{e^{\frac{\lambda(1+\Omega)y}{\beta}}}{(1-y)^{\frac{\varphi}{\beta}} y^{\frac{(\alpha-\beta)}{\beta}}}\left[\frac{(\alpha-\beta)}{\beta y}F_1(\mathrm{y})\mathrm{P}_{0,0} - \frac{\mu}{\beta(1-y)}F_2(\mathrm{y})P_{1,1}\ \right]$ (9)

Where

$F_1(\mathrm{y}) = \int_0^y e^{\frac{-\lambda(1+\Omega)u}{\beta}}(1-u)^{\frac{\varphi}{\beta}} u^{\frac{(\alpha-\beta)}{\beta}-1}\mathrm{du}$

$F_2(\mathrm{y}) = \int_0^y e^{\frac{-\lambda(1+\Omega)u}{\beta}}(1-u)^{\frac{\varphi}{\beta}-1} u^{\frac{(\alpha-\beta)}{\beta}}\mathrm{du}$

Since $0 \leq G_0(1) = \sum_{n=0}^{\infty}\mathrm{P}_{\mathrm{n},0} \leq 1$ and $\lim_{y\to 0}(1-y)^{\frac{\varphi}{\beta}} = 0$ it must be

$$\frac{(\alpha-\beta)}{\beta}F_1(1)\mathrm{P}_{0,0} - \frac{\mu}{\beta}F_2(1)P_{1,1} = 0$$

Which in turn gives

$P_{1,1} = \frac{(\alpha-\beta)}{\mu}\frac{F_1(1)}{F_2(1)}\mathrm{P}_{0,0}$ (10)

By solving (6) at y=1 and by using (10), we have

$\varphi G_0(1) = \mu y P_{1,1} = \frac{(\alpha-\beta)F_1(1)}{F_2(1)}\mathrm{P}_{0,0}$ (11)

Using (10), equation (9) becomes

$G_0(y) = \frac{(\alpha-\beta)e^{\frac{\lambda(1+\Omega)y}{\beta}}}{\beta(1-y)^{\frac{\varphi}{\beta}} y^{\frac{(\alpha-\beta)}{\beta}}}\left[F_1(\mathrm{y}) - \frac{F_1(1)}{F_2(1)}F_2(\mathrm{y})\right]\mathrm{P}_{0,0}$ (12)

From (6), we obtain for $y \neq 0$ and $y \neq 1$

$G_0'(y) = \frac{(1-y)(\alpha-\beta)\mathrm{P}_{0,0} - [\lambda(1+\Omega)y^2 - (\lambda(1+\Omega)+\varphi+\alpha-\beta)y + (\alpha-\beta)]G_0(y) - \mu y P_{1,1}}{\beta y(1-y)}$ (13)

we get $G_0'(1)$ by applying L'hospital's rule on (13),

$G_0'(1) = \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi}$ (14)

From (7) we have for $y \neq 1$

$G_1(y) = \frac{\varphi y G_0(y) - \mu(1-y)\mathrm{P}_{0,1} - \mu y P_{1,1}}{(1-y)(\lambda y-\mu)}$ (15)

We get $G_1(1)$ by applying L'hospital's rule on (15)

$G_1(1) = \frac{\varphi G_0'(1) + \mu P_{0,1}}{\mu\lambda(1+\Omega)}$ (16)

From (3), we obtain

$P_{0,1} = \frac{\varphi\mathrm{P}_{0,0}}{\lambda(1+\Omega)}$ (17)

Using normalization condition , we have

$$G_0(1) + G_1(1) = \sum_{n=0}^{\infty} \mathrm{P}_{n,0} + \sum_{n=0}^{\infty} \mathrm{P}_{n,1} = 1$$

Using equations (11), (14), (16) and (17), we obtain the following

$$\mathrm{P}_{0,0} = \left\{ \frac{(\alpha-\beta)F_1(1)}{\varphi F_2(1)} + \frac{(\lambda(1+\Omega)-(\alpha-\beta))(\alpha-\beta)F_1(1)}{(\beta+\varphi)(\mu-\lambda(1+\Omega)F_2(1)} + \frac{\varphi(\alpha-\beta)}{(\beta+\varphi)(\mu-\lambda(1+\Omega))} + \frac{\mu\varphi}{\lambda(1+\Omega)} \right\}^{-1} \quad (18)$$

IV. Performance measures

- Expected number of clients in the system during swv is given by

$$E(N_{swv}) = G_0'(1) = \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi} \quad (19)$$

- Expected number of clients in the system during regular busy session is given by

$$E(N_{rb}) = G_1'(1) = \frac{\varphi G_0''(1)}{2(\mu-\lambda(1+\Omega))} + \frac{\mu\varphi G_0'(1)}{(\mu-\lambda(1+\Omega))^2} + \frac{\mu\varphi \mathrm{P}_{0,0}}{(\mu-\lambda(1+\Omega))^2} \quad (20)$$

Where

$$G_0''(1) = \frac{2(\lambda(1+\Omega)-\varphi-\alpha)G_0'(1)+2\lambda(1+\Omega)G_0(1)}{2\alpha+\varphi} \quad (21)$$

- The total expected number of clients in the system is given as

$$E(N) = E(N_{swv}) + E(N_{rb})$$

Therefore

$$E(N) = \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi} + \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi}$$

- The expected rate of reneging is given as follows

$$E(R) = \sum_{n=1}^{\infty} \beta(n-1)\mathrm{P}_{n,0} = \beta(G_0'(1) - G_0(1) + \mathrm{P}_{0,0})$$

V. Numerical analysis

The numerical analysis shows the impact of parameters on system's performance measures. We consider the following parameters for numerical computation λ=2, μ=5, α=3, ψ=3 and β=0.7

Table 1: *Evaluation of performance measures with respect to varying arrival rate*

Performance measures	λ=2	$\lambda(1+\Omega)$ $\Omega = 10\%$	$\lambda(1+\Omega)$ $\Omega = 20\%$	$\lambda(1+\Omega)$ $\Omega = 30\%$
$E(N_{swv})$	0.12503	0.13081	0.14353	0.15442
$E(N_{rb})$	0.61054	0.73431	0.86898	1.02835
$E(N)$	0.73548	0.86424	1.00343	1.17371
$E(R)$	0.02030	0.02557	0.02981	0.03321
$\mathrm{P}_{0,0}$	0.2366	0.2238	0.21657	0.2785
$\mathrm{P}_{0,1}$	0.3424	0.30402	0.2785	0.2468

From table 1. We observe that the performance measures increases with increase in arrival rate. In other words, as the number of clients joining the firm increases the probability of system in swv and the probability of firm being in regular busy session decreases.

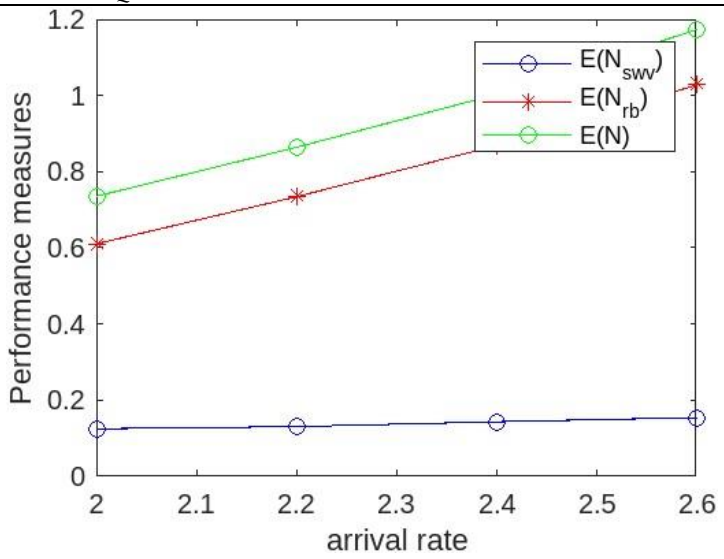

Figure 1. *Variation in performance measures with respect to arrival rate*

Table 2*: Evaluation of performance measures with respect to varying service rate during swv*

α	$E(N_{swv})$	$E(N_{rb})$	$E(N)$	$E(R)$	$P_{0,0}$	$P_{0,1}$
3	0.12503	0.61054	0.73548	0.02030	0.2366	0.3412
3.2	0.1125	0.6141	0.7184	0.0282	0.2383	0.3448
3.4	0.1180	0.60011	0.7182	0.0273	0.2314	0.3462
3.6	0.1148	0.6853	0.7012	0.01742	0.2323	0.3504
3.8	0.1037	0.6018	0.7837	0.0174	0.2346	0.3514
4	0.1003	0.5970	0.6974	0.01469	0.2358	0.3535

From table 2. We observe that the performance measures decreases with increase in service rate during swv.

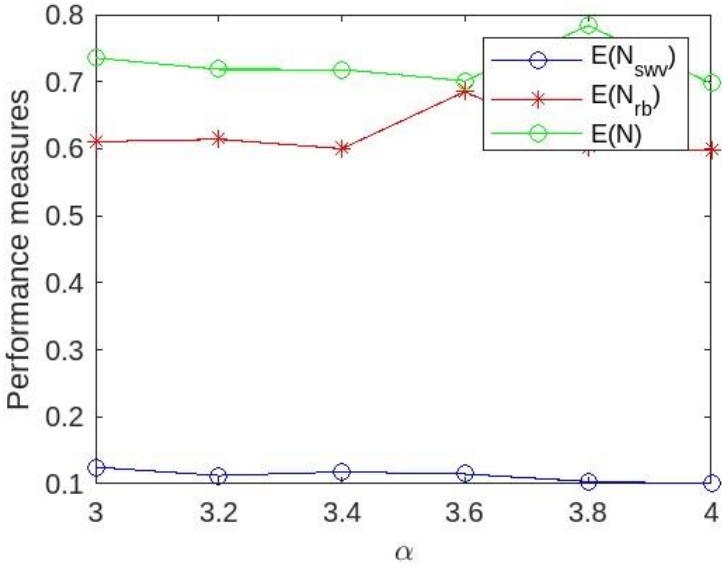

Figure 2. *Variation in performance measures with respect to* α

VI. Conclusion

In this paper, we consider a single server markovian queueing model with encouraged arrival, single working vacation, impatient clients and reneging of such impatient clients during working vacation period. We derived the performance measures using the probability generating function

of the system's steady state probabilities. The numerical analysis shows the impact of encouraged arrivals on the performance measures. As the arrival rate increases, the performance measures increases which benefits the firm .

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IMPACT OF DESIGN AND CONSTRUCTION ERRORS ON THE STRUCTURAL RELIABILITY OF STEEL INDUSTRIAL BUILDINGS

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Abstract

Errors in design and construction critically undermine the structural reliability of industrial buildings, putting property, the environment, and human safety at risk. In this regard, the present research work is intended to investigate how such mistakes influence the performance of the main structural components and the stability of steel industrial buildings. Detailed finite element analysis was performed using DIANA FEA for solid modeling and SAP2000 for beam modeling to assess global structural performance. This includes, among others, the insufficiency of local reinforcement in compressed members and eccentricity in column connections. It was performed to analyze the local and global buckling behaviors, deviations in symmetry, and inefficiency of the bracing systems. Consequently, it reveals a significant reduction in load-bearing capacity due to reinforcement deficiencies in the compressed elements and eccentricity, while a structural loss in integrity becomes highly significant at symmetry deviations, especially in horizontal loads. This study provides critical insights into mitigating design and construction errors to enhance the reliability of industrial steel buildings.

Keywords: steel structures, finite element method, finite element analysis, structural performance, structural design, reinforcement, buckling, symmetry deviation

I. Introduction

Steel structures have many advantages in the construction of industrial buildings in terms of tensile strength, ductility, and durability. That is, their versatility, design flexibility, and efficiencies realized from pre-engineered building systems make them necessary to meet the

demand for industrial requirements. The possibility of covering large areas without interruptions, with reasonable consumption of materials and resistance in the case of earthquakes, makes steel superior to other materials for construction in this industry [1]. The environmental benefits of industrial building construction are necessary, and steel structures are imperative. The sustainability and recyclability of steel, coupled with the industry's efforts to reduce carbon emissions and increase energy efficiency, reflect a greener construction sector. By leveraging the many advantages that steel has to offer, today's construction industry has the possibility of constructing environmentally responsible buildings that will meet any modern industrial process while minimizing the impact on the planet [2]. Although they offer several advantages for industrial building construction, steel structures encounter problems that should be considered during the planning, design, and maintenance stages. It is through the knowledge of these disadvantages associated with steel, such as its susceptibility to corrosion, fire resistance concerns, and initial cost considerations, that appropriate strategies for maximizing its benefits and mitigating the associated drawbacks can be designed by the project stakeholders at different levels. These shortcomings have been addressed earlier, making steel versatile, resilient, and able to stand up to or meet all modern requirements expected of contemporary building projects in any industrial construction [3].

Steel structures of large-span industrial buildings were designed as volumetric systems. In the soil, all columns were fixed to foundations, and the top was fixed to trusses, thus creating a transverse frame. Transverse frames are linked together by means of bracings and purlins, and such a design ensures volumetric rigidity of the frame. Large-span industrial buildings are designed to accommodate industrial processes and to shelter them from open environments. Therefore, the structures of industrial buildings are designed to withstand environmental impacts such as snow, wind, dust, seismic, and service loads considered by industrial operations inside these buildings, including live loads, crane loads, explosions, and temperature fluctuations [4].

The design of large-span industrial buildings and their loads and influences are regulated by national codes and general engineering practices [5]. Errors during the design and construction phases of large-span industrial buildings may contribute to the disruption of industrial operations, jeopardy for personnel health, and the total failure of structures. According to statistics, errors causing the malfunctioning and collapse of industrial buildings can be divided into two groups: errors during the design phase and errors during the construction phase. According to statistics from to 1993-2004, the following average distribution of causes of accidents was observed: defects in construction and erection works, 44%; violation of operation rules, 24%; poor quality of materials, 15%; excess loads and external impacts, 5%; erroneous design decision making, 4%; and other causes, 8%. Thus, approximately 60% of accidents are associated with the construction stage, including critical defects in the construction facilities and materials used [6]. The distribution of defects in a construction can be represented by the following statistics [7]: By the reasons of defect origin: design error - 4%; poor quality of materials and products - 17%; low quality of installation work - 42%; operation deficiency - 18%; combination of reasons - 19% [6].

Other researchers have focused on the structural failure of steel structures due to (quasi-static) loads during the erection-construction phase, service phase, and fatigue failure. In most cases, structural failure is caused by gross human errors. Human errors in the execution of steel structures have been identified as a cause of failure in comparison to failures originating from errors in the design process. The primary source of fatigue destruction in steel structures is insufficient welding size or inadequate quality of the welding joints. According to previous studies, fatigue cracks nearly always originate at the toes of a weld, namely at the transition point between the weld and base material, or at the edges between individual weld passes [8-10]. Although this may be accounted for by the combinations of an unlucky set of variations

influencing parameters affecting the actions and responses of the structures in probabilistic and semi-probabilistic design methods, significant human error is the main cause of the failure of steel structures.

Geis et al. studied 1029 cases of snow-induced building failures in the United States from 1989 to 2009 and 91 international cases in 16 countries across four continents from 1979 to 2009 [11]. Paper archives provided data for this study from 1345 articles from 883 distinct sources. They found that the primary causes of snow-related building failure were excessive snow (89% of total incidents), rain-on-snow events (13% of total incidents), and building problems (9% of incidents). As buildings age, their structural members deteriorate and may be damaged. A higher percentage of incidents was attributed to building problems in older buildings: 28% of historic buildings and 26% of middle-aged buildings, while only 5.4% of new buildings were part of the U.S. dataset. Other contributing factors listed in the surveys included melting and drifting snow, drainage, and people living on the roof. In addition, Alinaitwe and Ekolu identified several causes for the collapse of structures during the construction phase, including poor materials and workmanship, design and construction errors, lack of professional supervision of site work, improper implementation of construction methods, and neglecting design approval procedures. They argued that construction failures could be prevented if proper procedures were followed during the design, construction, and operation of structures [12].

Oloyede et al. conducted extensive research and collected data by administering questionnaires to professionals in the building industry, including contractors, builders, architects, estate surveyors and valuers, civil engineers, electrical engineers, structural engineers, and town planners [13]. The data were analyzed using descriptive and analytical statistics. Their findings indicated that the causes of building collapse included soil type, poor building design and planning, use of low-quality building materials, and employment of incompetent craftsmen, resulting in poor workmanship, weak supervision, natural disasters, and corruption.

It is envisioned that active involvement and quality input by professionals in the building industry from design to construction, including supervision at each stage, would be vital to ensure adherence to standards and procedures [14]. It is becoming increasingly difficult to learn from mistakes in the prevention, detection, and limitation of design errors. Kamara et al. [15]. observed that most failures and their ensuing damage are rooted in planning, design, construction, and use errors rather than in construction material variability, strengths, and structural loads. It should be borne in mind that design is the first step in construction, and design faults have been the fundamental cause of many disasters, leading to the death and injury of workers and the public [15].

Recent advances in computer-aided design and high-strength materials have optimized modern structures compared with their ancestors. However, optimization reduces the inherent margin of safety; thus, modern structures have a minimal excess capacity to handle unexpected loads. Therefore, modern structures are prone to unexpected loading [16]. Gross and McGuire also indicated that new construction forms contributed to increased vulnerability, which reduced costs but sacrificed the strength and continuity inherent in older construction forms [17].

A few studies have examined inaccurate results for the numerical modeling of truss nodes. Truss collapse was found to be a result of incorrect node design. The control results of numerical calculations by analytical calculations of statically determinate systems were proposed, which will help determine the error at the initial stages of design [18].

The possible modes of structural failure can be broadly classified into three types: large local plasticity, instability, or fracture. Large local plasticity indicates that deformation of the material occurs over and beyond the elastic limit and is often accompanied by the prospect of large plastic deformation. Instability is the occurrence of a sudden change in the response initiated by a load,

which results in bifurcation, branching, or non-bifurcation. It is usually initiated by direct tensile rupture, fatigue failure, and brittle fracture and is a critical failure mode. In real structural failures, a mixture of these failure modes occurs, with special consideration given to the interaction between the large local plasticity and instability. Normally, instability precedes or acts alternatively to plastic action, leading one member to have less strength and stiffness under given circumstances into a hinged mechanism. Both failures can be better explained by examining the load-displacement behavior characteristics of each type. Depending on the type of member, loading conditions, and type of support, the load-deflection curves or instantaneous stiffness of the member are perhaps the single most important parameters used in the prediction of load-carrying capacity and stability [19].

These notable cases can serve as valuable tools in engineering education; however, they might misleadingly suggest that only large and uncommon structures are prone to failure. Moreover, incidents that do not result in the loss of human lives, which constitute the majority, can offer valuable insights into engineering malpractice, demonstrating that any area of engineering practice can present complex challenges [18]. The general structural performance of steel industrial buildings can primarily be evaluated by analyzing the individual structural elements and overall structural performance. A case study was conducted to assess the performance of compressed structural elements and the overall performance of the structure.

II. Methodology

This study examined two cases of errors and their impact on the structural performance of industrial buildings. In the first case study, construction errors, such as a lack of local reinforcement, large eccentricities, and symmetry deviations, are analyzed; in the second case, design errors, such as insufficient bracings and insufficient cross-section size, are analyzed.

An industrial steel building, length 23.5 m, free span 20 m, clear height 20m, was designed and constructed in the Republic of Azerbaijan in Baku city. The purpose of this building is to serve as a painting and blasting chamber for the ship hull blocks. The building is designed and constructed as a volumetric structure. The transverse frame consisted of steel columns from the H section of 300 × 300 × 17 mm and a truss. The truss elements were all double angles: upper and lower chords 75x75x6 mm, vertical bars 75x75x6 mm, bracing 75x75x6 mm and 50x50x6 mm. Purlins Channel 150x75x10 mm and an angle of 95 × 75 × 7 mm. Vertical bracing between columns from cross angle + 100 × 100 × 10 mm, vertical bracing between trusses from double angle 75 × 75 × 6 mm. The steel used had a yield strength of 235 MPa and tensile strength of 400 MPa for all elements. The building was considered under the Azerbaijan National Construction and Design codes and standards [20].

2.1. Lack of local reinforcement and eccentricity connections in column

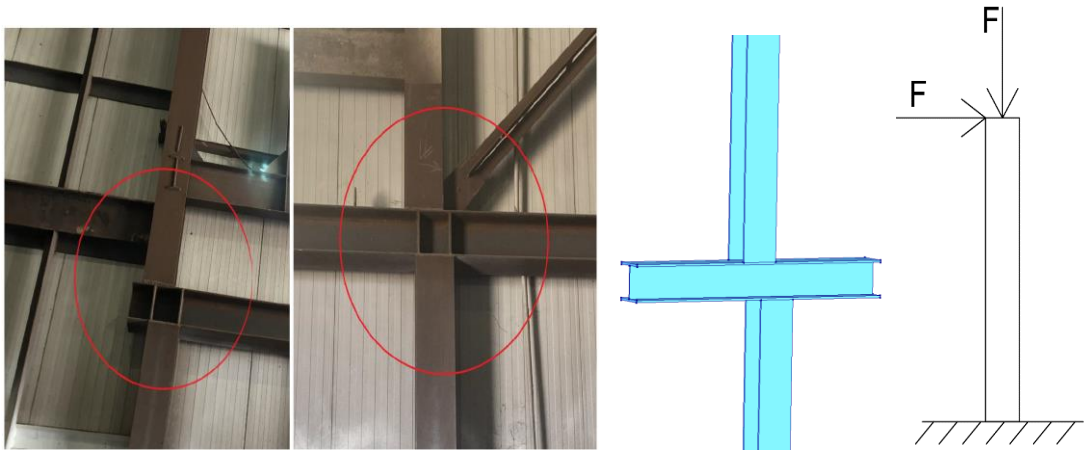
In engineering practice, tall steel columns are reinforced with brackets (Fig. 1). To study the performance of the columns, FE models were created and subjected to 3D performance analyses. The first model was created based on the actual on-site conditions without reinforcement (Fig. 1). The second model was created with brackets installed every 1 m [21,22].

Large eccentricities in the column end-to-end connection introduce two forces that create moments. This moment induced additional stresses in the column, which were superimposed on the axial compressive stress. To study the performance of the columns, three FE models were created and subjected to 3D performance analyses. The first model was created according to actual onsite conditions with an eccentricity of 100 mm (Fig. 2a). The second model was created

according to the actual onsite conditions with an eccentricity of 150 mm (Fig. 2b). The third model was created without eccentricity (Fig. 1b). In both cases, the boundary conditions were applied as a fixed supported column with axial and lateral forces at the top tip, as shown in (Fig. 2c).

(a) (b) (c)

Figure 1. *Column of industrial building: (a) actual conditions on site; (b) FE model of steel column without reinforcement; (c) FE model of steel column with reinforcement.*



(a) (b) (c)

Figure 2. *Column connection eccentricity: (a) actual conditions on site, (b) FE model and (c) analysis scheme*

2.2. Lack of Gusset plates in the compressed members of the truss

A double-angle truss, also known as a double-angle truss, uses pairs of angle sections connected to form truss members. Joining plates (gusset plates) between the angles are used to connect these members at the joints, ensuring structural integrity and proper load distribution. Compressed truss members are particularly susceptible to buckling. Gusset plates provide lateral support to

these members, reducing their effective length and increasing their buckling resistance. Without the gusset plates, the unsupported length of the compressed members increases, making them more prone to buckling under axial loads. The insufficient use of gusset plates may compromise the overall stability of the truss. To study the performance of all the compressed members, they were FE modeled and subjected to 3D performance analyses. Two different models were created for each compressed element: one model expressing real onsite conditions without gusset plates and the other with gusset plates. The boundary conditions in both cases were applied as a pin-supported element with an axial force applied at one end, as shown in (Fig. 3).

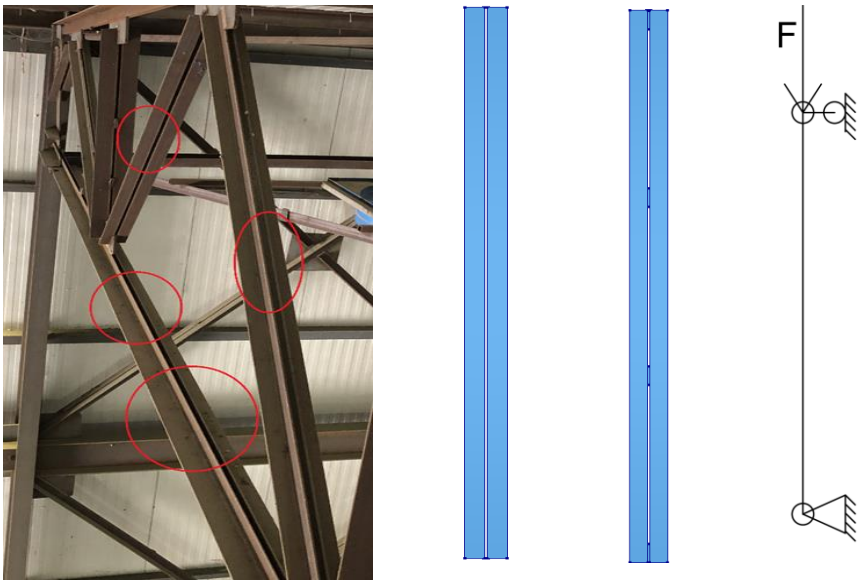
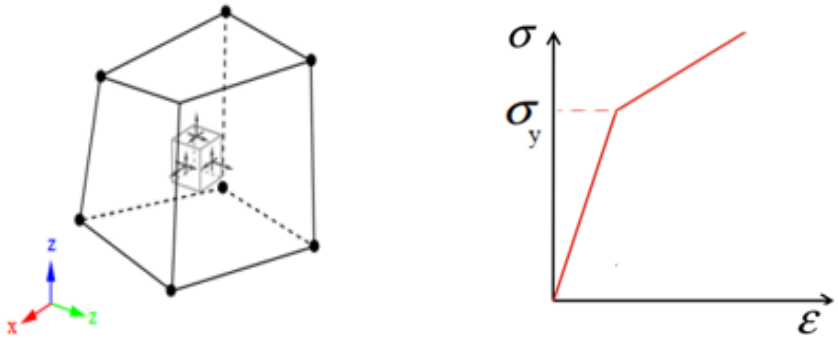


Figure 3. *Compressed truss elements: (a) actual condition on site, (b) without gusset plate, (c) with gusset plate and (d) analysis scheme*

Finite element analyses (FEA) of the structures were conducted using DIANA FEA software. The most comprehensive analysis was provided by the 3D solid element model (Fig. 4a), which examined both the longitudinal and transverse behavior and accounted for the entire three-dimensional response [21,22]. The steel material model was based on the Von Mises plasticity with linear plastic hardening (Fig. 4b). based on the experimental properties [23].
In the analysis stage, an increment in the point load was applied to the steel column up to the maximum tensile stress. Features associated with the load-displacement behavior were modeled in solid modeling, whereby at every step, 100 increments of point load equal to a factor of 3.5, were applied in the numerical models of the behavior under study for load–displacement behavior. The mesh topology used in the analysis was quadratically shaped. For the analysis of the columns, a mesh size ranging from 50 to 60 mm was employed, while a finer mesh size of 25 mm was utilized for the analysis of other structural elements. Essentially, it results in a spurious outcome when an incremental primary method is used through the DIANA program, unless very small steps are taken. This is because nonlinear systems are very sensitive and taking larger steps in an incremental analysis result in numerical instability and a loss of convergence. This limitation can be determined using an iterative approach. In the numerical simulation, the applied loads on the elements increased gradually up to the point of the tensile stress. First, the response of the column was linear elastic; there was a direct relationship between any applied load and the deflection thereof. An updated arc-length control method for the normal plane is used to capture the nonlinear response of the column. This control mechanism ensured that matching of the analysis was performed for the reply

because the element exhibited nonlinear behavior, such as peak tensile stress.

The control method allows the increment of the load at each step to adjust to the evolving element behavior in a dynamic manner through its automatic load step-scaling function. Thus, adaptive scaling improved the nonlinear analysis convergence and made it possible to simulate the reactions of the columns more realistically at failure. The numerical analysis in the present study was performed by employing the Newton-Raphson method adopted with the Line Search Approach, with a selected default value of 25 iterations. The selected numerical technique was unconditionally stable and numerically efficient for analysis.



(a) (b)

Figure 4. *Element and material model for the FE analysis: (a) 3D solid model, (b) Steel model*

2.3. Insufficient bracings and insufficient cross sections size

Inadequate and insufficient bracings were determined by inspection of the blasting chamber. Insufficient bracings and cross-sectional sizes in industrial steel structures can lead to serious structural issues, compromising the overall stability of the structure. This may increase vulnerability to buckling under loads. Without adequate bracing, the structure may experience excessive deflection, which can lead to misalignment and damage to structural and nonstructural elements. The bracings reinforcing the lower cord of the truss were completely missed. Vertical bracings connecting trusses in the middle of the span were applied incorrectly only in the 1st step, whereas vertical bracing between columns was applied in the 2nd and 4th steps. To study the impact of insufficient bracings and insufficient cross-section size on the performance of industrial buildings, two models were created using SAP 2000 software. The first model was created following actual on-site conditions with insufficient bracings and insufficient cross-sectional size. The second model was created with proper bracing according to AzDTN II-23-81 [20]. Both models were exposed to 3D performance analyzes. Pushover analyses were conducted for the overall structure, and the reading point for the results of the pushover curve was considered at the top of the column.

In this study, the method used for static pushover analysis was nonlinear static pushover analysis, which was conducted with the nonlinear version of SAP2000 software [26] applied to three-dimensional structural models. The model was designed as a beamline. The analyses were performed under displacement-controlled conditions until the specified displacement level was reached in the direction of the control and at the control point. In the case of SAP2000, the frame elements were modeled as linear elastic line elements, whereas the nonlinear force–displacement behavior for individual frame elements was represented by hinges through a series of linear segments [27, 28]. Hinge properties can be considered either as default or completely definable by the user [29-31]. Numerically, for user-defined steel moment and PMM hinges, the yield moment, yield rotation, and axial force-bending moment interaction diagrams could have been computed

based on the section and material properties [32,33]. Thus, although user-defined moment-rotation relationships can result in various plastic rotation capacities and strain hardening ratios, the default steel moment and PMM hinges were used in this study for the pushover analyses of steel frames because of their simplicity. By default, pushover analyses using SAP2000 prefer to consider default hinge properties based on the ATC-40 and FEMA-273 criteria [34-36]. This is because specifying the cross-sectional properties of all members of a structure can make pushover analysis challenging to implement, especially when dealing with a three-dimensional structure.

2.4. Symmetry deviations

A blasting chamber was constructed with symmetrical deviations at each step (Fig. 5). Symmetry deviations in the steps of industrial building steel structures can have significant implications for the stability, safety, and functionality of the structure. Asymmetry can lead to an uneven load distribution, causing certain parts of the structure to bear more weight than designed, potentially leading to structural failure. Deviations can affect the dynamic response of a structure to loads such as wind or seismic activities, increasing the risk of resonant vibrations or instability. To study the impact of symmetry deviation on the performance of industrial buildings under seismic and wind loads, two models were created using the SAP 2000 software. The first model was created based on actual on-site conditions with symmetrical deviations. The second model was created without any deviation in symmetry. Both the models were subjected to 3D performance analyses.

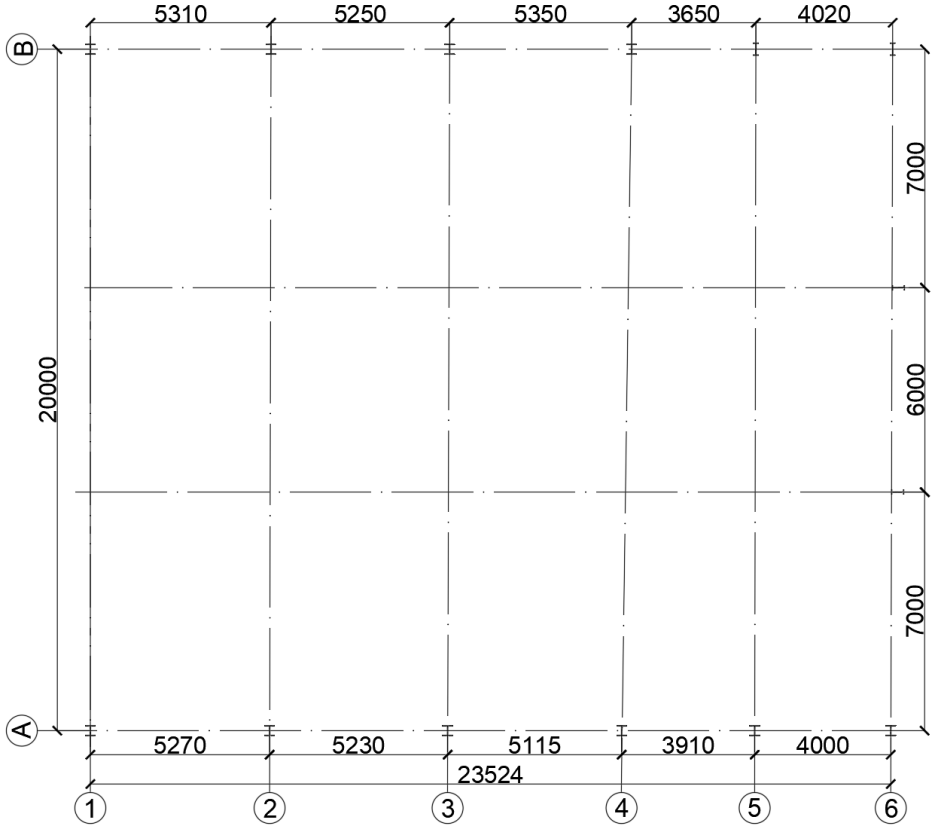


Figure 5. *Plan of existing Blasting Chamber*

III. Results and Discussion

In this research, using detailed case studies, the finite element modelling method was used to assess the load-carrying elements of an existing industrial building under several variables. The analyses focused on the load-bearing elements of existing structures. First, the current conditions of the industrial building were carefully checked, and the geometric and material properties of the structural elements were detailed. Subsequently, a data-driven finite element model was created, and performance analyses were performed under various loads. These were conducted for case studies representing the worst-case scenarios of different loads commonly encountered in industrial buildings. Detailed analyses were performed for the response of the load-bearing elements concerning various parts of the structure and conditions relevant to events when the critical performance limits were exceeded. The results from the analysis clearly demonstrate the performance of the structure under load and the capability of the structure to maintain its integrity. Specifically, cases of overloading and scenarios in which the structural responses exceeded the safety limits for deformation and stress distributions were investigated

3.1. Columns

The deficiencies of the local reinforcement and eccentric column connections were studied in this work. FE models were generated to investigate the column performance, and 3D performance analyses were carried out. First, a model without strengthening was created using existing site conditions. In the second model, strengthening was performed using additional brackets every 1 m. The results of the analyses showed that the peak tensile stress in the column without reinforcement was 400 kN, whereas that in the column with local reinforcement was approximately 500 kN. It can be seen from the load–displacement curve that after reaching the peak tensile stress, the column with local reinforcement did not collapse; although the displacement increased, it was still able to bear 500 kN. In contrast, for a reinforced column, a linear decrease in the bearing capacity with a peak value of the tensile stress was observed beyond the reinforced area, which was compensated by plate reinforcements, thereby improving the functioning of the column (Fig. 6). Hence, the lack of local reinforcement can reduce the structural performance of the column by approximately 20%.

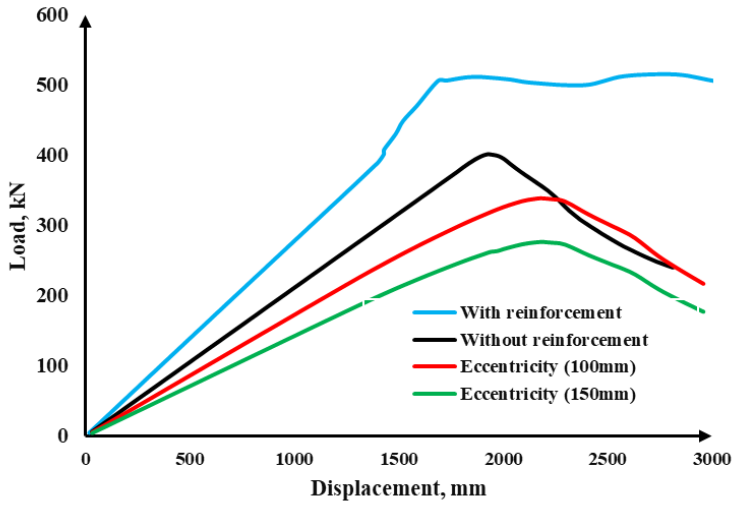


Figure 6. *Load displacement curve of column*

Large eccentricities in the end-to-end connections of the column generated large moments, which caused additional stresses on the column, adding to the original axial compressive stress. Column case studies under different applied eccentricity conditions and reinforcements returned results showing their behavior and structural performance. For example, as shown in Figure 6, in a similar column, the capacity decreased from 400 kN to 300 kN with an eccentricity of 100 mm, whereas for the other column with an eccentricity of 150 mm, the capacity decreased drastically to 240 kN. These produced moments, owing to the eccentricities, increased the stresses inside the column to dangerous values. Columns without reinforcement with 100 mm eccentricity lost 40% of their bearing capacity compared to the properly reinforced column without eccentricities. Columns without reinforcement with 150 mm eccentricity lose 52% of the bearing capacity compared to the properly reinforced column without eccentricities.

Figure 7 shows the stress distribution in the column for the three cases. The installation of the local reinforcement prevented the column web from buckling. Reinforcements provided additional support for the web, which helped distribute stress in a more uniform manner. Figure 7a shows that in the absence of reinforcement, the stress is concentrated in the web of the column. This occurred because the column lacked additional reinforcement, which increased its rigidity, allowing the stress to be distributed more evenly. Consequently, areas with high stress values developed weak points that were susceptible to damage and eventual collapse. In contrast, Figure 7b shows that in the reinforced column, the stress distribution between the two flanges is even. The stress spread uniformly over the cross-section, reducing the likelihood of local stress concentrations. This improves the overall stability of the column and minimizes the risk of buckling or other failures. Consequently, higher loads can be applied to the reinforced column, which remains structurally stable for a longer period.

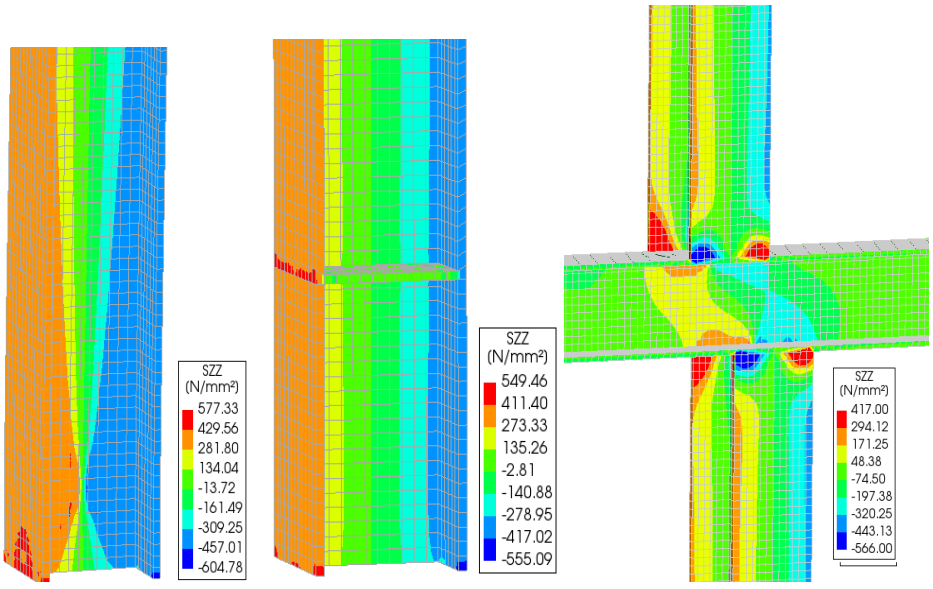


Figure 7. *Stress distribution diagram: (a) actual on-site conditions, (b) with reinforcement, (c) large eccentricities column*

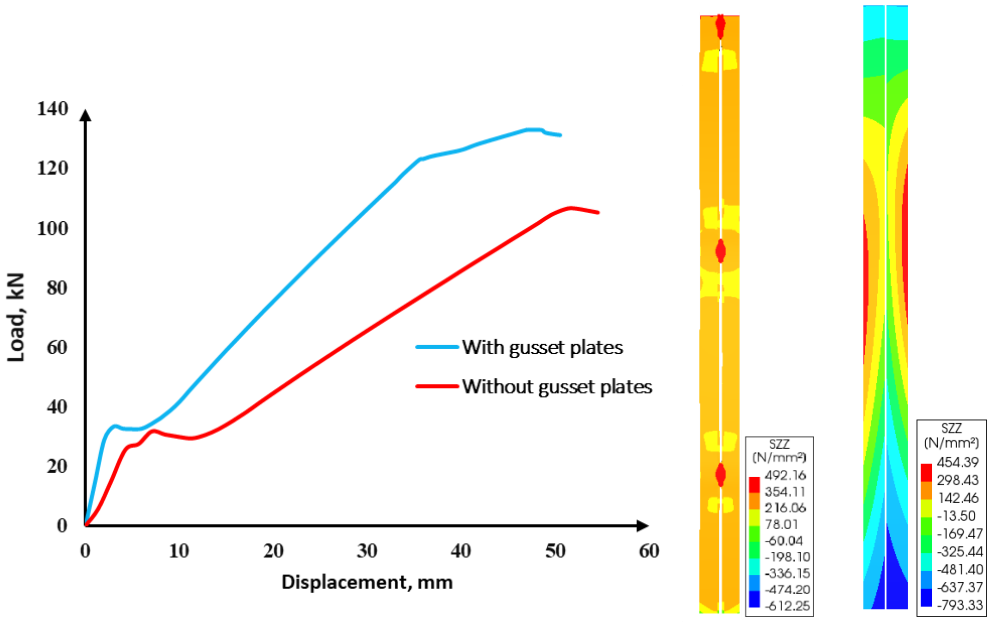
In the case of column end-to-end connections with eccentricity (Fig. 7c), the stress distribution at the critical section became somewhat complicated. Eccentricity is considered to be the lateral deviation of the load path from the central axis of the column, leading to additional stresses owing to the bending moment created. At the failure point, a highly irregular stress distribution prevails, leading to a compound failure mode. Consequently, several types of failures can occur

simultaneously or sequentially. The added eccentricity further complicates the already intricate stress distribution, making it more difficult to predict and mitigate the potential failure zones.

3.2. Truss compressed elements

In this study, detailed analyses were conducted to examine the performance of a double-angle truss. The analysis results explained the differences in the structural performance of the compressed elements with and without gusset plates. A significant increase in the buckling resistance of the elements was observed when they were supported by gusset plates and the integrity of the structure was retained. The obtained results for the load displacement showed that the rigidity value in the elastic region of the curve was higher when the truss was provided with gusset plates. For the models without gusset plates, the effective length of the elements was increased, which already brought danger by way of buckling and risked a decrease in structural stability. Specifically, as shown in Figure 8a, the behavioral change in both trusses came at 38-40 kN of load. In terms of performance, one can see that the element with gusset plates stood at a maximum force of 125-130 kN compared to the one without plates, which only reached a maximum of 105-110 kN. This proves that the gusset plates significantly improved the buckling resistance and increased the rigidity of the structure.

As shown in Figure 8b, the double-angle profiles with the gusset plates have an even stress distribution along the element. The application of the gusset plates significantly increased the buckling resistance by approximately 15-16%. In the truss elements without gusset plates, as shown in Figure 8c, buckling started at specific points of the element, and the profiles did not work efficiently together. This causes high stresses in some areas and buckling problems in others within compressed elements.



(a) *(b)* *(c)*

Figure 8. *Load displacement curve (a), stress distribution diagram of compression truss elements with (b) and without (c) gusset plates*

3.3. Symmetry deviation and inadequate bracing

The results of the static pushover analysis for the current state of the steel industrial building compared to its design according to standards provide valuable information about the volumetric performance of the structure. It has been observed that the structural performance of existing steel building is 550 kN. This value increased to 690 kN in the building designed according to the standards. This indicated a performance difference of approximately 20% (Figure 9). Furthermore, the load-displacement curve shows a significant difference in the formation of the initial hinges in both structures. In the examined existing structure, the first hinge formation in the first column occurred at a load of 450 kN, whereas in the building designed according to the standards, this value increased to 540 kN. This demonstrated an average performance difference of 17% between the two structures. Despite these differences, the collapse modes of both buildings exhibited relatively consistent behavior and similar patterns. These results demonstrate that designing according to the standards significantly improves the overall performance of the structure and provides a safer building.

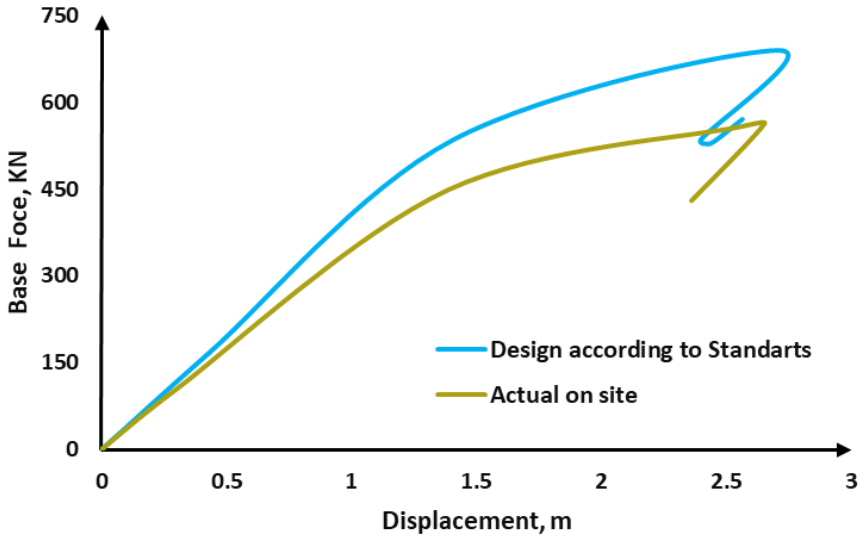


Figure 9. *Pushover curve of the overall steel structure*

IV. Conclusion

This study presents the impact of design and construction errors on the structural reliability of industrial steel buildings, focusing on critical issues, such as insufficient local reinforcement, eccentric connections, symmetry deviations, and inadequate bracing systems. The results indicate the serious consequences of these errors at both the individual structural component and overall building performance level. In the case study, the bearing capacity was reduced by approximately 20% in tall columns owing to insufficient local reinforcement. In the case of no reinforcement with an eccentricity of 100 mm, the bearing capacity was reduced by approximately 40% compared with properly reinforced columns without eccentricity. This was further reduced by approximately 52% for a 150 mm eccentricity. Furthermore, the proper application of gusset plates in the compressed truss elements increased the buckling resistance by approximately 15–16%. Deviations in symmetry and scanty bracing systems resulted in significant structural performance when horizontally loaded. An industrial building built with such a scarcity performed 20% below the

standards. These directly point to an urgent need to pay more emphasis in detail by design and during construction to establish how reliably safe such structures in the form of steel industrial buildings can be achieved, keeping up with common flaws in integrity and performance expectations.

Declaration

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Funding	:	*Not applicable*
Conflict Interests	:	*No conflict of interest*
Data availability	:	*Not applicable*
Authors contribution	:	*All the authors made substantial contribution*
Competing Interests:	:	*No known competing financial interests or personal relationship*

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COST AND RELIABILITY OPTIMIZATION OF A COMPLEX SYSTEM USING MULTI-OBJECTIVE GREY WOLF OPTIMIZATION TECHNIQUE

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Abstract

Modern engineering systems increasingly focus on multi-objective optimization. Nature-inspired optimization techniques have shown superior efficiency and effectiveness compared to many traditional methods across various parameters. This work demonstrates the reliability and cost optimization of a complex bridge system using the Multi-Objective Grey Wolf Optimization algorithm (MOGWO). The bridge system in question is a series-parallel system. A key performance highlight is the use of an archive for search agents to generate a Pareto optimal front (PoF) with a minimal number of iterations. Among the various solutions in the PoF, the solution set that best meets the multi-objective criteria is preferred. Additionally, statistical analyses are conducted to further validate the competitiveness of the results.

Keywords: Nature-inspired optimization techniques, Cost minimization, reliability optimization, multi-objective grey wolf optimization

I. Introduction

Addressing the challenges of real-world nonlinear problems requires models that achieve multiple objectives simultaneously. This necessity arises from the need to maximize reliability while minimizing costs within the expansive search space of reliability issues. Balancing these opposing objectives without compromise is crucial for optimal results. Therefore, multi-objective optimization techniques are employed as effective methods to achieve the desired outcomes under given constraints. Finding the optimal solution is challenging, but nature-inspired optimization techniques have proven highly effective, consistently producing competitive Pareto optimal solution (PoS) sets. In this article, we use an efficient MOGWO technique to optimize the reliability and cost of a complex bridge system.

As compared to single objective optimization problem (SOOP) producing only a single optimum solution, in multi-objective optimization problem (MOOP) a number of solutions are

obtained as a Pareto optimal set. Thus, MOOP determines the desirable set of trade-off solutions for the decision makers to choose their best trade-off solution. As a result, nature inspired optimization methods are being extensively used for achieving the opposing objectives for their efficiency in producing competitive results. Multi-objective optimization algorithms converge to a true approximate global optimal solution as it gives the option of choosing from among a set of Pareto optimal solutions (PoS) satisfying the desired trade-offs between the objectives. This is the result of multi-objective formulation of the problem which explores design parameters of the system with high variation. Better exploration ensures selection from more diversified large search space. This algorithm helps in dealing with local fronts, insolvable areas, separation of the optimum and less diverse nature of solution. This leads to (i) speedy attainment of the global optimum due to the quick sharing of information between the search agents and (ii) better exploration to choose from varied design characteristics and requirements of operations. Efficiency of the multi-objective optimization techniques is based on a number of PoS obtained during the optimization mechanism.

Kumar et al. [1] presented a brief description of the nature of reliability optimization problems along with the different terminologies involved. The authors briefed about various metaheuristic techniques of reliability optimization and also solved problems of complex bridge structure and life support system in a space capsule applying cuckoo search algorithm (CSA). Kumar et al. [2] calculated the availability cost optimization of the butter oil processing plant using GWO technique and compared with the results obtain by CSA. The authors established that Grey Wolf Optimizer (GWO) outperformed the results of CSA. Mirjalili et al. [3] proposed a novel GWO technique based on the social hierarchical behaviour of grey wolves used by them for the hunting mechanism. Nastasi G. et al. [4] applied three variants of genetic algorithm for the problem of multi-objective strategies optimization of steel making industry. The authors gave detailed statistical results with the significant outcomes of the applied techniques.

Zhang and Li [5] proposed an efficient decomposition method of dividing the MOOP into a number scalar optimization sub problems to reduce computational complexity. The authors showed that a set of evenly distributed solutions can be generated with the method used thus highlighted the scalability and sensitivity factor of the technique experimentally. So, to satisfy the multi-objectives covering a variety of design characteristics such computational algorithm is required which avoid local stagnation and also derivatives in the mathematical formulation of the problem. Multi-objective problems have therefore driven a lot of research towards development of meta-heuristics inspired by nature. Nebro et al. [6] proposed the speed-constrained Multi-objective PSO algorithm (SMPSO) and analysed different leader selection schemes. The author suggested based on tests that the hyper volume indicator to guide leader selection is the best for multi-objective PSO algorithms. Pradhan and Panda [7] introduced an extended Cat Swarm Optimization algorithm aimed at identifying non-dominated solutions throughout the search process by employing Pareto dominance principles. This algorithm utilizes an external archive for storage. Their findings suggest that this new method is a promising option for tackling MOOPs.

Shi and Kong [8] investigated enhancements to the multi-objective ACO and introduced the Elitist Multi-objective Ant Colony Optimization (EMOACO) method, which accelerates the parallel search for multiple objectives. Their results demonstrate that EMOACO improves global optimization capabilities and population diversity compared to the basic MOACO, as it quickly converges to PoS and offers a dependable foundation for decision-making. Mirjalili et al. [9] introduced the MOGWO, which incorporates a fixed-size external archive into the GWO for storing and retrieving PoS. This integration helps define the social hierarchy and simulate the hunting behavior of grey wolves. Additionally, Hancer et al. [10] developed a multi-objective artificial bee colony (MOABC) algorithm for feature selection in classification tasks. Their research demonstrated that among the three filter fitness evaluation criteria tested—mutual information, fuzzy mutual information, and a proposed fuzzy mutual information—the proposed fuzzy mutual information

yielded the best results in terms of classification accuracy and the number of features selected. Zhou et al. [11] surveyed the development of MOEAs such as decomposition based MOEA, coevolutionary variant of MOEA, MOEA variant for multimodal problems and MOPs of dynamic, noisy, combinatorial and discrete nature. The authors highlighted the advantages of MOEAs in terms of approximation of the Pareto optimal set from a population of solutions cover the conflicting objectives.

Marler & Arora [12] did an extensive survey of the non-linear multi-objective optimization (MOO) techniques consisting of priori, posteriori and the no articulation preferences method. The authors presented a detailed description of the advantages and the limitations of the MOO techniques including a detailed description of the Genetic algorithm. The survey also highlighted the often, ignored ideas and their utility in engineering problem solving with the emphasis on the fact that there is no best single approach for solving real world optimization problems. Zitzler [13] proposed a novel MOO approach called Strength Pareto Evolutionary Algorithm (SPEA) to investigate the development of heterogeneous hardware/systems and to explore software implementations of multidimensional nature for the digital signal processors. The authors also compared the MOO algorithms developed so far with the experimentally and quantitatively and also investigated the effect of elitism and population size. Deb [14] presented a framework of the principles, application and recent developments in the Evolutionary MOO. The authors discuss the Evolutionary MOO's applicability in multiple criterion decision making (MCDM) procedures to handle of a large number of objectives and also outlined the concepts of multi-objectification and innovation.

Zitzler et al. [15] introduced an enhanced version of the Strength Pareto Evolutionary Algorithm (SPEA), named SPEA2. This improved algorithm incorporates three novel strategies: a fine-grained fitness assignment method, a density estimation technique, and an advanced truncation technique. The comparison of the proposed improved SEPA algorithm with other latest methods reveals better performance of SEPA 2. Messac & Mattson [16] presented a Physical programming-based method for generation of well distributed PoS to obtain an Optimization-Based Design (OBD). The authors presented that the characteristics an OBD may possess are its ability generate all PoS with reasonable ease despite the changes in the parameters of the optimization method. Song et al. [17] presented MOO with parameter matching method based on NGA II algorithm. The authors obtained PoS using PHEV integrated optimization simulation platform with fuel economy effect is increased by 2.26%. Kumar et al. [18] proposed to compute various availability measures applying MOGWO in a nuclear power plant. The authors basically aim to optimize technical specifications for residual heat removal system for safety system of the plant. Tiwari et al. [19] proposed an improved version of the Archive-based Micro Genetic Algorithm called AMGA2 which incorporates a selection strategy for the reducing the chance of missing out on enough exploration of the desirable search space. The algorithm retains a collection of wide range of best solution along with a working population of small size.

Emary et al. [20] proposed MOGWO based feature selection strategy. The authors showed that the results of present version of MOGWO and better performance of present algorithm. Makhadmeh et al. [21] presented MOGWO for minimizing the electricity bill and peak-to-average ratio (PAR) and increasing the comfort level of users of smart homes. The authors established a better performance of the MOGWO for power scheduling problem as compared to GA. Dilip et al. [22] introduced a MOGWO aimed at optimizing the power flow problem. They addressed emission, fuel cost, and active power loss as individual objectives and derived Pareto-optimal solutions (PoS) for two multi-objective scenarios: minimizing fuel cost alongside emission value, and minimizing fuel cost along with active power loss. Their results showed significant competitiveness in these scenarios. Xia et al. [23] proposed a multi-objective optimal function for Hydraulic turbine governing system (HTGS) under multiple operation conditions by applying novel MOGWO with searching

factor called sMOGWO. The authors employed two improvements which include addition of more no-domain solutions with adjustment of control parameters for exploration in the latter period of the process of optimization to finally make the algorithm more effective.

Petrovic et al. [24] developed a MOGWO for scheduling material transport systems using a single mobile robot within an intelligent manufacturing system. They quantitatively assessed and compared the effectiveness of their algorithm against three other algorithms—MOGA, MOAOA, and MOPSO—using four metrics: Generational Distance (GD), Inverted Generational Distance (IGD), Spacing (SP), and Maximum Spread (MS). Experimental results demonstrated the efficiency of their proposed method. Additionally, Darvish [25] applied a non-dominated sorting MOGWO-based fractional-order sliding mode controller (FOSMC) to precisely regulate the active and reactive power of a DFIG-based wind turbine. The FOSMC was designed to handle uncertainties and unmodeled dynamics in the nonlinear, multivariable, time-varying system of the DFIG, showing valid performance.

The present paper optimizes the reliability cost of a complex bridge structure consisting a series parallel configuration. Section II describes the MOO technique, MOGWO algorithm along with the motivation for the algorithm. Section III describes the mathematical formulation of the problem. The discussion of numerical solution, along with graphical representation of the solution of the problem is presented in section IV. The conclusion and future scope are given in section V.

II. Multi-Objective Grey Wolf Optimization Optimizer (MOGWO)

General representation of a linear or nonlinear MOOP is given as

$$Maximize\ (\ Minimize): F(x) = \{f_1(x), f_2(x), \dots, f_h(x)\}. \quad (1)$$

Subject to:

$$p_i(x) \geq 0,\ i = 1,\ 2, \dots . o$$
$$q_i(x) = 0,\ i = 1,\ 2, \dots . n$$
$$H_i(x) \leq x_i \leq G_{i,}\ i = 1,\ 2, \dots . m.$$

There is inherent complexity of the reliability optimization problems having vast exploration area for finding the global optimum solution from among the large population of the candidate solutions which may have the risk of late or early convergence to near optimal solution. Apart from these problems there is a major problem of obtaining more than one objective with the choice trade-offs for suiting the different preferences of the decision makers. Here comes the role of MOO techniques. Also, MOO approach is divided into priori approach converting the different objectives into single objective by using weights the decision makers give to the objectives for the sake of preferences of the different objectives and the other approach being the posterior retaining the multi-objective nature of the problem giving a chance to the model parameters to shape the optimization to the fullest for attaining the global best solutions of pareto optimal set. To avoid the local stagnation problem of the conventional MOO techniques using the deterministic methods applying the mathematical and computer science study, the modern stochastic methods are producing much better results. MOWGO is one of the well-known recent stochastic optimization techniques for MOOP.

Proposed by Mirjalili et al. [3] Grey wolf optimization (GWO) technique has been extended to MOGWO technique by Mirjalili et al. [9]. GWO technique is an optimization method which involves the simulation of the unique hunting mechanism adopted by the grey wolves by following three steps of surveying, encircling and attacking with their social hierarchical behavior. In the technique the search space exploration is done for the candidate solutions and the they are divided into four categories like those of the alpha, beta, delta and the rest as the omega category in the decreasing order of their fitness (hierarchical ability of the wolves). At the end of every iteration the hierarchy is updated. Based on the unique hunting mechanism involving a balanced exploration and

exploitation approach the GWO technique has been developed into MOO technique to achieve the different conflicting objectives of increasing availability and reliability of complex systems along with the cost minimization objective.

The following terms are worth noting:

- Pareto Dominance: For two vectors $x = (x_1, x_2, \ldots . x_k)$ and y $= (y_1, y_2, \ldots . y_k)$, $x > y$ if $\forall\, j \in 1, 2, \ldots \ldots, k, \left[f(x_j) \geq f(y_j)\right] \wedge [\exists j \in 1, 2, 3 \ldots . . k: f(x_j)]$
- PoS: $x \in X$ is called PoS if and only if $\nexists\, y \in$ X for F(y) $>$ F(x). The Pareto optimal set P_s is the set of all PoS.
- PoF: The set of values of the objective functions for Pareto solution set that is $P_i = \{F(x): x \in P_s\}$. The PoF consisting of the values of the objectives for different POF the best suitable values are preferred to satisfy the operating conditions.

The GWO algorithm simulates the hunting mechanism of the wolves for a single optimal solution. MOGWO on the other hand produces a set of solutions called as POS which is a result of the following two strategies employed in MOGWO technique.

(i) An archive responsible for sorting non-dominated PoS

An archive is an ordinary collection of PoS. It has a maximum capacity so the entry of a new solution (new member) to the archive is possible only if the new solution dominates at least one member of the archive or if both the new solution and each of the members of the archive are equally dominating. In case the archive is full then the entry is possible only after the grid mechanism is run followed by the re-arrangement of the segmentation of the search space and omission of one the solutions of the most crowded segment (hypercube). The accommodation of the new solution in the least crowded segment or outside the segment increases the diversity of the final PoS.

(ii) Leader selection strategy that assists to choose (Roulette Wheel method)

As against the three best solutions obtained in the GWO to guide the other search for the global optimum solution, in MOGWO the Pareto optimality restricts the comparison of the solutions. To compensate for this aspect of MOGWO there is a leader selection strategy in which the least crowded segment is offered one of its non-dominated solutions as the alpha, beta or delta wolves.

Probability of selection is given by $P_i = \frac{c}{N_i}$;

$c > 1$ and N_i is the number of obtained PoS in the ith segment.

As three best solutions (or leaders) have to be selected so if there are less than three solutions in the least crowded segment then second least crowded segment is considered for the leader selection and the process continues if there is not enough non-dominated leaders in this segment as well. This process is important to maintain the selection of the different kinds of leaders and explore the un-explored areas of the search space.

Thus, the grid mechanism and leader selection strategies enhance the diversity of the archive as the optimization process advances. Also, the Roulette Wheel method helps to overcome the problem of local front for the MOGWO. MOGWO possesses almost same characteristics of GWO except for the fact that GWO tries to maintain and upgrade the three best solutions whereas the MOGWO does the sorting of the archive members in terms with respect to dominated and non-dominated solutions. Following Figure 1 shows code of the MOGWO [26].

Initialize the grey wolf population Xi (i = 1, 2, , k)

Initialize a, A, and C

Calculate the objective values for each search agent

Find the non-dominated solutions and initialized the archive with them Xα=Select Leader(archive)

Exclude alpha from the archive temporarily to avoid selecting the same leader Xβ= Select Leader(archive)

Exclude beta from the archive temporarily to avoid selecting the same leader Xδ= Select Leader(archive)

Add back alpha and beta to the archive t=1;

while (t < Max number of iterations)

for each search agent

Update the position of the current search agent

end for

Update a, A, and C

Calculate the objective values of all search agents Find the non-dominated solutions

Update the archive with respect to the obtained non-dominated solutions

If the archive is full

Run the grid mechanism to omit one of the current archive members Add the new solution to the archive

end if

If any of the new added solutions to the archive is located outside the hypercubes Update the grids to cover the new solution(s)

end if

Xα=Select Leader(archive)

Exclude alpha from the archive temporarily to avoid selecting the same leader Xβ= Select Leader(archive)

Exclude beta from the archive temporarily to avoid selecting the same leader Xδ= Select Leader(archive)

Add back alpha and beta to the archive t=t+1

end while

return **archive**

Figure 1: *Pseudo code for MOGWO*

III. Mathematical Formulation of Complex Bridge System (CBS)

The system has a total of five components (Fig. 2) each having component reliability r_j, $j = 1,2,3,4,5$.

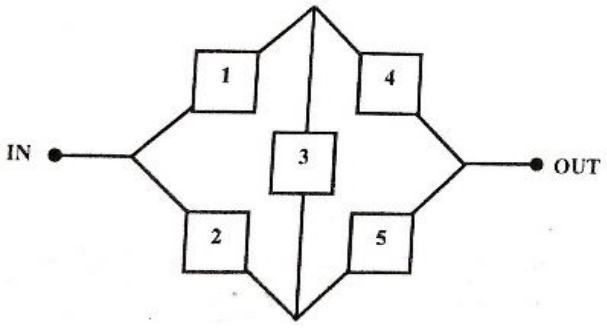

Figure 2: *CBS Block Diagram*

The overall reliability of system, which is probability of success of system, is given by

$$R_s = r_1r_4 + r_2r_5 + r_2r_3r_4 + r_1r_3r_5 + 2r_1r_2r_3r_4r_5 - r_1r_2r_4r_5 - r_1r_2r_3r_4 - r_2r_3r_4r_5 - r_1r_2r_3r_5 - r_1r_3r_4r_5 \quad (2)$$

The cost of j[th] component is taken as

$$c_j = a_j exp\left(\frac{b_j}{(1-r_j)}\right),\ j = 1,2,3,4,5 \quad (3)$$

Thus, the overall system cost is given by,

$$C_S = \sum_{i=1}^{5} a_i \exp\left[\frac{b}{(1-r_i)}\right] \quad (4)$$

The MOOP proposed here is to determine the reliability of components, which minimize both system unreliability and system cost is presented as follows.

To find $(r_1,\ r_2,\ r_3,\ r_4, r_5)$ to minimize (Q_S, C_S)

subject to,

$0.5 \le r_j \le 1,\ j = 1,\ 2,\ 3,\ 4,\ 5$ where

a_i=1 and b_i, =0.0003, $\forall i,\ i = 1,\ 2,\ \ldots.,5$

IV. Results and Discussion

The MOGWO technique used to is successfully used to achieve two opposing objectives of maximizing reliability R_s and minimizing cost C_s. This is done by the POF obtained in the course of the iterations using MATLAB (Fig. 3). The numerical results involve following parameter settings.

Grey wolves =500

Max Iterations= 1000

Archive size =100

Alpha wolves 0.1 % of the Grid Inflation Parameter

Beta wolves 4 % of the Leader selection pressure parameter.

Gamma = 2% (which could be deleted being extra)

N Grid = 10 % per each dimension of the hyper volume of the search space.

Table 1: *Examples of non-dominated optimal solution obtained by MOGWO*

Solutions (Sol.)		**Sol. 1**	**Sol. 2**	**Sol. 3**	**Sol. 4**
Optimum variables	r_1	0.647078	0.955759	0.874055	0.920607
	r_2	0.813646	0.985532	0.85428	0.823586
	r_3	0.666308	0.830469	0.550224	0.828131
	r_4	0.809893	0.968221	0.724388	0.851756
	r_5	0.757558	0.816819	0.918158	0.862526
Optimum system cost	C_s	5.006178	5.040651	5.009874	5.011445
Optimum system reliability	R_s	0.866563	0.992170	0.942139	0.961015
Solutions		**Sol. 5**	**Sol. 6**	**Sol. 7**	**Sol. 8**
Optimum variables	r_1	0.784118	0.967767	0.677376	0.816032
	r_2	0.894535	0.837209	0.721532	0.786603
	r_3	0.630719	0.574196	0.588316	0.624724

	r_4	0.607285	0.962570	0.692794	0.854300
	r_5	0.917022	0.890661	0.691226	0.745382
Optimum system cost	C_s	5.009438	5.022695	5.004686	5.007079
Optimum system reliability	R_s	0.930796	0.987236	0.786834	0.906191

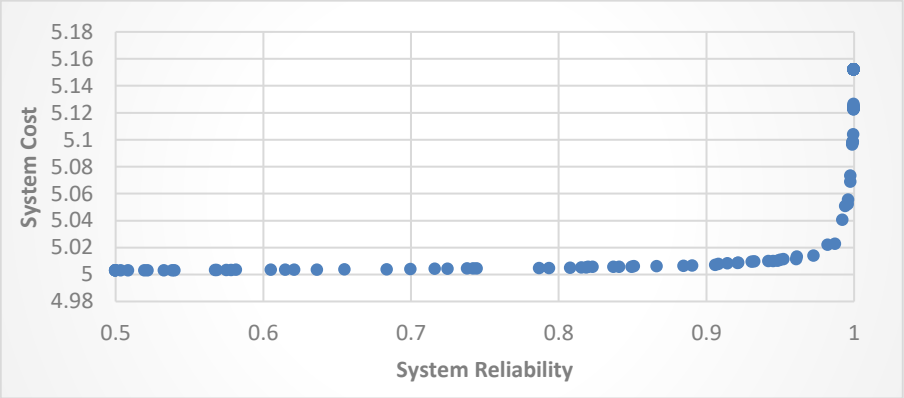

Figure 3: *PoF solutions for the Complex Bridge system by MOGWO*

Table 2: Convergence results of R_s *and* C_s (1000 runs) MOGWO

	Repository size = 100	**Mean**	**Median**	**S. D**	**Minimum**	**Maximum**
MOGWO	System Reliability (Rs)	0.854069	0.943724	0.176146	0.500000	0.999798
	System Cost (Cs)	5.054642	5.009944	0.062221	5.003001	5.152273

V. Conclusion and Further Scope

MOGWO technique with its two module strategies of an archive of solutions for storing and retrieving the best solutions during the progress of the optimization process along with the selection of leader gradually lead to achieve diverse PoS. On one hand the grid mechanism improved the non-dominated solutions in the archive the leader selection mechanism geared the best coverage and convergence. Thus, the exploration and exploitation balance is maintained.

- The above Table 1 presents the numerical results. It includes the estimation of the optimum reliabilities and costs in different run of the MATLAB.
- Total eight sets of PoS corresponding to the optimum reliabilities and cost using MOGWO have been presented in the Table.
- Table 2 presents the average values of optimum reliabilities and costs using simple statistical tools like mean, median and standard deviation.
- Table 2 gives the minimum and maximum values of the mean of all eight PoS. Minimum r= 0.500000 and maximum r= 0.999798 whereas costs range from minimum value 5.003001 to

maximum value of 5.15227.

- By and large the results are competitive as compared to the those of MOPSO.
- Figure 2 indicates very clearly that for the reliability less than r = 0.786834 the minimum cost is approximately same around 5 and less than 5.004686 and also that the minimum cost constantly increases with the increase in the reliability. The highest reliability 0.992170 corresponds to the cost of 5.009874.
- Figure 2 also shows that for reliability is almost same for the values of the minimum cost of 5.009874 approximately.

In the future, MOGWO can be instrumental in evaluating and prioritizing multiple objective problems for solving complex systems with redundant components, ensuring high performance, and achieving optimal cost and efficiency. This approach can be applied in various fields, including telecommunications, optimal power load transmission, artificial neural networks, space program reliability optimization, mutation processes, and other biological and medical areas. The Pareto optimal front (PoF) and multi-criteria decision-making (MCDM) techniques can be utilized to select the most suitable optimal solutions from the PoF, ensuring efficiency throughout the entire operation of complex multi-state systems.

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EXPLORING AN EXTENDED RAYLEIGH DISTRIBUTION: MODELING AND APPLICATIONS IN REAL LIFE SCENARIOS

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Abstract

In this manuscript, we propose a new extension of the Rayleigh distribution, named as Ratio Transformation Rayleigh Distribution (RTRD), which offers superior fits compared to the Rayleigh distribution and several of its known generalizations. We derive various properties of the proposed distribution, including moments, moment generating function, hazard rate, conditional moments, Bonferroni and Lorenz curves, mean residual life, mean waiting time, Renyi entropy and order statistics. The unknown parameters are estimated using the maximum likelihood estimation procedure. An extensive simulation study is conducted to illustrate the behavior of the maximum likelihood estimators (MLEs) based on Mean Square Errors. The flexibility of the new distribution is demonstrated by applying it to two real data sets. Comparative analysis with the Rayleigh distribution, Weighted Rayleigh distribution, Exponentiated Rayleigh distribution and Transmuted Rayleigh distribution reveals that RTRD outperforms these competing distributions based on Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion Corrected (AICC) and other goodness of fit measures.

Keywords: Ratio transformation, Rayleigh distribution , Moments, Conditional moments , Renyi entropy, Maximum Likelihood estimation

1. Introduction

Probability distributions are vital for statistical inference and data analysis, enabling meaningful interpretations and informed decision-making. While classical distributions are widely used across various fields, they often struggle to accurately model real-world data. Consequently, researchers have focused on extending classical distributions to improve their fit and adaptability in data modeling.

The Rayleigh distribution, originally introduced by Rayleigh [22] in the context of acoustics, has been extensively studied in the statistical literature. Several extensions and applications of the Rayleigh distribution (RD) have been proposed over time. Siddiqui [24] explored its genesis and origin, while Howlader and Hossain [14] examined its Bayesian estimation under type-II censored data. Lalitha and Mishra [17] discussed modified maximum likelihood estimation for the Rayleigh distribution. Surles and Padgett [25] introduced the two-parameter Burr type X distribution, referring to it as the exponentiated Rayleigh distribution (ERD) or generalized Rayleigh distribution. Kundu and Raqab [16] investigated parameter estimation techniques for the generalized Rayleigh distribution. Abd Elfattah et al. [1] studied the efficiency of maximum likelihood estimators under different censored sampling schemes. Dey and Tanujit [13] explored

Bayesian estimation of the scale parameter, while Ahmed et al. [4] employed the square error loss function and Al-Bayyati's loss function for Bayesian analysis of the Rayleigh distribution.

Ajami and Jhansi [5] focused on parameter estimation for the weighted Rayleigh distribution, while Ahmad et al. [3] introduced the Weibull-Rayleigh distribution, characterizing and estimating its parameters using the transformed transformer technique. Ardianti [7] applied classical and Bayesian methods to estimate Rayleigh distribution parameters. Bhat and Ahmad [12] proposed a novel extension of the exponentiated Rayleigh distribution, studied its properties, and demonstrated its applicability using various datasets. The same authors [11] investigated the mixture of Gamma and Rayleigh distributions. Kilai et al. [15] developed a versatile modification of the Rayleigh distribution for modeling COVID-19 mortality rates. Bhat et al. [9] proposed a new extension of the odd Lindley power Rayleigh distribution, analyzing its properties and parameter estimation using classical and Bayesian approaches. Bhat and Ahmad [10] introduced a generalization of the Rayleigh distribution using a power transformation technique, while Mir and Ahmad [20] proposed the sine power Rayleigh distribution, examining its properties and applications. Abdelall and Yassmen [2] studied the Marshall-Olkin power Rayleigh distribution with properties and engineering applications. Anis et al. [6] reviewed the Rayleigh distribution, discussing its properties, estimation techniques, and application to COVID-19 data.

This manuscript aims to present and analyze a new lifetime model, termed the Ratio Transformation Rayleigh Distribution (RTRD), developed using the Ratio Transformation (RT) method. A notable advantage of the RTRD is its additional parameter, which imparts desirable properties and enhances the flexibility of its density and hazard rate functions. Furthermore, the model demonstrates superior performance compared to several established distributions when applied to real-world datasets.

The structure of the paper is as follows: Section 2 introduces the RT method. Section 3 outlines the formulation of the RTRD, while Section 4 discusses its statistical properties in detail. The maximum likelihood approach for parameter estimation is addressed in Section 5. Sections 6 and 7 present the results of an extensive simulation study and demonstrate the model's practical applicability, respectively. Finally, Section 8 provides concluding remarks.

2. Ratio Transformation Method

The CDF and PDF of the Ratio Transformation (RT) Method proposed by [18] are defined by the following equations:

$$F_{RT}(x) = \frac{F(x)}{1+\alpha-\alpha^{F(x)}}\ ; \qquad \alpha > 0. \tag{1}$$

$$f_{RT}(x) = f(x)\frac{\left(1+\alpha-\alpha^{F(x)}\left(1-F(x)log\alpha\right)\right)}{\left(1+\alpha-\alpha^{F(x)}\right)^2}\ ; \qquad \alpha > 0. \tag{2}$$

Where $F(x)$ and $f(x)$ in Eq. (1) and Eq. (2) above are the CDF and PDF of the base line distribution respectively.
Rasool and Ahmad [21] explored the Ratio Transformation Lomax distribution and its applications.

3. Ratio Transformation Rayleigh Distribution (RTRD)

The Rayleigh distribution (RD), named after Lord Rayleigh [22] is prominent lifetime probability model concerned with describing skewed data. The probability density function (PDF) associated with random variable $x > 0$ having RD with scale parameter θ is given by

$$f(x;\theta) = \frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\ ; \quad x > 0, \theta > 0 \tag{3}$$

and the corresponding cumulative distribution function (CDF) is given as

$$F(x;\theta) = 1 - e^{-\frac{x^2}{2\theta^2}} \; ; \quad x > 0, \theta > 0 \tag{4}$$

Here we introduce, RT method. Considering F(x; θ) be the CDF of Rayleigh distribution. Then the CDF of RTRD can be obtained by inserting Eq. (4) in Eq.(1) and is given by

$$F(x;\alpha,\theta) = \begin{cases} \frac{1-e^{-\frac{x^2}{2\theta^2}}}{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}} & ;\alpha \neq 1, \alpha, \theta > 0 \\ 1 - e^{-\frac{x^2}{2\theta^2}} & ;\alpha = 1, \theta > 0 \end{cases} \tag{5}$$

The corresponding PDF of RTRD is obtained as

$$f(x;\alpha,\theta) = \begin{cases} \frac{\frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}} \left(1-\left(1-e^{-\frac{x^2}{2\theta^2}}\right) \log \alpha\right)\right)}{\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^2} & ,\alpha \neq 1, \alpha, \theta > 0 \\ \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & ,\alpha = 1, \theta > 0 \end{cases} \tag{6}$$

Figure 1 illustrates the probability density function (PDF) of the RTRD for various parameter combinations of α and θ.

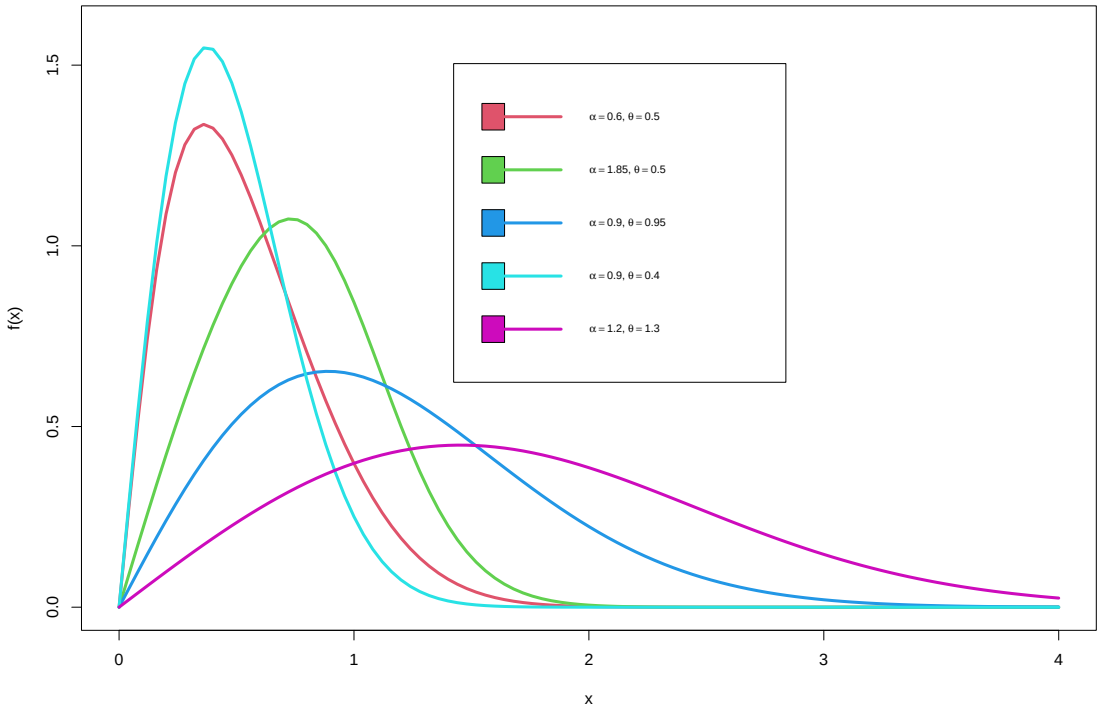

Figure 1: *Plots of the pdf of the RTRD*

3.1. Survival function

The survival function for the RTRD is given as

$$R_{RTRD}(x;\alpha,\theta) = \frac{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}}{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}} \; ; \; \alpha,\theta>0 \tag{7}$$

3.2. Hazard Rate

The hazard rate for RTRD is obtained as

$$h_{RTRD}(x) == \frac{\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x^2}{2\theta^2}}\right)\log\alpha\right)\right)}{\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)\left(\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}\right)} \; ; \; \alpha,\theta>0 \tag{8}$$

Figure 2 depicts graphs of the hazard rate of the RTRD for different parameter values. Figure 2 suggests that the proposed distribution is quite flexible in nature and can exhibit variety of shapes over the parameter space.

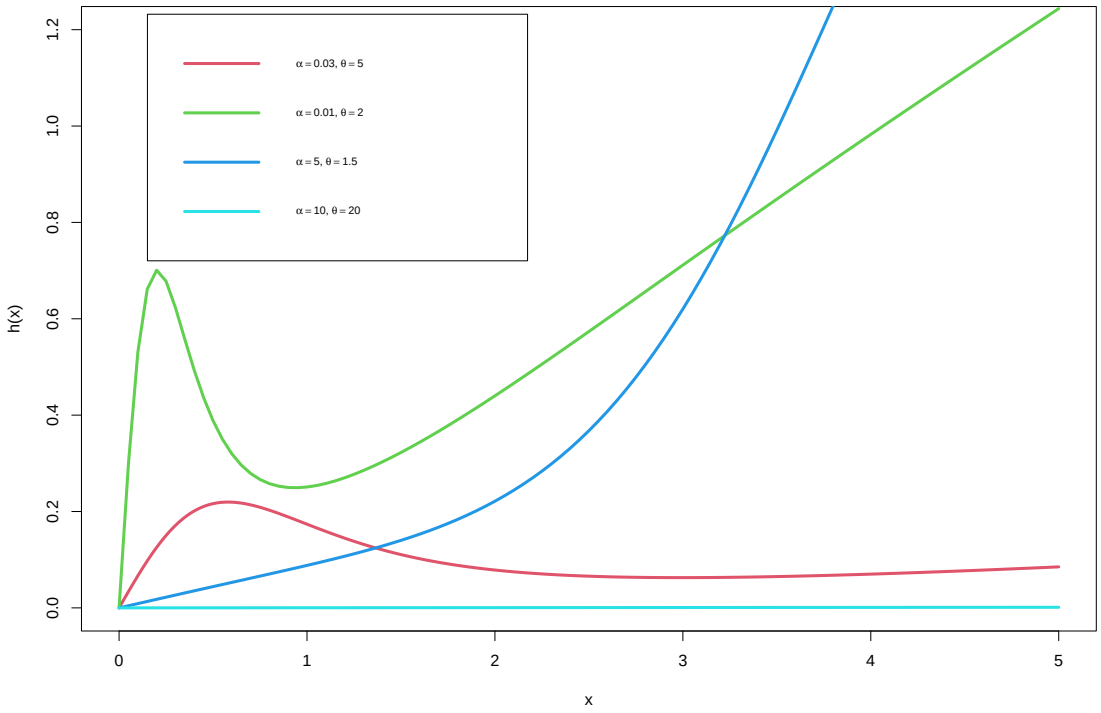

Figure 2: *Plots of the hazard rate of the model*

3.3. Reverse Hazard function

The reverse hazard rate for RTRD is obtained as

$$h_r(x;\alpha,\theta) = \frac{\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x^2}{2\theta^2}}\right)\log\alpha\right)\right)}{\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)\left(1-e^{-\frac{x^2}{2\theta^2}}\right)};\ \alpha,\theta>0 \tag{9}$$

3.4. Cumulative Hazard function

The cumulative hazard function for the RTRD is defined as

$$\Lambda_{RTRD}(x;\alpha,\theta) = \log\left\{\frac{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}}\right\}$$

3.5. Mills Ratio

The mills ratio for RTRD is given by

$$M.R = \frac{1-e^{-\frac{x^2}{2\theta^2}}}{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}} \tag{10}$$

4. Statistical Properties of RTRD

This section focuses on deriving several key mathematical properties, such as the r^{th} moment, moment generating function, conditional moments and their associated measures, entropy and order statistics.

4.1. Moments

The r^{th} moment of X can be obtained as

$$\begin{aligned}E(X^r) &= \int_0^\infty x^r f(x;\alpha,\theta)dx \\ &= \int_0^\infty x^r \frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-(1-e^{-\frac{x^2}{2\theta^2}})\log\alpha\right)\right)\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^{-2}dx.\end{aligned} \tag{11}$$

By substituting $1-e^{-\frac{x^2}{2\theta^2}} = y$ in (11), we get

$$E(X^r) = \sum_{j=0}^{\infty}\frac{1}{(1+\alpha)^{j+1}}\left(\int_0^1\left(-2\theta^2 log(1-y)\right)^{\frac{r}{2}}\left(\alpha^{yj}+\frac{\alpha^{(j+1)y}(j+1)log\alpha}{1+\alpha}y\right)dy\right). \tag{12}$$

Again, substituting $-2\theta^2 log(1-y) = x$ in (12), we get the final expression as

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{r}{2}+1)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\ log\alpha\ (j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{r}{2}+1}} - \frac{1}{(m+2)^{\frac{r}{2}+1}}\right)\right\}. \tag{13}$$

setting $r = 1$ in Eq. (13) the mean of the model is computed as

$$E(X) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{1}{2}}\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{3}{2})\left\{\frac{j^m}{(m+1)^{\frac{3}{2}}} + \frac{\alpha\ log\alpha\ (j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{3}{2}}} - \frac{1}{(m+2)^{\frac{3}{2}}}\right)\right\} \tag{14}$$

Similarly for r = 2, 3 and 4 in Eq. (13) , the second, third and fourth moment about origin are respectively.

4.2. Moment Generating function of RTRD

The following theorem provides the MGF for the RTRD .

Theorem 1. Let X follow the RTRD (α, θ) , then the moment generating function, $M_X(t)$, is

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^r}{r!}\frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma\left(\frac{r}{2}+1\right)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^3} - \frac{1}{(m+2)^3}\right)\right\} \tag{15}$$

Proof: The moment generating function of the RTRD is defined as

$$M_X(t) = \int_0^{\infty} e^{tx} f(x;\alpha,\theta)dx \tag{16}$$

Using the series representation of e^{tx}, we have

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r) \tag{17}$$

Substituting the value of Eq. (13) in Eq. (17) , we get

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^r}{r!}\frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma\left(\frac{r}{2}+1\right)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^3} - \frac{1}{(m+2)^3}\right)\right\} \tag{18}$$

4.3. Conditional moments and associated measures

In this sub section, the expression for conditional moments is acquired. But first we will introduce an important lemma which will be applied in the next sub section.

Lemma 1. Let us suppose a random variable X follows RTRD (α, θ) with PDF given in Eq. (6) and let $\varphi_r(z) = \int_0^z x^r f(x)\,dx$ denotes the r^{th} incomplete moment, then we have

$$\varphi_r(z) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right)\right.$$
$$\left. + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)z^2, \frac{r}{2}+1\right)\right]\right\} \tag{19}$$

where $\gamma(a,b) = \int_0^b z^{a-1}e^{-z}\,dz$ denotes the lower incomplete gamma function.

Proof. Using the PDF of RTRD given in Eq. (6), we have

$$\varphi_r(z) = \int_0^z x^r f(x;\alpha,\theta)dx \tag{20}$$

$$= \int_0^z x^r \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-(1-e^{-\frac{x^2}{2\theta^2}})\log\alpha\right)\right)\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^{-2} dx. \tag{21}$$

By using the same procedure as in the Eq. (13) above, we get

$$\varphi_r(z) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!} \left\{ \frac{j^m}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right) + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)z^2, \frac{r}{2}+1\right)\right]\right\} \tag{22}$$

Setting $r = 1$ in Eq. (22) will yield first incomplete moment as given by

$$\varphi_1(z) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{1}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!} \left\{ \frac{j^m}{(m+1)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{3}{2}\right) + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{3}{2}\right) - \frac{1}{(m+2)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)z^2, \frac{3}{2}\right)\right]\right\} \tag{23}$$

4.3.1 Lorenz and Bonferroni inequality Curves

The Lorenz and Bonferroni inequality curves represent significant applications of the first incomplete moment. For a given probability distribution, they are defined as follows.

$$L_p = \frac{1}{E(X)}\int_0^t xf(x;\alpha,\theta)\,dx = \frac{\varphi_1(t)}{E(X)}$$

$$L_p = \frac{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\{A_2+B_2\}}{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{3}{2})\{A_1+B_1\}}$$

Similarly,

$$B_P = \frac{1}{pE(X)}\int_0^t xf(x;\alpha,\theta)\,dx = \frac{\varphi_1(t)}{pE(X)}$$

$$B_P = \frac{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\{A_2+B_2\}}{p\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{3}{2})\{A_1+B_1\}}$$

Where,

$$A_1 = \frac{j^m}{(m+1)^{\frac{3}{2}}},$$

and

$$B_1 = \frac{\alpha\ log\alpha\ (j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{3}{2}}} - \frac{1}{(m+2)^{\frac{3}{2}}}\right).$$

$$A_2 = \frac{j^m}{(m+1)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)t^2, \frac{3}{2}\right),$$

and

$$B_2 = \frac{\alpha \log \alpha (j+1)^{m+1}}{1+\alpha} \left[\frac{1}{(m+1)^{\frac{3}{2}}} \gamma \left(\left(\frac{m+1}{2\theta^2} \right) t^2, \frac{3}{2} \right) - \frac{1}{(m+2)^{\frac{3}{2}}} \gamma \left(\left(\frac{m+2}{2\theta^2} \right) t^2, \frac{3}{2} \right) \right].$$

4.3.2 r-th Conditional Moment and r-th Reversed Conditional Moment of RTRD

The r^{th} conditional moment of the RTRD is calculated by

$$E\left[X^r | x > t\right] = \frac{1}{R(t)} \int_t^{\infty} x^r f(x;\alpha,\theta)\, dx = \frac{1}{R(t)} \left[E(X^r) - \varphi_r(t)\right]$$

where $R(t)$ is the reliability of RTRD at time t. Inserting the value of Eq.s (7), (13) and (22), we obtain

$$E\left[X^r \mid x > t\right] = \frac{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma\left(\frac{r}{2}+1\right)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{r}{2}+1}} - \frac{1}{(m+2)^{\frac{r}{2}+1}}\right)\right\} - \left(\frac{j^m}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\frac{(m+1)t^2}{2\theta^2}, \frac{r}{2}+1\right) + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\frac{(m+1)t^2}{2\theta^2}, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}}\gamma\left(\frac{(m+2)t^2}{2\theta^2}, \frac{r}{2}+1\right)\right]\right).$$

Similarly, the r^{th} reversed conditional moment of the RTRD is defined by

$$E\left[X^r | x \le t\right] = \frac{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}{1-e^{-\frac{x^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)t^2, \frac{r}{2}+1\right) + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)t^2, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)t^2, \frac{r}{2}+1\right)\right]\right\}.$$

4.3.3 Mean Residual Life (MRL) and Mean Waiting Time (MWT)

Mean Residual Life (MRL) is the expected remaining lifetime of an item that has already survived up to a certain time t. It provides a measure of the average future life expectancy of an item given that it has lasted until time t. The MRL is defined as

$$\mu(t) = \frac{1}{R(t)}\left[E(t) - \int_0^t x f(x;\alpha,\theta)\, dx\right] - t = \frac{1}{R(t)}\left[E(t) - \varphi_1(t)\right] - t$$

After inserting the value of Eq. (7), Eq. (14) and Eq. (23), we obtain the required expression for MRL as

$$\mu(t) = \frac{1+\alpha-\alpha^{1-e^{-\frac{t^2}{2\theta^2}}}}{\alpha\left(1-\alpha^{-e^{-\frac{t^2}{2\theta^2}}}\right)+e^{-\frac{t^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta)^{\frac{1}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left(\Gamma(\frac{3}{2})\left\{A_1+B_1\right\} - \left\{A_2+B_2\right\}\right) - t$$

The mean waiting time is crucial for analyzing the actual time of failure of an item that has already failed. It represents the elapsed time since the failure, assuming the failure happened within the interval $[0,t]$. This mean waiting time, denoted as $\overline{\mu}(t)$, is defined as

$$\overline{\mu}(t) = t - \frac{1}{F(t)}\int_0^t x f(x;\alpha,\theta)\, dx = t - \frac{\varphi_1(t)}{F(t)}$$

$$\overline{\mu}(t) = t - \frac{1+\alpha-\alpha^{1-e^{-\frac{t^2}{2\theta^2}}}}{1-e^{-\frac{t^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left\{A_2+B_2\right\}$$

4.4. Renyi entropy

Theorem 2. If $X \sim \text{RTRD}(\alpha, \theta)$, then the Renyi entropy of the RTRD is given as

$$R_\eta = \frac{1}{1-\eta} \log \left[\left(\frac{1}{\theta^2}\right)^{\eta-1} \left(\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^j (-\log \alpha)^m}{(1+\alpha)^{j+1} m!} \right)^{\eta} \left(2\theta^2\right)^{\frac{\eta+1}{2}-1} \Gamma\left(\frac{\eta+1}{2}\right) \times \left\{ \frac{j^{\eta m}}{(\eta(m+1))^{\frac{\eta+1}{2}}} + \left(\frac{\alpha \ \log \alpha \ (j+1)^{m+1}}{1+\alpha} \right)^{\eta} \left(\frac{1}{(\eta(m+1))^{\frac{\eta+1}{2}}} - \frac{1}{(\eta(m+2))^{\frac{\eta+1}{2}}} \right) \right\} \right]$$

Proof: The Renyi entropy, which Alfred Renyi introduced [23] and generalises Shannon's measure of information, is defined as

$$R_\eta = \frac{1}{1-\eta} \log \int_{-\infty}^{\infty} f^\eta(x; \alpha, \theta)\, dx, \quad \eta > 0, \quad \eta \neq 1$$

By using the same procedure as in the Eq. (13) , we get the final expression for Renyi entropy as

$$R_\eta = \frac{1}{1-\eta} \log \left[\left(\frac{1}{\theta^2}\right)^{\eta-1} \left(\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^j (-\log \alpha)^m}{(1+\alpha)^{j+1} m!} \right)^{\eta} \left(2\theta^2\right)^{\frac{\eta+1}{2}-1} \Gamma\left(\frac{\eta+1}{2}\right) \times \left\{ \frac{j^{\eta m}}{(\eta(m+1))^{\frac{\eta+1}{2}}} + \left(\frac{\alpha \ \log \alpha \ (j+1)^{m+1}}{1+\alpha} \right)^{\eta} \left(\frac{1}{(\eta(m+1))^{\frac{\eta+1}{2}}} - \frac{1}{(\eta(m+2))^{\frac{\eta+1}{2}}} \right) \right\} \right]$$

4.5. Order Statistics of RTRD

Let $x_{(r;n)}$ be the r^{th} order statistics with the random sample $x_{(1)}, x_{(2)}, x_{(3)}, ...x_{(n)}$ derived from the RTRD having the PDF $f(x;,\alpha,\theta)$ and CDF $F(x;\alpha,\theta)$. Therefore, the PDF and CDF of $x_{(r;n)}$ say $f_{(r;n)}(x)$ and $F_{(r;n)}(x)$ are respectively defined as

$$f_{(r;n)}(x) = \frac{1}{B(n, n-r+1)} \left[F(x;\alpha,\theta)\right]^{r-1} \left[1 - F(x;\alpha,\theta)\right]^{n-r} f(x;\alpha,\theta) \tag{24}$$

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[F(x;\alpha,\theta)\right]^{j} \left[1 - F(x;\alpha,\theta)\right]^{n-j} \tag{25}$$

Using Eq. (6) and Eq. (5) in Eq. (24) and Eq. (25), the PDF and CDF of r^{th} ordered statistics for the RTRD are derived and are expressed as

$$f_{r:n}(x) = \frac{\frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}} \left(1 - \left(1 - e^{-\frac{x^2}{2\theta^2}}\right) \log \alpha\right)\right)}{B(r, n-r+1) \left(1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^{n+1}} \left(1 - e^{-\lambda x^\beta}\right)^{r-1} \left(\alpha \left(1 - \alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right) + e^{-\frac{x^2}{2\theta^2}}\right)^{n-r}.$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[\frac{1 - e^{-\frac{x^2}{2\theta^2}}}{1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}\right]^{j} \left[\frac{\alpha \left(1 - \alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right) + e^{-\frac{x^2}{2\theta^2}}}{1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}\right]^{n-j}$$

5. Estimation

This section covers the maximum likelihood estimation method for determining the unknown parameters, α and θ, of the RTRD.

5.1. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from RTRD with parameters $\alpha, \theta > 0$. Then, the logarithm of the likelihood function of RTRD is given by

$$l = \sum_{i=1}^{n} \log x_i - 2n \log \theta - \frac{1}{2\theta^2} \sum_{i=1}^{n} x_i^2 - 2 \sum_{i=1}^{n} \log \left(1 + \alpha - \alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}} \right) + \sum_{i=1}^{n} \log \left[1 + \alpha - \alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}} \left(1 - \log \alpha \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) \right) \right] \tag{26}$$

The MLEs of α and θ are obtained by partially differentiating equation (26) with respect to the corresponding parameters and equating to zero. We have:

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n} \frac{1 + \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^2 \alpha^{-e^{-\frac{x_i^2}{2\theta^2}}} \log \alpha}{1 + \alpha - \alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}} \left(1 - \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) \log \alpha \right)} - 2 \sum_{i=1}^{n} \frac{1 - \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) \alpha^{-e^{-\frac{x_i^2}{2\theta^2}}}}{1 + \alpha - \alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}} \tag{27}$$

$$\frac{\partial l}{\partial \theta} = \frac{1}{\theta^3} \sum_{i=1}^{n} x_i^2 - \frac{2n}{\theta} + \frac{\alpha \log \alpha}{\theta^3} \sum_{i=1}^{n} x_i^2 e^{-\frac{x_i^2}{2\theta^2}} \alpha^{-e^{-\frac{x_i^2}{2\theta^2}}} \left[\frac{\left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) \log \alpha}{1 + \alpha - \alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}} \left(1 - \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) \log \alpha \right)} - \frac{2}{1 + \alpha - \alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}} \right] \tag{28}$$

The expressions in equations (27) and (28) do not possess a closed-form representation, posing a challenge for obtaining analytical solutions. Consequently, determining the parameter estimates for α and θ becomes intricate. Despite this complexity, numerical methods using R software can be employed to derive these estimates effectively.

6. Simulation Study

In this section, we carry out a simulation study using R software to examine the behavior of the MLEs for various sample sizes. We generate random samples of sizes 25, 75, 150, 300, and 500 from the RTRD and repeat the process 1000 times in R software. Various combinations of parameters are chosen as $(1.8, 2.2)$ and $(3.0, 3.5)$ in relation to the standard order (α, θ). The average MLE values, biases and related empirical mean squared errors (MSEs) were determined for each scenario. The results are presented in Tables 1 and 2. The estimates are stable and close to the true parameter values, as shown in Tables 1 and 2. Furthermore, in all scenarios, the MSE decreases as the sample size increases.

Table 1: *MLE, Bias, and MSE for the parameters α and θ*

Sample size	Parameters		MLE		Bias		MSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	1.8	2.2	2.42229	2.07894	0.62229	-0.12106	1.67886	0.08581
75			2.03232	2.14810	0.23232	-0.05189	0.38731	0.03327
150			1.90745	2.18024	0.10745	-0.01975	0.15921	0.01795
300			1.84770	2.19133	0.04770	-0.00866	0.08808	0.01115
500			1.83268	2.19420	0.03268	-0.00579	0.05256	0.00668

Table 2: *MLE, Bias, and MSE for the parameters α and θ*

Sample size	Parameters		MLE		Bias		MSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	3.0	3.5	3.87207	3.40580	0.87207	-0.09419	4.96014	0.21477
75			3.25054	3.48193	0.25054	-0.01806	1.01548	0.08117
150			3.11418	3.50042	0.11418	0.00042	0.48716	0.04540
300			3.03796	3.50412	0.03796	0.00412	0.21861	0.02087
500			3.01584	3.50210	0.01584	0.00210	0.12086	0.01244

7. Applications to Real Life Data

This section focuses on the application of the proposed model to real-life data sets. The potential of the proposed model is assessed by comparing its performance with several other models, namely Weighted Rayleigh Distribution (WRD) [5], Transmuted Rayleigh Distribution (TRD) [19], Exponentiated Rayleigh Distribution (ERD) [25] and Rayleigh Distribution (RD) [22]. Using two actual data sets, we demonstrate the utility of the RTRD in this section.

Data Set 1: The first data set pertains to the breaking stress of carbon fibers of 50 mm length (GPa). This data has been previously used by [10].

Data Set 2: The second data set represents the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at a gauge length of 20 mm. This data was originally reported by [8]. For illustrative purposes, we consider the same transformed data set as used by [16].

The results presented in Tables 5 and 6 reveal that the RTRD achieves the smallest values of AIC, BIC, and AICC compared to the other competing models. This demonstrates that the RTRD outperforms the base Rayleigh distribution as well as the mentioned competing models. Moreover, its strong performance across two engineering datasets underscores its practical utility and effectiveness in accurately modeling complex data patterns. The results are further supported by Figures 3 and 4.

Table 3: *MLEs of RTRD and competitive models with corresponding SE (given in parenthesis) for Data set 1*

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\eta}$
RTRD	6.9048	1.2180	-	-
	(2.2212)	(0.08174)		
WRD	-	1.3551	2.5727	-
		(0.1234)	(0.7452)	
TRD	-	1.6956	-	-0.9587
		(0.0824)		(0.0929)
ERD	2.3483	0.1919	-	-
	(0.4311)	(0.0245)		
RD	-	2.0491	-	-
		(0.1261)		

Table 4: *MLEs of RTRD and competitive models with corresponding SE (given in parenthesis) for Data set 2*

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\eta}$
RTRD	5.2366	0.6858	-	-
	(1.6048)	(0.0488)		
WRD	-	0.7457	2.2209	-
		(0.0667)	(0.6696)	
TRD	-	0.89478	-	-0.9610
		(0.0443)		(0.1193)
ERD	2.1746	0.6621	-	-
	(0.3875)	(0.0847)		
RD	-	1.0833	-	-
		(0.0652)		

Table 5: *Comparison of RTRD and competitive models for Data set 1*

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
RTRD	170.1694	174.1694	178.5487	174.3599	0.0635	0.9528
WRD	175.7107	179.7107	184.0900	179.9012	0.1104	0.3963
TRD	177.7488	181.7488	186.1282	181.9393	0.1410	0.1446
ERD	177.2735	181.2735	185.6528	181.4640	0.1205	0.2930
RD	196.4168	198.4168	200.6065	198.4793	0.2265	0.0022

Table 6: *Comparison of RTRD and competitive models for Data set 2*

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
RTRD	98.4043	102.4043	106.8725	102.5861	0.0599	0.9654
WRD	100.6399	104.6399	109.1081	104.8217	0.0664	0.9206
TRD	101.9050	105.9050	110.3732	106.0868	0.0887	0.6494
ERD	101.8098	105.8098	110.2780	105.9916	0.0752	0.8293
RD	118.8367	120.8367	123.0708	120.8964	0.1999	0.0080

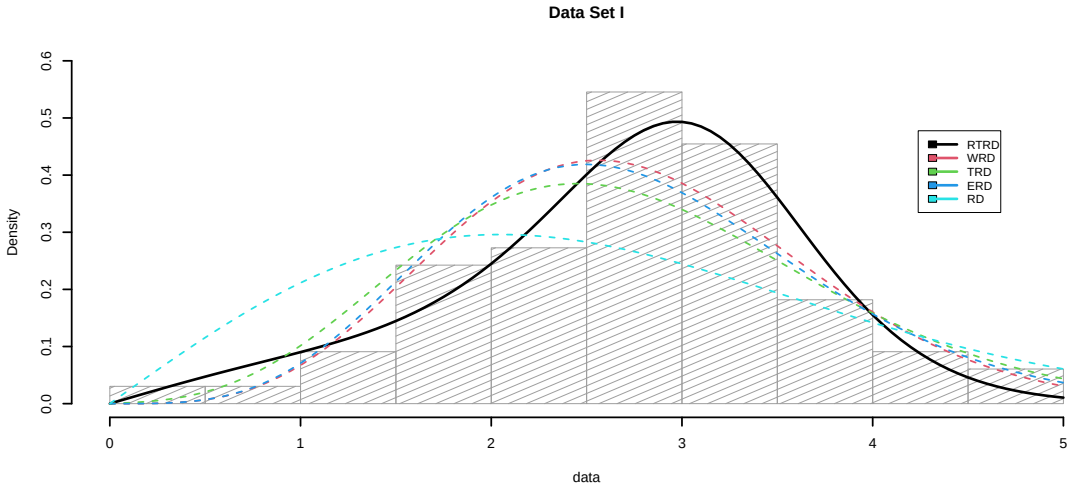

Figure 3: *Fitted density plots for data set 1*

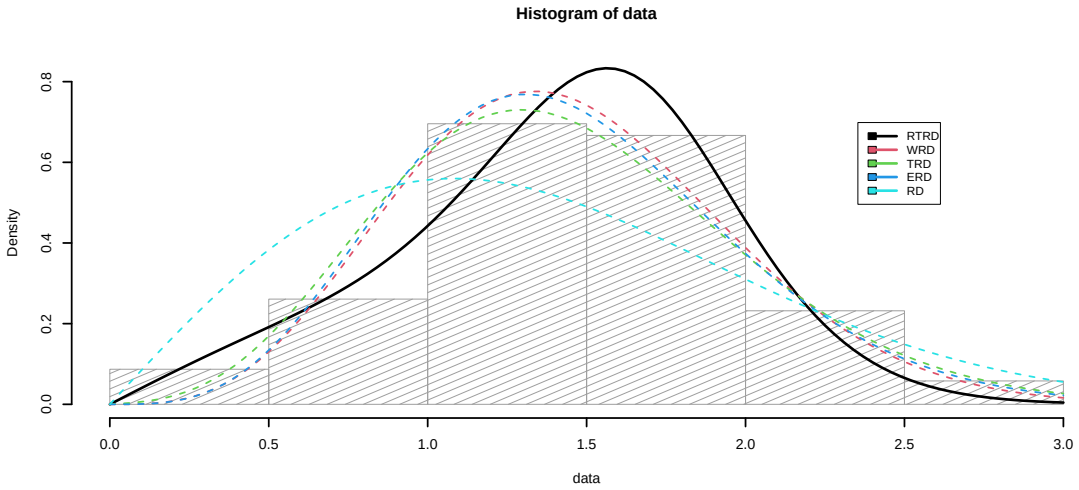

Figure 4: *Fitted density plots for data set 2*

8. Conclusion

In this manuscript, we introduce the Ratio Transformation Rayleigh Distribution (RTRD), a new model that extends the Rayleigh distribution for analyzing data with real support. The motivation behind this generalization is to enhance the flexibility of the standard distribution, thereby improving its ability to model real-world data. We derive key statistical properties of the proposed model and examine several reliability measures. The RTRD showcases greater flexibility, with its hazard rate function exhibiting a variety of complex shapes. A simulation study was conducted to assess the performance of the maximum likelihood estimate, demonstrating both its consistency and precision. Parameter estimation is performed using maximum likelihood estimation. Furthermore, we analyze two real datasets, showing that RTRD provides a superior fit compared to other competitive distributions.The RTRD model was applied in the engineering field to analyze material properties such as breaking stress and tensile strength, successfully capturing complex data patterns. We foresee the RTRD's broad applicability across statistics and other domains, with future research focused on extending the model to multidimensional frameworks.

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THE MARSHALL-OLKIN EXTENDED SHANKER DISTRIBUTION AND ITS APPLICATIONS

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Abstract

In this paper, we introduce the Marshall–Olkin extended Shanker distribution, as an extension of the Shanker distribution, using the Marshall-Olkin approach. Several important properties of the new distribution, such as the hazard rate function, moments, incomplete moments, mean deviations, Lorenz and Bonferroni curves, and Rényi entropy are explored. The estimation of the parameters is discussed with the help of the maximum likelihood method. The performance of the estimators is evaluated using a simulation study. Two real data applications are developed in order to assess the flexibility and power of the new distribution. The goodness of fit criteria reveal that the new model may provide a better fit than the Shanker distribution and other competing models that belong to the Marshall-Olkin G family of distributions.

Keywords: Shanker distribution, Hazard rate function, Moments, Mean deviations, Bonferroni and Lorenz curves, Rényi entropy, Estimation of parameter

1. Introduction

Marshall and Olkin [17] introduced an interesting method of adding a new parameter to an existing distribution. Let $F(x)$ and $\bar{F}(x) = 1 - F(x)$ be the CDF and survival function of the baseline distribution, respectively. Then, using the above-mentioned method, the survival function of the new distribution takes the following form

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}, \quad -\infty < x < \infty, \tag{1}$$

where $\alpha > 0$, and , $\bar{\alpha} = 1 - \alpha$. The corresponding PDF of (1) is given by

$$g(x) = \frac{\alpha f(x)}{[1 - \bar{\alpha}\bar{F}(x)]^2}, \quad -\infty < x < \infty, \tag{2}$$

We note resulting new distribution admits an additional shape parameter, which can affect the

behavior of the hazard rate function of the new distribution. The PDF in (2) is called the Marshall-Olkin extended *G* (MOE-G for short) distribution. Note that for $\alpha = 1$, $F(x) = G(x)$ and thus the new family includes the baseline distribution as its special case. Marshall and Olkin [17] discussed two special cases of (2), which are the MOE exponential and MOE Weibull distributions. Since then many authors implemented the above-mentioned method to obtain a new family of distributions from an existing baseline distribution. For example, Ghitany et al. [10] introduced the MOE Lindley distribution, MirMostafaee et al. [19] proposed the MOE generalized Rayleigh distribution, and Benkhelifa [6] defined the MOE generalized Lindley distribution. Examples of more recent studies include the Marshall-Olkin inverse Maxwell distribution (Yadav et al. [24]), the Marshall-Olkin Sujatha distribution (Ikechukwu and Eghwerido [14]), the Marshall-Olkin two-parameter Lindley distribution (Gillariose and Tomy [11]), the Marshall-Olkin length biased weighted generalized uniform distribution (Mathew [18]), the Marshall-Olkin alpha power inverse Rayleigh distribution (Adegbite et al. [2]). Some general results and mathematical properties of the MOE family of distributions have been discussed in detail by Barreto-Souza et al. [5], and Cordeiro et al. [8].

Shanker [22] introduced a new lifetime distribution, called the *Shanker distribution*, and showed that the new distribution can give closer fits to lifetime data sets than both exponential and Lindley distributions. The Shanker distribution possesses the following probability density function (PDF)

$$f(x,\theta) = \frac{\theta^2}{\theta^2+1}(\theta + x)\mathrm{e}^{-\theta x}, \quad x > 0, \ \theta > 0. \tag{3}$$

The corresponding cumulative distribution function (CDF) is also given by

$$F(x,\theta) = 1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}\mathrm{e}^{-\theta x}, \qquad x > 0, \ \theta > 0. \tag{4}$$

Shanker [22] showed that the PDF of the Shanker distribution is a mixture of an exponential distribution and a gamma distribution, and then discussed many mathematical properties of this distribution. Both Shanker and Lindley distributions involve increasing hazard rate functions (HRFs). There are several generalizations of the Shanker distribution in the literature, for example, Shanker and Shukla [23] presented the power Shanker distribution, Abdollahi Nanvapisheh et al. [1] and Jayakumar et al. [15] introduced the exponentiated Shanker distribution, Alzoubi et al. [3] proposed the transmuted Shanker distribution, Helal et al. [13] worked on the weighted Shanker distribution, and Ganaei et al. [9] suggested the weighted power Shanker distribution.

In this paper, we intend to introduce a new extension of the Shanker distribution using the method developed by Marshall and Olkin [17]. The new model is called the Marshall-Olkin extended Shaker (MOE-Sh for short) distribution. The MOE-Sh distribution involves increasing, increasing-decreasing-increasing and decreasing-increasing HRFs so that it can be a very flexible model in lifetime experiments. The new distribution can work better than some other lifetime distribution in a fitting data problem. The rest of the paper is organized as follows: The new distribution is defined in Section 2. The HRF of the new distribution is discussed in Section 3. Several mathematical properties of the new distribution are investigated in Section 4. Section 5 is devoted to the maximum likelihood (ML) estimation of the parameters. A Monte Carlo simulation is developed in Section 6. Two real data applications are given in Section 7. The paper ends with some remarks in Section 8.

2. The New Distribution

If we let $\bar{F}(x,\theta) = \frac{(\theta^2+1)+\theta x}{\theta^2+1}\mathrm{e}^{-\theta x}, x > 0$, i.e. the survival function (SF) of the Shanker distribution, in

equation (1), we arrive at the following SF

$$\bar{G}(x,\alpha,\theta)=\frac{\alpha(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}},\quad x>0,\ \ \alpha,\theta>0,\ \ \bar{\alpha}=1-\alpha, \tag{5}$$

which is the SF of the MOE-Sh distribution. If a random variable X possesses the SF (5) with parameters α and θ, then we write $X\sim$ MOE-Sh(α,θ). The PDF of the MOE-Sh distribution with parameters α and θ is given by

$$g(x,\alpha,\theta)=\frac{\alpha\theta^2(\theta+x)(\theta^2+1)\mathrm{e}^{-\theta x}}{(\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x})^2}\ ,x>0,\ \ \alpha,\theta>0,\ \ \bar{\alpha}=1-\alpha. \tag{6}$$

The graphs of the PDF of the MOE-Sh distribution for selected values of α and θ are given in Figure 1.

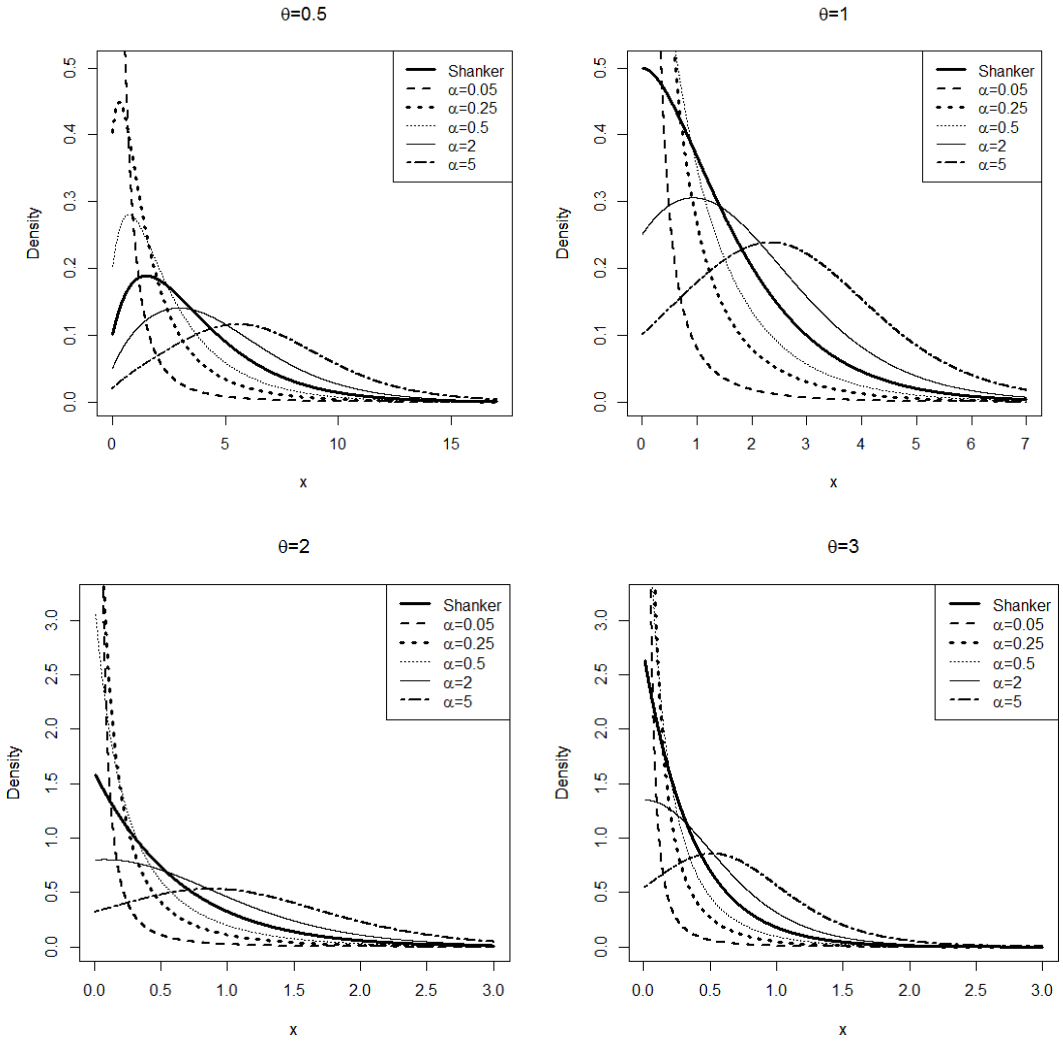

Figure 1: *PDFs of MOE-Sh(α,θ) distribution for selected values of α and θ.*

From Figure 1, we observe that the PDF of the MOE-Sh distribution can be decreasing or unimodal depending on the values of parameters.

The CDF of $X\sim$ MOE-Sh(α,θ) is also given by

$$G(x,\alpha,\theta)=\frac{\theta^2+1-(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}},\quad x>0,\ \ \alpha,\theta>0,\ \ \bar{\alpha}=1-\alpha. \tag{7}$$

3. Hazard Rate Function

The HRF of the MOE-Sh distribution with parameters α and θ is given by

$$h(x)=\frac{\theta^3(\theta^2+1+\theta x)+\theta^2 x}{(\theta^2+1+\theta x)[\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}]}.$$

We see that

$$h(0)=\frac{\theta^3}{\alpha(\theta^2+1)},\qquad \text{and}\qquad \lim_{x\to\infty}h(x)=\theta.$$

Therefore, the HRF of the MOE-Sh distribution is bounded. The graphs of the HRF of MOE-Sh distribution for selected values of α and θ are displayed in Figure 2. From Figure 2, we observe that the HRF of the new distribution can be increasing, decreasing-increasing, or increasing-decreasing-increasing depending on the values of parameters. Note that for example for the case when α =0.5 and θ =3, we see that the HRF decreases and then increases very slowly after it attains its minimum.

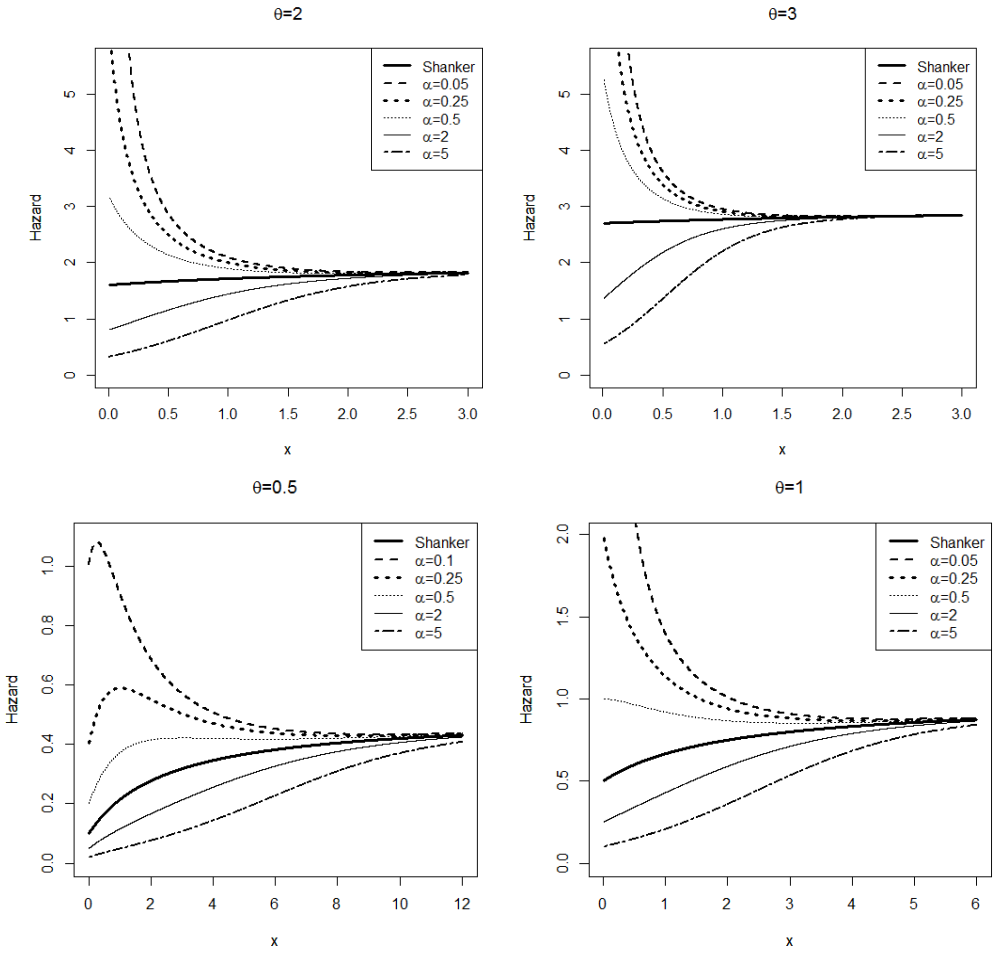

Figure 2. *HRFs of MOE-Sh*(α,θ) *distribution for selected values of* α *and* θ.

4. Several Mathematical Properties of the New Distribution

In this section, we discuss some mathematical properties of the new distribution such as the moment generating function, moments, incomplete moment, the mean deviation from the mean and the mean deviation from the median, Bonferroni and Lorenz curves, and Rényi entropy. First, we obtain an expansion for the density of the new distribution, which will be used to obtain general properties of this distribution in the next discussions.

For $|z| < 1$ and $\rho > 0$, we have

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho)j!} z^j, \tag{8}$$

where $\Gamma(.)$ is the gamma function. Applying (8) to (6), for $0 < \alpha < 1$, gives

$$g(x,\alpha,\theta) = \frac{\alpha\theta^2(\theta+x)\mathrm{e}^{-\theta x}}{\theta^2+1} \sum_{j=0}^{\infty} \frac{\Gamma(2+j)}{\Gamma(2)j!} (1-\alpha)^j \left[\left(1+\frac{\theta x}{\theta^2+1}\right)\mathrm{e}^{-\theta x}\right]^j. \tag{9}$$

Upon applying the binomial expansion to (9), for $x, \theta > 0$ and $0 < \alpha < 1$, we get

$$g(x,\alpha,\theta) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \alpha(j+1)(1-\alpha)^j \binom{j}{m} \frac{\theta^{2+m}(\theta+x)}{(\theta^2+1)^{m+1}} x^m \mathrm{e}^{-(j+1)\theta x}. \tag{10}$$

We can rewrite (2) as follows

$$g(x) = \frac{f(x)}{\alpha\left(1-\frac{(\alpha-1)}{\alpha}F(x)\right)^2}. \tag{11}$$

Therefore, using (8) and (11), and then using the binomial expansion for two times, we arrive at the following expansion for the density when $\alpha > 1$

$$g(x,\alpha,\theta) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \binom{j}{m}\binom{m}{k} \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^j (j+1)(-1)^m \frac{\theta^{2+k}(\theta+x)}{(\theta^2+1)^{k+1}} x^k \mathrm{e}^{-\theta(m+1)x}. \tag{12}$$

4.1. Moment Generating Function

Using (10), the moment generating function of the MOE-Sh distribution with parameters α and θ, denoted by $M_X(t)$, for $0 < \alpha < 1$ is given by

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \binom{j}{m} \frac{\alpha(1-\alpha)^j(j+1)\theta^{2+m}\Gamma(m+1)}{(\theta^2+1)^{m+1}(\theta(j+1)-t)^{m+2}} \big(\theta\big((j+1)\theta-t\big)+m+1\big), \quad t < \theta.$$

For $\alpha > 1$, using (12), we find

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \binom{j}{m}\binom{m}{k} \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^j \frac{(j+1)(-1)^m\theta^{2+k}\Gamma(k+1)}{(\theta^2+1)^{k+1}(\theta(m+1)-t)^{k+2}} (\theta(\theta(m+1)-t)+k+1), \quad t < \theta.$$

4.2. Moments and Related Measures

Some of the most important features and characteristics of a distribution can be studied through its moments. Using (10), the r-th moment of $X \sim$ MOE-Sh(α, θ) for $\alpha \in (0,1)$ has been obtained as

$$\mu_r = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \alpha(1-\alpha)^j \binom{j}{m} \frac{\Gamma(r+m+1)}{(\theta^2+1)^{m+1}(j+1)^{r+m}\theta^r} \left(\theta^2 + \frac{r+m+1}{j+1}\right).$$

Besides, using (12), the r-th moment of $X \sim$ MOE-Sh(α, θ) for $\alpha > 1$ is given by

$$\mu_r = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \binom{j}{m} \binom{m}{k} \frac{1}{\alpha} \left(1-\frac{1}{\alpha}\right)^j \frac{(j+1)(-1)^m \Gamma(r+k+1)}{(\theta^2+1)^{k+1}(m+1)^{r+k+1}\theta^r} \left(\theta^2 + \frac{r+k+1}{m+1}\right).$$

Thus, the mean of the new distribution for $0 < \alpha < 1$ can be expressed as

$$\mu = \mu_1 = E(X) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \alpha(1-\alpha)^j \binom{j}{m} \frac{(m+1)!}{(\theta^2+1)^{m+1}(j+1)^{m+1}\theta} \left(\theta^2 + \frac{m+2}{j+1}\right).$$

Moreover, the mean of the new model for $\alpha > 1$ is given by

$$\mu = \mu_1 = E(X) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \binom{j}{m} \binom{m}{k} \frac{1}{\alpha} \left(1-\frac{1}{\alpha}\right)^j \frac{(j+1)(-1)^m (k+1)!}{(\theta^2+1)^{k+1}(m+1)^{k+2}\theta} \left(\theta^2 + \frac{k+2}{m+1}\right).$$

Now, the skewness and kurtosis of the new distribution can be obtained with the help of the following equations, respectively

$$Skewness = \frac{E([X-E(X)]^3)}{\left(E([X-E(X)]^2)\right)^{3/2}} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{[\mu_2 - \mu_1^2]^{3/2}},$$

and

$$Kurtosis = \frac{E([X-E(X)]^4)}{\left(E([X-E(X)]^2)\right)^2} = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{[\mu_2 - \mu_1^2]^2},$$

where μ_r denotes the r-th moment of $X \sim$ MOE-Sh(α, θ).

Next, we work on finding an expression for the incomplete moment of $X \sim$ MOE-Sh(α, θ). Using (10), for $0 < \alpha < 1$, the incomplete moment of X is given by

$$\int_0^z x g(x, \alpha, \theta) \mathrm{d}x = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{\binom{j}{m}\alpha(1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}} \left(\theta^2 \Gamma(m+2, (j+1)\theta z) + \frac{\Gamma(m+3, (j+1)\theta z)}{j+1}\right), \quad (13)$$

where $\Gamma(a, z) = \int_0^z x^{a-1} \mathrm{e}^{-x} \mathrm{d}x$ is the incomplete gamma function.

Besides, using (12), for $\alpha > 1$, the incomplete moment of X is obtained to be

$$\int_0^z x g(x, \alpha, \theta) \mathrm{d}x = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \frac{\left(1-\frac{1}{\alpha}\right)^j \binom{j}{m}\binom{m}{k}(j+1)(-1)^m \left(\theta^2 \Gamma(k+2, \theta(m+1)z) + \frac{\Gamma(k+3, \theta(m+1)z)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}}. \quad (14)$$

4.3. Mean Deviations and Bonferroni and Lorenz Curves

The mean deviations from the mean and the mean deviation from the median are defined as

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\mathrm{d}x, \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x-M| f(x)\mathrm{d}x,$$

respectively, where $f(x)$ is the density of X, $\mu = E(X)$ and M denotes the median of X.

The measures $\delta_1(X)$ and $\delta_2(X)$ can be computed using the following expressions

$$\delta_1(X) = 2\mu F(\mu) - 2\int_0^\mu x f(x)\mathrm{d}x, \tag{15}$$

and

$$\delta_2(X) = \mu - 2\int_0^M x f(x)\mathrm{d}x, \tag{16}$$

respectively, where $F(x)$ is the CDF of X.

Let of $X \sim$ MOE-Sh(α,θ). Then, for $0<\alpha<1$, from (7), (13) and (15), the mean deviation from the mean becomes

$$\delta_1(X) = \frac{2\mu\left(\theta^2+1-(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}\right)}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}} - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{\binom{j}{m}\alpha(1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}}\left(\theta^2\Gamma(m+2\,,(j+1)\theta\mu)+\frac{\Gamma(m+3\,,(j+1)\theta\mu)}{j+1}\right).$$

Besides, for $\alpha>1$, from (7), (14) and (15), the mean deviation from the mean is given by

$$\delta_1(X) = \frac{2\mu\left(\theta^2+1-(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}\right)}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}} - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\left(1-\frac{1}{\alpha}\right)^j\binom{j}{m}\binom{m}{k}(j+1)(-1)^m\left(\theta^2\Gamma(k+2\,,\theta(m+1)\mu)+\frac{\Gamma(k+3\,,\theta(m+1)\mu)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}}.$$

Moreover, for $0<\alpha<1$, from (13) and (16), the mean deviation from the median becomes

$$\delta_2(X) = \mu - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{\binom{j}{m}\alpha(1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}}\left(\theta^2\Gamma(m+2\,,(j+1)\theta M)+\frac{\Gamma(m+3\,,(j+1)\theta M)}{j+1}\right).$$

Besides, for $\alpha>1$, from (14) and (16), the mean deviation from the median is given by

$$\delta_2(X) = \mu - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\left(1-\frac{1}{\alpha}\right)^j\binom{j}{m}\binom{m}{k}(j+1)(-1)^m\left(\theta^2\Gamma(k+2\,,\theta(m+1)M)+\frac{\Gamma(k+3\,,\theta(m+1)M)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}}.$$

Next, we focus on the formulas of the Bonferroni and Lorenz curves, which are important tools in various fields such as economics, reliability, medicine and insurance. The Bonferroni and Lorenz

curves are defined as

$$B_F(F(x)) = \frac{1}{\mu F(x)} \int_0^x u f(u) \mathrm{d}u, \tag{17}$$

and

$$L_F(F(x)) = F(x) B_F(F(x)) = \frac{1}{\mu} \int_0^x u f(u) \mathrm{d}u, \tag{18}$$

respectively, where $F(x)$ is the CDF of X, $f(x)$ is the density of X, and $\mu = E(X)$.

Let of $X \sim$ MOE-Sh(α, θ). Then, for $0 < \alpha < 1$, from (7), (13), (17) and (18), the Bonferroni and Lorenz curves are given by

$$B_F(F(x)) = \frac{\theta^2 + 1 - \bar{\alpha}(\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x}}{\mu(\theta^2 + 1 - (\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x})} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{\binom{j}{m} \alpha (1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}} \left(\theta^2 \Gamma(m+2, (j+1)\theta x) + \frac{\Gamma(m+3, (j+1)\theta x)}{j+1} \right),$$

and

$$L_F(F(x)) = \frac{1}{\mu} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{\binom{j}{m} \alpha (1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}} \left(\theta^2 \Gamma(m+2, (j+1)\theta x) + \frac{\Gamma(m+3, (j+1)\theta x)}{j+1} \right),$$

respectively.

Besides, for $\alpha > 1$, from (7), (14), (17) and (18), the Bonferroni and Lorenz curves are given by

$$B_F(F(x)) = \frac{\theta^2 + 1 - \bar{\alpha}(\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x}}{\mu(\theta^2 + 1 - (\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x})} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \frac{\left(1 - \frac{1}{\alpha}\right)^j \binom{j}{m} \binom{m}{k} (j+1)(-1)^m \left(\theta^2 \Gamma(k+2, \theta(m+1)x) + \frac{\Gamma(k+3, \theta(m+1)x)}{m+1} \right)}{\alpha \theta (\theta^2+1)^{k+1}(m+1)^{k+2}},$$

and

$$L_F(F(x)) = \frac{1}{\mu} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \frac{\left(1 - \frac{1}{\alpha}\right)^j \binom{j}{m} \binom{m}{k} (j+1)(-1)^m \left(\theta^2 \Gamma(k+2, \theta(m+1)x) + \frac{\Gamma(k+3, \theta(m+1)x)}{m+1} \right)}{\alpha \theta (\theta^2+1)^{k+1}(m+1)^{k+2}},$$

respectively.

4.4. Rényi Entropy

The entropy of a random variable X is the measure of variation of uncertainty. If X is a continuous random variable having PDF $f(x)$, then the Rényi entropy is defined as

$$T_R(q) = \frac{1}{1-q} \log \left\{ \int f^q(x) \mathrm{d}x \right\}, \tag{19}$$

where $q > 0$ and $q \neq 1$.

Let of $X \sim$ MOE-Sh(α, θ). Then, from (6) and (8) and using the binomial expansion, for $0 < \alpha < 1$, we have

$$g(x,\alpha,\theta)^q = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{\Gamma(2q+j)}{\Gamma(2q)j!}\alpha^q(1-\alpha)^j\binom{j}{m}\frac{\theta^{2q+m}(\theta+x)^q}{(\theta^2+1)^{m+q}}x^m\mathrm{e}^{-\theta(q+j)x}.$$

Therefore, the Rényi entropy is given by

$$T_R(q) = \frac{1}{1-q}\log\left\{\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\Gamma(2q+j)}{\Gamma(2q)j!}\binom{j}{m}\binom{m}{k}\frac{\alpha^q(1-\alpha)^j(-1)^{m-k}\mathrm{e}^{\theta^2(q+j)}\gamma\left(q+k+1,\theta^2(q+j)\right)}{\theta^{2k+1-q-2m}\left(\theta^2+1\right)^{m+q}(j+q)^{q+k+1}}\right\},$$

where $\gamma(a\,,z) = \int_z^{\infty} x^{a-1}\mathrm{e}^{-x}\mathrm{d}x = \Gamma(a) - \Gamma(a\,,z)$.

From (6) and (8) and using the binomial expansion for two times, for $\alpha > 1$, we have

$$g(x,\alpha,\theta)^q = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\Gamma(2q+j)}{\alpha^q\Gamma(2q)j!}\binom{j}{m}\binom{m}{k}\left(1-\frac{1}{\alpha}\right)^j(-1)^m\frac{\theta^{k+2q}(\theta+x)^q}{(\theta^2+1)^{q+k}}x^k\mathrm{e}^{-\theta(q+m)x}.$$

Thus, the Rényi entropy becomes

$$T_R(q) = \frac{1}{1-q}\log\left\{\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\sum_{i=0}^{k}\frac{\Gamma(2q+j)}{\alpha^q\Gamma(2q)j!}\binom{j}{m}\binom{m}{k}\binom{k}{i}\frac{\left(1-\frac{1}{\alpha}\right)^j(-1)^{m+k-i}\mathrm{e}^{\theta^2(q+m)}\gamma\left(q+i+1,\theta^2(q+m)\right)}{\theta^{2i+1-q-2k}(\theta^2+1)^{q+k}(q+m)^{q+i+1}}\right\}.$$

5. Maximum Likelihood Estimation

Let $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)$ be an observed random sample of size n from the MOE-Sh distribution with parameters α and θ. Then, the likelihood function of the parameters given $\boldsymbol{x}$ is given by

$$\mathcal{L}(\alpha,\theta|\boldsymbol{x}) = \frac{\alpha^n\theta^{2n}(\theta^2+1)^n\mathrm{e}^{-\theta\sum_{i=1}^n x_i}\prod_{i=1}^n(\theta+x_i)}{\prod_{i=1}^n(\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i})^2}.$$

Thus, the log-likelihood function takes the following form

$$\ell(\alpha,\theta|\boldsymbol{x}) = n\ln\alpha + 2n\ln\theta + n\ln(\theta^2+1) + \sum_{i=1}^n\ln(\theta+x_i) - \theta\sum_{i=1}^n x_i - 2\sum_{i=1}^n\ln\left(\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}\right).$$

Upon taking the derivatives from the log-likelihood function with respect to (w.r.t.) the parameters, we obtain the following equations that might help us to find the ML estimates of the unknown parameters

$$\frac{\partial\ln L}{\partial\alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^n\frac{(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}} = 0,$$

$$\frac{\partial\ln L}{\partial\theta} = \frac{2n}{\theta} + \frac{2n\theta}{\theta^2+1} + \sum_{i=1}^n\frac{1}{\theta+x_i} - \sum_{i=1}^n x_i - 2\theta\sum_{i=1}^n\frac{2-\bar{\alpha}(2-{x_i}^2-\theta x_i)\mathrm{e}^{-\theta x_i}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}} = 0.$$

Numerical procedures such as the Newton-Raphson may be implemented to solve the above

nonlinear equations.

6. A Simulation Study

In this section, we evaluate the performance of the ML estimators of the parameters of the MOE-Sh distribution by means of a simulation study. The inverse transform algorithm is used to generate random data from the MOE-Sh distribution. We generated $N = 10000$ samples of sizes $n = 50, 150, 300$ from the MOE-Sh distribution with the parameter combinations: $(\alpha, \theta) = (0.5, 0.5), (0.5, 4), (3, 2)$, and $(2, 0.5)$. The performance of the ML estimators is assessed by means of the estimated bias (bias for short), the estimated mean squared error (EMSE), and the estimated mean relative error (EMRE). Let $\hat{\alpha}$ be the ML estimator of α and $\hat{\alpha}_i$ be the ML estimator of α that is obtained in the i-th iteration, then the estimated bias, EMSE, and EMRE of $\hat{\alpha}$ can be obtained using the following equations

$$bias(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha), \qquad EMSE(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2, \qquad \text{and} \qquad EMRE(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\hat{\alpha}_i}{\alpha}\right),$$

respectively. We can obtain the estimated bias, MSE, and MRE of $\hat{\theta}$ (the ML estimator of θ) similarly. The numerical results of the simulation are given in Table 1. It is clear from Table 1 that the estimated biases and estimated MSEs decrease when the sample size n increases. Besides, the estimated MREs of all parameters are close to one and approach this nominal value when the sample size increases.

7. Applications

In this section, we provide two real data applications in order to demonstrate the flexibility of the MOE-Sh distribution. We check how well the MOESH distribution fits the data compared to several other lifetime distributions which are

1. The Shanker distribution with the following PDF

$$f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta + x)\mathrm{e}^{-\theta x}, \quad x > 0, \quad \theta > 0.$$

2. The Marshall-Olkin Sujatha (MOS) [14] distribution with the following PDF

$$f(x;\alpha,\theta) = \frac{\alpha\,\theta^3\mathrm{e}^{-\theta x}(\theta^2+\theta+2)(1+x+x^2)}{[(\theta^2+\theta+2)-(1-\alpha)((\theta^2+\theta+2)+\theta x(\theta x+\theta+2))\mathrm{e}^{-\theta x}]^2}, \qquad x > 0, \qquad \alpha,\theta > 0.$$

3. The Marshall-Olkin extended Lindley (MOE-L) [10] distribution with the following PDF

$$f(x;\alpha,\theta) = \frac{\alpha\,\theta^2(\theta+1)(1+x)\mathrm{e}^{-\theta x}}{[\theta+1-(1-\alpha)(\theta+1+\theta x)\mathrm{e}^{-\theta x}]^2}, \quad x > 0, \quad \alpha,\theta > 0.$$

The fitness performance of the considered distributions is investigated using the Akaike information criteria (AIC), Bayesian information criteria (BIC), and Kolmogorov-Smirnov (K-S) along with its p-value. The distribution with the smallest K-S, AIC and BIC values and the highest p-value is considered to possess the best fit to the data sets.

The first real data set, denoted by D I, reported by Chinedu et al. [7] is related to the infant mortality rate per 1000 live births for a few selected nations in 2021, see https://data.worldbank.org/indicator/SP.DYN.IMRT.IN (accessed on 2021). The data are

56, 10, 22, 3, 69, 6, 7, 11, 4, 4, 19, 13, 7, 27, 12, 3, 4, 11, 84, 27, 25, 6, 35, 14, 11, 12, 6

Table 1: *The simulation results*

$\alpha = 0.5$ and $\theta = 4$				
n	Parameters	bias	EMSE	EMRE
50	α	0.135	0.148	1.269
	θ	0.485	2.449	1.121
150	α	0.041	0.032	1.083
	θ	0.153	0.664	1.038
300	α	0.019	0.014	1.039
	θ	0.072	0.311	1.018
$\alpha = 3$ and $\theta = 2$				
n	Parameters	bias	EMSE	EMRE
50	α	0.726	5.057	1.242
	θ	0.079	0.143	1.039
150	α	0.211	1.012	1.070
	θ	0.025	0.042	1.012
300	α	0.105	0.447	1.035
	θ	0.012	0.021	1.006
$\alpha = 0.5$ and $\theta = 0.5$				
n	Parameters	bias	EMSE	EMRE
50	α	0.270	0.543	1.540
	θ	0.038	0.040	1.077
150	α	0.079	0.088	1.159
	θ	0.014	0.011	1.029
300	α	0.039	0.036	1.077
	θ	0.007	0.005	1.014
$\alpha = 2$ and $\theta = 0.5$				
n	Parameters	bias	EMSE	EMRE
50	α	0.678	4.143	1.339
	θ	0.021	0.012	1.042
150	α	0.189	0.685	1.094
	θ	0.006	0.004	1.013
300	α	0.084	0.287	1.042
	θ	0.003	0.002	1.005

The second data set, denoted by D II, was originally taken from Aydin [4]. This data set is related to the average daily wind speed collected in 2015 from meteorological Turkish services, see also Salahuddin et al. [21]. The data are

2.8, 1.8, 3.2, 5.0, 2.4, 4.8, 2.9, 2.9, 2.3, 3.2, 2.3, 2.0, 1.9, 3.3, 4.4, 6.7,

4.3, 1.9, 2.2, 3.3, 2.1, 4.0, 2.0, 3.1, 3.8, 3.1, 3.2, 3.4, 2.8, 2.1, 3.1

We compute the ML estimates of the parameters for the considered distributions. We also use the Kolmogorov-Smirnov (K-S) test, the Akaike information criterion (AIC), and the Bayesian

information criterion (BIC) for the purpose of comparing the fits of the distributions. We know that ties should not be present for the K-S test, when we analyze continuous data. However, ties may arise due to rounding numbers. Here, to avoid this problem, we added (and also subtracted when there are 3 equal numbers) a too small number, which is $z = 10^{-14}$, to one of the equal numbers, when we want to calculate K-S test statistics. For example, D II has been changed to the following data in this regard

$$2.8,\ 1.8,\ 3.2-z,\ 5.0,\ 2.4,\ 4.8,\ 2.9+z,\ 2.9,\ 2.3+z,\ 3.2,\ 2.3,\ 2.0+z,\ 1.9,\ 3.3,\ 4.4,\ 6.7,$$
$$4.3,\ 1.9+z,\ 2.2,\ 3.3+z,\ 2.1+z,\ 4.0,\ 2.0,\ 3.1+z,\ 3.8,\ 3.1,\ 3.2+z,\ 3.4,\ 2.8+z,\ 2.1,\ 3.1-z$$

The computed ML estimates, K-S test statistics along with their corresponding p-values, and the values of AIC and BIC for both data sets are given in Table 2. We note that the smaller values of AIC, BIC and K-S test statistics (and equivalently the larger p-values) indicate a better fit to a data set. Table 2 reveals that the MOE-Sh distribution possesses the best fits for both data sets among the considered distributions. Figures 3 and 4 include the probability-probability (P-P) plots for D I and D II, respectively. From Figures 3 and 4, we might conclude the superiority of the MOE-Sh distribution over the other considered models.

Table 2: *The ML estimates of the parameters, K-S test statistics along with their corresponding p-values, and the values of AIC and BIC for D I and D II*

Data set	Models	α	θ	AIC	BIC	K-S	p-value
D I	Shanker		0.10577	217.2489	218.5447	0.22721	0.1046
	MOS	0.04691	0.06489	214.1856	216.7773	0.13897	0.6245
	MOE-L	0.08771	0.03662	211.8659	214.4576	0.09468	0.9500
	MOE-Sh	0.07497	0.03690	**210.8241**	**213.4158**	**0.08978**	**0.9678**
D II	Shanker		0.54730	120.4348	121.8688	0.34356	0.0009
	MOS	68.37348	2.13868	95.0517	97.9196	0.12053	0.7139
	MOE-L	100.0365	1.90636	94.4033	97.2713	0.11641	0.7518
	MOE-Sh	123.2082	1.88271	**94.2711**	**97.1391**	**0.11478**	**0.7666**

12. Concluding Remarks

In this paper, we follow the Marshall-Olkin strategy of developing more flexible models to introduce a new two-parameter lifetime distribution, called the Marshall-Olkin extended Shanker (MOE-Sh) distribution. Several useful properties of the new distribution are discussed. A simulation study has been conducted to examine the performance of the ML estimators of the proposed MOE-Sh distribution. Two real data applications have been analyzed to illustrate the flexibility of the new distribution in comparison with several competitive distributions. The data analyses indicate that the MOE-Sh has the potential power to model real data quite well and it can be useful in the study of real-life phenomena. Still, there exist some other characteristics of the new distribution such as the reliability parameter, stochastic ordering, order statistics and so on that have not been investigated in this paper. Moreover, some inferential subjects for the new distribution such as the Bayesian estimation of the parameters, prediction of future observations and so on may be considered to be studied in the future.

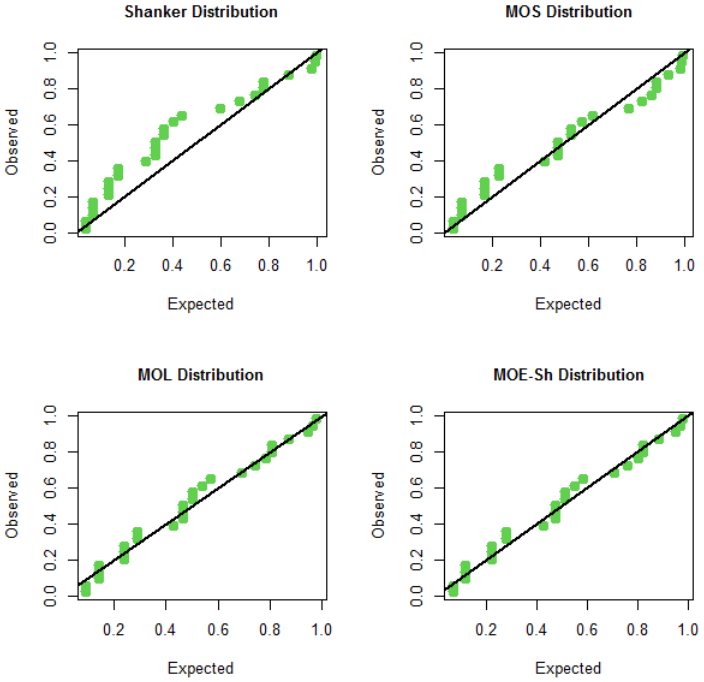

Figure 3. *P-P plots for D I.*

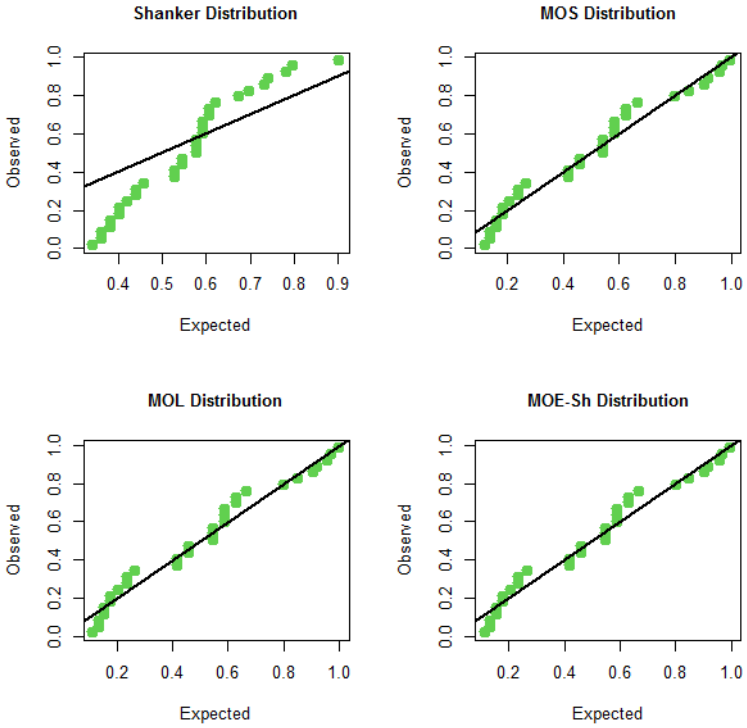

Figure 4. *P-P plots for D II.*

All the computations of the paper were performed using the statistical software R (R Core Team [20]) and the packages *nleqslv* (Hasselman [12]) and *AdequacyModel* (Marinho et al. [16]) therein.

Data Availability Statement

The data sets used in this paper are provided in the manuscript.

Declaration of Conflicting Interests

The Authors declare that there is no conflict of interests.

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ANALYSIS OF THERMAL PROCESSES IN A CONTROLLED ASYNCHRONOUS MOTOR

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Abstract

This article examines the reliability and risks associated with technical systems involved in the conversion of mechanical energy to electrical energy, focusing on the thermal dynamics of electric machines. It explores the processes of heat generation due to energy losses, primarily heat dissipation, and the effects of temperature increases on the longevity and performance of the machine. The cooling systems essential for managing heat transfer and minimizing overheating are analyzed, considering factors such as heat conduction, convection, and radiation, as well as the role of electrohydraulic and aerodynamic systems in optimizing heat exchange. Special attention is given to the impact of temperature fluctuations on the insulation materials of electric machines, with an emphasis on how overheating accelerates insulation degradation and reduces machine lifespan. The paper further discusses the intricate relationship between cooling efficiency, machine power, and the economic implications of designing effective thermal management systems. Moreover, the challenges of selecting and optimizing cooling strategies in electric machine design are highlighted, considering both technical and economic factors. Lastly, the study delves into ventilation calculations necessary to ensure efficient airflow and cooling, using practical equations and methods for determining pressure loss and fan performance, underscoring the complexity and importance of achieving optimal temperature conditions for long-term, reliable machine operation.

Key words: Thermal processes, controlled asynchronous motor, cooling systems, energy losses, insulation degradation, ventilation, power output

I. Introduction

The conversion of mechanical energy into electrical energy and vice versa is accompanied by losses, in which part of the converted energy is converted into heat. The thermal energy of the losses heats the electric machine, is transferred to the surrounding environment, dissipates and is lost without returning. The more intensive the output of heat losses in an electric machine, the more power it is possible to obtain from it for a given overall dimensions of the machine.

Electromechanical conversion of energy is based on many physical laws, the main of which are electromagnetism and mechanics. However, for normal operation, it is impossible to design engines without electrohydraulic and aerodynamic calculations, since these processes are based on the laws of thermo, aero and hydrodynamics.

The transfer of heat from the place of separation to the surrounding environment, that is, heat exchange or heat transfer, is the temperature increase achieved by the parts of the machine relative to the temperature of the surrounding environment. Heat transfer is carried out mainly by three methods: heat conduction, convection and radiation. In electric machines, heat flows sometimes travel along complex paths from the heat source to the surrounding environment through intermediate masses - solid, gaseous or liquid. This is due to the complexity of the design of the electric machine, since it consists of heterogeneous elements, and is also realized by the use of special cooling systems to improve heat transfer. In such systems, the cooling medium or heat

carrier moves, carrying out convective heat transfer. The driving elements as a source of motion: fan, pump or compressor are taken depending on the type and pressure of the medium.

The more heat energy the cooling system can remove from the machine at the permissible temperature of the heated parts, the greater the losses can be allowed and, accordingly, the greater the power load on the machine. Thus, the cooling system has a significant impact on the use of the machine in terms of power. The efficiency of the cooling of the machine is determined by the efficiency of heat exchange between its active parts and the surrounding environment.

One of the main requirements for the cooling system is that it should ensure that the heating of important systems and parts of the machine (windings, bearings, etc.) is within the permissible limit: this is necessary for the long-term and reliable operation of the machine. On the other hand, the cooling system must be effective, so that the costs incurred for its preparation and the operating costs associated with its good performance should be within the acceptable range. The design and calculation of the cooling system, verification of its efficiency by calculating the heating temperature of the main parts of the machine - one of the main important parts of the design process of electric machines, the heating test of the test sample of the machine - is a necessary condition for acceptance. Hydrogasdynamics and, in particular, its engineering hydraulics and the theory of fans can be taken as the theoretical basis for the design of the cooling system. Hydraulic methods are adopted to calculate the flow of liquids and gases, so that their compression and thermodynamic conditions are not taken into account.

The conditions for the transfer of heat flows, heat exchange with the surrounding environment and the formation of temperature fields on this basis are an element of the study of heat transfer theory, which forms the basis of thermal calculations of various electrical machines.

The high temperature of heating of electric machines affects the durability of insulation of coils, pads, collector-brush junction and other elements. High heating temperatures cause temperature wear of the insulation of the windings, which leads to an irreversible decrease in electrical and mechanical strength. According to Montzinger's rule, an increase in temperature of 8^0S - 10^0S reduces the life of the insulation by 2 times. This rule makes it possible to determine the required service life of the machine based on the permissible value of the temperature of the insulation. It is taken into account that the local temperature increase can also cause the insulation to break down in a certain place. In the process of operating the machine, accidents and overloading are also encountered, during which a short-term high temperature regime is observed. The transition process due to temperature in the working process is one of the influencing factors for the normal operation of the machine. For example, it is shown as an example that 5 sec. If the short-circuited insulation cools at a rate of 30^0S/min, its effect on wear is equal to the result of 3 years of normal operation. Thermo-mechanical stresses can also have a significant effect on the work of insulation in circuit-insulation constructions. These are caused by heating of materials and especially rapid changes in temperature - heat shocks during short circuits, start-ups, etc. can be.

II. Formulation of the problem

The indicated situations force to choose the allowable temperature of the insulation with a certain caution, taking into account the experience of operating the machines. In this case, different types of insulating materials and their combinations are divided into classes according to their heat resistance.

The heat resistance classes of insulation A, E, B, F, H have allowable temperatures $V_{bb} = 105^0$S, 120^0S, 130^0S, 155^0S, 180^0S. For electrical machines, the main indicator of heating is the temperature increase of the machine elements relative to the temperature of the surrounding environment - $\Delta\theta$. For normal climatic conditions, the temperature of the surrounding environment is taken to be $\nu_e = 40^0$S. The allowable temperature increase $\Delta\theta_{bb}$ of insulating materials is related to the allowable value of the outer limit, which is determined by the following expression (1):

$$\Delta\theta_{bb} = \nu_k - \nu_e - \Delta\theta_r \quad (1)$$

where $\Delta\theta_r$ is a reserve taking into account the inviscidity of the heat considered for the element of the machine's design.

According to state standards, the following $\Delta\theta e$ values are considered for the windings of electric machines: $\Delta\theta_{bbe}=60^0S$, 75^0S, 80^0S, 100^0S, 125^0S and, accordingly, the change according to the thermometer $\Delta\theta_{bb}T=50^0S$, 65^0S, 75^0S, 85^0S, 105^0S. Of course, as a rule, the values cannot be increased even at $\nu_e<40^0S$. When designing, an additional constructive reserve can be taken: $\Delta\theta_{bb}\approx5\%\div10\%$ (due to errors in thermal calculations and technological errors that arose during the preparation of experimental samples) Uninsulated short-circuited windings and cores that are not in direct contact with insulation and are not directly connected must have a temperature that is not dangerous for the materials in contact with them. For oscillating bearings, the temperature should not exceed 100^0S, and for sliding bearings - 80^0S.

The wide range of rotation frequencies and power of electric machines requires a variety of cooling systems, changing requirements for them and operating conditions. With an increase in the power of the machine, the process of increasing the heat flux is inevitable. As a result, the density of the heat flux in the heat transfer areas and insulation layers increases, which is prevented by increasing the cooling efficiency as the machine power increases [1].

A large number of micromachines operate without any special cooling system. Starting from a few tens of watts of power, the use of a fan is required. At a power of several tens of kilovolts and higher, special ventilation channels are already being used.To protect against the negative effects of the surrounding environment (dust, moisture, etc.), most machine designs are equipped with an external blowing system with internal ventilation. This system is a primary requirement of the designed machine, which works with a speed control program in various cases.

In the modern design process of electric machines, the issues of choosing a cooling system, its calculation and optimization often arise as a complex issue. A broad knowledge base is necessary to overcome these issues.

The cooling system remains one of the important issues for increasing the technical and economic indicators of many types of electric machines. How effective this issue is is clearly seen from the following analysis. The effectiveness of machines with a power of 0.5 ÷ 5 kW and a speed of 1500 rpm over the years is clearly visible are given in Table 1.

Table 1. Analysis of the dependence of the efficiency of electrical machines on the cooling system

Years	1910	1935	1950	1965	1975	2000
Mass reduction, %	120	60	42	38	33	27

Although the efficiency obtained was due to improvements in the characteristics of materials, filling coefficient, design and methods, the increase in the efficiency of cooling was of primary importance.

The efficiency of heat transfer of machines can be increased by various methods, but the economic idea should be the main one in this matter. It should be borne in mind that the machine designed using optimal calculations, based on the minimum prices, will be the most efficient machine [2].

Such an electric machine will have an optimal useful work coefficient (U.W.C) and, accordingly, will meet the optimal loss level. The requirement for the efficiency of the cooling system makes it difficult to eliminate losses while ensuring the permissible value of the permissible temperature of the main parts. The fact that the cooling intensity exceeds the permissible value cannot be justified economically, since as a result it leads to an increase in

ventilation losses and noise generated by the fan; on the other hand, the cost of the cooling system also increases.

Regardless of the requirements set for the designer: maximizing the efficiency of heat transfer or setting the optimal level for it - the fulfillment of the technical and economic requirements for the designed machine of ventilation (or hydraulic) and thermal calculations is very high.

All the above requirements and instructions are fully accepted and effective design, ventilation and optimal U.W.C are created for the designed two-rotor machine.

III. Problem solution

Temperature characteristics of the general structure and its elements.

It is known that the thermal characteristics of electric machines have a special character. Here, the main thing is not only the general structure of the machine or its parts, but also the temperature difference between individual elements of the structure, as well as the cooling medium and the windings. This feature is due to the long-term operating conditions of insulating materials.

When the machine is connected to the load, the temperature increase primarily affects the condition of the insulation and steel of the windings. Under the influence of heat and the mechanical forces associated with it, the properties of insulating materials deteriorate, and over time their quality is lost. As a result of the decrease in quality, the insulating properties are lost and, as a result, thermal or electrical breakdown of the insulation occurs. Thus, the service life of the insulation (the period of trouble-free operation at the rated load) is one of the most important parameters, but it does not depend on the operating temperature, but on the fact that the temperature increase of the active parts of the machine exceeds the temperature of the surrounding air [4].

The negative effect of the absolute temperature increase on the endurance of insulating materials has been relatively well studied. Numerous studies and operational experience show that for each class of insulation materials, a certain temperature level has been determined. A temperature increase of just a few degrees leads to a significant reduction in operating time. The duration of the temperature increase also plays a significant role here.

For some classes of insulation materials, there is a rate law of aging. According to this law, with a temperature increase of $\Delta\theta$ degrees during operation, the operating time of the insulation is reduced by half (2):

$$D_\theta = D_v \cdot 2^{\frac{v-\theta}{\Delta\theta}} \quad (3)$$

where D_θ – operating time at temperature rise; D_v – operating time determined experimentally at temperature θ (for example, 7 years at v=105^0 S for class A insulation materials); $\Delta\theta$ – constant temperature rise. This can also be seen from the characteristics are shown in Table 2.

For class A insulation materials, $\Delta\theta$=8K is usually assumed. For thermosetting insulation (class B), $\Delta\theta$=10K.

Table 2. Dependence of insulation service life on temperature

Year, D_v	0,875	1,75	3,5	7	14	28	56
Temprerature,θ^0S	127	120	113	105	100	90	80

The logarithmic nature of the dependence (3), which requires compliance with strict operating rules for electric machines. Of course, for the thermal conductivity of an electric machine to be good, it must be performed during design. Determining the local temperature value in the machine is an important step. The local temperature is sometimes also called the peak temperature.

It can be said that the peak temperature determines the practical duration of the machine's operation. Therefore, it can be noted that the lower the ratio of the peak temperature to the average temperature, the higher the service life of the machine, and the more useful it is to use the active volume of the machine.

In the design of the presented design, attention should also be paid to the known stator design, and when applying a new ventilation system, the temperature issue in the stator winding should also be considered. Here, the effect of the difference in temperature between the insulation and the cooling medium should be accurately assessed. The temperature difference between the winding copper and the body steel causes the winding to slip in the housing, which results in copper expansion. The winding slippage has a greater effect on the deterioration of the insulation than the temperature increase of the winding copper [5-7].

The temperature process in the electric machine, the design of which is presented, is significantly complicated by the change in the season and also the speed of rotation. If the engine is located in the atmospheric air zone and operates in the speed regulation mode, then at low temperatures of the surrounding environment the local temperature values will also change. Therefore, the temperature should be controlled in all cases. At low speeds, the torque value should not be lower than the ventilation system. Therefore, when considering the operation of the electric machine, the maximum torque value corresponding to the specified speeds should be seriously considered so that the temperature limit in individual insulation elements is normal.

Based on the above considerations, it can be concluded that the operation of the electric machine should be carried out at a constant temperature [3]. Temperature changes damage the integrity of the insulation and, in general, affect the service life of the electric machine. Accordingly, it is always necessary to prevent temperature changes with changes in load and climatic conditions. Of course, a strong connection should be established between the ventilation system and the above changes. The rational operating temperature of the electric machine under given design and power conditions can be achieved by a rational cooling system of the machine. This should be the main goal for the engineer designing the electric machine. In organizing the cooling system of the machine, such heat exchange with the cooling medium should be carried out so that the temperature and temperature increases of the active parts of the machine do not exceed the intended norm relative to the temperature of the cooling medium.

Issues considered in ventilation calculations [8].

The main purpose of the ventilation calculations of the designed electric machine is to determine the selection of the ventilation scheme as a whole, as well as the operation of the air intake elements, thereby ensuring the necessary volume of the cooling medium per unit time, or in other words, to create the necessary air consumption.

The volumetric air consumption (or consumption for short) is the volume of the medium passing through the cross section of the channel per unit time. This is the volume that, as applied to the entire electric machine, passes either through all parallel paths of the ventilation path or through the cross section of the leading (leading) gaps per unit time [14].

The consumption Q is expressed in meters cubed per second and is simply related to the average speed of air movement in the channel ω (3):

$$Q = \omega S \quad (3)$$

where in (3) S- is the cross-sectional area of the channel.

Thus, it is possible to compare the air consumption with the current in the electric circuit, while the speed, that is, the consumption per unit cross-section of the channel, can be considered as the density of the electric current. Since the cooling air takes into account the losses of the electric machine and connects it with the cold environment, the required amount of air is determined by the volume of the removed losses, that is, the design of the electric machine is the result of electromagnetic and thermal calculations. Thus, the nominal consumption Q within the framework of ventilation calculations is a given value.

The air circulating in the ventilation duct channels of the machine overcomes resistance in the direction of its movement. In other words, it is necessary to expend mechanical energy to ensure the circulation of air. This work is performed by air-turning elements, creating a pressure difference at the inlet and outlet cross-sections of the ventilation duct [11-13].

Each ventilation area has resistance, and therefore the total air pressure at the end of the area is always lower (relative to the initial area); the decrease is equal to the pressure loss that has completely disappeared; we denote the pressure loss by Δp. Regardless of the form of air movement (laminar or turbulent), the pressure loss is calculated by the following expression (4):

$$\Delta p = zQ^2 \quad (4)$$

This expression should be understood as follows:

When the pressure loss is proportional to the second power of the flow, the proportionality coefficient Z will be a constant value in the expression (4). If the indicated ratio is not actually observed, the coefficient z will be such that the expression (4) will be true.

The coefficient z is determined for each aerodynamic resistance not only by the dimensions of the duct and the properties of the air, but also by the local resistance ξ. This statement shows that it is necessary to take only the local resistance in a suitable form (for example, from experience).

As a result, the relative coefficient z (called aerodynamic resistance) is calculated by the following expression (5):

$$Z = \xi \frac{\rho}{2S^2} \quad (5)$$

The selection of the air blowing elements of the electric machine, i.e., the determination of the required nominal pressure Pn of the fans, taking into account the given nominal consumption Q_n, can be calculated using the following expression (6):

$$P_n = zQ_n^2 \quad (6)$$

where z is the total aerodynamic resistance of the electric machine.

It can be seen from this statement that the pressure loss in the machine is equal to the pressure created by the air-pumping elements, so equation (6) confirms the fact of equality.

Thus, in the ventilation calculations of the electric machine, its aerodynamic resistance z and equation (6) should be solved. After that, the calculation of consumption distribution in separate branches of the scheme can be made [10].

Based on the known value of the aerodynamic resistance z, the pressure drop in the machine$\Delta P_n = zQ_n^2$ can be determined, but later it should be compared with the nominal pressure of the fan due to the given consumption Q_n.

Here the complexity of the issue arises, as the pressure of the fan is a complex function of the consumption P=φ(Q), and this is called the aerodynamic characteristic of the fan.

Characteristics obtained experimentally or by calculations from a model cannot always be subjected to simple analytical writing. For this reason, writing the equation of equality in analytical form creates many difficulties.

In the practice of designing electrical machines, the graphical solution of equation (7) is widely used. This method is very simple and clear.

$$\varphi(Q) = zQ^2 \quad (7)$$

In the coordinates Q and p, two consumption functions are written: P=φ(Q) and $\Delta p=zQ^2$. Their intersection point is the point of equality: the point is the point of mutual correspondence of the pressure loss and the fan pressure (figure 1).

As a result of ventilation calculations, the working flow rate Qi is never equal to the nominal flow rate Q_n. This happens for two reasons. On the one hand, because when selecting the air blowing elements, the intersection of their characteristics with the pressure curve occurred at the required point (Q_n, ΔP_n). On the other hand, it is taken into account that the fan characteristics and the pressure loss curve are determined with some error, which leads to an unpleasant air shortage, although the electric machine is calculated for the value of Qn. In this regard, the air blowing elements should be selected so that the working flow rate Q_i is 10% higher than the nominal. Such a margin will compensate for 20% of the error made in determining the pressure loss [12].

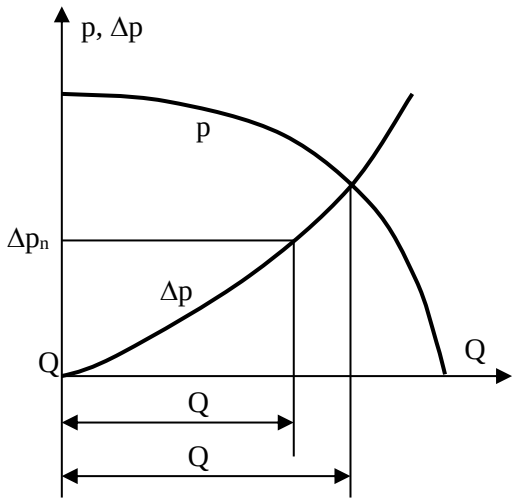

Figure 1. *Schematic determination of the working flow rate*

Information on the operation of ventilation.

Since the designed machine has a wide range of changes in the value and direction of the rotation frequency, the processes, parameters, torque changes, and the heating processes caused by the losses during operation are also quite complex. Speed regulation requires the correct design of the cooling system. The approximate stability of the cooling process in the design of an electric machine, the fact that the working rotor has a wide rotation frequency zone, and also the creation of variable losses, further complicate the cooling process [15-17].

The placement of two rotors ensures that the additional rotor performs the ventilation operation in the system with high efficiency, regardless of the operation of the working rotor. Although the rotation frequency of the working rotor varies within wide limits, the additional rotor continues to operate in asynchronous mode. At minimum rotation frequencies of the working rotor, the speed of the additional rotor may decrease by several percent of the slip, which will not significantly affect the operation of the cooling system.

The designed two-rotor asynchronous motor can be manufactured for various operating modes and in the program of changing the direction of rotation of the working rotor. For machines designed with the same and changing directions of rotation, a high-quality and high-capacity ventilation system with high air delivery capacity should be designed. If the operating mode of the designed machine is two-directional, both internal and external fans should blow air in only one direction. This direction should be from the auxiliary rotor to the working rotor. The point is that the power of the heat source in the high-resistance short-circuit loop of the copper winding of the working rotor placed in the opposite direction of the auxiliary rotor is high, and from there there is a large amount of heat flow relative to the ventilation system. For this purpose, the designed centrifugal fans (both internal and external) blow the cooling ambient air into the cooling channels in one direction in both directions of rotation. If the designed engine is created for the same direction of rotation, the ventilation system must be provided with fan devices with a high useful operating mode. In the considered design, the internal and external fans are driven by a certain rotating (auxiliary rotor) element, which indicates that both fans have a maximum operating coefficient. For this, the design of centrifugal fans is possible by selecting a design with a high air blowing capacity. The air blowing elements of such fans can be made forward or backward inclined [11].

IV. Conclusions

1. In conclusion, the thermal management of electric machines plays a crucial role in ensuring their long-term reliability, efficiency, and optimal performance. The conversion of mechanical energy to electrical energy inevitably leads to heat losses, which must be effectively managed to prevent overheating and deterioration of key components like insulation. The design of the cooling system is integral to this process, as it directly influences the machine's power output, operational lifespan, and safety. Efficient heat transfer, achieved through methods such as conduction, convection, and radiation, alongside well-calculated ventilation and hydraulic systems, helps maintain the permissible temperature limits for critical machine parts.

2. As the power and complexity of electric machines increase, so too does the need for advanced cooling solutions. Insufficient cooling can accelerate the aging of insulating materials and lead to electrical and mechanical failures. The optimization of cooling systems not only improves the thermal efficiency of the machine but also reduces the associated costs and energy losses. Furthermore, understanding the relationship between temperature increases, insulation life, and cooling system efficiency is essential for designing machines that meet both technical and economic objectives.

3. Ultimately, the integration of reliable cooling systems is vital for maximizing the lifespan and performance of electric machines, making it a key focus in modern machine design. By ensuring that thermal management is prioritized throughout the design process, engineers can enhance the durability, safety, and operational effectiveness of electric machines across a wide range of applications.

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ENHANCING ENERGY SYSTEM RELIABILITY: MODERN APPROACHES AND SOLUTIONS

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Abstract

The article analyzes methods for improving the reliability of energy systems considering the SAIDI and SAIFI indicators, which reflect the duration and frequency of power outages. Approaches are discussed, including the implementation of intelligent monitoring systems, Automated Distribution Management Systems (ADMS), as well as distributed generation and redundancy. The study confirms that the integrated use of these technologies significantly enhances network reliability, reducing SAIDI and SAIFI indices, and evaluates the economic efficiency of these solutions, demonstrating their long-term profitability.

Keywords: energy system reliability, SAIDI, SAIFI, Automated Distribution Management Systems (ADMS), redundancy, economic efficiency.

I. Introduction

The reliability of energy systems plays a key role in ensuring the sustainable operation of both industrial enterprises and the residential sector. Modern energy systems face numerous challenges, including equipment wear, the impact of climatic factors, rising energy consumption, and the need to integrate renewable energy sources. As society becomes increasingly dependent on uninterrupted power supply, reliability issues take on strategic importance, as power outages can lead to significant economic losses, decreased productivity, and reduced quality of life [1-3].

To quantitatively assess the reliability of energy systems, key performance indicators are used, with the most common being SAIDI (System Average Interruption Duration Index) and SAIFI (System Average Interruption Frequency Index). The first indicator reflects the average duration of outages per consumer over the course of a year, while the second measures the average number of outages per consumer during the same period. These metrics allow for the evaluation of the performance of energy grids and the development of strategies for their modernization.

The consequences of unreliable power supply extend beyond technical aspects and have a significant impact on the economy. Emergency shutdowns and energy supply instability result in decreased production capacity, losses in industry, worsened business conditions, and increased operational costs. In the residential sector, power outages can lead to social discontent, disrupt critical systems, and raise the costs of alternative power sources. Thus, improving energy system reliability is not only a technical but also a socio-economic challenge.

This study focuses on analyzing methods for improving the reliability of energy systems, taking into account SAIDI and SAIFI indicators [4-6]. Modern approaches to optimizing network infrastructure, implementing intelligent monitoring systems, automated solutions for managing distribution networks, and reservation strategies are considered. Special attention is given to

finding balanced solutions that minimize the frequency and duration of outages while keeping the costs of energy system modernization reasonable.

II. Formulation of the problem

The reliability of energy systems is determined by a variety of factors, with key roles played by the technical condition of equipment, operating conditions, and the influence of external factors. In modern energy systems, one of the main issues remains the wear and obsolescence of equipment, which leads to an increase in failure frequency and a reduction in overall power supply efficiency. A large portion of electrical grids was built decades ago and is operating at the limits of its capabilities, making them particularly vulnerable to overloads and emergency situations.

In addition to internal wear, significant influence is also exerted by external factors, including climatic conditions [7]. Extreme temperatures, hurricanes, floods, and other natural phenomena contribute to the destruction of infrastructure and increase the time needed to restore power supply. Such events can significantly degrade the reliability indicators of energy systems, raising the values of SAIDI (System Average Interruption Duration Index) and SAIFI (System Average Interruption Frequency Index). These indices reflect both the frequency and duration of power outages, allowing for an evaluation of the performance of energy systems in different regions [8, 9].

To analyze the current state of electrical network reliability, the SAIDI and SAIFI indicators for various energy systems can be compared. Below is Table 1, illustrating the average values of these indices in several countries.

Table 1.: *Average values of SAIDI and SAIFI indicators in some countries*

Country	SAIDI (minutes/year)	SAIFI (outages/year)
Germany	12	0.3
USA	240	1.5
France	50	0.8
Brazil	600	7.2
India	900	12.5

As seen from the data, significant differences in the reliability of power supply are due not only to the state of infrastructure but also to the approaches used in managing energy systems. For instance, Germany's energy system demonstrates a high level of reliability due to regular network modernization, the implementation of intelligent monitoring systems, and the use of automated distribution management systems [10]. On the other hand, countries with less developed infrastructure exhibit high values of SAIDI and SAIFI, indicating the need for network modernization and increased resilience.

To visually represent the differences in energy system reliability, we will perform a graphical comparison of SAIDI and SAIFI values across countries (figure 1.).

Despite the development of technologies, current methods for ensuring the reliability of energy systems have certain limitations. The implementation of intelligent power management systems and automated solutions can reduce the frequency of outages, but it requires significant investments. Many countries face a shortage of financial resources for modernizing network infrastructure, which leads to the need to choose between the costs of improving reliability and the economic efficiency of energy system operations.

Furthermore, existing strategies are often focused on reactive measures, such as addressing the

consequences of emergency outages, rather than preventing them. It is crucial to shift towards predictive management models that use big data analysis and machine learning to forecast potential failures.

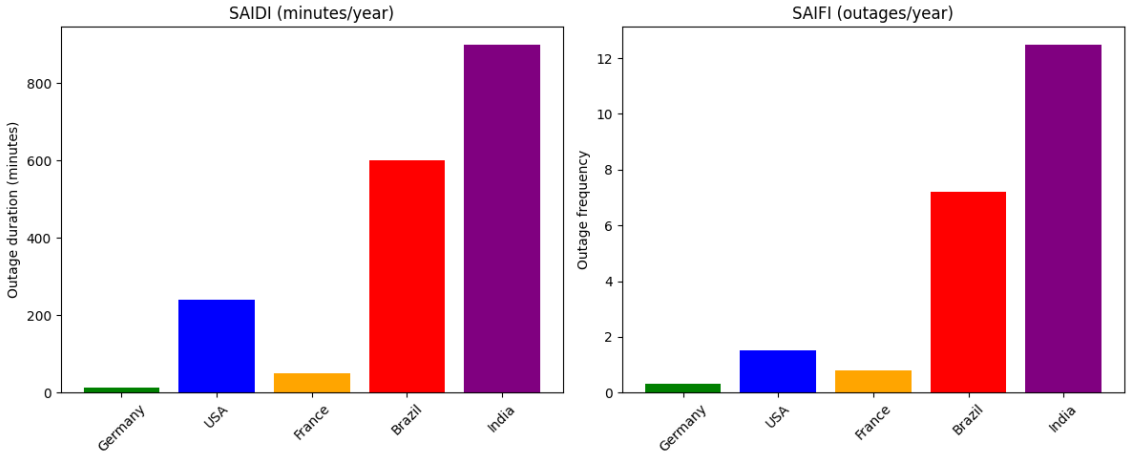

Figure 1.: *Comparison of SAIDI and SAIFI values in different countries*

One of the main challenges remains balancing the costs of upgrading energy systems with improving their reliability. Investing substantial funds in equipment modernization and the implementation of digital technologies must be economically justified. In developed countries, Automated Distribution Management Systems (ADMS) and Smart Grids are widely used; however, their deployment requires a comprehensive approach and long-term planning.

Thus, the issue of enhancing the reliability of energy systems demands a thorough analysis and a well-balanced approach that considers economic, technical, and climatic factors [11]. Addressing this challenge is only possible through a set of comprehensive measures, including infrastructure modernization, the integration of intelligent management systems, and strategic investment planning.

III. Problem solution

One of the main challenges remains balancing the costs of upgrading energy systems with improving their reliability. Investing significant funds in equipment modernization and the implementation of digital technologies must be economically justified. In developed countries, Advanced Distribution Management Systems (ADMS) and Smart Grids are widely used; however, their implementation requires a comprehensive approach and long-term planning.

Thus, the issue of improving the reliability of energy systems requires a thorough analysis and a well-balanced approach that considers economic, technical, and climatic factors [12]. Addressing this challenge is only possible through comprehensive measures, including infrastructure modernization, the deployment of intelligent management systems, and strategic investment planning (figure 2).

The obtained results help identify the load ranges in which the energy system is most vulnerable to failures. Such an analysis enables energy system operators to develop failure prevention strategies, optimize load management, and implement predictive maintenance of equipment. The visualization confirms that intelligent analysis methods can effectively assess the reliability of energy systems and predict potential failures, contributing to the reduction of SAIDI and SAIFI indices.

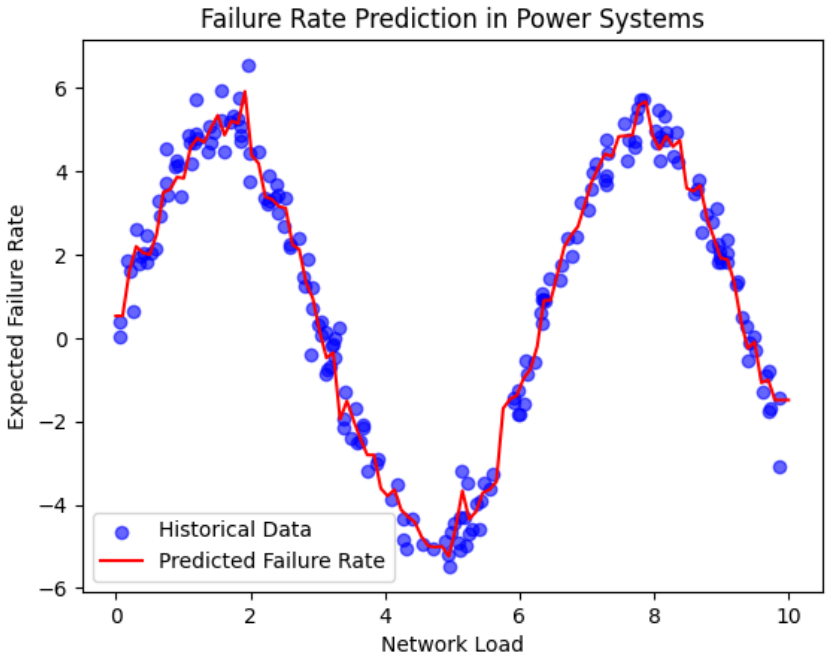

Figure 2: *Failure rate prediction in power systems*

Modern energy systems require efficient management of electricity flows to minimize emergency situations. One of the key solutions is the implementation of Advanced Distribution Management Systems (ADMS). These systems enable automatic network topology adjustments, isolation of damaged sections, and load redistribution, thereby reducing the impact of failures on consumers.

The effectiveness of ADMS can be evaluated by comparing SAIDI and SAIFI indicators before and after system implementation:

Table 2.: *Comparison of SAIDI and SAIFI indicators before and after the implementation of the system*

Parameter	**Before ADMS Implementation**	**After ADMS Implementation**
Average SAIDI (minutes/year)	300	120
Average SAIFI (outages/year)	3.5	1.2

As shown in the table, the application of ADMS significantly reduces both the duration and frequency of outages, improving the overall reliability of energy systems.

Another crucial aspect of enhancing power supply reliability is the implementation of redundancy mechanisms, dynamic network reconfiguration, and distributed generation:

- Redundancy ensures the availability of backup power sources that automatically activate in case of a failure. This can include transformer substation-level redundancy or the integration of local battery storage systems.
- Network reconfiguration allows for automatic changes in power supply routes, which is particularly effective in local outage scenarios.
- Distributed Generation (DG), including local solar and wind power stations, reduces dependence on centralized energy sources and decreases the load on power grids.

To evaluate the impact of distributed generation, a graph can be plotted showing the dependence of SAIDI on the share of renewable energy sources in the power system (figure 2).

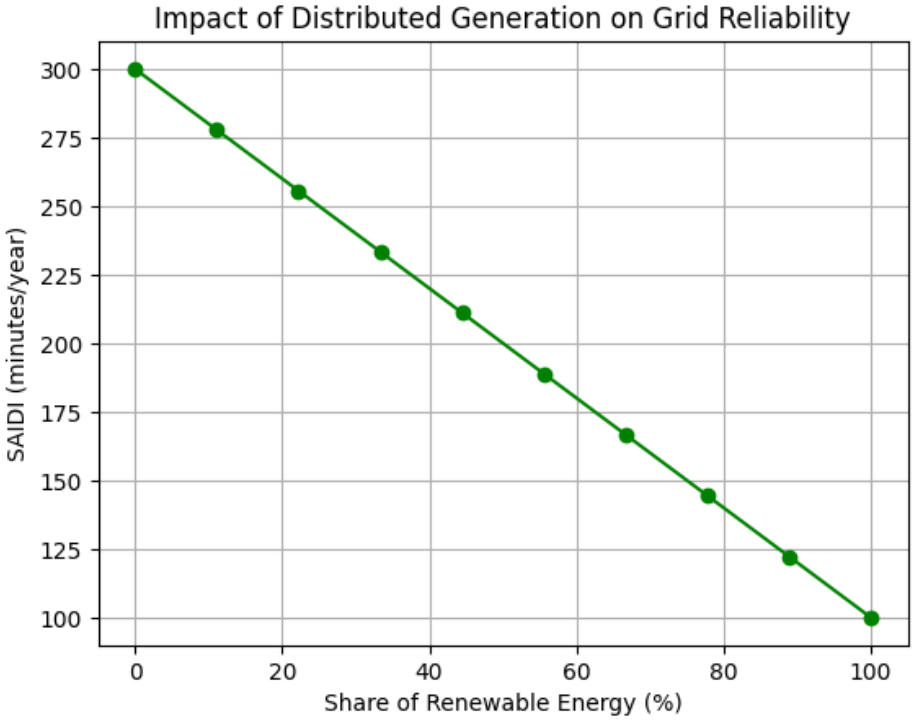

Figure 2: *Impact of distributed generation on grid reliability*

This graph demonstrates that an increase in the share of distributed generation reduces SAIDI, as local energy sources enhance the resilience of the grid against failures.

Developed countries are actively implementing comprehensive strategies to improve the reliability of energy systems. For example:

- *Germany* utilizes Smart Grids and decentralized energy sources, minimizing power outages.
- *The United States* is deploying ADMS and active network management to ensure rapid emergency response.
- *France* is modernizing its power grids to reduce the impact of weather-related disruptions.

A comparative analysis indicates that a combined approach, integrating predictive analytics, automated management, and local generation, is the most effective solution.

Despite the high initial costs associated with the deployment of intelligent management systems and grid modernization, the long-term benefits significantly outweigh the investments. For instance, the economic efficiency of ADMS implementation can be estimated as follows (table 3).

Table 3.: *Example of calculating the economic efficiency of implementation*

Indicator	Before Implementation	After Implementation
Annual losses due to outages ($ million)	50	15
Investment in modernization ($ million)	100	-
Savings over 10 years ($ million)	-	350

As shown, the adoption of advanced technologies enhances the reliability of power systems and reduces economic losses, making them cost-effective in the long run.

A comprehensive approach to improving power system reliability includes predictive analytics, automated management systems, redundancy mechanisms, and distributed generation. These measures contribute to reducing SAIDI and SAIFI, increasing grid resilience against failures, and ensuring economic efficiency in power supply management.

IV. Conclusions

The conclusion summarizes the key findings of the analysis on methods to improve power system reliability, focusing on SAIDI and SAIFI indicators. The study confirmed that the implementation of intelligent monitoring systems, automated distribution management systems (ADMS), and distributed generation plays a crucial role in reducing the frequency and duration of power supply interruptions. Modern strategies not only minimize outages but also enhance the overall resilience of power systems to external factors, including climate conditions and grid overloads.

Optimization of operational processes, the use of predictive analytics, and equipment modernization significantly reduce downtime for fault recovery. Improved coordination between power system elements enables a rapid response to emergencies and shortens restoration times. Hybrid approaches, combining redundancy, network reconfiguration, and local generation, contribute to enhanced reliability for all categories of consumers.

Future research in this field should focus on developing advanced failure prediction algorithms, integrating machine learning for automated grid management, and refining distributed energy technologies. The future of power systems lies in the comprehensive application of digital solutions and innovative technical advancements, ensuring a balance between reliability and economic efficiency.

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SELECTION OF A BAYESIAN DOUBLE SAMPLING PLAN THROUGH MARKOV DEPENDENCE METHOD IN DRUG DISCOVERY

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Abstract

Most of the pharmaceutical firms have worked hard to maintain quality in their manufacturing products like medicines and biological instruments using the principles of statistical quality control to optimize the fault model. In this field, one of the pioneering statistical methods is acceptance sampling by attributes. A sampling plan is used to assess the quality of goods, keep an eye on the quality of the materials, and confirm whether or not the yields are defect-free or not. When posterior knowledge about the parameter is known, the Bayesian strategy provides a more robust statistical method for reaching a suitable conclusion. In this article a new Bayesian double sampling plan under stochastic modeling was established. This is achieved by various characteristics of sampling plan explicit by its random variable and its probability function. This plan is studied through the Gamma- Poisson model to safeguard both the producer and consumer by minimizing the Average Sample Number and Total Cost. Necessary tables and figures are constructed for the selection of optimal plan parameters and suitable illustrations are provided that are applicable under pharmaceutical industries.

Keywords: Acceptance Sampling, Bayesian double sampling plan, Gamma-Poisson, Pharmaceutical Industry, Stochastic Process, Transition Probability Matrix.

1. Introduction

The pharmaceutical industry directly deals with human's health by its medical needs and drug development towards human safety. All countries formulates policies to safeguard public health and implements regulations for monitoring the production of pharmaceutical products. Developing countries puts significant effort to deliver quality goods to their citizens. The book "Quality management in the medicines industry: philosophy and essential elements"[21] describes the fundamental principles of quality assurance (QA) and the core elements of Good Manufacturing Practices (GMP), which are integral to both production and quality control management. These include of persons, premises, materials, documentation, validation, self-inspection, hygiene, and equipment. "Good practices in production and quality control" offers instructions on specific steps that quality control and production staff should independently undertake to implement core principles of Quality Assurance. Commonly, World Health Organization (WHO) suggests three types of sampling plans: n, p and r. The primary objective of the n-plan is the determination

of the sample size, whereas the *p*-plan emphasizes the maximum number of defects and the *r*-plan focuses on the rejection number.

Many industries commonly use acceptance sampling plan to meet the inspection quality requirements for ever increasing global demand. The nature of inspection may be destructive or non-destructive depends on its characteristics of its products to fulfil the inspections is often impractical that resulting the lot is accepted/rejected, therefore move on to acceptance sampling plan is more efficient. Acceptance sampling is divided into two primary categories: (1) attribute sampling plan and (2) variable sampling plan. Lot-by-lot acceptance sampling for attributes was studied by many authors under a variety of sampling designing methods, such as single, double, sequential, continuous, chain, skip-lot and other related sampling plans each having its own advantage and its limitations. Hald [7] and Schilling [17] developed the sampling plan parameters based on the unity value approach. Several authors have contributed to the foundational principles of quality control, including Calvin T.M [1], Cameron [2], Dodge H.F. and Roming H.G. [3], Edwin B wilson *et. al.*[4], Golub [5], Peach P.[16], Gunether [20]. The theory behind traditional sampling plans by attributes is based on the basic assumption that the nonconforming process fraction or lot is a constant, implying that the manufacturing process is steady. Conventional sampling strategies cannot be used in these situations since the judgement about the submitted lots must take into account and it's not consider between lots variations and within lots variations. Bayesian sampling plans are a sophisticated technique to statistical sampling that uses Bayesian probability theory to make conclusions and judgments about observed data. Choosing a prior distribution for the lot fraction nonconforming has been discussed by Vijayaraghavan et al. [19], selecting a prior distribution for a metric quality characteristic of the production process has been studied by Loganathan et al. [11]. When sampling from a Poisson distribution, the gamma distribution is a conjugate prior, according to Bayesian theory. The gamma distribution is the conjugate prior to the average number of nonconformities per unit. When sample units are randomly selected from a process, the number of nonconformities in the sample is distributed based on the Poisson equation. Under these circumstances, the operating characteristic (OC) function of SSP by characteristics labelled as gamma-Poisson SSP has been mathematically expressed by Hald [7]. Implication of production and monitoring techniques in Bayesian Single sampling plan using Gamma- Zero Inflated Poisson distribution studied by Kaviyarasu and Sivakumar [9]. Bayesian single sampling plan using gamma prior were provided by Suresh K.K and Latha [18]. The fundamental concepts of quality control have been explored by several researchers, including Guptha S.K. *et. al.*[6], Kaviyarau *et. al.*[8] [10], Lord *et. al.*[12], Moskowitz *et. al.*[15]. A novel control strategy was presented by Fallahnezhad and Nasab [13] addressing the acceptance sampling problem. Fallahnezhad [14] provided a minimal angle method-based Markov chain acceptance sampling plan. In this work gives tables to determine the c_1, c_2, c_3, c_4, n (minimum sample size) and minimum cost of the Bayesian Acceptance Sampling Plan under the circumstances of using the Gamma-Poisson distribution.

2. Transition Probability Matrix for Acceptance Sampling

A stochastic process can have several branches. A markov chain is a type of stochastic process that is used to determine the best threshold value to accept or reject a lot. This concept actually gave rise to the idea of modelling the acceptance sampling using a markovin method. Let P= p_{ij} represent the transition probability matrix for the finite state space S with discrete-time. A transition probability matrix depicts the single-step process of moving from one state to another. It includes all of the information required to understand how the process moves across the states.

2.1. Assumptions

- The entries of matrixes are all non-negative.
- In the matrix each row equals to one.

$$p_{11} + p_{12} + p_{13} + ... + p_{1n} = 1 \tag{1}$$

The three states of the problem are designed as follows:

- Accept the lot when the batch is in good state.
- Reject the lot when the batch is in bad state.
- Continue the inspection when it is in between the upper and lower thresholds.

Every phase is viewed as a state of the transition matrix P, which comprises the probabilities of accepting the batch p_{12}, rejecting the batchp_{13}, and continues the processp_{11}. The matrix P is an absorbing Markov chain where the states are known to be transient (continue inspection) and absorbing (reject or accept the lot). Finding the optimal threshold values for the sampling plan is the aim of this Markovian technique. Wilson and Burgess [19], Specified the block form for the transition probability matrix is arranged as follows

$$P = \begin{bmatrix} A & O \\ R & Q \end{bmatrix} \tag{2}$$

Here

A = An identity matrix representing the probability of remains in a state.

O = The matrix representing the probabilities of escaping an absorbing state (always zero).

R = The matrix containing all probabilities of going from a non-absorbing state to an absorbing state. (i.e., finished or scrapped product).

Q = The square matrix contacting the transition probability of going from a non-absorbing state to another non-absorbing state.

2.2. Total Cost

The expected total cost associated with the inspection policy can be expressed using the equation3, which is employed to calculate the total cost. The acceptance cost multiplied by the probability of accepting the batch at stage 1. The rejection cost is calculated by multiplying the rejection cost by the probability of rejecting the batch at stage 2.
The average total cost for double sampling plan is given as

$$E(TC) = Acceptancecost + Rejectioncost + Inspectioncost \tag{3}$$

where,
Acceptance cost = cost of defective item $\times$ lot size $\times$ probability of accepting the lot
Rejection cost = cost of batch rejection $\times$ probability of rejecting lot
Inspection cost = ((Number of times the lot inspected with n_1 items $\times$ Number of samples in stage 1) + (Number of times the lot inspected with n_2 items $\times$ Number of samples in stage 2 $\times$ probability that the second sample is taken during the first sample inspected)) $\times$ the cost of the defective items.

3. Construction of a Bayesian Double Sampling Plan using the Markov Chain Method

Using a Markov chain in a Bayesian double sampling enables the modelling of sequential decision-making, allowing for the integration of prior knowledge and the updating of beliefs as new data becomes available. This dynamic approach enhances decision-making under uncertainty, reducing risks in the sampling process. Additionally, Markov chains help to accurately compute the posterior probabilities of accepting or rejecting a lot by considering previous sample outcomes, leading to more efficient.

3.1. Markov Chain Procedure for Bayesian Double Sampling Plan

- Initially, inspect n_1 items from the lot, this items collected through random sampling
- Set the threshold values (lower threshold r_1, upper threshold r_2) for the n_1 samples
- If the defective items (d_1) items from the sample are below or equal to the lower threshold (r_1) value, the lot will be accepted
- If the defective items (d_1) items from the sample are above the upper threshold (r_2) value, the lot will be rejected
- If defective items (d_1) in between the two threshold values, then take a second sample of size n_2
- Set the threshold values (lower threshold r_3, upper threshold r_4) for the n_2 samples
- If defective items (d_2) of the sample are below the lower threshold (r_3) value, the lot will be accepted
- If defective items (d_2) of the sample are above the upper threshold (r_4) value, the lot will be rejected
- If defective items (d_2) in between threshold values, return to first sample size of n_1 items are to be inspected, such a continuous process is called as a double sampling decision-making process under markov decision

Each stages of the plan is defined as a states.

Step 1: For the lot, the first stage acceptance policy ought to be implemented.

Step 2: For the lot, the Second stage acceptance policy ought to be implemented.

Step 3: The lot should be accepted.

Step 4: The lot should be rejected.

3.2. Transition Probability Matrix

The lot's transition probability matrix is given as follows,

$$\begin{bmatrix} 0 & P_{12} & P_{13} & P_{14} \\ P_{21} & 0 & P_{23} & P_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rearranging the matrix gives the fundamental matrix.

$$N_{ij} = \begin{bmatrix} \frac{1}{1-P_{12}P_{21}} & \frac{P_{21}}{1-P_{12}P_{21}} \\ \frac{P_{12}}{1-P_{12}P_{21}} & \frac{1}{1-P_{12}P_{21}} \end{bmatrix} = m_{ij}$$

The probability of long-term absorption ζ is

$$\zeta = \begin{bmatrix} \frac{1+P_{12}P_{23}}{1-P_{12}P_{21}} & \frac{P_{21}P_{13}+P_{23}}{1-P_{12}P_{21}} \\ \frac{P_{14}+P_{12}P_{24}}{1-P_{12}P_{21}} & \frac{P_{21}P_{14}+P_{24}}{1-P_{12}P_{21}} \end{bmatrix}$$

The elements of ζ matrix, ζ_{13} is probability of accepting the batch at stage 1 and ζ_{14} is probability of rejecting the batch at stage 1.

3.3. Transition Probability Matrix for Gamma – Poisson

The Poisson distribution with parameter $np = \lambda$, where λ is the average number of nonconformities per unit, is used to distribute the random variable, when it is taken at random from a production process that generates from a continuous stream of lots.

The Poisson distribution is a suitable model for the number of nonconforming units in the sample, as noted by Hald [7] for $p < 0.10$ and by Schilling [17] when n is large and P is small such that $np < 5$.

$$P_{(n,p)}(d) = \frac{e^{-np}(np)^d}{d!} \quad \text{for } d = 0, 1, 2, \ldots$$

A random variable, say L, must be introduced to represent the randomly varying distribution parameter when the distribution parameter λ is not constant but rather varies at random from lot to lot.

This random variable is distributed according to a gamma distribution, which is the Poisson distribution's conjugate distribution.
The prior distribution of L's density function can be found as follows:

$$f_{(a,m)}(\lambda) = \frac{e^{-a\lambda}a^m\lambda^{m-1}}{\Gamma(m)} \quad \text{for } 0 \leq \lambda < \infty, a, m > 0$$

Where $a = \frac{m}{\lambda}$, with E[L] = $\bar{\lambda}$ and m is the shape

When production is not stable, L is independently distributed, and according to Hald [7] the Gamma-Poisson probability distribution can be determine by the following

$$p = \sum_{d=0}^{n} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m$$

Where

n = number of observations

p = proportion defectives

m = shape parameter for gamma prior

d = number of defectives

To determine the probability of reaching each state, the cumulative Gamma-Poisson distribution is used as

$$p(r_1 < d < r_2) = p_{11} = \sum_{d=0}^{n_1} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m - p_{12} \tag{4}$$

$$p(d > r_2) = p_{13} = \sum_{d=0}^{r_2} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m = 1 - p_{12} \tag{5}$$

$$p(d \leq r_1) = p_{12} = \sum_{d=0}^{r_1} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m \quad ; x = 0, 1, \ldots r_1 \tag{6}$$

$$p(r_3 < d < r_4) = p_{21} = \sum_{d=0}^{n_2} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m - p_{23} \tag{7}$$

$$p(d > r_4) = p_{24} = \sum_{d=0}^{r_4} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m = 1 - p_{23} \tag{8}$$

$$p(d \leq r_3) = p_{23} = \sum_{d=0}^{r_3} \frac{(m+d+1)!}{d!(m-1)!} \left(\frac{np}{np+m}\right)^d \left(\frac{m}{np+m}\right)^m \quad ; x = 0, 1, \ldots r_3 \tag{9}$$

Where

p_{12} = The probability that the second sample from the lot will be taken while the first sample is being inspected.

p_{13} = Probability that a lot will be accepted after the initial sample inspection.

p_{14} = Probability that a lot will be rejected after the initial sample inspection.

p_{21}= Probability that the first sample is taken during second sample inspection.

p_{23}= Probability that a lot will be accepted after the second sample inspection.

p_{24} = Probability that a lot will be rejected after the second sample inspection.

The following expression represents the average total cost associated with the lot acceptance policy of the double sampling plan:

$$E(Total\ cost) = (cs \times N \times p \times \zeta_{13}) + (rc \times \zeta_{14}) + (m_{11} \times n_1 + m_{22} \times n_2 \times p_{12})I$$

Where

$\zeta_{13} = \frac{p_{13}+p_{12}p_{23}}{1-p_{12}p_{21}}$

ζ_{14}= 1-ζ_{13}

cs= Cost of defective item

N= Lot Size

p = Incoming quality

m_{11} = Number of times the lot inspected with n_1 items

m_{22} = Number of times the lot inspected with n_2 items

4. Different Prior Values for Different Threshold Values

This analysis is carried out to showcase the implementation of the proposed methodology in the formulation of an acceptance sample design. Consider the following problem: AQL = 0.05, LQL = 0.1. The alternate values of m among the present ones provide various combinations.

In the case N = 1500, cs = 1, p = 0.05, I = 2, rc = 100, n_1 = 75, n_2 = 65.

Table 1: *The feasible values of r_1, r_2, r_3, r_4, E(n) and average cost values for m = 1*

When p_1 = 0.02 Probability of accepting the lot	When p_2 = 0.10 Probability of rejecting the lot	r_1	r_2	r_3	r_4	E(n)	Average Cost
0.66	0.85	0	2	0	2	92	266.56
0.74	0.84	0	3	0	2	99	279.16
0.84	0.73	1	3	1	3	89	257.23
0.90	0.70	1	4	1	4	97	271.65
0.91	0.62	2	4	2	4	87	251.31
0.91	0.62	2	4	2	5	88	253.09
0.95	**0.54**	**3**	**5**	**3**	**5**	**85**	**256.89**
0.95	0.54	3	5	3	6	86	248.22

Table 2: *The feasible values of r_1, r_2, r_3, r_4, E(n) and average cost values for m = 3*

When p_1 = 0.02 Probability of accepting the lot	When p_2 = 0.10 Probability of rejecting the lot	r_1	r_2	r_3	r_4	E(n)	Average Cost
0.61	**0.97**	**0**	**2**	**0**	**2**	**85**	**253.67**
0.71	0.97	0	3	0	2	91	264.85
0.87	0.91	1	3	1	3	88	253.77
0.94	0.90	1	4	1	4	98	272.58
0.95	0.82	2	4	2	4	90	255.58
0.95	0.81	2	4	2	5	91	258.81
0.98	0.71	3	5	3	5	90	255.82
0.98	0.70	3	5	3	6	91	257.57

Table 3: *The feasible values of r_1, r_2, r_3, r_4, E(n) and average cost values for m = 5*

When p_1 = 0.02 Probability of accepting the lot	When p_2 = 0.10 Probability of rejecting the lot	r_1	r_2	r_3	r_4	E(n)	Average Cost
0.60	0.99	0	2	0	2	81	247.84
0.72	0.99	0	3	0	2	87	256.62
0.88	0.95	1	3	1	3	86	249.16
0.95	**0.95**	**1**	**4**	**1**	**4**	**96**	**267.82**
0.95	0.88	2	4	2	4	89	254.48
0.96	0.88	2	4	2	5	91	258.11
0.90	0.77	3	5	3	5	92	258.28
0.99	0.77	3	5	3	6	94	262.30

The odds of accepting and rejecting the lot probability values are shown in Table 1 as m = 1. The least sample size obtained in this table is 85, with the minimum total cost is 248.22. However, the probability of acceptance and rejection the lot fails to satisfy the said conditions to align with the requirements. Further, Table 2 displays m = 3 for both the lot's acceptance and rejection probabilities are studied and obtained the smallest sample size in this table 2 is 85, and the minimum total cost is 253.67.

However, the probability of accepting and rejecting the lot fails again to satisfy the basic conditions to align with the requirements. In Table 3, threshold values shows a significant improvement in the said conditions 1, 4, 1 and 4 and meets its basic conditions, so the feasible sample is 96 and the total cost is 267.82. Thus the sample size is fixed as 96 with its corresponding probability values.

5. Numerical Illustration

In a pharmaceutical company with continuous production, a Quality Engineer (QE) plan to check the solubility of capsules, in the lot, there are 1500 capsules, and the capsules dissolve within three seconds.

QE fixed the AQL = 0.05, LQL = 0.1, the cost of a defective item *(cs)* = 1, the proportion of the defective items in a batch is *(p)* = 0.05, the cost of inspecting an item in the batch *(I)* = 2, the cost of batch rejection *(rc)* = 100, and the first and second samples are n_1 = 75, n_2 = 65.

Table 4: *Capsules dissolving time for first sample*

Time	0	1	1	0	2	3	1	0	1	1	0	0	2	1	0	0	1	1	1	2	0	1	3	2	0
P/F	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P
Time	1	2	1	0	0	1	1	0	5	1	3	0	2	1	2	0	1	3	1	2	2	1	2	2	1
P/F	P	P	P	P	P	P	P	P	F	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P
Time	0	1	0	0	1	1	1	3	0	1	0	2	3	1	0	4	1	1	1	2	0	2	2	1	2
P/F	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	F	P	P	P	P	P	P	P	P	P

Table 5: *Capsules dissolving time for second sample*

Time	2	3	1	0	1	1	1	0	1	1	0	0	0	1	1	1	0	0	1	3	0	1
P/F	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P
Time	0	1	1	1	4	0	1	0	0	1	3	1	2	2	1	0	1	3	1	2	2	1
P/F	P	P	P	P	F	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P
Time	0	0	1	1	1	3	1	3	0	1	0	2	3	0	1	0	2	3	1	2	0	
P/F	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P	

Where
P/F = Pass or Fail
R software is utilized to generate the Gamma-Poisson data (mean = 0.405 < Standard deviation = 0.667) and to draw samples through random sampling. The samples are presented in Table 4 and Table 5. In this table, dissolving seconds ranging from 0 to 3 seconds are deemed acceptable, whereas times exceeding 3 seconds are considered to have failed the inspection.

Proceed to the second sample if the previous one has two defects. The lot gets accepted because the second sample only contains one defect. For the first sample, the lower threshold value is 1, and the upper threshold value is 4. The lower threshold value is 1 for the second sample, and the upper threshold value is 4 for the second sample, where m = 5.

6. Conclusion

In this article, a new sampling plan methodology is developed under Markovian matrix method as an improved method of inspection under attribute quality characteristics of Gamma-Poisson distribution. Bayesian double sampling plan incorporates more benefits through sample size and its acceptance number which are determined with prior knowledge towards improving decision-making under uncertainty. Transition probability matrix allows for a probabilistic view of decision routes by calculating the likelihood of changing from one state to another state. By using this matrix, the inspection cost is optimized, leading to a better comprehension of potential results. Suitable tables are developed with numerical examples are given to demonstrate Pharmaceutical industry application for the drug discovery of this study.

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OPTIMIZATION ANALYSIS OF UNRELIABLE MULTI-SERVER QUEUEING SYSTEM WITH BERNOULLI SCHEDULE WORKING VACATION, THRESHOLD-BASED RECOVERY POLICY, AND IMPATIENCE

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Abstract

This paper analyzes an unreliable multi-server queueing system incorporating working vacations, Bernoulli interruptions, breakdowns with a threshold recovery policy, balking, abandonment, and retention. During the break period, if there are customers in the queue, the servers may either resume normal service or continue their vacation. Customers arriving while the system is saturated are rejected. Failures occur unexpectedly but only when at least one customer is present in the system. Recovery procedures remain in effect until the total number of customers surpasses a predefined threshold. Using matrix-analytic methods, we derive steady-state solutions and explicit formulas for various performance indicators. Further, we explore cost parameter optimization.

Keywords: unreliable queueing systems, threshold-based recovery policy, working vacation, impatience

1. Introduction

With growth of communication systems and networks, manufacturing systems, transportation systems, etc, queueing systems with breakdowns have received growing significance [14, 17, 18].

Queueing models incorporating threshold policies, specifically the N-policy and F-policy, have garnered significant attention in recent years. The former policy dictates that a server activates only when N (where $N \geq 1$) or more customers accumulate in the system [11, 19, 5, 25]. Conversely, the latter policy restricts customer entry into the system once it reaches its capacity. When the queue length decreases to a threshold parameter value F, the server then permits customers to enter [9, 4, 12].

The literature on N and F policies is extensive. However, research on queueing models with breakdowns, repairs, and a threshold-based recovery policy, where the server remains unrepaired until the number of customers in the system reaches a predetermined threshold value, is limited. Notable works include [22, 10, 15].

Vacation queueing models have attracted substantial interest from researchers over the past decades, owing to their ubiquitous applications across diverse fields. These applications span production/manufacturing systems, telecommunication systems and computer networks. Notably, comprehensive surveys on this subject have been conducted by [6, 7, 20, 21].

The concept of working vacations was introduced by [16], proposing a model where the server processes jobs with varying intensities based on the incoming traffic. The primary objective is twofold: better control of queue lengths and reduction of customer loss. Additionally, working vacations enable servers to be strategically redirected for maintenance purposes. As a result, these models have gained significant popularity, leading to a wealth of analytical results in the literature, such as [26, 8, 24, 23].

In recent times, queueing systems that account for customer impatience have garnered significant attention. These models find realistic applications in various service systems and e-commerce domains. For a comprehensive overview of the literature on this theme, readers can refer to studies by [13, 2, 3, 1].

In this paper, we delve into the analysis of a multi-server Markovian queue that integrates several crucial practical features including breakdowns, threshold-based recovery policy, working vacations, Bernoulli interruption schedule, impatient customers, and retention of reneged customers. The contributions and advantages of this paper are as follows:

1. ***The model.*** Unlike existing literature that predominantly focuses on single-server queueing models, our study embraces a multi-server queue. By incorporating the diverse features mentioned above, our proposed model offers greater flexibility in characterizing complex stochastic phenomena within multi-server machining systems.

2. ***Methodology and results.*** Leveraging the Q-matrix method, we provide a detailed theoretical analysis. We derive steady-state probabilities and various performance measures. Our chosen method is well-suited for analyzing quasi-birth-and-death (QBD) processes in steady-state.

3. ***Numerical illustrations.*** We develop a cost function to optimize service rates during both working vacation and normal busy periods. Additionally, we determine the optimal number of servers and explore threshold-based recovery policies. These insights empower system managers and decision-makers to regulate the system economically.

The manuscript is structured concisely in the following manner: Section 2 presents the main motivation and practical applications of the current research work. Section 3 briefly describes the model under consideration. Section 4 comprises the analysis of the model in the stationary state. Section 5 enlists important performance measures. Section 6 develops a cost model for the proposed system and introduces cost optimization methods, namely, the direct search method and the quasi-Newton method. Section 7 deals with a cost optimization problem and provides numerical examples to illustrate the effects of different system parameters on performance measures, total expected cost, and total expected profit. Section 8 presents the conclusions of the study.

2. Main motivation and practical application

The motivating context for our model is analysis of automated teller machine (ATM) manufacturing systems. Such facilities commonly face machine failures and repairs, congestion issues, operator unavailability, impatient customers, and more that can significantly hamper production efficiency.

Specifically, we consider a production system with c parallel machines and finite finished goods capacity. Upon arrival of failed parts/subassemblies for repair, they immediately occupy any available operator. Otherwise failed units wait in queue for a random duration. Once all repairs are completed, operators take group vacations, relying on substitutes with slower service rates, and may have their breaks interrupted if failures resume.

Moreover, operators undergo their own failures following a breakdown process. Repairs only initiate after M failed machines have accumulated via a threshold policy. Newly arriving failures may balk from the repair queue or later renege after prolonged waits.

All such issues–breakdowns, vacations, congestion, balking and reneging–are commonly faced by real ATM manufacturers. By mathematically capturing these dynamics in a closed-form queueing model, we aim to evaluate the complex tradeoffs between maintainability, throughput, and customer impatience. The model can help optimize the number of machines, the threshold-based recovery policy, and service rates, to control costs in ATM production systems through resilience to inevitable disruptions.

3. Model description

Consider an Automated Manufacturing System modeled as an unreliable $M/M/c/L$ queueing system. The model formulation necessitates several distinct assumptions, which can be summarized as follows:

(i) Arrival process: Customers arrive following Poisson process with parameter α.

(ii) Service and working vacation processes:

(a) Upon arrival, customers are served if any servers are available.

(b) After serving all existing customers, servers synchronously switch to a vacation period.

(c) Upon returning from vacation, if the system remains empty, servers immediately begin another synchronous vacation.

(d) The vacation duration follows an exponential distribution with parameter τ.

(e) During vacation, substitute servers take over from the main servers to serve new customers.

(f) Service times during regular busy periods (RBP) and vacations follow exponential distributions with parameters μ and ν, respectively. We assume that $\nu < \mu$.

(g) If a customer arrives and finds any of the c servers free (during busy or working vacation), they immediately occupy that server. If all servers are busy, the customer joins the end of the queue in the buffer and is served later according to the First-Come-First-Served (FCFS) discipline.

(iii) Bernoulli interruption scheme:

(a) During the working vacation period (WVP), the server operates under the Bernoulli rule. Specifically, at the instant of service completion during this period, if there are customers in the system:

- With probability β, the server interrupts the vacation and switches to the regular working period.
- With probability $\beta' = 1 - \beta$, the server continues the vacation.

(b) Notably, the service during WVP is applied only to the first customer who arrives during this period.

Then, we can write

$$\delta_n = n\beta\nu\mathbb{1}_{2\leq n\leq c-1} + c\beta\nu\mathbb{1}_{c\leq L}.$$

(iv) Breakdown process: The system is susceptible to unreliability at any given time. During regular busy periods, servers are vulnerable to breakdowns. Specifically, a server break down only if there is at least one customer in the system. The occurrence of breakdowns follows a stationary Poisson process with parameter φ. Importantly, during repair periods (RP), customers cannot be served.

(v) The threshold-based recovery policy and repair process: The recovery can be performed when M $(1 \leq M \leq c-1)$ or more customers are present. The repair period has exponential distribution with parameter γ. Customers arriving during the repair time are ignored by the system.

(vi) Balking: When a customer arrives, their actions depend on server availability:

- - If some servers are working and others are free, the customer is directly served.
- - Otherwise, during working vacation, regular busy, or repair periods:
  (a) The customer may join the queue with probability θ_n.
  (b) Customers faced with joining a queue have an alternative: they may balk, choosing not to enter. The balking probability is denoted as: $\theta'_n = 1 - \theta_n$, where in the case of working vacation/regular busy period, we have : $0 \leq \theta_{n+1} \leq \theta_n \leq 1$. Consider the following scenarios:
    i. For working vacation/regular busy period case, we have:
      – $0 \leq \theta_{n+1} \leq \theta_n \leq 1$ for $c \leq n \leq L-1$;
      – $\theta_0 = 1, \ldots, \theta_{c-1} = 1$.
    ii. For repair period, we observe:
      – $0 \leq \theta_{n+1} \leq \theta_n \leq 1$ for $1 \leq n \leq L-1$.
      – $\theta_0 = 1$ (no balking when the system is empty).
    iii. In both cases, we have: $\theta_L = 0$ (no entering when the system is at full capacity).

Shortly, we have for working vacation and regular normal busy:

$$\alpha_n = \alpha \mathbb{1}_{n<c} + \theta_n \alpha \mathbb{1}_{c \leq n \leq L},$$

and for breakdown period: $\alpha_n = \theta_n \alpha, 1 \leq n \leq L$.

(vii) Reneging and retention:

(a) Upon arrival, customers exhibit different behaviors based on the server status:
- If servers are in regular working mode or working vacation period:
  – The customer activates an impatience timer T_1 (for regular working) or T_0 (for working vacation). If the customer's service is not completed before the timer expires, they may abandon the system.
- During the reparation period:
  – A new arrival activates its own timer T_2. If service is unavailable before the expiration of the impatience timer, the customer may give up.

(b) The impatience time T_j follows an exponentially distributed random variable with rates $\varsigma_j > 0$ (where $j = 0, 1, 2$).

(c) Impatient customers have two options:
- They may quit the system without receiving service with probability κ.
- Alternatively, they may be kept in the system with probability $\kappa' = 1 - \kappa$.

Then, we can put:

$$\epsilon_{n,j} = n\kappa\varsigma_0 \mathbb{1}_{j=0} + n\kappa\varsigma_1 \mathbb{1}_{j=1} + n\kappa\varsigma_2 \mathbb{1}_{j=2},$$
$$\upsilon_n = (n\mu + \epsilon_{n,1}) \mathbb{1}_{1 \leq n \leq c-1} + (c\mu + \epsilon_{n,1}) \mathbb{1}_{c \leq n \leq L}$$

and

$$\zeta_n = (\nu + \epsilon_{1,0}) \mathbb{1}_{n=1} + (n\beta'\nu + \epsilon_{n,0}) \mathbb{1}_{2 \leq n \leq c-1} + (c\beta'\nu + \epsilon_{n,0}) \mathbb{1}_{c \leq n \leq L}.$$

The customers timers are independent and identically distributed (i.i.d.) random variables and independent of the number of customers currently waiting.

(viii) The various stochastic processes within the system are assumed to be mutually independent.

4. Equilibrium probability analysis

We employ the Markov process approach, utilizing the Q-matrix, to establish the steady-state distribution for our proposed queueing model. Our primary focus lies in deriving the steady-state probabilities of the system, specifically as a function of the probability $\pi_{1,1}$, rather than relying on $\pi_{0,j}$ or $\pi_{L,j}$ for $j = 0, 1, 2$.

The system under consideration can be modeled as a continuous-time Markov process, denoted by $\{\mathfrak{X}(t), \mathfrak{Y}(t); t \geq 0\}$, where $\mathfrak{X}(t)$ represents the number of customers present in the system at time t, and $\mathfrak{Y}(t)$ characterizes the operational state of the servers at time t. The possible states for $\mathfrak{Y}(t)$ are as follows:

$$\mathfrak{Y}(t) = \begin{cases} 0, & \text{Servers are in a WVP} \\ 1, & \text{Servers are in a RBP} \\ 2, & \text{Servers are in a RP} \end{cases}$$

Let $\pi_{n,j}$ denote the steady-state probability that the system has n customers and the servers are in state j, such that: $\pi_{n,j} = \lim_{t\to\infty} P\{\mathfrak{X}(t) = n, \mathfrak{Y}(t) = j\}$, where
$(n, j) \in \{\{(n, 0) : n = 0, 1, ..., L\} \cup \{(n, 1) : n = 1, 2, ..., L\} \cup \{(n, 2) : n = 1, 2, ..., L\}\}$.The state transition rate diagram is depicted in Figure 1.

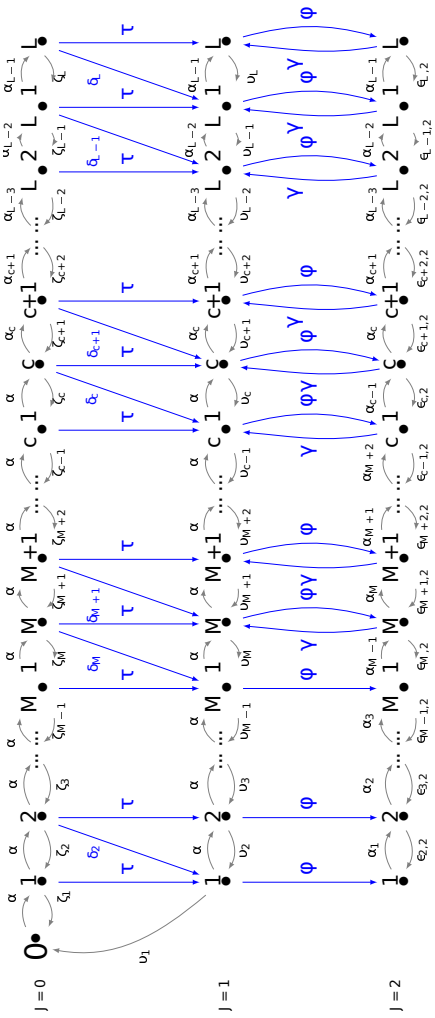

Figure 1: *State transition diagram for the proposed model*

4.1. Governing equations

The steady-state balance equations that govern our system are expressed as follows:

$$\begin{aligned}
\alpha\pi_{0,0} &= (\mu+\kappa\varsigma_1)\pi_{1,1}+(\nu+\kappa\varsigma_0)\pi_{1,0}, \ n=0,\\
(\alpha+n(\nu+\kappa\varsigma_0)+\tau)\pi_{n,0} &= \alpha\pi_{n-1,0}+(n+1)(\beta'\nu+\kappa\varsigma_0)\pi_{n+1,0},\\
&\quad 1\le n\le c-1,\\
(\alpha\theta_n+c\nu+n\kappa\varsigma_0+\tau)\pi_{n,0} &= \alpha\pi_{n-1,0}+(c\beta'\nu+(n+1)\kappa\varsigma_0)\pi_{n+1,0},\\
&\quad n=c,\\
(\alpha\theta_n+c\nu+n\kappa\varsigma_0+\tau)\pi_{n,0} &= \alpha\theta_{n-1}\pi_{n-1,0}+(c\beta'\nu+(n+1)\kappa\varsigma_0)\pi_{n+1,0},\\
&\quad c+1\le n\le L-1,\\
(\tau+c\nu+L\kappa\varsigma_0)\pi_{L,0} &= \alpha\theta_{L-1}\pi_{n-1,0}, \ n=L,\\
(\alpha+\mu+\kappa\varsigma_1+\varphi)\pi_{1,1} &= \tau\pi_{1,0}+2\beta\nu\pi_{2,0}+2(\mu+\kappa\varsigma_1)\pi_{2,1}, \ n=1,
\end{aligned}$$

$$\begin{aligned}
(\alpha+n(\mu+\kappa\varsigma_1)+\varphi)\pi_{n,1} &= \alpha\pi_{n-1,1}+(n+1)\beta\nu\pi_{n+1,0}+(n+1)(\mu+\kappa\varsigma_1)\pi_{n+1,1}\\
&\quad +\tau\pi_{n,0}, \ 2\le n\le M-1,\\
(\alpha+n(\mu+\kappa\varsigma_1)+\varphi)\pi_{n,1} &= \alpha\pi_{n-1,1}+(n+1)\beta\nu\pi_{n+1,0}+(n+1)(\mu+\kappa\varsigma_1)\pi_{n+1,1}\\
&\quad +\tau\pi_{n,0}+\gamma\pi_{n,2}, \ M\le n\le c-1,\\
(\theta_n\alpha+n(\mu+\kappa\varsigma_1)+\varphi)\pi_{n,1} &= \alpha\pi_{n-1,1}+c\beta\nu\pi_{n+1,0}+(c\mu+(n+1)\kappa\varsigma_1)\pi_{n+1,1}\\
&\quad +\tau\pi_{n,0}+\gamma\pi_{n,2}, \ n=c,\\
(\alpha\theta_n+n(\mu+\kappa\varsigma_1)+\varphi)\pi_{n,1} &= \alpha\theta_{n-1}\pi_{n-1,1}+c\beta\nu\pi_{n+1,0}+(c\mu+(n+1)\kappa\varsigma_1)\pi_{n+1,1}\\
&\quad +\tau\pi_{n,0}+\gamma\pi_{n,2}, \ c+1\le n\le L-1,\\
(c\mu+L\kappa\varsigma_1+\varphi)\pi_{L,1} &= \alpha\theta_{L-1}\pi_{L-1,1}+\tau\pi_{L,0}+\gamma\pi_{L,2}, \ n=L,\\
\theta_1\alpha\pi_{1,2} &= \varphi\pi_{1,1}+2\kappa\varsigma_2\pi_{2,2}, \ n=1,\\
(\alpha\theta_n+n\kappa\varsigma_2)\pi_{n,2} &= \alpha\theta_{n-1}\pi_{n-1,2}+(n+1)\kappa\varsigma_2\pi_{n+1,2}+\varphi\pi_{n,1},\\
&\quad 2\le n\le M-1,\\
(\alpha\theta_n+n\kappa\varsigma_2+\gamma)\pi_{n,2} &= \alpha\theta_{n-1}\pi_{n-1,2}+(n+1)\kappa\varsigma_2\pi_{n+1,2}+\varphi\pi_{n,1},\\
&\quad M\le n\le L-1,\\
(L\kappa\varsigma_2+\gamma)\pi_{L,2} &= \alpha\theta_{L-1}\pi_{L-1,2}+\varphi\pi_{L,1}, \ n=L.
\end{aligned}$$

The normalizing condition is expressed as:

$$\sum_{n=0}^{L}\pi_{n,0}+\sum_{n=1}^{L}\pi_{n,1}+\sum_{n=1}^{L}\pi_{n,2}=1. \tag{1}$$

Let's introduce the necessary notations for the subsequent sections of the paper:

$$\dot{\zeta}_n=\begin{cases} n\nu+\epsilon_{n,0}, & 1\le n\le c-1,\\ c\nu+\epsilon_{n,0}, & c\le n\le L,\end{cases}$$

$$\varrho_n=\begin{cases} -(\alpha+\upsilon_n+\varphi), & 1\le n\le c-1,\\ -(\alpha_c+\upsilon_n+\varphi), & n=c,\\ -(\alpha_n+\upsilon_n+\varphi), & c+1\le n\le L-1,\\ -(\upsilon_L+\varphi), & n=L,\end{cases}$$

$$\vartheta_n=\begin{cases} -(\alpha+\dot{\zeta}_n+\tau), & 1\le n\le c-1,\\ -(\alpha_c+\dot{\zeta}_n+\tau), & n=c,\\ -(\alpha_n+\dot{\zeta}_n+\tau), & c+1\le n\le L-1,\\ -(\dot{\zeta}_L+\tau), & n=L,\end{cases}$$

$$\sigma_n = \begin{cases} -\alpha_1, & n = 1, \\ -(\alpha_n + \epsilon_{n,2}), & 2 \le n \le M-1, \\ -(\alpha_n + \epsilon_{n,2} + \gamma), & M \le n \le L-1, \\ -(\epsilon_{L,2} + \gamma), & n = L. \end{cases}$$

4.2. Matrix solution

To obtain the steady-state solution, let the steady-state probability vector of the infinitesimal generator Q be denoted as $P = (\pi_0, \pi_1, \pi_2)$, where $\pi_0 = (\pi_{0,0}, \pi_{1,0}, ..., \pi_{L,0})$, $\pi_1 = (\pi_{1,1}, \pi_{2,1}, ..., \pi_{L,1})$, and $\pi_2 = (\pi_{1,2}, \pi_{2,2}, ..., \pi_{L,2})$. The steady-state equations $PQ = 0$ must be satisfied by P along with the normalization condition $Pe = 1$, where 0 is a zero row vector, and $e = (e_0, e_1, e_2)$ is a $(3L+1)$ column vector of ones, with e_0 being an $(L+1)$ column vector and e_1 and e_2 being L column vectors. The block structure of the infinitesimal generator Q is as follows:

$$Q = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 \\ \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \end{pmatrix},$$

where

$$\mathcal{A}_1 = \begin{pmatrix} -\alpha & \alpha & & & & & & \\ \zeta_1 & \vartheta_1 & \alpha & & & & & \\ & \zeta_2 & \vartheta_2 & \alpha & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \zeta_{c-1} & \vartheta_{c-1} & \alpha & & \\ & & & & \zeta_c & \vartheta_c & \alpha_c & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \zeta_{L-1} & \vartheta_{L-1} & \alpha_{L-1} \\ & & & & & & 0 & \zeta_L & \vartheta_L \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & ... & ... & 0 \\ \tau & 0 & ... & ... & 0 \\ 0 & \tau & ... & ... & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & ... & ... & \tau \end{pmatrix},$$

$$\mathcal{E}_2 = \begin{pmatrix} \varrho_1 & \alpha & & & & & & \\ \upsilon_2 & \varrho_2 & \alpha & & & & & \\ & \upsilon_3 & \varrho_3 & \alpha & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \upsilon_{c-1} & \varrho_{c-1} & \alpha & & \\ & & & & \upsilon_c & \varrho_c & \alpha_c & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \upsilon_{L-1} & \varrho_{L-1} & \alpha_{L-1} \\ & & & & & & 0 & \upsilon_L & \varrho_L \end{pmatrix},$$

$$\mathcal{D}_3 = \begin{pmatrix} \sigma_1 & \alpha_1 & & & & \\ \epsilon_{2,2} & \sigma_2 & \alpha_2 & & & \\ & \epsilon_{3,2} & \sigma_3 & \alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \epsilon_{L-1,2} & \sigma_{L-1} & \alpha_{L-1} \\ & & & 0 & \epsilon_{L,2} & \sigma_L \end{pmatrix},$$

$$\mathcal{E}_1 = \begin{bmatrix} v_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}, \quad \mathcal{E}_3 = \begin{bmatrix} \varphi & & \\ & \ddots & \\ & & \varphi \end{bmatrix}, \quad \text{and} \quad \mathcal{D}_2 = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \gamma & \\ & & & & \ddots \\ & & & & & \gamma \end{bmatrix}.$$

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are matrices of dimensions $(L+1)\times(L+1)$ and $(L+1)\times L$, respectively. $\mathcal{A}_3$ is a zero matrix with dimensions $(L+1)\times L$. $\mathcal{E}_1$ is an $L\times(L+1)$ matrix. $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are square matrices of order $L\times L$. Additionally, $\mathcal{D}_1$ is a zero matrix with dimensions $L\times(L+1)$.

4.3. System state probabilities

We present the steady state equations $PQ = 0$ and the normalization condition $Pe = 1$, as

$$\begin{cases} \pi_0\mathcal{A}_1 + \pi_1\mathcal{E}_1 + \pi_2\mathcal{D}_1 = 0, \\ \pi_0\mathcal{A}_2 + \pi_1\mathcal{E}_2 + \pi_2\mathcal{D}_2 = 0, \\ \pi_0\mathcal{A}_3 + \pi_1\mathcal{E}_3 + \pi_2\mathcal{D}_3 = 0, \\ \pi_0 e_1 + \pi_1 e_2 + \pi_2 e_3 = 1. \end{cases} \tag{2}$$

As $\mathcal{A}_3$ and $\mathcal{D}_1$ are null matrices, then Equation (2) can be rewritten as

$$\pi_0\mathcal{A}_1 + \pi_1\mathcal{E}_1 = 0, \tag{3}$$
$$\pi_0\mathcal{A}_2 + \pi_1\mathcal{E}_2 + \pi_2\mathcal{D}_2 = 0, \tag{4}$$
$$\pi_1\mathcal{E}_3 + \pi_2\mathcal{D}_3 = 0, \tag{5}$$
$$\pi_0 e_1 + \pi_1 e_2 + \pi_2 e_3 = 1. \tag{6}$$

Next, put $\mathcal{A}_2 = \begin{pmatrix} O_1 \\ \tau I_L \end{pmatrix}$, $\mathcal{E}_1 = \begin{pmatrix} v_1 & O_1 \\ O_2 & O_3 \end{pmatrix}$, $\mathcal{E}_3 = (\varphi I_n)$, $\mathcal{D}_2 = \begin{pmatrix} O_4 & \\ \dot{O}_4 & \gamma I_{L-M+1} \end{pmatrix}$, with I_L denotes the identity matrix. Further, O_1 is a $1\times L$ matrix. O_2 and O_3 are both of order $(L-1)\times 1$ and $(L-1)\times L$, respectively. O_4 has dimensions $(M-1)\times L$. $\dot{O}_4$ is of order $(L-M+1)\times(M-1)$. I_{L-M+1} represents the identity matrix of order $L-M+1$.

Let $\mathcal{A}1^{-1}$ and $\mathcal{D}3^{-1}$ denote the inverse matrices of $\mathcal{A}1$ and $\mathcal{D}3$, respectively. By referring to Eq. (3), we obtain the following result:

$$\begin{aligned} \pi_0 &= -\pi_1\mathcal{E}_1\mathcal{A}_1^{-1} \\ &= -\pi_1 \begin{pmatrix} v_1 o \\ O_5 \end{pmatrix} \\ &= -\pi_{1,1} v_1 o, \end{aligned} \tag{7}$$

where $o = (o_0, \tilde{o})$, such that $\tilde{o} = (o_1, ..., o_L)$ be an L row vector of the matrix $\mathcal{A}_1^{-1}$, and O_5 is $(L-1)\times(L+1)$. From Eq. (5), we have

$$\pi_2 = -\pi_1\mathcal{E}_3\mathcal{D}_3^{-1} = -\pi_1\varphi\mathcal{D}_3^{-1}. \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (4), obtain

$$-\pi_{1,1} v_1 \tilde{o}\tau + \pi_1(\mathcal{E}_2 - \varphi\mathcal{D}_3^{-1}\mathcal{D}_2) = 0. \tag{9}$$

As $\mathcal{E}_2$ and $\mathcal{D}_3^{-1}$ are both square matrices of order L, we can affirm the existence of the matrix:

$$\tilde{\mathcal{E}} = (\mathcal{E}_2 - \varphi\mathcal{D}_3^{-1}\mathcal{D}_2)^{-1}.$$

Thus

$$\pi_1 = (v_1\tilde{o}\tau\tilde{\mathcal{E}})\pi_{1,1}. \tag{10}$$

Consequently, we can deduce easily

$$\pi_2 = -(v_1\tilde{o}\tau\tilde{\mathcal{E}}\varphi\mathcal{D}_3^{-1})\pi_{1,1}. \tag{11}$$

Then, using Eqs. (7)–(11), we get:

$$\begin{cases} \pi_{n,0} = -v_1 o_n \pi_{1,1}, \\ \pi_{n,1} = (v_1\tau\sum_{i=1}^{L} o_{i+1}\tilde{\psi}_{in})\pi_{1,1}, \\ \pi_{n,2} = -\left(v_1\tau\varphi\sum_{j=1}^{L}\sum_{i=1}^{L} o_{i+1}\tilde{\psi}_{ij}\tilde{\omega}_{jn}\right)\pi_{1,1}, \end{cases}$$

where $\tilde{\omega}_{ij}$ are the elements of matrix $\tilde{\mathcal{D}} = \mathcal{D}_3^{-1}$, and $\tilde{\psi}_{ij}$ are the elements of the matrix $\tilde{\mathcal{E}} = (\mathcal{E}_2 - \varphi\mathcal{D}_3^{-1}\mathcal{D}_2)^{-1}$. Finally, to determine $\pi_{1,1}$, we apply the normalizing condition (as described in Equation (1)):

$$\pi_{1,1} = \left(-v_1\sum_{n=0}^{L} o_n + v_1\tau\sum_{n=1}^{L}\sum_{i=1}^{L} o_{i+1}\tilde{\psi}_{in} - v_1\tau\varphi\sum_{n=1}^{L}\sum_{j=1}^{L}\sum_{i=1}^{L} o_{i+1}\tilde{\psi}_{ij}\tilde{\omega}_{jn}\right)^{-1}.$$

5. Performance measures

In this section, we delve into the derivation of crucial system indices, leveraging the probabilities associated with the system distribution.

Result 1: The servers are in busy period with probability

$$P_{busy} = \sum_{n=1}^{L}\pi_{n,1} = v_1\tau\sum_{n=1}^{L}\sum_{i=1}^{L} o_{i+1}\tilde{\psi}_{in}\pi_{1,1}. \tag{12}$$

Result 2: The servers are in working vacation period with probability

$$P_{wv} = \sum_{n=1}^{L}\pi_{n,0} = -\left(v_1\sum_{n=0}^{L} o_n\right)\pi_{1,1}. \tag{13}$$

Result 3: The servers are in breakdown period with probability

$$P_{bp} = \sum_{n=1}^{L}\pi_{n,2} = -v_1\tau\varphi\left(\sum_{n=1}^{L}\sum_{j=1}^{L}\sum_{i=1}^{L} o_{i+1}\tilde{\psi}_{ij}\tilde{\omega}_{jn}\right)\pi_{1,1}. \tag{14}$$

Result 3: The probability of system reliability

$$P_{re} = 1 - \pi_{pb}.$$

Result 4: The mean system size is

$$\begin{aligned} E_s &= \sum_{n=1}^{L} n(\pi_{n,0} + \pi_{n,1} + \pi_{n,2}) \\ &= v_1\left(-\sum_{n=1}^{L} no_n + \tau\sum_{n=1}^{L}\sum_{i=1}^{L} no_{i+1}\tilde{\psi}_{in} - \tau\varphi\sum_{n=1}^{L}\sum_{j=1}^{L}\sum_{i=1}^{L} no_{i+1}\tilde{\psi}_{ij}\tilde{\omega}_{jn}\right)\pi_{1,1}. \end{aligned} \tag{15}$$

Result 5: The effective arrival rate

$$\alpha' = \alpha\pi_{0,0} + \sum_{n=1}^{L}\alpha_n\pi_{n,0} + \sum_{n=1}^{L}\alpha_n\pi_{n,1} + \sum_{n=1}^{L}\alpha_n\pi_{n,2}.$$

Result 6: The mean waiting time of customers in the system

$$W_s = E_s/\alpha^{'}.$$

Result 7: The average balking rate

$$R_{balk} = \alpha - \alpha^{'}. \tag{16}$$

Result 8: The average reneging rate

$$\begin{aligned} R_{ren} &= \kappa\varsigma_0 \sum_{n=1}^{L} n\pi_{n,0} + \kappa\varsigma_1 \sum_{n=1}^{L} n\pi_{n,1} + \kappa\varsigma_2 \sum_{n=2}^{L} n\pi_{n,2} \\ &= \upsilon_1\kappa \left(-\varsigma_0 \sum_{n=1}^{L} no_n + \tau\varsigma_1 \sum_{n=1}^{L}\sum_{i=1}^{L} no_{i+1}\tilde{\psi}_{in} - \tau\varphi\varsigma_2 \sum_{n=2}^{L}\sum_{j=1}^{L}\sum_{i=1}^{L} no_{i+1}\tilde{\psi}_{ij}\tilde{\omega}_{jn} \right) \pi_{1,1}. \end{aligned} \tag{17}$$

Result 9: The average retention rate

$$\begin{aligned} R_{ret} &= \kappa^{'}\varsigma_0 \sum_{n=1}^{L} n\pi_{n,0} + \kappa^{'}\varsigma_1 \sum_{n=1}^{L} n\pi_{n,1} + \kappa^{'}\varsigma_2 \sum_{n=2}^{L} n\pi_{n,2} \\ &= \upsilon_1\kappa^{'} \left(-\varsigma_0 \sum_{n=1}^{L} no_n + \tau\varsigma_1 \sum_{n=1}^{L}\sum_{i=1}^{L} no_{i+1}\tilde{\psi}_{in} - \tau\varphi\varsigma_2 \sum_{n=2}^{L}\sum_{j=1}^{L}\sum_{i=1}^{L} no_{i+1}\tilde{\psi}_{ij}\tilde{\omega}_{jn} \right) \pi_{1,1}. \end{aligned} \tag{18}$$

Result 10: The mean number of customers served per unit time

$$C_s = \nu \sum_{n=1}^{L} n\pi_{n,0} + \mu \sum_{n=1}^{L} n\pi_{n,1}.$$

6. Cost model and optimization

For our queueing model, we consider he cost components as outlined below:

1. C_{busy}: unit time cost for system being in busy period.
2. C_{wv}: unit time cost for system being is in working vacation.
3. C_{break}: unit time cost for system being in breakdown period.
4. C_{sq}: Holding unit time cost when a customer enters the queue.
5. C_{s_1}: Cost per service per unit time in regular working period.
6. C_{s_2}: Cost per service per unit time in working vacation period.
7. C_l: unit time cost when a customer is lost.
8. C_t: unit time cost when the system retains a customer.
9. C_f: Fixed purchase cost of the server per unit.

The formulation of the cost per unit time function for the queueing system is as follows:

$$\begin{aligned} \mathcal{T}_c &= C_{busy}P_{busy} + C_{wv}P_{wv} + C_{break}P_{pb} + C_{sq}E_s + C_l(R_{ren} + R_{balk}) \\ &\quad + C_tR_{ret} + c(\mu C_{s_1} + \nu C_{s_2}) + cC_f. \end{aligned} \tag{19}$$

Expressing the expected cost function $\mathcal{T}_c$ explicitly by substituting Equations (12)-(18) into (19) would result in an extremely complex formulation. Consequently, studying the analytical behavior of $\mathcal{T}_c$ becomes a big challenge. Furthermore, due to the nonlinearity and intricacy of the

expected cost function, deriving the optimal solution (c^*, M^*, μ^*, ν^*) in closed form would be an arduous task.

To circumvent these difficulties and perform the optimization analysis, we employ direct search and Newton's methods as numerical optimization techniques to search for the optimal solution (c^*, M^*, μ^*, ν^*). Initially, the direct search method is utilized to determine the optimal values of the variables (c^*, M^*). Subsequently, with these variables fixed, Newton's method is applied to find the optimal values of the variables (μ^*, ν^*).

6.1. Numerical cost optimum parameter

We consider a practical problem concerning the automated teller machine (ATM) production facility mentioned in Section 2. In the context of the considered practical example, the system parameters are delineated as follows: failed machines arrive according to a Poisson process with $\alpha = 7$. The system capacity is considered finite with $L = 20$. If the system is in operation, the failure occurs in which the breakdown times are exponential distribution with $\varphi = 0.1$. The service and repair times of the machines obey exponential distributions with parameters $\mu = 3.0$, $\nu = 0.9$, and $\gamma = 0.3$, respectively. Once the system gets empty, it goes on vacation period, the vacation period follows exceptional distribution with parameter $\tau = 0.4$. The failed machines during both period may get impatient and leave the system with being served. The impatience timers follow exponential distribution with $\varsigma_0 = 0.5$, $\varsigma_1 = 0.3$, $\varsigma_2 = 0.9$. Further, during working vacation period, the failed machines service may be continue their service during working vacation period with probability $\beta' = 0.6$, and they can leave the system with probability $\kappa = 0.7$. The joining probability is taken as $\theta_n = 1 - \frac{n}{L}$.

An efficient algorithm based on the direct search method is employed to determine the optimal discrete values (c^*, M^*) that optimize the expected cost function. The effectiveness of this approach hinges on the convexity (or unimodality) of the cost function. Throughout the numerical analysis, the following cost elements are considered: $C_{busy} = \$20$, $C_{wv} = \$20$, $C_{bp} = \$50$, $C_{sq} = \$10$, $C_{s1} = \$5$, $C_{s2} = \$5$, $C_l = \$30$, $C_t = \$25$, $C_f = \$1$ and $R = 50$.

Figure 2 illustrates the behavior of the expected cost function $\mathcal{T}_c(c^*, M^*)$ for varying values of c and M. The plotted curve exhibits a convex shape, indicating the existence of a single relative minimum. Consulting Table 1, it is evident that the minimum expected cost per unit time, which amounts to **182.5710**, is attained when $c^* = 6$ and $M^* = 1$.

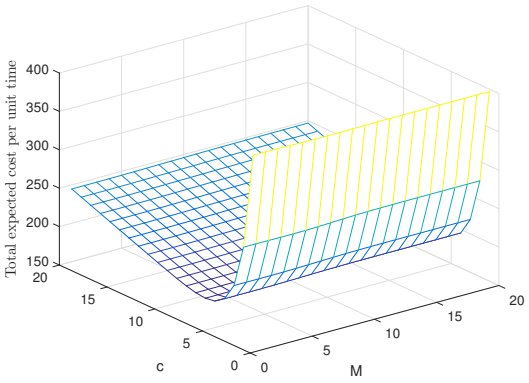

Figure 2: *The expected cost $\mathcal{T}_c$ for different values of c and M.*

Table 1: *c and M vs.* $\mathcal{T}_c(c, M)$

c / M	1	2	3	4	5	6	7	8
2	271.9823	-	-	-	-	-	-	-
3	216.2269	216.2388	-	-	-	-	-	-
4	193.3445	193.3762	193.3707	-	-	-	-	-
5	184.5919	184.6346	184.6462	184.7100	-	-	-	-
6	**182.5710**	182.6202	182.6410	182.7123	182.8470	-	-	-
7	183.9711	184.0241	184.0501	184.1262	184.2599	184.4512	-	-
8	187.1491	187.2043	187.2331	187.3123	187.4466	187.6304	187.8610	-
9	191.2406	191.2970	191.3274	191.4083	191.5437	191.7250	191.9432	192.1971

Once the optimal values (c^*, M^*) are determined, Newton's method is employed to locate the minimum value of $\mathcal{T}_c(c^*, M^*, \mu^*, \nu^*)$ by iteratively optimizing the continuous variables μ and ν. Newton's method is an efficient iterative technique for finding the optimum of a nonlinear function by computing the search direction at each iteration.

The Quasi-Newton method, a variant of Newton's method, is utilized to numerically determine μ^* and ν^*. Numerical results obtained through this optimization process are presented in Tables 2-7 for various system parameter settings.

Table 2: $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ *and* (M^*, c^*, μ^*, ν^*) *while adjusting* τ *and* α *(*$\varsigma_0 = 0.5$, $\varsigma_1 = 0.3$, $\varsigma_2 = 0.9$, $L = 20$, $\kappa = 0.7$, $\beta' = 0.6$, $\varphi = 0.1$, $\gamma = 0.3$*)*

(τ , α)	(2,5)	(2,8)	(2.5,5)	(2.5,8)	(3.0,5)	(3.0,8)
(M^*, c^*)	(1,2)	(4,7)	(1,2)	(4,7)	(1,2)	(4,7)
μ^*	3.3953	1.4294	3.5066	1.4697	3.5893	1.4991
ν^*	1.7352	0.4679	1.3760	0.3242	1.0157	0.1853
$\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$	125.1117	182.8660	124.4980	180.6144	123.3919	177.7679

Table 3: $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ *and* (M^*, c^*, μ^*, ν^*) *while adjusting* β' *(*$\varsigma_0 = 0.5$, $\varsigma_1 = 0.3$, $\varsigma_2 = 0.9$,$L = 20$, $\kappa = 0.7$, $\alpha = 7$, $\varphi = 0.1$, $\tau = 0.4$, $\gamma = 0.3$*)*

β'	0.75	0.8	0.85	0.9	0.95
(M^*, c^*)	(3,4)	(4,5)	(4,6)	(5,7)	(6,8)
μ^*	1.6362	1.3007	1.0810	0.9233	0.8065
ν^*	1.5967	1.2149	0.9672	0.7970	0.6748
$\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$	147.4804	150.3079	153.4443	156.8410	160.5053

Table 4: $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ *and* (M^*, c^*, μ^*, ν^*) *while adjusting L and* κ *(*$\varsigma_0 = 0.5$, $\varsigma_1 = 0.3$, $\varsigma_2 = 0.9$, $\alpha = 7$, $\tau = 0.4$, $\beta' = 0.6$, $\varphi = 0.1$ $\gamma = 0.3$*)*

(L,κ)	(10,0.6)	(40,0.6)	(10,0.7)	((40,0.7)	(10,0.9)	(40,0.9)
(c^*, M^*)	(7,1)	(7,4)	(7,1)	(7,4)	(7,1)	(7,4)
μ^*	0.9190	1.0397	0.8876	0.9821	0.8333	0.8757
ν^*	0.9180	0.8733	0.8866	0.8594	0.8323	0.8333
$\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$	167.7709	167.6681	165.5115	165.0619	161.7871	160.6461

Table 5: $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ *and* (M^*, c^*, μ^*, ν^*) *while adjusting* φ *and* γ *(*$\varsigma_0 = 0.5$, $\varsigma_1 = 0.3$, $\varsigma_2 = 0.9$, $\kappa = 0.7$,$\beta' = 0.6$, $\tau = 0.4$, $\alpha = 7$,$L = 20$*)*

$(\varphi\ ,\ \gamma)$	(0.4 1)	(0.4 5)	(0.6 1)	(0.6 5)	(0.8 1)	(0.8 5)
(7,3)						
μ^*	0.9571	0.9275	0.9299	0.8955	0.9691	0.9342
ν^*	0.8955	0.8907	0.9178	0.8945	0.9075	0.9030
$\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$	170.3847	162.9654	172.9348	163.4767	177.8815	165.2887

Table 6: $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ *and* (M^*, c^*, μ^*, ν^*) *while adjusting* ς_0 *,* ς_1 *and* $\varsigma_2 = 1.9$ *(*$\kappa = 0.7$,$L = 20$ *,* $\alpha = 7$, $\tau = 0.4$,$\varphi = 0.1$, $\beta' = 0.6$,$\gamma = 0.3$*)*

$(\varsigma_1\ \varsigma_0)$	(0.2,0.6)	(0.4,0.6)	(0.2,0.8)	(0.4,0.8)	(0.2,1.0)	(0.4,1.0)
$(c^*\ ,M^*)$	(8,2)	(8,4)	(8,2)	(8,3)	(8,5)	(8,7)
μ^*	0.8287	0.7616	0.8079	0.7640	0.8041	0.7644
ν^*	0.7697	0.7606	0.8056	0.7630	0.8031	0.7634
$\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$	171.4859	172.4115	179.1606	180.3308	186.5172	188.2080

Table 7: $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ *and* (M^*, c^*, μ^*, ν^*) *while adjusting* ς_0 *,* ς_1 *and* $\varsigma_2 = 3.1$ *(*$\kappa = 0.7$,$L = 20$ *,* $\alpha = 7$, $\tau = 0.4$,$\varphi = 0.1$, $\beta' = 0.6$,$\gamma = 0.3$*)*

$(\varsigma_1\ \varsigma_0)$	(0.2,0.6)	(0.4,0.6)	(0.2,0.8)	(0.4,0.8)	(0.2,1.0)	(0.4,1.0)
$(c^*\ ,M^*)$	(8,2)	(8,4)	(8,2)	(8,3)	(8,5)	(8,7)
μ^*	0.8211	0.7583	0.8032	0.7607	0.8006	0.7621
ν^*	0.7712	0.7573	0.8022	0.7597	0.7996	0.7611
$\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$	170.7178	171.7406	178.4237	179.6992	185.8408	187.5095

Tables 2-7 illustrate the relationships between various system parameters and the optimal service rates (μ^*, ν^*) that minimize the expected cost $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$:

- As the arrival rate of failed machines (α) increases, the expected cost $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$ rises substantially. Similarity for β'. This is understandable, as a higher influx of failures naturally strains the service system, leading to longer queues, more congestion, and ultimately increased costs.

- Conversely, higher operator vacation rate (τ) and greater customer non-retention probability (κ) decrease the expected cost. Obviously, more frequent vacations provide more opportunities to serve customers during vacation periods, alleviating congestion. Likewise, allowing more customers to renege without service reduces the queue length and wait times.

- The positive effect of operator breakdown rate (φ) on expected cost is expected, since more breakdowns directly degrade service capability and capacity. In contrast and as anticipated, faster operator repair rate (γ) significantly improves system performance and reduces costs by restoring capacity quicker after failures.

- Larger system capacity (L) and impatience rates $(\varsigma_j, j = 0, 1, 2)$ increase service abandonments, lowering congestion and $\mathcal{T}_c(M^*, c^*, \mu^*, \nu^*)$. However, excessive abandonments negatively impact customer service. An optimal balance is required.

7. Conclusion

In this paper, based on the characteristics of the repair machine, we presented a $M/M/c/L$ queue with breakdowns, repairs, threshold-based recovery policy, working vacation, Bernoulli interruption, balking, reneging, and retention. We established the steady-state solution of the system using Q-matrix. Then, we studied important system characteristics based on the steady-state probabilities. Finally, we presented the sensitivity and cost optimization analysis; we discussed an economic analysis as well as the optimal threshold, the optimal number of servers as well as the service rates μ and ν under a given cost assumption because determining these parameters to achieve the minimum cost is very important in queueing theory. As further potential future study, we can generalize this queueing model with to some different cases, as follows:

- (i) Considering the feedback phenomenon within the queueing systems, it is pertinent to examine the scenario involving feedback customers in the proposed queueing model.
- (ii) It will be interesting to incorporate retrial policy and preemptive resume priority, this makes the system closer to real-life congestion scenarios and the study can provide potentially practical application in flexible manufacturing systems, transportation system, telecommunication systems, and so on.
- (iii) One could also extend the present study by considering multi-optional services.

Declarations

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Consent to Participate Not applicable.
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CLASSICAL AND BAYESIAN ESTIMATION OF EXPONENTIATED INVERSE RAYLEIGH DISTRIBUTION BASED ON RECORD VALUES

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Abstract

In this article explores two approaches for estimating the parameters of the exponentiated inverse Rayleigh distribution (EIRD) using record values: Classical estimation and Bayesian estimation. In classical estimation, maximum likelihood estimators (MLE's) and the asymptotic confidence intervals are derived based on the observed Fisher information matrix of the parameters. In Bayesian estimation, estimators of the parameters are obtained under the square error loss function. This involves using Tierney-Kadane's approximation (TK) and Markov chain Monte Carlo (MCMC) methods for Bayesian computation. Additionally, the article constructs the highest posterior credible intervals of the parameters using the MCMC method. To evaluate the performance of these estimators, a Monte Carlo simulation study is conducted to compare their behavior. Finally, a real data analysis is presented to illustrate the application of the methods discussed in the article.

Keywords: :Exponentiated inverse Rayleigh distribution, Maximum likelihood estimators, Bayes estimators, Square error loss function, MCMC, TK, Record values, and Real data.

1. Introduction

The Rayleigh distribution was introduced by Lord Rayleigh (1880) and is used in the field of acoustics. This distribution possesses the properties of some well-known distributions, such as Weibull, chi-square, and extreme value distribution, which makes it even more useful for different areas of science and technology. There are several authors who have studied the application of the Rayleigh distribution, such as Beckmann[1] study the generalization of rayleigh distribution, Hoffman and Karst[2] mentioned that theroy and application of Rayleigh Distribution, Lee et al.[3] estimated the scale parameters of the Rayleigh distribution. Based on censored data, Soliman et al.[4] study the inference and application of the Rayleigh model. Let us suppose that a random variable Z follows the Rayleigh distribution, then $X = \frac{1}{Z}$ follows the Inverse Rayleigh distribution. The inverse Rayleigh distribution (IRD) is widely applied in reliability studies and other related fields. For more information about Inverse Rayleigh distribution studies, see more papers such as Voda[5],El-Helbawy and Abd-El-Monem[6],Shawky and Majdah M[7],Sindhu et al.[8],C. Tans[9].

Let us suppose X is a random variable for the Inverse Rayleigh distribution with scale parameters σ. Then its probability density function (pdf) and cumulative distribution function (cdf) are

respectively given as

$$q(x;\sigma) = \begin{cases} \frac{2\sigma^2}{x^3} e^{-(\frac{\sigma}{x})^2}, & \text{if } x > 0,\ \sigma > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

$$Q(x;\sigma) = \begin{cases} e^{-(\frac{\sigma}{x})^2}, & \text{if } x > 0,\ \sigma > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

There are many researchers who have suggested that the exponentiated inverse Rayleigh distribution is a generalized case of the inverse Rayleigh distribution, such as Nadarajah and Kotz [10] and Srinivasa et al. [11], who studied the estimation of multicomponent stress-strength reliability from the exponentiated inverse Rayleigh distribution. The cumulative distribution function (cdf) of the exponentiated inverse Rayleigh distribution is

$$F(x;\alpha,\sigma) = 1 - (1 - e^{-(\frac{\sigma}{x})^2})^{\alpha}, x \geq 0,\ \alpha,\sigma > 0 \tag{3}$$

and the corresponding probability density function (pdf) is

$$f(x,\alpha,\sigma) = \begin{cases} \frac{2\alpha\sigma^2}{x^3} e^{-(\frac{\sigma}{x})^2} (1 - e^{-(\frac{\sigma}{x})^2})^{\alpha-1}, & \text{if } x \geq 0,\ \alpha,\sigma > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

In numerous real-life applications, particularly within industries and reliability studies, products frequently fail under stress. For instance, a wooden beam may fracture when subjected to sufficient perpendicular force, an electronic component might cease functioning at excessively high temperatures, or a battery could expire over time. However, the precise threshold of failure can vary, even among identical items. Therefore, in such experiments, measurements are often taken sequentially, with only the record values-either the lowest or highest-being observed. These record values naturally emerge across various domains, including weather tracking, sports analytics, economic data analysis, and life-test assessments.

Let $(X_n, n \geq 1)$ be a series of independent and identically distributed (i.i.d.) random variables with distribution function $F(x)$ and probability function $f(x)$. An observation X_j is called an upper record value if $X_j > X_i$ for every $j > i$. Let us suppose $X_1, X_2, ..., X_n$ be upper record values and $x_1, x_2, ..., x_n$ be observed values of upper record values. Then the joint density function of upper record values is given by

$$f_{\underline{X}}(\underline{x}) = f(x_n) \prod_{i=1}^{n-1} \frac{f(x_i)}{1 - F(x_i)},\ x_1 < x_2 < ..., x_n \tag{5}$$

In recent times, utilizing record values for parameter estimation across various lifetime models has garnered significant attention among researchers. A multitude of studies have explored employing the MCMC and TK procedures to derive Bayes estimates in this context, such as Janss and Gerben [12], Andrieu et al. [13], Solimanet.al[14], Hassan et al. [15], Singh et al. [16], Sana and Faizan [17].

The paper are arrange in following order: In Section 2, the maximum likelihood estimation and asymptotic confidence intervals is presented. In Section 3, Bayesian estimation and MCMC algorithm are presented. In section 4, TK approximation is presented. In Section 5, simulation study is presented . In Section 6, application of real data sets. Finally, the conclusion of this paper is discussed in section 7.

2. Maximum likelihood estimation

Let us suppose that we have m upper record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ from the exponentiated inverse Rayleigh distribution with (cdf) (3) and (pdf) (4). The maximum likelihood function for record values is given by Ahsanullah [18]

$$f_{1,2,3,\ldots,m}(X_{L(1)}, X_{L(2)}, X_{L(3)}, \ldots, X_{L(m)}) = f(x_{L(m)}) \prod_{i=1}^{m-1} \frac{f(x_{L(i)})}{1 - F(x_{L(i)})} \tag{6}$$

The likelihood function based on the upper records observed from the exponentiated inverse Rayleigh distribution is given by

$$\begin{aligned} L(\sigma, \alpha; x) &= \frac{2\alpha\sigma^2}{x_m^3} e^{-(\frac{\sigma}{x_m})^2} (1 - e^{-(\frac{\sigma}{x_m})^2})^{\alpha-1} \prod_{i=1}^{m-1} \frac{2\alpha\sigma^2 e^{-(\frac{\sigma}{x_i})^2} (1 - e^{-(\frac{\sigma}{x_i})^2})^{\alpha-1}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})^{\alpha}} \\ &= 2^m \alpha^m \sigma^{2m} (1 - e^{-(\frac{\sigma}{x_m})^2})^{\alpha} \prod_{i=1}^{m} \frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})} \\ &= 2^m \alpha^m \sigma^{2m} e^{\alpha ln(1 - e^{-(\frac{\sigma}{x_m})^2})} \prod_{i=1}^{m} \frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})} \\ L(\sigma, \alpha; x) &= 2^m \alpha^m \sigma^{2m} e^{\alpha ln(1 - e^{-(\frac{\sigma}{x_m})^2})} \prod_{i=1}^{m} \frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})} \end{aligned} \tag{7}$$

Now, taking the log on both sides, we get

$$\begin{aligned} l = \ln L(\sigma, \alpha; x) = m \ln 2 + m \ln \alpha + 2m \ln \sigma + \alpha \ln \left(1 - e^{-\left(\frac{\sigma}{x_m}\right)^2}\right) - \\ \sigma^2 \sum_{i=1}^{m} \left(\frac{1}{x_i}\right)^2 - \sum_{i=1}^{m} \ln \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right) + \frac{1}{\sum_{i=1}^{m} \ln(x_i)^3} \end{aligned} \tag{8}$$

Differentiating Eq. (8) with respect to α and σ and equating to zero, we get

$$\frac{\partial}{\partial \alpha} \ln L(\alpha, \sigma; x) = \frac{m}{\alpha} + ln(1 - e^{-(\frac{\sigma}{x_m})^2}) = 0 \tag{9}$$

$$\frac{\partial}{\partial \sigma} \ln L(\alpha, \sigma; x) = \frac{2m}{\sigma} + \frac{2\alpha\sigma e^{-(\frac{\sigma}{x_m})^2}}{x_m^2 (1 - e^{-(\frac{\sigma}{x_m})^2})} - 2\sigma \sum_{i=1}^{m} \frac{1}{x_i^2} - \sum_{i=1}^{m} \frac{2\sigma}{x_i^2 (1 - e^{-(\frac{\sigma}{x_i})^2})} = 0. \tag{10}$$

Here, equations (9) and (10) are not in exact form, so we cannot obtain the maximum likelihood estimation easily. So the Newton-Ramphson method is used to find the maximum likelihood estimation of $\hat{\alpha}$ and $\hat{\sigma}$. To solve these non-linear equations, an R-package is used to find the mle of $\hat{\alpha}$ and $\hat{\sigma}$.

2.1. Asymptotic confidence intervals

The MLE's of unknown parameter cannot be obtained in closed from, it is not easy to derive the exact distribution of the MLE's. Therefore, we obtain the asymptotic confidence intervalof the parameter based on observed Fisher information matrix. Let ($\hat{\alpha}$,$\hat{\sigma}$) be the MLE's of (α,σ). The observed Fisher information matrix is given by

$$I(\hat{\alpha}, \hat{\sigma}) = - \begin{bmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \sigma} \\ \frac{\partial^2 l}{\partial \sigma \partial \alpha} & \frac{\partial^2 l}{\partial \sigma^2} \end{bmatrix}_{(\hat{\alpha}, \hat{\sigma})}$$

where,

$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{m}{\alpha^2}$, $\frac{\partial^2 l}{\partial \alpha \sigma} = \frac{\partial^2 l}{\partial \sigma \alpha} = \frac{e^{-(\frac{\sigma}{x_m})^2} \cdot 2\left(\frac{\sigma}{x_m^2}\right)}{(1 - e^{-(\frac{\sigma}{x_m})^2})^2}$,

$$\frac{\partial^2 l}{\partial \sigma^2} = \alpha \left(\frac{e^{-\left(\frac{\sigma}{x_m}\right)^2}\left(\frac{2}{x_m^2} - e^{-\left(\frac{\sigma}{x_m}\right)^2}\frac{2\sigma}{x_m^2}\frac{2\sigma}{x_m^2}\right)}{1-e^{-\left(\frac{\sigma}{x_m}\right)^2}} - \frac{e^{-\left(\frac{\sigma}{x_m}\right)^2}\frac{2\sigma}{x_m^2}e^{-\left(\frac{\sigma}{x_m}\right)^2}\frac{2\sigma}{x_m^2}}{(1-e^{-\left(\frac{\sigma}{x_m}\right)^2})^2} \right) - 2\sum_{i=1}^{m}\frac{1}{x_i^2}$$

$$- \sum_{i=1}^{m} \left(\frac{e^{-\left(\frac{\sigma}{x_i}\right)^2}\left(\frac{2}{x_i^2} - e^{-\left(\frac{\sigma}{x_i}\right)^2}\frac{2\sigma}{x_i^2}\frac{2\sigma}{x_i^2}\right)}{1-e^{-\left(\frac{\sigma}{x_i}\right)^2}} - \frac{e^{-\left(\frac{\sigma}{x_i}\right)^2}\frac{2\sigma}{x_i^2}e^{-\left(\frac{\sigma}{x_i}\right)^2}\frac{2\sigma}{x_i^2}}{(1-e^{-\left(\frac{\sigma}{x_i}\right)^2})^2} \right)$$

Thus, the observed variance-covariance matrix becomes $I^{-1}(\hat{\alpha}, \hat{\sigma})$. To obtain the asymptotic confidence interval of the unknown parameters the MLE's estimate follow a bivariate normal distribution with mean (α, σ) and variance- covariance matrix is $I^{-1}(\hat{\alpha}, \hat{\sigma})$. The asymptotic normality of the MLE's can be used to compute approximate $100(1-\eta)\%$ confidence intervals for the parameters α and σ, as follows:
$\hat{\alpha} \pm z_{\eta/2}\sqrt{var(\hat{\alpha})}$ and $\hat{\sigma} \pm z_{\eta/2}\sqrt{var(\hat{\sigma})}$; where $z_{\eta/2}$ is the upper $(\eta/2)$ point of standard normal distribution.

3. Bayes estimation

In this portion, we explore the Bayesian estimation to derive parameter estimates for the EIRD based on upper record values. In the Bayesian estimation framework, decisions regarding the prior distribution and the loss function are of the utmost importance. In the existing literature, various prior distributions have been proposed for the unknown parameters of a particular distribution of interest. For example,Kizilaslan and Nadar[19] consider the gamma prior for generalized exponential distribution,Doostparast et al. [20] consider the normal prior, Fan[21] consider non informative prior, Singh and Tripathi [22] considered the conditional prior for the lognormal distribution. Hu and Ren[23] considered conditional prior for the Inverse Weibull distribution. However, Arnold and Press [24] stated that it's evident that no definitive method exists to determine the superiority of one prior over another. In the context of the preceding discussions, we consider non informative prior $g_1(\alpha) = \frac{1}{\alpha}$ and gamma priors of the EIRD such that

$$g_2(\sigma|a,b) = \frac{b^a \sigma^{a-1} e^{-b\sigma}}{\Gamma a}, \alpha, \sigma > 0; a, b > 0.$$

Now the joint prior distribution of α and σ is,

$$g(\alpha, \sigma) = g_1(\alpha) \times g_2(\sigma|a,b) = \frac{b^a \sigma^{a-1} e^{-b\sigma}}{\alpha \Gamma a}. \tag{11}$$

Here a,b show the hyperparameter, and Γ is the gamma function.

To demonstrate the versatility of our findings and to encompass a wide spectrum of real-world scenarios, we introduce both symmetric and asymmetric loss functions. The inclusion of a symmetric loss function is motivated by its equitable penalization of both underestimation and overestimation, proving advantageous in many instances. However, practical situations often involve scenarios where positive loss holds greater severity than negative loss, and vice versa. In such cases, the need for asymmetric loss functions arises. In our study, we encompass one symmetric option, namely the squared error loss function (SELF).
The mathematical expressions for these loss functions and their corresponding Bayes estimators are given as:
The square error loss function is defined as

$$L_1(\hat{\alpha}, \alpha) = (\hat{\alpha} - \alpha)^2, \alpha > 0.$$

where $\hat{\alpha}$ is the estimate of parameter α.The Bayes estimator under sqare error loss function is

posterior mean ($\hat{\alpha}_{SEL}$).
Now, the joint posterior distribution, obtained using equations (7) and (11), is given as

$$\pi(\alpha,\sigma|x) \propto \alpha^{m-1}\sigma^{2m+a-1}e^{-b\sigma}e^{\alpha ln(1-e^{-(\frac{\sigma}{x_m})^2})}\prod_{i=1}^{m}\frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3(1-e^{-(\frac{\sigma}{x_i})^2})^2} \tag{12}$$

We observe that the joint posterior distribution given in equation (12) cannot be simplified into a closed form expression. So by making use of some approximation methods, we can derive explicit expressions for these estimators. To tackle this situation, two widely applicable approximation methods, i.e., the Tierney-Kadane approximation and the Markov chain Monte Carlo method, are applied. In the existing literature, Lindley's method [25] has been extensively taken into account for such situations. However, this method requires third derivatives of the log-likelihood function. Instead, we consider another approximation method proposed by Tierney and Kadane (TK) [26], in which derivatives only up to second order are required to compute the desired Bayes estimates.

3.1. MCMC Algorithm.

In this specific section, we employ the Markov Chain Monte Carlo (MCMC) methodology to obtain an estimated Bayesian approximation for the parameters α and σ under the square error loss function. With the help of posterior densities, the MCMC method can be used to generate a random sample of unknown quantities.The generated sample is used to obtain the Bayes estimator for the loss functions. The marginal densities of α and σ are given as

$$\pi(\alpha|\sigma,x) \propto Gamma\left(m, \frac{1}{ln(1-e^{-(\frac{\sigma}{x_m})^2})}\right)$$

$$\pi(\sigma|\alpha,x) \propto \sigma^{2m+a-1}e^{-b\sigma}e^{\alpha ln(1-e^{-(\frac{\sigma}{x_m})^2})}\prod_{i=1}^{m}\frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3(1-e^{-(\frac{\sigma}{x_i})^2})^2}$$

The marginal posterior density of α, a closed form of which follows the Gamma distribution, So,the Gibbs sampling [27] method is used to generate the sample of α. The marginal posterior density of σ is not an exact form of any distribution, so we used the M-H algorithm to generate a sample of σ. For more information on the algorithm, methods and steps are given in [28]. This algorithm combines the Metropolis-Hastings scheme with the Gibbs sampling scheme under the Gaussian proposal distribution.
The steps in which the M-H approach performs to simulate the posterior sample are as follows:
Step 1: Take some initial guess values for the parameters α and σ be (α^0,σ^0).
Step 2: Set t=1.
Step 3: Generate $\sigma^{(t)}$ from $\pi(\sigma|\alpha^{(t)},a,b)$ using the M-H algorithm with the proposal that the distribution is normal distribution.
Step 4: Generate $\alpha^{(t)}$ from $\pi(\alpha|\sigma^{(t-1)},a,b)$.
Step 5: Set t=t+1.
Step 6: Repeat steps 2–5 up to N times and obtain the posterior sample (α^t,σ^t) for t=1,2,...,N.
Using the posterior sample, we obtain the Bayesian estimates for the parameters α and σ under the squared error function, given by,

$$\hat{\alpha}_{SELF} = \frac{1}{N-M}\sum_{t=M+1}^{N}\alpha^t$$

$$\hat{\sigma}_{SELF} = \frac{1}{N-M}\sum_{t=M+1}^{N}\sigma^t$$

where M is the burn period of MCMC.

4. Tierney-Kadane Approximation

The TK approximation method was first proposed by Tierney and Kadane in 1986 [26] as a way to estimate the posterior expectation that involves the ratio of two integrals. The process of applying the TK technique is simple and straight-forward. This section deals with the use of TK's method to approximate the Bayes estimates. Suppose our objective is to estimate the expression $E(u(\alpha,\sigma)|x)$ using the TK method. Then,we first consider the following functions:

$$I(x) = E[u(\alpha,\sigma|x)] = \frac{\int_0^\infty \int_0^\infty u(\alpha,\sigma)e^{[L(\alpha,\sigma|x)+\rho(\alpha,\sigma|x)]}d\alpha d\sigma}{\int_0^\infty \int_0^\infty e^{[L(\alpha,\sigma|x)+\rho(\alpha,\sigma|x)]}d\alpha d\sigma}.$$

where u(α,σ) is a function of α and σ, L(α,σ) can defined in equation (8).
$\rho(\alpha,\sigma)$ is logarithm of joint prior distribution which is given in equation (11) and defined as :
$\rho(\alpha,\sigma) = ln(g(\alpha,\sigma)) = aln(b) + (a-1)ln(\sigma) - b\sigma - ln(\Gamma a) - ln(\alpha)$.

We can approximate the function I(x) into an explicit expression by applying the TK approximation method. We first consider the following function:

$$\delta(\alpha,\sigma) = \frac{L(\alpha,\sigma|x) + \rho(\alpha,\sigma|x)}{n},$$

and

$$\delta_\theta{}^*(\alpha,\sigma) = \delta(\alpha,\sigma) + \frac{lnu(\alpha,\sigma)}{n},$$

Now, we assume that $(\hat{\alpha_\delta},\hat{\sigma_\delta})$ and $(\hat{\alpha_{\delta^*}},\hat{\sigma_{\delta^*}})$ maximize the function $\delta(\alpha,\sigma)$ and $\delta_\theta{}^*(\alpha,\sigma)$,respectively.
We then approximate $I(x)$ as

$$I(x) = \sqrt{\frac{|\Sigma_\theta{}^*|}{|\Sigma|}}e^{[n(\delta_\theta{}^*(\hat{\alpha_{\delta^*}},\hat{\sigma_{\delta^*}}))-\delta(\hat{\alpha_\delta},\hat{\sigma_\delta})]},$$

Here,$|\Sigma_\theta|$ and $|\Sigma_\theta^*|$ are the negative Inverse of Hessian matrices of $\delta(\alpha,\sigma)$and $\delta_\theta{}^*(\alpha,\sigma)$respectively.
$|\Sigma| = [\frac{\partial^2\delta}{\partial\alpha^2}\frac{\partial^2\delta}{\partial\sigma^2} - \frac{\partial^2\delta}{\partial\alpha\partial\sigma}\frac{\partial^2\delta}{\partial\sigma\partial\alpha}]^{-1}$ and $|\Sigma_\theta{}^*| = [\frac{\partial^2\delta_\theta{}^*}{\partial\alpha^2}\frac{\partial^2\delta_\theta{}^*}{\partial\sigma^2} - \frac{\partial^2\delta_\theta{}^*}{\partial\alpha\partial\sigma}\frac{\partial^2\delta_\theta{}^*}{\partial\sigma\partial\alpha}]^{-1}$ Now,
The prior information is

$$\rho(\alpha,\sigma|x) = alnb + (a-1)ln\sigma - b\sigma - ln\Gamma a - ln\alpha.$$

The likelihood function is
$lnL(\sigma,\alpha;x) = mln2 + mln\alpha + 2mln\sigma + \alpha ln(1-e^{-(\frac{\sigma}{x_m})^2}) - \sigma^2\sum_{i=1}^m(\frac{1}{x_i})^2 - \sum_{i=1}^m ln(1-e^{-(\frac{\sigma}{x_i})^2}) + \frac{1}{\sum_{i=1}^m \ln(x_i)^3}$.

Now,

$$\delta(\alpha,\sigma) = \frac{L(\alpha,\sigma|x) + \rho(\alpha,\sigma|x)}{n}$$

$= \frac{1}{n}[mln2 + mln\alpha + 2mln\sigma + \alpha ln(1-e^{-(\frac{\sigma}{x_m})^2}) - \sigma^2\sum_{i=1}^m(\frac{1}{x_i})^2 - \sum_{i=1}^m ln(1-e^{-(\frac{\sigma}{x_i})^2}) + \frac{1}{\sum_{i=1}^m \ln(x_i)^3} + alnb + (a-1)ln\sigma - b\sigma - ln\Gamma a - ln\alpha]$.
It's important to observe that

$$\frac{\partial\delta}{\partial\alpha} = \frac{1}{n}\big[\frac{m}{\alpha} + ln(1-e^{-(\frac{\sigma}{x_m})^2}) - \frac{1}{\alpha}\big]$$

and

$$\frac{\partial^2\delta}{\partial\alpha^2} = \frac{1}{n}\big[-\frac{m}{\alpha^2} + \frac{1}{\alpha^2}\big],$$

$$\frac{\partial \delta^2}{\delta\alpha\delta\sigma} = \frac{\partial \delta^2}{\delta\sigma\delta\alpha} = \frac{2e^{-(\frac{\sigma}{x_m})}\frac{\sigma}{x_m}}{n(1-e^{-(\frac{\sigma}{x_m})^2})},$$

$$\frac{\partial \delta}{\partial \sigma} = \frac{2m}{\sigma} + \frac{2\sigma\alpha e^{-(\frac{\sigma}{x_m})^2}}{x_m(1-e^{-(\frac{\sigma}{x_m})^2})} - 2\sigma\sum_{i=1}^{m}\frac{1}{x_i^2} - \sum_{i=1}^{m}\frac{2\sigma e^{-(\sigma/x_i)}}{x_i(1-e^{-(\sigma/x_i)^2})} + \frac{(a-1)}{\sigma} - b.$$

Further, we use the derived quantities to obtain the Bayes estimators under the square error loss functions. It is evident that quantities except $\delta(\alpha,\sigma)$ and its derivatives are common in each form of Bayes estimators. The $\delta^*_\theta(\alpha,\sigma)$ quantity is given as the square error loss function:

(i) If $u(\alpha,\sigma) = \alpha$, then

$$\hat{\alpha}_{SEL} = \sqrt{\frac{|\Sigma^*_{\alpha_{SEL}}|}{|\Sigma|}} e^{[n(\delta^*_{\alpha_{SEL}}(\alpha_{\hat{\delta}^*},\sigma_{\hat{\delta}^*}))-\delta(\hat{\alpha_\delta},\hat{\sigma_\delta})]},$$

In order to compute $|\Sigma^*_{\alpha SEL}|$, we first obtain the following expression:

$$\delta^*_{\alpha_{SEL}} = \delta(\alpha,\sigma) + \frac{1}{n}ln(\alpha)$$

$$\frac{\partial \delta^*}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} + \frac{1}{n\alpha},$$

$$\frac{\partial \delta^*}{\partial \alpha^2} = \frac{\partial^2 \delta}{\partial \alpha^2} - \frac{1}{n\alpha^2},$$

$$\frac{\partial^2 \delta^*}{\partial \sigma^2} = \frac{\partial^2 \delta}{\partial \sigma^2},$$

$$\frac{\partial^2 \delta^*}{\partial \alpha \partial \sigma} = \frac{\partial^2 \delta}{\partial \alpha \partial \sigma}.$$

(ii) If $u(\alpha,\sigma) = \sigma$ then

$$\hat{\sigma}_{SEL} = \sqrt{\frac{|\Sigma^*_{\sigma SEL}|}{|\Sigma|}} e^{[n(\delta^*_{\sigma SEL}(\alpha_{\hat{\delta}^*},\sigma_{\hat{\delta}^*}))-\delta(\hat{\alpha_\delta},\hat{\sigma_\delta})]},$$

In order to compute $|\Sigma^*_{\sigma_{SEL}}|$, we first obtain the following expression.

$$\delta^*_{\sigma_{SEL}} = \delta(\alpha,\sigma) + \frac{1}{n}ln(\sigma),$$

$$\frac{\partial \delta^*}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha},$$

$$\frac{\partial \delta^*}{\partial \alpha^2} = \frac{\partial^2 \delta}{\partial \alpha^2},$$

$$\frac{\partial^2 \delta^*}{\partial \sigma^2} = \frac{\partial^2 \delta}{\partial \sigma^2} - \frac{1}{n\sigma^2},$$

$$\frac{\partial^2 \delta^*}{\partial \alpha \partial \sigma} = \frac{\partial^2 \delta}{\partial \alpha \partial \sigma}.$$

5. Simulation

In this section, we present the simulation result comparing the performance of the MLE's estimator and Bayes estimator for the parameters of the EIRD using upper record values. We consider the two sets of parameter values (2, 1) and (1,1) of the EIRD , generate the random sample using the inverse CDF method, and select the record values (6,7,8,9) from the generated sample for each parameter value.

The Monte Carlo simulation study compares various estimators using different sample sizes and true parameter values. Here are the key points summarized from the result:

1: Two sets of parameter values are used as (2,1), (1,1) and two pairs of hyperparameter values (0.2,0.2),(0.5,0.5).
2: There are four different sample sizes (6,7,8,9) considered in the simulation.
3: There are two methods of estimation: one is the classical method, such as MLE's, and the other is the Bayesian estimation (MCMC,TK) method.
4: Here we consider the square error loss function, which is used to compute the Bayes estimate.
5: We generate 10000 posterior samples with a burn period of 2000 sample are used.
6: Confidence interval based on observed Fisher information matrix and Highest Posterior Density (HPD) credible interval are computed for 95 %.

Table 1: *Estimate of MLE's and MSE (in parenthesis) for α ,σ along with confidence interval when $\alpha = 2$, σ=1*

m	$\hat{\alpha}_{\text{MLE}}$	$\hat{\sigma}_{\text{MLE}}$	$\text{CI}_{\hat{\alpha}}$	$\text{LCI}_{\hat{\alpha}}$	$\text{CI}_{\hat{\sigma}}$	$\text{LCI}_{\hat{\sigma}}$
6	2.3136(7.3692)	0.8780 (0.8532)	(0.3515,7.1391)	6.7875	(0.4649,2.4442)	1.9793
7	1.9778 (5.0171)	0.7579 (0.7859)	(0.1058,6.0209)	5.9151	(0.4580,2.3812)	1.9232
8	1.8063 (3.4510)	0.7925 (0.7225)	(0.3828,5.2598)	4.8770	(0.4475,2.3299)	1.8824
9	1.7981 (2.4550)	0.7931(0.6868)	(0.5546,4.7534)	4.1988	(0.4396,2.2933)	1.8537

Table 2: *Estimate of MLE and MSE (in parenthesis) for α, σ along with confidence interval when $\alpha = 1$ and $\sigma = 1$*

m	$\hat{\alpha}_{\text{MLE}}$	$\hat{\sigma}_{\text{MLE}}$	$\text{CI}_{\hat{\alpha}}$	$\text{LCI}_{\hat{\alpha}}$	$\text{CI}_{\hat{\sigma}}$	$\text{LCI}_{\hat{\sigma}}$
6	1.2427(1.9978)	0.9327 (4.3135)	(0.0464,3.1658)	3.1194	(0.1935,3.7162)	3.5227
7	0.8527 (1.1597)	0.7835 (4.0323)	(0.2089,2.6796)	2.4707	(0.1698,3.6325)	3.4627
8	0.9565 (0.7039)	0.8269 (3.8289)	(0.3002,2.3759)	2.0757	(0.1467,3.5706)	3.4239
9	0.9590(0.4689)	0.8293(3.6965)	(0.3552,2.1821)	1.8269	(0.1303,3.5289)	3.3986

Table 3: *MSE's(in parentheses) of TK and MCMC Bayes estimates of parameter values based on record values for prior (0.2,0.2) at (1,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	1.1039(5.6130)	0.7300(1.5430)	1.7841(1.0749)	1.1316(0.1894)	(0.5900,3.1165)	2.5265	(0.3773,1.9681)	1.5908
7	0.7510(1.9584)	0.5738(1.4414)	1.1619(0.1904)	0.9902(0.1598)	(0.4396,2.9633)	2.5237	(0.2812,1.7995)	1.5183
8	0.8711(0.6620)	0.6420(1.3094)	1.2595(0.2350)	0.9966(0.1554)	(0.5422,2.0560)	1.5138	(0.2486,1.7505)	1.5019
9	0.8762(0.3964)	0.6482(1.2667)	1.2064(0.1860)	0.9843(0.1860)	(0.4841,1.9342)	1.4501	(0.2622,1.6971)	1.4349

Table 4: *MSE's(in parentheses) of TK and MCMC Bayes estimates of paramter values based on record values for prior (0.5,0.5) at (1,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	1.1430(3.9937)	0.7701(0.8875)	1.7934(1.0677)	1.1166(0.1723)	(0.6164,3.0766)	2.4602	(0.3224,1.8573)	1.5349
7	0.7953(1.4146)	0.6489(0.8168)	1.1442(0.1622)	0.9646(0.1494)	(0.4872,2.8896)	2.4024	(0.2576,1.7041)	1.4465
8	0.8985(0.5799)	0.6943(0.7549)	1.2675(0.2363)	1.0064(0.1492)	(0.5259,2.0663)	1.5404	(0.2751,1.7057)	1.4306
9	0.8984(0.3504)	0.6963(0.7253)	1.2220(0.1992)	1.0021(0.1420)	(0.5107,1.9473)	1.4366	(0.2617,1.6775)	1.4158

Table 5: *MSE's(in parentheses) of TK and MCMC Bayes estimates of paramter values based on record values for prior (0.2,0.2) at (2,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	2.1127(6.8287)	0.7480(0.4823)	2.5948(0.7996)	1.0486(0.0960)	(1.2007,6.6411)	5.4404	(0.4850,1.6617)	1.1767
7	1.9563(4.4619)	0.6261(0.4425)	2.1602(0.6114)	0.8843(0.09384)	(0.8237,4.6751)	3.8514	(0.3691,1.4662)	1.0971
8	1.6683(3.0084)	0.6797(0.4030)	2.4131(0.8801)	0.9173(0.0931)	(0.8816,4.0792)	3.1976	(0.3311,1.4743)	1.1432
9	1.6834(2.0903)	0.6862(0.3803)	2.2680(0.6123)	0.9261(0.0916)	(0.9144,3.7480)	2.8336	(0.3390,1.320)	0.9810

Table 6: *MSEs(in parentheses) of TK and MCMC Bayes estimates of paramter values based on record values for prior (0.5,0.5) at (2,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	2.1524(6.5179)	0.7685(0.4027)	2.1645(5.1493)	1.0567(0.0959)	(0.9770,6.7309)	5.7539	(0.4583,1.6389)	1.1806
7	1.4706(4.2555)	0.6536(0.3725)	2.1839(0.9530)	0.8829(0.0951)	(0.8348,3.7032)	2.8684	(0.3062,1.4533)	1.1471
8	1.6937(2.8772)	0.7007(0.3397)	2.1308(0.8968)	0.9114(0.0949)	(0.8590,3.0922)	2.2332	(0.3320,1.4685)	1.1365
9	1.7044(1.9899)	0.7054(0.3215)	2.1052(0.8002)	0.9261(0.0674)	(0.9150,2.8668)	1.9518	(0.3858,1.4851)	1.0993

From Tables 1, 2, 3, 4, 5 , and 6 the following conclusions are given as:
In cases where the sample size increases, the mean square error of the maximum likelihood decreases, and the length of the asymptotic confidence interval also decreases in all cases. Bayes estimates are better than the maximum likelihood function as compared to MSEs. In the case of Bayesian estimation, MCMC methods are better than T-K approximation methods. The length of HPD intervals also decreases as the sample size increases.

Therefore, in situations where prior knowledge about parameters is known or where non-informative priors are being used, we advise utilizing the Bayes estimators. In other circumstances, ML estimators could be utilized to get an immediate outcome.

6. Application

ALAF Industry, a part of the Safal Group, is a leading producer of steel roofing in Tanzania. The Safal Group is renowned for its trusted steel roofing brand and operates in 11 countries across Eastern and Southern Africa. The group has introduced advanced coating technology to Africa, with four coating mills located in Kenya, Uganda, Tanzania, and South Africa. ALAF Industry, as one of Safal Group's coating mills, focuses on enhancing the quality of steel roofing.

One crucial process in improving steel roofing quality is the coating process, where ALAF Industry utilizes aluminum-zinc galvanization technology. Two datasets were analyzed to demonstrate the effectiveness of the coating process. The first dataset comprises 72 observations on coating weight using chemical methods on the top center side (Tcs), while the second dataset consists of 72 observations on coating weight using chemical methods on the bottom center side (Bcs), the two Data sets are given as:
Data set1(Tcs):36.8 47.2 35.6 36.7 55.8 58.7 42.3 37.8 55.4 45.2 31.8 48.3 45.3 48.5 52.8 45.4 49.8 48.2 54.5 50.1 48.4 44.2 41.2 47.2 39.1 40.7 40.3 41.2 30.4 42.8 38.9 34.0 33.2 56.8 52.6 40.5 40.6 45.8 58.9 28.7 37.3 36.8 40.2 58.2 59.2 42.8 46.3 61.2 58.4 38.5 34.2 41.3 42.6 43.1 42.3 54.2 44.9 42.8 47.1 38.9 42.8 29.4 32.7 40.1 33.2 31.6 36.2 33.6 32.9 34.5 33.7 39.9.

Data Set2(Bcs):45.5 37.5 44.3 43.6 47.1 52.9 53.6 42.9 40.6 34.1 42.6 38.9 35.2 40.8 41.8 49.3 38.2 48.2 44.0 30.4 62.3 39.5 39.6 32.8 48.1 56.0 47.9 39.6 44.0 30.9 36.6 40.2 50.3 34.3 54.6 52.7 44.2 38.9 31.5 39.6 43.9 41.8 42.8 33.8 40.2 41.8 39.6 24.8 28.9 54.1 44.1 52.7 51.5 54.2 53.1 43.9 40.8 55.9 57.2 58.9 40.8 44.7 52.4 43.8 44.2 40.7 44.0 46.3 41.9 43.6 44.9 53.6

To check whether the data set follows the EIRD, the K-S test, emperical cdf, and P-P plot are applied to the test. Data set I supports the EIRD for alpha and beta, with a K-S distance of 0.0523 and p values of 0.8325. Similarly, data set II also supports the EIRD determination with a K-S distance of 0.0731 and p-value of 0.7602. Furthermore, the empirical and theoretical CDFs, as well as the P-P plot (probability-probability plot) displayed in Figure 1, confirm that the EIRD provides a good fit for both the data sets.

Overall, based on the statistical analysis and visual inspection of the data, it can be concluded that the EIRD is suitable for analyzing the both the Data sets. Now the upper record values generated from the data sets I and II are (36.8,47.2,55.8,58.7,58.9,59.2,61.2) and (45.5,52.9,53.6,62.3), respectively.

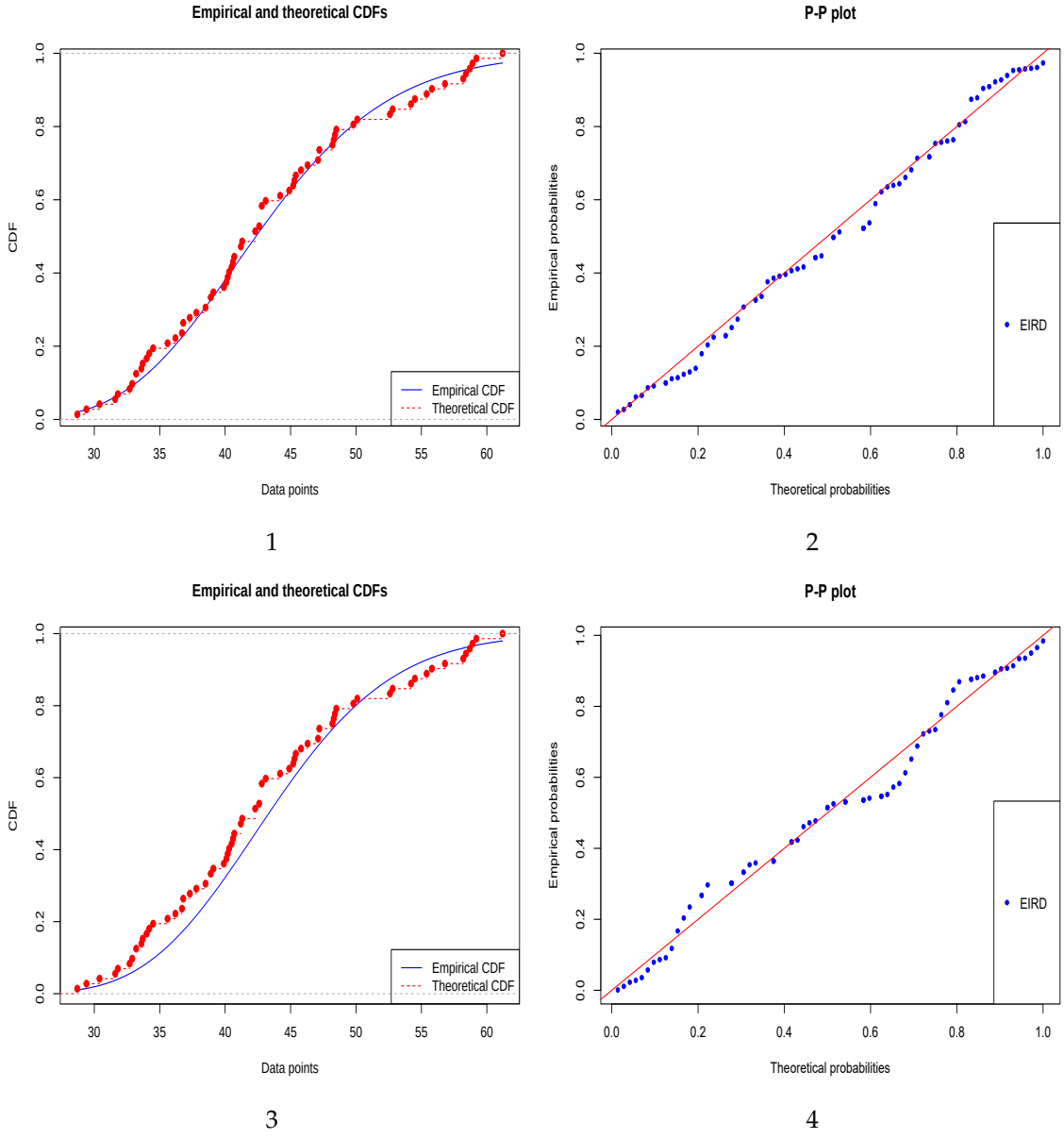
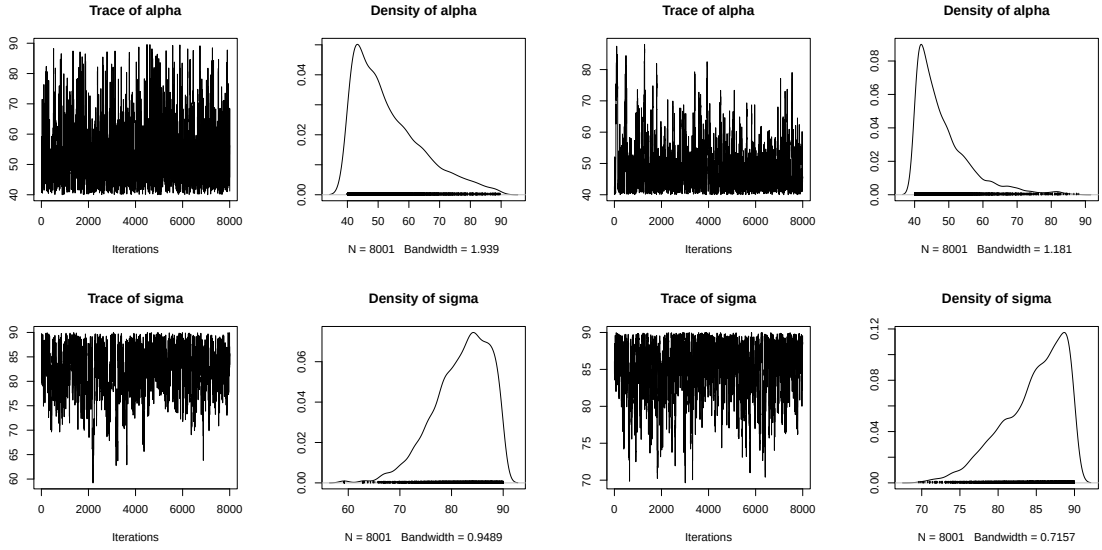


Figure 1. The emperical and theoratical CDF plot and P-P plot for the real data set.

The article describes obtaining maximum likelihood (ML) estimates and Bayesian estimates using a square error loss function, following the procedure outlined earlier. The results are presented in Table 7. Additionally, trace plots and posterior density plots for the parameters α and σ are depicted in Figures 2 .



Data set I Data set II

Figure 2. The iteration plot and posterior sample plot of Data set I and Data set II.

Table 7: *MLE's, MCMC and TK approximation methods to estimates of parameters for real data set.*

	MLE		MCMC		TK	
Data	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$
Data set I	64.5901	92.3745	52.8539	81.9270	58.3177,	83.6670
Data set II	42.7885	97.2838	49.3734	84.84805	52.4409	93.4215

From these Figure 2, it's concluded that the Markov Chain Monte Carlo (MCMC) samples exhibit good mixing, indicating effective exploration of the parameter space. Moreover, the skewed posterior density suggests a preference for higher parameter values. This observation supports the conclusion that the MCMC chain is stationary, meaning that it has reached a stable distribution.

7. Conclusions

In this study, we have examined the EIRD in a situation where the data are available as upper record values. We follow on the task of estimating the unknown parameter of the EIRD distribution and obtain the maximum likelihood estimators and corresponding confidence intervals for the distribution parameters. In the simulation study, we noticed that the behaviour of estimations in terms of mean square error improved with an increase in the sample size of record values. Additionally, the true value and estimate values are contained in the asymptotic confidence interval. Next, we discussed the problems of computing Bayes estimates under the square error loss function using the TK and MCMC methods in Bayesian estimation. We discovered that the MCMC approach performs better than TK in our simulation study. Still, HPD interval estimates were computed with the help of the MCMC approach.We have used real data sets to demonstrate each of the suggested estimation techniques.

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THE POISSON-SUJA DISTRIBUTION AND ITS APPLICATIONS IN BIOLOGICAL COUNT DATA SETS

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Abstract

The Poisson-Suja distribution which is a Poisson mixture of Suja distribution has been proposed. The descriptive statistics based on moments including coefficient of variation, skewness, kurtosis and index of dispersion has been derived and studied. Over-dispersion, unimodality and increasing hazard rate properties of the distribution have been studied. The method of moment and the method of maximum likelihood have been discussed for estimating parameters. Applications and the goodness of fit the distribution and its comparison with other one-parameter discrete distributions have also been presented. It was found more closer fit than other compared distributions. So, it can be considered as good discrete distribution for count datasets.

Keywords: Suja distribution, compounding, descriptive statistics, statistical properties, estimation of parameter, goodness of fit.

I. Introduction

The statistical analysis and modeling of count data are essential in almost all fields of knowledge including biological science, insurance, medical science, and finance, are some among others. Count data are generated from different discrete phenomena such as the number of insurance claimants in insurance, the number of yeast cells in biological science, the number of chromosomes in genetics, etc. It has been observed that, in general, count data follows under-dispersion (variance greater than mean), equi-dispersion (variance equal to mean), or over-dispersion (variance less than mean). The over-dispersion of count data has been well addressed and discussed using mixed Poisson distributions by different researchers including Raghavachari et al [1], Karlis and Xekalaki [2], Panjeer [3], are some among others. Mixed Poisson distributions arise when the parameter of the Poisson distribution is a random variable having some specified distribution and the distribution of the parameter of the Poisson distribution is known as mixing distribution. It has been observed that the general characteristics of the mixed Poisson distribution follow some characteristics of its mixing distributions also. The field of distribution theory is flooded with mixed Poisson distributions based on some proper mixing distributions.

The classical negative binomial distribution (NBD) derived by Greenwood and Yule [4]is the mixed Poisson distribution where the parameter of the Poisson random variable is distributed as a gamma random variable. The NBD has been used to model over-dispersed count data. However, the NBD may not be appropriate for some over-dispersed count data due to its theoretical or applied point of view. Other mixed Poisson distributions arise from a choice of alternative mixing distributions. For example, the Poisson-Lindley distribution, introduced by Sankaran [5], is a Poisson mixture of Lindley [6] distribution. The Poisson-Akash distribution, introduced by Shanker [7], is a Poisson mixture of Akash distribution proposed by Shanker [8]. The discrete Poisson-Ishita distribution (PID) introduced by Shukla and Shanker [9] is a Poisson mixture of Ishita distribution proposed by Shanker and Shukla [10]. The generalized Poisson-Lindley distribution, introduced by Mahmoudi and Zakerzadeh [11], is a mixed Poisson distribution where the mixing distribution is the generalized Lindley distribution proposed by Zakerzadeh and Dolati [12]. Other mixed distributions are the Poisson-weighted exponential distribution (P-WED) introduced by Zamani et al [13]and the negative binomial-Lindley distribution (NB-LD) introduced by Zamani and Ismail [14]where the mixing distributions were weighted exponential distribution and Lindley [6] distribution, respectively. The Poisson-weighted Lindley distribution (P-WLD) introduced by Abd EL-Monsef and Sohsah [15] is a Poisson mixture of weighted Lindley distribution proposed by Ghitany et al [16].

It has been observed by Karlis and Xekalaki [2] that there are some natural situations where a good fit is not obtainable with a particular mixed Poisson distribution in case of over-dispersed count data. This shows that there is a requirement for new mixed Poisson distribution which gives better fit as compared with the existing mixed Poisson distributions. In the recent decade, there were some one parameter lifetime distributions whose Poisson mixture has not been derived and studied. For example, Shanker [17] suggested a one parameter lifetime distribution named Suja distribution for modeling lifetime data from engineering and biomedical sciences. The Suja distribution is defined by its probability density function (pdf)

$$f(x,\theta) = \frac{\theta^5}{\theta^4+24}(1+x^4)e^{-\theta x}; x>0, \theta>0 \quad (1)$$

The Suja distribution is a two-component mixture of exponential distribution having scale parameter θ and a gamma distribution having shape parameter 5 and scale parameterθ with their mixing proportions $\frac{\theta^4}{\theta^4+24}$ and $\frac{24}{\theta^4+24}$ respectively. Various statistical properties of the Suja distribution including its shape, moments, skewness , kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability have been discussed by Shanker [17].

In the present paper, a Poisson mixture of Suja distribution named, "Poisson-Suja distribution (PSD) has been proposed. Its various mathematical and statistical properties including its shape, moments, coefficient of variation, skewness, kurtosis and index of dispersion have been discussed. The estimation of its parameter has been discussed using maximum likelihood estimation and method of moments. The goodness of fit of PSD along with equi-dispersed Poisson distribution (PD), and over-dispersed PLD and PID have been studied and presented with some count datasets.

II. Poisson-Suja distribution

Suppose the parameter λ of Poisson distribution follows Suja distribution (1). Then the Poisson mixture of Suja distribution (1) can be obtained as

$$P(X=x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^5}{\theta^4+24}(1+\lambda^4)e^{-\theta\lambda}d\lambda \quad (2)$$
$$= \frac{\theta^5}{(\theta^4+24)x!}\left[\int_0^\infty e^{-(1+\theta)\lambda}\lambda^x d\lambda + \int_0^\infty e^{-(1+\theta)\lambda}\lambda^{x+4}d\lambda\right]$$

$$= \frac{\theta^5}{(\theta^4+24)x!}\left[\frac{\Gamma(x+1)}{(\theta+1)^{x+1}} + \frac{\Gamma(x+5)}{(\theta+1)^{x+5}}\right]$$

$$= \frac{\theta^5}{(\theta^4+24)(\theta+1)^{x+5}}\left[(\theta+1)^4 + (x+1)(x+2)(x+3)(x+4)\right]; \theta > 0, x = 0,1,2,\ldots$$

$$= \frac{\theta^5}{\theta^{24}+24} \cdot \frac{x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)}{(\theta+1)^{x+5}}; x = 0,1,2,\ldots,\theta > 0. \quad (3)$$

As this is the Poisson mixture of Suja distribution, we name this distribution "Poisson-Suja distribution (PSD)".

The graphs of the pmf of PSD for different parameter values are shown in figure 1. It is obvious that as the parameter value increases, the PSD shapes change from negatively skewed to positively skewed.

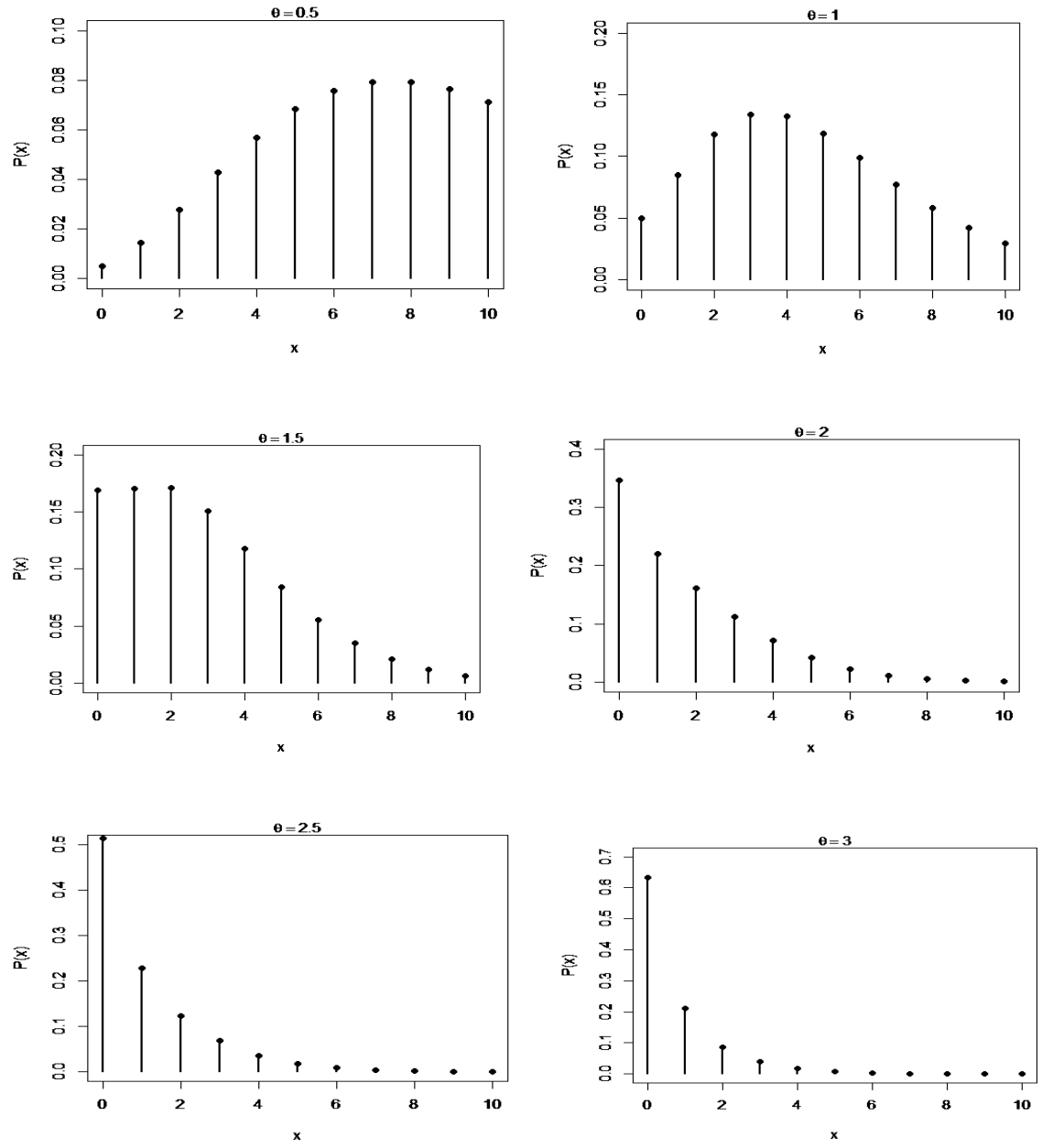

Figure1: *Graphs of probability mass function of PSD for different values of the parameter* θ.

III. Moments and associated measures

The rth factorial moment about origin of PSD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}|\lambda\right)\right], \quad \text{where} \quad X^{(r)} = X(X-1)(X-2)\ldots(X-r+1).$$

Using (2.1), the rth factorial moment about origin of PSD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}|\lambda\right)\right] = \frac{\theta^5}{\theta^4+24}\int_0^\infty \left[\sum_{x=0}^\infty x^{(r)}\frac{e^{-\lambda}\lambda^x}{x!}\right](1+\lambda^4)e^{-\theta\lambda}d\lambda$$
$$= \frac{\theta^5}{\theta^4+24}\int_0^\infty \lambda^r\left[\sum_{x=r}^\infty \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right](1+\lambda^4)e^{-\theta\lambda}d\lambda$$

Taking $x+r$in place of $x$within the bracket, we get

$$\mu_{(r)}' = \frac{\theta^5}{\theta^4+24}\int_0^\infty \lambda^r\left[\sum_{x=0}^\infty \frac{e^{-\lambda}\lambda^y}{y!}\right](1+\lambda^4)e^{-\theta\lambda}d\lambda$$

The expression within the bracket is clearly unity and hence we have

$$\mu_{(r)}' = \frac{\theta^5}{\theta^4+24}\int_0^\infty \lambda^r\,(1+\lambda^4)e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, we get finally, a general expression for the rth factorial moment of PSD (2.2) as

$$\mu_{(r)}' = \frac{r!\{\theta^4+(r+1)(r+2)(r+3)(r+4)\}}{\theta^r(\theta^4+24)}; r = 1,2,3,4,\ldots \tag{4}$$

Substituting $r = 1,2,3,$and4 in (4), the first four factorial moments about origin can be obtained and using the relationship between factorial moments about origin and moments about origin, the first four moment about origin of the PSD (3) are obtained as

$$\mu'_1 = \mu'_{(1)} = \frac{\theta^4+120}{\theta(\theta^4+24)} \tag{5}$$

$$\mu'_2 = \frac{\theta^5+2\theta^4+120\theta+720}{\theta^2(\theta^4+24)} \tag{6}$$

$$\mu'_3 = \frac{\theta^6+6\theta^5+6\theta^4+120\theta^2+2160\theta+5040}{\theta^3(\theta^4+24)} \tag{7}$$

$$\mu'_4 = \frac{\theta^7+14\theta^6+36\theta^5+24\theta^4+120\theta^3+5040\theta^2+30240\theta+40320}{\theta^4(\theta^4+24)} \tag{8}$$

Using the relationship between moments about mean and the moments about origin, the moments about mean of the PSD (3) are obtained as

$$\mu_2 = \sigma^2 = \frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta^2(\theta^{24}+24)^2} \tag{9}$$

$$\mu_3 = \frac{\left(\begin{matrix}\theta^{14}+3\theta^{13}+2\theta^{12}+168\theta^{10}+1656\theta^9+3024\theta^8+6336\theta^6+46656\theta^5\\ +3456\theta^4+132\theta^3+69120\theta^2+207360\theta+138240\end{matrix}\right)}{\theta^3(\theta^4+24)^3} \tag{10}$$

$$\mu_4 = \frac{\left(\begin{matrix}\theta^{19}+10\theta^{18}+18\theta^{17}+9\theta^{16}+192\theta^{15}+4896\theta^{14}+22464\theta^{13}+23904\theta^{12}+10368\theta^{11}\\ +281088\theta^{10}+946944\theta^9+528768\theta^8+221184\theta^7+5584896\theta^6+12939264\theta^5\\ +11114496\theta^4+1658880\theta^3+36495360\theta^2+69672960\theta+34836480\end{matrix}\right)}{\theta^4\left(\theta^4+24\right)^4} \tag{11}$$

The coefficient of variation$(C.V)$, coefficient of Skewness$\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis(β_2), and index of dispersion (γ) of the PSD (3) are thus given by

$$C.V = \frac{\sigma}{\mu'_1} = \frac{\sqrt{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}}{\theta^4+120} \tag{12}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{\left(\begin{matrix}\theta^{14}+3\theta^{13}+2\theta^{12}+168\theta^{10}+1656\theta^9+3024\theta^8+6336\theta^6+46656\theta^5\\ +3456\theta^4+132\theta^3+69120\theta^2+207360\theta+138240\end{matrix}\right)}{(\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880)^{\frac{3}{2}}} \tag{13}$$

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{\begin{pmatrix}\theta^{19}+10\theta^{18}+18\theta^{17}+9\theta^{16}+192\theta^{15}+4896\theta^{14}+22464\theta^{13}+23904\theta^{12}+10368\theta^{11}\\ +2810880\theta^{10}+946944\theta^{9}+528768\theta^{8}+221184\theta^{7}+55848960\theta^{6}+12939264\theta^{5}\\ +11114496\theta^{4}+1658880\theta^{3}+364953600\theta^{2}+69672960\theta+34836480\end{pmatrix}}{(\theta^{9}+\theta^{8}+144\theta^{5}+528\theta^{4}+2880\theta+2880)^2} \quad (14)$$

$$\gamma = \frac{\sigma^2}{\mu'_1} = \frac{\theta^{9}+\theta^{8}+144\theta^{5}+528\theta^{4}+2880\theta+2880}{\theta(\theta^{4}+24)(\theta^{4}+120)}. \quad (15)$$

The shapes of coefficient of variation (CV), skewness, kurtosis and index of dispersion (ID) of PSD are presented in figure 2 with varying values of parameter.

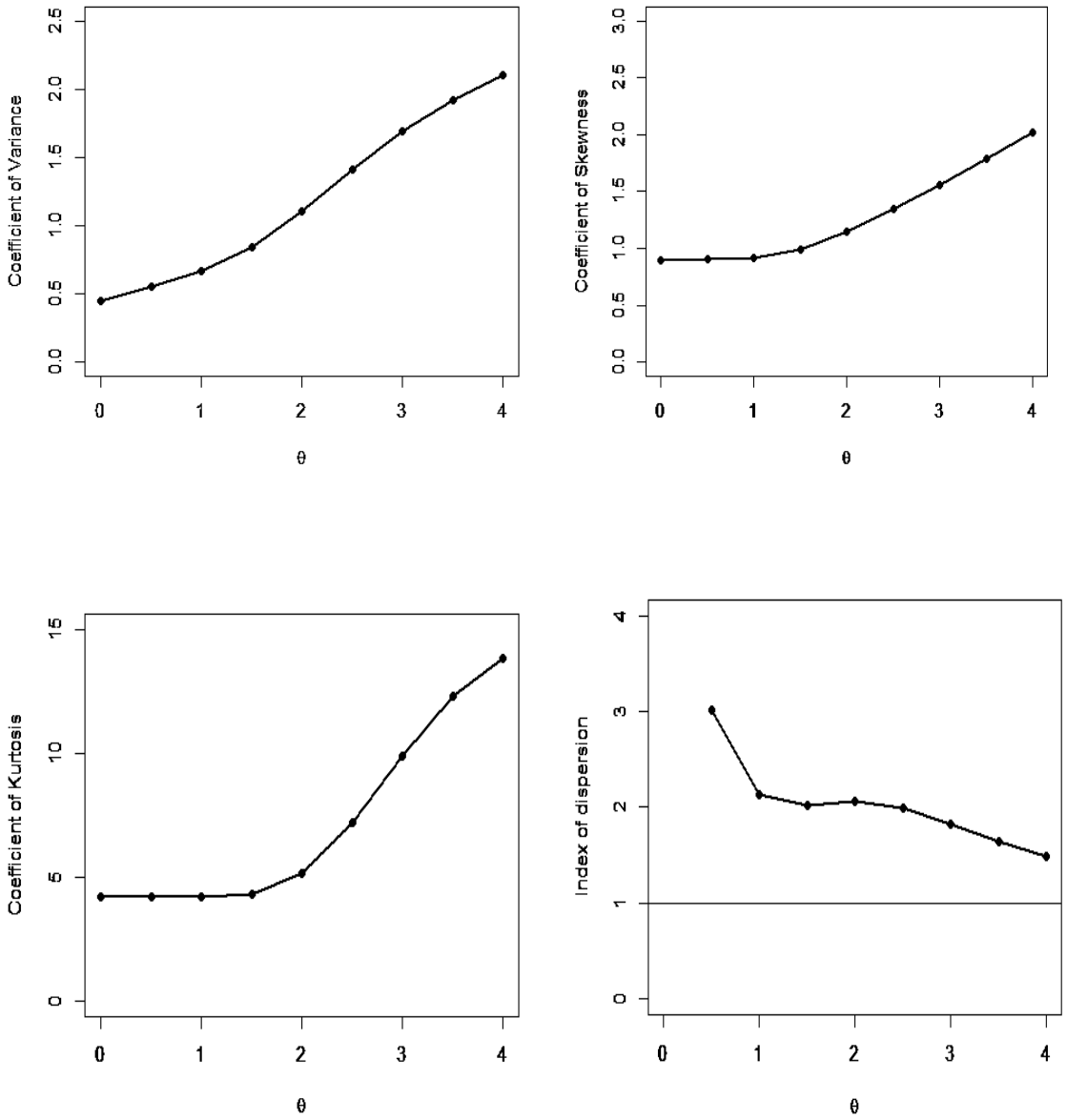

Figure2: *Graphs of coefficient of variation, coefficient of skewness, coefficient of kurtosis and Index of dispersion of PSD for different values of the parameter θ*

IV. Statistical properties

The PSD has two important properties namely increasing hazard rate with unimodality and over-dispersion which has been discussed below

I. Increasing Hazard Rate and Unimodality

The PSD has an increasing hazard rate (IHR) and unimodal. Since

$$\frac{P(x+1;\theta)}{P(x;\theta)} = \frac{1}{\theta+1}\left[1 + \frac{4x^3+36x^2+104x+116}{x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)}\right] \tag{16}$$

is decreasing function in x, $P(x;\theta)$is log-concave. Therefore, the PSD has an increasing hazard rate and unimodal. The interrelationship among log-concavity, unimodality and increasing hazard rate (IHR) of discrete distributions has been discussed in Grandell [18].

II. Over-dispersion

The PSD is always over-dispersed$(\sigma^2 > \mu)$ **.** We have

$$\begin{aligned}\sigma^2 &= \frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta^2(\theta^4+24)^2}\\ &= \frac{\theta^4+120}{\theta(\theta^4+24)}\left[\frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta(\theta^4+24)(\theta^4+120)}\right]\\ &= \frac{\theta^4+120}{\theta(\theta^4+24)}\left[1+\frac{\theta^8+528\theta^4+2880}{\theta(\theta^4+24)(\theta^4+120)}\right]\\ &= \mu\left[1+\frac{\theta^8+528\theta^4+2880}{\theta(\theta^4+24)(\theta^4+120)}\right] > \mu.\end{aligned}$$

This shows that PSD is always over-dispersed.

V. Estimation of the parameter

I. Method of Moment Estimate (MOME): Let $(x_1, x_2, \ldots, x_n)$be a random sample of size $n$from the PSD (3). Equating the first population moment about origin to the corresponding sample moment, the MOME $\tilde{\theta}$of θ of PSD (3) is the solution of the following fifth degree polynomial equation in θ

$$\bar{x}\theta^5 - \theta^4 + 24\bar{x}\theta - 120 = 0\,,$$

where $\bar{x}$ is the sample mean. This can easily be solved using Newton-Raphson method.

II. Maximum Likelihood Estimate (MLE): Let $(x_1, x_2, \ldots, x_n)$be a random sample of size n from the PSD (3) and let $f_x$be the observed frequency in the sample corresponding to $X = x(x = 1,2,3,\ldots,k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the PSD (3) is given by

$$L = \left(\frac{\theta^5}{\theta^{24}+24}\right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k}(x+5)f_x}} \prod_{x=1}^{k}[x^4 + 10x^3 + 35x^2 + 50x + (\theta^4 + 4\theta^3 + 6\theta^2 + 4\theta + 25)]^{f_x}$$

The log likelihood function is thus obtained as

$$log\ L = n\ log\left(\frac{\theta^5}{\theta^{24}+24}\right) - \sum_{x=1}^{k}(x+5)\ f_x\ log(\theta+1) + \sum_{x=1}^{k} f_x\ log\begin{bmatrix} x^4+10x^3+35x^2+50x \\ +(\theta^4+4\theta^3+6\theta^2+4\theta+25)\end{bmatrix}$$

The first derivative of the log likelihood function is given by

$$\frac{d\ log\ L}{d\theta} = \frac{5n}{\theta} - \frac{4n\theta^3}{\theta^4+24} - \sum_{x=1}^{k}\frac{(x+5)f}{\theta+1} + \sum_{x=1}^{k}\frac{(4\theta^3+12\theta^2+12\theta+4)f_x}{[x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)]}$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate (MLE), $\hat{\theta}$ of θ of PSD is the solution of the equation $\frac{d\ log\ L}{d\theta} = 0$ and is given by the solution of the following non-linear equation

$$\frac{5n}{\theta} - \frac{4n\theta^3}{\theta^4+24} - \sum_{x=1}^{k}\frac{(x+5)f}{\theta+1} + \sum_{x=1}^{k}\frac{(4\theta^3+12\theta^2+12\theta+4)f_x}{[x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)]} = 0$$

This non-linear equation can be solved by any numerical iteration methods such as Newton-Raphson, Bisection method, Regula –Falsi method etc.

VI. Goodness of fit of PSD

As we know that the count data arising in real life, in general, are over-dispersed or under-dispersed. We have seen that PSD is over-dispersed. In this section, an attempt has been made to test the goodness of fit of PSD with some over-dispersed count data and the goodness of fit has been compared with other available over-dispersed distributions. The goodness of fit of PSD has been compared with the goodness of fit of PLD, PAD, PID and PSD. The goodness of fit is based on the maximum likelihood estimates of a parameter of the considered distributions.

In this section, four examples of observed count datasets, for which the PLD, PAD, PID, and PSD have been fitted, are presented. The first data-set is due to Kemp and kemp [19] regarding the distribution of mistakes in copying groups of random digits, the second dataset is due to Beall [20]regarding the distribution of Pyrausta nublilalis, the third dataset is the distribution of red mites per leaf on apple leaves, available in Fisher et al [21], and the fourth dataset is the distribution of number of Chromatid aberrations, available in Loeschke and Kohler [22] and Janardan and Schaeffer [23].

Table 1: *Distribution of mistakes in copying groups of random digits*

No. of errors per group	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PSD
0	35	27.4	33.0	33.5	33.7	34.6
1	11	21.5	15.3	14.7	14.5	13.3
2	8	8.4	6.8	6.6	6.5	6.3
3	4	2.2	2.9	2.9	2.9	3.1
4	2	0.5	2.0	2.3	2.4	2.7
Total	60	60.0	60.0	60.0	60.0	60.0
ML estimate $(\hat{\theta})$		0.7833	1.7434	2.0779	1.8643	2.7379
χ^2		7.98	2.20	1.40	1.33	0.86
d.f.		1	1	2	2	2
p-value		0.0047	0.1380	0.4966	0.5140	0.6472

Table 2: Distribution of Pyrausta nublilalis

No. of insects	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PSD
0	33	26.4	31.5	32.0	32.2	33.1
1	12	19.8	14.2	13.6	13.4	12.3
2	6	7.4	6.1	5.9	5.8	5.6
3	3	1.8	2.5	2.6	2.6	2.7
4	1	0.3	1.0	1.1	1.1	1.3
5	1	0.3	0.7	0.8	0.9	1.0
Total	56	56.0	56.0	56.0	56.0	56.0
ML estimate $(\hat{\theta})$		0.7500	1.8081	2.1446	1.9186	2.8014
χ^2		4.87	0.53	0.24	0.20	0.03
d.f.		1	1	1	1	2
p-value		0.0273	0.4666	0.6242	0.6547	0.9821

Table 3: *Distribution of number of red mites on Apple leaves, Fisher et al (1943)*

Number of red mites per leaf	Observed Frequency	Expected Frequency			
		PD	PLD	PAD	PSD
0	38	25.3	35.8	36.3	37.6
1	17	29.1	20.7	20.1	18.3
2	10	16.7	11.4	11.2	10.7
3	9	6.4	6.0	6.1	6.3
4	3	1.8	3.1	3.2	3.5
5	2	0.4	1.6	1.6	1.8
6	1	0.2	0.8	0.8	0.9
7+	0	0.1	0.6	0.7	0.9
Total	80	80.0	80.0	80.0	80.0
ML estimate $(\hat{\theta})$		1.1500	1.2559	1.6206	2.3602
χ^2		18.27	2.47	2.07	1.47
d.f.		2	3	3	3
p-value		0.0001	0.4807	0.5580	0.6894

Table 4: *Distribution of number of Chromatid aberrations (0.2 g chinon 1, 24 hours)*

No. of chromatid aberrations	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PSD
0	268	231.3	257.0	260.4	260.8	268.3
1	87	126.7	93.4	89.7	89.3	81.5
2	26	34.7	32.8	32.1	31.8	29.9
3	9	6.3	11.2	11.5	11.5	12.1
4	4	0.8	3.8	4.1	4.2	4.9
5	2	0.1	1.2	1.4	1.5	1.9
6	1	0.1	0.4	0.5	0.6	0.7
7+	3	0.1	0.2	0.3	0.3	0.7
Total	400	400.0	400.0	400.0	400.0	400
ML estimate $(\hat{\theta})$		0.5475	2.3804	2.6594	2.3362	3.2184
χ^2		38.21	6.21	4.17	3.61	2.07
d.f.		2	3	3	3	3
p-value		0.0000	0.1018	0.2437	0.3067	0.5582

VII. Concluding Remarks

In this paper, Poisson-Suja distribution (PSD) has been proposed. The PSD has been obtained by compounding the Poisson distribution with the Suja distribution. The expression for the r th factorial moment has been derived and hence the first four moments about the origin and the moments about the mean have been given. The descriptive measures including coefficient of variation, skewness, kurtosis, and index of dispersion have been obtained. Both the method of moments and the method of maximum likelihood have been discussed for estimating a parameter of the proposed distribution. The goodness of fit of the PSD has been discussed with four examples of count data sets that are over-dispersed and the goodness of fit of the PSD has been compared with the goodness of fit given by PLD, PAD, and PID. In these datasets, the PSD shows a much closer fit than other considered distributions.

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A NEW TRANSMUTED PROBABILITY MODEL: PROPERTIES AND APPLICATIONS

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Abstract

In this article, we introduced a new three parameter continuous probability model by extending a two parameter log-logistic distribution using the quadratic rank transmutation map technique. We provide a comprehensive description of the statistical properties of the newly introduced model. Robust measures of skewness and kurtosis of the proposed model have also been derived along with the moment generating function, characteristic function, reliability function and hazard rate function of the proposed model. The estimation of the model parameters is performed by maximum likelihood method followed by a Monte Carlo simulation procedure. The applicability of this distribution to modeling real life data is illustrated by two real life examples and the results of comparison to base distribution in modeling the data are also exhibited.

Keywords: Transmuted Probability Model, Survival Analysis, Reliability Measures, Monte Carlo Simulation.

1. Introduction

The quality of procedures that are put to use in a statistical analysis relies greatly upon the assumed probability model or distribution. As a consequence of this, significant effort has been directed over the course of history towards the development of large classes of standard distributions along with relevant statistical methodologies. These happen to be designed for serving as models for a wide variety of real-world phenomena. However, many important situations exist where real data does not follow any of the classical or standard models. In the work that follows, we have obtained a three-parameter Generalized Log-Logistic Distribution (GLLD) by utilizing the Quadratic Rank Transmutation Map (QRTM) technique proposed by Shaw and Buckley [1]. The field of transmutation has seen a lot of research recently. Ashour and Eltehiwy [2] introduced a new generalized distribution of the exponentiated modified Weibull distribution using the transmutation technique. Aryal et al. [3] introduced the transmuted extreme value distribution. Merovci et al. [4, 5] studied the transmuted Lindley and Rayleigh distributions. Now we will study the three-parameter Generalized Log-Logistic Distribution (GLLD) and obtain and understand its different characteristics as well as its structural properties.

According to the Quadratic Rank Transmutation Map (QRTM) technique for generalization, the cumulative distribution function (CDF) must satisfy the relationship:

$$F_t(x) = (1+\lambda)F_b(x) - \lambda[F_b(x)]^2 \quad (1)$$

which upon differentiation yields,

$$f_t(x) = f_b(x)[1+\lambda-2\lambda F_b(x)] \quad (2)$$

where $f_b(x)$and $f_t(x)$ are the probability density functions corresponding to $F_b(x)$ and

$F_t(x)$respectively and$|\lambda| \leq 1$. $F_b(x)$is the CDF of the base distribution. If we put $\lambda = 0$, we get the base distribution.

The log-logistic distribution is a continuous probability distribution particularly useful in dealing with survival data. It is specifically used as a parametric model for events whose rate increases initially and later diminishes. For example, mortality rate from a certain cancer post diagnosis or treatment. The probability density function (pdf) of the two-parameter log-logistic distribution is given as:

$$f(x;\alpha,\beta) = \frac{\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^{\beta})^2} \tag{3}$$

The corresponding cumulative distribution function (CDF) is given as:

$$F(x) = \Pr(X \leq x) = \frac{(\alpha x)^{\beta}}{1+(\alpha x)^{\beta}} \tag{4}$$

where α is a scale parameter while β is a shape parameter.

The remaining paper is organized as follows. In subSection 1, the three-parameter Generalized Log-Logistic Distribution is demonstrated. The various statistical properties of the generalized distribution such as the moments, moment generating function, characteristic function, order statistics, quantile function, etc. are summarized in Section 2. The MLE of the distribution parameters are illustrated in Section 3 of this paper and contains an exhibition of the Monte Carlo simulation procedure. Robust measures of skewness and Kurtosis along with graphical illustrations are presented in Section 4. Section 5 deals with the applicability of this generalized distribution in modeling real life data which is illustrated by two real-life data sets.

1.1 Three-Parameter Generalized Log-Logistic Distribution (GLLD)

This section deals with the study of the three-parameter Generalized Log-Logistic Distribution. Using (1) and (4), the CDF of GLLD is obtained as follows:

$$F_t(x) = (1+\lambda)F_b(x) - \lambda[F_b(x)]^2$$

$$\Rightarrow F_t(x) = (1+\lambda)\left[\frac{(\alpha x)^{\beta}}{1+(\alpha x)^{\beta}}\right] - \lambda\left[\frac{(\alpha x)^{\beta}}{1+(\alpha x)^{\beta}}\right]^2$$

After simplifying, we obtain the CDF of three-parameter Generalized Log-Logistic Distribution as

$$\therefore F(x;\alpha,\beta,\lambda) = \frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}, \qquad x,\alpha,\beta > 0 \;\&\; -1 \leq \lambda \leq 1 \tag{5}$$

Hence, the pdf of GLLD with parameters α, β and λ is obtained using (5) as follows:

$$f(x;\alpha,\beta,\lambda) = \frac{d}{dx}\frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}$$

$$\therefore f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^3}, \quad x,\alpha,\beta > 0 \;\&\; -1 \leq \lambda \leq 1 \tag{6}$$

The CDF and pdf plots for (5) and (6) respectively for different values of the parameters involved is illustrated through figure 1 and 2 respectively. The plots reveal quite evidently that the distribution of the three-parameter generalized log-logistic random variable X is right skewed.

2. Statistical Properties of GLLD

This section deals with the various structural properties of the three-parameter GLLD such as moments (non-central and central), moment generating function, characteristic function, order statistics, quantile function as well as the survival measures. All these have been obtained and discussed in the sub-sections that follow.

2.1 Moments

Moments refer to a set of statistical parameters that are useful in measuring a distribution. They are the crucial measures to calculate mean, variance, skewness and kurtosis of the data. Skewness deals with symmetry of a distribution, or in more precise terms, the lack of symmetry of a distribution. Kurtosis enables us to measure the peakedness or flatness of a distribution. Another interpretation of kurtosis is concerned with the heavy or light-tailed nature of the data relative to a normal distribution.

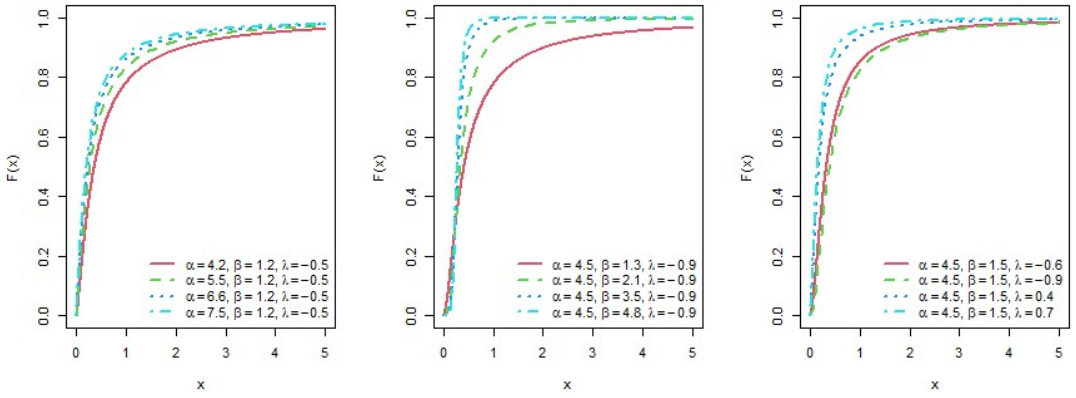

Fig 1: *CDF plots of three parameter GLLD*

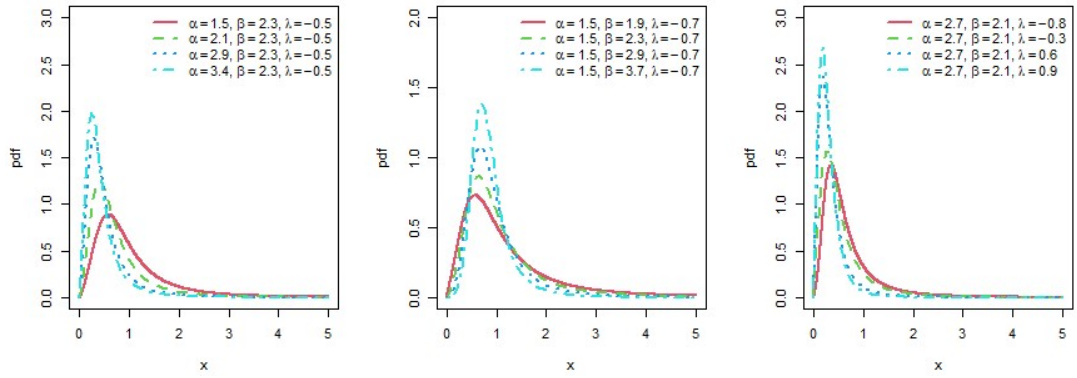

Fig 1: *pdf plots of three parameter GLLD*

The theorem 1.1 is used to arrive at the rthnon-central moment of the three parameter GLLD.

Theorem 1.1: If a random variable X follows GLLD with parameters α, β and λ such that $\alpha, \beta >$ 0and $|\lambda| \leq 1$, then the rth non-central moment is given by

$$\mu_r' = \frac{(1+\lambda)}{\alpha^r}\beta\left(1+\frac{r}{\beta}, 1-\frac{r}{\beta}\right) - \frac{2\lambda}{\alpha^r}\beta\left(2+\frac{r}{\beta}, 1-\frac{r}{\beta}\right) \tag{7}$$

Proof:

We know by the definition of the rth raw moment that

$$\mu_r' = \mathbb{E}(X^r)$$

$$\Rightarrow \mu_r' = \int_0^\infty x^r f(x;\alpha,\beta,\lambda)dx$$

$$\Rightarrow \mu_r' = \int_0^\infty x^r \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^\beta\right) - 2\lambda(\alpha x)^\beta\}}{(1+(\alpha x)^\beta)^3}dx$$

$$\Rightarrow \mu_r' = \int_0^\infty x^r \frac{(1+\lambda)\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^\beta)^2}dx - \int_0^\infty x^r \frac{2\lambda(\alpha x)^\beta \alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^\beta)^3}dx$$

Put $(\alpha x)^\beta = t$, we obtain $x = \frac{t^{\frac{1}{\beta}}}{\alpha}$ and $\alpha\beta(\alpha x)^{\beta-1}dx = dt$

Also, as $x \to 0, t \to 0$ and as $x \to \infty, t \to \infty$

$$\therefore \mu_r' = (1+\lambda)\int_0^\infty \frac{\left(\frac{t^{\frac{1}{\beta}}}{\alpha}\right)^r}{(1+t)^2}dt - 2\lambda\int_0^\infty \frac{\left(\frac{t^{\frac{1}{\beta}}}{\alpha}\right)^r t}{(1+t)^3}dt$$

$$\Rightarrow \mu_r' = \frac{(1+\lambda)}{\alpha^r}\int_0^\infty \frac{t^{\left(\frac{r}{\beta}+1\right)-1}}{(1+t)^{\left(\frac{r}{\beta}+1\right)+\left(1-\frac{r}{\beta}\right)}}dt - \frac{2\lambda}{\alpha^r}\int_0^\infty \frac{t^{\left(\frac{r}{\beta}+2\right)-1}}{(1+t)^{\left(\frac{r}{\beta}+2\right)+\left(1-\frac{r}{\beta}\right)}}dt$$

$$= \frac{(1+\lambda)}{\alpha^r}\beta\left(1+\frac{r}{\beta},1-\frac{r}{\beta}\right) - \frac{2\lambda}{\alpha^r}\beta\left(2+\frac{r}{\beta},1-\frac{r}{\beta}\right)$$

where

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\therefore \mu_r' = \frac{(1+\lambda)}{\alpha^r}\frac{\Gamma\left(1+\frac{r}{\beta}\right)\Gamma\left(1-\frac{r}{\beta}\right)}{\Gamma(2)} - \frac{2\lambda}{\alpha^r}\frac{\Gamma\left(2+\frac{r}{\beta}\right)\Gamma\left(1-\frac{r}{\beta}\right)}{\Gamma(3)}$$

$$\Rightarrow \mu_r' = \frac{(1+\lambda)}{\alpha^r}\Gamma\left(\frac{\beta+r}{\beta}\right)\Gamma\left(\frac{\beta-r}{\beta}\right) - \frac{\lambda}{\alpha^r}\Gamma\left(\frac{2\beta+r}{\beta}\right)\Gamma\left(\frac{\beta-r}{\beta}\right)$$

Thus, the rth non-central moment is given by the expression

$$\mu_r' = \frac{1}{\alpha^r}\Gamma\left(\frac{\beta-r}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+r}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+r}{\beta}\right)\right] \tag{8}$$

Using expression (8), the first two raw moments for three-parameter GLLD can be easily obtained. These are given by:

$$\mu_1' = \frac{1}{\alpha}\Gamma\left(\frac{\beta-1}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+1}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+1}{\beta}\right)\right] \tag{9}$$

$$\mu_2' = \frac{1}{\alpha^2}\Gamma\left(\frac{\beta-2}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+2}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+2}{\beta}\right)\right] \tag{10}$$

Besides, we know that variance is given by

$$\mu_2 = \mu_2' - (\mu_1')^2$$

Thus, the variance of the three-parameter GLLD is given by:

$$\mu_2 = \frac{1}{\alpha^2}\Gamma\left(\frac{\beta-2}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+2}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+2}{\beta}\right)\right] - \left[\frac{1}{\alpha}\Gamma\left(\frac{\beta-1}{\beta}\right)\left\{(1+\lambda)\Gamma\left(\frac{\beta+1}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+1}{\beta}\right)\right\}\right]^2 \tag{11}$$

It is important note that for the convergence of the rth moment, $\left(1-\frac{r}{\beta}\right)$ in (8) must be greater than zero. In other words, convergence of rth moment is possible only if $\beta > r$. Thus, existence of mean of the proposed distribution requires that β is greater than 1. For variance, β must be greater than 2. Similarly, for skewness and kurtosis, β must be greater than 3 and 4 respectively. Any situation of divergence of the statistical measures is dealt with by employing robust measures.

2.2 Moment generating function (mgf) and characteristic function (cf)

This sub-section contains the derivation of the mgf and cf of the three-parameter GLLD. The following theorem gives the mgf and cf of the distribution under study.

Theorem 3.2:*If a random variable X follows GLLD with parameters α, β and λ such that $\alpha, \beta > 0$and $|\lambda| \leq 1$, then the mgf denoted by $M_X(t)$ and the cf denoted by $\psi_X(t)$ has the following form:*

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda\, \Gamma\left(\frac{2\beta + j}{\beta}\right)\right]$$

and

$$\psi_X(t) = \sum_{j=0}^{\infty} \frac{(\iota t)^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda\, \Gamma\left(\frac{2\beta + j}{\beta}\right)\right]$$

Proof:

We know from the definition of mgf that

$$M_X(t) = \mathbb{E}(e^{tx})$$
$$= \int_0^{\infty} \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} f(x;\alpha,\beta,\lambda)dx$$
$$= \int_0^{\infty} \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f(x;\alpha,\beta,\lambda)dx = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j'$$

From (8), we know

$$\mu_j' = \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda\, \Gamma\left(\frac{2\beta + j}{\beta}\right)\right]$$

$$\therefore M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda\, \Gamma\left(\frac{2\beta + j}{\beta}\right)\right] \tag{12}$$

which is the required *mgf* of the three-parameter GLLD.

Also, we know that

$$\psi_X(t) = \mathbb{E}(e^{\iota t x})$$
$$\Rightarrow \psi_X(t) = \mathbb{E}(e^{(\iota t)x})$$

$$\Rightarrow \psi_X(t) = \sum_{j=0}^{\infty} \frac{(\iota t)^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda\, \Gamma\left(\frac{2\beta + j}{\beta}\right)\right] \tag{13}$$

which is the required *cf* of the three-parameter GLLD.

2.3 Order Statistics

Stated in the simplest of terms, order statistics refer to sampling values arranged in an ascending order. If $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$ denote the order statistics of a random sample $X_1, X_2, X_3, \dots, X_n$drawn from a continuous population having CDF $F_X(x)$ and pdf $f_X(x)$, then the pdf of the rth order statistics $X_{(r)}$ is given by:

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)[F_X(x)]^{r-1}[1 - F_X(x)]^{n-r}, \qquad \forall\, r = 1,2, \dots, n$$

Using (5) and (6), the formula for the pdf of the rth order statistic $X_{(r)}$ for the three-parameter GLLD is obtained and is given as under:

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^3} \times \left[\frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{r-1} \left[1 - \frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{n-r} \tag{14}$$

For $r = n$, we get the pdf of the nth or the largest order statistic $X_{(n)}$ for the three-parameter GLLD which is obtained as follows:

$$f_n(x) = \frac{n!}{(n-1)!\,(n-n)!} \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^3}$$
$$\times \left[\frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{n-1} \left[1 - \frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{n-n}$$

$$= \frac{n!}{(n-1)!}\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1}$$

$$\therefore f_n(x) = \frac{n\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}}\left[\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1} \quad (15)$$

Also, for $r = 1$, we get the pdf of the first or the smallest order statistic $X_{(1)}$ for the three-parameter GLLD which is obtained as follows:

$$f_1(x) = \frac{n!}{(1-1)!\,(n-1)!}\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{1-1}\left[1-\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1}$$

$$= \frac{n!}{(n-1)!}\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[1-\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1}$$

$$\therefore f_1(x) = \frac{n\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[1-\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1} \quad (16)$$

Quite evidently, for $\lambda = 0$, the order statistics of the base distribution i.e., the Log-Logistic Distribution, are yielded.

2.4. Quantile function and random number generation

A prominent method that is put to use for the sake of generating random numbers from a specified distribution is the inverse CDF method. This method generates random numbers from a particular distribution by equating the CDF of the distribution to a number u where u itself follows continuous uniform distribution, $U(0,1)$. Solving the equation yields the quantile function of the distribution. Employing this inverse CDF method, we proceed to obtain the quantile function of the three-parameter GLLD as follows:

$$F(x;\alpha,\beta,\lambda) = u$$

$$\Rightarrow \frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}} = u$$

$$\Rightarrow (\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta} = u\big(1+(\alpha x)^{\beta}\big)^{2}$$

After simplifying, we obtain

$$x^{\beta} = \frac{-\alpha^{\beta}(1+\lambda-2u) \pm \sqrt{\alpha^{2\beta}(1+\lambda)^2-4u\lambda\alpha^{2\beta}}}{2\alpha^{2\beta}(1-u)}$$

$$= \frac{-\alpha^{\beta}(1+\lambda-2u) \pm \sqrt{(\alpha^{\beta})^2}\sqrt{(1+\lambda)^2-4u\lambda}}{2\alpha^{2\beta}(1-u)}$$

$$= \alpha^{\beta}\left\{\frac{-(1+\lambda-2u) \pm \sqrt{(1+\lambda)^2-4u\lambda}}{2\alpha^{\beta}\alpha^{\beta}(1-u)}\right\}$$

$$= \frac{-(1+\lambda-2u) \pm \sqrt{(1+\lambda)^2-4u\lambda}}{2\alpha^{\beta}(1-u)}$$

$$\therefore x = \left[\frac{-(1+\lambda-2u)+\sqrt{(1+\lambda)^2-4u\lambda}}{2\alpha^{\beta}(1-u)}\right]^{\frac{1}{\beta}}, \quad (17)$$

Equation (17) is the required quantile function of three-parameter GLLD. Note that the negative root of (17) has been discarded since x only takes values greater than 0. Equation (17) yields random numbers from three-parameter GLLD. For $u = 0.25, 0.50$ and 0.75, the values of x obtained represent the first, second and third quartiles of the distribution, respectively. In a similar fashion, deciles and percentiles of different orders are obtained by assigning different values to u.

2.5. Survival measures of three-parameter GLLD

This sub-section deals with the survival measures of three-parameter GLLD such as the survival function and the hazard function. The survival function, also known as the survivorship function, refers to the probability that a life, system or a component will survive beyond a specified time. In mathematical terms, it happens to be the complement of the CDF and is given by:

$$S(x) = \Pr(X > x) = 1 - F(x) \tag{18}$$

Using (5) in (18), we obtain the survival function of three-parameter GLLD as follows:

$$S(x;\alpha,\beta,\lambda) = \frac{\left(1+(\alpha x)^{\beta}\right)^2 - (\alpha x)^{2\beta} - (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}$$

$$S(x;\alpha,\beta,\lambda) = \frac{\left(1+2(\alpha x)^{\beta} + (\alpha x)^{2\beta}\right) - (\alpha x)^{2\beta} - (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}$$

$$\therefore S(x;\alpha,\beta,\lambda) = \frac{1+(1-\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}, \qquad x,\alpha,\beta > 0 \;\&\; -1 \le \lambda \le 1 \tag{19}$$

The hazard function, also known as the hazard rate or failure rate or force of mortality, happens to be an important quantity used for the characterization of life phenomenon. Hazard function is defined as the conditional probability that a life, system or a component that survives up to a specified time, will undergo failure or succumb in the immediate, infinitesimally small interval of time that follows. In mathematical terms, the hazard rate or the hazard function is given by:

$$h(x) = \lim_{\Delta t \to 0} \frac{\Pr[t \le X < t + \Delta t \mid X \ge t]}{\Delta t}$$

which upon simplification yields

$$h(x) = \frac{f(x)}{S(x)} \tag{20}$$

Using (6) and (19) in (20), we obtain the hazard function of three-parameter GLLD as follows:

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})\{1+(1-\lambda)(\alpha x)^{\beta}\}}, \quad x,\alpha,\beta > 0 \;\&|\lambda| \le 1 \tag{21}$$

The survival function and the hazard function plots for (19) and (21) respectively for different values of the parameters involved is illustrated through figure 3 and 4 respectively.

3. Maximum Likelihood Estimation

One of the most useful frameworks in parameter estimation is the Maximum Likelihood estimation (MLE). This method obtains the unknown population parameters by the virtue of likelihood maximization.

In this section, the parameters α, β and $\lambda$of the three-parameter GLLD are estimated using the method of maximum likelihood estimation (MLE). The procedure is given as follows: Consider a random sample $X_1, X_2, \ldots, X_n$of size n taken from the three-parameter GLLD. The likelihood function based on this sample is therefore given as:

$$L(x|\alpha,\beta,\lambda) = \prod_{i=1}^{n} \frac{\alpha\beta(\alpha x_i)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x_i)^{\beta}\right) - 2\lambda(\alpha x_i)^{\beta}\}}{(1+(\alpha x_i)^{\beta})^3} \tag{22}$$

$$\Rightarrow L = \left(\alpha^{\beta}\beta\right)^n \frac{\prod_{i=1}^{n} x_i^{\beta-1} \prod_{i=1}^{n}\{(1+\lambda)\left(1+(\alpha x_i)^{\beta}\right) - 2\lambda(\alpha x_i)^{\beta}\}}{\prod_{i=1}^{n}(1+(\alpha x_i)^{\beta})^3} \tag{23}$$

Taking logarithm on both sides of (23), we obtain the log likelihood function as follows:

$$\Rightarrow \log L = \log\left[\left(\alpha^{\beta}\beta\right)^n \frac{\prod_{i=1}^{n} x_i^{\beta-1} \prod_{i=1}^{n}\{(1+\lambda)\left(1+(\alpha x_i)^{\beta}\right) - 2\lambda(\alpha x_i)^{\beta}\}}{\prod_{i=1}^{n}(1+(\alpha x_i)^{\beta})^3}\right]$$

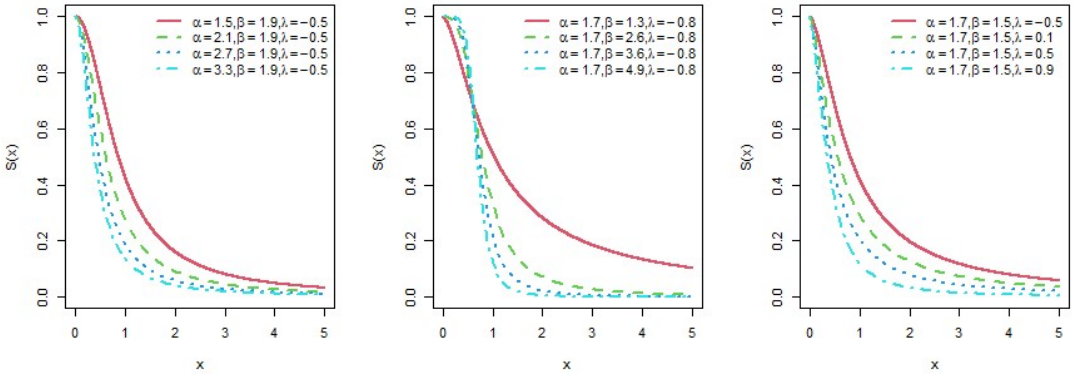

Fig. 3: *Survival function plot for three parameter GLLD*

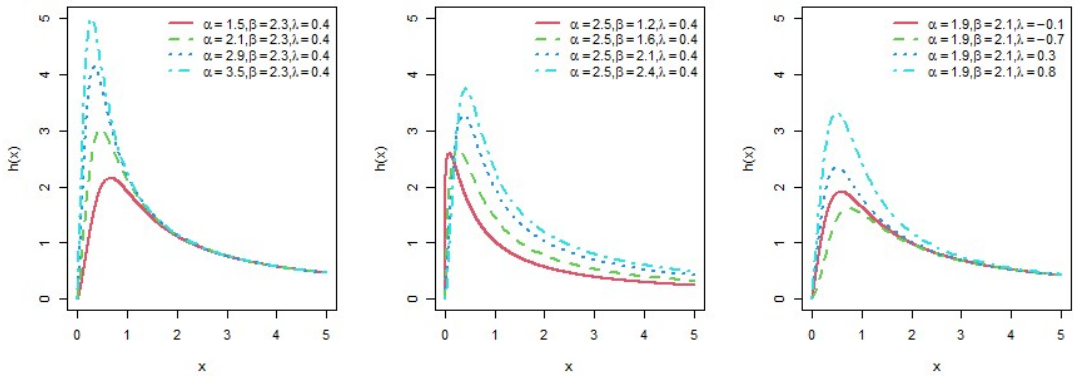

Fig. 4: *Hazard rate function plot for three parameter GLLD*

$$\Rightarrow \log L \log L = n\beta \log \alpha + n \log \beta + (\beta - 1) \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\{(1+\lambda)\big(1+(\alpha x_i)^{\beta}\big) - 2\lambda(\alpha x_i)^{\beta}\} - 3 \sum_{(i=1)}^{n} \log(1+(\alpha x_i)^{\beta}) \tag{24}$$

which is the required log-likelihood function.

The MLEs of the parameters α, β and λ of GLLD are obtained by differentiation of the log-likelihood function (24) w.r.t α, β and λ.The partial derivatives used for estimating the parameters are obtained as follows:

$$\frac{\partial}{\partial \alpha} \log L = \frac{n\beta}{\alpha} + \sum_{i=1}^{n} \left[\frac{\left\{(1+\lambda)\big(\beta \alpha^{\beta-1} x_i^{\beta}\big) - 2\lambda\beta\alpha^{\beta-1}x_i^{\beta}\right\}}{\{(1+\lambda)(1+(\alpha x_i)^{\beta}) - 2\lambda(\alpha x_i)^{\beta}\}} \right] - 3 \sum_{i=1}^{n} \left[\frac{\beta \alpha^{\beta-1} x_i^{\beta}}{(1+(\alpha x_i)^{\beta})} \right] \tag{25}$$

$$\frac{\partial}{\partial \beta} \log L = n \log \alpha + \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left[\frac{\{(1+\lambda)(\alpha x_i)^{\beta} \log(\alpha x_i) - 2\lambda(\alpha x_i)^{\beta} \log(\alpha x_i)\}}{\{(1+\lambda)(1+(\alpha x_i)^{\beta}) - 2\lambda(\alpha x_i)^{\beta}\}} \right] - 3 \sum_{i=1}^{n} \left[\frac{(\alpha x_i)^{\beta} \log(\alpha x_i)}{(1+(\alpha x_i)^{\beta})} \right] \tag{26}$$

$$\frac{\partial}{\partial \lambda} \log L = \sum_{i=1}^{n} \left[\frac{1-(\alpha x_i)^{\beta}}{\{(1+\lambda)(1+(\alpha x_i)^{\beta}) - 2\lambda(\alpha x_i)^{\beta}\}} \right] \tag{27}$$

The derivative equations (25), (26) and (27) cannot be analytically solved and thereby estimates of the parameters α, β and λ denoted by $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are obtained by maximization of log-likelihood function through the employment of powerful iterative numerical methods such as the Newton-Raphson method. The second order partial derivatives are computed which are helpful in

obtaining the Fisher's Information Matrix in the following manner:

$$I_x(\alpha,\beta,\lambda)=\begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) & -E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) & -E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial \beta \partial \lambda}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right) & -E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \beta}\right) & -E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) \end{bmatrix} \tag{28}$$

It can be shown that the three-parameter GLLD satisfies the regularity conditions and thereby the MLE vector $\Theta=\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)^T$ is consistent as well as asymptotically normal, i.e., $\sqrt{n}\left[\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)^T-(\alpha,\beta,\lambda)^T\right]$ converges to a normal distribution with mean vector 0 and the identity covariance matrix. Fisher's Information matrix in (28) is calculated by virtue of the following approximation:

$$I_x\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)\approx\begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} \\ -E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial \beta \partial \lambda}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} \\ -E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \beta}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} \end{bmatrix} \tag{29}$$

Where $\hat{\alpha},\hat{\beta}$ and $\hat{\lambda}$ are the MLEs of α,β and $\hat{\lambda}$ respectively. This approximation is useful in the construction of the confidence intervals for the parameters of three-parameter GLLD. The approximate $100(1-\alpha)\%$ confidence intervals for α,β and λ are respectively given by:

$$\hat{\alpha}\pm z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}\left(\widehat{\Theta}\right)},\hat{\beta}\pm z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}\left(\widehat{\Theta}\right)}\ and\ \hat{\lambda}\pm z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}\left(\widehat{\Theta}\right)} \tag{30}$$

3.1. Monte Carlo Simulation Study of ML Estimates

Monte Carlo simulation refers to a wide range of computational algorithms aimed at obtaining numerical results by using repeated random sampling. This sub-section contains a behavioral analysis of the maximum likelihood estimates of three-parameter GLLD for a finite sample of size n. A MC simulation study for different values of parameters α,β and γ is employed for this purpose with random numbers being generated using the quantile function (17) obtained earlier. The procedure undertaken involves a simulation study for each triplet (α,β,λ) for the parameter combinations $(\alpha=0.7,\beta=0.5,\lambda=0.4)$ and $(\alpha=1.2,\beta=0.8,\lambda=0.5)$.The iterative process is carried out 100 times for samples of size n, where $n=25,75,150,200$ and 500, generating 100 samples of the mentioned sample sizes. ML estimates for each sample generated are then obtained and their average bias, variance and MSE is calculated. The results have been tabulated in Table 1 and clearly indicate that with the increase in the sample size n, agreement between theory and practice improves significantly. MSE and variance of estimates of α,β and λ indicate consistency and that the ML method performs well for estimation of parameters of the three-parameter GLLD.

Table 1: *Average Bias, Variance and MSE for simulated results of MLEs*

Sample size n	Parameters	$(\alpha=0.7,\beta=0.5,\lambda=0.4)$			$(\alpha=1.2,\beta=0.8,\lambda=0.5)$		
		Bias	Variance	***MSE***	Bias	Variance	***MSE***
25	α	0.026299	2.645356	2.646048	-0.15184	0.633681	0.656737
	β	0.008205	0.009891	0.009958	0.012896	0.052401	0.052567
	λ	0.284835	0.18187	0.263	0.160373	0.173618	0.199338
75	α	-0.184323	0.841443	0.875418	-0.16281	0.495852	0.522358
	β	-0.009204	0.003322	0.003407	-0.04699	0.026425	0.028633
	λ	0.237382	0.142089	0.198439	0.156228	0.153889	0.178296

150	α	-0.357382	0.54484	0.672562	-0.05974	0.366872	0.370441
	β	-0.020729	0.001617	0.002047	-0.03842	0.012117	0.013593
	λ	0.271836	0.148146	0.22204	0.067127	0.140693	0.145199
200	α	-0.276202	0.540957	0.617245	0.037218	0.422209	0.423594
	β	-0.018463	0.001689	0.00203	-0.00735	0.012184	0.012238
	λ	0.241823	0.135421	0.193899	0.03889	0.128701	0.130213
500	α	-0.377683	0.375987	0.518631	0.095593	0.332553	0.341691
	β	-0.01319	0.000977	0.001151	-0.02082	0.006032	0.006466
	λ	0.252524	0.10096	0.164728	-0.01595	0.102887	0.103141

4. Robust Skewness and Kurtosis Measures for three-parameter GLLD

This section deals with the study of skewness and kurtosis measures for the proposed distribution. Skewness and kurtosis both deal with the shape of the distribution with the former concerned with symmetry while latter with the tailedness and peakedness of the distribution. The effect of parameters on the skewness and kurtosis of the distribution is studied in this section by considering measures based on quantiles.

Bowley[6] proposed a coefficient of skewness based on quantiles which is well known in statistical literature and is one of the earliest measures of skewness. It is defined as the average of the first and third quartiles minus the median divided by half the interquartile range. It is given by:

$$B = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1} = \frac{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{31}$$

Bowley's coefficient of skewness lies between $+1$ and -1.

Moors[7] proposed a robust alternative to the conventional measure of kurtosis in order to overcome the shortcomings of the latter. For many heavy tailed distributions, the conventional measure is infinite and uninformative as such. The new measure of kurtosis based on quantiles, however, is less sensitive to outliers and even exists for distributions for which there are not any defined moments. The Moors' kurtosis based on octiles is given by:

$$M = \frac{(E_3 - E_1) + (E_7 - E_5)}{E_6 - E_2} = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{32}$$

For distributions that are symmetrical to 0, the Moors' kurtosis reduces to:

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)} \tag{33}$$

Table 2: *Bowley'sskewness for GLLD* $(x; \alpha, \beta, \lambda)$ *for different parameter combinations*

Parameters		$\alpha = 1.3$						
		β						
		0.7	1.4	1.9	2.6	3.3	4.4	5.6
λ	-0.9	0.62570	0.36985	0.28923	0.22568	0.18823	0.15298	0.13014
	-0.7	0.63217	0.36683	0.28241	0.21574	0.17642	0.13942	0.11543
	-0.6	0.63601	0.36633	0.28009	0.21192	0.17169	0.13383	0.10929
	-0.3	0.64848	0.36931	0.27877	0.20697	0.16455	0.12460	0.09870
	0.3	0.64453	0.36414	0.27336	0.20143	0.15894	0.11895	0.09303
	0.6	0.60956	0.33252	0.24491	0.17596	0.13538	0.09727	0.07260
	0.7	0.59349	0.31787	0.23156	0.16380	0.12400	0.08664	0.06248
	0.9	0.55777	0.28555	0.20196	0.13671	0.09850	0.06271	0.03959

Table 3: *Moors' kurtosis for GLLD* $(x; \alpha, \beta, \lambda)$ *for different parameter combinations*

Parameters		$\alpha = 2.6$						
		β						
		0.7	1.4	1.9	2.6	3.3	4.4	5.6
λ	-0.9	3.02678	1.75277	1.56318	1.45452	1.40469	1.36648	1.34594
	-0.7	3.05154	1.74699	1.55687	1.45018	1.40247	1.36694	1.34849
	-0.6	3.06594	1.74405	1.55301	1.44675	1.39978	1.36525	1.34763
	-0.3	3.11493	1.73857	1.54241	1.43495	1.38840	1.35504	1.33856
	0.3	3.07158	1.72426	1.53261	1.42809	1.38310	1.35111	1.33550
	0.6	2.76317	1.64151	1.48248	1.39720	1.36142	1.33684	1.32545
	0.7	2.60943	1.59528	1.45225	1.37657	1.34545	1.32469	1.31549
	0.9	2.27908	1.48840	1.37979	1.32506	1.30423	1.29197	1.28772

For standard normal distribution, it is easy to compute that

$$E_1 = -E_7 = -1.15, E_2 = -E_6 = -0.67 \; and \; E_3 = -E_5 = -0.32$$

Therefore, $M = 1.23$. The centered Moors' coefficient is thus given by:

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} - 1.23 \tag{34}$$

Using *R* software, the values of Bowley'sskewness and Moors' kurtosis for the three-parameter GLLD for different parameter values have been numerically calculated and tabulated in Tables 2 and 3 respectively. Clearly, Bowley'sskewness as well as Moors' kurtosis are decreasing function of β for a fixed value of the transmuted parameter λ. However, for a fixed value of the scale parameter β, both Bowley's skewness and Moors' kurtosis reflect both increasing and decreasing behavior for different values of the transmuted parameter λ.

5. Applications of three-parameter GLLD

In this particular section, the performance of the proposed generalized log-logistic model is put to test by comparing it with base model. Two real life data sets, one based on survival times and the other on strength data, that are already available in the literature have been used to carry out the comparisons. The procedure involves the computation of MLEs of the transmuted model as well the base model based on both data sets using *R* software. The various goodness of fit statistics for the two models are then calculated and comparisons carried out. These statistics include AIC (Akaike's Information Criterion) provided by Akaike[8], AICC (AIC Corrected) and BIC (Bayesian Information Criterion) given by Schwarz[9]. AIC, AICC and BIC for a model with k parameters are calculated using the following generic functions:

$$AIC = 2k - 2\log L$$
$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$
$$BIC = k\log n - 2\log L$$

Kolmogorov-Smirnov test is also carried out for testing model significance based on the two mentioned real-life data sets.

Data Set I: The data set reported by Efron[10] is analyzed for carrying out comparisons between three-parameter GLLD and LLD. Efron [10] reported the data set in which observations represent the survival times of a group of patients suffering from head and neck cancer disease and are

treated using radiotherapy. The data set is given in Table 4.

Table 4: *Survival times of 58 patients suffering from head and neck cancer disease*

6.53	7	10.42	14.48	16.10	22.70	34	41.55	42	45.28	49.40	53.62
63	64	83	84	91	108	112	129	133	133	139	140
140	146	149	154	157	160	160	165	146	149	154	157
160	160	165	173	176	218	225	241	248	273	277	297
405	417	420	440	523	583	594	1101	1146	1417		

The MLEs, model functions alongside the standard errors based on the above data set are tabulated in Table 5.

The Table 6 contains various goodness of fit measures for models fitted to data given in Table 4. From the table, it is evident that the AIC, AICC and BIC values for the transmuted model (GLLD) are better as compared to the base model (LLD), thereby suggesting that the new model is a better performer. Furthermore, the KS p-value is also greater than 0.05 for GLLD as such reiterating the statistical significance of the new transmuted model over the base model.

Table 5: *MLEs with standard errors of parameters for GLLD and LLD for data set I*

Model	Model function	MLEs	Standard Error
Transmuted Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.55$ $\hat{\lambda}=-0.47$	$SE(\hat{\alpha})=0.002$ $SE(\hat{\beta})=0.203$ $SE(\widehat{\lambda})=0.344$
Base Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^{\beta})^{2}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.52$	$SE(\hat{\alpha})=0.002$ $SE(\hat{\beta})=0.196$

Table 6: *Goodness of fit measures for models fitted to data set I*

Model	$-\log L$	**AIC**	**AICC**	**BIC**	**KS Distance**	**KS p-value**	**LR Statistic**
GLLD	371.1943	748.3887	748.8331	754.5700	0.15548	0.1211	5.01905
LLD	373.7039	751.4077	751.6259	755.5286	0.26802	0.0004	

The GLLD and LLD plots fitted to the survival times of the 58 patients suffering from head and neck cancer disease are illustrated through Figure 5. The graphical overview of the empirical and theoretical (GLLD) CDFs and survival functions for data set I is illustrated through Figures 6 and 7 respectively.

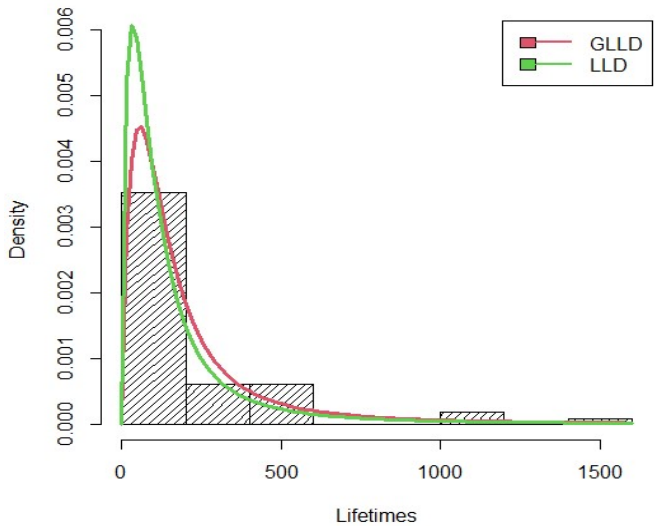

Fig. 5: *Curve fitting GLLD vz LLD for data set I*

Data Set II: The data set reported by Lawless [11] is analyzed for carrying out comparisons between three-parameter GLLD and LLD. Lawlessreported the data set in which the observations represent the number of cycles to failure for 25-100 cm specimens of yarn tested at a particular strain level. The data set is given in Table 7.

Table 7: *Cycles to failure for 25-100 cm specimens of yarn at a specific strain level*

15	20	38	42	61	76	86	98	121	146
149	157	175	176	180	180	198	220	224	251
175	176	180	180	198	653				

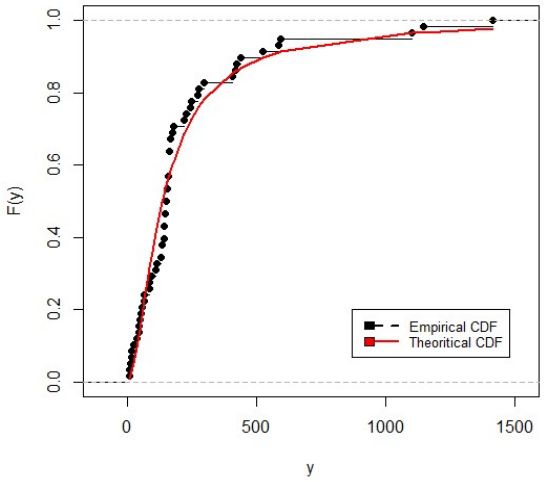

Fig. 6: *Empirical and Theoritical CDF for data set I*

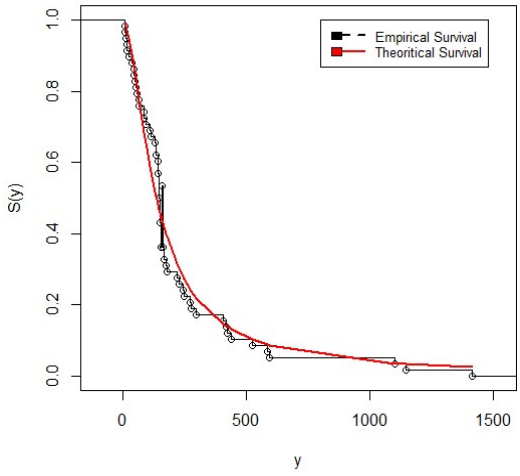

Fig. 7:*Empirical and Theoretical Survival Function for data set I*

The MLEs, model functions alongside the standard errors based on the above data set are tabulated in Table 8 below:

Table 8: *MLEs with standard errors of parameters for GLLD and LLD for data set II*

Model	Model function	MLEs	Standard Error
Transmuted Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.89$ $\hat{\lambda}=-0.56$	$SE(\hat{\alpha})=0.004$ $SE(\hat{\beta})=0.450$ $SE(\hat{\lambda})=0.491$
Base Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^{\beta})^{2}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.85$	$SE(\hat{\alpha})=0.003$ $SE(\hat{\beta})=0.423$

From the table 9, it is evident that the AIC, AICC and BIC values for the transmuted model (GLLD) are better as compared to the base model (LLD), thereby suggesting that the new model is a better performer. Furthermore, the KS p-value>0.05 for GLLD as such reiterating the statistical significance of the new transmuted model over the base model. In other words, GLLD is a better fit for data given in Table 7 as compared to LLD.

Table 9: *Goodness of fit measures for models fitted to data set II*

Model	$-\log L$	AIC	AICC	BIC	KS Distance	KS *p*-value	LR Statistic
GLLD	154.2395	314.4790	315.6219	318.1356	0.18704	0.346	3.440033
LLD	155.9595	315.9191	316.4645	318.3568	0.30815	0.01734	

The GLLD and LLD plots fitted to the number of cycles to failure for 25 100-cm specimens of yarn tested at a particular strain level are illustrated through Figure 8.The graphical overview of the empirical and theoretical (GLLD) CDFs and survival functions for data set II is illustrated through Figures 9 and 10 respectively.

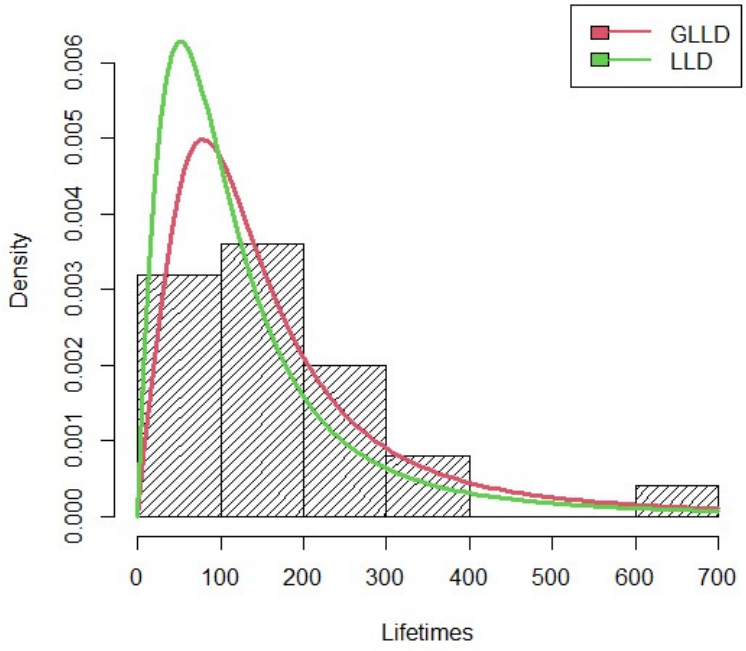

Fig. 8: *Curve fitting GLLD vz LLD for data set I*

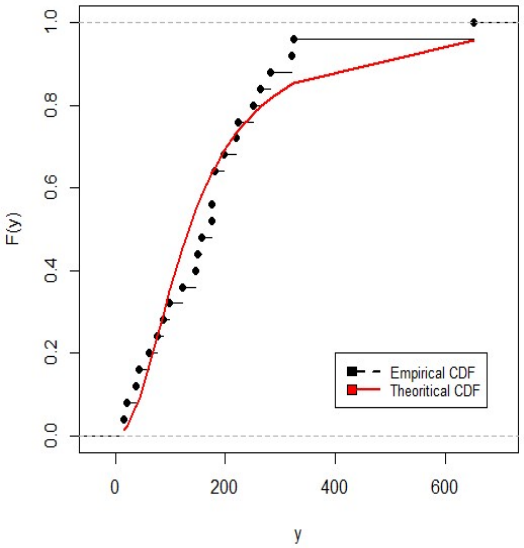

Fig. 9: *Empirical and Theoritical CDF for data set II*

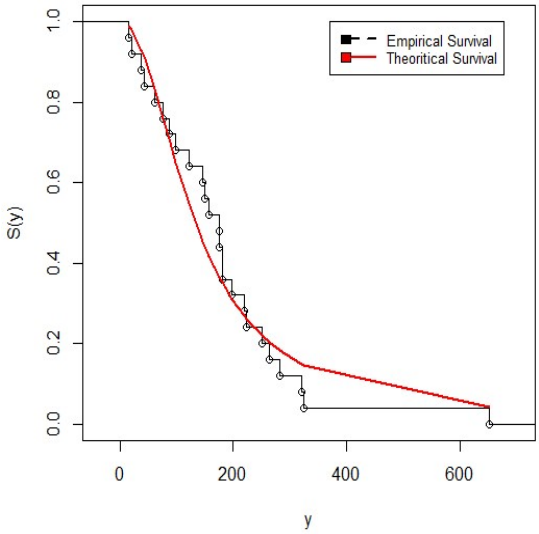

Fig. 10:*Empirical and Theoretical Survival Function for data set II*

6. Concluding Remarks

A new three parameter transmuted probability model namely is introduced by using the quadratic rank transmutation map technique. Comprehensive description of the statistical properties of the newly introduced model are introduced. Robust measures of skewness and kurtosis of the proposed model have also been derived along with the moment generating function, characteristic function, reliability function and hazard rate function of the said model. The estimation of the model parameters is performed by maximum likelihood method followed by a Monte Carlo simulation procedure. The applicability of this distribution to modeling real life data is illustrated by two real life examples and the results of comparison to base distribution in modeling the data are also exhibited.

Conflict of Interest

The Authors declare that there is no conflict of Interest.

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MULTI-OBJECTIVE PROBLEM WITH MULTIPLE JOBS ASSIGNED TO A SINGLE MACHINE WITHIN AVAILABLE COST UNDER UNCERTAIN ENVIRONMENT

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Abstract

The assignment problem is a key challenge in optimization and operations research, finding applications in diverse real-world scenarios. The Hungarian method is a widely employed algorithm for solving this problem, especially in its balanced form. However, for unbalanced assignment problems, where tasks outnumber resources (or vice versa), an extension is necessary. One common approach introduces a dummy resource, but this may leave tasks unassigned. The Modified Hungarian method improves upon the standard algorithm for unbalanced problems, ensuring that all tasks are assigned to real resources. This is achieved by modifying the cost matrix and algorithm steps to accommodate additional tasks and resources. Triangular fuzzy numbers are discussed when exact parameter information is undefined, and fuzzy programming is applied to determine a compromise result. Incorporating cost and profit per resource, the Modified Hungarian algorithm addresses the problem of unspecified job allocations to a single machine by introducing a cost parameter for each machine. The methodology is demonstrated on a numerical example for better comprehension.

Keywords: Triangular Fuzzy Number, Unbalanced Assignment Problem, Modified Hungarian Technique, Fuzzy Programming Approach

I. Introduction

In the contemporary landscape of industrial and operational management, the allocation of tasks to machines stands as a pivotal challenge, especially when confronted with the intricacies of handling multiple objectives concurrently. This challenge becomes particularly nuanced when multiple jobs must be assigned to a single machine, governed by predefined cost limitations, and set against the backdrop of an environment characterized by uncertainties. The fusion of these elements gives rise to a multifaceted problem, demanding sophisticated optimization techniques to strike an equilibrium among conflicting goals while ensuring resource efficiency. The optimization of job allocation on a single machine is a critical concern across various industries, spanning manufacturing, services, and beyond. The efficient utilization of resources in the face of dynamic and uncertain conditions is essential for organizations striving to enhance productivity, reduce costs, and maintain adaptability in an ever-evolving business environment. This intricate dance of optimization unfolds against the backdrop of challenges such as varying processing times, resource availability fluctuations, and external environmental factors, all of which

contribute to the complexity of the multi-objective problem at hand.

A foundational element in the exploration of the multi-objective problem is the extensive body of research on multi-objective optimization in job scheduling. Studies [1] and [2] have pioneered methodologies for handling conflicting objectives, including cost reduction, and resource utilization. These works have laid the groundwork for understanding the trade-offs involved in optimizing multiple objectives simultaneously, providing essential concepts for application in the context of job allocation on a single machine. The integration of uncertainty into optimization models is a crucial aspect of addressing real-world operational challenges. Researchers, including [3, 4], have explored various approaches for modeling uncertainty in scheduling and optimization problems. Techniques such as stochastic programming, fuzzy logic, and robust optimization have been employed to account for uncertainties in processing times, resource availability, and external factors. Understanding these methodologies is vital for adapting optimization models to the uncertain environment inherent in the multi-objective problem under consideration. The interplay between cost constraints and optimization objectives has been a focal point in operations research. Works such as the study by [5] have investigated cost-sensitive optimization models, aiming to strike a balance between achieving objectives and adhering to budget limitations. These insights are particularly relevant in the context of allocating multiple jobs to a single machine, where cost constraints play a pivotal role in decision-making. Recent advancements have seen the emergence of hybrid and metaheuristic approaches in solving complex optimization problems. Research by [6, 7] exemplifies the application of genetic algorithms, simulated annealing, and other metaheuristic techniques to address combinatorial optimization problems. These approaches offer promise in handling the intricate nature of the multi-objective problem, providing effective means to navigate the solution space efficiently. To bridge the gap between theoretical models and practical implementation, several studies have presented real-world applications and case studies. Works by authors like [8, 9] have demonstrated the applicability of optimization models in industries such as manufacturing, healthcare, and logistics. Examining these cases provides valuable insights into the challenges faced by practitioners and the effectiveness of proposed methodologies in diverse operational contexts.

Our ultimate goal is to contribute to the evolving landscape of operations research and optimization by presenting novel insights and frameworks that not only tackle the complexities of the multi-objective problem but also address the uncertainties inherent in real-world industrial settings. By doing so, we aspire to furnish decision-makers and practitioners with a comprehensive toolkit that enables them to navigate the labyrinth of multi-objective optimization within the constraints of cost and uncertainty, fostering resilience and agility in their operational strategies. Through this exploration, we aim to illuminate pathways towards a more efficient, adaptive, and sustainable operational paradigm.

The assignment problem and the transportation problem [10] are both types of optimization problems in operations research and linear programming, but they are not exactly the same. The assignment problem is indeed a special case of the transportation problem, with certain constraints and characteristics that make it more specific. There are various methods to solve the assignment problem, including enumeration methods, the simplex method, and the Hungarian method [11]. In a balanced assignment problem, the number of jobs equals the number of machines, and the Hungarian method is generally very convenient and capable of finding the optimal assignment. In an unbalanced assignment problem, where the number of jobs and machines is not equal, it may not be possible to assign all jobs to machines. In such cases, some jobs may remain unassigned. It's important to note that real-world scenarios may indeed involve unbalanced assignment problems. In such situations, it might be necessary to address the unassigned jobs differently, perhaps by revising the problem formulation or considering additional constraints to handle the unbalance. Suppose, in a production factory there are five machines and eight numbers of jobs. If one machine can do only one job than remaining jobs not

executed, that's create problem in manufacturing process. The Hungarian approach is improved by [12]. Kumar [13] provided a strategy for overcoming the imbalanced assignment problem that involves executing all jobs. Rabbani *et al.* [14] changed the formulation of the Hungarian approach by allowing the user to allocate several jobs to a single machine while without limiting any machine to a maximum number of jobs. The focus of this exploration lies in dissecting the intricacies of the "Multi-Objective Problem with Multiple Jobs Assigned to a Single Machine within Available Cost under Uncertain Environment." As we embark on this journey, our aim is to delve deep into the complexities of this operational puzzle, surveying existing methodologies, pinpointing gaps in current approaches, and proposing innovative frameworks that offer robust solutions to the multifaceted challenges faced by industries today.

In this paper, we will navigate through the theoretical foundations of multi-objective optimization, exploring its applications in the context of job allocation on a single machine. We will scrutinize the influence of uncertain variables on this optimization process, emphasizing the dynamic nature of real-world operational scenarios. Additionally, we will investigate the existing tools and methodologies employed in addressing similar challenges, critically evaluating their strengths and limitations. This work focuses on cost and profit management for jobs performed by machines, using the concept of [14] and providing cost parameters for constraints on each machine. Jobs are assigned to each machine based on the cost of each machine. For better comprehension, a step-by-step approach is provided and solved using a numerical example.

The motivation for studying the assignment problem originates from its importance in optimization and operations research, with numerous applications in real-life situations. While the Hungarian approach works well for balanced assignments, it struggles with imbalanced problems in which one job or resource exceeds another. The Modified Hungarian technique is unique in that it improves on the normal algorithm's ability to manage unbalanced assignments. By adjusting the cost matrix and algorithm stages, it assures that all jobs find actual resource allocations, eliminating the problem of tasks remaining unassigned while applying dummy resources.

The Modified Hungarian algorithm introduces a novel solution to the problem of unspecified job allocations to a single machine. By assigning a cost parameter to each machine and considering both cost and profit per resource, it provides a more comprehensive and realistic model for solving assignment problems. The methodology's practicality is demonstrated through a numerical example, enhancing understanding and showcasing the applicability of the Modified Hungarian algorithm in real-world scenarios. Furthermore, the use of triangular fuzzy numbers addresses cases where accurate parameter information is unavailable, resulting in a more flexible method. In such cases, the use of fuzzy programming might help identify a compromise solution.

II. Assumption of unbalanced assignment problem

- Here we consider the numbers of machines are always less than the numbers of jobs.
- Each machine is capable of performing multiple jobs.
- Each job can assign to only one machine, and no job can be assigned to multiple machines simultaneously.
- Every machine is assigned at least one task, and there are no tasks that remain unallocated to a machine.
- If a machine has multiple tasks to complete, it can perform them sequentially or in succession.
- The sum of the costs associated with the tasks assigned to each machine does not exceed the machine's available budget or cost limit.

III. Formulation

In a scenario with 'm' machines and 'n' jobs (where 'n' is greater than 'm'), each machine has a specific effectiveness or cost associated with it. The challenge here is to efficiently assign each job to one machine, ensuring that if there are more jobs than machines, the excess jobs are queued and processed subsequently.

Let's denote c_{ij} as the cost of assigning the i^{th} machine to the j^{th} job, where $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$. Our aim to find an optimal assignment for the problem, determining the job that will be allocated to each machine, in a way to minimize the overall cost and maximize the profit incurred while performing all the tasks. It's important to note that the sum of jobs exceeds the sum of available machines, and each machine can handle multiple jobs, but a single job cannot be assigned on two machines. Additionally, each machine has a specified budget or cost that is utilized for executing the assigned jobs. This problem can be represented using a $m \times n$ cost matrix $[c_{ij}]$ (Table 1) and a profit matrix $[p_{ij}]$ (Table 2).

Table 1: *The cost matrix c_{ij} in the form of $m \times n$*

	J_1	J_2	...	J_n	Cost
M_1	c_{11}	c_{12}	...	c_{1n}	C_1
M_2	c_{21}	c_{22}	...	c_{2n}	C_2
⋮	⋮	⋮	⋮	⋮	⋮
M_m	c_{m1}	c_{m2}	...	c_{mn}	C_m

Table 2: *The profit matrix p_{ij} in the form of $m \times n$*

	J_1	J_2	...	J_n
M_1	p	p_{12}	...	p_{1n}
M_2	p_{21}	p_{22}	...	p_{2n}
⋮	⋮	⋮	⋮	⋮
M_m	p_{m1}	p_{m2}	...	p_{mn}

Let x_{ij} denote the i^{th} machine is assigned for j^{th}job such that

$$x_{ij} = \begin{cases} 1, & if\ i^{th}\ machine\ is\ assigned\ to\ j^{th} job \\ 0, & if\ i^{th}\ machine\ is\ not\ assigned\ to\ j^{th} job \end{cases}$$

The Mathematical model is stated as

$$\text{Minimize } Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \quad (1)$$

$$\text{Maximize } Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}x_{ij} \quad (2)$$

subject to constraints

$$\sum_{j=1}^{n} x_{ij} \geq 1; for\ i = 1,2,\ldots,m \quad (3)$$

$$\sum_{i=1}^{m} x_{ij} = 1; for\ j = 1,2\ldots,n \quad (4)$$

$$\sum_{j=1}^{n} c_{ij}x_{ij} \leq C_i;\ for\ i = 1,2,\ldots,m \quad (5)$$

$$x_{ij} = 0\ or\ 1 \quad (6)$$

Eq [1-2] represents the objective functions, while Eq [3] specifies that a machine has the capability to handle multiple jobs. Eq [4] enforces the constraint that no identical job can be assigned to multiple machines, and Eq [5] signifies that the total assignment cost for any machine must not exceed its available budget. Lastly, Eq [6] indicates that only binary values are permissible in this context.

IV. Method

Solve multiple goal optimization problems by focusing on one objective at a time and disregarding the others with the restrictions provided. The following are the step-by-step procedures:

1. Examine the problem, Is the problem certain? If so, skip the defuzzification approach. Otherwise, change the problem into a neat model.
2. To reduce fuzzy problems into crisp equivalent forms, the defuzzification approach of the ranking function is applied.
   Using the defuzzification approach, we turn it into the corresponding crisp form. To get a similar crisp form, the ranking function [15, 16] was applied.

   Let$\tilde{P}_j = \left(P_j^{(1)}, P_j^{(2)}, P_j^{(3)}\right)$be a triangular fuzzy number, then the following equation must be used to compute its magnitude:

$$M\left(\tilde{P}_j\right) = \frac{P_j^{(1)} + 4P_j^{(2)} + P_j^{(3)}}{6} \tag{7}$$

3. Input m, n
4. Find the lowest cost in each row and deduct it from the relevant row, resulting in at least one zero in each row.
5. Check all columns; if any column remains without producing a zero, choose the lowest cost of that column and subtract it from all of the values for that column to produce a zero in that column.
6. Draw the fewest lines possible to cover the zeros in order to get the optimum matrix.
7. If the number of lines does not match the number of machines, choose the least uncovered cost and deduct it from each uncovered cost before adding the intersection of lines.
8. Repeat steps 6 and 7 until the number of lines equals the number of rows.
9. To assign the job, identify the smallest number of zeros in each row or column, assign that zero to the appropriate machine, and remove the actual cost of the assigned job from the available cost for that machine.
10. We cross the remaining zeros in the relevant column after allocation, and if there will be availability of cost for that machine is completed, also cross the remaining zeros in that row (the total cost of allotted jobs to a single machine cannot exceed the availability cost of that machine).
11. In the event of a tie, i.e., two rows or columns with the same number of zeros; assign the zero with the lowest cost in the original problem. There will be no duplicate jobs assigned two separate machines, and no machine will be left without assigning a job.
12. Repeat steps 9–11 until each job have been allocated.
13. End of algorithm.

The resulting solution is the idle solution. Using the idle solution, we constructed the payoff matrix. The payoff matrix will help in the development of the desired level for each objective function.

Fuzzy Goal Programming

The Fuzzy Goal Programming is a strong and adaptable approach that may be used to a wide range of decision-making issues with multiple objectives [17]. As a result, we can take advantage of this method to get the most effective solution for the specified models. The following are the step-by-step procedure:

- The resulting solution is the idle solution. The idle solution will help in constructing the payoff matrix. Finally, the payoff matrix helps in the development of the desired level for each objective function.

- The target value is established as the objective function's goals level $(g_k, k = 1,2)$.
- We constructed the fuzzy linear membership function for the fuzzy goal of $Z(X) \preceq g$ (*i.e.,* fuzzy-min) is as follows:

$$\mu(Z(X)) = \begin{cases} 1, & if\ Z(X) \le g \\ \dfrac{U - Z(X)}{U - g}, & if\ g \le Z(X) \le U \\ 0, & if\ Z(X) \ge U \end{cases}$$

where, $U$is the fuzzy goal's the highest tolerance limit of $Z(X)$.

Moreover, if the fuzzy goal $Z(X) \succeq g$ (*i.e.,* fuzzy-max), then, the membership function is as follows:

$$\mu(Z(X)) = \begin{cases} 1, & if \quad Z(X) \ge g \\ \dfrac{Z(X) - L}{g - L}, & if \quad L \le Z(X) \le g \\ 0, & if \quad Z(X) \le L \end{cases}$$

Where L is the fuzzy goal's lower tolerance limit of $Z(X)$.

- Finally, we use the linear membership function to convert multi-objective problem into single objective problem which can be solved by using a suitable traditional optimization method. The fuzzy achievement function μ is maximized.

V. Numerical Illustration

We consider a numerical example to illustrate in which five machines are offered to complete eight jobs with related cost (in USD).
The input parameters are in the form of Triangular fuzzy number for transportation cost (Table 3) and profit (Table 4).

Table 3: *The transportation cost in the form of triangular fuzzy number*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	26, 30, 34	24, 25, 26	38, 40, 42	45, 50, 55	33, 35, 37	24, 25, 26	39, 40, 41	24, 25, 26	26, 30, 34
M_2	38, 40, 42	26, 30, 34	22, 20, 24	22, 20, 24	22, 20, 24	41, 45, 49	22, 20, 24	24, 25, 26	38, 40, 42
M_3	22, 20, 24	38, 40, 42	26, 30, 34	38, 40, 42	26, 30, 34	45, 50, 55	26, 30, 34	38, 40, 42	45, 50, 55
M_4	24, 25, 26	22, 20, 24	33, 35, 37	26, 30, 34	24, 25, 26	26, 30, 34	33, 35, 37	26, 30, 34	57, 60, 63
M_5	33, 35, 37	33, 35, 37	45, 50, 55	38, 40, 42	38, 40, 42	57, 60, 63	45, 50, 55	38, 40, 42	74, 80, 86

Table 4: *The transportation profit in the form of triangular fuzzy number*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
M_1	8, 10, 12	11, 12, 13	14, 15, 16	21, 23, 25	15, 17, 19	10, 11, 12	15, 19, 23	8, 9, 10
M_2	13, 14, 15	6, 8, 10	5, 7, 9	4, 5, 6	2, 4, 6	20, 22, 24	1, 3, 5	5, 6, 7
M_3	5, 7, 9	15, 16, 17	11, 13, 15	15, 17, 19	11, 13, 15	25, 27, 29	10, 11, 12	14, 15, 16
M_4	6, 8, 10	1, 3, 5	12, 14, 16	11, 12, 13	8, 9, 10	6, 8, 10	20, 21, 22	8, 10, 12
M_5	11, 13, 15	10, 11, 12	21, 23, 25	12, 14, 16	20, 21, 22	27, 29, 31	15, 19, 23	14, 15, 16

The defuzzication approache ranking function (Equation [7]) is used to convert it into a crisp equivalent form for Table 3 & 4 will be shown in table 5 (Transportation cost) & Table 6 (Transportation profit).

Table 5: *Crisp equivalent form of transportation cost*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	30	25	40	50	35	25	40	25	30
M_2	40	30	20	20	20	45	20	25	40
M_3	20	40	30	40	30	50	30	40	50
M_4	25	20	35	30	25	30	35	30	60
M_5	35	35	50	40	40	60	50	40	80

Table 6: *Crisp equivalent form of transportation profit*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
M_1	10	12	15	23	17	11	19	9
M_2	14	8	7	5	4	22	3	6
M_3	7	16	13	17	13	27	11	15
M_4	8	3	14	12	9	8	21	10
M_5	13	11	23	14	21	29	19	15

The cost associated with machines for jobs will be represented in crisp form as shown in Table 7.

Table 7: *The cost associated with machines for jobs (in USD)*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	30	25	40	50	35	25	40	25	30
M_2	40	30	20	20	20	45	20	25	40
M_3	20	40	30	40	30	50	30	40	50
M_4	25	20	35	30	25	30	35	30	60
M_5	35	35	50	40	40	60	50	40	80

Find the simplest cost in each row and subtract it from the row that results in at least one zero in each row.
If any column remains without a zero, choose the lowest cost in that column and subtract it from all of the items in that column to obtain zero (Table 8).

Table 8: *Zeros row and column*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	5	0	15	25	10	0	15	0	30
M_2	20	10	0	0	0	25	0	5	40
M_3	0	20	10	20	10	30	10	20	50
M_4	5	0	15	5	5	10	15	10	60
M_5	0	0	15	5	5	25	15	5	80

Draw the number of lines required covering all zeros; in this case, four lines are required to cover all zeros, and the number of lines is not equal to the number of machines (Table 9). Go to step 7.

Table 9: *Line covered to all zeros*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	5	0	15	25	10	0	15	0	30
M_2	20	10	0	0	0	25	0	5	40
M_3	0	20	10	20	10	30	10	20	50
M_4	5	0	15	5	5	10	15	10	60
M_5	0	0	15	5	5	25	15	5	80

Select the least uncovered cost, i.e. 5, and subtract it from each uncovered cost and add it to the intersection point (Table 10). Also, check that there are zeros in each row and column. Draw lines to cover all of the zeros. In this case, five lines are drawn that are equal to the number of rows to generate the needed matrix.

Table 10: *New line on updated values that covering all zeros*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	10	5	15	25	10	0	15	0	30
M_2	25	15	0	0	0	25	0	5	40
M_3	0	20	5	15	5	25	5	15	50
M_4	5	0	10	0	0	5	10	5	60
M_5	0	0	10	0	0	20	10	0	80

Begin by assigning jobs to the rows. Find a row with only one zero, assign that zero, then cross the other zeros in that column i.e. J_1 assign to M_3 and subtract the cost of the allocated job from the available cost. Here, the available cost for machine M_3 is 50 USD, and after assigning the work, the remaining cost is 30 USD. Now, verify the columns that have one zero allocated to the relevant machine i.e.,J_3,J_7 assign to M_2 and J_6 assign to M_1 and subtract the cost from the corresponding machine's available cost (Table 11).

Table 11: *Available cost of corresponding machine*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	10	5	15	25	10	0	15	0	5
M_2	25	15	0	0	0	25	0	5	0
M_3	0	20	5	15	5	25	5	15	30
M_4	5	0	10	0	0	5	10	5	40
M_5	0	0	10	0	0	20	10	0	0

In the event of a tie, i.e., two rows or columns with the same number of zeros, we allocate the zero with the lowest cost in the problem. There will be no duplicate jobs assigned two separate machines, and no machine will be left without assigning at least one job.
If the cost is not accessible to do any more jobs for that machine after it has been assigned, and there are still any position related to that machine, then cross it for not assigning any other job.
Continue step 8 to 10 until all jobs are assigned.

Table 12: *Jobs assigned to all machine*

	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	Cost (USD)
M_1	15	5	15	25	10	0	15	0	5
M_2	30	15	0	0	0	25	0	5	0
M_3	0	15	0	10	0	20	0	10	0
M_4	10	0	10	0	0	5	10	5	15
M_5	5	0	10	0	0	20	10	0	40

Table 13 shows the task assignment that minimizes overall cost, and Figure 1 shows a graph of job allocation to the respective machine.

Table 13: *Jobs assigned to respected machines with the associated cost*

Machine	*Jobs*	*Cost (USD)*
M_1	J_6	*25*
M_2	J_3, J_4	*20+20=40*
M_3	J_1, J_7	*20+30=50*
M_4	J_2, J_5	*20+25=45*
M_5	J_8	*40*
Total Cost		*200*

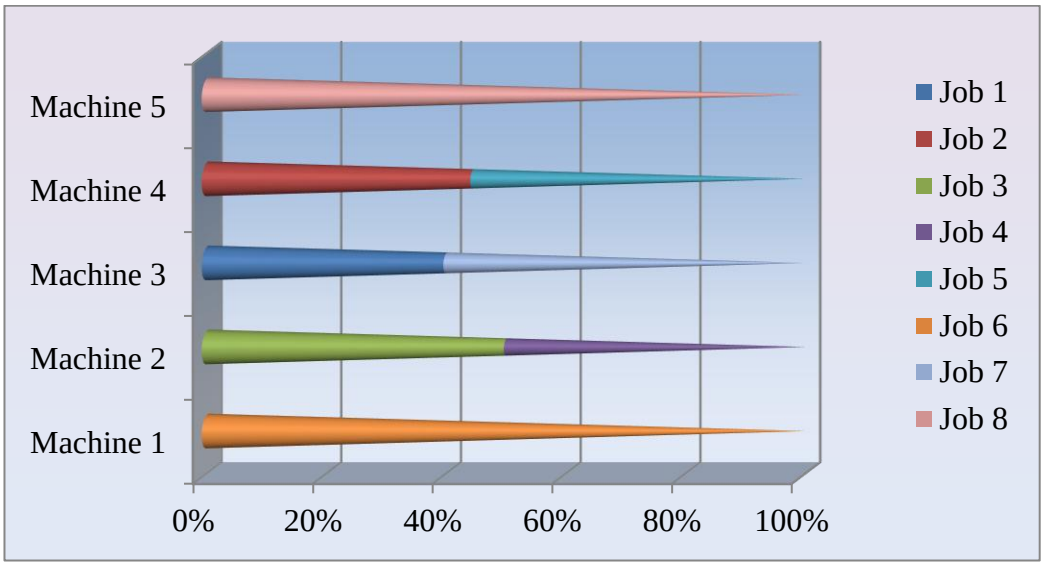

Figure 1: *Allocation of Jobs to the Machines for cost*

The second objective is maximizing type, so we select the maximum value from the Table 6 and subtract each element of the Table 6 from the selected value. Similarly, Table 14 shows the assignment of work that maximizes profit, and Figure 2 shows a graph of job allocation to the respective machine.

Table 14: *Jobs assigned to respected machines with the associated profit*

Machine	Jobs	Profit (USD)
M_1	J_2	12
M_2	J_3, J_4	7+5=12
M_3	J_6	27
M_4	J_5, J_7	9+21=30
M_5	J_1, J_8	13+15=28
Total Profit		109

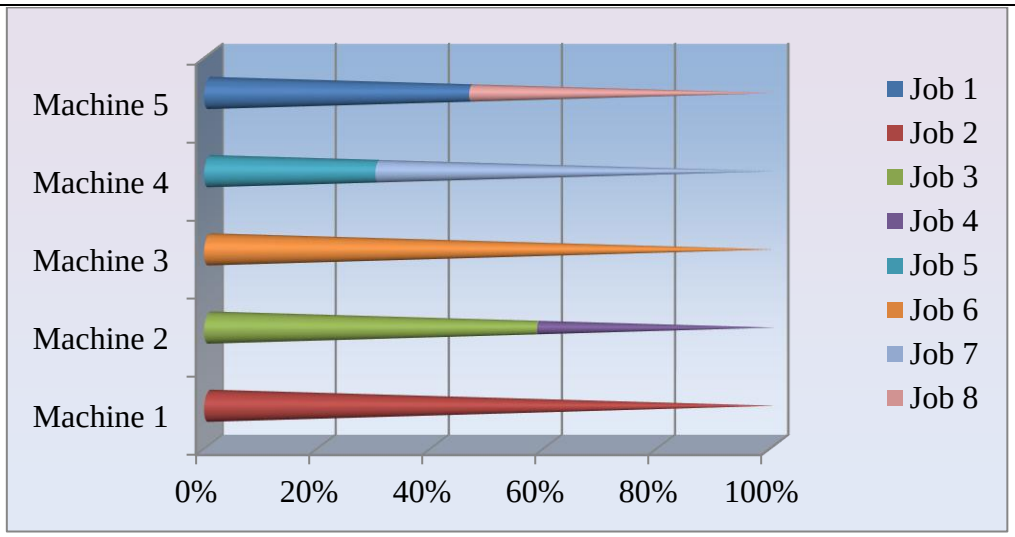

Figure 2: *Allocation of Jobs to the Machines for the profit*

After solving each objective, we have lower and upper bound of each objective such as: $200 \leq Z_1(x) \leq 250, 68 \leq Z_2(x) \leq 109$. Then, we construct the membership function of each objective are as follows:

$$\mu_1(Z_1(x)) = \begin{cases} 1, & Z_1(x) \leq 200 \\ \dfrac{250 - Z_1(x)}{250 - 200} & 200 < Z_1(x) < 250 \\ 0 & Z_1(x) \geq 250 \end{cases}$$

and

$$\mu_2(Z_2(x)) = \begin{cases} 0, & Z_2(x) \leq 68 \\ \dfrac{Z_2(x) - 68}{109 - 68}, & 68 < Z_2(x) < 109 \\ 1, & Z_2(x) \geq 109 \end{cases}$$

Use the fuzzy programming approach with these memberships, we get a compromise solution for the assignment of jobs as shown in Table 15 and graphical illustrating work allocation to the relevant machines in terms of cost and profit are shown in Figure 3 and 4 respectively.

Table 15: *Jobs assigned to respected machines with the associated profit*

Machine	Jobs	Cost (USD)	Profit (USD)
M_1	J_6	25	11
M_2	J_3, J_4	20+20=40	7+5=12
M_3	J_1, J_5	20+30=50	7+13=20
M_4	J_2, J_7	20+35=55	3+21=24
M_5	J_8	40	15
Total Profit		210	82

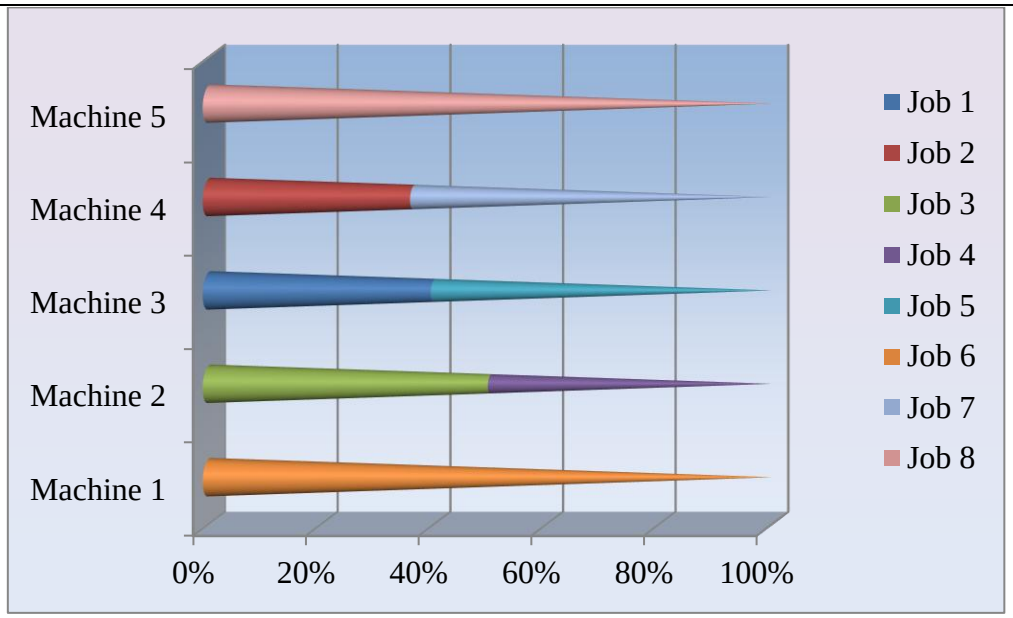

Figure 3: *Compromise allocation of Jobs to the Machines for the cost*

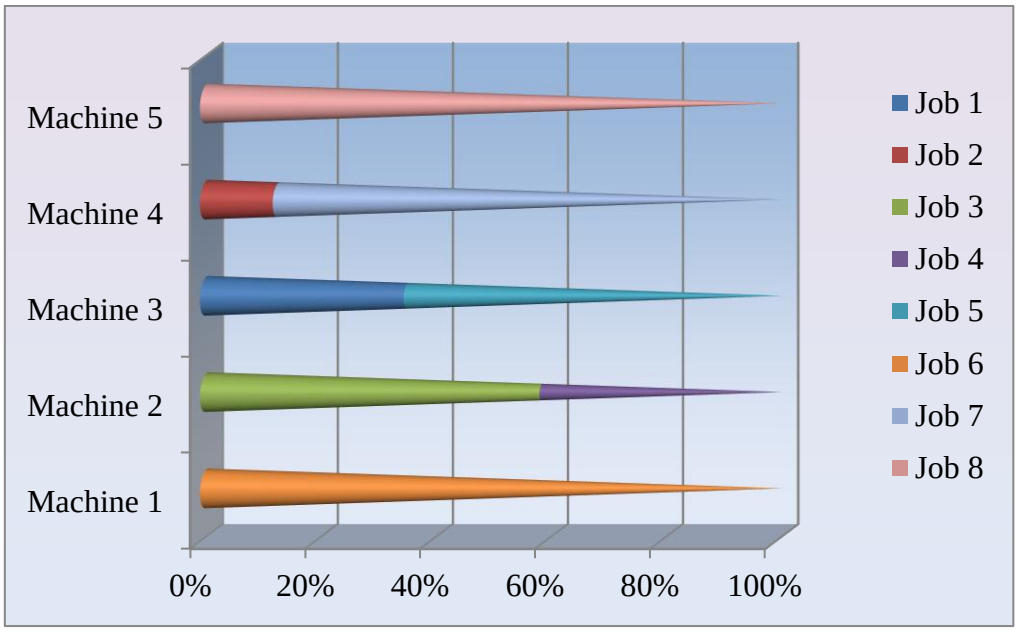

Figure 4: *Compromise allocation of Jobs to the Machines for profit*

VI. Discussion

Acknowledgment of the assignment problem's critical role in optimization and operations research, with broad applicability in real-world scenarios. Recognition of the Hungarian method's effectiveness in addressing the assignment problem in its balanced form. Expansion of the exploration to encompass unbalanced scenarios where tasks or resources outnumber each other. Introduction of a new and straightforward strategy for assigning multiple activities within given resources to achieve specific goals. Addition of two parameters, cost and profit for each resource,

in the formulation to meet the demand of allocating more jobs to one resource within a specified time frame. Utilization of fuzzy programming to find compromise solutions for multi-objective problems, offering adaptability in scenarios with undefined parameter information. In-depth exploration of triangular fuzzy numbers when the exact information of parameters is not defined, showcasing a flexible approach in problem-solving. Recognition of the Modified Hungarian method as a significant advancement in rectifying the shortcomings of leaving tasks unassigned. Explanation of how modifications to the cost matrix and algorithm steps ensure the comprehensive allocation of tasks to real resources.

The role of triangular fuzzy numbers in providing adaptability in scenarios with undefined parameters. Discussion on the contribution of fuzzy programming to compromise result determination, enhancing the methodology's effectiveness. Emphasis on the incorporation of cost and profit considerations per resource, adding a practical and comprehensive dimension to the algorithm. Explanation of how the Modified Hungarian algorithm adeptly handles unspecified job allocations by introducing a cost parameter for each machine. Recognition of the methodology's practical demonstration on a numerical example as a key element for enhancing comprehension. Emphasis on how the envisioned future scope aims to ensure the continued evolution and broader applicability of the Modified Hungarian algorithm in effectively addressing intricate assignment problems across diverse and dynamic contexts.

I. Needs

- Address the fundamental need for optimization in various real-life applications.
- Fulfill the need for effective solutions in scenarios where tasks significantly outnumber resources or vice versa.
- Cater to the need for flexibility when dealing with scenarios lacking exact parameter information through the discussion of triangular fuzzy numbers.
- Meet the need for a comprehensive approach by incorporating both cost and profit considerations for each resource in the formulation.

II. Limitations

- Acknowledge the limitation of dependency on the Hungarian method, particularly in its balanced form.
- Recognize the potential limitation of leaving tasks unassigned when utilizing the common approach of introducing a dummy resource.
- Understand that the Modified Hungarian method, while addressing unbalanced problems, may introduce complexity in modifying cost matrices and algorithm steps.
- Acknowledge the subjective nature of fuzzy programming in finding a compromise result, as it relies on fuzzy logic and human judgment.

VII. Conclusion

The assignment problem, crucial in optimization and operations research, spans diverse real-world applications. While the Hungarian method adeptly tackles the problem in its balanced form, the exploration extends to address unbalanced scenarios where tasks or resources outnumber each other. In this work, a new and easy strategy for assigning multiple activities within the given resources for a particular goal is proposed. Two more parameters as cost and profit for each resource is included in the formulation. This strategy meets the demand of allocating more jobs to one resource within a certain time frame while producing the most effective results. The fuzzy programming approach is used to find the compromise solution to a

multi-objective problem. We also discussed triangular fuzzy number if the exact information of the parameter is not defined. In future, researcher can used different fuzzy number for the allocation of the jobs to the machine and also used different techniques under the different circumstances.

The Modified Hungarian method, a significant advancement, rectifies the shortcomings of leaving tasks unassigned. By modifying the cost matrix and algorithm steps, it ensures all tasks find allocation to real resources. The discussion on triangular fuzzy numbers provides adaptability in scenarios with undefined parameters, and fuzzy programming contributes to compromise result determination. Incorporating cost and profit considerations per resource, the Modified Hungarian algorithm adeptly handles unspecified job allocations by introducing a cost parameter for each machine. The methodology's practical demonstration on a numerical example enhances comprehension. Looking forward, the future scope involves algorithmic refinements, dynamic resource models, integration with machine learning, validation across industries, and robustness to noisy data, ensuring the continued evolution and applicability of the Modified Hungarian algorithm in solving complex assignment problems across diverse and dynamic contexts.

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RELATIONSHIP BETWEEN THE LEIMKUHLER CURVE AND RELIABILITY MEASURE CONCEPTS IN DOUBLE TRUNCATED VARIABLES

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Abstract

This paper investigates the application of Leimkuhler curve and doubly truncated distributions in informetrics. Leimkuhler curve, ranking sources in descending order, emerges as a key tool for identifying efficient information sources. The study introduces a random variable representing the age of cited articles, influencing the probability distribution in retrospective citation analysis. Reliability measures, including mean residual life function and mean past residual life function are employed to analyze engineering and reliability aspects in informometric data. Truncation in probability distributions, particularly the doubly truncated distribution, is explored, revealing its broad applicability. The relationship between the Leimkuhler curve and truncated distributions will also be examined.

Keywords: leimkuhler curve, mean residual life function, mean past life function, double truncated random variable, risk measures

1. Introduction

The Leimkuhler curve and Lorenz curve serve as valuable tools in both information processing and economics. In economics, they are utilized to graphically represent the cumulative distribution of productivity versus resources. Moreover, they find application in analyzing the concentration of bibliometric distributions within the field of information sciences. The key distinction between the Lorenz [23] curve and the Leimkuhler curve lies in their ranking order: the Lorenz curve ranks sources (or individuals) in ascending order of productivity (income), while the Leimkuhler curve ranks them in descending order. In informetrics, where the focus often lies on identifying the most efficient sources of information, the Leimkuhler curve (LKC) serves as the equivalent graphical representation (see Burrell [9, 10, 13]). Its general definition can be found in Sarabia's work [28], and Balakrishnan et al. [3] have highlighted the relationships between the reliability function and the Leimkuhler curve. In retrospective citation studies within informetrics, interest is drawn to the age at which an article is cited, referring to the elapsed time from its publication to inclusion in the examined collection.

To conduct such studies, a single random variable X indicating article age determines the probability distribution of X. Burrell's research [12] has linked the data types reported in retrospective citation analysis with reliability models. Hazard rate, mean residual life function, reversed hazard rate, mean past life function, and vitality function are commonly employed tools for analyzing engineering and reliability aspects (Barlow and Proschan [4]). Truncation in probability distributions often arises in studies like reliability analysis, where unit failure may only be observed within specific time frames.(Abdul Sathar and Nair [1]) The broader utility of

truncated distributions has been explored in many references such as Bernardic and Candel [8], Belzunce et al. [6], Kupka and Loo [21] , Ato and Bernardic [2], Coffey and Muller [14] and Nair et al. [25] were analyze truncated data in various disciplines, necessitating the examination of truncated versions of the standard distribution, particularly in relation to reliability issues and economic inequality.

The doubly truncated distribution encompasses right truncated, left truncated, and non-truncated distributions as special cases. Notably, Belzunce et al. [6] have identified properties of concentration truncated distribution curves, while Behdani et al. [5] have explored the properties and applications of doubly truncated distributions in income inequality.

The subsequent sections of the paper are organized as follows: preliminary information is presented in section 2. Section 3 covers measures of reliability and risk related to the Leimkuhler curve, section 4 specifies the relationship between the mean past life function and the Leimkuhler curve, section 5 explores the Leimkuhler curve of doubly truncated distributions, section 6 discusses the relation between the geometric vitality function and the Leimkuhler curve, and finally, section 7 presents some conclusions.

2. Preliminaries

Let X be a non-negative random variable with finite and positive mean $E(X) = \mu$. The distribution function and survival function of X are symbolized by F and $\bar{F} = 1 - F$, respectively. The quantile function is defined as $F^{-1}(t) = \inf\{x : F(x) \geq t, t \in (0,1)\}$. The Lorenz curve, introduced by Lorenz, is a widely used graphical tool for illustrating and examining size distribution and wealth. For a random income variable X, the Lorenz curve is defined as:

$$L(p) = \frac{\int_0^p F^{-1}(t)dt}{\int_0^1 F^{-1}(t)dt}, \quad 0 \leq p \leq 1. \tag{1}$$

Here, the function $L(p)$ represents the cumulative percentage of total income earned by the lowest $100p\%$ earners. This paper presents the main result in the form of the Leimkuhler curve $K(p)$, as proposed by Sarabia [28] . The Leimkuhler curve is defined as:

$$K(p) = \frac{\int_{1-p}^1 F^{-1}(t)dt}{\int_0^1 F^{-1}(t)dt} = \frac{\int_{F^{-1}(1-p)}^\infty tf(t)dt}{\int_0^1 F^{-1}(t)dt}, \quad 0 \leq p \leq 1. \tag{2}$$

This curve indicates the share of total productivity returning to sources with productivity $100p\%$ greater. The Leimkuhler curve is essentially an inverted image of the Lorenz curve reflected along the diagonal line at 45 degrees.

The definitions of the Lorenz and Leimkuhler curves, $L(p)$ and $K(p)$ respectively, imply that these curves are linked by the relationship:

$$K(p) + L(1-p) = 1. \tag{3}$$

It is evident that the Leimkuhler curve acts as a distribution function, exhibiting continuity on $[0,1]$, with a second derivative $K''(p) \leq 0$, $K(0) = 0$, $K(1) = 1$, $K'(1^-) \geq 0$ and $K(p) \geq p$, among other trivial properties. The Gini index G, representing the area between the Leimkuhler and Lorenz curves, serves as a measure of income inequality. The Gini coefficient theoretically ranges from 0 (complete equality) to 1 (complete inequality), expressed as:

$$G = \int_0^1 [K(p) - L(p)]dp = 1 - \frac{\int_0^\infty \bar{F}(x)^2 dx}{E(x)}. \tag{4}$$

A low Gini index suggests a more equitable distribution of productivity, whereas a high index indicates a more unequal distribution. To illustrate, let's consider a classical Pareto distribution with a distribution function.

$$F(x) = 1 - (\frac{x}{\sigma})^{-\alpha}, \quad x \geq \sigma \tag{5}$$

where $\sigma > 0$ is a scale parameter and $\alpha > 0$ is a shape parameter. The Lorenz and the Leimkuhler curves of the classical Pareto distribution are each given by

$$L(p;\alpha) = 1 - (1-p)^{1-\frac{1}{\alpha}},\ 0 \le p \le 1,$$

$$K(p;\alpha) = p^{1-\frac{1}{\alpha}},\ 0 \le p \le 1. \tag{6}$$

Using relation (4), the Gini index of the classical Pareto distribution is

$$G(\alpha) = \frac{1}{2\alpha - 1}.$$

Figure 1 illustrates how the Lorenz curve, Leimkuhler curve, and Gini index are visualized for a Pareto distribution. The Gini coefficient is represented by the ratio of the area between the Lorenz curve and the line of equality to the total area under the line of equality. In this plot, the Gini coefficient is indicated by the relative size of the yellow-shaded area, demonstrating that the classical Pareto distribution for $\alpha = 2$ has significant inequality.

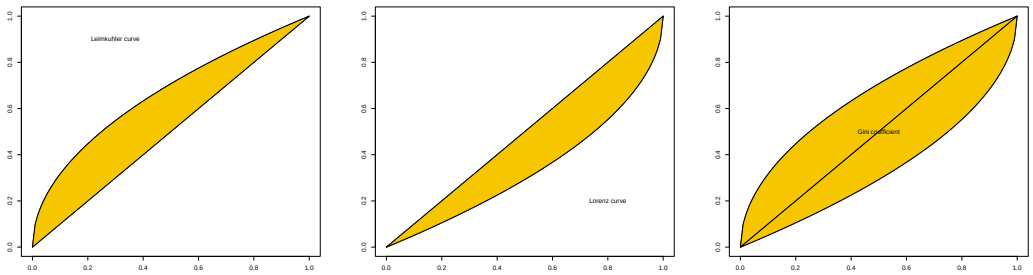

Figure 1: *Plot of the Lorenz and Leimkuhler curves and Gini index of the classical Pareto distribution for* $\alpha = 2$.

In prospective studies, there exists a time span until the initial citation occurs, as discussed by Burrell [11]. Assuming X represents a continuous random variable denoting the age of an object, the random variable $X(t,\infty) = \{X - t \mid X > t\}$ signifies the remaining lifespan of an entity aged t. The anticipated additional lifespan of an item, provided it has endured until time t, forms a function dependent on t, referred to as the mean residual life (MRL) function. This function, introduced by Knight [19], represents the average remaining life of a component that has survived until time t, and is given by:

$$m(t) = E(X - t \mid X > t) = \frac{1}{\bar{F}(t)} \int_t^\infty (x - t) dF(x). \tag{7}$$

In retrospective studies, we define the random variable $X(.,t) = \{t - X \mid X < t\}$, termed the inactivity time or elapsed lifetime of X. This variable signifies the duration since the expiration of a unit with a lifetime of at most t. We express this as the mean past life (MPL) function:

$$m^*(t) = E(t - X \mid X < t) = \frac{1}{F(t)} \int_0^t (t - x) dF(x). \tag{8}$$

In many practical situations, lifespan information is only available between two points in time, which necessitates the examination of reliability measures under the condition of truncated random variables. In reliability theory and survival analysis, often only individuals whose event time falls within a certain time interval are observed, and we have information on the lifetime between two points in time. Doubly truncated variables are the most general case, as they include right-truncated, left-truncated and non-truncated variables.

If X denotes the lifetime of a unit, then the random variable is $X(t_1,t_2)=\{X-t_1|t_1<X<t_2\}$ means remaining lifetime truncated twice. Note that the random variable, $X(t_1,\infty)$ is the special case of $X(t_1,t_2)$ when t_2 tends to ∞. Also, doubly truncated past lifetime is the random variable $X^*(t_1,t_2)=(t_2-X|t_1<X<t_2)$), which is special case $t_1=0$, it is the past lifetime random variable $X(.,t)$.

The mean residual life function for a doubly truncated variable is defined as:

$$m(t_1,t_2)=E(X-t_1|t_1<X<t_2)=\frac{1}{F(t_2)-F(t_1)}\int_{t_1}^{t_2}(x-t_1)dF(x). \tag{9}$$

Similarly, the mean past life function for a doubly truncated variable can be written as:

$$m^*(t_1,t_2)=E(t_2-X|t_1<X<t_2)=\frac{1}{F(t_2)-F(t_1)}\int_{t_1}^{t_2}(t_2-x)dF(x). \tag{10}$$

In engineering and other branches related to reliability, $\nu(t)=E(X|X>t)$ is referred to as the vitality function and $\nu^*(t)=E(X|X<t))$ is termed the past vitality function. Additionally, the vitality function for a doubly truncated variable is defined as:

$$\nu(t_1,t_2)=E(X\mid t_1<X<t_2)=\frac{1}{F(t_2)-F(t_1)}\int_{t_1}^{t_2}xdF(x) \tag{11}$$

which is related to MRL and MPL functions via

$$\nu(t_1,t_2)=m(t_1,t_2)+t_1, \tag{12}$$
$$=t_2-m^*(t_1,t_2). \tag{13}$$

3. Leimkuhler curve and reliability measures

As is commonly understood, the hazard rate function $r(t)=\frac{f(t)}{\bar{F}(x)}$ of F indicates whether the random variable X exhibits an increasing failure rate (IFR) or a decreasing failure rate (DFR) if $r(x)$ is respectively an increasing or decreasing function on the interval $(0,\infty)$.

Barlow and Proschan [4] demonstrated that if F is $IFR(DFR)$, then under certain conditions:

$$\begin{cases}\bar{F}(t)\geq(\leq)e^{-at}, & t\leq F^{-1}(p)\\ \bar{F}(t)\leq(\geq)e^{-at}, & t\geq F^{-1}(p)\end{cases}$$

where $a=-\frac{\ln(1-p)}{F^{-1}(p)}$.

Proposition 1. Let X be a non-negative random variable with distribution function F. If X is $IFR(DFR)$, then

$$K(p)\leq(\geq)\mu^{-1}[pF^{-1}(1-p)(1-(\ln p)^{-1})]. \tag{14}$$

Proof. By using (2), we obtain:

$$\begin{aligned}K(p)&=\frac{1}{\mu}\int_{F^{-1}(1-p)}^{\infty}tf(t)dt\\&=\frac{1}{\mu}[pF^{-1}(1-p)+\int_{F^{-1}(1-p)}^{\infty}\bar{F}(t)dt]\\&\leq(\geq)\frac{1}{\mu}[pF^{-1}(1-p)+\int_{F^{-1}(1-p)}^{\infty}e^{\frac{t\ln p}{F^{-1}(1-p)}}dt]\\&=\mu^{-1}[pF^{-1}(1-p)(1-(\ln p)^{-1})].\end{aligned}$$

This result is thus derived. ■

The following assertions are also noticeable:

- The stop loss transformation

$$\begin{aligned}\pi(t) = E(X-t)_+ &= \int_t^\infty \bar{F}(x)dx \\ &= -t\bar{F}(t) + \int_t^\infty xf(x)dx \\ &= -t\bar{F}(t) + \int_{F^{-1}(F(t))}^\infty xf(x)dx \\ &= -t\bar{F}(t) + \mu K(\bar{F}(t)),\end{aligned}$$

where $(X-t)_+$ means $X \geq t$. On taking $t = F^{-1}(p)$ then

$$K(p) = \frac{1}{\mu}(\pi_X(F^{-1}(p)) + pF^{-1}(1-p)).$$

- The total time on test transformation (TTT)

$$\begin{aligned}T(p) &= \int_0^{F^{-1}(p)} \bar{F}(t)dt \\ &= (1-p)F^{-1}(p) + \mu - \mu K(1-p),\end{aligned}$$

leading to

$$K(p) = \frac{1}{\mu}[pF^{-1}(1-p) - T(1-p)] + 1.$$

- $e(t) = E(\frac{X}{t} \mid X > t)$), representing the expected proportion of income up to t for incomes greater than t, can be expressed in terms of LKC

$$\begin{aligned}e(t) &= \frac{1}{t\bar{F}(t)} \int_t^\infty xf(x)dx \\ &= \frac{\mu K(\bar{F}(t))}{t\bar{F}(t)}.\end{aligned}$$

- The vertical diameter inequality index introduced in Eliazar [15] is given by

$$\begin{aligned}\varepsilon_{Vdiam} &= E(\frac{X}{\mu} \mid X \leq median) \\ &= \frac{\int_0^{median} \frac{x}{\mu} f(x)dx}{F(median)} \\ &= \frac{2}{\mu}[\mu - \int_{median}^\infty xf(x)dx] \\ &= 2[1 - K(\frac{1}{2})].\end{aligned}$$

In insurance and risk, the value at risk (VaR) for risk of X with distribution F is defined as $VaR(x;p) = F^{-1}(p) = \inf\{x \in R \mid F(x) \geq p\};\ p \in (0,1)$. Some of the measures related to risks based on VaR as mentioned in Belzunce et al. [7] arranged in table 1.

Example 1. Let X have a classical Pareto distribution (5) with Leimkuhler curve (6), $F^{-1}(x) =$

Table 1: *Some risk measures*

Name of measure	formula	LKC	reliability function
Tail value at risk	$TVaR(X;p) = \frac{1}{1-p}\int_p^1 F^{-1}(u)du$	$\frac{\mu}{1-p}K(1-p)$	$\nu(F^{-1}(p))$
Conditional tail expectation	$CTE(X;p) = E(X \mid X > F^{-1}(p))$	$\frac{\mu}{1-p}K(1-p)$	$\nu(F^{-1}(p))$
Conditional value at risk	$CVaR(X;p) = E(X - F^{-1}(p) \mid X > F^{-1}(p))$	$\frac{\mu}{1-p}K(1-p) - F^{-1}(p)$	$m(F^{-1}(p))$
Expected shortfall	$ES(X;p) = E(X - F^{-1}(p))_+$[1]	$\mu K(1-p) - (1-p)F^{-1}(p)$	$\pi(F^{-1}(p))$
Expected proportional shortfall	$EPS(X;p) = E[(\frac{X - F^{-1}(p)}{F^{-1}(p)})_+]$	$\frac{\mu K(1-p) - (1-p)F^{-1}(p)}{F^{-1}(p)}$	$\frac{\pi(F^{-1}(p))}{F^{-1}(p)}$

[1] $(x)_+ = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$

$\sigma(1-x)^{\frac{-1}{\alpha}}$ and $\mu = \frac{\alpha\sigma}{1-\alpha}$, $\alpha > 1$, based on Table 1 we have

$$\begin{aligned} TVaR(X;p) &= \frac{\alpha\sigma}{\alpha-1}(1-p)^{\frac{-1}{\alpha}}, \\ CTE(X;p) &= \frac{\alpha\sigma}{\alpha-1}(1-p)^{\frac{-1}{\alpha}}, \\ CVaR(X;p) &= \sigma(1-p)^{\frac{-1}{\alpha}}, \\ ES(X;p) &= \sigma(1-p)^{1-\frac{1}{\alpha}}, \\ EPS(X;p) &= 1-p. \end{aligned}$$

4. Mean past life function and Leimkuhler curve

This section commences by delineating the connection between the Leimkuhler curve and the mean residual life function through a theorem:

Theorem 1. (Balakrishnan et al. [3]) Let X be a random variable with cumulative distribution function F , $\bar{F}(x) = 1 - F(x) = P(X > x)$, Leimkuhler curve $K(p)$ and expectation μ. There exists a relationship between the Leimkuhler curve and the MRL function expressed as follows:

$$m(t) = \frac{\mu}{\bar{F}(t)}K[\bar{F}(t)] - t, \quad t > 0. \tag{15}$$

The MPL function and LKC can also be connected similarly to the theorem:

Theorem 2. Let X be a random variable with cumulative distribution function F, survival function $\bar{F}(x) = 1 - F(x)$, Leimkuhler curve $K(p)$ and expectation μ. The relationship between the Leimkuhler curve and the mean past life function is described as follows:

$$m^*(t) = t - \frac{\mu}{F(t)}[1 - K(\bar{F}(t)], \quad t > 0. \tag{16}$$

Proof. Beginning with the definition of the MPL function as (8), we express:

$$\begin{aligned} m^*(t) &= t - \frac{1}{F(t)}[\mu - \int_t^\infty xf(x)dx] \\ &= t - \frac{1}{F(t)}[\mu - \mu K(\bar{F}(t))] \\ &= t - \frac{\mu}{F(t)}[1 - K(\bar{F}(t))]. \end{aligned}$$

■

5. Double Truncated Distributions and Leimkuhler Curve

Truncated data holds significant importance in statistical analysis, representing variables that have been limited or constrained due to specific selection criteria. When observations falling outside certain ranges or conditions are disregarded in analysis, truncated variables emerge. Examples of this include distributions like doubly truncated exponential, normal, and Cauchy distributions. A truncated variable undergoes restriction to a defined range or set of conditions, leading to alterations in its probability density function (PDF) and cumulative distribution function (CDF) compared to the untruncated variable.

Let's consider a random variable X with PDF $f(x)$ and CDF $F(x)$. We want to find the PDF and CDF of the truncated variable

$$X_{(t_1,t_2)} = \{X|t_1 < X < t_2\}.$$

The PDF and CDF of the truncated variable can be expressed as:

$$f(x|t_1,t_2) = \frac{f(x)}{F(t_2)-F(t_1)}, \quad x > 0, \quad t_1,t_2 > 0. \tag{17}$$

$$F(x|t_1,t_2) = \begin{cases} 0 & x < t_1 \\ \dfrac{F(x)-F(t_1)}{F(t_2)-F(t_1)} & t_1 \le x \le t_2 \\ 1 & x > t_2. \end{cases} \tag{18}$$

The quantile and the survival function of $X_{(t_1,t_2)}$ is

$$F^{-1}(p|t_1,t_2) = F^{-1}(pF(t_2) + (1-p)F(t_1)),$$

$$\bar{F}(x|t_1,t_2) = \frac{\bar{F}(x)-\bar{F}(t_2)}{\bar{F}(t_1)-\bar{F}(t_2)}.$$

The LKC of $X_{(t_1,t_2)}$ has the following form:

$$K(p|t_1,t_2) = \frac{K[p(F(t_2)-F(t_1)) + \bar{F}(t_2)] - K(\bar{F}(t_2))}{K(\bar{F}(t_1)) - K(\bar{F}(t_2))}, \quad 0 \le p \le 1.$$

The Gini index for the double truncation is given by:

$$2\int_0^1 K(p|t_1,t_2)dp - 1.$$

Truncation to the right is a special case of double truncation when $t_1 \to 0$. This is evident from the following:

$$F^{-1}(p|t_2) = F^{-1}(pF(t_2)),$$

and its Leimkuhler curve is:

$$K(p|t_2) = \frac{K[pF(t_2) + \bar{F}(t_2)] - K(\bar{F}(t_2))}{1 - K(\bar{F}(t_2))}, \quad 0 \le p \le 1.$$

When $t_2 \to \infty$ in the doubly truncated distribution, we have a left truncated distribution:

$$F^{-1}(p|t_1) = F^{-1}(p\bar{F}(t_1) + F(t_1)),$$

and is the Leimkuhler curve

$$K(p|t_1) = \frac{K(p\bar{F}(t_1))}{K(\bar{F}(t_1))}, \quad 0 \le p \le 1.$$

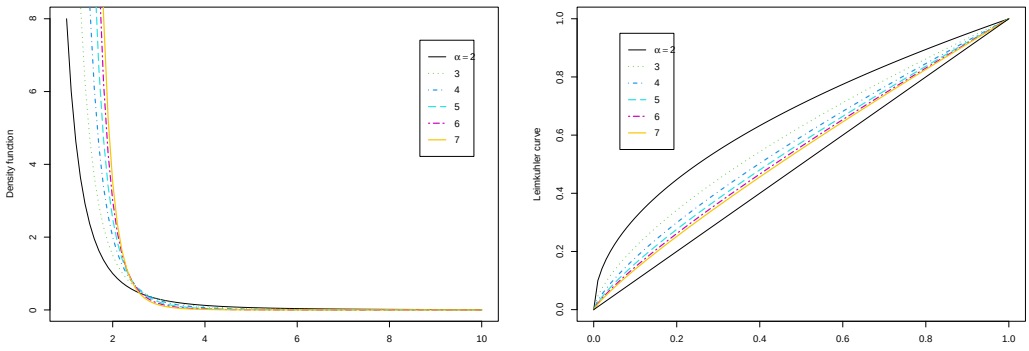

Figure 2: *The density function and Leimkuhler curve for Pareto (2, α)*

The density function and the Leimkuhler curve of the Pareto distribution with various parameters ($\alpha = 2, 3, \ldots, 7$) are depicted in Figure 2. It is evident that the density function of the Pareto distribution consistently decreases. Therefore, it proves advantageous in modeling distributions of high or moderate productivity. Pareto Leimkuhler curves never intersect, for $X_i \sim Pareto(\sigma, \alpha), i = 1, 2$

$$X_1 \leq_{LKC} X_2 \Longleftrightarrow \alpha_1 \leq \alpha_2.$$

Figure 3 illustrates the density function and Leimkuhler curves for the original Pareto distribution (O), left-truncated (L), right-truncated (R), and double-truncated (D) for Pareto (1, 3), $t_1 = 3, t_2 = 6$. It's noteworthy that the Leimkuhler curve of the left-truncated distribution remains unaffected by the truncation point.

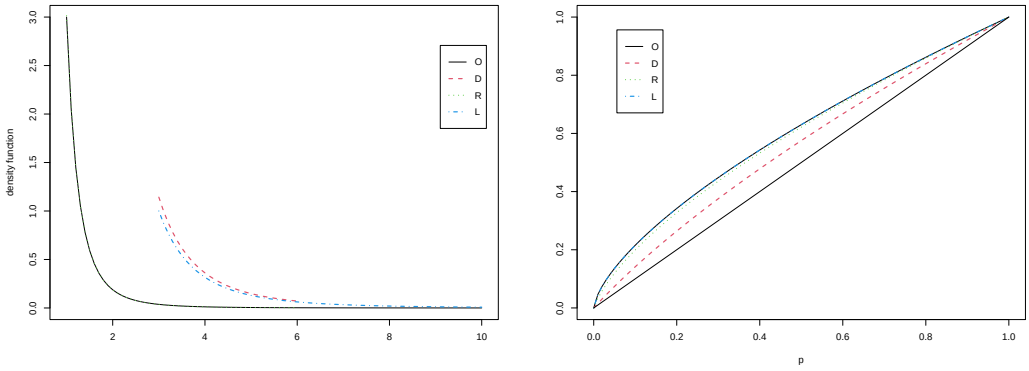

Figure 3: *Density function and Leimkuhler curve for the truncated and original Pareto distribution for Pareto (2,3),* $t_1 = 3, t_2 = 6$

The following theorem, an extended version by Balakrishnan et al. [3], pertains to the doubly truncated case:

Theorem 3. Let X be a non-negative random variable with cumulative distribution function $F(x)$, survival function $\bar{F}(x) = 1 - F(x)$, Leimkuhler curve $K(p)$ and expectation μ. Then, the relations between the Leimkuhler curve and the mean residual (past) lifetime of a doubly truncated random variable are given by:

$$m(t_1, t_2) = \frac{\mu(K[\bar{F}(t_1)] - K[\bar{F}(t_2)])}{\bar{F}(t_1) - \bar{F}(t_2)} - t_1, \tag{19}$$

$$m^*(t_1,t_2) = t_2 - \frac{\mu(K[\bar{F}(t_1)] - K[\bar{F}(t_2)])}{\bar{F}(t_1) - \bar{F}(t_2)}, \tag{20}$$

$$\nu(t_1,t_2) = \frac{\mu(K[\bar{F}(t_1)] - K[\bar{F}(t_2)])}{\bar{F}(t_1) - \bar{F}(t_2)}, \tag{21}$$

which holds for all $0 < t_1 < t_2$.

Proof. Via (9)

$$m(t_1,t_2) = E(X - t_1 \mid t_1 < X < t_2) = \int_{t_1}^{t_2} \frac{(x-t_1)f(x)}{F(t_2)-F(t_1)}dx$$
$$= \frac{1}{F(t_2)-F(t_1)}\left(\int_{t_1}^{\infty} xf(x)dx - \int_{t_2}^{\infty} xf(x)dx\right) - t_1.$$

Taking $t_1 = F^{-1}(1-p_1)$ and $t_2 = F^{-1}(1-p_2)$ we can express
$m(t_1,t_2) = \frac{1}{p_1-p_2}[\int_{F^{-1}(1-p_1)}^{\infty} xf(x)dx - \int_{F^{-1}(1-p_2)}^{\infty} xf(x)dx] - F^{-1}(1-p_1)$,
By changing variables to $z_1 = F^{-1}(1-p_1)$ and $z_2 = F^{-1}(1-p_2)$), we derive the relation in (19). Similarly, utilizing (10) $m^*(t_1,t_2)$ can be expressed as:

$$m^*(t_1,t_2) = E(t_2 - X \mid t_1 < X < t_2) = \int_{t_1}^{t_2} \frac{(t_2-x)f(x)}{F(t_2)-F(t_1)}dx$$
$$= t_2 - \frac{1}{F(t_2)-F(t_1)}\left(\int_{t_1}^{\infty} xf(x)dx - \int_{t_2}^{\infty} xf(x)dx\right).$$

By performing a similar change of variables, we arrive at the relation in (20) Finally, using relation (12) we establish the relationship in (21). ■

Theorem 22 is a special case of Theorem 3 when $t_2 \to \infty$. Additionally, when $t_1 = 0$ in Theorem 3, we obtain $m^*(t)$ as presented in Theorem 2.

Example 2. Let X be random variable with classical Pareto distribution (5) and Leimkuhler curve (6), the density function and cumulative distribution function for the right-truncated and doubly truncated variable (with $t_1 = \sigma$ and $t_2 = t$) are given by:

$$f(x \mid \sigma < X < t) = \frac{\alpha\sigma x^{-(1+\alpha)}}{1 - \left(\frac{t}{\sigma}\right)^{-\alpha}},$$

$$F(x \mid \sigma \le X \le t) = \frac{1 - \left(\frac{x}{\sigma}\right)^{-\alpha}}{1 - \left(\frac{t}{\sigma}\right)^{-\alpha}}, \quad \alpha > 0.$$

Now, if we assume $\sigma = 1$, $t = \beta + 1$, and $\alpha = 1$, the cumulative distribution function of the right-truncated and doubly truncated Pareto variable is the same as that of the Bradford distribution (see Leimkuhler [22]):

$$f(x \mid 1 < X < 1+\beta) = \frac{1+\beta}{\beta x^2}, \quad 1 < x < 1+\beta,$$

$$F(x \mid 1 < X < 1+\beta) = \frac{1+\beta}{\beta}\left(1 - \frac{1}{x}\right).$$

The mean of the truncated Pareto random variable is equal to the survival function of this variable:

$$\mu = \frac{1+\beta}{\beta}\ln(1+\beta).$$

Using equation (21), we have:

$$\nu(1,(1+\beta)) = \frac{1}{F(1+\beta)-F(1)}\int_1^{1+\beta} \frac{1}{x}\,dx = \frac{1+\beta}{\beta}\ln(1+\beta) = \mu.$$

6. Geometric Vitality Functions and their Links with Leimkuhler Curve

In recent years, special attention has been given to various forms of conditional distribution functions. The function

$$\phi(t_1, t_2) = E(h(x)|t_1 < X < t_2), \tag{22}$$

is called the geometric vitality function. This function is similar to the survival function and is used in the analysis of lifetime data. The goal is to establish the relationship between this function and the Limkohler curve, which we derive in the following theorem. If $\{h(X)|t_1 < X < t_2\}$ is an increasing (decreasing) function, then
$\{h(X)|t_1 < X < t_2\} = \{Y|h^{-1}(t_1) < Y < h^{-1}(t_2)\}$ $(\{Y|h^{-1}(t_2) < Y < h^{-1}(t_1)\})$.

Theorem 4. Assume that the random variable X has a distribution F, and the function $\{h(X)|t_1 < X < t_2\}$ is a continuous and increasing function with respect to X. Then

$$\phi(t_1, t_2) = \frac{\mu_g(y)[K(\bar{F}(t_1)) - K(\bar{F}(t_2))]}{F(t_2) - F(t_1)}$$

where $\mu_g(z) = \int_{h(a)}^{h(b)} h(z)f(z)dz, \quad z \in (a, b)$.

Proof. Assuming $Y = h(X)$, the proof is as follows: $Y = h(X)$ implies
$G(y) = F(h^{-1}(y))$, and

$$\begin{aligned}
\phi(t_1, t_2) &= E(h(X)|t_1 < X < t_2) \\
&= E(Y|t_1 < h^{-1}(y) < t_2) \\
&= E(Y|h(t_1) < Y < h(t_2)) \\
&= \frac{\int_{h(t_1)}^{h(t_2)} yg(y)dy}{G(h(t_2)) - G(h(t_1))} \\
&= \frac{\mu_g(y)[K(\bar{G}(h(t_1))) - K(\bar{G}(h(t_2)))]}{G(h(t_2)) - G(h(t_1))} \\
&= \frac{\mu_g(y)[K(\bar{F}(t_1)) - K(\bar{F}(t_2))]}{F(t_2) - F(t_1)}.
\end{aligned}$$

■

Various special cases of Theorem 4 are noteworthy:

(i) When h is decreasing, then

$$\phi(t_1, t_2) = \frac{\mu_g^*(y)[K(\bar{F}(t_2)) - K(\bar{F}(t_1))]}{F(t_1) - F(t_2)}, \tag{23}$$

where $\mu_g^*(z) = \int_{h(b)}^{h(a)} h(z)f(z)dz, \quad z \in (a, b)$.

(ii) When $h(x) = x - t_1$, it results in a doubly truncated residual life denoted as $X(t_1, t_2) = \{X - t_1|t_1 < X < t_2\}$ as discussed in Sankaran and Sunoj [29], indicating its association with the Leimkuhler curve. As $t_2 \to \infty$, , the connection between the Mean Residual Life (MRL) function and Leimkuhler curve becomes evident.

(iii) The MPL function for doubly truncated variables, as outlined in Khorashadizadeh et al. [18] and Ruiz and Navarro [26, 27] for $h(x) = t_2 - x$ in Theorem 4 , reveals a specific case where $t = 0$ leads to the MPL function. This relationship becomes apparent when put into relation (9) , demonstrating their connection to the Leimkuhler curve.

(iv) The doubly truncated geometric vitality is defined as

$$\phi(t_1, t_2) = E(\ln X|t_1 < X < t_2), \tag{24}$$

while a special case of Theorem 4, through $h(x) = \ln x$ is also the link with LKC specified. When $t_2 \to \infty$, tends to infinity, it leads to (10) in Nair and Rajesh [24] illustrating the relationship between geometric vitality and the Leimkuhler curve.

(v) When $h(x) = -\log \frac{f(x)}{\bar{F}(t_1) - \bar{F}(t_2)}$, then Theorem 4 leads to doubly truncated dynamic entropy, as discussed in Khorashadizadeh et al. [17].

7. Conclusions

This paper has demonstrated the versatility and significance of the Leimkuhler curve across various domains, including information processing, economics, and reliability analysis. The integration of reliability measures, such as the mean past life function, with the Leimkuhler curve, adds depth to the analysis by connecting informetric data with engineering principles, providing a novel approach to understanding data longevity and efficiency. The exploration of doubly truncated distributions further expands the applicability of the Leimkuhler curve, showing its relevance in contexts where data is naturally bounded or limited. Additionally, the study of geometric vitality functions and their relationship with the Leimkuhler curve highlights the broader implications of these mathematical tools, especially in areas related to resource distribution and productivity analysis. By bridging concepts from different fields, this paper underscores the Leimkuhler curve's potential as a comprehensive tool for both theoretical and applied research.

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MATHEMATICAL ANALYSIS OF THE MECHANICAL PART OF THE DESIGN SCHEME OF THE ELECTRIC DRIVE OF A HYBRID ELECTRIC MACHINE

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Abstract

The paper analyzes the mechanical part of the design scheme of the electric drive of a hybrid electric machine, which is a key stage in the design and research of automatic control systems. The main elements of a mechanical system, a model of a real mechanical system connected to an electric drive, including all moving elements, transmission mechanisms, and actuators that convert electrical energy into mechanical work, are considered. The presented calculation scheme allows you to analyze dynamic processes, i.e. to study the system's behavior over time, to determine stability, fluctuations, and other characteristics. Calculations of various mass systems are performed using the capabilities of the MATLAB/Simulink software package for a three-mass and two-mass system. These models can be used for different systems with different parameters. To draw up a structural diagram, the elements of the mechanical part and the connections between the elements, the types of these connections (rigid, elastic) and the directions of motion transmission are determined. Structural diagrams are used to analyze the dynamic characteristics of the system, determine transients, stability, and vibrations.

Key words: hybrid electric machine, mechanical system, electric drive, inertial element, moment of inertia, elastic element, transfer function, mathematical model, disturbing effect.

I. Introduction

The mathematical analysis of the mechanical part of the design scheme of the electric drive of a hybrid electric machine (HEM) includes various methods of theoretical mechanics, dynamics, control, and electronics to evaluate and optimize the operation of the electric drive. In the context of a hybrid electric machine, where both a traditional internal combustion engine (ICE) and an electric motor are used, it is important to carry out complex calculations for all system elements to interact correctly [2, 4, 10]. The main steps for the mathematical analysis of the mechanical part of the electric drive are:

1. Modeling of a mechanical system - the mechanical part of the electric drive of a hybrid electric machine consists of various elements such as engines (internal combustion engine and electric), transmission, wheels, etc. For each of these elements, a mathematical model is created describing their movement, interaction, and reactions to external forces.

The main elements of the mechanical system include:

- Models of drive motors (for both electric and internal combustion engines).
- Transmission models (if the hybrid system uses a manual transmission).

– Wheel and suspension models to account for their inertial characteristics and impact on the road surface.

Mathematically, this can be represented using differential equations describing the motion and interaction of these elements.

2. System dynamics - mathematical analysis of the dynamics of the mechanical part of an electric drive includes the study of the system's response to various external influences, such as: acceleration; load change; transitions between different operating modes (for example, from working on an electric motor to working on an internal combustion engine).

To describe these processes, a system of second-order differential equations is usually used, which simulate the movement of a body under the influence of forces. An important element here is to take into account inertia, friction and other factors that affect the operation of the system.

3. Energy analysis - energy analysis allows you to evaluate the efficiency of the system. For a hybrid car, it is important to switch between the operating modes of the electric motor and the internal combustion engine, which is associated with energy consumption management.

For the purpose of mathematical analysis, the following are required: power calculation: an estimate of the power transferred from the engine to the wheels; transmission efficiency: calculation of energy losses in transmission from the motor to the wheels; energy consumption optimization: a system operation strategy that uses the optimal balance between the internal combustion engine and the electric motor to minimize fuel consumption and maximize efficiency [1, 3, 7].

4. Control models - for the efficient operation of the electric drive of a hybrid machine, a control system is required that will optimize the use of an electric and mechanical drive, for example, torque control on wheels depending on the operating modes of the machine, a system that optimizes the transition between an electric motor and an internal combustion engine depending on the load and battery condition.

5. Transient modeling - for a HEM, it is also important to take into account transients such as: switching between the electric motor and the internal combustion engine; changes in the speed of rotation of the motor when the load changes; the effect of accelerations and decelerations.

For this purpose, methods of numerical solution of differential equations and modeling using specialized software packages can be used.

8. Vibration and noise calculation - The hybrid electric machine must operate with minimal vibrations and noises, which is also part of the analysis. This includes modeling vibrations arising from the operation of the engine and other transmission elements, as well as noise characteristics, which are important to take into account when designing. As a result of mathematical analysis of the mechanical part of the electric drive of a hybrid electric machine, it is possible to obtain optimal system operation parameters, evaluate its dynamic characteristics, energy efficiency, and make predictions about the durability and wear of key system components.

To perform calculations of various mass systems, it is advisable to use the capabilities of the MATLAB/Simulink software package [2, 4, 6, 12]. For three- and two-mass systems. These models can be used for different systems with different parameters. To draw up a structural diagram, it is necessary to determine all the elements of the mechanical part, establish connections between the elements, types of connections and directions of motion transmission. With the help of structural diagrams, it is possible to analyze the dynamic characteristics of the system, determine transients, stability, and vibrations. At the same time, it is necessary to take into account the factors influencing the choice of a block diagram: the type of engine, the nature of the load. Drawing up and analyzing structural diagrams is an important stage in the design and operation of electric drives. A properly selected and constructed scheme allows you to ensure the effective operation of the system and avoid undesirable phenomena.

II. Formulation of the problem

The mechanical part of the design scheme of the electric drive is a model of a real mechanical system connected to an electric drive. It includes all moving elements, transmission mechanisms and actuators that convert electrical energy into mechanical work. The main elements of the mechanical part consist of the following [9, 13]:

– Inertial elements with mass and moment of inertia, such as engine rotors, working bodies of machines and mechanisms.

– Elastic elements that can deform under the influence of external forces and restore their shape after removing the load (for example, shafts, springs).

– Damping elements that convert mechanical energy into thermal energy, reducing system vibrations (e.g. friction).

The calculation scheme allows you to analyze dynamic processes, that is, to study the behavior of the system over time, to determine stability, fluctuations and other characteristics. In addition, it is possible to determine the parameters of the electric drive, select an engine, calculate its power, select a control system, optimize the operation of the system, minimize energy losses, improve positioning accuracy, reduce vibrations [8, 11]. To build a calculation scheme, it is necessary to pay attention to the following factors:

1. Identification of elements, i.e. determination of all elements of a mechanical system and their characteristics (masses, moments of inertia, stiffness, coefficients of friction).

2. Connecting the elements according to their physical connection, forming a chain.

3. Simplifying calculations, for example, neglecting some masses or friction.

There are various types of calculation schemes: kinematic schemes reflecting only the geometric connections between the elements of the system and dynamic circuits that take into account not only geometric connections, but also inertial, elastic and damping properties of the elements.

Various mathematical methods are used to analyze computational circuits, such as: the method of differential equations, in which the differential equations of motion of the system are compiled and solved analytically or numerically, the method of transfer functions, the method of state matrices, which is described by a system of first-order differential equations in matrix form.

III. Problem solution

Simplification methods can be used when constructing the design scheme of an electric drive. The system of equations for a three-mass system is described as follows [7, 9, 12]:

$$\left.\begin{aligned} M_m - \Delta M_1 - M_{12} &= J_1 \frac{d\omega_1}{dt} \\ M_{12} - \Delta M_2 - M_{23} &= J_2 \frac{d\omega_2}{dt} \\ M_{23} - \Delta M_3 - M_d &= J_3 \frac{d\omega_3}{dt} \end{aligned}\right\} \quad (1)$$

In these equations, M_m - moment of the engine; J_1, J_2, J_3 - moments of inertia; ω_1, ω_2, ω_3 -rotational velocities; M_{12} and M_{23} - elastic moments acting between masses; ΔM_1, ΔM_2, ΔM_d, - disturbing effects. For the first mass, the moment M_{12} is used as the moment of resistance. This moment of motion for the second mass is used as a torque:

$$M_{12} = C_{12}\left(\varphi_1 - \varphi_2\right) \tag{2}$$

$$M_{23} = C_{23}\left(\varphi_2 - \varphi_3\right) \tag{3}$$

Here, φ_1, φ_2, φ_3 – rotation angles during rotation of the mass; C_{12}, C_{23} – stiffness of elastic bonds. The stiffness of elastic bond is determined as follows:

$$C_h = \frac{M_{sh}}{\Delta\varphi_i}, \tag{4}$$

where M_{sh} - torque of the motor shaft, $\Delta\varphi_i$ - difference in the angles of rotation at the ends of the elastic elements, and the deformation of the part. Similarly, we can write a system of equations for a two-mass system:

$$\left.\begin{aligned} M_m - \Delta M_1 - M_{12} &= J_1 \frac{d\omega_1}{dt} \\ M_{12} - \Delta M_2 - M_{23} &= J_2 \frac{d\omega_2}{dt} \\ M_{12} &= C_{12}\left(\varphi_1 - \varphi_2\right) \end{aligned}\right\} \tag{5}$$

For a single-mass system, the equation is represented as follows:

$$M_m - M_d = J_\Sigma \frac{d\omega}{dt} \tag{6}$$

To perform calculations of various mass systems, it is advisable to use the capabilities of the MATLAB/Simulink software package. Figure 1 shows a Simulink model of the equation system for a three-mass system, and Figure 2 for a two-mass system. These models can be used for different systems with different parameters.

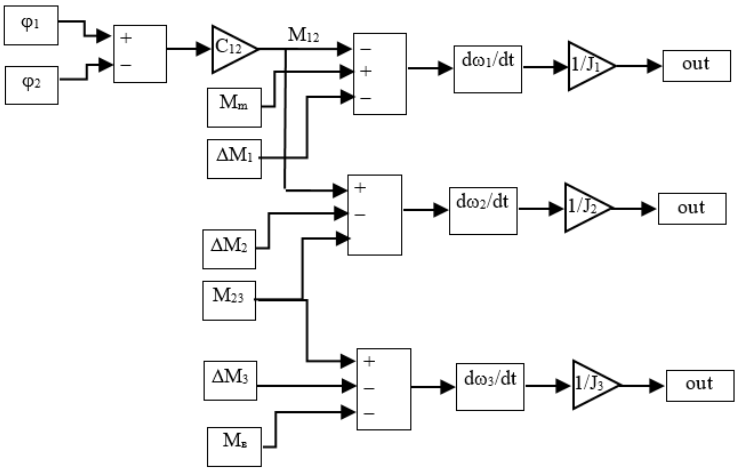


Figure 1. Simulink model of the equation system for a three-mass system

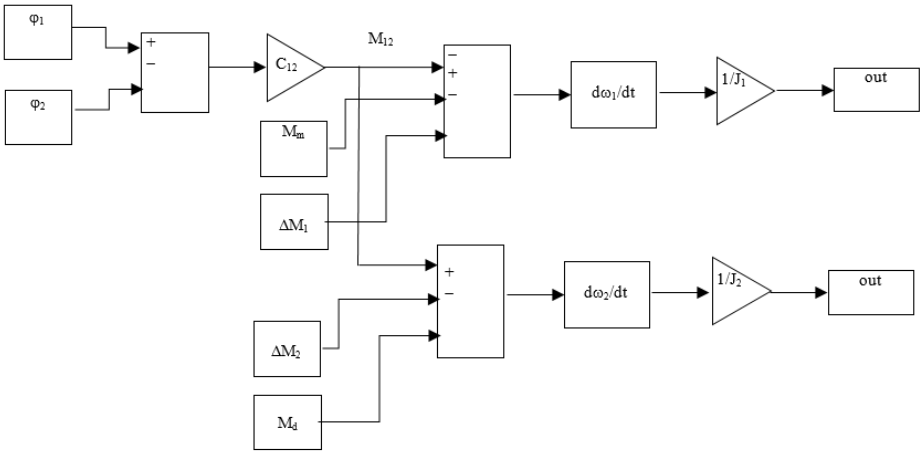


Figure 2. Simulink model of the equation system for a two-mass system

The block diagram of the mechanical part of an electric drive of the HEM is a graphical representation of the relationships between the elements of a mechanical system that converts the electrical energy of the engine into mechanical work. It allows you to visualize and analyze the processes taking place in the system, evaluate its dynamic characteristics and select the optimal parameters [6, 9, 10]. To draw up a structural diagram, it is necessary to determine all the elements of the mechanical part, the establishment of connections between the elements, types of connections (rigid, elastic) and directions of motion transmission. With the help of structural diagrams, it is possible to analyze the dynamic characteristics of the system, determine transients, stability, and vibrations. At the same time, it is necessary to take into account the factors influencing the choice of a block diagram: the type of engine, and the nature of the load (constant, variable, shock).

Drawing up and analyzing structural diagrams is an important stage in the design and operation of electric drives. A properly selected and constructed scheme allows you to ensure the effective operation of the system and avoid undesirable phenomena.

Figure 3 shows a block diagram of the mechanical part of a two-mass electric drive of the HEM.

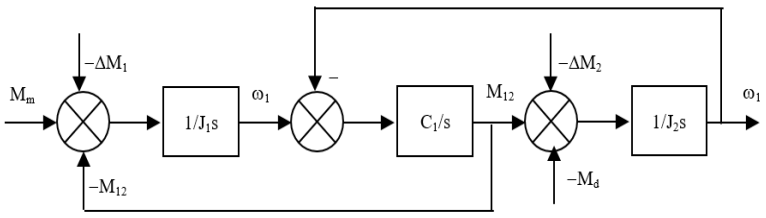


Figure 3. Block diagram of the mechanical part of a two-mass electric drive

Equivalent transfer functions for the mechanical part of the drive, depending on the adjustable parameters, can be expressed in various forms [3, 12, 14]. For example, the output value of the transfer function $\omega_1(s)$ for control is expressed in this way:

$$W_{\omega_1}(s)=\frac{\omega_1(s)}{M_m(s)}=\frac{s^2+\frac{J_1}{J_1+J_2}\Omega_{12}^2}{J_1 s(s^2+\Omega_{12}^2)}, \tag{7}$$

where, Ω_{12} - specific oscillation frequency of a two - mass mechanical system:

$$\Omega_{12}=\sqrt{\frac{C_{12}(J_1+J_2)}{J_1 J_2}} \tag{8}$$

Similarly, for $\omega_2(s)$ we get:

$$W_{\omega_2}(s)=\frac{\omega_2(s)}{M_m(s)}=\frac{\Omega_{12}^2}{s(J_1+J_2)(s^2+\Omega_{12}^2)} \tag{9}$$

The transfer function for a two-mass electric drive of the HEM is determined by solving the ratio of the polynomials of the system of equations of motion written in operator form. Similarly, the transfer functions of the perturbation can be expressed as follows:

$$W_{\omega_1}(s)=\frac{\omega_2(s)}{(\Delta M_2+M_d)(s)}=\frac{\Omega_{12}^2}{s(J_1+J_2)(s^2+\Omega_{12}^2)}=W_{\omega_2}(s) \tag{10}$$

$$W_{\omega_2}(s)=\frac{\omega_3(s)}{(\Delta M_3+M_d)(s)} \tag{11}$$

It should be noted that the transfer function of the system does not depend on either the shape or the amplitude of the input signal. This function depends only on the internal structure of the system and the circuit parameters.

These steps will help you create and configure a three-mass electric drive model in Simulink and visualize its behavior. If the system has more complex interactions or additional elements, you may need to adapt the model depending on specific requirements [4, 10].

Consider the standard model of a two-mass electric drive. It is usually described by the following equation:

$$\frac{\Theta_1(s)}{U(s)}=\frac{K_m}{s((J_1J_2s^2+J_1+J_2)B_2s+K_mK_t)} \tag{12}$$

Where, $\Theta_1(s)$ - angle of rotation at the output (in radians); $U(s)$ - control voltage; K_m - torque constant of the electric motor; K_t - constant of the torque of the load; J_1 and J_2 - moments of inertia of the electric motor and the load, respectively; B_2 - coefficient of friction of the load.

Next, the following algorithm (code) of equation (12) is written in the MATLAB program indicated below. This code will create a transfer function, plot its frequency and transient response (Figure 4). Substituting the real values of the parameters into the code, depending on the required task, you can create a transfer function, plot its frequency and transient response. To construct the amplitude-phase characteristic of the system, it is necessary to replace the symbol of the operator s with the symbol $j\Omega$ in the expression (9) of the transfer function of the system.

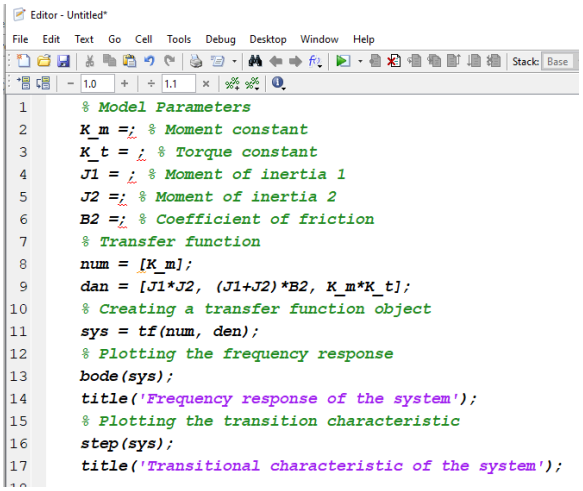


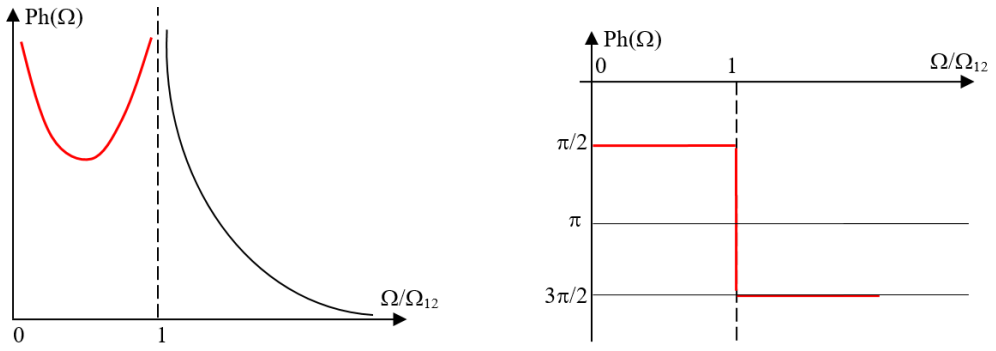
Figure 4. Algorithm (code) of equation (12)

$$W_{\omega_1}(j\Omega) = \frac{1}{\Omega_{12}(J_1 + J_2)} + \frac{j}{(\Omega / \Omega_{12})(1-(\Omega\Omega_{12})^2)} \tag{13}$$

$$W_{\omega_1}(j\Omega) = Re(\Omega) + jIm(\Omega)$$

here, $Re(\Omega)$ – real number; $Im(\Omega)$ – imaginary number.

Using expression (13), we obtain the amplitude-phase (Figure 5,a) and phase-frequency (Figure 5,b) characteristics of a two-mass system.



a) b)

Figure 5. Amplitude-phase and phase-frequency characteristics of a two-mass system

When the frequency value changes in the range $0 \div \Omega_{12}$, the amplitude of the output signal has a U-shaped appearance [2, 9]. But in the range of $0 \div \infty$, the amplitude of the variable value will tend to infinity. In subsequent increases, the amplitude of the variable always decreases. When changing the phase variable in the range $0 \div \Omega_{12}$ this value abruptly changes from $-\pi/2$ to $3\pi/2$, and will be a constant in phase $3\pi/2$: $\Omega > \Omega_{12}$.

Suppose that the excitation of the electric drive under study affects the formation of the amplitude of the electromagnetic moment, which varies according to the harmonic law of time, and the amplitude value of this electromagnetic moment is slightly more than the value of the static torque. When the frequency of the electromagnetic torque is close to zero, the difference between static and dynamic torques, that is, the dynamic torque will be positive. As a result, after a certain time you will return the value of the angular velocity, which is the output value, will be equal to infinity.

With an increase in the frequency of change in the angle of rotation of the electromagnetic torque, its speed decreases due to the action of moments of inertia on the mechanical parts of the electric drive of the HEM. Here, the inertia of the mechanical parts plays the role of a damper. When the amplitude of the forced frequency fluctuations approaches a certain frequency of the system, this value begins to increase again. In this case, a mechanical resonance is detected, and at a value of $\Omega=\Omega_{12}$, the amplitude of the oscillations again takes on an infinite value [7, 9, 13].

With a subsequent increase in the amplitude of the forced frequency, the inertia of the mechanical parts leads to a decrease in the amplitude of the output value, which plays the role of a damper that prevents movement. Under the condition $\Omega>>\Omega_{12}$, the inertia of the parts of the mechanism completely disables the oscillatory motion. Vibrations during movement, along with dynamic influences, dissipating forces arise (for example, friction against the lining), the speed also increases as the resistance force increases, as a result of which the dynamic torque decreases to zero, and the speed naturally reaches a large value.

IV. Conclusions

1. The article presents a mathematical analysis of the mechanical part of the design circuit of an electric drive, which is a key stage in the design and research of automatic control systems, which allows to obtain an accurate description of the dynamic processes occurring in the system and evaluate its characteristics.

2. The presented calculation scheme allows analyzing dynamic processes, i.e. studying the behavior of the system over time, determining stability, fluctuations and other characteristics. At the same time, it is possible to determine the parameters of the electric drive, select an engine, calculate its power, select a control system, optimize the operation of the system, minimize energy losses, improve positioning accuracy, reduce vibrations.

3. To draw up a structural diagram, the elements of the mechanical part and the connections between the elements, the types of these connections (rigid, elastic) and the directions of motion transmission are determined. The analysis of the dynamic characteristics of the system, the determination of transients, stability, and vibrations was performed using structural diagrams.

4. With an increase in the frequency of change in the angle of rotation of the electromagnetic torque, its speed decreases due to the action of moments of inertia on the mechanical parts of the electric drive. When the amplitude of the forced frequency fluctuations approaches a certain frequency of the system, this value begins to increase again. With a subsequent increase in the amplitude of the forced frequency, the inertia of the mechanical parts leads to a decrease in the amplitude of the output value, which plays the role of a damper that prevents movement.

5. Vibrations during movement, along with dynamic influences, dissipating forces occur (for example, friction against the lining), the speed also increases as the resistance force increases, as a result of which the dynamic torque decreases to zero, and the speed naturally reaches a large value.

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ZERO TRUNCATED POISSON REGRESSION MODEL FOR REPRODUCTIVE PATTERNS ON COUNT DATA

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Abstract

The number of children ever born is an important measure for understanding fertility patterns, which impact demographic structures and population growth. The problem relates to the modeling of count data that includes the truncation of zero values, specifically focusing on women who have experienced childbirth at least once. This study analyzes the factors that influence the number of children ever born (CEB) among women aged 15 to 50 in Andhra Pradesh, utilizing data from the National Family Health Survey (NFHS-5) conducted from 2019 to 2021. The study used Zero-Truncated Poisson (ZTP) and Zero-Truncated Generalized Poisson (ZTGP) models to identify major determinants, including religion, kind of cooking fuel used, place of delivery, wealth, age, and fertility choices. The ZTP regression model was found to be the best model and identifies significant determinants such as religion, wealth, age, and fertility preferences. The results show that rural residence, Muslim faith, and older age groups are associated with higher CEB, while wealthier women tend to have fewer children. The study shows the importance of implementing focused reproductive health activities, specifically in rural regions, to manage population growth and enhance the health outcomes of both mothers and children.

Keywords: Number of children ever born, Fertility Patterns, Zero-Truncated Poisson Model, NFHS, Reproductive Health, Under-dispersion

I. Introduction

The number of children ever born (CEB) quantifies the total count of live births among women aged 15 to 50 [19]. CEB, a summary of birth histories, is a quantitative measure of all women's live births during their lifetime. The CEB is a significant factor in shaping global population trends [5,23]. Population growth is not only influenced by it, but it also plays a crucial role in shaping the demographic age distribution. Fertility is a major component of Demography, which has three primary categories and refers to the natural ability for reproduction. Evaluating fertility trends [6] and prospective opportunities is vital to economic and social planning, workforce accessibility, and advancement [2]. Examining variations in reproduction rates among Indian states based on socioeconomic and demographic factors indicates significant variety [7]. The fertility rate in India has had a gradual decrease over the years, reaching 2.47 in 2012, 2.41 in 2013, 2.31 in 2014, 2.29 in 2015, 2.27 in 2016, 2.2 in 2017, 2.18 in 2018, 2.11 in 2019, 2.05 in 2020, and 2.03 in 2021 children per woman. The National Family Health Survey 2019-2021 (NFHS-5) offers comprehensive data on India's population, health, and nutrition [18]. This study aims to evaluate the prevalence of CEB (Children Ever Born) and its determining factors among women of reproductive age in Andhra

Pradesh.

Counting data with an excess of zeros is common in various fields, including engineering, biomedical research, public health, demography [1], economics, and social science. The basic Poisson regression model is the best strategy for analyzing a random variable Y expressing counts with equal sample mean and variance [8,21]. Count data displays significant variability when the sample variance is either smaller or bigger than the sample mean and is categorized as under-dispersion or over-dispersion [4,11]. Several models [27] have been suggested to address these variations, such as the negative binomial model [13], extended Poisson model [8], hurdle Poisson model [3,22], and truncated models [10]. The Generalized Poisson model was designed to analyze family fertility [15,27] and injury data [31]. However, it is frequently seen that count data exhibits a low frequency of zeros and is under-dispersed, indicating the absence of zero inflation in fertility [25]. This study examines the impact of several socioeconomic and demographic characteristics, such as site of residence, kind of cooking fuel used, place of delivery, wealth index, marital status [26], and caste, on the outcomes of women not experiencing infertility. To obtain the fertility rate of women aged 15–50, truncate the zero values in the count variable of the dataset [20,30]. The model-building procedure employs data from the NFHS-5 survey, specifically focusing on 8087 women from Andhra Pradesh who had given birth at least once. This paper examines the zero-truncated Poisson (ZTP) model [12], which accounts for both over and under-dispersion, as well as the zero-truncated generalized Poisson (ZTGP) model [28], which accounts for under-dispersion.

This study used secondary data as its basis. The data for this investigation was gathered from the fifth round of the NFHS, the most extensive sample survey representing the entire nation [17]. The NFHS is an Indian dataset derived from the seventh phase of the Demographic and Health Surveys (DHS) Program, carried out under the supervision of the National Institute for Population Research and Training of the Ministry of Health and Family Welfare. The NFHS Subject Reports are concise summaries of secondary data analysis from the 1992–93 National Family Health Survey (NFHS) conducted in India [16]. The National Family Health Survey (NFHS) collected information from around 90,000 women in India, covering all aspects of demographics and health. The Indian Ministry of Health and Family Welfare conducted this survey, which provides in-depth information on maternal and child health, family planning practices, infant and child mortality, and the use of mothers' and children's services at the national and state levels [14]. IIPS performed the survey in collaboration with consultancy organizations and 18 population research centres across India. The East-West Center and Macro International, a U.S.-based consulting firm, offered technical help, while the United States Agency for International Development (USAID) provided financial support.

II. Methodology

2.1 Regression Models

The Poisson regression model is a widely used non-linear regression model for counting data [8]. Let Y represent the number of children ever born (CEB) to a woman of reproductive age in Andhra Pradesh. This variable follows a Poisson distribution, determined by the independent variables X_1, X_2, ..., X_{10}.

$$P(Y = y) = \frac{e^{-\mu}\mu^{y}}{y!} \quad ; for\ \mu > 0, y = 0,1,2, \ldots \tag{1}$$

It implies that μ is the exponential function of independent variables,

$$\mu = e^{\alpha+\beta_1 X_1+\beta_2 X_2+\cdots+\beta_{10} X_{10}} = e^{x'\beta} \tag{2}$$

Here, α is the intercept, and β's are the Poisson regression coefficients.

For subject i,

$$ln(\mu_i) = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_{10} X_{10i} = x_i'\beta \quad (3)$$

where, $x' = [1 \quad X_1 \quad X_2 \ldots \quad X_{10}]$ (4)

Since the variable CEB represents a count and all observations are greater than zero, this study aimed to develop a regression model using the zero-truncated Poisson and zero-truncated generalized Poisson models [12].

2.1.1 Zero Truncated Poisson (ZTP) Model

For the Poisson distribution with the probability mass function (pmf) (1), the pmf for ZTP distribution is given by

$$P(Y = y|Y > 0) = \frac{\mu^y}{y![e^{\mu}-1]} \quad ; for\ \mu > 0, y = 1,2, \ldots. \quad (5)$$

The mean and variance of the ZTP random variable are as follows:

$$E(Y_i) = \frac{\mu e^{\mu_i}}{e^{\mu_i}-1}, \quad (6)$$

$$V(Y_i) = \frac{\mu e^{\mu_i}}{e^{\mu_i}-1}\left[1 - \left(\frac{\mu e^{\mu_i}}{e^{\mu_i}-1}\right)\right] \quad (7)$$

2.1.2 Zero Truncated Generalized Poisson (ZTGP) Model

Let a count response Yi ~ GP(μ,α); i =1, 2, …, n, then Yi has a probability function [29]:

$$f(y_i; \mu_i, \alpha|Y > 0) = \frac{1}{\left[exp\left(-\frac{\mu_i}{1+\alpha\mu_i}\right)-1\right]}\left[\frac{\mu_i}{1+\alpha\mu_i}\right]^{y_i}\frac{(1+\alpha y_i)^{y_i-1}}{y_i!}exp\left(-\frac{\alpha\mu_i y_i}{1+\alpha\mu_i}\right) \quad (8)$$

2.2 Accessing model adequacy and model comparisons

Subsequently, the loglikelihood, Akaike Information Criterion (AIC) [9], and Bayesian Information Criterion (BIC) were compared across all models to assess and choose the most appropriate model. The statistical tests were conducted using the professional statistical program R [32] and SPSS. The final model for the analysis was selected based on the greater loglikelihood and the minimum information criteria value [24].

III. Results

Table 1: *Descriptive statistics of the number of CEB*

Variable	N	Mean	Variance	Minimum	Maximum
CEB	8087	2.203	0.601	1	4

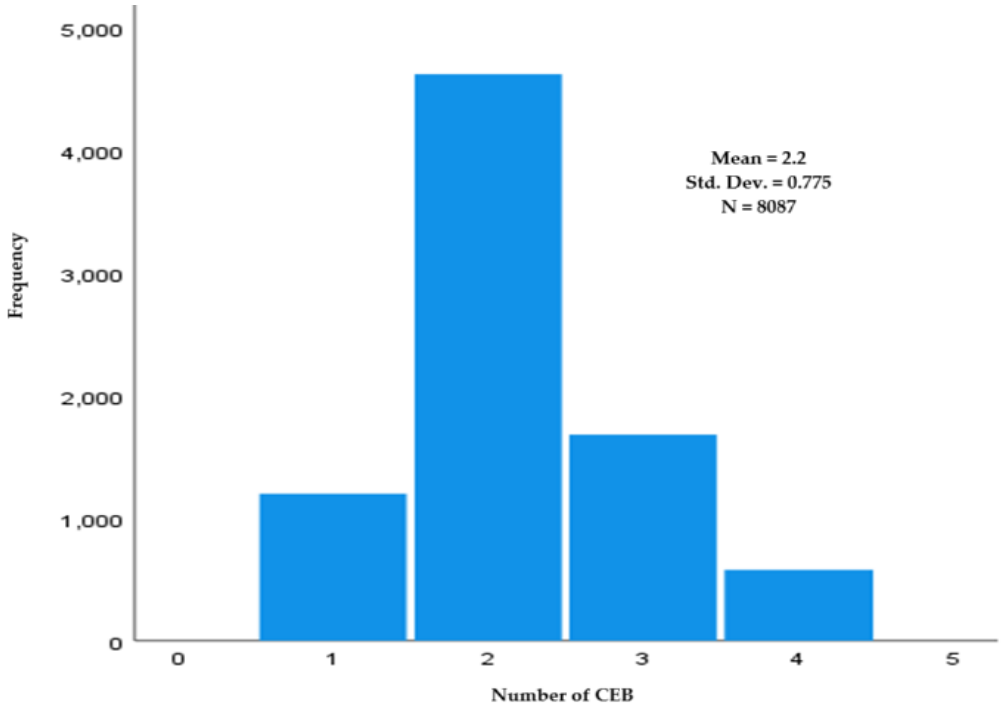

Figure 1: *Histogram of the Number of CEB*

Table 1 displays the descriptive statistics for the count of CEB, which is used as the response variable. The quantity of CEBs varies between 1 and 4. The dataset consisted of 8087 observations. The mean and variance of the number of CEB were calculated to be 2.203 and 0.601, respectively. These values indicate the data set exhibits under-dispersion. Figure 1 shows that the smallest number of children was 1, whereas the maximum number was 4.

Table 2: *Frequency distribution of CEB*

CEB	1	2	3	4
Frequency	1200	4627	1682	578
Percent	11.4	44.0	16.0	5.5

Table 2 displays the number of CEBs, their frequencies, and corresponding percentages. According to the table, 44% of women had two children, the most common number. In addition, 16% of women had three children, the second-highest percentage. Moreover, it is evident that in AP, the population of women with 2-3 children exceeded those with only one child and more than four children.

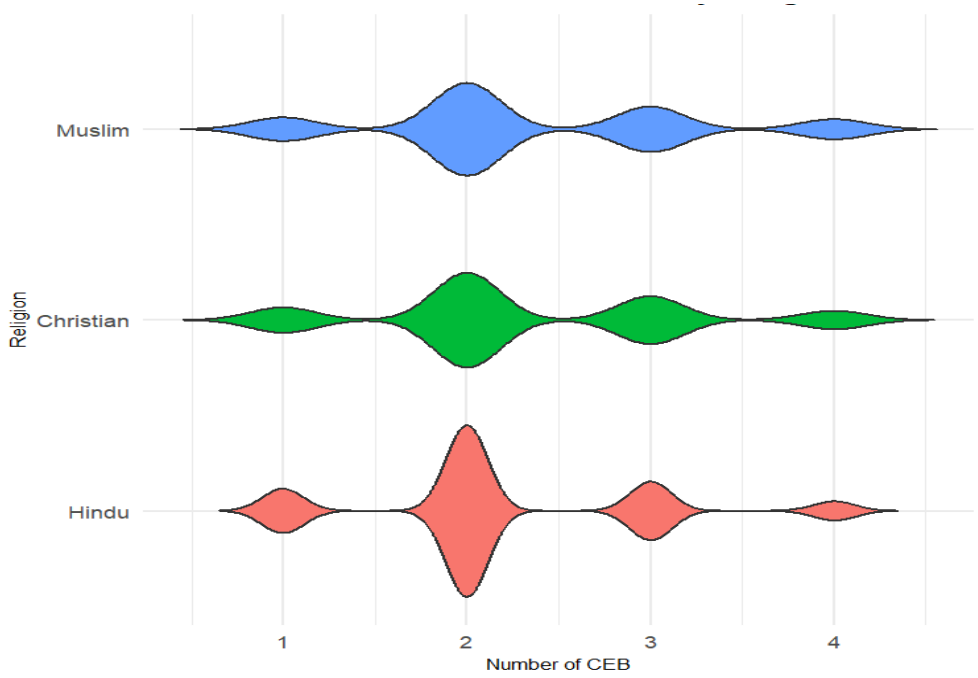

Figure 2: *Distribution of Number of CEB by Religion*

Figure 2 shows the distribution of the number of CEB among various religious groups. Most women had two children, corresponding to the maximum width of each violin plot, emphasizing the disparities in fertility rates among different religious groups.

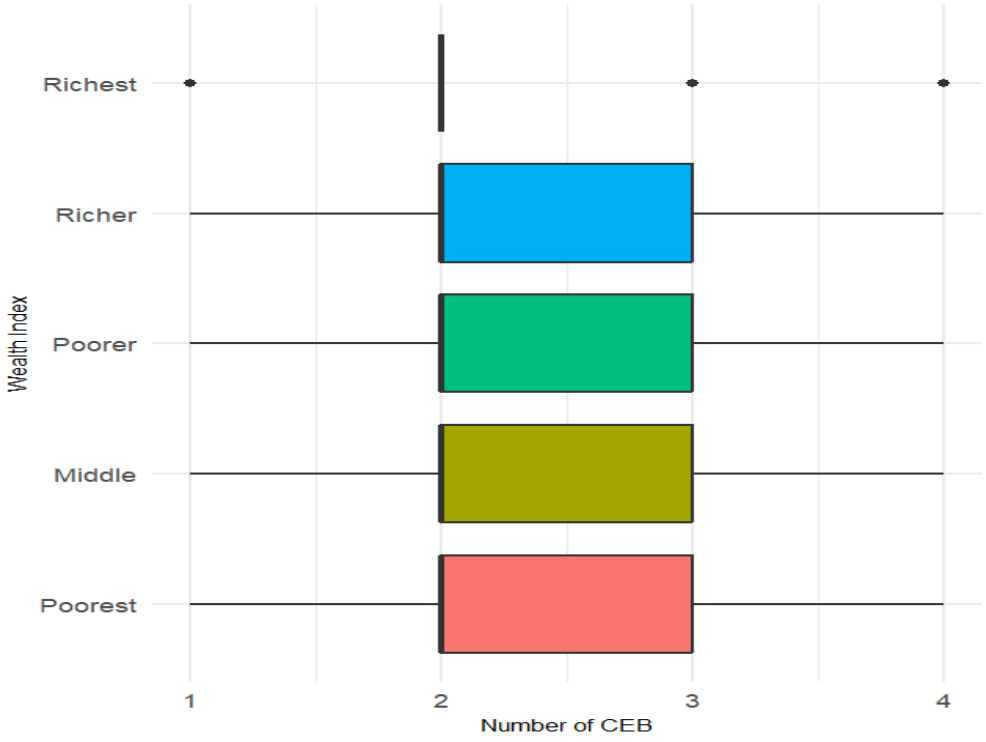

Figure 3: *Distribution of Number of CEB by Wealth Index*

Figure 3 demonstrates the relationship between the number of CEB and the wealth index. The median number of children decreases as we move from poorer to wealthier households.

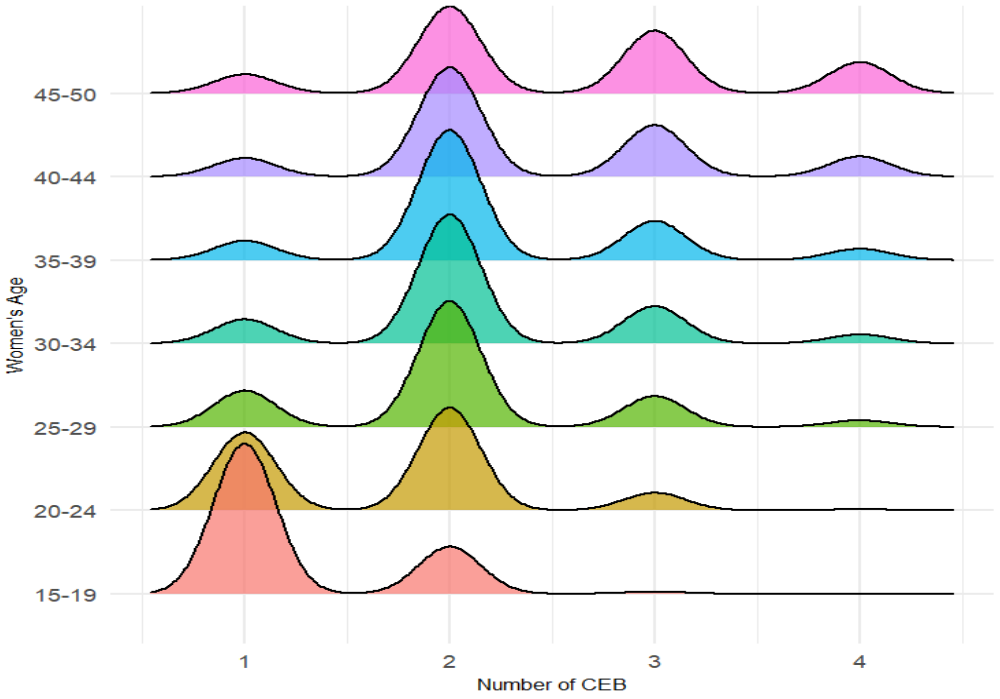

Figure 4: *Distribution of Number of CEB by Women's Age*

Figure 4 shows the distribution of the number of CEB across different age groups of women. It provides a clear visual representation of how fertility patterns differ by age, with older women generally having more children than younger women.

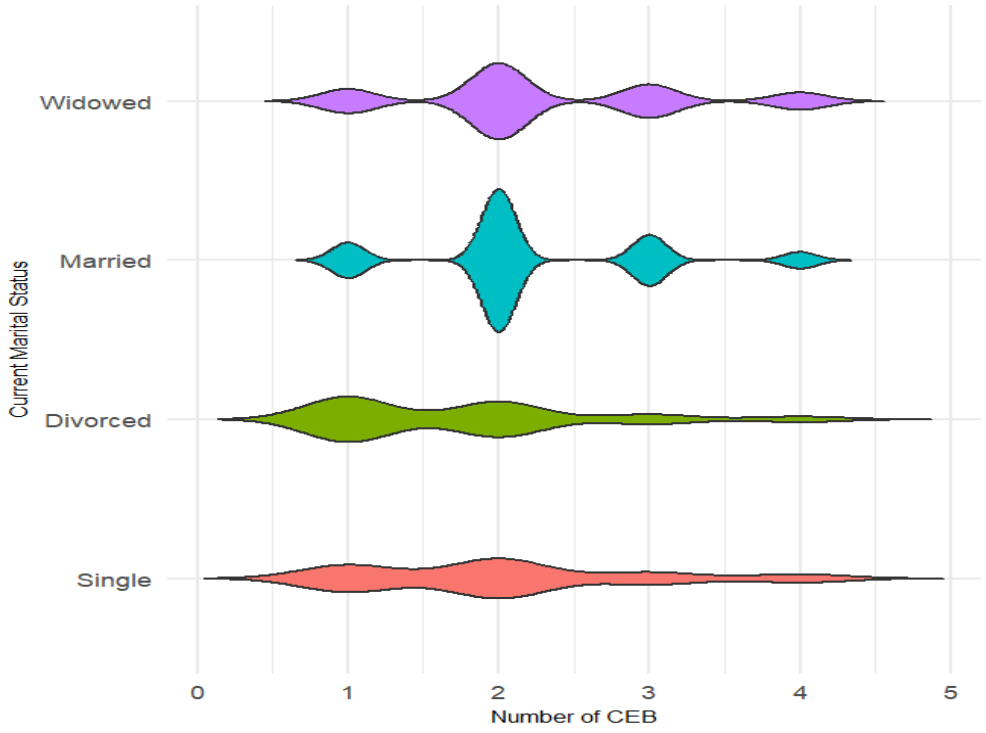

Figure 5: *Distribution of Number of CEB by Current Marital Status*

Figure 5 depicts the distribution of the number of children by current marital status. It shows the differences in fertility rates among single, married, divorced, and widowed women.

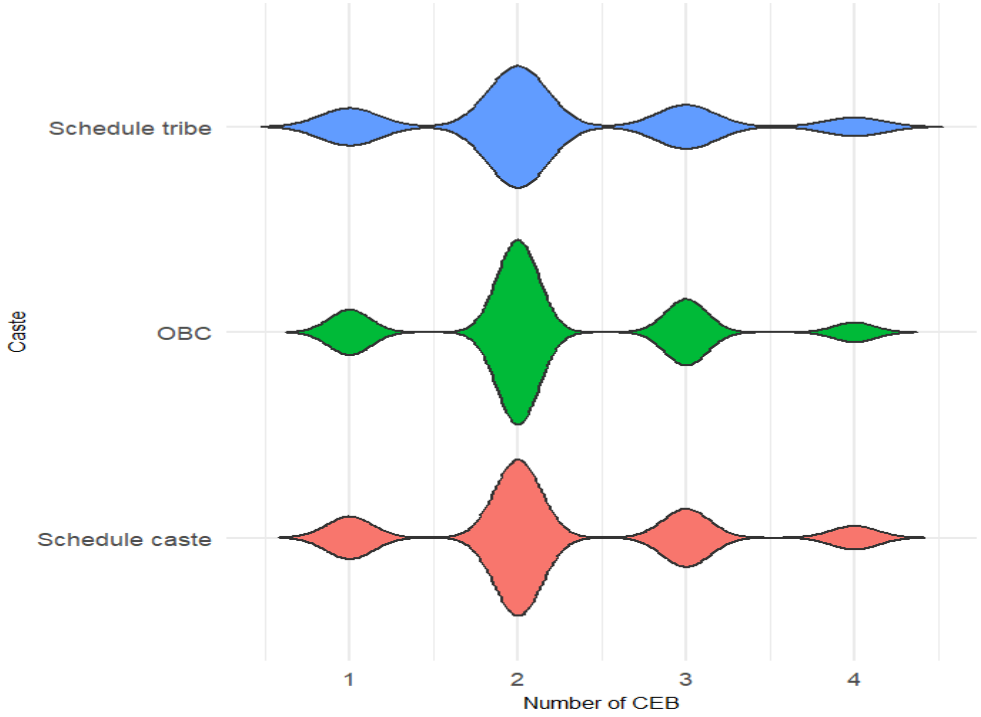

Figure 6: *Distribution of Number of CEB by Caste*

Figure 6 shows the violin plot of the distribution of the number of CEB within each caste group. Most women had two children in various caste groups, providing a clear picture.

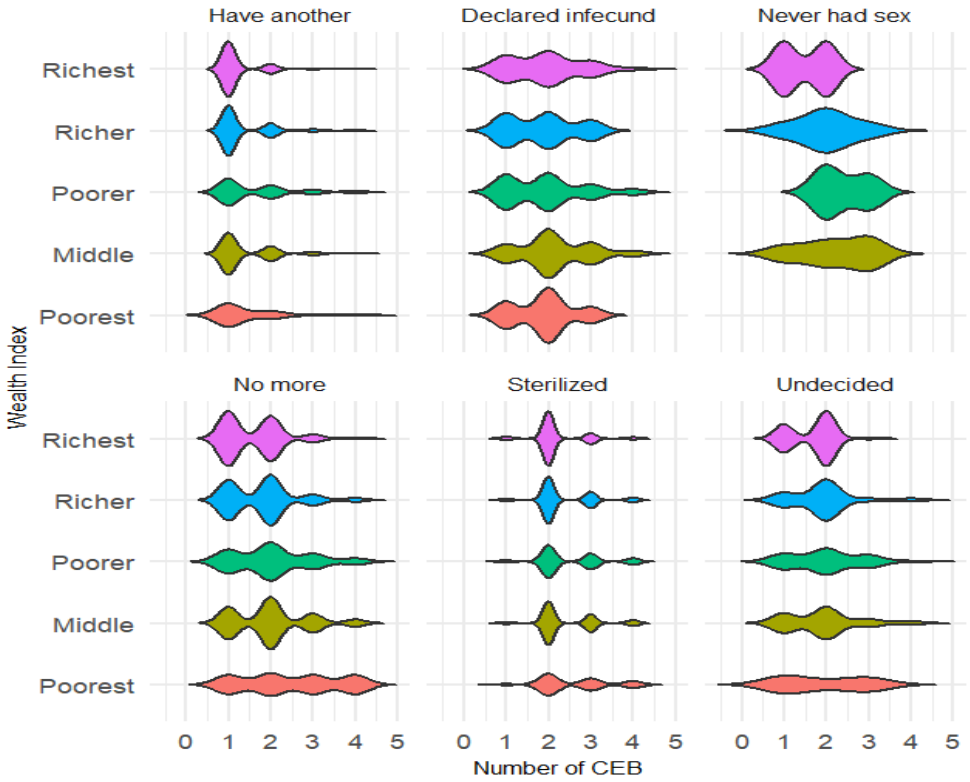

Figure 7: *Violin Plot of Number of CEB by Wealth Index and Fertility Preference*

Figure 7 visualizes the flow of the number of children across different wealth index categories and fertility preferences. It helps to understand how different wealth categories and fertility attitudes influence family size, with the flow's thickness indicating the transition's magnitude.

Table 3: *Overall model comparison by model fit characteristics*

Test Statistics	ZTP	ZTGP
Log Likelihood	-10191.22	-11520.35
AIC	20462.45	23122.7
BIC	20742.37	23409.61

Table 3 clearly shows that the loglikelihood of the ZTP model (-10191.22) is greater than that of the ZTGP model (-11520.35). According to this study, the ZTP model provides a more accurate fit for the data. The Akaike Information Criterion (AIC) of the Zero-Truncated Poisson (ZTP) model (20462.45) is lower than that of the Zero-Truncated Generalized Poisson (ZTGP) model (23122.7), indicating that the ZTP model is a better fit and has less complexity. The BIC value for the ZTP model (20742.37) is lower than the BIC value for the ZTGP model (23409.61), indicating that the ZTP model outperforms the ZTGP model in terms of BIC. BIC is known for imposing greater penalties on model complexity than AIC. Thus, compared to the ZTGP model, the ZTP model outperforms it on these criteria.

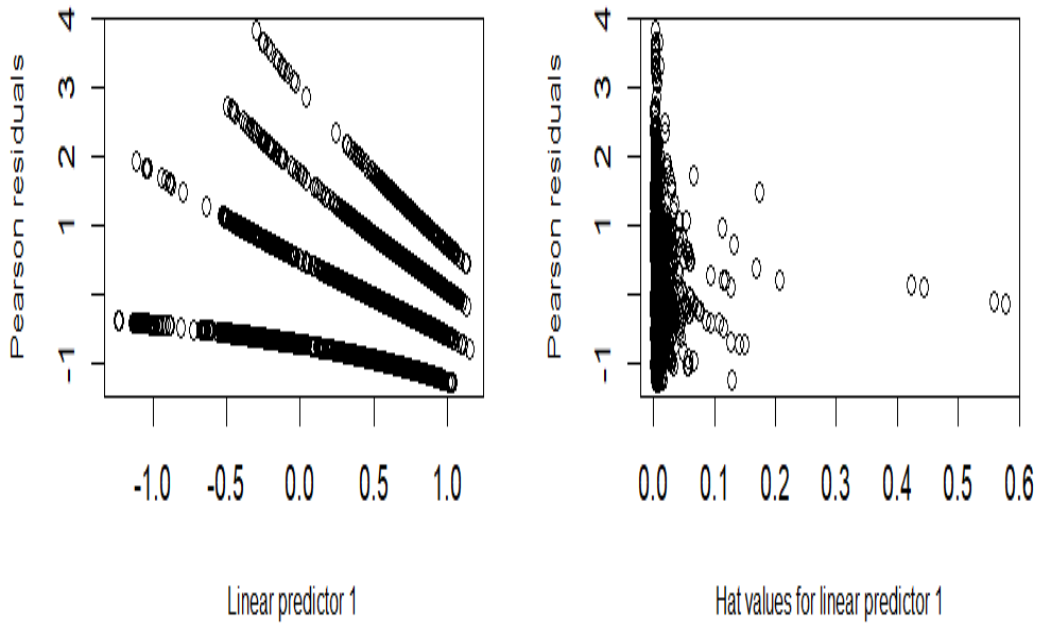

Figure 8: *Linear Predictor of ZTP model*

Table 4: *Sociodemographic, socioeconomic, and environmental factors affecting the number of children born among reproductive-aged women in AP; data from NFHS-5*

Variables	Category	N	Percentage
Place of residence	Urban	2235	27.6
	Rural	5852	72.4
Religion	Hindu	6780	83.8
	Muslim	615	7.6
	Christian	692	8.6

Type of cooking fuel	Electricity	57	0.7
	LPG	6692	82.8
	Biogas	7	0.1
	Kerosene	11	0.1
	Coal, lignite	16	0.2
	Charcoal	102	1.3
	Wood	1104	13.7
	Straw/shrubs/grass	35	0.4
	Agricultural crop	59	0.7
	Animal dung	2	0
	Other	2	0
Wealth index combined	Poorest	336	4.2
	Poorer	1556	19.2
	Middle	2633	32.6
	Richer	2380	29.4
	Richest	1182	14.6
Place of delivery	Home	3551	43.9
	Public	3925	48.5
	Private	611	7.6
Women age	15-19	93	1.1
	20-24	703	8.7
	25-29	1438	17.8
	30-34	1417	17.5
	35-39	1583	19.6
	40-44	1310	16.2
	45-50	1543	19.1
Current marital status	Single	42	0.5
	Married	7340	90.8
	Widowed	637	7.9
	Divorced	68	0.8
Fertility preference	Have another	859	10.6
	Undecided	182	2.3
	No more	651	8.0
	Sterilized	6146	76.0
	Declared infecund	228	2.8
	Never had sex	21	0.3
Caste	Schedule caste	2579	31.9
	Schedule tribe	801	9.9
	OBC	4707	58.2
Husband age	18-27	1667	20.6
	28-37	1513	18.7
	38-47	1684	20.8
	48-57	1699	21.0
	58 & above	1524	18.8
	Total	10522	100

Table 4 shows the percentage distribution of women in AP and provides an overview of the demographic and socioeconomic parameters that affect the number of CEBs. 72.4% of women live in rural areas, while 27.6% stay in urban areas. Most women, specifically 83.8%, belong to the Hinduism religion. The majority of respondents, approximately 82.8%, utilized LPG as their cooking fuel. However, 13.7% of respondents used wood, and 1.3% used charcoal. According to the wealth index, 29.4% of respondents belong to the wealthier category, while only 4.2% fall into the poorest category. Public facilities account for most deliveries, with 48.5%, followed by house deliveries at 43.9%. 19.6% of the participants fell within the age range of 35-39 years, while a close second was the group of respondents aged between 45-50 years, accounting for 19.1%.

Almost 90.8% of the marital status occurred by women who were married. Approximately 76% of the women have undergone sterilization, reflecting their lack of desire to have any more children. The percentage of respondents from Other Backward Classes (OBC) was 58.2%, while the percentage of respondents from Scheduled Tribes was 9.9%. The age distribution of husbands exhibits distinct age categories, with the largest proportion (21%) lying within the range of 48-57 years, followed by 20.8% coming within the range of 38-47 years.

Table 5: *Cross-tabulation of predictor factors with AP's (n=8087) children ever born count among women in their reproductive years*

Variables	Response	1	2	3	4
Place of residence	Urban	371 (30.9)	1353 (29.2)	367 (21.8)	144 (24.9)
	Rural	829 (69.1)	3274 (70.8)	1315 (78.2)	434 (75.1)
Religion	Hindu	1026 (85.5)	3962 (85.6)	1350 (80.3)	442 (76.5)
	Muslim	80 (6.7)	313 (6.8)	154 (9.2)	68 (11.8)
	Christian	94 (7.8)	352 (7.6)	178 (10.6)	68 (11.8)
Type of cooking fuel	Electricity	7 (0.6)	33 (0.7)	12 (0.7)	5 (0.9)
	LPG	1005 (83.8)	3913 (84.6)	1341 (79.7)	433 (74.9)
	Biogas	4 (0.3)	2 (0.0)	1 (0.1)	0 (0.0)
	Kerosene	3 (0.3)	3 (0.1)	3 (0.2)	2 (0.3)
	Coal, lignite	2 (0.2)	11 (0.2)	2 (0.1)	1 (0.2)
	Charcoal	14 (1.2)	45 (1.0)	28 (1.7)	15 (2.6)
	Wood	156 (13.0)	561 (12.1)	272 (16.2)	115 (19.9)
	Straw/shrubs/grass	2 (0.2)	21 (0.5)	8 (0.5)	4 (0.7)
	Agricultural crop	7 (0.6)	34 (0.7)	15 (0.9)	3 (0.5)

	Animal dung	0 (0.0)	2 (0.0)	0 (0.0)	0 (0.0)
	Other	0 (0.0)	2 (0.0)	0 (0.0)	0 (0.0)
Wealth index combined	Poorest	53 (4.4)	150 (3.2)	85 (5.1)	48 (8.3)
	Poorer	217 (18.1)	804 (17.4)	379 (22.5)	156 (27.0)
	Middle	344 (28.7)	1500 (32.4)	590 (35.1)	199 (34.4)
	Richer	358 (29.8)	1405 (30.4)	476 (28.3)	141 (24.4)
	Richest	228 (19.0)	768 (16.6)	152 (9.0)	34 (5.9)
Place of delivery	Home	523 (43.6)	2051 (44.3)	712 (42.3)	265 (45.8)
	Public	589 (49.1)	2227 (48.1)	838 (49.8)	271 (46.9)
	Private	88 (7.3)	349 (7.5)	132 (7.8)	42 (7.3)
Women age	15-19	70 (5.8)	22 (0.5)	1 (0.1)	0 (0.0)
	20-24	275 (22.9)	362 (7.8)	62 (3.7)	4 (0.7)
	25-29	262 (21.8)	907 (19.6)	222 (13.2)	47 (8.1)
	30-34	172 (14.3)	916 (19.8)	265 (15.8)	64 (11.1)
	35-39	153 (12.8)	1031 (22.3)	310 (18.4)	89 (15.4)
	40-44	122 (10.2)	717 (15.5)	338 (20.1)	133 (23.0)
	45-50	146 (12.2)	672 (14.5)	484 (28.8)	241 (41.7)
Current marital status	Single	13 (1.1)	19 (0.4)	6 (0.4)	4 (0.7)
	Married	1053 (87.8)	4262 (92.1)	1528 (90.8)	497 (86.0)
	Widowed	102 (8.5)	321 (6.9)	141 (8.4)	73 (12.6)
	Divorced	32 (2.7)	25 (0.5)	7 (0.4)	4 (0.7)
Fertility preference	Have another	602 (50.2)	196 (4.2)	46 (2.7)	15 (2.6)
	Undecided	55 (4.6)	98 (2.1)	23 (1.4)	6 (1.0)
	No more	226 (18.8)	291 (6.3)	94 (5.6)	40 (6.9)

	Sterilized	242 (20.2)	3930 (84.9)	1466 (87.2)	508 (87.9)
	Declared infecund	71 (5.9)	102 (2.2)	46 (2.7)	9 (1.6)
	Never had sex	4 (0.3)	10 (0.2)	7 (0.4)	0 (0.0)
Caste	Schedule caste	391 (32.6)	1436 (31.0)	535 (31.8)	217 (37.5)
	Schedule tribe	135 (11.3)	440 (9.5)	159 (9.5)	67 (11.6)
	OBC	674 (56.2)	2751 (59.5)	988 (58.7)	294 (50.9)
Husband age	18-27	251 (20.9)	966 (20.9)	337 (20.0)	113 (19.6)
	28-37	202 (16.8)	875 (18.9)	333 (19.8)	103 (17.8)
	38-47	265 (22.1)	954 (20.6)	345 (20.5)	120 (20.8)
	48-57	262 (21.8)	959 (20.7)	353 (21.0)	125 (21.6)
	58 & above	220 (18.3)	873 (18.9)	314 (18.7)	117 (20.2)

According to the data in Table 5, rural women have a greater percentage of larger families. Specifically, 78.2% of rural women have three children, while 75.1% have four or more children. Meanwhile, only 21.8% of urban women have three children, and 24.9% have four or more children. According to this study, living in rural areas is linked to greater fertility rates. Among all religious groups, Hindu women have the greatest percentage in all categories of childbearing age. Muslim women exhibit a greater proportion in the higher CEB categories, specifically 9.2% (154) for three children and 11.8% (68) for four or more children, in comparison to Christian women who had 10.6% (178) for three children and 11.8% (68) for four or more children.

Among women, 84.6% use LPG as their main cooking fuel, which affects all CEB groups. Those who use wood as fuel have greater percentages in bigger family sizes, with 16.2% having three children and 19.9% having four or more children. Women with two children have a greater proportion (22.3%) in the CEB categories for the age group of 35-39 years. Among all CEB categories, married women had the greatest percentage, specifically 92.1% (4262). The majority of women who undergo sterilization belong to the higher CEB groups, with 87.2% having three children and 87.9% having four or more children. Most women in the Other Backward Classes (OBC) in all Central Employment Bureau (CEB) categories have higher percentages. The spouses of these women are between the ages of 38-47 and 48-57. Additionally, more women with two children (954, 20.6%) and (959, 20.7%) fall into this category.

Table 6: *Results from the ZTP Model*

Variables	Category	Estimate	Std. Error	z-value	P-value	IRR	95% Wald Confidence Interval IRR	
Intercept		-1.088	0.271	-4.012	0.000***	0.337	0.198	0.573
Place of residence (Ref: Urban)	Rural	0.003	0.023	0.117	0.907@	1.003	0.958	1.050
Religion (Ref: Hindu)	Muslim	0.149	0.034	4.410	0.000***	1.161	1.087	1.241
	Christian	0.064	0.034	1.892	0.059*	1.066	0.998	1.139
Type of cooking fuel (Ref: Electricity)	LPG	-0.039	0.108	-0.364	0.716@	0.961	0.778	1.188
	Biogas	-0.619	0.481	-1.287	0.198@	0.539	0.210	1.382
	Kerosene	0.169	0.258	0.654	0.513@	1.184	0.714	1.963
	Coal, lignite	-0.134	0.241	-0.557	0.578@	0.874	0.545	1.402
	Charcoal	0.048	0.132	0.366	0.714@	1.049	0.811	1.359
	Wood	-0.001	0.111	-0.010	0.992@	0.999	0.804	1.241
	Straw/shrubs/grass	0.042	0.169	0.248	0.804@	1.043	0.749	1.452
	Agricultural crop	-0.036	0.152	-0.240	0.811@	0.964	0.716	1.298
	Animal dung	-0.140	0.658	-0.213	0.831@	0.869	0.239	3.159
	Other	-0.360	0.659	-0.546	0.585@	0.698	0.192	2.538
Wealth index combined (Ref: Poorest)	Poorer	-0.071	0.048	-1.477	0.140@	0.931	0.848	1.024
	Middle	-0.120	0.049	-2.461	0.014**	0.887	0.806	0.976
	Richer	-0.180	0.051	-3.517	0.000***	0.835	0.756	0.923
	Richest	-0.319	0.057	-5.628	0.000***	0.727	0.650	0.812
Place of delivery (Ref: Home)	Public	-0.014	0.019	-0.751	0.453@	0.986	0.949	1.024
	Private	0.003	0.036	0.072	0.943@	1.003	0.934	1.077
Women age (Ref: 15-19)	20-24	0.603	0.202	2.990	0.003***	1.827	1.231	2.711
	25-29	0.755	0.199	3.787	0.000***	2.128	1.440	3.146
	30-34	0.776	0.200	3.886	0.000***	2.173	1.469	3.214
	35-39	0.809	0.200	4.053	0.000***	2.246	1.519	3.321
	40-44	0.905	0.200	4.531	0.000***	2.472	1.671	3.657
	45-50	1.001	0.200	5.016	0.000***	2.721	1.840	4.023
Current marital status (Ref: Single)	Married	0.215	0.140	1.531	0.126@	1.240	0.942	1.633
	Widowed	0.142	0.143	0.989	0.323@	1.152	0.870	1.527
	Divorced	-0.208	0.191	-1.090	0.276@	0.812	0.559	1.181
Fertility preference (Ref: Have another)	Undecided	0.625	0.091	6.898	0.000***	1.868	1.564	2.230
	No more	0.654	0.066	9.971	0.000***	1.923	1.691	2.186
	Sterilized	0.934	0.056	16.809	0.000***	2.544	2.282	2.837
	Declared infecund	0.609	0.083	7.337	0.000***	1.838	1.562	2.163
	Never had sex	0.763	0.195	3.912	0.000***	2.144	1.463	3.141
Caste (Ref:	Schedule tribe	-0.042	0.035	-1.194	0.232@	0.959	0.896	1.027

Schedule caste)	OBC	-0.030	0.022	-1.405	0.160@	0.970	0.930	1.012
Husband age (Ref: 18-27)	28-37	0.021	0.030	0.713	0.476@	1.021	0.964	1.082
	38-47	0.017	0.029	0.575	0.565@	1.017	0.961	1.076
	48-57	0.018	0.029	0.641	0.522@	1.019	0.963	1.078
	58 & above	0.023	0.030	0.781	0.435@	1.023	0.966	1.084

***1% Level of Significant (p-value<0.01)

** 5% Level of Significant (p-value<0.05)

* 10% Level of Significant (p-value<0.1)

@ Not Significant

Table 6 displays the Zero Truncated Poisson (ZTP) model, which analyzes the factors that influence the number of Children Ever Born (CEB) among women of reproductive age. The results indicate that the Muslim faith has a favourable effect, whereas income index categories (excluding poorer women) have a negative effect. A positive relationship exists between women's age (from 20-24 to 45-50) and fertility preference. Women who used Biogas as a cooking fuel observed a 46% lower risk (IRR = 0.539, 95% CI: 0.210-1.382), whereas those who used kerosene had an 18% greater risk (IRR = 1.184, 95% CI: 0.714-1.963), in comparison to women who depended on electricity. Women from lower socioeconomic backgrounds had a 6.9% lower risk (IRR = 1.02, 95% CI: 0.97-1.024) compared to women who give birth at home.

Women aged 20-24, 25-29, 30-34, 35-39, 40-44, and 45-50 had significantly higher incidences compared to women aged 15-19, with incidence rate ratios (IRR) of 1.827 (95% CI: 1.231-2.711), 2.128 (95% CI: 1.440-3.146), 2.173 (95% CI: 1.469-3.214), 2.246 (95% CI: 1.519-3.321), 2.472 (95% CI: 1.671-3.657), and 2.721 (95% CI: 1.840-4.023), respectively. Married women had a 24% decrease in risk (IRR = 1.240, 95% CI: 0.942-1.633) compared to single women. Divorced women had an 18.8% decrease in risk (IRR = 0.812, 95% CI: 0.559-1.181) compared to single women.

Women who have had sterilization have a 1.544 times greater risk (IRR = 2.544, 95% CI: 2.282-2.837) compared to women who have given birth to another child. Women belonging to the Scheduled Caste had a 4.1% reduced risk of fertility compared to women belonging to the Scheduled Tribe caste, with an incidence rate ratio (IRR = 0.959, 95% CI: 0.896-1.027). Respondents whose husbands were aged 58 and above and 28-37 had a 2.3% (IRR = 1.023, 95% CI: 0.966-1.084) and 2.1% (IRR = 1.021, 95% CI: 0.964-1.082) higher risk of fertility compared to respondents whose husbands were aged 18-27.

Table 7: *Results from the ZTGP Model*

Variables	Category	Estimate	Std. Error	z-value	P-value	IRR	95% Wald Confidence Interval IRR	
Intercept1		0.089	0.175	0.507	0.612@	1.093	0.776	1.538
Intercept2		-30.490	8.245	-0.004	0.997@	0.000	0.000	Inf
Place of residence (Ref: Urban)	Rural	0.002	0.019	0.088	0.930@	1.002	0.965	1.040
Religion (Ref: Hindu)	Muslim	0.099	0.028	3.531	0.000***	1.104	1.045	1.167
	Christian	0.044	0.028	1.578	0.115@	1.045	0.989	1.104
Type of cooking fuel (Ref: Electricity)	LPG	-0.026	0.089	-0.288	0.773@	0.975	0.819	1.160
	Biogas	-0.330	0.315	-1.047	0.295@	0.719	0.388	1.333
	Kerosene	0.114	0.216	0.531	0.596@	1.121	0.735	1.710
	Coal, lignite	-0.088	0.193	-0.456	0.649@	0.916	0.627	1.337

	Charcoal	0.036	0.109	0.331	0.741@	1.037	0.837	1.285
	Wood	0.000	0.091	0.000	1.000@	1.000	0.836	1.196
	Straw/shrubs/grass	0.028	0.141	0.202	0.840@	1.029	0.781	1.356
	Agricultural crop	-0.024	0.124	-0.196	0.845@	0.976	0.765	1.245
	Animal dung	-0.088	0.508	-0.174	0.862@	0.915	0.338	2.480
	Other	-0.233	0.509	-0.459	0.647@	0.792	0.292	2.146
Wealth index combined (Ref: Poorest)	Poorer	-0.048	0.040	-1.199	0.231@	0.953	0.881	1.031
	Middle	-0.080	0.041	-1.970	0.049**	0.923	0.852	1.000
	Richer	-0.120	0.043	-2.822	0.005***	0.887	0.816	0.964
	Richest	-0.207	0.047	-4.450	0.000***	0.813	0.742	0.891
Place of delivery (Ref: Home)	Public	-0.010	0.016	-0.639	0.523@	0.990	0.960	1.021
	Private	0.001	0.030	0.032	0.975@	1.001	0.945	1.061
Women age (Ref: 15-19)	20-24	0.169	0.098	1.735	0.083*	1.184	0.978	1.434
	25-29	0.250	0.096	2.605	0.009**	1.284	1.064	1.550
	30-34	0.263	0.096	2.730	0.006**	1.301	1.077	1.572
	35-39	0.286	0.096	2.961	0.003***	1.330	1.101	1.607
	40-44	0.352	0.097	3.633	0.000***	1.421	1.176	1.718
	45-50	0.420	0.097	4.355	0.000***	1.523	1.260	1.840
Current marital status (Ref: Single)	Married	0.137	0.109	1.256	0.209@	1.147	0.926	1.420
	Widowed	0.088	0.112	0.784	0.433@	1.092	0.877	1.359
	Divorced	-0.107	0.142	-0.750	0.453@	0.899	0.680	1.188
Fertility preference (Ref: Have another)	Undecided	0.251	0.062	4.019	0.000***	1.285	1.137	1.452
	No more	0.268	0.042	6.420	0.000***	1.307	1.204	1.418
	Sterilized	0.446	0.033	13.607	0.000***	1.561	1.464	1.665
	Declared infecund	0.236	0.057	4.112	0.000***	1.266	1.131	1.416
	Never had sex	0.330	0.153	2.154	0.031**	1.390	1.030	1.877
Caste (Ref: Schedule caste)	Schedule tribe	-0.028	0.028	-0.982	0.326@	0.973	0.920	1.028
	OBC	-0.020	0.018	-1.116	0.264@	0.981	0.947	1.015
Husband age (Ref: 18-27)	28-37	0.014	0.024	0.574	0.566@	1.014	0.967	1.063
	38-47	0.011	0.023	0.462	0.644@	1.011	0.966	1.058
	48-57	0.012	0.023	0.517	0.605@	1.012	0.967	1.060
	58 & above	0.015	0.024	0.612	0.541@	1.015	0.968	1.064

***1% Level of Significant (p-value<0.01)

** 5% Level of Significant (p-value<0.05)

* 10% Level of Significant (p-value<0.1)

@ Not Significant

The analysis of the ZTGP model in Table 7 provides the impact of various determinants on fertility count, specifically focusing on the number of CEB. The results indicate that factors such as the positive influence of the Muslim faith, the negative impact of being richer or richest women, the positive influence of women's age ranging from 25-29 to 45-50, and the positive influence of

fertility preference categories are all significant.

The occurrence rates of CEB (Childbearing Ever) increased by 10.4% (IRR = 1.104, 95% CI:1.045-1.167) and 4.5% (IRR = 1.045, 95% CI:0.989-1.104) for women who held Muslim and Christian beliefs, respectively, compared to those who practis loglikelihooded Hinduism. Women who use biogas and other fuel sources observe a 28% decrease in the occurrence of CEB (IRR = 0.719, 95% CI:0.388-1.333) and a 21% reduction (IRR = 0.792, 95% CI:0.292-2.146) compared to women who depend on electricity. The wealthiest women exhibited a reduced rate of childbearing (IRR = 0.813, 95% CI: 0.742-0.891) compared to the poorest women.

In comparison to women aged 15-19 years, the rates of the number of CEB among women aged 20-24, 25-29, 30-34, 35-39, 40-44, and 45-50 increased by 18%, 28%, 30%, 33%, 42%, and 52% respectively. Compared to single women, divorced women had a 10% reduced incidence risk of CEB (IRR = 0.899, 95% CI: 0.680-1.188). Similarly, the incidence rates of CEB among sterilized women increased by 56% (IRR = 1.561, 95% CI:1.464-1.665) in comparison to women with different fertility preferences. The incidence rate of CEB among women who belong to the Scheduled Tribe was reduced by 3% (IRR = 0.973, 95% CI: 0.920-1.028) compared to women from the Scheduled Caste. Similarly, women from the Other Backward Classes (OBC) had a 2% decrease in the incidence rate of CEB (IRR = 0.981, 95% CI: 0.947-1.015) compared to women from the Scheduled Caste.

IV. Discussion

This study included 8087 women between the ages of 15 and 50. The majority of these women, representing 44%, had given birth to two children. This study utilized the ZTP and ZTGP regression models to examine reproductive patterns in Andhra Pradesh. The analysis focused on the number of CEBs, which was influenced by several variables, including various sociodemographic, socioeconomic, and environmental factors.

The study found that a significant proportion of women (72.4%) live in rural areas, while a smaller proportion (27.6%) live in urban areas. The generality of the rural regions is associated with higher fertility rates, as women living in rural areas tend to have bigger families due to the limited availability of family planning services and distinct socio-cultural norms. Hinduism is the most prevalent religion, with 83.8% of women identifying as Hindus. There is variation in fertility rates among different religious groups, with Muslim women having greater fertility rates in higher CEB categories compared to Hindu and Christian women. The majority of women (82.8%) utilize LPG as their primary cooking fuel, although traditional fuels such as wood (13.7%) and charcoal (1.3%) are less prevalent. The utilization of traditional fuels has been associated with higher household sizes, most likely due to socioeconomic limitations and lifestyle aspects in lower-income households.

According to the wealth index, 29.4% of respondents are classified as rich, while only 4.2% are categorized as the poorest. Women with higher wealth tend to have fewer children, indicating the negative correlation between economic status and fertility rates. Public facilities are the most common places for delivery (48.5%), followed by home deliveries (43.9%). The selection of the birth location is impacted by factors such as ease of access, cost-effectiveness, and cultural inclinations, which subsequently affect the health results of both mothers and children. The age distribution reveals that 19.6% of participants fall within the age range of 35-39 years, while 19.1% fall within the age range of 45-50 years. A majority (90.8%) of women are married, and this is strongly correlated with reproductive patterns since married women tend to have greater rates of fertility. The prevalence of sterilization is high, with 76% of women undergoing the procedure, which indicates a clear and final decision to cease childbearing. The caste division reveals that 58.2% of the participants belong to the Other Backward Classes (OBC), while 9.9% are from

scheduled tribes, thus highlighting the socio-cultural diversity within the community.

Rural women exhibit higher fertility rates, with 78.2% having three children and 75.1% having four or more children, compared to 21.8% and 24.9% for urban women. This study highlights the impact of living in rural areas on increased fertility rates. Hindu women make up the largest proportion of all categories of CEB. In contrast, Muslim women have a greater proportion in the upper CEB categories (9.2% for three children and 11.8% for four or more children) compared to Christian women. Women who use LPG have a significant presence in all CEB categories. However, those that depend on traditional fuels such as wood tend to have higher proportions of bigger family sizes, indicating that economic and social factors affect fertility. Women between the ages of 35 and 39 have more children in the CEB categories, while married women make up the majority of all CEB categories with a percentage of 92.1%. Women who have had sterilization tend to have a larger number of CEB, which indicates their previous high fertility before treatment. Women in the Other Backward Classes (OBC) exhibit greater proportions in all CEB categories, suggesting higher fertility rates within this demographic.

The ZTP regression model study identifies significant factors, including the Muslim faith, wealth index, women's age, and fertility preference. Muslim women, women with higher wealth, women in older age groups (20-50), and those who prefer sterilization have higher rates of CEB. This analysis, like the ZTP model, additionally highlights the beneficial impact of the Muslim faith and women's age on CEB. However, the data reveals a more prominent adverse effect of money, as wealthier and wealthiest women have far lower rates of childbearing. The ZTP model indicates that the Muslim faith is positively associated with higher CEB, whereas wealth index categories, except for poorer women, show a negative association. Age is an important variable, as women between 20 and 50 have significantly higher CEB than those in the 15-19 age group. Fertility desire, namely sterilization, is an additional influential factor since sterilized women tend to have a larger number of CEB.

To manage population growth to improve mother and child health outcomes, rural areas with higher fertility rates require focused family planning and reproductive health care. The significant impact of religion on reproductive patterns highlights the necessity for culturally sensitive interventions that respond to the distinct requirements and opinions of various religious communities. The negative relationship between wealth and fertility underscores the significance of economic advancement and education in lowering childbearing rates. There is a relationship between the financial status of women and the number of children they have, indicating that enhancing economic status may result in decreased fertility rates. The strong relationship between age, marital status, and fertility suggests that reproductive health programs must focus on specific age groups and marital statuses to manage fertility rates effectively.

V. Conclusion

This study aimed to examine reproductive patterns and the determinants that influence the number of children ever born (CEB) among women aged 15 to 50 in Andhra Pradesh using count data regression models based on NFHS conducted between 2019-2021. The ZTP regression model most effectively identified the key factors influencing CEB. These factors include the Muslim religion, the wealth index of the richest individuals, women aged between 20 and 50, and fertility preferences based on sterilization. Furthermore, reproductive health programs should be customized to the specific needs of different demographic groups, particularly in rural regions, such as religion, wealth, age, and marital status, to effectively manage population growth and improve the health outcomes of mothers and children. By addressing the specific determinants of fertility identified, policymakers can develop more effective strategies to encourage sustainable population growth and improve the overall health of women and children in the region.

Ethics approval and consent to participate

The Demographic and Health Surveys (DHS) Program permits the authors to download survey data. The publicly available data do not contain personal information.

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AN M/G/1 RETRIAL QUEUE WITH WORKING VACATION, NON PERSISTENT CUSTOMERS AND A WAITING SERVER

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Abstract

An M/G/1 retrial queue with working vacation, non persistent customers and a waiting server is taken into consideration in this study. Both retrial times and service times are assumed to follow general distribution and the waiting server follows an exponential distribution. Before switching over to a vacation the server waits for some arbitrary amount of time and so is called a waiting server. During the working vacation period customers are served at a lesser rate of service. We obtain the PGF for the number of customers and the mean number of customers in the invisible waiting area which is acquired by utilizing the supplementary variable technique. We compute the waiting time distribution. Out of interest a few special cases are conferred. Numerical outcomes are exhibited.

Keywords: Retrial queue, working vacation, supplementary variable technique, non persistent customers, waiting server

1. Introduction

In order to work with queues, we will need some basis on stochastic processes for a detailed deliberated about stochastic models in queueing theory by Medhi [1]. Retrial queues are expressed by the fact that if a customer observes that the server is occupied, then they are entered into the invisible waiting area called an orbit. In recent years, numerous researchers have examined the retrial queue. For a more in-depth analysis of the retrial queues, refer to[2, 3, 4, 5, 6]. In queueing theory, queueing models with server vacation have the most impactful application. Whenever the system becomes empty, the server leaves from the regular service period (RS) and goes on vacation, but in a waiting server model, the server will wait for an arbitrary amount of time before going on vacation. For a detailed study on waiting servers with vacation, debated in [7, 8, 9, 10, 11]. In addition to the vacation strategy, we developed the newest vacation strategy, called Working Vacation (WV). In the WV period, the server provides a lesser rate of service to the customers than during the regular service period. A survey on working vacation in queueing models by Chandrasekaran et al. [12]. The M/M/1 queue with single and multiple working vacations was discussed by [13, 14]. Similarly, the same discussion for M/G/1 queue was done by Wu and Gao [15, 16].

Kalyanaraman and Pazhani Bala Murugan [17] discussed retrial queue with vacation and presented operating characteristics results. Further the same retrial queue with single and multiple working vacations by Pazhani Bala Murugan and Santhi [18, 19]. Murugan and Keerthana [20] conducted a study on the $M/G/1$ retrial queue, which included both multiple working vacation and waiting server. Additionally, the same authors studied a similar problem with feedback in [21].

When a primary customer finds a server busy, the customer becomes unsatisfied and may quit the system without service permanently. For a more in-depth analysis of the non-persistent customers discussed in $M|G|1$ retrial queue with non persistent customers and orbital search by Krishnamoorthy et al. [22], Murugan and Vijaykrishnaraj [23] discussed a bulk arrival retrial queue with nonpersistent customers and exponentially distributed multiple working vacations and presented the results about probability generating function(PGF) for the number of customers in the orbit. Based on the above studies, $M|G|1$ retrial queue with nonpersistent customers and waiting server in working vacation were not discussed. In this article it is assumed that the waiting server has an exponential distribution and that the retrial and service times follow a general distribution. The server is referred to as a waiting server as it waits for a random amount of time before going into a vacation state. Based on that, the waiting time distribution, the performance measures, and some numerical results are discussed.

2. Model Description

We examine an $M/G/1$ retrial queue with WV, non persistent customers and a waiting server where the primary customers arrival follows a Poison process with arrival rate λ. If an approaching customer discovers that the server is occupied then they exit the service area because we assume that there is no waiting area and they joins the orbit. At a service completion instant, if the number of customer is one at the extreme front end of the orbit, is permitted to approach the server with a distribution function $G(x)$ and the retrial time follows a general distribution. For the normal service period, let $g(x)$ and $G^*(\theta)$ signify the distribution function, pdf and LST respectively, and for WV period, let $L(x), l(x), L^*(\theta)$ signify the pdf and LST respectively. On the service completion epoch of each customer, if there is a contest between primary customer and an orbit customer, then it will be determined with $R_s(x)$, $r_s(x)$, $R_s^*(\theta)$ as its distribution function, pdf, LST with general distribution. The service delivered among the WV period follows general distribution with $W_v(x)$, $w_v(x)$, $W_v^*(\theta)$ as its distribution function, pdf, LST. The arriving (or primary) customer receives service instantly if the server is idle. If not, he will choose whether to leave the system without service with probability $(1-\nu)$ or returning again later with probability ν.

The server waits for a arbitrary period of time once the orbit turns empty which follows an (exp.) distribution with rate α. After completion of waiting time the server goes for WV which follows an (exp.) distribution with rate β and Inter-arrival times, retrial periods, RS periods, and WV periods are all presumed to be independent of one another.

Let's use the subsequent random variables
$O(t)$- Size of the orbit at "t",
$R_s^0(t), G^0(t)$- the RST and RRT in RS period,
$W_v^0(t), L^0(t)$- the RST and RRT in WV period.
At time "t" the four distinct states of the server are

$$E(t) = \begin{cases} 0 & \text{if the server is not occupied in WV} \\ 1 & \text{if the server is not occupied in RS period} \\ 2 & \text{if the server is occupied in WV} \\ 3 & \text{if the server is occupied in RS period} \end{cases}$$

To generate bivariate Markov Processs, further variables are introduced that $\{(O(t), B(t)); t \geq 0\}$, where
$B(t) = L^0(t)$, if $E(t) = 0; G^0(t)$, if $E(t) = 1; W_v^0(t)$, if E(t) = 2; $R_s^0(t)$, if E(t) = 3.

$$\begin{aligned} W_{0,0} &= \lim_{t\to\infty} P[O(t) = 0,\ E(t) = 0] \\ R_{0,0} &= \lim_{t\to\infty} P[O(t) = 0,\ E(t) = 1] \\ W_{0,h} &= \lim_{t\to\infty} P[O(t) = h,\ E(t) = 0,\ x < L^0(t) \leq x + dx];\ h \geq 1 \end{aligned}$$

$$
\begin{aligned}
R_{0,h} &= \lim_{t\to\infty} P[O(t)=h,\ E(t)\ =1,\ x < G^0(t)\ \le x+dx];\ h\ge 1\\
W_{1,h} &= \lim_{t\to\infty} P[O(t)=h,\ E(t)\ =2,\ x < W_v^0(t)\ \le x+dx];\ h\ge 0\\
R_{1,h} &= \lim_{t\to\infty} P[O(t)=h,\ E(t)\ =3,\ x < R_s^0(t)\ \le x+dx];\ h\ge 0
\end{aligned}
$$

The above mentioned are the limiting probabilities which we have defined. Let us define common LST and PGF's are $F^*(s)=\int\limits_0^\infty e^{-st}dF(t)$ and $F^*(z,x)=\sum\limits_{h=0}^{\infty} F_h^*(x)z^h$.
In steady state the system was illustrated by the subsequent differential difference equations:

$$\lambda W_{0,0} = W_{1,0}(0)+\alpha R_{0,0} \tag{1}$$

$$-\frac{d}{dx}W_{0,h}(x) = -(\beta+\lambda)W_{0,h}(x)+W_{1,h}(0)l(x);\ \ h\ge 1 \tag{2}$$

$$-\frac{d}{dx}W_{1,0}(x) = -(\beta+\lambda\nu)W_{1,0}(x)+W_{0,1}(0)w_v(x)+\lambda W_{0,0}w_v(x) \tag{3}$$

$$
\begin{aligned}
-\frac{d}{dx}W_{1,h}(x) &= -(\beta+\lambda\nu)W_{1,h}(x)+\lambda\nu W_{1,h-1}(x)+W_{0,h+1}(0)w_v(x)\\
&\quad +\lambda\int\limits_0^\infty W_{0,h}(x)dx\ w_v(x); h\ge 1
\end{aligned} \tag{4}
$$

$$(\lambda+\alpha)R_{0,0} = R_{1,0}(0) \tag{5}$$

$$-\frac{d}{dx}R_{0,h}(x) = -\lambda R_{0,h}(x)+R_{1,h}(0)g(x)+\beta\int\limits_0^\infty W_{0,h}(x)dx\ g(x);\quad h\ge 1 \tag{6}$$

$$-\frac{d}{dx}R_{1,0}(x) = -\lambda\nu R_{1,0}(x)+R_{0,1}(0)r_s(x)+\beta\int\limits_0^\infty W_{1,0}(x)r_s(x)dx \tag{7}$$

$$
\begin{aligned}
-\frac{d}{dx}R_{1,h}(x) &= -\lambda\nu R_{1,h}(x)+\lambda\nu R_{1,h-1}(x)+\beta r_s(x)\int\limits_0^\infty W_{1,h}(x)dx\\
&\quad +R_{0,h+1}(0)r_s(x)+\lambda r_s(x)\int\limits_0^\infty R_{0,h}(x)dx;\quad h\ge 1
\end{aligned} \tag{8}
$$

Taking the LST from (2) to (8) on both sides results

$$\theta W_{0,h}^*(\theta)-W_{0,h}(0) = (\beta+\lambda)W_{0,h}^*(\theta)-W_{1,h}(0)L^*(\theta);\quad h\ge 1 \tag{9}$$

$$\theta W_{1,0}^*(\theta)-W_{1,0}(0) = (\beta+\lambda\nu)W_{1,0}^*(\theta)-W_{0,1}(0)W_v^*(\theta)-\lambda W_{0,0}W_v^*(\theta) \tag{10}$$

$$
\begin{aligned}
\theta W_{1,h}^*(\theta)-W_{1,h}(0) &= (\beta+\lambda\nu)W_{1,h}^*(\theta)-\lambda\nu W_{1,h-1}^*(\theta)\\
&\quad -W_{0,h+1}(0)W_v^*(\theta)-\lambda W_{0,h}^*(0)W_v^*(\theta);\quad h\ge 1
\end{aligned} \tag{11}
$$

$$\theta R_{0,h}^*(\theta)-R_{0,h}(0) = \lambda R_{0,h}^*(\theta)-R_{1,h}(0)G^*(\theta)-\beta G^*(\theta)W_{0,h}^*(0);\quad h\ge 1 \tag{12}$$

$$
\begin{aligned}
\theta R_{1,0}^*(\theta)-R_{1,0}(0) &= \lambda\nu R_{1,0}^*(\theta)-R_{0,1}(0)R_s^*(\theta)-\beta R_s^*(\theta)W_{1,0}^*(0)\\
&\quad -\lambda R_{0,0}R_s^*(\theta)
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\theta R_{1,h}^*(\theta)-R_{1,h}(0) &= \lambda\nu R_{1,h}^*(\theta)-\lambda\nu R_{1,h-1}^*(\theta)-R_s^*(\theta)R_{0,h+1}(0)\\
&\quad -\beta R_s^*(\theta)W_{1,h}^*(0)-\lambda R_s^*(\theta)R_{0,h}^*(0);\qquad h\ge 1
\end{aligned} \tag{14}
$$

Summing over h from 1 to infinity $\times$ (9) with z^h and results

$$W_0^*(z,\theta)[\theta-(\beta+\lambda)] = W_0(z,0)-L^*(\theta)[W_1(z,0)-W_{1,0}(0)] \tag{15}$$

Summing over h from 1 to infinity $\times$ (11) with z^h and comprise with (10) results

$$W_1^*(z,\theta)[\theta-(\beta+\lambda\nu-\lambda\nu z)] = W_1(z,0)-\frac{W_v^*(\theta)}{z}W_0(z,0)-\lambda W_{0,0}W_v^*(\theta) - \lambda W_v^*(\theta)W_0^*(z,0) \tag{16}$$

Placing $\theta=\beta+\lambda$ in (15), results

$$W_0(z,0)=L^*(\beta+\lambda)[W_1(z,0)-W_{1,0}(0)] \tag{17}$$

Placing $\theta=0$ and (Sub.) (17) in (15), results

$$W_0^*(z,0)=\frac{(1-L^*(\beta+\lambda))(W_1(z,0)-W_{1,0}(0))}{\beta+\lambda} \tag{18}$$

Placing $\theta=\beta+\lambda\nu-\lambda\nu z$ and (Sub.) (17) and (18) in (16), results

$$W_1(z,0)=\frac{W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z(\beta+\lambda)W_{0,0}-[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]W_{1,0}(0)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]} \tag{19}$$

(Sub.)(19) in (17), results

$$W_0(z,0)=\frac{zL^*(\beta+\lambda)(\beta+\lambda)[\lambda W_v^*(\beta+\lambda\nu-\lambda\nu z)W_{0,0}-W_{1,0}(0)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]} \tag{20}$$

Let $f(z)=(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]$, for $f(z)=0$ we obtain $f(0)<0$ and $f(1)>0$ which $\Rightarrow$ that $\exists$ a real root $z_1\in(0,1)$.
At $z=z_1$(20) seems

$$W_{1,0}(0)=\lambda W_v^*(\lambda\nu-\lambda\nu z_1+\beta)W_{0,0} \tag{21}$$

(Sub.) (21) in (19), results

$$W_1(z,0)=\frac{\lambda W_v^*(\beta+\lambda\nu-\lambda\nu z)UP(z)}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]}W_{0,0} \tag{22}$$

where,

$$UP(z)=z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)[\lambda z+L^*(\beta+\lambda)(\beta-\lambda z+\lambda)]$$

(Sub.) (21) in (20), results

$$W_0(z,0)=\frac{\lambda\nu z(\beta+\lambda)L^*(\beta+\lambda)[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]}W_{0,0} \tag{23}$$

(Sub.) (21) and (22) in (18), results

$$W_0^*(z,0)=\frac{(1-L^*(\beta+\lambda))\lambda z[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]}W_{0,0} \tag{24}$$

Placing $\theta=0$ and (Sub.) (22), (23) and (24) in (16), results

$$W_1^*(z,0)=\frac{\lambda(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))UP(z)}{(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)}W_{0,0} \tag{25}$$

We define $W_v(z)=W_0^*(z,0)+W_1^*(z,0)+W_{0,0}$; it represents the PGF for the number of customers in the orbit during WV period.

$$\begin{aligned}W_v(z) = {} & \frac{W_{0,0}}{(\beta+\lambda\nu-\lambda\nu z)D_1(z)}\Big\{(\beta+\lambda\nu-\lambda\nu z)(W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\\ & \times\lambda z(1-L^*(\beta+\lambda))+\lambda(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))[z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)\\ & \times(\lambda z+L^*(\beta+\lambda)(\beta+\lambda-\lambda z))]+(\beta+\lambda\nu-\lambda\nu z)[z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)\\ & \times(\lambda z+L^*(\beta+\lambda)(\beta+\lambda-\lambda z))]\Big\}\end{aligned} \tag{26}$$

$$Dr_1(z) = z(\beta+\lambda) - W_v^*(\beta+\lambda\nu-\lambda\nu z)(\lambda z + L^*(\beta+\lambda)(\beta+\lambda-\lambda z))$$

Summing over h from 1 to infinity $\times$ (12) with z^h and results

$$R_0^*(z,\theta)(\theta-\lambda) = R_0(z,0) - G^*(\theta)[R_1(z,0) - R_{1,0}(0)] - W_0^*(z,0)\beta G^*(\theta) \tag{27}$$

(Sub.) $W_{1,0}(0) = \lambda W_v^*(\beta+\lambda\nu-\lambda\nu z_1)W_{0,0}$ in (1), we get $\alpha R_{0,0} = \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0}$.

Placing $\theta = \lambda$ and (Sub.) $R_{1,0}(0) = \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} - \lambda R_{0,0}$ in (27), results

$$R_0(z,0) = G^*(\lambda)[R_1(z,0) - \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} - \lambda R_{0,0} + \beta W_0^*(z,0)] \tag{28}$$

Summing over h from 1 to infinity $\times$ (14) with z^h and comprise with (13) results

$$R_1^*(z,\theta)[\theta-\lambda\nu+\lambda\nu z)] = R_1(z,0) - \left\{\frac{R_0(z,0)}{z} + \beta W_1^*(z,0) + \lambda R_0^*(z,0) + \lambda R_{0,0}\right\} R_s^*(\theta) \tag{29}$$

Placing $\theta = 0$ and (Sub.) (28) and $R_{1,0}(0) = (1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\lambda W_{0,0} - \lambda R_{0,0}$ in (27), results

$$R_0^*(z,0) = \frac{(1-G^*(\lambda))}{\lambda}[R_1(z,0) - (1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\lambda W_{0,0} - \lambda R_{0,0} + \beta W_0^*(z,0)] \tag{30}$$

Placing $\theta = \lambda\nu - \lambda\nu z$ and (Sub.) (28) and (30) in (29), results

$$\begin{aligned} R_1(z,0) =& \frac{1}{Dr_2(z)}\Big\{R_s^*(\lambda\nu-\lambda\nu z)\big[\beta z W_1^*(z,0) + \lambda z R_{0,0} + \beta[(1-z)G^*(\lambda)+z]W_0^*(z,0) \\ & - [(1-z)G^*(\lambda)+z][(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\lambda W_{0,0} + \lambda R_{0,0}]\big]\Big\} \end{aligned} \tag{31}$$

(Sub.) (31) in (28), results

$$\begin{aligned} R_0(z,0) =& \frac{1}{Dr_2(z)}\Big\{[zG^*(\lambda)\ [\beta R_s^*(\lambda\nu-\lambda\nu z)W_1^*(z,0) - \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} \\ & + \beta W_0^*(z,0) - \lambda(1 - R_s^*(\lambda\nu-\lambda\nu z))R_{0,0}]\Big\} \end{aligned} \tag{32}$$

(Sub.) (31) in (30), results

$$\begin{aligned} R_0^*(z,0) =& \frac{1}{\lambda Dr_2(z)}\Big\{[\beta W_1^*(z,0)R_s^*(\lambda\nu-\lambda\nu z) - \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} \\ & + \beta W_0^*(z,0) - \lambda(1 - R_s^*(\lambda\nu-\lambda\nu z))R_{0,0}](1-G^*(\lambda))z\Big\} \end{aligned} \tag{33}$$

Placing $\theta = 0$ and (Sub.) (31), (32) and (33) in (29), results

$$\begin{aligned} R_1^*(z,0) =& \frac{1}{\lambda\nu(1-z)Dr_2(z)}\Big\{(1 - R_s^*(\lambda\nu-\lambda\nu z))\big\{\lambda z R_{0,0} + \beta W_0^*(z,0)[(1-z)G^*(\lambda)+z] \\ & + \beta W_1^*(z,0)z - [\lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} + \lambda R_{0,0}][G^*(\lambda)(1-z)+z]\big\}\Big\} \end{aligned} \tag{34}$$

(Sub.) (24) and (25) in (33), results

$$\begin{aligned} R_0^*(z,0) \quad = \quad & \frac{z(1-G^*(\lambda))W_{0,0}}{(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)Dr_2(z)}\Big\{\beta R_s^*(\lambda\nu-\lambda\nu z)(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z))\big\{(\beta+\lambda)z \\ & -W_v^*(\beta+\lambda\nu-\lambda\nu z_1)[\lambda z + (\beta+\lambda-\lambda z)G^*(\beta+\lambda)]\big\} + \beta z(W_v^*(\beta+\lambda\nu-\lambda\nu z) \\ & -W_v^*(\beta+\lambda\nu-\lambda\nu z_1))(\beta+\lambda\nu-\lambda\nu z)(1-L^*(\beta+\lambda)) - (1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\ & \times(\beta+\lambda\nu-\lambda\nu z)\big\{(\beta+\lambda)z - W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z + (\beta+\lambda-\lambda z)L^*(\beta+\lambda)]\big\} \\ & -(\beta+\lambda\nu-\lambda\nu z)(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))(1 - R_s^*(\lambda\nu-\lambda\nu z))\big\{(\beta+\lambda)z \\ & -W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z + (\beta+\lambda-\lambda z)L^*(\beta+\lambda)]\big\}\frac{\lambda}{\alpha}\Big\} \end{aligned} \tag{35}$$

(Sub.) (24), (25) in (34), results

$$\begin{aligned} R_1^*(z,0) &= \frac{(1-R_s^*(\lambda-\lambda z))W_{0,0}}{\nu Dr_2(z)(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)}\Big\{\beta z[\lambda\nu z+G^*(\lambda)(\beta+\lambda\nu-\lambda\nu z)][1-L^*(\beta+\lambda)] \\ &\times[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]-[\lambda\nu z+G^*(\lambda)(\beta+\lambda\nu-\lambda\nu z)] \\ &\times\big\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z+(\beta+\lambda-\lambda z)L^*(\beta+\lambda)]\big\} \\ &\times(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))+\beta z(\beta+\lambda)(W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\ &\times L^*(\beta+\lambda)-\frac{\lambda}{\alpha}(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))[(\beta+\lambda-\lambda z)G^*(\lambda)]\big\{(\beta+\lambda)z \\ &-W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z+(\beta+\lambda-\lambda z)L^*(\beta+\lambda)\big\}\Big\} \end{aligned} \tag{36}$$

We define $R_S(z)=R_0^*(z,0)+R_1^*(z,0)+R_{0,0}$; it represents the PGF for the number of customers in the orbit during RS period.

$$Dr_1(z) = z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)(\lambda z+L^*(\beta+\lambda)(\beta+\lambda-\lambda z)) \tag{37}$$

$$Dr_2(z) = z-R_s^*(\lambda\nu-\lambda\nu z)[G^*(\lambda)(1-z)+z] \tag{38}$$

where $Dr_1(z)$ and $Dr_2(z)$ are given in (37) and (38). Again, we define $R(z)=R_S(z)+W_v(z)$ as the PGF for the number of customers in the orbit. Make use of the normalising condition $R(1)=1$ to find out that $W_{0,0}$ is raised in (39). Using L'Hospitals rule and (sub.) $z=1$ in $R(z)$ results,

$$W_{0,0} = \frac{1-\rho_s}{\left[\frac{M_{r_1}}{\beta K_r}-\frac{M_{r_2}}{K_r}+\frac{M_{r_3}}{K_r}+\frac{(1-\nu)N_p}{\beta K_r}+\frac{\lambda M_{r_4}}{\alpha G^*(\lambda)}\right]} \tag{39}$$

$$R_{0,0} = \frac{\lambda}{\alpha}(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} \tag{40}$$

where,

$$\begin{aligned} M_{r_1} &= (\lambda-\lambda W_v^*(\beta+\lambda-\lambda z_1)+\beta)[\beta+\lambda\nu G^*(\lambda)-W_v^*(\beta)(\beta+\lambda\nu L^*(\beta+\lambda))] \\ M_{r_2} &= \lambda\nu E(R_s)W_v^*(\beta)[\beta+\lambda-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)(\beta+\lambda L^*(\beta+\lambda))] \\ M_{r_3} &= \beta W_v^*(\beta+\lambda\nu-\lambda\nu z_1)L^*(\beta+\lambda)(1-G^*(\lambda)) \\ M_{r_4} &= (1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))[\lambda G^*(\lambda)E(R_s)(1-\nu)+G^*(\lambda)] \\ K_r &= G^*(\lambda)[\beta+\lambda-W_v^*(\beta)(\beta+\lambda L^*(\beta+\lambda))] \\ N_p &= \lambda G^*(\lambda)(1-W_v^*(\beta))+\lambda\beta E(R_s)[L^*(\beta+\lambda)W_v^*(\beta+\lambda\nu-\lambda\nu z_1)+G^*(\lambda)(1-W_v^*(\beta))] \\ &\times(\beta+\lambda)-(1-W_v^*(\beta))\lambda\beta G^*(\lambda)E(R_s)[\lambda W_v^*(\beta)+\beta W_v^*(\beta+\lambda\nu-\lambda\nu z_1)L^*(\beta+\lambda)] \\ &-\lambda\beta W_v^*(\beta)L^*(\beta+\lambda)+(1-G^*(\lambda))\lambda\beta L^*(\beta+\lambda)W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\ &\times\beta G^*(\lambda)[\beta+\lambda-W_v^*(\beta)(\beta+\lambda L^*(\beta+\lambda))] \\ \rho_s &= \frac{\lambda\nu E(R_s)}{G^*(\lambda)} \end{aligned}$$

$E(R_s)$ is the mean service time and the system's stability condition $\rho_s<1$ is obtained from (39).

3. Performance Measures

Mean System Length:
We assume that
W_v, R_s - mean orbit size in WV period, RS period.

W_{vw}, R_{sw} - mean waiting time of the customer in the orbit during WV period, RS period. Then

$$\begin{aligned} W_v &= \frac{d}{dz}W_v(z)\Big|_{z=1} \\ &= \frac{d}{dz}[W_0^*(z,0)+W_1^*(z,0)]\Big|_{z=1} \\ &= \frac{d}{dz}\left[\frac{S(z)}{(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)}+\frac{K(z)}{Dr_1(z)}\right]W_{0,0}\Big|_{z=1} \end{aligned}$$

where,

$$\begin{aligned} S(z) &= \lambda(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))[z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\ &\quad (L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z)] \\ K(z) &= \lambda z(1-L^*(\beta+\lambda))(W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\ Dr_1(z) &= z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)(L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z) \end{aligned}$$

At $z=1$ W_v turns,

$$= \left\{\frac{\beta Dr_1(1)S'(1)-S(1)[\beta Dr_1'(1)-\lambda Dr_1(1)]}{(\beta Dr_1(1))^2}+\frac{Dr_1(1)K'(1)-K(1)Dr_1'(1)}{(Dr_1(1))^2}\right\}W_{0,0}$$

By Little's formula $W_{vw}=\dfrac{W_v}{\lambda}$,

$$\begin{aligned} R_s &= \frac{d}{dz}R_S(z)\Big|_{z=1} \\ &= \frac{d}{dz}[R_1^*(z,0)+R_0^*(z,0)]\Big|_{z=1} \\ &= \frac{d}{dz}\left[\frac{Nr_1(z)(1-G^*(\lambda))+Nr_2(z)Nr_3(z)}{Dr_1(z)(\lambda\nu-\lambda\nu z+\beta)Dr_2(z)}\right]W_{0,0}\Big|_{z=1} \end{aligned}$$

where,

$$\begin{aligned} Nr_1(z) &= \beta z R_s^*(\lambda\nu-\lambda\nu z)(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\ &\quad \times[(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z]\}+(\beta+\lambda\nu-\lambda\nu z)\beta z^2(1-L^*(\beta+\lambda)) \\ &\quad \times[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]-(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))z \\ &\quad \times(\lambda\nu-\lambda\nu z)\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z]\} \\ &\quad -(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))(\beta+\lambda\nu-\lambda\nu z)(1-R_s^*(\lambda\nu-\lambda\nu z))\{(\beta+\lambda)z \\ &\quad -W_v^*(\beta+\lambda\nu-\lambda\nu z)[(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z]\}\frac{z\lambda}{\alpha} \\ Nr_2(z) &= (1-R_s^*(\lambda\nu-\lambda\nu z)) \\ Nr_3(z) &= \beta z[(\beta+\lambda\nu-\lambda\nu z)G^*(\lambda)+\lambda\nu z](1-L^*(\beta+\lambda))[W_v^*(\beta+\lambda\nu-\lambda\nu z) \\ &\quad -W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]-(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))[(\beta+\lambda\nu-\lambda\nu z)G^*(\lambda) \\ &\quad +\lambda\nu z]\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z)\} \\ &\quad +\beta z(\lambda+\beta)[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]L^*(\beta+\lambda) \\ &\quad -\frac{\lambda}{\alpha}(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))G^*(\lambda)(\beta+\lambda\nu-\lambda\nu z)\{(\beta+\lambda)z \\ &\quad -W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z+(\beta+\lambda-\lambda z)L^*(\beta+\lambda)\} \\ Dr_1(z) &= (\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z] \\ Dr_2(z) &= z-R_s^*(\lambda\nu-\lambda\nu z)[(1-z)G^*(\lambda)+z] \end{aligned}$$

At $z = 1$ R_s turns,

$$\begin{aligned} R_s &= \frac{M_{r_4}}{2(\beta\eta Dr_1(1)Dr_2'(1))^2} W_{0,0} \\ M_{r_4} &= (1 - G^*(\lambda))\left[2Nr_1'(1)Dr_2'(1)(\lambda\eta Dr_1(1) - \beta\eta Dr_1'(1)) + \beta\eta Dr_1(1)(Dr_2'(1)Nr_1''(1)\right. \\ &\quad \left. -Nr_1'(1)Dr_2''(1))\right] + 2\beta\eta Nr_2'(1)Dr_2'(1)(Dr_1(1)Nr_3'(1) - Nr_3(1)Dr_1'(1)) \\ &\quad +Nr_3(1)Dr_1(1)[2\lambda\eta Nr_2'(1)Dr_2'(1) + \beta\eta Dr_2'(1)Nr_2''(1) - \beta\eta Nr_2'(1)Dr_2''(1)] \end{aligned}$$

By Little's formula $R_{sw} = \dfrac{R_s}{\lambda}$,

4. Special cases

(a) If the service time distribution follows an exponential distribution, $\nu = 1$ and no service during the vacation period, then the present model will be remodelled as a time-dependent analysis of the $M/M/1$ queue with server vacations and a waiting server.
(b) If the server does not wait after the completion of the RS period, $\nu = 1$ and there is no retrial time in the system then the present model will be remodeled as an $M/G/1$ queue with multiple working vacation.
(c) If the server does not wait after the completion of the RS period, $\nu = 1$ and the server never takes a vacation, then the present model will be remodelled as an $M/G/1$ retrial queue.

5. Numerical results

The curved graph constructed in Figure 1 and the values tabulated in the Table 1 are obtained by setting the fixed values $\mu_v = 1.5$, $\mu_s = 9$, $\mu_{vr} = 1.5$, $\mu_{sr} = 4.5$, $\alpha = 0.6$, $\nu = 0.5$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 1 to 2 in steps of 0.5, we observed that as λ rises W_v also rises and hence the stability of the model is verified.

Table 1: *R_s with turn over of λ*

λ	$\beta = 3$	$\beta = 5$	$\beta = 7$
1.0	0.0055	0.0081	0.0087
1.2	0.0082	0.0117	0.0125
1.4	0.0116	0.0161	0.0172
1.6	0.0156	0.0226	0.0226
1.8	0.0290	0.0271	0.0290
2.0	0.0262	0.0343	0.0363

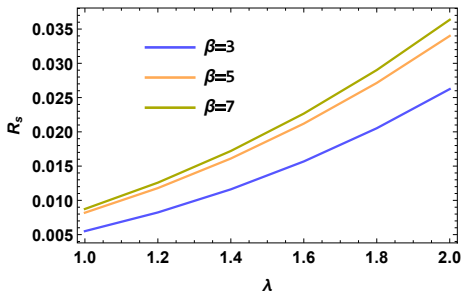

Figure 1: *R_s with turn over of λ*

Table 2: *R_{sw} with turn over of λ*

λ	$\beta = 1.5$	$\beta = 2$	$\beta = 2.5$
1.0	0.0062	0.0109	0.0131
1.2	0.0087	0.0134	0.0157
1.4	0.0114	0.0160	0.0184
1.6	0.0142	0.0188	0.0212
1.8	0.0174	0.0218	0.0242
2.0	0.0208	0.0251	0.0275

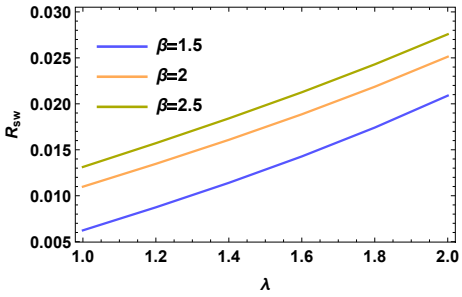

Figure 2: *R_{sw} with turn over of λ*

Table 3: *W_v with turn over of λ*

λ	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
1.0	0.0510	0.0352	0.0247
1.2	0.0680	0.0460	0.0321
1.4	0.0862	0.0573	0.0397
1.6	0.1051	0.0688	0.0474
1.8	0.1245	0.0804	0.0551
2.0	0.1440	0.0920	0.0628

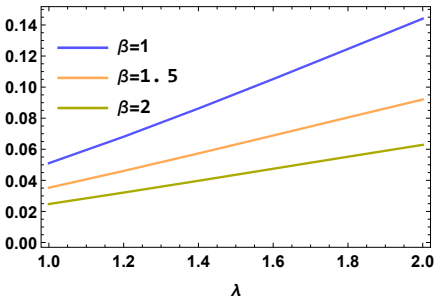

Figure 3: *W_v with turn over of λ*

The curved graph constructed in Figure 2 and the values tabulated in the Table 2 are obtained by setting the fixed values $\mu_v = 2.6$, $\mu_s = 9.7$, $\mu_{vr} = 1.5$, $\mu_{sr} = 3.9$, $\alpha = 0.5$, $\nu = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 1.2 to 1.6 in steps of 0.2. We observed that as λ rises W_{vw} also rises which is expected. The curved graph constructed in Figure 3 and the values tabulated in the Table 3 are obtained by setting the fixed values $\mu_v = 3.5$, $\mu_s = 7.5$, $\mu_{vr} = 2.5$, $\mu_{sr} = 6.5$, $\alpha = 0.7$, $\nu = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 3 to 7 in steps of 2. We observed that as λ rises R_s also rises which shows the stability of the model. The curved graph

Table 4: *W_{vw} with turnover of λ*

λ	$\beta = 1.2$	$\beta = 1.4$	$\beta = 1.6$
1.0	0.0198	0.0178	0.0159
1.2	0.0217	0.0196	0.0174
1.4	0.0234	0.0211	0.0187
1.6	0.0249	0.0223	0.0198
1.8	0.0263	0.0235	0.0207
2.0	0.0274	0.0244	0.0216

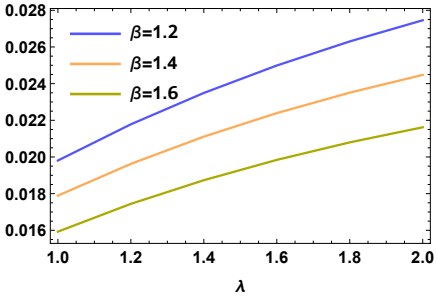

Figure 4: *W_{vw} with turnover of λ*

constructed in Figure 4 and the values tabulated in the Table 4 are obtained by setting the fixed values $\mu_v = 2.2$, $\mu_s = 9.9$, $\mu_{vr} = 1.2$, $\mu_{sr} = 3.3$, $\alpha = 0.2$, $\nu = 0.3$ and altering the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 1.5 to 2.5 in steps of 0.5. From the graph, we studied that as λ rises R_{sw} also rises which shows the stability of the model.

6. Conclusion

In this paper, an $M/G/1$ retrial queue with working vacation, nonpersistent customers, and a waiting server is evaluated. We obtain the PGF for the number of customers and the mean number of customers in the orbit. Further with the waiting time distribution and the performance measures are derived. The current problem is verified with some existing models through adding and assuming some particular values. We illustrate the mean orbit size and mean waiting time of customers during working vacation and regular service periods against the arrival rate of the customers. This model's results are very useful for better service management in various fields.

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REPETITIVE SAMPLING INSPECTION PLAN UNDER TRUNCATED LIFETEST BASED ON ONE PARAMETER POLYNOMIAL EXPONENTIAL DISTRIBUTION

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Abstract

This article constructs a Repetitive Sampling Inspection Plan under Truncated life test (RSIPTL) when the lifetime follows the One Parameter Polynomial Exponential (OPPE) family of distributions. In RSIPTL, a lot can be accepted or rejected in the first, second, and so on, based on the number of defective items in each sample. The OPPE has infinite support. It has transformed into its unit form to utilize finite support, i.e., having the support (0, 1). The Lindley distribution, a particular choice of the OPPE, has been studied in detail. We obtained the minimum number of items required in a lot to satisfy the consumer risk. Extensive tables are prepared for easy understanding and use of the plan for industrial workers. The RSIPTL is compared with a single sampling plan (SSP) and a two-stage reliability acceptance sampling plan (TSRASP) for Lindley and Exponential distributions. Two data sets are discussed and comparative statements are made with respect to the proposed plan.

Keywords: Consumer's risk, Operating characteristic function, Scale-invariant family of distributions, Truncated life test, unit-Lindley distribution.

1. Introduction

When a product is prepared to be sold to a consumer, it is crucial to verify its quality. Because reputation on the global market largely depends on product quality, it is imperative to confirm the quality of each product. This means that the manufacturers should focus more on each product's quality. It is possible to use an acceptance sampling plan to guarantee the items' quality. Several types of acceptance sampling are there, like attribute sampling plans, variable sampling plans, sequential sampling plans, etc. In these acceptance sampling plans, we aim to reduce sample size and make decisions about the lot to protect both the producer and consumer interests. We also know that any experiment is not free from error; here, we face two types of errors: 1st type of error known as producer error(α), where a good lot can reject; 2nd type of error known as consumer error (β), where a lousy lot can be accepted. So, in the sampling plan, we have to minimize these risks.

In real-life experiments like in medical or life testing experiments, we can see two types of censoring. In the first case, we fixed the time duration of the test, and in the second case, the number of failures was fixed. To reduce the cost of the experiment and testing time, we performed the 1st type of censoring, i.e., fixed the test duration. In the life testing process, researchers truncated the testing procedure at a specific time to minimize the cost and time required for the experiment. The product's lifetime is the quality of interest in reliability acceptance sampling

strategies. All consumers anticipate that their products will last for an extended period. In other words, a product should be expected to last a long time. Furthermore, the buyer will accept the goods with trust if it is demonstrated that their real life exceeds its stated lifetime. Because of the expense and time involved, waiting until every sampled item fails in a life test is not ideal. A time-truncated life test, which ends at a particular time, can be helpful in this situation to guarantee a lifetime with the least amount of money and effort.

Many authors, including [10], [29], [12], [14], [13], [17], [24], [21], [1], [8], [2], [28], [4] proposed acceptance sampling for truncated life tests. Few studies have been done on acceptance sampling inspection plans assuming the Lindley distribution [see [22], [25]. [3] and [30] devised an acceptance sampling technique based on the assumptions of a two-parameter Lindley distribution and a truncated life test. [27] first designed the attribute repetitive sampling plan for normal distribution. According to him, this sampling scheme gives producers and consumers the minimum sample size and the required protection. He also said that a repetitive sampling plan is more effective than a single one and less effective than a sequential one. [6] and[7] designed the repetitive group sampling (RGS) plan for variable inspection. Some other authors have also studied the design of RGS plans under various situations, including reliability concepts like [11], [15], [16]. In this paper, we construct a repetitive sampling plan for One Parameter Polynomial Exponential (OPPE) distribution.

Most works on repetitive sampling inspection plans for truncated life (RSIPTL) tests are done for scale-invariant distributions. Minimum sample size (n) are determined for different times per unit mean $(\frac{t}{\mu_0})$. Our study aims to determine the RSIPTL for the OPPE distribution and compare it to that for the exponentially distributed quality characteristics. The OPPE distribution does not belong to a scale-invariant family of distributions. Therefore, the utilization of time per mean is beyond our scope. We may directly chalk the plan with plan parameters $(n,\ c_1,\ c_2,\ t)$. Since the OPPE distribution has support (0,∞), we may utilize it with the finite support by transforming into its unit form, i.e., having the support (0,1) with the transformation $V = e^{-T}$. Using the unit-Lindley form, we make tables by choosing V and the mean μ_v in the interval (0,1). Utilizing this benefit, we choose optimal $(n,\ c_1,\ c_2)$ and then revert to plan parameter t from the relation of the transformation. So, in a nutshell, our objective is to develop an RSIPTL for the OPPE distributed quality characteristic. Based on the time-truncated life test, the plan saves the organization's time and cost while being very helpful in determining whether to accept or reject a lot. The OC is derived for choosing the optimal plan based on the consumer's confidence level. Tables of minimum sample sizes are examples for easy understanding and execution of the proposed plan. It is implemented for real-life experimental data, and the OC surface is depicted to provide a clear picture of the plan.

The following is the arrangement of the rest of the paper. The OPPE and unit-OPPE distributions are described in section 2. Section 3 describes the sampling design, operating characteristics function, and operating procedure. The sampling results for the Lindley and exponential distributions, in particular, are presented in tabular form. Section 4 compares the sample size for the repetitive, two-stage, and single-sampling plans. In section 5, we use the said sampling plan to work on real-world data. Section 6 concludes.

2. The One Parameter Polynomial Exponential Distribution And Its Unit Version

The probability density function (PDF) of a random variable T of the OPPE distribution can be written as

$$f_T(t;\theta) = h(\theta)p(t)e^{-\theta t}\, t > 0, \theta > 0, \tag{1}$$

where, $h(\theta) = \frac{1}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}}$, $p(t) = \sum_{k=0}^{r} a_k t^k$, $a_k's$ are known non-negative constants and r is known non-negative integer. The distribution is also known as

$$\begin{aligned} f_T(t,\theta) &= h(\theta)\sum_{k=0}^{r} a_k t^k e^{-\theta t} \\ &= \frac{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}} f_{GA}(t;k+1,\theta)}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}}, \end{aligned} \tag{2}$$

where $f_{GA}(t;k+1,\theta)$ is the PDF of a gamma distribution with shape parameter (k+1) and scale parameter θ. The distribution is a finite mixture of (r+1) gamma distributions. The cumulative density function (CDF) is given by

$$F_T(t,\theta) = 1 - \left(\frac{\sum_{k=0}^{r} \frac{a_k \Gamma(k+1)\Gamma(k+1,\theta t)}{\theta^{k+1}}}{\sum_{k=0}^{r} a_k \frac{k!}{\theta^{k+1}}} \right),\ t,\ \theta > 0, \tag{3}$$

where $\Gamma(m,t) = \frac{1}{\Gamma(m)} \int_t^{\infty} e^{-u} u^{m-1} du$.
The s-th order raw moment of OPPE is given by

$$\mu_s' = E(T^s) = \frac{\sum_{k=0}^{r} a_k \frac{\Gamma(k+s+1)}{\theta^{k+s+1}}}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}}. \tag{4}$$

Now, if we take a transformation V= e^{-T}, then the OPPE turns into unit-OPPE in the range of (0,1). The PDF and CDF of unit-OPPE are given by,

$$\begin{aligned} f_V(v,\theta) &= h(\theta)\sum_{k=0}^{r} a_k (-\ln v)^k v^{\theta-1} \\ &= \frac{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}} f_{UGA}(v;k+1,\theta)}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}},\ 0 < v < 1, \end{aligned} \tag{5}$$

where $f_{UGA}(v;k+1,\theta) = \frac{\theta^{k+1}}{\Gamma(k+1)}(-\ln v)^{k+1} v^{\theta-1}$ is the PDF of the unit-gamma distribution with shape parameter (k+1) and scale parameter θ, and

$$F_v(v,\theta) = 1 - \left(\frac{\sum_{k=0}^{r} \frac{a_k \Gamma(k+1)\Gamma(k+1,-\theta ln v)}{\theta^{k+1}}}{\sum_{k=0}^{r} a_k \frac{k!}{\theta^{k+1}}} \right), 0 < v < 1, \theta > 0, \tag{6}$$

respectively.
The s-th order raw moment of unit-OPPE is given by

$$\mu_s' = E(V^s) = \frac{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{(s+\theta)^{k+1}}}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}}. \tag{7}$$

The Lindley distribution (for r=1,$a_0 = a_1 = 1$), introduced by [9] to analyze failure time data, has the PDF, CDF, and hazard rate function (HRF) as

$$f_T(t;\theta) = \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t} \quad t > 0, \theta > 0, \tag{8}$$

$$F_T(t;\theta) = 1 - \frac{1+\theta+\theta t}{\theta+1} e^{-\theta t} \tag{9}$$

and

$$h_T(t;\theta) = \frac{\theta^2(1+t)}{1+\theta+\theta t} \quad t > 0, \theta > 0, \tag{10}$$

respectively.

The mean of the random variable T is

$$\mu = \frac{\theta+2}{\theta(1+\theta)}. \tag{11}$$

The unit-Lindley distribution with parameter θ has the PDF , CDF and HRF respectively , as follows:

$$f(v;\theta) = \frac{\theta^2}{1+\theta}(1-logv)\left(v^{\theta-1}\right) \quad 0 < v < 1, \theta > 0, \tag{12}$$

$$F(v;\theta) = \frac{v^\theta(1+\theta(1-logv))}{1+\theta} \quad 0 < v \leq 1, \theta > 0, \tag{13}$$

$$h(v;\theta) = \frac{\theta^2(1-logv)}{v(\theta logv - (1+\theta)(1-v^{-\theta}))} \quad 0 < v < 1, \theta > 0. \tag{14}$$

We will choose the exponential distribution and its corresponding unit version for comparison purposes. The unit version of the exponential distribution with parameter θ has the PDF, CDF, and HRF as

$$f_v(v,\theta) = \theta v^{\theta-1}, \quad 0 < v < 1, \theta > 0, \tag{15}$$

$$F_v(v,\theta) = v^\theta, \quad 0 < v \leq 1, \theta > 0, \tag{16}$$

$$h_v(v,\theta) = \frac{\theta v^{\theta-1}}{1-v^\theta}, \quad 0 < v < 1, \theta > 0, \tag{17}$$

respectively. In this case , $\mu_v = \frac{\theta}{1+\theta}$ implying $\theta = \frac{\mu_v}{1-\mu_v}$.

3. Sampling Process of Repetitive Sampling Inspection Plan under Truncated life test (RSIPTL) for OPPE distribution

According to the product's mean life, this sampling plan labels a lot (of products) as good or bad. The RSIPTL has the plan parameters n, c_1, c_2, and t. Engineers and practitioners use the tabulated value or algorithm to implement the plan properly. Tables are presented for fixed t and c_1, c_2, the optimal value of n. Since the value of $t \in (0, \infty)$, fixing t is tedious, whereas choosing $v \in (0, 1)$ is easy and comprehensive.

3.1. Design of the sampling plan

A product is defective if it fails before truncation time v. The fraction defective, i.e., the probability that a product is defective, is

$$p(v) = F(v;\theta) = 1 - \left(\frac{\sum_{k=0}^{r} \frac{a_k \Gamma(k+1)\Gamma(k+1,v^\theta)}{\theta^{k+1}}}{\sum_{k=0}^{r} a_k \frac{k!}{\theta^{k+1}}} \right), \ 0 < v < 1, \ \theta > 0. \tag{18}$$

In particular, for the unit-Lindley distribution,

$$p(v) = F(v;\theta) = \frac{v^\theta(1+\theta(1-logv))}{1+\theta} \quad 0 < v < 1, \theta > 0. \tag{19}$$

In this equation, we replace the shape parameter θ as the function of the product's mean life (μ). We can say that θ = g(μ), and we get the value of θ by solving the equation by numerical method, and hence we have $p(v) = F(v, \mu_v)$.

The Operating Characteristic (OC) function, i.e., the probability of accepting the lot according to the RSIPTL plan (also see [27]), is

$$L(p) = \frac{P_a}{P_r + P_a} \tag{20}$$

and the average sample number (ASN) of RSP is obtained as

$$ASN = \frac{n}{P_r + P_a} \tag{21}$$

where the lot acceptance probability is

$$P_a = \sum_{i=0}^{c_1} \binom{n}{i} p^i (1-p)^{n-i}, \tag{22}$$

The lot rejection probability is

$$P_r = 1 - \sum_{i=0}^{c_2} \binom{n}{i} p^i (1-p)^{n-i}, \tag{23}$$

where p=$F(v, \mu_v)$.

3.2. Sampling Procedure

We propose the sampling procedure as follows.

1. Put n items in the test for t_0 times.
2. Count the number of defectives (d) and, if the number of defectives is less than or equal to the acceptance number, c_1, then accept the lot. If the number of defectives exceeds the acceptance number, c_2, then reject the lot.
3. If $c_1 < d \leq c_2$, then repeat the test.

The triplet (n, c_1, c_2) is the plan parameters of the proposed plan, which are to be estimated. The minimum sample size n is estimated subject to constraints

$$L(p) \leq 1 - P^*, \tag{24}$$

$$0 \leq c_1 < c_2, \tag{25}$$

$$n > 1, c_1, c_2 \text{ are integers.} \tag{26}$$

3.3. Sampling plan result

The minimum values of n satisfying the inequality are obtained and shown in Tables 1-4 for the unit-Lindley (Lindley) and that for the unit-exponential (exponential) in Tables 5-6 for P^*= 0.95 and 0.99, and the truncation time $t = 2, 3, 4, 5$ and $\mu_0 = 2, 3, 4, 5$.

Table 1: *Determination of optimal sample size for Lindley set up*

	$P^* = 0.95, c_1 = 1, c_2 = 2$			
$\mu_v^0(\mu_0)$ | v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3002(2)	4	6	7	9
0.2065(3)	3	4	5	6
0.1515(4)	3	3	4	4
0.1162(5)	-	3	3	4
	$P^* = 0.95, c_1 = 1, c_2 = 3$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	5	6	7	10
0.2065(3)	-	4	5	6
0.1515(4)	-	-	4	5
0.1162(5)	-	-	-	4
	$P^* = 0.95, c_1 = 1, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	5	6	8	10
0.2065(3)	-	5	6	7
0.1515(4)	-	-	-	5
0.1162(5)	-	-	-	-

Table 2: *Determination of optimal sample size for Lindley set up*

	$P^* = 0.95, c_1 = 2, c_2 = 3$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3002(2)	6	8	9	11
0.2065(3)	5	6	7	8
0.1515(4)	-	5	5	7
0.1162(5)	-	-	5	7
	$P^* = 0.95, c_1 = 2, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	6	8	9	12
0.2065(3)	-	6	7	8
0.1515(4)	-	-	6	6
0.1162(5)	-	-	-	-
	$P^* = 0.95, c_1 = 3, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	7	9	11	14
0.2065(3)	7	7	9	10
0.1515(4)	-	-	7	8
0.1162(5)	-	-	-	7

Table 3: *Determination of optimal sample size for Lindley set up*

	$P^* = 0.99, c_1 = 1, c_2 = 2$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	5	7	9	11
0.2065(3)	4	5	6	7
0.1515(4)	4	4	5	6
0.1162(5)	3	4	4	5
	$P^* = 0.99, c_1 = 1, c_2 = 3$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	5	7	9	11
0.2065(3)	4	5	6	7
0.1515(4)	4	4	5	6
0.1162(5)	-	4	4	5
	$P^* = 0.99, c_1 = 1, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	6	7	9	11
0.2065(3)	5	6	7	8
0.1515(4)	-	5	5	6
0.1162(5)	-	-	5	5

Table 4: *Determination of optimal sample size for Lindley set up*

	$P^* = 0.99, c_1 = 2, c_2 = 3$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	7	9	10	16
0.2065(3)	6	7	8	9
0.1515(4)	5	6	6	7
0.1162(5)	5	5	6	8
	$P^* = 0.99, c_1 = 2, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	7	9	12	17
0.2065(3)	6	7	8	10
0.1515(4)	-	6	7	7
0.1162(5)	-	-	6	6
	$P^* = 0.99, c_1 = 3, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3001(2)	9	11	14	18
0.2065(3)	7	9	10	12
0.1515(4)	7	7	8	9
0.1162(5)	-	-	7	8

Table 5: *Determination of optimal sample size for Exponential set up*

	$P^* = 0.95, c_1 = 1, c_2 = 2$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3334(2)	12	20	34	58
0.25(3)	8	12	17	24
0.2(4)	6	9	12	15
0.1667(5)	6	7	9	12
	$P^* = 0.95, c_1 = 1, c_2 = 3$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3334(2)	12	21	36	60
0.25(3)	8	12	18	25
0.2(4)	7	9	12	16
0.1667(5)	6	8	10	12
	$P^* = 0.95, c_1 = 1, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3334(2)	13	22	38	74
0.25(3)	8	13	19	27
0.2(4)	7	10	13	17
0.1667(5)	6	8	10	13

Table 6: *Determination of optimal sample size for Exponential set up*

	$P^* = 0.95, c_1 = 2, c_2 = 3$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3334(2)	16	27	46	76
0.25(3)	11	16	23	32
0.2(4)	9	12	16	21
0.1667(5)	7	10	12	16
	$P^* = 0.95, c_1 = 2, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3334(2)	16	28	47	78
0.25(3)	11	16	23	33
0.2(4)	9	12	16	21
0.1667(5)	8	10	13	16
	$P^* = 0.95, c_1 = 3, c_2 = 4$			
$\mu_v^0(\mu_0)$v(t)	0.1353(2)	0.0498(3)	0.0183(4)	0.0067(5)
	n	n	n	n
0.3334(2)	19	33	56	94
0.25(3)	13	19	28	40
0.2(4)	11	14	19	25
0.1667(5)	9	12	15	19

Few observations:

1. If c_1 increases, the sample size increases. For example, Tables 1 and 2.
2. If c_2 increases, the sample size decreases. For example, Tables 1 and 2.

3. If the confidence level increases, the sample size increases. For example, Tables 1 and 3.

4. The sample required is less in this plan for the Lindley distributed quality characteristic than the exponential one. For example, Tables 2 and 6.

4. Comparison of sample sizes among different sampling plans

We compared this RSIPTL with a single sampling plan (SSP) and two-stage reliability acceptance sampling plan (TSRASP) at a fixed time point 5, 10, and 15 and mean lives at 5, 10, and 15, respectively, for Lindley and Exponential distributions. The optimal sample size comparison for confidence level $P^* = 0.95$ is shown in Table 7. It is observed that RSIPTL has a smaller sample size than others if it exists for a specified confidence level. Moreover, the sample sizes for the Lindley distribution are less than that of the exponential distribution.

Table 7: *Optimal Sample Size Comparison for Different Sampling Plans*

		$P^* = 0.95$							
		RSIPTL			TSRASP			SSP	
Distribution	μ	$6.74 \times 10^{-3}(5)$			$6.74 \times 10^{-3}(5)$			$6.74 \times 10^{-3}(5)$	
		n	c_1	c_2	n	c_1	c_2	n	c
Lindley	0.1162(5)	3	0	2	4	0	2	5	2
	0.0447(10)	-	0	2	-	0	2	4	2
	0.0236(15)	-	0	2	-	0	2	3	2
Exponential	0.1667(5)	8	0	2	9	0	2	15	2
	0.0909(10)	4	0	2	8	0	2	8	2
	0.0625(15)	4	0	2	6	0	2	7	2
Distribution	μ	$4.54 \times 10^{-5}(10)$			$4.54 \times 10^{-5}(10)$			$4.54 \times 10^{-5}(10)$	
		n	c_1	c_2	n	c_1	c_2	n	c
Lindley	0.1162(5)	4	0	2	6	0	2	8	2
	0.0447(10)	-	0	2	-	0	2	4	2
	0.0236(15)	-	0	2	-	0	2	4	2
Exponential	0.1667(5)	24	0	2	52	0	2	45	2
	0.0909(10)	8	0	2	14	0	2	15	2
	0.0625(15)	5	0	2	10	0	2	10	2
Distribution	μ	$3.059 \times 10^{-7}(15)$			$3.059 \times 10^{-7}(15)$			$3.059 \times 10^{-7}(15)$	
		n	c_1	c_2	n	c_1	c_2	n	c
Lindley	0.1162(5)	6	0	2	8	0	2	12	2
	0.0447(10)	-	0	2	-	0	2	5	2
	0.0625(10)	-	0	2	-	0	2	4	2
Exponential	0.1667(5)	67	0	2	150	0	2	197	2
	0.0909(10)	14	0	2	26	0	2	34	2
	0.0625(15)	8	0	2	14	0	2	8	2

5. Real-life data analysis

This section considers two cited data sets for applying the proposed RSIPTL under the exponential and Lindley setup. Since these are cited data sets, the sample sizes are known. Therefore, we have found out optimal $(c_1,\ c_2)$ for given n and P^* and decision tables are shown different choices of $v(t)$ and $\mu_v^0(\mu_0)$.

Using the maximum likelihood technique, we estimate the parameters of the distributions. The trial-and-error method determines the non-negative constants a_i of the OPPE. The selection criteria for each data set is Akaike's Information Criterion ($AIC = -2\log -likelihood + 2k$, where k is the number of parameters in the model). The lower the value of AIC, the better the model fit.

The goodness-of-fit of the distribution to a data set is made through Kolmogorov-Smirnov (K-S) test.

Data set 1: Let us consider data on the mileages at which 19 military personnel carriers failed in service. There is no censoring, and the mileages are ([19], page 194) 162, 200, 271, 320, 393, 508, 539, 629, 706, 777, 884, 1008, 1101, 1182, 1463, 1603, 1984, 2355, 2880. Table 8 shows that the Lindley distribution better fits the data based on the AIC value and the p-value. It is reflected in the Histogram, fitted distributions, and the P-P plot in Figure 1.

Table 8: *Comparison of Exponential and Lindley Distribution for Data Set 1.*

Distribution	Estimate of θ	Negative Log-likelihood	AIC	K-S Statistic	p-value
Exponential	0.001000977	150.2123	302.4247	0.14983	0.7328
Lindley	0.0020019	148.4087	298.8174	0.075969	0.9995

The failure times of the military carriers are 400,800,1600 and 2400. To make the decision table, we select the specified mean life as 400, 800, 1600, and 2400. The average failure time of the military carriers is 998.1579. To find the optimal c_1 and c_2 (the acceptance and rejection number of items for a lot), Table 9 and 10 are constructed for the Lindley and exponential distributions.

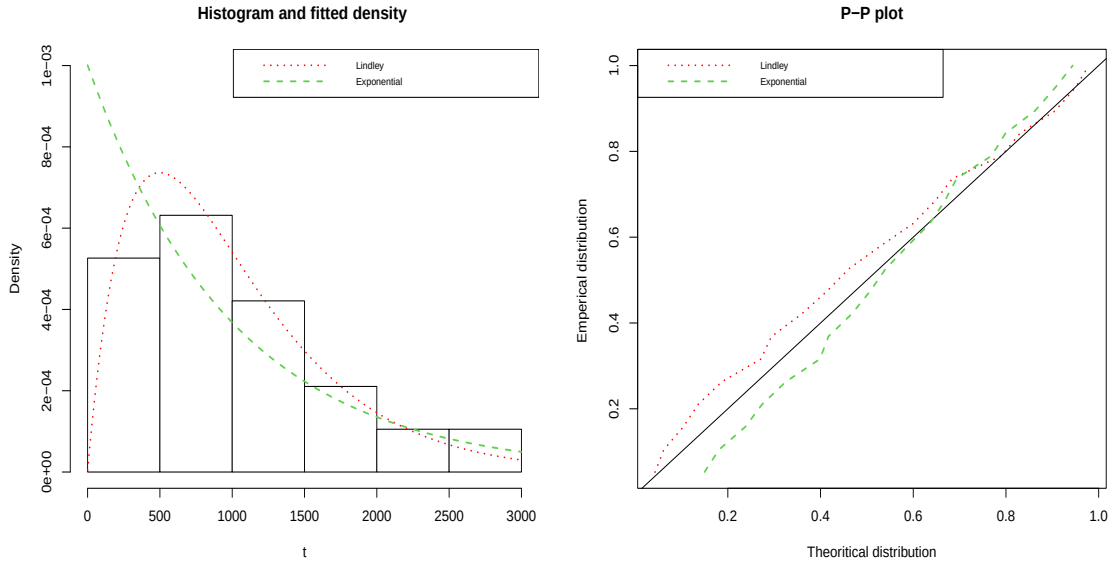

Figure 1: *Histogram, fitted distributions, and P-P plot for Data Set 1.*

If $d \leq c_1$, we accept the lot; if $d > c_2$, we reject the lot, and if d lies between c_1 and c_2, we repeat the plan until we get any result. In table 9, we choose a failure time point 1600 and a specified mean life of 800. Since the number of defective, $d = 10$ and c_1=10, we accept the lot. The OC surface of RSIPTL under Lindley set up has been shown in Figure 2 for n=19, $c_1 = 12$, $c_2 = 14$.

Table 9: *Determination of optimal* $(c_1,\ c_2)$ *for Lindley set up when n=19*

$P^* = 0.95$												
$\mu_v^0(\mu_0)$v(t)		$1.8312 \times 10^{-2}(400)$			$3.3546 \times 10^{-4}(800)$			$1.1253 \times 10^{-6}(1600)$			$3.7751 \times 10^{-11}(2400)$	
	d	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D
0.15155(400)	5	12	14	A	7	10	R	2	4	R	2	3	R
0.06207(800)	10	16	17	A	13	14	A	10	12	A	6	8	R
0.02122(1600)	14	17	18	A	16	17	A	16	17	A	14	15	Rep
0.01062(2400)	17	-	-	No	17	18	Rep	17	18	Rep	16	17	Rep

a

Table 10: *Determination of optimal* $(c_1,\ c_2)$ *for Exponential distribution set up when n=19*

$P^* = 0.95$												
$\mu_v^0(\mu_0)$v(t)		$1.8312 \times 10^{-2}(400)$			$3.3546 \times 10^{-4}(800)$			$1.1253 \times 10^{-6}(1600)$			$3.7751 \times 10^{-11}(2400)$	
	d	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D
0.2(400)	5	-	-	No	-	-	No	-	-	No	-	-	No
0.1111(800)	10	7	8	R	-	-	No	-	-	No	-	-	No
0.05882(1600)	14	11	12	R	7	9	R	4	5	R	-	-	No
0.040(2400)	17	12	14	R	9	12	R	6	8	R	2	6	R

b

[a]R=Reject, A=Accept, Rep= Repetition, No= No Decision
[b]R=Reject, A=Accept, Rep= Repetition, No= No decision

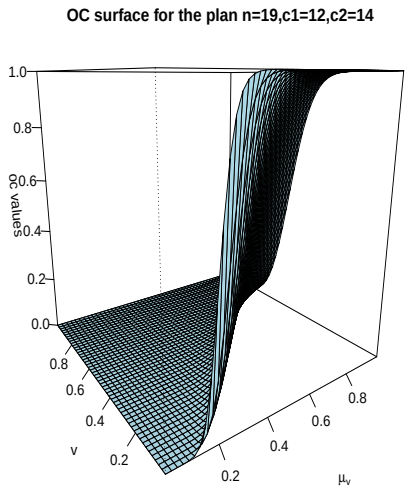

Figure 2: *OC surface of RSIPTL under Lindley set up for n=19,* $c_1 = 12$, $c_2 = 14$

Data set 2: Sixteen units are drawn randomly from a population and put on a time-truncated life test. After testing for 92 hours, which is prespecified, the test is terminated, by which time 15 fail. The failure times, in hours, are 13.4, 14.2, 28.8, 29, 29.8, 33, 37.8, 39.6, 43.4, 49.8, 54.8, 58.2, 70.2, 91.2, from [18]. Table 11 indicates the OPPE ($a_0 = 9,\ a_1 = 4,\ a_2 = 0.1$) distribution better fits the data. Histogram, fitted distributions, and P-P plot in Figure 3 are in the same tune.

Table 11: *Comparison of Exponential and OPPE (*$a_0 = 9,\ a_1 = 4,\ a_2 = 0.1$*) Distribution for Data Set 2.*

Distribution	Estimate of θ	Negative Log-likelihood	AIC	K-S Statistic	p-value
Exponential	0.02359375	66.45064	134.9013	0.35038	0.04846
OPPE(9,4,0.1)	0.05615	62.82411	127.64822	0.24319	0.3246

To make the decision table, we select the specified mean life as 20,40,60,80. The average failure time is 42.37143. To find the optimal c_1 and c_2 , table 12 and 13 are constructed for the OPPE ($a_0 = 9,\ a_1 = 4,\ a_2 = 0.1$) and exponential distributions. In table 13, if we choose a failure time point 20 and specified mean life 40, we can say that the lot is accepted because d (=8) is less than $c_1(= 10)$. The OC surface of RSIPTL under OPPE ($a_0 = 9,\ a_1 = 4,\ a_2 = 0.1$) set up has been shown in Figure 4 for n=15, $c_1 = 7$, $c_2 = 9$.

For all these data sets, the RSIPTL under the exponential set-up, the optimal sample sizes are enormous and are not recommended from an economic point of view.

Table 12: *Determination of optimal* $(c_1,\ c_2)$ *for Exponential set up when n=15*

$P^* = 0.95$												
$\mu_v^0(\mu_0)$v(t)		0.1353(20)			0.0183(40)			0.0025(60)			0.0003(80)	
	d	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D
0.3333(20)	2	-	-	No	-	-	No	-	-	No	-	-	No
0.2(40)	8	5	6	R	-	-	No	-	-	No	-	-	No
0.1428(60)	12	7	8	R	3	5	R	1	5	R	-	-	No
0.1111(80)	14	8	10	R	5	6	R	3	5	R	-	-	No

a

Table 13: *Determination of optimal* $(c_1,\ c_2)$ *for OPPE* $(a_0 = 9,\ a_1 = 4,\ a_2 = 0.1)$ *set up when n=15*

$P^* = 0.95$												
$\mu_v^0(\mu_0)$v(t)		0.1353(20)			0.0183(40)			0.0025(60)			0.0003(80)	
	d	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D	c_1	c_2	D
0.3164(20)	2	7	9	A	4	5	A	1	5	A	-	-	No
0.1764(40)	8	10	11	A	8	9	A	6	7	R	4	7	R
0.1089(60)	12	12	13	A	10	11	R	9	10	R	8	9	R
0.0760(80)	14	13	14	Rep	12	13	R	11	12	R	10	11	R

b

[a]R=Reject, A= Accept , Rep= Repetition, No= No decision
[b]R=Reject, A= Accept , Rep= Repetition, No= No decision

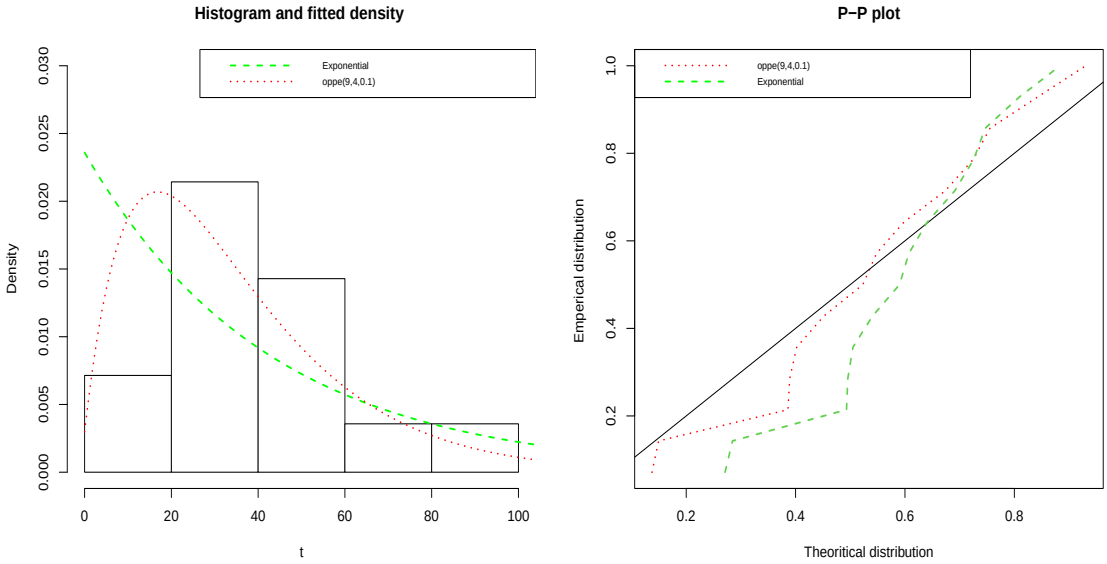

Figure 3: *Histogram, fitted distribution, and P-P plot for data set 2*

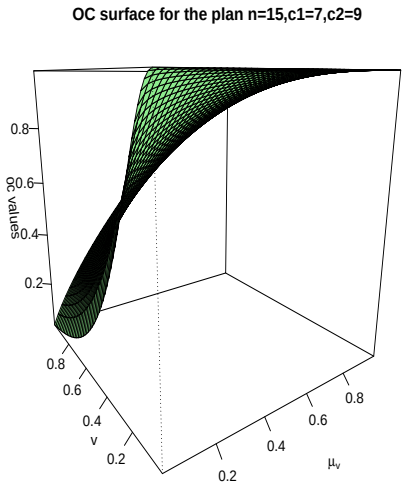

Figure 4: *OC surface of RSIPTL under OPPE* ($a_0 = 9,\ a_1 = 4,\ a_2 = 0.1$) *set up for n=15,* $c_1 = 7$, $c_2 = 9$

6. Concluding remarks

This article proposes a Repetitive Sampling Inspection Plan under Truncated life test (RSIPTL) when the lifetime follows the One Parameter Polynomial Exponential (OPPE) family of distributions. It is a more general setup than a two-stage reliability acceptance sampling plan, where a lot can be accepted or rejected based only on the first or second sample. Since the OPPE has infinite support, it has transformed into its unit form to utilize finite support, i.e., having the support (0, 1) for preparing tables for industrial workers to determine optimal sample sizes. The Lindley distribution, a particular choice of the OPPE ($r = 1, a_0 = a_1 = 1$), has been studied in detail. The Lindley distribution does not belong to the scale-invariant family, whereas the works so far on RSIPTL are chalked out for the scale-invariant family in the literature. The optimal

plan parameters are estimated by transforming the Lindley distribution into its unit form. A few representative examples are presented for an easy and comprehensive understanding of the scientists and quality practitioners for implementation to find the optimal sample size of the lot to satisfy the consumer risk. Two data sets are analyzed to implement the proposed plan and compared with the widely used exponential model. The approach may be adopted to construct RSIPTL for other lifetime quality characteristic distributions that do not belong to the scale-invariant family.

Declarations

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Ethical Approval: This article does not contain any studies with human participants performed by any of the authors.
Code availability: Codes are available on request.

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