A NEW TRANSMUTED PROBABILITY MODEL: PROPERTIES AND APPLICATIONS

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Abstract

In this article, we introduced a new three parameter continuous probability model by extending a two parameter log-logistic distribution using the quadratic rank transmutation map technique. We provide a comprehensive description of the statistical properties of the newly introduced model. Robust measures of skewness and kurtosis of the proposed model have also been derived along with the moment generating function, characteristic function, reliability function and hazard rate function of the proposed model. The estimation of the model parameters is performed by maximum likelihood method followed by a Monte Carlo simulation procedure. The applicability of this distribution to modeling real life data is illustrated by two real life examples and the results of comparison to base distribution in modeling the data are also exhibited.

Keywords: Transmuted Probability Model, Survival Analysis, Reliability Measures, Monte Carlo Simulation.

1. Introduction

The quality of procedures that are put to use in a statistical analysis relies greatly upon the assumed probability model or distribution. As a consequence of this, significant effort has been directed over the course of history towards the development of large classes of standard distributions along with relevant statistical methodologies. These happen to be designed for serving as models for a wide variety of real-world phenomena. However, many important situations exist where real data does not follow any of the classical or standard models. In the work that follows, we have obtained a three-parameter Generalized Log-Logistic Distribution (GLLD) by utilizing the Quadratic Rank Transmutation Map (QRTM) technique proposed by Shaw and Buckley [1]. The field of transmutation has seen a lot of research recently. Ashour and Eltehiwy [2] introduced a new generalized distribution of the exponentiated modified Weibull distribution using the transmutation technique. Aryal et al. [3] introduced the transmuted extreme value distribution. Merovci et al. [4, 5] studied the transmuted Lindley and Rayleigh distributions. Now we will study the three-parameter Generalized Log-Logistic Distribution (GLLD) and obtain and understand its different characteristics as well as its structural properties.

According to the Quadratic Rank Transmutation Map (QRTM) technique for generalization, the cumulative distribution function (CDF) must satisfy the relationship:

$$F_t(x) = (1+\lambda)F_b(x) - \lambda[F_b(x)]^2 \tag{1}$$

which upon differentiation yields,

$$f_t(x) = f_b(x)[1+\lambda-2\lambda F_b(x)] \tag{2}$$

where $f_b(x)$and $f_t(x)$ are the probability density functions corresponding to $F_b(x)$ and

$F_t(x)$respectively and$|\lambda| \leq 1$. $F_b(x)$is the CDF of the base distribution. If we put $\lambda = 0$, we get the base distribution.

The log-logistic distribution is a continuous probability distribution particularly useful in dealing with survival data. It is specifically used as a parametric model for events whose rate increases initially and later diminishes. For example, mortality rate from a certain cancer post diagnosis or treatment. The probability density function (pdf) of the two-parameter log-logistic distribution is given as:

$$f(x;\alpha,\beta) = \frac{\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^{\beta})^2} \tag{3}$$

The corresponding cumulative distribution function (CDF) is given as:

$$F(x) = \Pr(X \leq x) = \frac{(\alpha x)^{\beta}}{1+(\alpha x)^{\beta}} \tag{4}$$

where α is a scale parameter while β is a shape parameter.

The remaining paper is organized as follows. In subSection 1, the three-parameter Generalized Log-Logistic Distribution is demonstrated. The various statistical properties of the generalized distribution such as the moments, moment generating function, characteristic function, order statistics, quantile function, etc. are summarized in Section 2. The MLE of the distribution parameters are illustrated in Section 3 of this paper and contains an exhibition of the Monte Carlo simulation procedure. Robust measures of skewness and Kurtosis along with graphical illustrations are presented in Section 4. Section 5 deals with the applicability of this generalized distribution in modeling real life data which is illustrated by two real-life data sets.

1.1 Three-Parameter Generalized Log-Logistic Distribution (GLLD)

This section deals with the study of the three-parameter Generalized Log-Logistic Distribution. Using (1) and (4), the CDF of GLLD is obtained as follows:

$$F_t(x) = (1+\lambda)F_b(x) - \lambda[F_b(x)]^2$$

$$\Rightarrow F_t(x) = (1+\lambda)\left[\frac{(\alpha x)^{\beta}}{1+(\alpha x)^{\beta}}\right] - \lambda\left[\frac{(\alpha x)^{\beta}}{1+(\alpha x)^{\beta}}\right]^2$$

After simplifying, we obtain the CDF of three-parameter Generalized Log-Logistic Distribution as

$$\therefore F(x;\alpha,\beta,\lambda) = \frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}, \qquad x,\alpha,\beta > 0 \;\&\; -1 \leq \lambda \leq 1 \tag{5}$$

Hence, the pdf of GLLD with parameters α, β and λ is obtained using (5) as follows:

$$f(x;\alpha,\beta,\lambda) = \frac{d}{dx}\frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}$$

$$\therefore f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\big(1+(\alpha x)^{\beta}\big) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^3}, \quad x,\alpha,\beta > 0 \;\&\; -1 \leq \lambda \leq 1 \tag{6}$$

The CDF and pdf plots for (5) and (6) respectively for different values of the parameters involved is illustrated through figure 1 and 2 respectively. The plots reveal quite evidently that the distribution of the three-parameter generalized log-logistic random variable X is right skewed.

2. Statistical Properties of GLLD

This section deals with the various structural properties of the three-parameter GLLD such as moments (non-central and central), moment generating function, characteristic function, order statistics, quantile function as well as the survival measures. All these have been obtained and discussed in the sub-sections that follow.

2.1 Moments

Moments refer to a set of statistical parameters that are useful in measuring a distribution. They are the crucial measures to calculate mean, variance, skewness and kurtosis of the data. Skewness deals with symmetry of a distribution, or in more precise terms, the lack of symmetry of a distribution. Kurtosis enables us to measure the peakedness or flatness of a distribution. Another interpretation of kurtosis is concerned with the heavy or light-tailed nature of the data relative to a normal distribution.

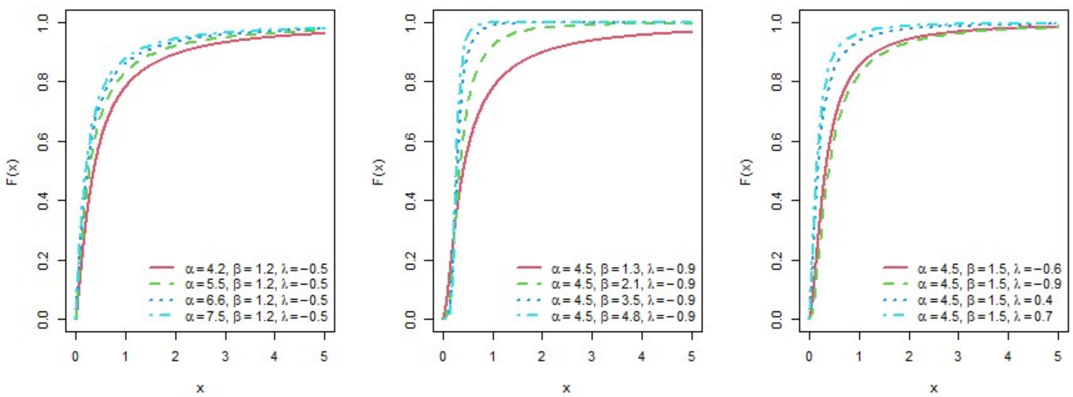

Fig 1: *CDF plots of three parameter GLLD*

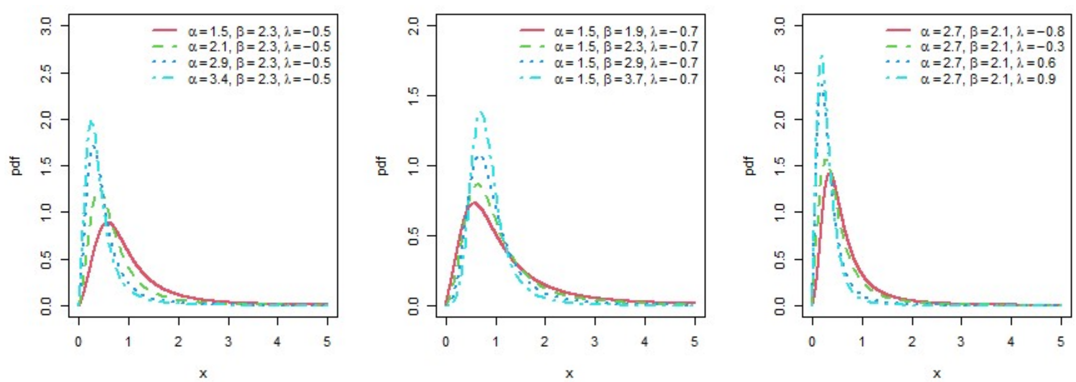

Fig 1: *pdf plots of three parameter GLLD*

The theorem 1.1 is used to arrive at the rthnon-central moment of the three parameter GLLD.

Theorem 1.1: If a random variable X follows GLLD with parameters α, β and λ such that $\alpha, \beta > 0$and $|\lambda| \leq 1$, then the rth non-central moment is given by

$$\mu_r' = \frac{(1+\lambda)}{\alpha^r}\beta\left(1+\frac{r}{\beta}, 1-\frac{r}{\beta}\right) - \frac{2\lambda}{\alpha^r}\beta\left(2+\frac{r}{\beta}, 1-\frac{r}{\beta}\right) \quad (7)$$

Proof:

We know by the definition of the rth raw moment that

$$\mu_r' = \mathbb{E}(X^r)$$

$$\Rightarrow \mu_r' = \int_0^\infty x^r f(x;\alpha,\beta,\lambda)dx$$

$$\Rightarrow \mu_r' = \int_0^\infty x^r \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^\beta\right) - 2\lambda(\alpha x)^\beta\}}{(1+(\alpha x)^\beta)^3}dx$$

$$\Rightarrow \mu_r' = \int_0^\infty x^r \frac{(1+\lambda)\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^\beta)^2}dx - \int_0^\infty x^r \frac{2\lambda(\alpha x)^\beta \alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^\beta)^3}dx$$

Put $(\alpha x)^\beta = t$, we obtain $x = \frac{t^{\frac{1}{\beta}}}{\alpha}$ and $\alpha\beta(\alpha x)^{\beta-1}dx = dt$

Also, as $x \to 0, t \to 0$ and as $x \to \infty, t \to \infty$

$$\therefore \mu_r' = (1+\lambda)\int_0^\infty \frac{\left(\frac{t^{\frac{1}{\beta}}}{\alpha}\right)^r}{(1+t)^2}dt - 2\lambda\int_0^\infty \frac{\left(\frac{t^{\frac{1}{\beta}}}{\alpha}\right)^r t}{(1+t)^3}dt$$

$$\Rightarrow \mu_r' = \frac{(1+\lambda)}{\alpha^r}\int_0^\infty \frac{t^{\left(\frac{r}{\beta}+1\right)-1}}{(1+t)^{\left(\frac{r}{\beta}+1\right)+\left(1-\frac{r}{\beta}\right)}}dt - \frac{2\lambda}{\alpha^r}\int_0^\infty \frac{t^{\left(\frac{r}{\beta}+2\right)-1}}{(1+t)^{\left(\frac{r}{\beta}+2\right)+\left(1-\frac{r}{\beta}\right)}}dt$$

$$= \frac{(1+\lambda)}{\alpha^r}\beta\left(1+\frac{r}{\beta}, 1-\frac{r}{\beta}\right) - \frac{2\lambda}{\alpha^r}\beta\left(2+\frac{r}{\beta}, 1-\frac{r}{\beta}\right)$$

where

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\therefore \mu_r' = \frac{(1+\lambda)}{\alpha^r}\frac{\Gamma\left(1+\frac{r}{\beta}\right)\Gamma\left(1-\frac{r}{\beta}\right)}{\Gamma(2)} - \frac{2\lambda}{\alpha^r}\frac{\Gamma\left(2+\frac{r}{\beta}\right)\Gamma\left(1-\frac{r}{\beta}\right)}{\Gamma(3)}$$

$$\Rightarrow \mu_r' = \frac{(1+\lambda)}{\alpha^r}\Gamma\left(\frac{\beta+r}{\beta}\right)\Gamma\left(\frac{\beta-r}{\beta}\right) - \frac{\lambda}{\alpha^r}\Gamma\left(\frac{2\beta+r}{\beta}\right)\Gamma\left(\frac{\beta-r}{\beta}\right)$$

Thus, the rth non-central moment is given by the expression

$$\mu_r' = \frac{1}{\alpha^r}\,\Gamma\left(\frac{\beta-r}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+r}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+r}{\beta}\right)\right] \tag{8}$$

Using expression (8), the first two raw moments for three-parameter GLLD can be easily obtained. These are given by:

$$\mu_1' = \frac{1}{\alpha}\,\Gamma\left(\frac{\beta-1}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+1}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+1}{\beta}\right)\right] \tag{9}$$

$$\mu_2' = \frac{1}{\alpha^2}\,\Gamma\left(\frac{\beta-2}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+2}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+2}{\beta}\right)\right] \tag{10}$$

Besides, we know that variance is given by

$$\mu_2 = \mu_2' - (\mu_1')^2$$

Thus, the variance of the three-parameter GLLD is given by:

$$\mu_2 = \frac{1}{\alpha^2}\,\Gamma\left(\frac{\beta-2}{\beta}\right)\left[(1+\lambda)\Gamma\left(\frac{\beta+2}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+2}{\beta}\right)\right] - \left[\frac{1}{\alpha}\,\Gamma\left(\frac{\beta-1}{\beta}\right)\left\{(1+\lambda)\Gamma\left(\frac{\beta+1}{\beta}\right) - \lambda\,\Gamma\left(\frac{2\beta+1}{\beta}\right)\right\}\right]^2 \tag{11}$$

It is important note that for the convergence of the rth moment, $\left(1-\frac{r}{\beta}\right)$ in (8) must be greater than zero. In other words, convergence of rth moment is possible only if $\beta > r$. Thus, existence of mean of the proposed distribution requires that β is greater than 1. For variance, β must be greater than 2. Similarly, for skewness and kurtosis, β must be greater than 3 and 4 respectively. Any situation of divergence of the statistical measures is dealt with by employing robust measures.

2.2 Moment generating function (mgf) and characteristic function (cf)

This sub-section contains the derivation of the mgf and cf of the three-parameter GLLD. The following theorem gives the mgf and cf of the distribution under study.

Theorem 3.2: *If a random variable X follows GLLD with parameters α, β and λ such that $\alpha, \beta > 0$ and $|\lambda| \leq 1$, then the mgf denoted by $M_X(t)$ and the cf denoted by $\psi_X(t)$ has the following form:*

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda \Gamma\left(\frac{2\beta + j}{\beta}\right)\right]$$

and

$$\psi_X(t) = \sum_{j=0}^{\infty} \frac{(\iota t)^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda \Gamma\left(\frac{2\beta + j}{\beta}\right)\right]$$

Proof:

We know from the definition of mgf that

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tx}) \\ &= \int_0^{\infty} \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} f(x;\alpha,\beta,\lambda)dx \\ &= \int_0^{\infty} \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f(x;\alpha,\beta,\lambda)dx = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j' \end{aligned}$$

From (8), we know

$$\mu_j' = \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda \Gamma\left(\frac{2\beta + j}{\beta}\right)\right]$$

$$\therefore M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda \Gamma\left(\frac{2\beta + j}{\beta}\right)\right] \tag{12}$$

which is the required *mgf* of the three-parameter GLLD.

Also, we know that

$$\psi_X(t) = \mathbb{E}(e^{\iota tx})$$
$$\Rightarrow \psi_X(t) = \mathbb{E}(e^{(\iota t)x})$$

$$\Rightarrow \psi_X(t) = \sum_{j=0}^{\infty} \frac{(\iota t)^j}{j!} \frac{1}{\alpha^j} \Gamma\left(\frac{\beta - j}{\beta}\right) \left[(1+\lambda)\Gamma\left(\frac{\beta + j}{\beta}\right) - \lambda \Gamma\left(\frac{2\beta + j}{\beta}\right)\right] \tag{13}$$

which is the required *cf* of the three-parameter GLLD.

2.3 Order Statistics

Stated in the simplest of terms, order statistics refer to sampling values arranged in an ascending order. If $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$ denote the order statistics of a random sample $X_1, X_2, X_3, \dots, X_n$ drawn from a continuous population having CDF $F_X(x)$ and pdf $f_X(x)$, then the pdf of the rth order statistics $X_{(r)}$ is given by:

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)[F_X(x)]^{r-1}[1 - F_X(x)]^{n-r}, \qquad \forall\, r = 1,2,\dots,n$$

Using (5) and (6), the formula for the pdf of the rth order statistic $X_{(r)}$ for the three-parameter GLLD is obtained and is given as under:

$$\begin{aligned} f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} & \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^3} \\ & \times \left[\frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{r-1} \left[1 - \frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{n-r} \end{aligned} \tag{14}$$

For $r = n$, we get the pdf of the nth or the largest order statistic $X_{(n)}$ for the three-parameter GLLD which is obtained as follows:

$$\begin{aligned} f_n(x) = \frac{n!}{(n-1)!\,(n-n)!} & \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^3} \\ \times & \left[\frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{n-1} \left[1 - \frac{(\alpha x)^{2\beta} + (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}\right]^{n-n} \end{aligned}$$

$$= \frac{n!}{(n-1)!} \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1}$$

$$\therefore f_n(x) = \frac{n\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \left[\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1} \tag{15}$$

Also, for $r = 1$, we get the pdf of the first or the smallest order statistic $X_{(1)}$ for the three-parameter GLLD which is obtained as follows:

$$f_1(x) = \frac{n!}{(1-1)!\,(n-1)!} \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}}$$

$$\times \left[\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{1-1} \left[1-\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1}$$

$$= \frac{n!}{(n-1)!} \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[1-\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1}$$

$$\therefore f_1(x) = \frac{n\alpha\beta(\alpha x)^{\beta-1}\left\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\right\}}{(1+(\alpha x)^{\beta})^{3}} \times \left[1-\frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}}\right]^{n-1} \tag{16}$$

Quite evidently, for $\lambda = 0$, the order statistics of the base distribution i.e., the Log-Logistic Distribution, are yielded.

2.4. Quantile function and random number generation

A prominent method that is put to use for the sake of generating random numbers from a specified distribution is the inverse CDF method. This method generates random numbers from a particular distribution by equating the CDF of the distribution to a number u where u itself follows continuous uniform distribution, $U(0,1)$. Solving the equation yields the quantile function of the distribution. Employing this inverse CDF method, we proceed to obtain the quantile function of the three-parameter GLLD as follows:

$$F(x;\alpha,\beta,\lambda) = u$$

$$\Rightarrow \frac{(\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^{2}} = u$$

$$\Rightarrow (\alpha x)^{2\beta}+(1+\lambda)(\alpha x)^{\beta} = u\left(1+(\alpha x)^{\beta}\right)^{2}$$

After simplifying, we obtain

$$x^{\beta} = \frac{-\alpha^{\beta}(1+\lambda-2u) \pm \sqrt{\alpha^{2\beta}(1+\lambda)^{2}-4u\lambda\alpha^{2\beta}}}{2\alpha^{2\beta}(1-u)}$$

$$= \frac{-\alpha^{\beta}(1+\lambda-2u) \pm \sqrt{(\alpha^{\beta})^{2}}\sqrt{(1+\lambda)^{2}-4u\lambda}}{2\alpha^{2\beta}(1-u)}$$

$$= \alpha^{\beta}\left\{\frac{-(1+\lambda-2u) \pm \sqrt{(1+\lambda)^{2}-4u\lambda}}{2\alpha^{\beta}\alpha^{\beta}(1-u)}\right\}$$

$$= \frac{-(1+\lambda-2u) \pm \sqrt{(1+\lambda)^{2}-4u\lambda}}{2\alpha^{\beta}(1-u)}$$

$$\therefore x = \left[\frac{-(1+\lambda-2u) + \sqrt{(1+\lambda)^{2}-4u\lambda}}{2\alpha^{\beta}(1-u)}\right]^{\frac{1}{\beta}}, \tag{17}$$

Equation (17) is the required quantile function of three-parameter GLLD. Note that the negative root of (17) has been discarded since x only takes values greater than 0. Equation (17) yields random numbers from three-parameter GLLD. For $u = 0.25, 0.50$ and 0.75, the values of x obtained represent the first, second and third quartiles of the distribution, respectively. In a similar fashion, deciles and percentiles of different orders are obtained by assigning different values to u.

2.5. Survival measures of three-parameter GLLD

This sub-section deals with the survival measures of three-parameter GLLD such as the survival function and the hazard function. The survival function, also known as the survivorship function, refers to the probability that a life, system or a component will survive beyond a specified time. In mathematical terms, it happens to be the complement of the CDF and is given by:

$$S(x) = \Pr(X > x) = 1 - F(x) \quad (18)$$

Using (5) in (18), we obtain the survival function of three-parameter GLLD as follows:

$$S(x;\alpha,\beta,\lambda) = \frac{\left(1+(\alpha x)^{\beta}\right)^2 - (\alpha x)^{2\beta} - (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}$$

$$S(x;\alpha,\beta,\lambda) = \frac{\left(1+2(\alpha x)^{\beta} + (\alpha x)^{2\beta}\right) - (\alpha x)^{2\beta} - (1+\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}$$

$$\therefore S(x;\alpha,\beta,\lambda) = \frac{1+(1-\lambda)(\alpha x)^{\beta}}{(1+(\alpha x)^{\beta})^2}, \qquad x,\alpha,\beta > 0 \ \& -1 \le \lambda \le 1 \quad (19)$$

The hazard function, also known as the hazard rate or failure rate or force of mortality, happens to be an important quantity used for the characterization of life phenomenon. Hazard function is defined as the conditional probability that a life, system or a component that survives up to a specified time, will undergo failure or succumb in the immediate, infinitesimally small interval of time that follows. In mathematical terms, the hazard rate or the hazard function is given by:

$$h(x) = \lim_{\Delta t \to 0} \frac{\Pr[t \le X < t + \Delta t \mid X \ge t]}{\Delta t}$$

which upon simplification yields

$$h(x) = \frac{f(x)}{S(x)} \quad (20)$$

Using (6) and (19) in (20), we obtain the hazard function of three-parameter GLLD as follows:

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right) - 2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})\{1+(1-\lambda)(\alpha x)^{\beta}\}}, \quad x,\alpha,\beta > 0 \ \& |\lambda| \le 1 \quad (21)$$

The survival function and the hazard function plots for (19) and (21) respectively for different values of the parameters involved is illustrated through figure 3 and 4 respectively.

3. Maximum Likelihood Estimation

One of the most useful frameworks in parameter estimation is the Maximum Likelihood estimation (MLE). This method obtains the unknown population parameters by the virtue of likelihood maximization.

In this section, the parameters α, β and $\lambda$of the three-parameter GLLD are estimated using the method of maximum likelihood estimation (MLE). The procedure is given as follows: Consider a random sample $X_1, X_2, \ldots, X_n$of size n taken from the three-parameter GLLD. The likelihood function based on this sample is therefore given as:

$$L(x|\alpha,\beta,\lambda) = \prod_{i=1}^{n} \frac{\alpha\beta(\alpha x_i)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x_i)^{\beta}\right) - 2\lambda(\alpha x_i)^{\beta}\}}{(1+(\alpha x_i)^{\beta})^3} \quad (22)$$

$$\Rightarrow L = \left(\alpha^{\beta}\beta\right)^n \frac{\prod_{i=1}^{n} x_i^{\beta-1} \prod_{i=1}^{n}\{(1+\lambda)\left(1+(\alpha x_i)^{\beta}\right) - 2\lambda(\alpha x_i)^{\beta}\}}{\prod_{i=1}^{n}(1+(\alpha x_i)^{\beta})^3} \quad (23)$$

Taking logarithm on both sides of (23), we obtain the log likelihood function as follows:

$$\Rightarrow \log L = \log\left[\left(\alpha^{\beta}\beta\right)^n \frac{\prod_{i=1}^{n} x_i^{\beta-1} \prod_{i=1}^{n}\{(1+\lambda)\left(1+(\alpha x_i)^{\beta}\right) - 2\lambda(\alpha x_i)^{\beta}\}}{\prod_{i=1}^{n}(1+(\alpha x_i)^{\beta})^3}\right]$$

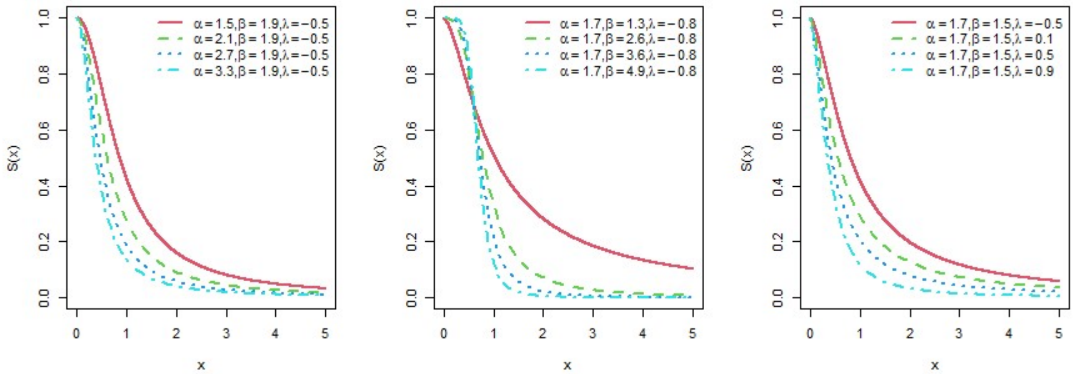

Fig. 3: *Survival function plot for three parameter GLLD*

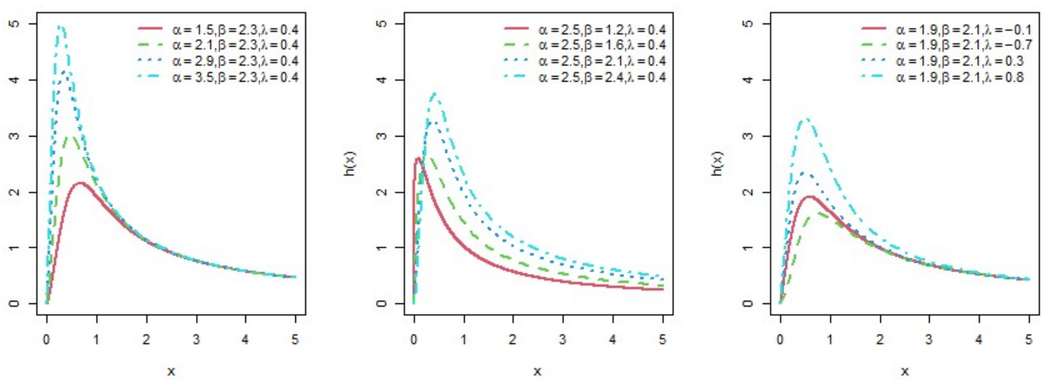

Fig. 4: *Hazard rate function plot for three parameter GLLD*

$$\Rightarrow \log L \log L = n\beta \log \alpha + n \log \beta + (\beta - 1)\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\{(1+\lambda)\big(1+(\alpha x_i)^{\beta}\big) - 2\lambda(\alpha x_i)^{\beta}\} - 3\sum_{(i=1)}^{n} \log(1+(\alpha x_i)^{\beta}) \tag{24}$$

which is the required log-likelihood function.

The MLEs of the parameters α, β and λ of GLLD are obtained by differentiation of the log-likelihood function (24) w.r.t α, β and λ.The partial derivatives used for estimating the parameters are obtained as follows:

$$\frac{\partial}{\partial \alpha}\log L = \frac{n\beta}{\alpha} + \sum_{i=1}^{n}\left[\frac{\{(1+\lambda)\big(\beta\alpha^{\beta-1}x_i^{\beta}\big) - 2\lambda\beta\alpha^{\beta-1}x_i^{\beta}\}}{\{(1+\lambda)(1+(\alpha x_i)^{\beta}) - 2\lambda(\alpha x_i)^{\beta}\}}\right] - 3\sum_{i=1}^{n}\left[\frac{\beta\alpha^{\beta-1}x_i^{\beta}}{(1+(\alpha x_i)^{\beta})}\right] \tag{25}$$

$$\frac{\partial}{\partial \beta}\log L = n\log\alpha + \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\left[\frac{\{(1+\lambda)(\alpha x_i)^{\beta}\log(\alpha x_i) - 2\lambda(\alpha x_i)^{\beta}\log(\alpha x_i)\}}{\{(1+\lambda)(1+(\alpha x_i)^{\beta}) - 2\lambda(\alpha x_i)^{\beta}\}}\right] - 3\sum_{i=1}^{n}\left[\frac{(\alpha x_i)^{\beta}\log(\alpha x_i)}{(1+(\alpha x_i)^{\beta})}\right] \tag{26}$$

$$\frac{\partial}{\partial \lambda}\log L = \sum_{i=1}^{n}\left[\frac{1-(\alpha x_i)^{\beta}}{\{(1+\lambda)(1+(\alpha x_i)^{\beta}) - 2\lambda(\alpha x_i)^{\beta}\}}\right] \tag{27}$$

The derivative equations (25), (26) and (27) cannot be analytically solved and thereby estimates of the parameters α, β and λ denoted by $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are obtained by maximization of log-likelihood function through the employment of powerful iterative numerical methods such as the Newton-Raphson method. The second order partial derivatives are computed which are helpful in obtaining the Fisher's Information Matrix in the following manner:

$$I_x(\alpha,\beta,\lambda)=\begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial\alpha^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right) & -E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\lambda}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right) & -E\left(\frac{\partial^2 \log L}{\partial\beta^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial\beta\partial\lambda}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial\lambda\partial\alpha}\right) & -E\left(\frac{\partial^2 \log L}{\partial\lambda\partial\beta}\right) & -E\left(\frac{\partial^2 \log L}{\partial\lambda^2}\right) \end{bmatrix} \tag{28}$$

It can be shown that the three-parameter GLLD satisfies the regularity conditions and thereby the MLE vector $\Theta=(\hat{\alpha},\hat{\beta},\hat{\lambda})^T$ is consistent as well as asymptotically normal, i.e., $\sqrt{n}\left[(\hat{\alpha},\hat{\beta},\hat{\lambda})^T-(\alpha,\beta,\lambda)^T\right]$ converges to a normal distribution with mean vector 0 and the identity covariance matrix. Fisher's Information matrix in (28) is calculated by virtue of the following approximation:

$$I_x(\hat{\alpha},\hat{\beta},\hat{\lambda})\approx\begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial\alpha^2}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\lambda}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} \\ -E\left(\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial\beta^2}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial\beta\partial\lambda}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} \\ -E\left(\frac{\partial^2 \log L}{\partial\lambda\partial\alpha}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial\lambda\partial\beta}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} & -E\left(\frac{\partial^2 \log L}{\partial\lambda^2}\right)\Big|_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} \end{bmatrix} \tag{29}$$

Where $\hat{\alpha},\hat{\beta}$ and $\hat{\lambda}$ are the MLEs of α,β and $\hat{\lambda}$ respectively. This approximation is useful in the construction of the confidence intervals for the parameters of three-parameter GLLD. The approximate $100(1-\alpha)\%$ confidence intervals for α,β and λ are respectively given by:

$$\hat{\alpha}\pm z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}(\widehat{\Theta})},\hat{\beta}\pm z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}(\widehat{\Theta})}\ and\ \hat{\lambda}\pm z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}(\widehat{\Theta})} \tag{30}$$

3.1. Monte Carlo Simulation Study of ML Estimates

Monte Carlo simulation refers to a wide range of computational algorithms aimed at obtaining numerical results by using repeated random sampling. This sub-section contains a behavioral analysis of the maximum likelihood estimates of three-parameter GLLD for a finite sample of size n. A MC simulation study for different values of parameters α,β and γ is employed for this purpose with random numbers being generated using the quantile function (17) obtained earlier. The procedure undertaken involves a simulation study for each triplet (α,β,λ) for the parameter combinations $(\alpha=0.7,\beta=0.5,\lambda=0.4)$ and $(\alpha=1.2,\beta=0.8,\lambda=0.5)$.The iterative process is carried out 100 times for samples of size n, where $n=25,75,150,200$ and 500, generating 100 samples of the mentioned sample sizes. ML estimates for each sample generated are then obtained and their average bias, variance and MSE is calculated. The results have been tabulated in Table 1 and clearly indicate that with the increase in the sample size n, agreement between theory and practice improves significantly. MSE and variance of estimates of α,β and λ indicate consistency and that the ML method performs well for estimation of parameters of the three-parameter GLLD.

Table 1: *Average Bias, Variance and MSE for simulated results of MLEs*

Sample size n	Parameters	$(\alpha=0.7,\beta=0.5,\lambda=0.4)$			$(\alpha=1.2,\beta=0.8,\lambda=0.5)$		
		Bias	Variance	***MSE***	Bias	Variance	***MSE***
25	α	0.026299	2.645356	2.646048	-0.15184	0.633681	0.656737
	β	0.008205	0.009891	0.009958	0.012896	0.052401	0.052567
	λ	0.284835	0.18187	0.263	0.160373	0.173618	0.199338
75	α	-0.184323	0.841443	0.875418	-0.16281	0.495852	0.522358
	β	-0.009204	0.003322	0.003407	-0.04699	0.026425	0.028633
	λ	0.237382	0.142089	0.198439	0.156228	0.153889	0.178296

150	α	-0.357382	0.54484	0.672562	-0.05974	0.366872	0.370441
	β	-0.020729	0.001617	0.002047	-0.03842	0.012117	0.013593
	λ	0.271836	0.148146	0.22204	0.067127	0.140693	0.145199
200	α	-0.276202	0.540957	0.617245	0.037218	0.422209	0.423594
	β	-0.018463	0.001689	0.00203	-0.00735	0.012184	0.012238
	λ	0.241823	0.135421	0.193899	0.03889	0.128701	0.130213
500	α	-0.377683	0.375987	0.518631	0.095593	0.332553	0.341691
	β	-0.01319	0.000977	0.001151	-0.02082	0.006032	0.006466
	λ	0.252524	0.10096	0.164728	-0.01595	0.102887	0.103141

4. Robust Skewness and Kurtosis Measures for three-parameter GLLD

This section deals with the study of skewness and kurtosis measures for the proposed distribution. Skewness and kurtosis both deal with the shape of the distribution with the former concerned with symmetry while latter with the tailedness and peakedness of the distribution. The effect of parameters on the skewness and kurtosis of the distribution is studied in this section by considering measures based on quantiles.

Bowley[6] proposed a coefficient of skewness based on quantiles which is well known in statistical literature and is one of the earliest measures of skewness. It is defined as the average of the first and third quartiles minus the median divided by half the interquartile range. It is given by:

$$B = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1} = \frac{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{31}$$

Bowley's coefficient of skewness lies between $+1$ and -1.

Moors[7] proposed a robust alternative to the conventional measure of kurtosis in order to overcome the shortcomings of the latter. For many heavy tailed distributions, the conventional measure is infinite and uninformative as such. The new measure of kurtosis based on quantiles, however, is less sensitive to outliers and even exists for distributions for which there are not any defined moments. The Moors' kurtosis based on octiles is given by:

$$M = \frac{(E_3 - E_1) + (E_7 - E_5)}{E_6 - E_2} = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{32}$$

For distributions that are symmetrical to 0, the Moors' kurtosis reduces to:

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)} \tag{33}$$

Table 2: *Bowley'sskewness for GLLD $(x; \alpha, \beta, \lambda)$ for different parameter combinations*

Parameters		$\alpha = 1.3$						
		β						
		0.7	1.4	1.9	2.6	3.3	4.4	5.6
$\boldsymbol{\lambda}$	-0.9	0.62570	0.36985	0.28923	0.22568	0.18823	0.15298	0.13014
	-0.7	0.63217	0.36683	0.28241	0.21574	0.17642	0.13942	0.11543
	-0.6	0.63601	0.36633	0.28009	0.21192	0.17169	0.13383	0.10929
	-0.3	0.64848	0.36931	0.27877	0.20697	0.16455	0.12460	0.09870
	0.3	0.64453	0.36414	0.27336	0.20143	0.15894	0.11895	0.09303
	0.6	0.60956	0.33252	0.24491	0.17596	0.13538	0.09727	0.07260
	0.7	0.59349	0.31787	0.23156	0.16380	0.12400	0.08664	0.06248
	0.9	0.55777	0.28555	0.20196	0.13671	0.09850	0.06271	0.03959

Table 3: *Moors' kurtosis for GLLD $(x;\alpha,\beta,\lambda)$ for different parameter combinations*

Parameters		$\alpha = 2.6$						
		β						
		0.7	1.4	1.9	2.6	3.3	4.4	5.6
λ	-0.9	3.02678	1.75277	1.56318	1.45452	1.40469	1.36648	1.34594
	-0.7	3.05154	1.74699	1.55687	1.45018	1.40247	1.36694	1.34849
	-0.6	3.06594	1.74405	1.55301	1.44675	1.39978	1.36525	1.34763
	-0.3	3.11493	1.73857	1.54241	1.43495	1.38840	1.35504	1.33856
	0.3	3.07158	1.72426	1.53261	1.42809	1.38310	1.35111	1.33550
	0.6	2.76317	1.64151	1.48248	1.39720	1.36142	1.33684	1.32545
	0.7	2.60943	1.59528	1.45225	1.37657	1.34545	1.32469	1.31549
	0.9	2.27908	1.48840	1.37979	1.32506	1.30423	1.29197	1.28772

For standard normal distribution, it is easy to compute that

$$E_1 = -E_7 = -1.15, E_2 = -E_6 = -0.67 \; and \; E_3 = -E_5 = -0.32$$

Therefore, $M = 1.23$. The centered Moors' coefficient is thus given by:

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} - 1.23 \quad (34)$$

Using R software, the values of Bowley'sskewness and Moors' kurtosis for the three-parameter GLLD for different parameter values have been numerically calculated and tabulated in Tables 2 and 3 respectively. Clearly, Bowley'sskewness as well as Moors' kurtosis are decreasing function of β for a fixed value of the transmuted parameter λ. However, for a fixed value of the scale parameter β, both Bowley's skewness and Moors' kurtosis reflect both increasing and decreasing behavior for different values of the transmuted parameter λ.

5. Applications of three-parameter GLLD

In this particular section, the performance of the proposed generalized log-logistic model is put to test by comparing it with base model. Two real life data sets, one based on survival times and the other on strength data, that are already available in the literature have been used to carry out the comparisons. The procedure involves the computation of MLEs of the transmuted model as well the base model based on both data sets using R software. The various goodness of fit statistics for the two models are then calculated and comparisons carried out. These statistics include AIC (Akaike's Information Criterion) provided by Akaike[8], AICC (AIC Corrected) and BIC (Bayesian Information Criterion) given by Schwarz[9]. AIC, AICC and BIC for a model with k parameters are calculated using the following generic functions:

$$AIC = 2k - 2\log L$$
$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$
$$BIC = k\log n - 2\log L$$

Kolmogorov-Smirnov test is also carried out for testing model significance based on the two mentioned real-life data sets.

Data Set I: The data set reported by Efron[10] is analyzed for carrying out comparisons between three-parameter GLLD and LLD. Efron [10] reported the data set in which observations represent the survival times of a group of patients suffering from head and neck cancer disease and are treated using radiotherapy. The data set is given in Table 4.

Table 4: *Survival times of 58 patients suffering from head and neck cancer disease*

6.53	7	10.42	14.48	16.10	22.70	34	41.55	42	45.28	49.40	53.62
63	64	83	84	91	108	112	129	133	133	139	140
140	146	149	154	157	160	160	165	146	149	154	157
160	160	165	173	176	218	225	241	248	273	277	297
405	417	420	440	523	583	594	1101	1146	1417		

The MLEs, model functions alongside the standard errors based on the above data set are tabulated in Table 5.

The Table 6 contains various goodness of fit measures for models fitted to data given in Table 4. From the table, it is evident that the AIC, AICC and BIC values for the transmuted model (GLLD) are better as compared to the base model (LLD), thereby suggesting that the new model is a better performer. Furthermore, the KS p-value is also greater than 0.05 for GLLD as such reiterating the statistical significance of the new transmuted model over the base model.

Table 5: *MLEs with standard errors of parameters for GLLD and LLD for data set I*

Model	Model function	MLEs	Standard Error
Transmuted Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.55$ $\hat{\lambda}=-0.47$	$SE(\hat{\alpha})=0.002$ $SE(\hat{\beta})=0.203$ $SE(\widehat{\lambda})=0.344$
Base Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^{\beta})^{2}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.52$	$SE(\hat{\alpha})=0.002$ $SE(\hat{\beta})=0.196$

Table 6: *Goodness of fit measures for models fitted to data set I*

Model	$-\log L$	**AIC**	**AICC**	**BIC**	**KS Distance**	**KS p-value**	**LR Statistic**
GLLD	371.1943	748.3887	748.8331	754.5700	0.15548	0.1211	5.01905
LLD	373.7039	751.4077	751.6259	755.5286	0.26802	0.0004	

The GLLD and LLD plots fitted to the survival times of the 58 patients suffering from head and neck cancer disease are illustrated through Figure 5. The graphical overview of the empirical and theoretical (GLLD) CDFs and survival functions for data set I is illustrated through Figures 6 and 7 respectively.

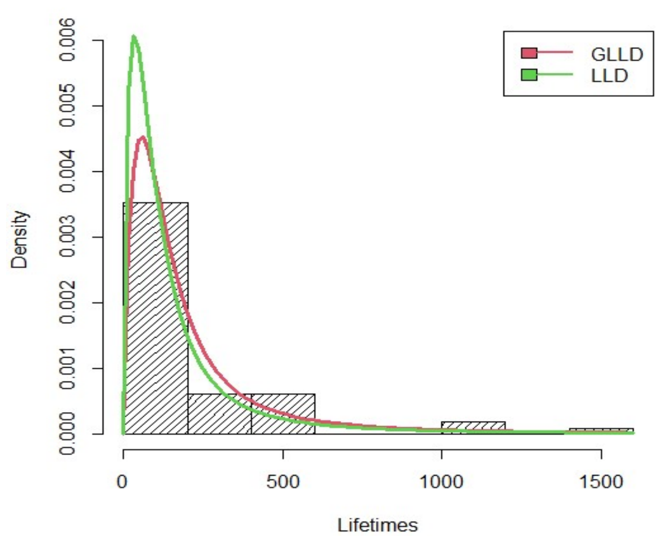

Fig. 5: *Curve fitting GLLD vz LLD for data set I*

Data Set II: The data set reported by Lawless [11] is analyzed for carrying out comparisons between three-parameter GLLD and LLD. Lawlessreported the data set in which the observations represent the number of cycles to failure for 25-100 cm specimens of yarn tested at a particular strain level. The data set is given in Table 7.

Table 7: *Cycles to failure for 25-100 cm specimens of yarn at a specific strain level*

15	20	38	42	61	76	86	98	121	146
149	157	175	176	180	180	198	220	224	251
175	176	180	180	198	653				

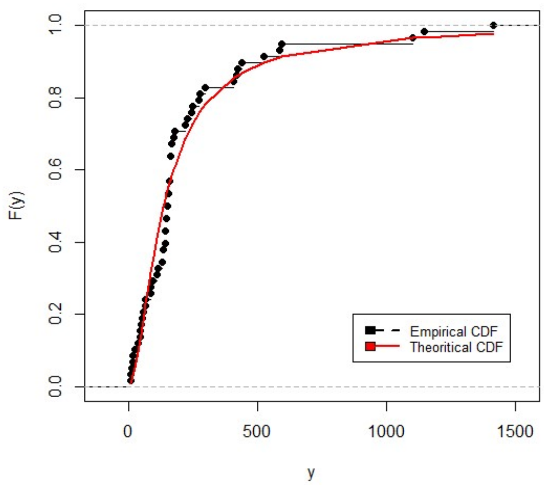

Fig. 6: *Empirical and Theoritical CDF for data set I*

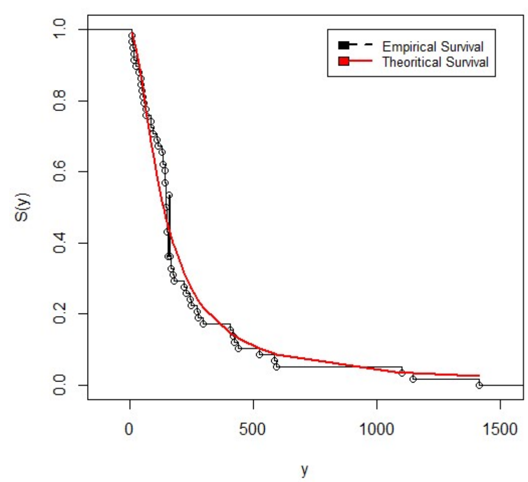

Fig. 7:*Empirical and Theoretical Survival Function for data set I*

The MLEs, model functions alongside the standard errors based on the above data set are tabulated in Table 8 below:

Table 8: *MLEs with standard errors of parameters for GLLD and LLD for data set II*

Model	Model function	MLEs	Standard Error
Transmuted Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}\{(1+\lambda)\left(1+(\alpha x)^{\beta}\right)-2\lambda(\alpha x)^{\beta}\}}{(1+(\alpha x)^{\beta})^{3}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.89$ $\hat{\lambda}=-0.56$	$SE(\hat{\alpha})=0.004$ $SE(\hat{\beta})=0.450$ $SE(\hat{\lambda})=0.491$
Base Model	$\frac{\alpha\beta(\alpha x)^{\beta-1}}{(1+(\alpha x)^{\beta})^{2}}$	$\hat{\alpha}=0.01$ $\hat{\beta}=1.85$	$SE(\hat{\alpha})=0.003$ $SE(\hat{\beta})=0.423$

From the table 9, it is evident that the AIC, AICC and BIC values for the transmuted model (GLLD) are better as compared to the base model (LLD), thereby suggesting that the new model is a better performer. Furthermore, the KS p-value>0.05 for GLLD as such reiterating the statistical significance of the new transmuted model over the base model. In other words, GLLD is a better fit for data given in Table 7 as compared to LLD.

Table 9: *Goodness of fit measures for models fitted to data set II*

Model	$-\log L$	AIC	AICC	BIC	KS Distance	KS *p*-value	LR Statistic
GLLD	154.2395	314.4790	315.6219	318.1356	0.18704	0.346	3.440033
LLD	155.9595	315.9191	316.4645	318.3568	0.30815	0.01734	

The GLLD and LLD plots fitted to the number of cycles to failure for 25 100-cm specimens of yarn tested at a particular strain level are illustrated through Figure 8.The graphical overview of the empirical and theoretical (GLLD) CDFs and survival functions for data set II is illustrated through Figures 9 and 10 respectively.

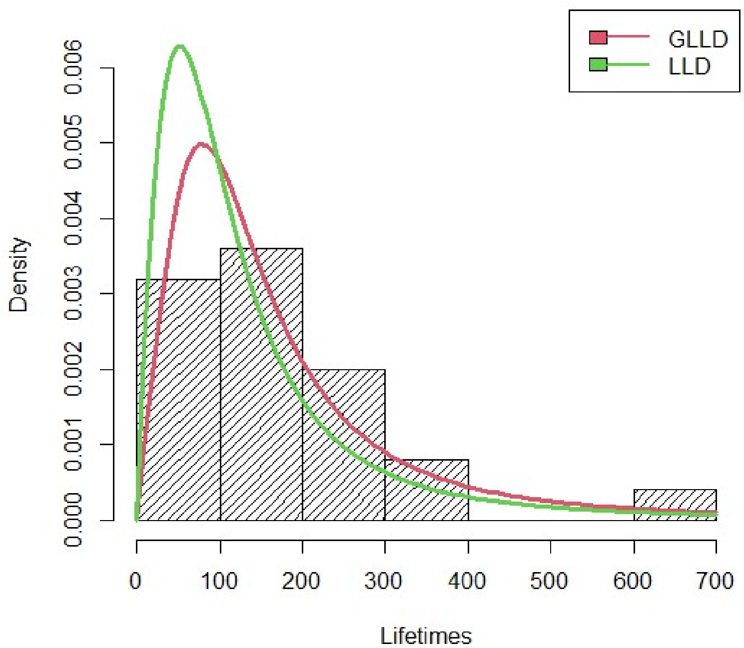

Fig. 8: *Curve fitting GLLD vz LLD for data set I*

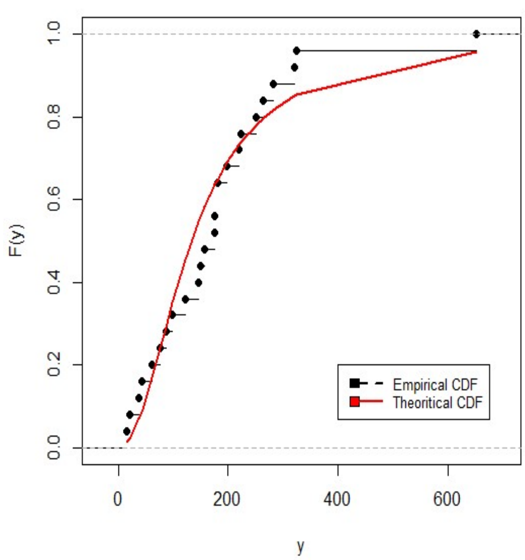

Fig. 9: *Empirical and Theoritical CDF for data set II*

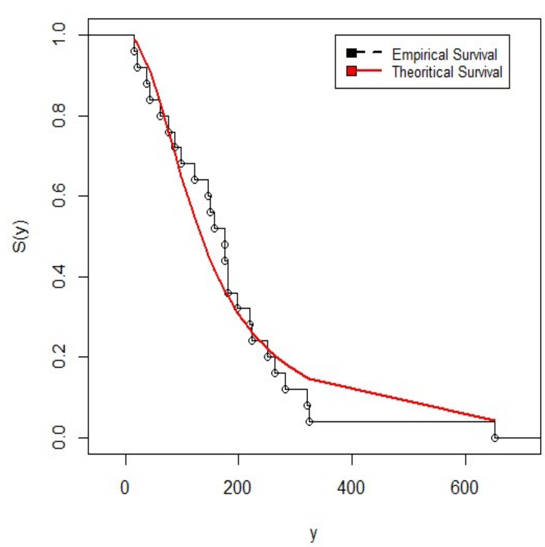

Fig. 10:*Empirical and Theoretical Survival Function for data set II*

6. Concluding Remarks

A new three parameter transmuted probability model namely is introduced by using the quadratic rank transmutation map technique. Comprehensive description of the statistical properties of the newly introduced model are introduced. Robust measures of skewness and kurtosis of the proposed model have also been derived along with the moment generating function, characteristic function, reliability function and hazard rate function of the said model. The estimation of the model parameters is performed by maximum likelihood method followed by a Monte Carlo simulation procedure. The applicability of this distribution to modeling real life data is illustrated by two real life examples and the results of comparison to base distribution in modeling the data are also exhibited.

Conflict of Interest

The Authors declare that there is no conflict of Interest.

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