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AN M/G/1 RETRIAL QUEUE WITH WORKING VACATION, NON PERSISTENT CUSTOMERS AND A WAITING SERVER

R.Keerthana

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V.S.B. Engineering College, India
keerthirn1996@gmail.com

Abstract

An M/G/1 retrial queue with working vacation, non persistent customers and a waiting server is taken into consideration in this study. Both retrial times and service times are assumed to follow general distribution and the waiting server follows an exponential distribution. Before switching over to a vacation the server waits for some arbitrary amount of time and so is called a waiting server. During the working vacation period customers are served at a lesser rate of service. We obtain the PGF for the number of customers and the mean number of customers in the invisible waiting area which is acquired by utilizing the supplementary variable technique. We compute the waiting time distribution. Out of interest a few special cases are conferred. Numerical outcomes are exhibited.

Keywords: Retrial queue, working vacation, supplementary variable technique, non persistent customers, waiting server

1. Introduction

In order to work with queues, we will need some basis on stochastic processes for a detailed deliberated about stochastic models in queueing theory by Medhi [1]. Retrial queues are expressed by the fact that if a customer observes that the server is occupied, then they are entered into the invisible waiting area called an orbit. In recent years, numerous researchers have examined the retrial queue. For a more in-depth analysis of the retrial queues, refer to[2, 3, 4, 5, 6]. In queueing theory, queueing models with server vacation have the most impactful application. Whenever the system becomes empty, the server leaves from the regular service period (RS) and goes on vacation, but in a waiting server model, the server will wait for an arbitrary amount of time before going on vacation. For a detailed study on waiting servers with vacation, debated in [7, 8, 9, 10, 11]. In addition to the vacation strategy, we developed the newest vacation strategy, called Working Vacation (WV). In the WV period, the server provides a lesser rate of service to the customers than during the regular service period. A survey on working vacation in queueing models by Chandrasekaran et al. [12]. The M/M/1 queue with single and multiple working vacations was discussed by [13, 14]. Similarly, the same discussion for M/G/1 queue was done by Wu and Gao [15, 16].

Kalyanaraman and Pazhani Bala Murugan [17] discussed retrial queue with vacation and presented operating characteristics results. Further the same retrial queue with single and multiple working vacations by Pazhani Bala Murugan and Santhi [18, 19]. Murugan and Keerthana [20] conducted a study on the $M/G/1$ retrial queue, which included both multiple working vacation and waiting server. Additionally, the same authors studied a similar problem with feedback in [21].

When a primary customer finds a server busy, the customer becomes unsatisfied and may quit the system without service permanently. For a more in-depth analysis of the non-persistent customers discussed in $M|G|1$ retrial queue with non persistent customers and orbital search by Krishnamoorthy et al. [22], Murugan and Vijaykrishnaraj [23] discussed a bulk arrival retrial queue with nonpersistent customers and exponentially distributed multiple working vacations and presented the results about probability generating function(PGF) for the number of customers in the orbit. Based on the above studies, $M|G|1$ retrial queue with nonpersistent customers and waiting server in working vacation were not discussed. In this article it is assumed that the waiting server has an exponential distribution and that the retrial and service times follow a general distribution. The server is referred to as a waiting server as it waits for a random amount of time before going into a vacation state. Based on that, the waiting time distribution, the performance measures, and some numerical results are discussed.

2. Model Description

We examine an $M/G/1$ retrial queue with WV, non persistent customers and a waiting server where the primary customers arrival follows a Poison process with arrival rate λ. If an approaching customer discovers that the server is occupied then they exit the service area because we assume that there is no waiting area and they joins the orbit. At a service completion instant, if the number of customer is one at the extreme front end of the orbit, is permitted to approach the server with a distribution function $G(x)$ and the retrial time follows a general distribution. For the normal service period, let $g(x)$ and $G^*(\theta)$ signify the distribution function, pdf and LST respectively, and for WV period, let $L(x), l(x), L^*(\theta)$ signify the pdf and LST respectively. On the service completion epoch of each customer, if there is a contest between primary customer and an orbit customer, then it will be determined with $R_s(x)$, $r_s(x)$, $R_s^*(\theta)$ as its distribution function, pdf, LST with general distribution. The service delivered among the WV period follows general distribution with $W_v(x)$, $w_v(x)$, $W_v^*(\theta)$ as its distribution function, pdf, LST. The arriving (or primary) customer receives service instantly if the server is idle. If not, he will choose whether to leave the system without service with probability $(1-\nu)$ or returning again later with probability ν.

The server waits for a arbitrary period of time once the orbit turns empty which follows an (exp.) distribution with rate α. After completion of waiting time the server goes for WV which follows an (exp.) distribution with rate β and Inter-arrival times, retrial periods, RS periods, and WV periods are all presumed to be independent of one another.
Let's use the subsequent random variables
$O(t)$- Size of the orbit at "t",
$R_s^0(t), G^0(t)$- the RST and RRT in RS period,
$W_v^0(t), L^0(t)$- the RST and RRT in WV period.
At time "t" the four distinct states of the server are

$$E(t) = \begin{cases} 0 & \text{if the server is not occupied in WV} \\ 1 & \text{if the server is not occupied in RS period} \\ 2 & \text{if the server is occupied in WV} \\ 3 & \text{if the server is occupied in RS period} \end{cases}$$

To generate bivariate Markov Processs, further variables are introduced that $\{(O(t), B(t)); t \geq 0\}$, where
$B(t) = L^0(t)$, if $E(t) = 0; G^0(t)$, if $E(t) = 1; W_v^0(t)$, if E(t) = 2; $R_s^0(t)$, if E(t) = 3.

$$\begin{aligned} W_{0,0} &= \lim_{t\to\infty} P[O(t) = 0,\ E(t) \ = 0] \\ R_{0,0} &= \lim_{t\to\infty} P[O(t) = 0,\ E(t) \ = 1] \\ W_{0,h} &= \lim_{t\to\infty} P[O(t) = h,\ E(t) \ = 0,\ x < L^0(t) \ \leq x + dx];\ h \geq 1 \end{aligned}$$

$$R_{0,h} = \lim_{t\to\infty} P[O(t)=h,\ E(t)\ =1,\ x<G^0(t)\ \le x+dx];\ h\ge 1$$
$$W_{1,h} = \lim_{t\to\infty} P[O(t)=h,\ E(t)\ =2,\ x<W_v^0(t)\ \le x+dx];\ h\ge 0$$
$$R_{1,h} = \lim_{t\to\infty} P[O(t)=h,\ E(t)\ =3,\ x<R_s^0(t)\ \le x+dx];\ h\ge 0$$

The above mentioned are the limiting probabilities which we have defined. Let us define common LST and PGF's are $F^*(s)=\int_0^\infty e^{-st}dF(t)$ and $F^*(z,x)=\sum_{h=0}^{\infty} F_h^*(x)z^h$.
In steady state the system was illustrated by the subsequent differential difference equations:

$$\lambda W_{0,0} = W_{1,0}(0)+\alpha R_{0,0} \tag{1}$$
$$-\frac{d}{dx}W_{0,h}(x) = -(\beta+\lambda)W_{0,h}(x)+W_{1,h}(0)l(x);\quad h\ge 1 \tag{2}$$
$$-\frac{d}{dx}W_{1,0}(x) = -(\beta+\lambda\nu)W_{1,0}(x)+W_{0,1}(0)w_v(x)+\lambda W_{0,0}w_v(x) \tag{3}$$
$$-\frac{d}{dx}W_{1,h}(x) = -(\beta+\lambda\nu)W_{1,h}(x)+\lambda\nu W_{1,h-1}(x)+W_{0,h+1}(0)w_v(x) +\lambda\int_0^\infty W_{0,h}(x)dx\ w_v(x);h\ge 1 \tag{4}$$
$$(\lambda+\alpha)R_{0,0} = R_{1,0}(0) \tag{5}$$
$$-\frac{d}{dx}R_{0,h}(x) = -\lambda R_{0,h}(x)+R_{1,h}(0)g(x)+\beta\int_0^\infty W_{0,h}(x)dx\ g(x);\quad h\ge 1 \tag{6}$$
$$-\frac{d}{dx}R_{1,0}(x) = -\lambda\nu R_{1,0}(x)+R_{0,1}(0)r_s(x)+\beta\int_0^\infty W_{1,0}(x)r_s(x)dx \tag{7}$$
$$-\frac{d}{dx}R_{1,h}(x) = -\lambda\nu R_{1,h}(x)+\lambda\nu R_{1,h-1}(x)+\beta r_s(x)\int_0^\infty W_{1,h}(x)dx +R_{0,h+1}(0)r_s(x)+\lambda r_s(x)\int_0^\infty R_{0,h}(x)dx;\quad h\ge 1 \tag{8}$$

Taking the LST from (2) to (8) on both sides results

$$\theta W_{0,h}^*(\theta)-W_{0,h}(0) = (\beta+\lambda)W_{0,h}^*(\theta)-W_{1,h}(0)L^*(\theta);\quad h\ge 1 \tag{9}$$
$$\theta W_{1,0}^*(\theta)-W_{1,0}(0) = (\beta+\lambda\nu)W_{1,0}^*(\theta)-W_{0,1}(0)W_v^*(\theta)-\lambda W_{0,0}W_v^*(\theta) \tag{10}$$
$$\theta W_{1,h}^*(\theta)-W_{1,h}(0) = (\beta+\lambda\nu)W_{1,h}^*(\theta)-\lambda\nu W_{1,h-1}^*(\theta) -W_{0,h+1}(0)W_v^*(\theta)-\lambda W_{0,h}^*(0)W_v^*(\theta);\quad h\ge 1 \tag{11}$$
$$\theta R_{0,h}^*(\theta)-R_{0,h}(0) = \lambda R_{0,h}^*(\theta)-R_{1,h}(0)G^*(\theta)-\beta G^*(\theta)W_{0,h}^*(0);\quad h\ge 1 \tag{12}$$
$$\theta R_{1,0}^*(\theta)-R_{1,0}(0) = \lambda\nu R_{1,0}^*(\theta)-R_{0,1}(0)R_s^*(\theta)-\beta R_s^*(\theta)W_{1,0}^*(0) -\lambda R_{0,0}R_s^*(\theta) \tag{13}$$
$$\theta R_{1,h}^*(\theta)-R_{1,h}(0) = \lambda\nu R_{1,h}^*(\theta)-\lambda\nu R_{1,h-1}^*(\theta)-R_s^*(\theta)R_{0,h+1}(0) -\beta R_s^*(\theta)W_{1,h}^*(0)-\lambda R_s^*(\theta)R_{0,h}^*(0);\quad h\ge 1 \tag{14}$$

Summing over h from 1 to infinity $\times$ (9) with z^h and results

$$W_0^*(z,\theta)[\theta-(\beta+\lambda)] = W_0(z,0)-L^*(\theta)[W_1(z,0)-W_{1,0}(0)] \tag{15}$$

Summing over h from 1 to infinity $\times$ (11) with z^h and comprise with (10) results

$$W_1^*(z,\theta)[\theta-(\beta+\lambda\nu-\lambda\nu z)] = W_1(z,0)-\frac{W_v^*(\theta)}{z}W_0(z,0)-\lambda W_{0,0}W_v^*(\theta) - \lambda W_v^*(\theta)W_0^*(z,0) \quad (16)$$

Placing $\theta=\beta+\lambda$ in (15), results

$$W_0(z,0)=L^*(\beta+\lambda)[W_1(z,0)-W_{1,0}(0)] \quad (17)$$

Placing $\theta=0$ and (Sub.) (17) in (15), results

$$W_0^*(z,0)=\frac{(1-L^*(\beta+\lambda))(W_1(z,0)-W_{1,0}(0))}{\beta+\lambda} \quad (18)$$

Placing $\theta=\beta+\lambda\nu-\lambda\nu z$ and (Sub.) (17) and (18) in (16), results

$$W_1(z,0)=\frac{W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z(\beta+\lambda)W_{0,0}-[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]W_{1,0}(0)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]} \quad (19)$$

(Sub.)(19) in (17), results

$$W_0(z,0)=\frac{zL^*(\beta+\lambda)(\beta+\lambda)[\lambda W_v^*(\beta+\lambda\nu-\lambda\nu z)W_{0,0}-W_{1,0}(0)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]} \quad (20)$$

Let $f(z)=(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]$, for $f(z)=0$ we obtain $f(0)<0$ and $f(1)>0$ which $\Rightarrow$ that $\exists$ a real root $z_1\in(0,1)$.
At $z=z_1$(20) seems

$$W_{1,0}(0)=\lambda W_v^*(\lambda\nu-\lambda\nu z_1+\beta)W_{0,0} \quad (21)$$

(Sub.) (21) in (19), results

$$W_1(z,0)=\frac{\lambda W_v^*(\beta+\lambda\nu-\lambda\nu z)UP(z)}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]}W_{0,0} \quad (22)$$

where,

$$UP(z)=z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)[\lambda z+L^*(\beta+\lambda)(\beta-\lambda z+\lambda)]$$

(Sub.) (21) in (20), results

$$W_0(z,0)=\frac{\lambda\nu z(\beta+\lambda)L^*(\beta+\lambda)[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]}W_{0,0} \quad (23)$$

(Sub.) (21) and (22) in (18), results

$$W_0^*(z,0)=\frac{(1-L^*(\beta+\lambda))\lambda z[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]}{z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z]}W_{0,0} \quad (24)$$

Placing $\theta=0$ and (Sub.) (22), (23) and (24) in (16), results

$$W_1^*(z,0)=\frac{\lambda(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))UP(z)}{(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)}W_{0,0} \quad (25)$$

We define $W_v(z)=W_0^*(z,0)+W_1^*(z,0)+W_{0,0}$; it represents the PGF for the number of customers in the orbit during WV period.

$$\begin{aligned} W_v(z) \;=\; & \frac{W_{0,0}}{(\beta+\lambda\nu-\lambda\nu z)D_1(z)}\Big\{(\beta+\lambda\nu-\lambda\nu z)(W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\ & \times\lambda z(1-L^*(\beta+\lambda))+\lambda(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))[z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\ & \times(\lambda z+L^*(\beta+\lambda)(\beta+\lambda-\lambda z))]+(\beta+\lambda\nu-\lambda\nu z)[z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z) \\ & \times(\lambda z+L^*(\beta+\lambda)(\beta+\lambda-\lambda z))]\Big\} \end{aligned} \quad (26)$$

$$Dr_1(z) = z(\beta+\lambda) - W_v^*(\beta+\lambda\nu-\lambda\nu z)(\lambda z + L^*(\beta+\lambda)(\beta+\lambda-\lambda z))$$

Summing over h from 1 to infinity $\times$ (12) with z^h and results

$$R_0^*(z,\theta)(\theta-\lambda) = R_0(z,0) - G^*(\theta)[R_1(z,0) - R_{1,0}(0)] - W_0^*(z,0)\beta G^*(\theta) \tag{27}$$

(Sub.) $W_{1,0}(0) = \lambda W_v^*(\beta+\lambda\nu-\lambda\nu z_1)W_{0,0}$ in (1), we get $\alpha R_{0,0} = \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0}$.

Placing $\theta = \lambda$ and (Sub.) $R_{1,0}(0) = \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} - \lambda R_{0,0}$ in (27), results

$$R_0(z,0) = G^*(\lambda)[R_1(z,0) - \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} - \lambda R_{0,0} + \beta W_0^*(z,0)] \tag{28}$$

Summing over h from 1 to infinity $\times$ (14) with z^h and comprise with (13) results

$$R_1^*(z,\theta)[\theta-\lambda\nu+\lambda\nu z)] = R_1(z,0) - \left\{\frac{R_0(z,0)}{z} + \beta W_1^*(z,0) + \lambda R_0^*(z,0) + \lambda R_{0,0}\right\} R_s^*(\theta) \tag{29}$$

Placing $\theta = 0$ and (Sub.) (28) and $R_{1,0}(0) = (1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\lambda W_{0,0} - \lambda R_{0,0}$ in (27), results

$$R_0^*(z,0) = \frac{(1-G^*(\lambda))}{\lambda}[R_1(z,0) - (1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\lambda W_{0,0} - \lambda R_{0,0} + \beta W_0^*(z,0)] \tag{30}$$

Placing $\theta = \lambda\nu - \lambda\nu z$ and (Sub.) (28) and (30) in (29), results

$$\begin{aligned} R_1(z,0) =& \frac{1}{Dr_2(z)}\Big\{R_s^*(\lambda\nu-\lambda\nu z)\big[\beta z W_1^*(z,0) + \lambda z R_{0,0} + \beta[(1-z)G^*(\lambda)+z]W_0^*(z,0) \\ &- [(1-z)G^*(\lambda)+z][(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))\lambda W_{0,0} + \lambda R_{0,0}]\big]\Big\} \end{aligned} \tag{31}$$

(Sub.) (31) in (28), results

$$\begin{aligned} R_0(z,0) =& \frac{1}{Dr_2(z)}\Big\{[zG^*(\lambda)\ [\beta R_s^*(\lambda\nu-\lambda\nu z)W_1^*(z,0) - \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} \\ &+ \beta W_0^*(z,0) - \lambda(1 - R_s^*(\lambda\nu-\lambda\nu z))R_{0,0}]\Big\} \end{aligned} \tag{32}$$

(Sub.) (31) in (30), results

$$\begin{aligned} R_0^*(z,0) =& \frac{1}{\lambda Dr_2(z)}\Big\{[\beta W_1^*(z,0)R_s^*(\lambda\nu-\lambda\nu z) - \lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} \\ &+ \beta W_0^*(z,0) - \lambda(1 - R_s^*(\lambda\nu-\lambda\nu z))R_{0,0}](1-G^*(\lambda))z\Big\} \end{aligned} \tag{33}$$

Placing $\theta = 0$ and (Sub.) (31), (32) and (33) in (29), results

$$\begin{aligned} R_1^*(z,0) =& \frac{1}{\lambda\nu(1-z)Dr_2(z)}\Big\{(1 - R_s^*(\lambda\nu-\lambda\nu z))\big\{\lambda z R_{0,0} + \beta W_0^*(z,0)[(1-z)G^*(\lambda)+z] \\ &+ \beta W_1^*(z,0)z - [\lambda(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} + \lambda R_{0,0}][G^*(\lambda)(1-z)+z]\big\}\Big\} \end{aligned} \tag{34}$$

(Sub.) (24) and (25) in (33), results

$$\begin{aligned} R_0^*(z,0) \quad = \quad & \frac{z(1-G^*(\lambda))W_{0,0}}{(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)Dr_2(z)}\Big\{\beta R_s^*(\lambda\nu-\lambda\nu z)(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z))\big\{(\beta+\lambda)z \\ &- W_v^*(\beta+\lambda\nu-\lambda\nu z_1)[\lambda z + (\beta+\lambda-\lambda z)G^*(\beta+\lambda)]\big\} + \beta z(W_v^*(\beta+\lambda\nu-\lambda\nu z) \\ &- W_v^*(\beta+\lambda\nu-\lambda\nu z_1))(\beta+\lambda\nu-\lambda\nu z)(1 - L^*(\beta+\lambda)) - (1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\ &\times(\beta+\lambda\nu-\lambda\nu z)\big\{(\beta+\lambda)z - W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z + (\beta+\lambda-\lambda z)L^*(\beta+\lambda)]\big\} \\ &- (\beta+\lambda\nu-\lambda\nu z)(1 - W_v^*(\beta+\lambda\nu-\lambda\nu z_1))(1 - R_s^*(\lambda\nu-\lambda\nu z))\big\{(\beta+\lambda)z \\ &- W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z + (\beta+\lambda-\lambda z)L^*(\beta+\lambda)]\big\}\frac{\lambda}{\alpha}\Big\} \end{aligned} \tag{35}$$

(Sub.) (24), (25) in (34), results

$$\begin{aligned}
R_1^*(z,0) &= \frac{(1-R_s^*(\lambda-\lambda z))W_{0,0}}{\nu Dr_2(z)(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)}\Big\{\beta z[\lambda\nu z+G^*(\lambda)(\beta+\lambda\nu-\lambda\nu z)][1-L^*(\beta+\lambda)] \\
&\times[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]-[\lambda\nu z+G^*(\lambda)(\beta+\lambda\nu-\lambda\nu z)] \\
&\times\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z+(\beta+\lambda-\lambda z)L^*(\beta+\lambda)]\} \\
&\times(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))+\beta z(\beta+\lambda)(W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\
&\times L^*(\beta+\lambda)-\frac{\lambda}{\alpha}(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))[(\beta+\lambda-\lambda z)G^*(\lambda)]\{(\beta+\lambda)z \\
&-W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z+(\beta+\lambda-\lambda z)L^*(\beta+\lambda)\}\Big\} \qquad (36)
\end{aligned}$$

We define $R_S(z)=R_0^*(z,0)+R_1^*(z,0)+R_{0,0}$; it represents the PGF for the number of customers in the orbit during RS period.

$$Dr_1(z) = z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)(\lambda z+L^*(\beta+\lambda)(\beta+\lambda-\lambda z)) \qquad (37)$$

$$Dr_2(z) = z-R_s^*(\lambda\nu-\lambda\nu z)[G^*(\lambda)(1-z)+z] \qquad (38)$$

where $Dr_1(z)$ and $Dr_2(z)$ are given in (37) and (38). Again, we define $R(z)=R_S(z)+W_v(z)$ as the PGF for the number of customers in the orbit. Make use of the normalising condition $R(1)=1$ to find out that $W_{0,0}$ is raised in (39). Using L'Hospitals rule and (sub.) $z=1$ in $R(z)$ results,

$$W_{0,0} = \frac{1-\rho_s}{\left[\frac{M_{r_1}}{\beta K_r}-\frac{M_{r_2}}{K_r}+\frac{M_{r_3}}{K_r}+\frac{(1-\nu)N_p}{\beta K_r}+\frac{\lambda M_{r_4}}{\alpha G^*(\lambda)}\right]} \qquad (39)$$

$$R_{0,0} = \frac{\lambda}{\alpha}(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))W_{0,0} \qquad (40)$$

where,

$$\begin{aligned}
M_{r_1} &= (\lambda-\lambda W_v^*(\beta+\lambda-\lambda z_1)+\beta)[\beta+\lambda\nu G^*(\lambda)-W_v^*(\beta)(\beta+\lambda\nu L^*(\beta+\lambda))] \\
M_{r_2} &= \lambda\nu E(R_s)W_v^*(\beta)[\beta+\lambda-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)(\beta+\lambda L^*(\beta+\lambda))] \\
M_{r_3} &= \beta W_v^*(\beta+\lambda\nu-\lambda\nu z_1)L^*(\beta+\lambda)(1-G^*(\lambda)) \\
M_{r_4} &= (1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))[\lambda G^*(\lambda)E(R_s)(1-\nu)+G^*(\lambda)] \\
K_r &= G^*(\lambda)[\beta+\lambda-W_v^*(\beta)(\beta+\lambda L^*(\beta+\lambda))] \\
N_p &= \lambda G^*(\lambda)(1-W_v^*(\beta))+\lambda\beta E(R_s)[L^*(\beta+\lambda)W_v^*(\beta+\lambda\nu-\lambda\nu z_1)+G^*(\lambda)(1-W_v^*(\beta))] \\
&\times(\beta+\lambda)-(1-W_v^*(\beta))\lambda\beta G^*(\lambda)E(R_s)[\lambda W_v^*(\beta)+\beta W_v^*(\beta+\lambda\nu-\lambda\nu z_1)L^*(\beta+\lambda)] \\
&-\lambda\beta W_v^*(\beta)L^*(\beta+\lambda)+(1-G^*(\lambda))\lambda\beta L^*(\beta+\lambda)W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\
&\times\beta G^*(\lambda)[\beta+\lambda-W_v^*(\beta)(\beta+\lambda L^*(\beta+\lambda))] \\
\rho_s &= \frac{\lambda\nu E(R_s)}{G^*(\lambda)}
\end{aligned}$$

$E(R_s)$ is the mean service time and the system's stability condition $\rho_s<1$ is obtained from (39).

3. Performance Measures

Mean System Length:
We assume that
W_v, R_s - mean orbit size in WV period, RS period.

W_{vw}, R_{sw} - mean waiting time of the customer in the orbit during WV period, RS period. Then

$$\begin{aligned} W_v &= \frac{d}{dz}W_v(z)\Big|_{z=1} \\ &= \frac{d}{dz}[W_0^*(z,0)+W_1^*(z,0)]\Big|_{z=1} \\ &= \frac{d}{dz}\left[\frac{S(z)}{(\beta+\lambda\nu-\lambda\nu z)Dr_1(z)}+\frac{K(z)}{Dr_1(z)}\right]W_{0,0}\Big|_{z=1} \end{aligned}$$

where,

$$\begin{aligned} S(z) &= \lambda(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))[z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\ &\quad (L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z)] \\ K(z) &= \lambda z(1-L^*(\beta+\lambda))(W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)) \\ Dr_1(z) &= z(\beta+\lambda)-W_v^*(\beta+\lambda\nu-\lambda\nu z)(L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z) \end{aligned}$$

At $z=1$ W_v turns,

$$= \left\{\frac{\beta Dr_1(1)S'(1)-S(1)[\beta Dr_1'(1)-\lambda Dr_1(1)]}{(\beta Dr_1(1))^2}+\frac{Dr_1(1)K'(1)-K(1)Dr_1'(1)}{(Dr_1(1))^2}\right\}W_{0,0}$$

By Little's formula $W_{vw}=\dfrac{W_v}{\lambda}$,

$$\begin{aligned} R_s &= \frac{d}{dz}R_S(z)\Big|_{z=1} \\ &= \frac{d}{dz}[R_1^*(z,0)+R_0^*(z,0)]\Big|_{z=1} \\ &= \frac{d}{dz}\left[\frac{Nr_1(z)(1-G^*(\lambda))+Nr_2(z)Nr_3(z)}{Dr_1(z)(\lambda\nu-\lambda\nu z+\beta)Dr_2(z)}\right]W_{0,0}\Big|_{z=1} \end{aligned}$$

where,

$$\begin{aligned} Nr_1(z) &= \beta z R_s^*(\lambda\nu-\lambda\nu z)(1-W_v^*(\beta+\lambda\nu-\lambda\nu z))\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z_1) \\ &\quad \times[(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z]\}+(\beta+\lambda\nu-\lambda\nu z)\beta z^2(1-L^*(\beta+\lambda)) \\ &\quad \times[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]-(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))z \\ &\quad \times(\lambda\nu-\lambda\nu z)\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z]\} \\ &\quad -(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))(\beta+\lambda\nu-\lambda\nu z)(1-R_s^*(\lambda\nu-\lambda\nu z))\{(\beta+\lambda)z \\ &\quad -W_v^*(\beta+\lambda\nu-\lambda\nu z)[(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z]\}\frac{z\lambda}{\alpha} \\ Nr_2(z) &= (1-R_s^*(\lambda\nu-\lambda\nu z)) \\ Nr_3(z) &= \beta z[(\beta+\lambda\nu-\lambda\nu z)G^*(\lambda)+\lambda\nu z](1-L^*(\beta+\lambda))[W_v^*(\beta+\lambda\nu-\lambda\nu z) \\ &\quad -W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]-(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))[(\beta+\lambda\nu-\lambda\nu z)G^*(\lambda) \\ &\quad +\lambda\nu z]\{(\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)(\beta+\lambda-\lambda z)L^*(\beta+\lambda)+\lambda z)\} \\ &\quad +\beta z(\lambda+\beta)[W_v^*(\beta+\lambda\nu-\lambda\nu z)-W_v^*(\beta+\lambda\nu-\lambda\nu z_1)]L^*(\beta+\lambda) \\ &\quad -\frac{\lambda}{\alpha}(1-W_v^*(\beta+\lambda\nu-\lambda\nu z_1))G^*(\lambda)(\beta+\lambda\nu-\lambda\nu z)\{(\beta+\lambda)z \\ &\quad -W_v^*(\beta+\lambda\nu-\lambda\nu z)[\lambda z+(\beta+\lambda-\lambda z)L^*(\beta+\lambda)\} \\ Dr_1(z) &= (\beta+\lambda)z-W_v^*(\beta+\lambda\nu-\lambda\nu z)[L^*(\beta+\lambda)(\beta+\lambda-\lambda z)+\lambda z] \\ Dr_2(z) &= z-R_s^*(\lambda\nu-\lambda\nu z)[(1-z)G^*(\lambda)+z] \end{aligned}$$

At $z = 1$ R_s turns,

$$R_s = \frac{M_{r_4}}{2(\beta\eta Dr_1(1)Dr_2'(1))^2} W_{0,0}$$

$$M_{r_4} = (1-G^*(\lambda))\left[2Nr_1'(1)Dr_2'(1)(\lambda\eta Dr_1(1) - \beta\eta Dr_1'(1)) + \beta\eta Dr_1(1)(Dr_2'(1)Nr_1''(1) - Nr_1'(1)Dr_2''(1))\right] + 2\beta\eta Nr_2'(1)Dr_2'(1)(Dr_1(1)Nr_3'(1) - Nr_3(1)Dr_1'(1)) + Nr_3(1)Dr_1(1)[2\lambda\eta Nr_2'(1)Dr_2'(1) + \beta\eta Dr_2'(1)Nr_2''(1) - \beta\eta Nr_2'(1)Dr_2''(1)]$$

By Little's formula $R_{sw} = \dfrac{R_s}{\lambda}$,

4. Special cases

(a) If the service time distribution follows an exponential distribution, $\nu = 1$ and no service during the vacation period, then the present model will be remodelled as a time-dependent analysis of the $M/M/1$ queue with server vacations and a waiting server.
(b) If the server does not wait after the completion of the RS period, $\nu = 1$ and there is no retrial time in the system then the present model will be remodeled as an $M/G/1$ queue with multiple working vacation.
(c) If the server does not wait after the completion of the RS period, $\nu = 1$ and the server never takes a vacation, then the present model will be remodelled as an $M/G/1$ retrial queue.

5. Numerical results

The curved graph constructed in Figure 1 and the values tabulated in the Table 1 are obtained by setting the fixed values $\mu_v = 1.5$, $\mu_s = 9$, $\mu_{vr} = 1.5$, $\mu_{sr} = 4.5$, $\alpha = 0.6$, $\nu = 0.5$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 1 to 2 in steps of 0.5, we observed that as λ rises W_v also rises and hence the stability of the model is verified.

Table 1: *R_s with turn over of λ*

λ	$\beta = 3$	$\beta = 5$	$\beta = 7$
1.0	0.0055	0.0081	0.0087
1.2	0.0082	0.0117	0.0125
1.4	0.0116	0.0161	0.0172
1.6	0.0156	0.0226	0.0226
1.8	0.0290	0.0271	0.0290
2.0	0.0262	0.0343	0.0363

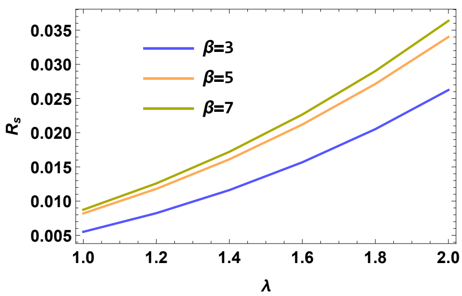

Figure 1: *R_s with turn over of λ*

Table 2: *R_{sw} with turn over of λ*

λ	$\beta = 1.5$	$\beta = 2$	$\beta = 2.5$
1.0	0.0062	0.0109	0.0131
1.2	0.0087	0.0134	0.0157
1.4	0.0114	0.0160	0.0184
1.6	0.0142	0.0188	0.0212
1.8	0.0174	0.0218	0.0242
2.0	0.0208	0.0251	0.0275

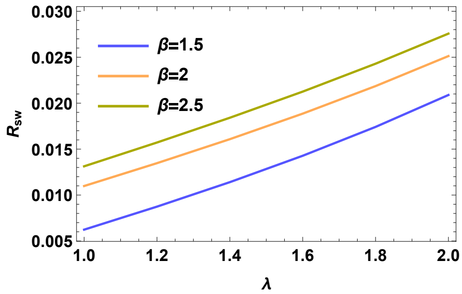

Figure 2: *R_{sw} with turn over of λ*

Table 3: *W_v with turn over of λ*

λ	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
1.0	0.0510	0.0352	0.0247
1.2	0.0680	0.0460	0.0321
1.4	0.0862	0.0573	0.0397
1.6	0.1051	0.0688	0.0474
1.8	0.1245	0.0804	0.0551
2.0	0.1440	0.0920	0.0628

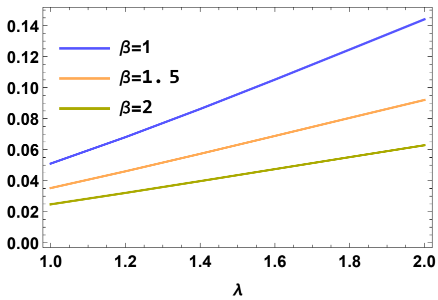

Figure 3: *W_v with turn over of λ*

The curved graph constructed in Figure 2 and the values tabulated in the Table 2 are obtained by setting the fixed values $\mu_v = 2.6$, $\mu_s = 9.7$, $\mu_{vr} = 1.5$, $\mu_{sr} = 3.9$, $\alpha = 0.5$, $\nu = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 1.2 to 1.6 in steps of 0.2. We observed that as λ rises W_{vw} also rises which is expected. The curved graph constructed in Figure 3 and the values tabulated in the Table 3 are obtained by setting the fixed values $\mu_v = 3.5$, $\mu_s = 7.5$, $\mu_{vr} = 2.5$, $\mu_{sr} = 6.5$, $\alpha = 0.7$, $\nu = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 3 to 7 in steps of 2. We observed that as λ rises R_s also rises which shows the stability of the model. The curved graph

Table 4: *W_{vw} with turnover of λ*

λ	$\beta = 1.2$	$\beta = 1.4$	$\beta = 1.6$
1.0	0.0198	0.0178	0.0159
1.2	0.0217	0.0196	0.0174
1.4	0.0234	0.0211	0.0187
1.6	0.0249	0.0223	0.0198
1.8	0.0263	0.0235	0.0207
2.0	0.0274	0.0244	0.0216

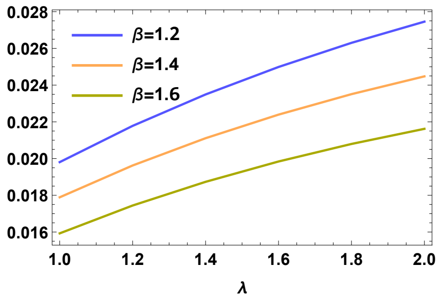

Figure 4: *W_{vw} with turnover of λ*

constructed in Figure 4 and the values tabulated in the Table 4 are obtained by setting the fixed values $\mu_v = 2.2$, $\mu_s = 9.9$, $\mu_{vr} = 1.2$, $\mu_{sr} = 3.3$, $\alpha = 0.2$, $\nu = 0.3$ and altering the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 1.5 to 2.5 in steps of 0.5. From the graph, we studied that as λ rises R_{sw} also rises which shows the stability of the model.

6. Conclusion

In this paper, an $M/G/1$ retrial queue with working vacation, nonpersistent customers, and a waiting server is evaluated. We obtain the PGF for the number of customers and the mean number of customers in the orbit. Further with the waiting time distribution and the performance measures are derived. The current problem is verified with some existing models through adding and assuming some particular values. We illustrate the mean orbit size and mean waiting time of customers during working vacation and regular service periods against the arrival rate of the customers. This model's results are very useful for better service management in various fields.

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