THE POISSON-SUJA DISTRIBUTION AND ITS APPLICATIONS IN BIOLOGICAL COUNT DATA SETS

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Abstract

The Poisson-Suja distribution which is a Poisson mixture of Suja distribution has been proposed. The descriptive statistics based on moments including coefficient of variation, skewness, kurtosis and index of dispersion has been derived and studied. Over-dispersion, unimodality and increasing hazard rate properties of the distribution have been studied. The method of moment and the method of maximum likelihood have been discussed for estimating parameters. Applications and the goodness of fit the distribution and its comparison with other one-parameter discrete distributions have also been presented. It was found more closer fit than other compared distributions. So, it can be considered as good discrete distribution for count datasets.

Keywords: Suja distribution, compounding, descriptive statistics, statistical properties, estimation of parameter, goodness of fit.

I. Introduction

The statistical analysis and modeling of count data are essential in almost all fields of knowledge including biological science, insurance, medical science, and finance, are some among others. Count data are generated from different discrete phenomena such as the number of insurance claimants in insurance, the number of yeast cells in biological science, the number of chromosomes in genetics, etc. It has been observed that, in general, count data follows under-dispersion (variance greater than mean), equi-dispersion (variance equal to mean), or over-dispersion (variance less than mean). The over-dispersion of count data has been well addressed and discussed using mixed Poisson distributions by different researchers including Raghavachari et al [1], Karlis and Xekalaki [2], Panjeer [3], are some among others. Mixed Poisson distributions arise when the parameter of the Poisson distribution is a random variable having some specified distribution and the distribution of the parameter of the Poisson distribution is known as mixing distribution. It has been observed that the general characteristics of the mixed Poisson distribution follow some characteristics of its mixing distributions also. The field of distribution theory is flooded with mixed Poisson distributions based on some proper mixing distributions.

The classical negative binomial distribution (NBD) derived by Greenwood and Yule [4]is the mixed Poisson distribution where the parameter of the Poisson random variable is distributed as a gamma random variable. The NBD has been used to model over-dispersed count data. However, the NBD may not be appropriate for some over-dispersed count data due to its theoretical or applied point of view. Other mixed Poisson distributions arise from a choice of alternative mixing distributions. For example, the Poisson-Lindley distribution, introduced by Sankaran [5], is a Poisson mixture of Lindley [6] distribution. The Poisson-Akash distribution, introduced by Shanker [7], is a Poisson mixture of Akash distribution proposed by Shanker [8]. The discrete Poisson-Ishita distribution (PID) introduced by Shukla and Shanker [9] is a Poisson mixture of Ishita distribution proposed by Shanker and Shukla [10]. The generalized Poisson-Lindley distribution, introduced by Mahmoudi and Zakerzadeh [11], is a mixed Poisson distribution where the mixing distribution is the generalized Lindley distribution proposed by Zakerzadeh and Dolati [12]. Other mixed distributions are the Poisson-weighted exponential distribution (P-WED) introduced by Zamani et al [13]and the negative binomial-Lindley distribution (NB-LD) introduced by Zamani and Ismail [14]where the mixing distributions were weighted exponential distribution and Lindley [6] distribution, respectively. The Poisson-weighted Lindley distribution (P-WLD) introduced by Abd EL-Monsef and Sohsah [15] is a Poisson mixture of weighted Lindley distribution proposed by Ghitany et al [16].

It has been observed by Karlis and Xekalaki [2] that there are some natural situations where a good fit is not obtainable with a particular mixed Poisson distribution in case of over-dispersed count data. This shows that there is a requirement for new mixed Poisson distribution which gives better fit as compared with the existing mixed Poisson distributions. In the recent decade, there were some one parameter lifetime distributions whose Poisson mixture has not been derived and studied. For example, Shanker [17] suggested a one parameter lifetime distribution named Suja distribution for modeling lifetime data from engineering and biomedical sciences. The Suja distribution is defined by its probability density function (pdf)

$$f(x,\theta) = \frac{\theta^5}{\theta^4+24}(1+x^4)e^{-\theta x};x>0,\theta>0 \tag{1}$$

The Suja distribution is a two-component mixture of exponential distribution having scale parameter θ and a gamma distribution having shape parameter 5 and scale parameterθ with their mixing proportions $\frac{\theta^4}{\theta^4+24}$ and $\frac{24}{\theta^4+24}$ respectively. Various statistical properties of the Suja distribution including its shape, moments, skewness , kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability have been discussed by Shanker [17].

In the present paper, a Poisson mixture of Suja distribution named, "Poisson-Suja distribution (PSD) has been proposed. Its various mathematical and statistical properties including its shape, moments, coefficient of variation, skewness, kurtosis and index of dispersion have been discussed. The estimation of its parameter has been discussed using maximum likelihood estimation and method of moments. The goodness of fit of PSD along with equi-dispersed Poisson distribution (PD), and over-dispersed PLD and PID have been studied and presented with some count datasets.

II. Poisson-Suja distribution

Suppose the parameter λ of Poisson distribution follows Suja distribution (1). Then the Poisson mixture of Suja distribution (1) can be obtained as

$$P(X=x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^5}{\theta^4+24}(1+\lambda^4)e^{-\theta\lambda}d\lambda \tag{2}$$

$$= \frac{\theta^5}{(\theta^4+24)x!}\left[\int_0^\infty e^{-(1+\theta)\lambda}\lambda^x d\lambda + \int_0^\infty e^{-(1+\theta)\lambda}\lambda^{x+4}d\lambda\right]$$

$$= \frac{\theta^5}{(\theta^4+24)x!}\left[\frac{\Gamma(x+1)}{(\theta+1)^{x+1}} + \frac{\Gamma(x+5)}{(\theta+1)^{x+5}}\right]$$

$$= \frac{\theta^5}{(\theta^4+24)(\theta+1)^{x+5}}\left[(\theta+1)^4 + (x+1)(x+2)(x+3)(x+4)\right];\theta > 0, x = 0,1,2,\ldots$$

$$= \frac{\theta^5}{\theta^{24}+24} \cdot \frac{x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)}{(\theta+1)^{x+5}}; x = 0,1,2,\ldots,\theta > 0. \quad (3)$$

As this is the Poisson mixture of Suja distribution, we name this distribution "Poisson-Suja distribution (PSD)".

The graphs of the pmf of PSD for different parameter values are shown in figure 1. It is obvious that as the parameter value increases, the PSD shapes change from negatively skewed to positively skewed.

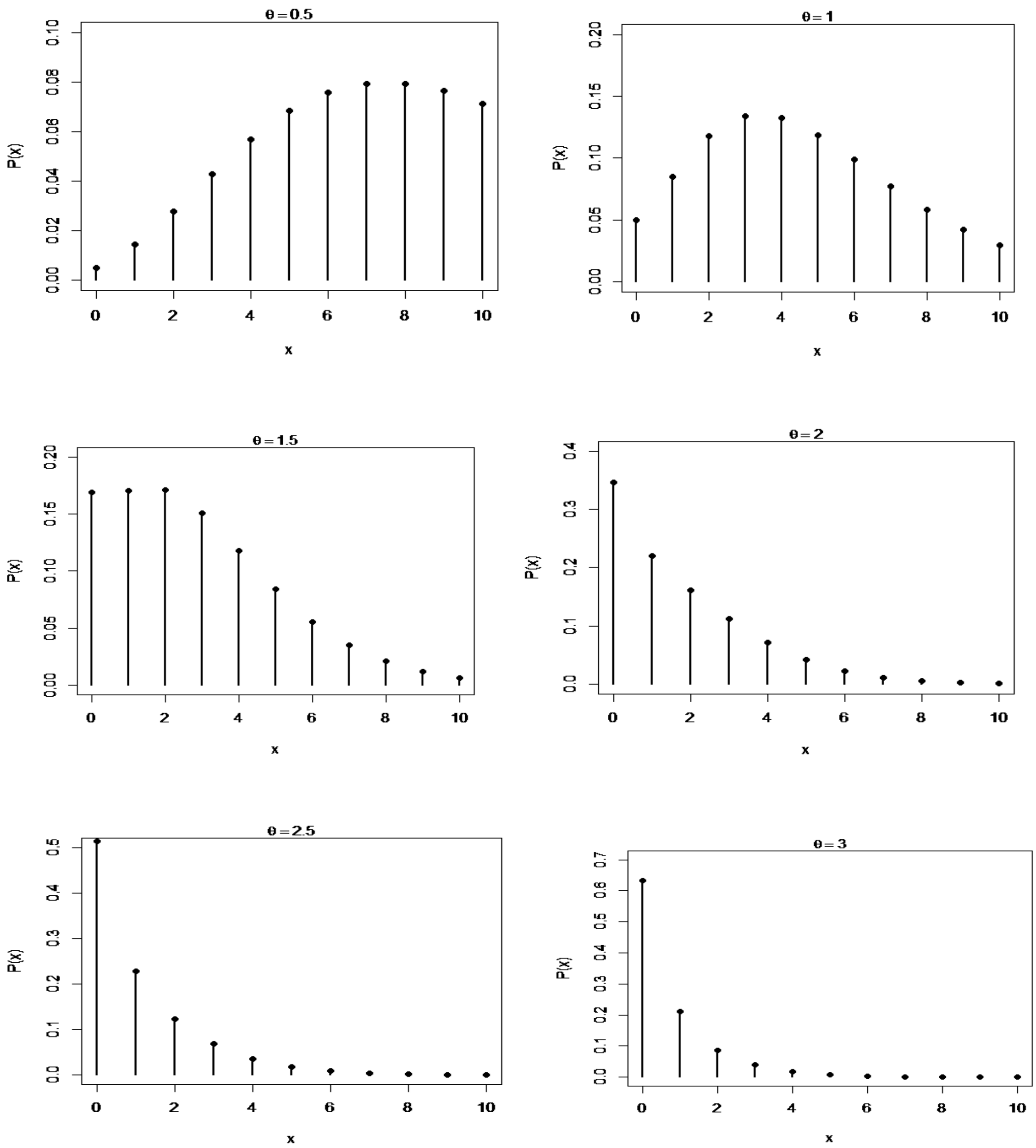

Figure1: *Graphs of probability mass function of PSD for different values of the parameter* θ.

III. Moments and associated measures

The rth factorial moment about origin of PSD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}|\lambda\right)\right], \text{ where } X^{(r)} = X(X-1)(X-2)\dots(X-r+1).$$

Using (2.1), the rth factorial moment about origin of PSD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}|\lambda\right)\right] = \frac{\theta^5}{\theta^4+24}\int_0^\infty \left[\sum_{x=0}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!}\right](1+\lambda^4)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^5}{\theta^4+24}\int_0^\infty \lambda^r \left[\sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right](1+\lambda^4)e^{-\theta\lambda}d\lambda$$

Taking $x+r$in place of $x$within the bracket, we get

$$\mu_{(r)}' = \frac{\theta^5}{\theta^4+24}\int_0^\infty \lambda^r \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!}\right](1+\lambda^4)e^{-\theta\lambda}d\lambda$$

The expression within the bracket is clearly unity and hence we have

$$\mu_{(r)}' = \frac{\theta^5}{\theta^4+24}\int_0^\infty \lambda^r (1+\lambda^4)e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, we get finally, a general expression for the rth factorial moment of PSD (2.2) as

$$\mu_{(r)}' = \frac{r!\{\theta^4+(r+1)(r+2)(r+3)(r+4)\}}{\theta^r(\theta^4+24)}; r = 1,2,3,4,\dots \quad (4)$$

Substituting $r = 1,2,3,$and4 in (4), the first four factorial moments about origin can be obtained and using the relationship between factorial moments about origin and moments about origin, the first four moment about origin of the PSD (3) are obtained as

$$\mu'_1 = \mu'_{(1)} = \frac{\theta^4+120}{\theta(\theta^4+24)} \quad (5)$$

$$\mu'_2 = \frac{\theta^5+2\theta^4+120\theta+720}{\theta^2(\theta^4+24)} \quad (6)$$

$$\mu'_3 = \frac{\theta^6+6\theta^5+6\theta^4+120\theta^2+2160\theta+5040}{\theta^3(\theta^4+24)} \quad (7)$$

$$\mu'_4 = \frac{\theta^7+14\theta^6+36\theta^5+24\theta^4+120\theta^3+5040\theta^2+30240\theta+40320}{\theta^4(\theta^4+24)} \quad (8)$$

Using the relationship between moments about mean and the moments about origin, the moments about mean of the PSD (3) are obtained as

$$\mu_2 = \sigma^2 = \frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta^2(\theta^{24}+24)^2} \quad (9)$$

$$\mu_3 = \frac{\left(\begin{matrix}\theta^{14}+3\theta^{13}+2\theta^{12}+168\theta^{10}+1656\theta^9+3024\theta^8+6336\theta^6+46656\theta^5\\+3456\theta^4+132\theta^3+69120\theta^2+207360\theta+138240\end{matrix}\right)}{\theta^3(\theta^4+24)^3} \quad (10)$$

$$\mu_4 = \frac{\left(\begin{matrix}\theta^{19}+10\theta^{18}+18\theta^{17}+9\theta^{16}+192\theta^{15}+4896\theta^{14}+22464\theta^{13}+23904\theta^{12}+103668\theta^{11}\\+2810088\theta^{10}+9469440\theta^9+5287680\theta^8+221184\theta^7+55848960\theta^6+129392640\theta^5\\+11114496\theta^4+16588800\theta^3+364953600\theta^2+696729600\theta+34836480\end{matrix}\right)}{\theta^4\left(\theta^4+24\right)^4} \quad (11)$$

The coefficient of variation$(C.V)$, coefficient of Skewness$\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis(β_2), and index of dispersion (γ) of the PSD (3) are thus given by

$$C.V = \frac{\sigma}{\mu'_1} = \frac{\sqrt{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}}{\theta^4+120} \quad (12)$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{\left(\begin{matrix}\theta^{14}+3\theta^{13}+2\theta^{12}+168\theta^{10}+1656\theta^9+3024\theta^8+6336\theta^6+46656\theta^5\\+3456\theta^4+132\theta^3+69120\theta^2+207360\theta+138240\end{matrix}\right)}{(\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880)^{\frac{3}{2}}} \quad (13)$$

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{\begin{pmatrix}\theta^{19}+10\theta^{18}+18\theta^{17}+9\theta^{16}+192\theta^{15}+4896\theta^{14}+22464\theta^{13}+23904\theta^{12}+10368\theta^{11} \\ +281088\theta^{10}+946944\theta^{9}+528768\theta^{8}+221184\theta^{7}+5584896\theta^{6}+12939264\theta^{5} \\ +11114496\theta^{4}+1658880\theta^{3}+36495360\theta^{2}+69672960\theta+34836480\end{pmatrix}}{(\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880)^2} \quad (14)$$

$$\gamma = \frac{\sigma^2}{{\mu'}_1} = \frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta(\theta^4+24)(\theta^4+120)}. \quad (15)$$

The shapes of coefficient of variation (CV), skewness, kurtosis and index of dispersion (ID) of PSD are presented in figure 2 with varying values of parameter.

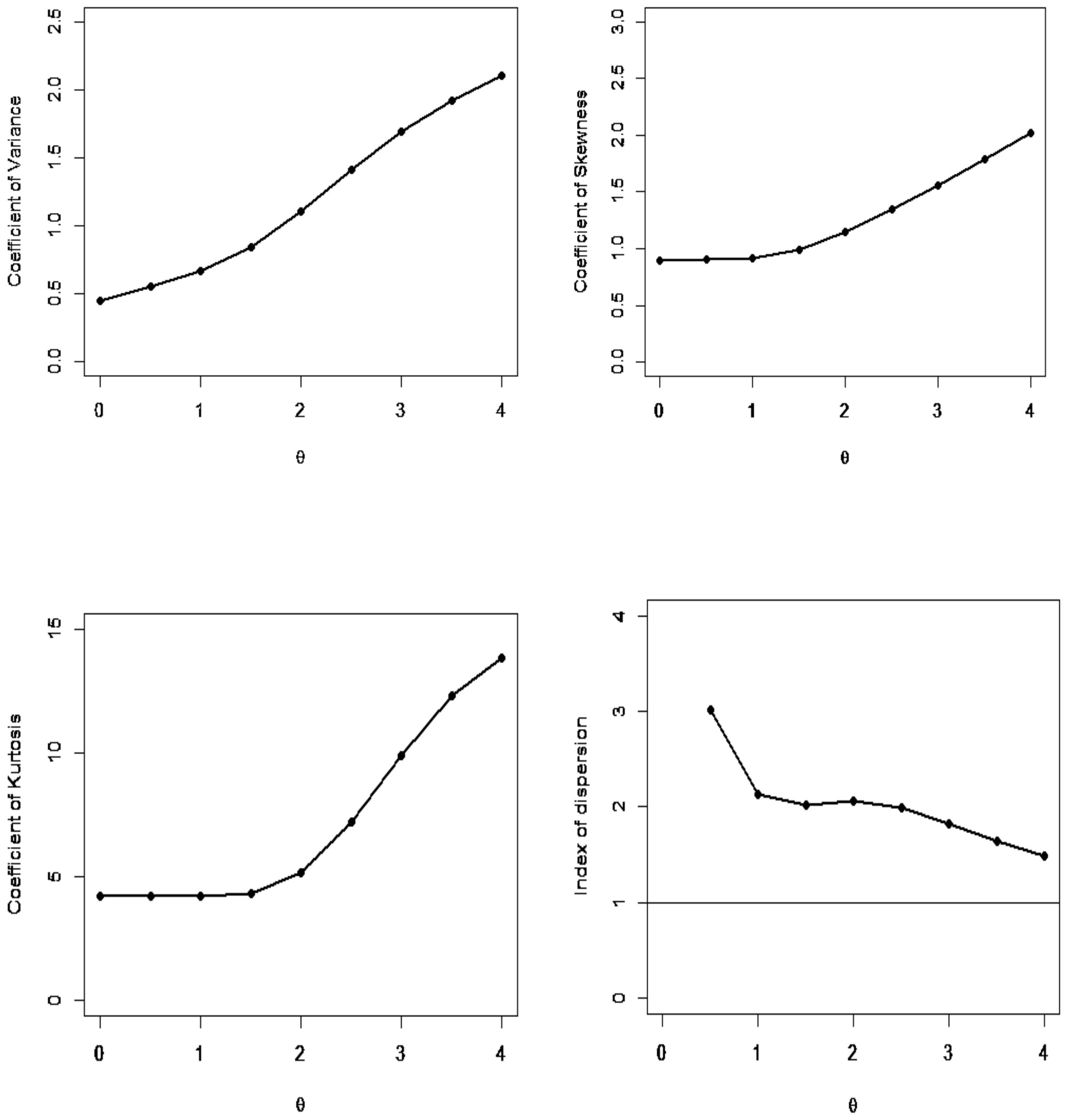

Figure2: *Graphs of coefficient of variation, coefficient of skewness, coefficient of kurtosis and Index of dispersion of PSD for different values of the parameter θ*

IV. Statistical properties

The PSD has two important properties namely increasing hazard rate with unimodality and over-dispersion which has been discussed below

I. Increasing Hazard Rate and Unimodality

The PSD has an increasing hazard rate (IHR) and unimodal. Since

$$\frac{P(x+1\,;\theta)}{P(x\,;\theta)}=\frac{1}{\theta+1}\left[1+\frac{4x^3+36x^2+104x+116}{x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)}\right] \qquad (16)$$

is decreasing function in x, $P(x\,;\theta)$is log-concave. Therefore, the PSD has an increasing hazard rate and unimodal. The interrelationship among log-concavity, unimodality and increasing hazard rate (IHR) of discrete distributions has been discussed in Grandell [18].

II. Over-dispersion

The PSD is always over-dispersed$(\sigma^2>\mu)$ **.** We have

$$\begin{aligned}\sigma^2&=\frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta^2(\theta^4+24)^2}\\&=\frac{\theta^4+120}{\theta(\theta^4+24)}\left[\frac{\theta^9+\theta^8+144\theta^5+528\theta^4+2880\theta+2880}{\theta(\theta^4+24)(\theta^4+120)}\right]\\&=\frac{\theta^4+120}{\theta(\theta^4+24)}\left[1+\frac{\theta^8+528\theta^4+2880}{\theta(\theta^4+24)(\theta^4+120)}\right]\\&=\mu\left[1+\frac{\theta^8+528\theta^4+2880}{\theta(\theta^4+24)(\theta^4+120)}\right]>\mu.\end{aligned}$$

This shows that PSD is always over-dispersed.

V. Estimation of the parameter

I. Method of Moment Estimate (MOME): Let $(x_1,x_2,\ldots,x_n)$be a random sample of size $n$from the PSD (3). Equating the first population moment about origin to the corresponding sample moment, the MOME $\tilde{\theta}$of θ of PSD (3) is the solution of the following fifth degree polynomial equation in θ

$$\bar{x}\theta^5-\theta^4+24\bar{x}\theta-120=0\,,$$

where $\bar{x}$ is the sample mean. This can easily be solved using Newton-Raphson method.

II. Maximum Likelihood Estimate (MLE): Let $(x_1,x_2,\ldots,x_n)$be a random sample of size n from the PSD (3) and let $f_x$be the observed frequency in the sample corresponding to $X=x(x=1,2,3,\ldots,k)$ such that $\sum_{x=1}^{k}f_x=n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the PSD (3) is given by

$$L=\left(\frac{\theta^5}{\theta^{24}+24}\right)^n\frac{1}{(\theta+1)^{\sum_{x=1}^{k}(x+5)f_x}}\prod_{x=1}^{k}[x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)]^{f_x}$$

The log likelihood function is thus obtained as

$$log\, L = n\, log\left(\frac{\theta^5}{\theta^{24}+24}\right) - \sum_{x=1}^{k}(x+5)\, f_x\, log(\theta+1) + \sum_{x=1}^{k} f_x\, log\begin{bmatrix} x^4+10x^3+35x^2+50x \\ +(\theta^4+4\theta^3+6\theta^2+4\theta+25) \end{bmatrix}$$

The first derivative of the log likelihood function is given by

$$\frac{d\, log\, L}{d\theta} = \frac{5n}{\theta} - \frac{4n\theta^3}{\theta^4+24} - \sum_{x=1}^{k}\frac{(x+5)f}{\theta+1} + \sum_{x=1}^{k}\frac{(4\theta^3+12\theta^2+12\theta+4)f_x}{[x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)]}$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate (MLE), $\hat{\theta}$ of θ of PSD is the solution of the equation $\frac{d\, log\, L}{d\theta} = 0$ and is given by the solution of the following non-linear equation

$$\frac{5n}{\theta} - \frac{4n\theta^3}{\theta^4+24} - \sum_{x=1}^{k}\frac{(x+5)f}{\theta+1} + \sum_{x=1}^{k}\frac{(4\theta^3+12\theta^2+12\theta+4)f_x}{[x^4+10x^3+35x^2+50x+(\theta^4+4\theta^3+6\theta^2+4\theta+25)]} = 0$$

This non-linear equation can be solved by any numerical iteration methods such as Newton-Raphson, Bisection method, Regula –Falsi method etc.

VI. Goodness of fit of PSD

As we know that the count data arising in real life, in general, are over-dispersed or under-dispersed. We have seen that PSD is over-dispersed. In this section, an attempt has been made to test the goodness of fit of PSD with some over-dispersed count data and the goodness of fit has been compared with other available over-dispersed distributions. The goodness of fit of PSD has been compared with the goodness of fit of PLD, PAD, PID and PSD. The goodness of fit is based on the maximum likelihood estimates of a parameter of the considered distributions.

In this section, four examples of observed count datasets, for which the PLD, PAD, PID, and PSD have been fitted, are presented. The first data-set is due to Kemp and kemp [19] regarding the distribution of mistakes in copying groups of random digits, the second dataset is due to Beall [20]regarding the distribution of Pyrausta nublilalis, the third dataset is the distribution of red mites per leaf on apple leaves, available in Fisher et al [21], and the fourth dataset is the distribution of number of Chromatid aberrations, available in Loeschke and Kohler [22] and Janardan and Schaeffer [23].

Table 1: *Distribution of mistakes in copying groups of random digits*

No. of errors per group	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PSD
0	35	27.4	33.0	33.5	33.7	34.6
1	11	21.5	15.3	14.7	14.5	13.3
2	8	8.4	6.8	6.6	6.5	6.3
3	4	2.2	2.9	2.9	2.9	3.1
4	2	0.5	2.0	2.3	2.4	2.7
Total	60	60.0	60.0	60.0	60.0	60.0
ML estimate $(\hat{\theta})$		0.7833	1.7434	2.0779	1.8643	2.7379
χ^2		7.98	2.20	1.40	1.33	0.86
d.f.		1	1	2	2	2
p-value		0.0047	0.1380	0.4966	0.5140	0.6472

Table 2: Distribution of Pyrausta nublilalis

No. of insects	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PSD
0	33	26.4	31.5	32.0	32.2	33.1
1	12	19.8	14.2	13.6	13.4	12.3
2	6	7.4	6.1	5.9	5.8	5.6
3	3	1.8	2.5	2.6	2.6	2.7
4	1	0.3	1.0	1.1	1.1	1.3
5	1	0.3	0.7	0.8	0.9	1.0
Total	56	56.0	56.0	56.0	56.0	56.0
ML estimate $(\hat{\theta})$		0.7500	1.8081	2.1446	1.9186	2.8014
χ^2		4.87	0.53	0.24	0.20	0.03
d.f.		1	1	1	1	2
p-value		0.0273	0.4666	0.6242	0.6547	0.9821

Table 3: *Distribution of number of red mites on Apple leaves, Fisher et al (1943)*

Number of red mites per leaf	Observed Frequency	Expected Frequency			
		PD	PLD	PAD	PSD
0	38	25.3	35.8	36.3	37.6
1	17	29.1	20.7	20.1	18.3
2	10	16.7	11.4	11.2	10.7
3	9	6.4	6.0	6.1	6.3
4	3	1.8	3.1	3.2	3.5
5	2	0.4	1.6	1.6	1.8
6	1	0.2	0.8	0.8	0.9
7+	0	0.1	0.6	0.7	0.9
Total	80	80.0	80.0	80.0	80.0
ML estimate $(\hat{\theta})$		1.1500	1.2559	1.6206	2.3602
χ^2		18.27	2.47	2.07	1.47
d.f.		2	3	3	3
p-value		0.0001	0.4807	0.5580	0.6894

Table 4: *Distribution of number of Chromatid aberrations (0.2 g chinon 1, 24 hours)*

No. of chromatid aberrations	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PSD
0	268	231.3	257.0	260.4	260.8	268.3
1	87	126.7	93.4	89.7	89.3	81.5
2	26	34.7	32.8	32.1	31.8	29.9
3	9	6.3	11.2	11.5	11.5	12.1
4	4	0.8	3.8	4.1	4.2	4.9
5	2	0.1	1.2	1.4	1.5	1.9
6	1	0.1	0.4	0.5	0.6	0.7
7+	3	0.1	0.2	0.3	0.3	0.7
Total	400	400.0	400.0	400.0	400.0	400
ML estimate $(\hat{\theta})$		0.5475	2.3804	2.6594	2.3362	3.2184
χ^2		38.21	6.21	4.17	3.61	2.07
d.f.		2	3	3	3	3
p-value		0.0000	0.1018	0.2437	0.3067	0.5582

VII. Concluding Remarks

In this paper, Poisson-Suja distribution (PSD) has been proposed. The PSD has been obtained by compounding the Poisson distribution with the Suja distribution. The expression for the r th factorial moment has been derived and hence the first four moments about the origin and the moments about the mean have been given. The descriptive measures including coefficient of variation, skewness, kurtosis, and index of dispersion have been obtained. Both the method of moments and the method of maximum likelihood have been discussed for estimating a parameter of the proposed distribution. The goodness of fit of the PSD has been discussed with four examples of count data sets that are over-dispersed and the goodness of fit of the PSD has been compared with the goodness of fit given by PLD, PAD, and PID. In these datasets, the PSD shows a much closer fit than other considered distributions.

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