CLASSICAL AND BAYESIAN ESTIMATION OF EXPONENTIATED INVERSE RAYLEIGH DISTRIBUTION BASED ON RECORD VALUES

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Abstract

In this article explores two approaches for estimating the parameters of the exponentiated inverse Rayleigh distribution (EIRD) using record values: Classical estimation and Bayesian estimation. In classical estimation, maximum likelihood estimators (MLE's) and the asymptotic confidence intervals are derived based on the observed Fisher information matrix of the parameters. In Bayesian estimation, estimators of the parameters are obtained under the square error loss function. This involves using Tierney-Kadane's approximation (TK) and Markov chain Monte Carlo (MCMC) methods for Bayesian computation. Additionally, the article constructs the highest posterior credible intervals of the parameters using the MCMC method. To evaluate the performance of these estimators, a Monte Carlo simulation study is conducted to compare their behavior. Finally, a real data analysis is presented to illustrate the application of the methods discussed in the article.

Keywords: :Exponentiated inverse Rayleigh distribution, Maximum likelihood estimators, Bayes estimators, Square error loss function, MCMC, TK, Record values, and Real data.

1. Introduction

The Rayleigh distribution was introduced by Lord Rayleigh (1880) and is used in the field of acoustics. This distribution possesses the properties of some well-known distributions, such as Weibull, chi-square, and extreme value distribution, which makes it even more useful for different areas of science and technology. There are several authors who have studied the application of the Rayleigh distribution, such as Beckmann[1] study the generalization of rayleigh distribution, Hoffman and Karst[2] mentioned that theroy and application of Rayleigh Distribution, Lee et al.[3] estimated the scale parameters of the Rayleigh distribution. Based on censored data, Soliman et al.[4] study the inference and application of the Rayleigh model. Let us suppose that a random variable Z follows the Rayleigh distribution, then $X = \frac{1}{Z}$ follows the Inverse Rayleigh distribution. The inverse Rayleigh distribution (IRD) is widely applied in reliability studies and other related fields. For more information about Inverse Rayleigh distribution studies, see more papers such as Voda[5],El-Helbawy and Abd-El-Monem[6],Shawky and Majdah M[7],Sindhu et al.[8],C. Tans[9].

Let us suppose X is a random variable for the Inverse Rayleigh distribution with scale parameters σ. Then its probability density function (pdf) and cumulative distribution function (cdf) are respectively given as

$$q(x;\sigma) = \begin{cases} \frac{2\sigma^2}{x^3} e^{-(\frac{\sigma}{x})^2}, & \text{if } x > 0, \sigma > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

$$Q(x;\sigma) = \begin{cases} e^{-(\frac{\sigma}{x})^2}, & \text{if } x > 0, \sigma > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

There are many researchers who have suggested that the exponentiated inverse Rayleigh distribution is a generalized case of the inverse Rayleigh distribution, such as Nadarajah and Kotz [10] and Srinivasa et al. [11], who studied the estimation of multicomponent stress-strength reliability from the exponentiated inverse Rayleigh distribution. The cumulative distribution function (cdf) of the exponentiated inverse Rayleigh distribution is

$$F(x;\alpha,\sigma) = 1 - (1 - e^{-(\frac{\sigma}{x})^2})^{\alpha}, x \geq 0, \alpha, \sigma > 0 \tag{3}$$

and the corresponding probability density function (pdf) is

$$f(x,\alpha,\sigma) = \begin{cases} \frac{2\alpha\sigma^2}{x^3} e^{-(\frac{\sigma}{x})^2} (1 - e^{-(\frac{\sigma}{x})^2})^{\alpha-1}, & \text{if } x \geq 0, \alpha, \sigma > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

In numerous real-life applications, particularly within industries and reliability studies, products frequently fail under stress. For instance, a wooden beam may fracture when subjected to sufficient perpendicular force, an electronic component might cease functioning at excessively high temperatures, or a battery could expire over time. However, the precise threshold of failure can vary, even among identical items. Therefore, in such experiments, measurements are often taken sequentially, with only the record values-either the lowest or highest-being observed. These record values naturally emerge across various domains, including weather tracking, sports analytics, economic data analysis, and life-test assessments.

Let $(X_n, n \geq 1)$ be a series of independent and identically distributed (i.i.d.) random variables with distribution function $F(x)$ and probability function $f(x)$. An observation X_j is called an upper record value if $X_j > X_i$ for every $j > i$. Let us suppose $X_1, X_2, ..., X_n$ be upper record values and $x_1, x_2, ..., x_n$ be observed values of upper record values. Then the joint density function of upper record values is given by

$$f_{\underline{X}}(\underline{x}) = f(x_n) \prod_{i=1}^{n-1} \frac{f(x_i)}{1 - F(x_i)}, \; x_1 < x_2 < ..., x_n \tag{5}$$

In recent times, utilizing record values for parameter estimation across various lifetime models has garnered significant attention among researchers. A multitude of studies have explored employing the MCMC and TK procedures to derive Bayes estimates in this context, such as Janss and Gerben [12], Andrieu et al. [13], Solimanet.al[14], Hassan et al. [15], Singh et al. [16], Sana and Faizan [17].

The paper are arrange in following order: In Section 2, the maximum likelihood estimation and asymptotic confidence intervals is presented. In Section 3, Bayesian estimation and MCMC algorithm are presented. In section 4, TK approximation is presented. In Section 5, simulation study is presented . In Section 6, application of real data sets. Finally, the conclusion of this paper is discussed in section 7.

2. Maximum likelihood estimation

Let us suppose that we have m upper record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ from the exponentiated inverse Rayleigh distribution with (cdf) (3) and (pdf) (4). The maximum likelihood function for record values is given by Ahsanullah [18]

$$f_{1,2,3,...,m}(X_{L(1)}, X_{L(2)}, X_{L(3)}, ..., X_{L(m)}) = f(x_{L(m)}) \prod_{i=1}^{m-1} \frac{f(x_{L(i)})}{1 - F(x_{L(i)})} \tag{6}$$

The likelihood function based on the upper records observed from the exponentiated inverse Rayleigh distribution is given by

$$L(\sigma, \alpha; x) = \frac{2\alpha\sigma^2}{x_m^3} e^{-(\frac{\sigma}{x_m})^2} (1 - e^{-(\frac{\sigma}{x_m})^2})^{\alpha-1} \prod_{i=1}^{m-1} \frac{2\alpha\sigma^2 e^{-(\frac{\sigma}{x_i})^2} (1 - e^{-(\frac{\sigma}{x_i})^2})^{\alpha-1}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})^{\alpha}}$$

$$= 2^m \alpha^m \sigma^{2m} (1 - e^{-(\frac{\sigma}{x_m})^2})^{\alpha} \prod_{i=1}^{m} \frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})}$$

$$= 2^m \alpha^m \sigma^{2m} e^{\alpha ln(1 - e^{-(\frac{\sigma}{x_m})^2})} \prod_{i=1}^{m} \frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})}$$

$$L(\sigma, \alpha; x) = 2^m \alpha^m \sigma^{2m} e^{\alpha ln(1 - e^{-(\frac{\sigma}{x_m})^2})} \prod_{i=1}^{m} \frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3 (1 - e^{-(\frac{\sigma}{x_i})^2})} \tag{7}$$

Now, taking the log on both sides, we get

$$\begin{aligned} l = \ln L(\sigma, \alpha; x) = m \ln 2 + m \ln \alpha + 2m \ln \sigma + \alpha \ln \left(1 - e^{-\left(\frac{\sigma}{x_m}\right)^2}\right) - \\ \sigma^2 \sum_{i=1}^{m} \left(\frac{1}{x_i}\right)^2 - \sum_{i=1}^{m} \ln \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right) + \frac{1}{\sum_{i=1}^{m} \ln(x_i)^3} \end{aligned} \tag{8}$$

Differentiating Eq. (8) with respect to α and σ and equating to zero, we get

$$\frac{\partial}{\partial \alpha} \ln L(\alpha, \sigma; x) = \frac{m}{\alpha} + ln(1 - e^{-(\frac{\sigma}{x_m})^2}) = 0 \tag{9}$$

$$\frac{\partial}{\partial \sigma} \ln L(\alpha, \sigma; x) = \frac{2m}{\sigma} + \frac{2\alpha\sigma e^{-(\frac{\sigma}{x_m})^2}}{x_m^2 (1 - e^{-(\frac{\sigma}{x_m})^2})} - 2\sigma \sum_{i=1}^{m} \frac{1}{x_i^2} - \sum_{i=1}^{m} \frac{2\sigma}{x_i^2 (1 - e^{-(\frac{\sigma}{x_i})^2})} = 0. \tag{10}$$

Here, equations (9) and (10) are not in exact form, so we cannot obtain the maximum likelihood estimation easily. So the Newton-Ramphson method is used to find the maximum likelihood estimation of $\hat{\alpha}$ and $\hat{\sigma}$. To solve these non-linear equations, an R-package is used to find the mle of $\hat{\alpha}$ and $\hat{\sigma}$.

2.1. Asymptotic confidence intervals

The MLE's of unknown parameter cannot be obtained in closed from, it is not easy to derive the exact distribution of the MLE's. Therefore, we obtain the asymptotic confidence intervalof the parameter based on observed Fisher information matrix. Let ($\hat{\alpha}$,$\hat{\sigma}$) be the MLE's of (α,σ). The observed Fisher information matrix is given by

$$I(\hat{\alpha}, \hat{\sigma}) = - \begin{bmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \sigma} \\ \frac{\partial^2 l}{\partial \sigma \partial \alpha} & \frac{\partial^2 l}{\partial \sigma^2} \end{bmatrix}_{(\hat{\alpha}, \hat{\sigma})}$$

where,

$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{m}{\alpha^2}$, $\frac{\partial^2 l}{\partial \alpha \sigma} = \frac{\partial^2 l}{\partial \sigma \alpha} = \frac{e^{-(\frac{\sigma}{x_m})^2} \cdot 2\left(\frac{\sigma}{x_m^2}\right)}{(1 - e^{-(\frac{\sigma}{x_m})^2})^2}$,

$$\frac{\partial^2 l}{\partial \sigma^2} = \alpha \left(\frac{e^{-\left(\frac{\sigma}{x_m}\right)^2}\left(\frac{2}{x_m^2} - e^{-\left(\frac{\sigma}{x_m}\right)^2}\frac{2\sigma}{x_m^2}\frac{2\sigma}{x_m^2}\right)}{1-e^{-\left(\frac{\sigma}{x_m}\right)^2}} \quad - \frac{e^{-\left(\frac{\sigma}{x_m}\right)^2}\frac{2\sigma}{x_m^2}e^{-\left(\frac{\sigma}{x_m}\right)^2}\frac{2\sigma}{x_m^2}}{(1-e^{-\left(\frac{\sigma}{x_m}\right)^2})^2} \right) - 2\sum_{i=1}^{m}\frac{1}{x_i^2}$$

$$-\sum_{i=1}^{m} \left(\frac{e^{-\left(\frac{\sigma}{x_i}\right)^2}\left(\frac{2}{x_i^2} - e^{-\left(\frac{\sigma}{x_i}\right)^2}\frac{2\sigma}{x_i^2}\frac{2\sigma}{x_i^2}\right)}{1-e^{-\left(\frac{\sigma}{x_i}\right)^2}} - \frac{e^{-\left(\frac{\sigma}{x_i}\right)^2}\frac{2\sigma}{x_i^2}e^{-\left(\frac{\sigma}{x_i}\right)^2}\frac{2\sigma}{x_i^2}}{(1-e^{-\left(\frac{\sigma}{x_i}\right)^2})^2} \right)$$

Thus, the observed variance-covariance matrix becomes $I^{-1}(\hat{\alpha},\hat{\sigma})$. To obtain the asymptotic confidence interval of the unknown parameters the MLE's estimate follow a bivariate normal distribution with mean (α,σ) and variance- covariance matrix is $I^{-1}(\hat{\alpha},\hat{\sigma})$. The asymptotic normality of the MLE's can be used to compute approximate $100(1-\eta)\%$ confidence intervals for the parameters α and σ, as follows:
$\hat{\alpha} \pm z_{\eta/2}\sqrt{var(\hat{\alpha})}$ and $\hat{\sigma} \pm z_{\eta/2}\sqrt{var(\hat{\sigma})}$; where $z_{\eta/2}$ is the upper $(\eta/2)$ point of standard normal distribution.

3. Bayes estimation

In this portion, we explore the Bayesian estimation to derive parameter estimates for the EIRD based on upper record values. In the Bayesian estimation framework, decisions regarding the prior distribution and the loss function are of the utmost importance. In the existing literature, various prior distributions have been proposed for the unknown parameters of a particular distribution of interest. For example,Kizilaslan and Nadar[19] consider the gamma prior for generalized exponential distribution,Doostparast et al. [20] consider the normal prior, Fan[21] consider non informative prior, Singh and Tripathi [22] considered the conditional prior for the lognormal distribution. Hu and Ren[23] considered conditional prior for the Inverse Weibull distribution. However, Arnold and Press [24] stated that it's evident that no definitive method exists to determine the superiority of one prior over another. In the context of the preceding discussions, we consider non informative prior $g_1(\alpha) = \frac{1}{\alpha}$ and gamma priors of the EIRD such that

$$g_2(\sigma|a,b) = \frac{b^a\sigma^{a-1}e^{-b\sigma}}{\Gamma a}, \alpha,\sigma > 0; a,b > 0.$$

Now the joint prior distribution of α and σ is,

$$g(\alpha,\sigma) = g_1(\alpha) \times g_2(\sigma|a,b) = \frac{b^a\sigma^{a-1}e^{-b\sigma}}{\alpha\Gamma a}. \tag{11}$$

Here a,b show the hyperparameter, and Γ is the gamma function.

To demonstrate the versatility of our findings and to encompass a wide spectrum of real-world scenarios, we introduce both symmetric and asymmetric loss functions. The inclusion of a symmetric loss function is motivated by its equitable penalization of both underestimation and overestimation, proving advantageous in many instances. However, practical situations often involve scenarios where positive loss holds greater severity than negative loss, and vice versa. In such cases, the need for asymmetric loss functions arises. In our study, we encompass one symmetric option, namely the squared error loss function (SELF).
The mathematical expressions for these loss functions and their corresponding Bayes estimators are given as:
The square error loss function is defined as

$$L_1(\hat{\alpha},\alpha) = (\hat{\alpha}-\alpha)^2, \alpha > 0.$$

where $\hat{\alpha}$ is the estimate of parameter α.The Bayes estimator under sqare error loss function is

posterior mean $(\hat{\alpha}_{SEL})$.
Now, the joint posterior distribution, obtained using equations (7) and (11), is given as

$$\pi(\alpha,\sigma|x) \propto \alpha^{m-1}\sigma^{2m+a-1}e^{-b\sigma}e^{\alpha ln(1-e^{-(\frac{\sigma}{x_m})^2})}\prod_{i=1}^{m}\frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3(1-e^{-(\frac{\sigma}{x_i})^2})^2} \quad (12)$$

We observe that the joint posterior distribution given in equation (12) cannot be simplified into a closed form expression. So by making use of some approximation methods, we can derive explicit expressions for these estimators. To tackle this situation, two widely applicable approximation methods, i.e., the Tierney-Kadane approximation and the Markov chain Monte Carlo method, are applied. In the existing literature, Lindley's method [25] has been extensively taken into account for such situations. However, this method requires third derivatives of the log-likelihood function. Instead, we consider another approximation method proposed by Tierney and Kadane (TK) [26], in which derivatives only up to second order are required to compute the desired Bayes estimates.

3.1. MCMC Algorithm.

In this specific section, we employ the Markov Chain Monte Carlo (MCMC) methodology to obtain an estimated Bayesian approximation for the parameters α and σ under the square error loss function. With the help of posterior densities, the MCMC method can be used to generate a random sample of unknown quantities.The generated sample is used to obtain the Bayes estimator for the loss functions. The marginal densities of α and σ are given as

$$\pi(\alpha|\sigma,x) \propto Gamma\left(m, \frac{1}{ln(1-e^{-(\frac{\sigma}{x_m})^2})}\right)$$

$$\pi(\sigma|\alpha,x) \propto \sigma^{2m+a-1}e^{-b\sigma}e^{\alpha ln(1-e^{-(\frac{\sigma}{x_m})^2})}\prod_{i=1}^{m}\frac{e^{-(\frac{\sigma}{x_i})^2}}{x_i^3(1-e^{-(\frac{\sigma}{x_i})^2})^2}$$

The marginal posterior density of α, a closed form of which follows the Gamma distribution, So,the Gibbs sampling [27] method is used to generate the sample of α. The marginal posterior density of σ is not an exact form of any distribution, so we used the M-H algorithm to generate a sample of σ. For more information on the algorithm, methods and steps are given in [28]. This algorithm combines the Metropolis-Hastings scheme with the Gibbs sampling scheme under the Gaussian proposal distribution.
The steps in which the M-H approach performs to simulate the posterior sample are as follows:
Step 1: Take some initial guess values for the parameters α and σ be (α^0,σ^0).
Step 2: Set t=1.
Step 3: Generate $\sigma^{(t)}$ from $\pi(\sigma|\alpha^{(t)},a,b)$ using the M-H algorithm with the proposal that the distribution is normal distribution.
Step 4: Generate $\alpha^{(t)}$ from $\pi(\alpha|\sigma^{(t-1)},a,b)$.
Step 5: Set t=t+1.
Step 6: Repeat steps 2–5 up to N times and obtain the posterior sample (α^t,σ^t) for t=1,2,...,N.
Using the posterior sample, we obtain the Bayesian estimates for the parameters α and σ under the squared error function, given by,

$$\hat{\alpha}_{SELF} = \frac{1}{N-M}\sum_{t=M+1}^{N}\alpha^t$$

$$\hat{\sigma}_{SELF} = \frac{1}{N-M}\sum_{t=M+1}^{N}\sigma^t$$

where M is the burn period of MCMC.

4. Tierney-Kadane Approximation

The TK approximation method was first proposed by Tierney and Kadane in 1986 [26] as a way to estimate the posterior expectation that involves the ratio of two integrals. The process of applying the TK technique is simple and straight-forward. This section deals with the use of TK's method to approximate the Bayes estimates. Suppose our objective is to estimate the expression $E(u(\alpha,\sigma)|x)$ using the TK method. Then,we first consider the following functions:

$$I(x) = E[u(\alpha,\sigma|x)] = \frac{\int_0^\infty \int_0^\infty u(\alpha,\sigma) e^{[L(\alpha,\sigma|x)+\rho(\alpha,\sigma|x)]} d\alpha d\sigma}{\int_0^\infty \int_0^\infty e^{[L(\alpha,\sigma|x)+\rho(\alpha,\sigma|x)]} d\alpha d\sigma}.$$

where u(α,σ) is a function of α and σ, L(α,σ) can defined in equation (8).
$\rho(\alpha,\sigma)$ is logarithm of joint prior distribution which is given in equation (11) and defined as :
$\rho(\alpha,\sigma) = ln(g(\alpha,\sigma)) = aln(b) + (a-1)ln(\sigma) - b\sigma - ln(\Gamma a) - ln(\alpha)$.

We can approximate the function I(x) into an explicit expression by applying the TK approximation method. We first consider the following function:

$$\delta(\alpha,\sigma) = \frac{L(\alpha,\sigma|x) + \rho(\alpha,\sigma|x)}{n},$$

and

$$\delta_\theta{}^*(\alpha,\sigma) = \delta(\alpha,\sigma) + \frac{lnu(\alpha,\sigma)}{n},$$

Now, we assume that $(\hat{\alpha_\delta}, \hat{\sigma_\delta})$ and $(\hat{\alpha_{\delta^*}}, \hat{\sigma_{\delta^*}})$ maximize the function $\delta(\alpha,\sigma)$ and $\delta_\theta{}^*(\alpha,\sigma)$,respectively.
We then approximate $I(x)$ as

$$I(x) = \sqrt{\frac{|\Sigma_\theta{}^*|}{|\Sigma|}} e^{[n(\delta_\theta{}^*(\hat{\alpha_{\delta^*}},\hat{\sigma_{\delta^*}})) - \delta(\hat{\alpha_\delta},\hat{\sigma_\delta})]},$$

Here,$|\Sigma_\theta|$ and $|\Sigma_\theta^*|$ are the negative Inverse of Hessian matrices of $\delta(\alpha,\sigma)$and $\delta_\theta{}^*(\alpha,\sigma)$respectively.
$|\Sigma| = [\frac{\partial^2\delta}{\partial\alpha^2}\frac{\partial^2\delta}{\partial\sigma^2} - \frac{\partial^2\delta}{\partial\alpha\partial\sigma}\frac{\partial^2\delta}{\partial\sigma\partial\alpha}]^{-1}$ and $|\Sigma_\theta{}^*| = [\frac{\partial^2\delta_\theta{}^*}{\partial\alpha^2}\frac{\partial^2\delta_\theta{}^*}{\partial\sigma^2} - \frac{\partial^2\delta_\theta{}^*}{\partial\alpha\partial\sigma}\frac{\partial^2\delta_\theta{}^*}{\partial\sigma\partial\alpha}]^{-1}$ Now,
The prior information is

$$\rho(\alpha,\sigma|x) = alnb + (a-1)ln\sigma - b\sigma - ln\Gamma a - ln\alpha.$$

The likelihood function is
$lnL(\sigma,\alpha;x) = mln2 + mln\alpha + 2mln\sigma + \alpha ln(1 - e^{-(\frac{\sigma}{x_m})^2}) - \sigma^2 \sum_{i=1}^m (\frac{1}{x_i})^2 - \sum_{i=1}^m ln(1 - e^{-(\frac{\sigma}{x_i})^2}) + \frac{1}{\sum_{i=1}^m \ln(x_i)^3}$.

Now,

$$\delta(\alpha,\sigma) = \frac{L(\alpha,\sigma|x) + \rho(\alpha,\sigma|x)}{n}$$

$= \frac{1}{n}[mln2 + mln\alpha + 2mln\sigma + \alpha ln(1 - e^{-(\frac{\sigma}{x_m})^2}) - \sigma^2 \sum_{i=1}^m (\frac{1}{x_i})^2 - \sum_{i=1}^m ln(1 - e^{-(\frac{\sigma}{x_i})^2}) + \frac{1}{\sum_{i=1}^m \ln(x_i)^3} + alnb + (a-1)ln\sigma - b\sigma - ln\Gamma a - ln\alpha]$.
It's important to observe that

$$\frac{\partial\delta}{\partial\alpha} = \frac{1}{n}\big[\frac{m}{\alpha} + ln(1 - e^{-(\frac{\sigma}{x_m})^2}) - \frac{1}{\alpha}\big]$$

and

$$\frac{\partial^2\delta}{\partial\alpha^2} = \frac{1}{n}\big[-\frac{m}{\alpha^2} + \frac{1}{\alpha^2}\big],$$

$$\frac{\partial\delta^2}{\delta\alpha\delta\sigma} = \frac{\partial\delta^2}{\delta\sigma\delta\alpha} = \frac{2e^{-(\frac{\sigma}{x_m})}\frac{\sigma}{x_m}}{n(1-e^{-(\frac{\sigma}{x_m})^2})},$$

$$\frac{\partial\delta}{\partial\sigma} = \frac{2m}{\sigma} + \frac{2\sigma\alpha e^{-(\frac{\sigma}{x_m})^2}}{x_m(1-e^{-(\frac{\sigma}{x_m})^2})} - 2\sigma\sum_{i=1}^{m}\frac{1}{x_i^2} - \sum_{i=1}^{m}\frac{2\sigma e^{-(\sigma/x_i)}}{x_i(1-e^{-(\sigma/x_i)^2})} + \frac{(a-1)}{\sigma} - b.$$

Further, we use the derived quantities to obtain the Bayes estimators under the square error loss functions. It is evident that quantities except $\delta(\alpha,\sigma)$ and its derivatives are common in each form of Bayes estimators. The $\delta^*_\theta(\alpha,\sigma)$ quantity is given as the square error loss function:

(i) If $u(\alpha,\sigma) = \alpha$, then

$$\hat{\alpha}_{SEL} = \sqrt{\frac{|\Sigma^*_{\alpha SEL}|}{|\Sigma|}}e^{[n(\delta^*_{\alpha_{SEL}}(\hat{\alpha_{\delta^*}},\hat{\sigma_{\delta^*}}))-\delta(\hat{\alpha_\delta},\hat{\sigma_\delta})]},$$

In order to compute $|\Sigma^*_{\alpha SEL}|$, we first obtain the following expression:

$$\delta^*_{\alpha_{SEL}} = \delta(\alpha,\sigma) + \frac{1}{n}ln(\alpha)$$

$$\frac{\partial\delta^*}{\partial\alpha} = \frac{\partial\delta}{\partial\alpha} + \frac{1}{n\alpha},$$

$$\frac{\partial\delta^*}{\partial\alpha^2} = \frac{\partial^2\delta}{\partial\alpha^2} - \frac{1}{n\alpha^2},$$

$$\frac{\partial^2\delta^*}{\partial\sigma^2} = \frac{\partial^2\delta}{\partial\sigma^2},$$

$$\frac{\partial^2\delta^*}{\partial\alpha\partial\sigma} = \frac{\partial^2\delta}{\partial\alpha\partial\sigma}.$$

(ii) If $u(\alpha,\sigma) = \sigma$ then

$$\hat{\sigma}_{SEL} = \sqrt{\frac{|\Sigma^*_{\sigma SEL}|}{|\Sigma|}}e^{[n(\delta^*_{\sigma SEL}(\hat{\alpha_{\delta^*}},\hat{\sigma_{\delta^*}}))-\delta(\hat{\alpha_\delta},\hat{\sigma_\delta})]},$$

In order to compute $|\Sigma^*_{\sigma_{SEL}}|$, we first obtain the following expression.

$$\delta^*_{\sigma_{SEL}} = \delta(\alpha,\sigma) + \frac{1}{n}ln(\sigma),$$

$$\frac{\partial\delta^*}{\partial\alpha} = \frac{\partial\delta}{\partial\alpha},$$

$$\frac{\partial\delta^*}{\partial\alpha^2} = \frac{\partial^2\delta}{\partial\alpha^2},$$

$$\frac{\partial^2\delta^*}{\partial\sigma^2} = \frac{\partial^2\delta}{\partial\sigma^2} - \frac{1}{n\sigma^2},$$

$$\frac{\partial^2\delta^*}{\partial\alpha\partial\sigma} = \frac{\partial^2\delta}{\partial\alpha\partial\sigma}.$$

5. Simulation

In this section, we present the simulation result comparing the performance of the MLE's estimator and Bayes estimator for the parameters of the EIRD using upper record values. We consider the two sets of parameter values (2, 1) and (1,1) of the EIRD , generate the random sample using the inverse CDF method, and select the record values (6,7,8,9) from the generated sample for each parameter value.

The Monte Carlo simulation study compares various estimators using different sample sizes and true parameter values. Here are the key points summarized from the result:

1: Two sets of parameter values are used as (2,1), (1,1) and two pairs of hyperparameter values (0.2,0.2),(0.5,0.5).
2: There are four different sample sizes (6,7,8,9) considered in the simulation.
3: There are two methods of estimation: one is the classical method, such as MLE's, and the other is the Bayesian estimation (MCMC,TK) method.
4: Here we consider the square error loss function, which is used to compute the Bayes estimate.
5: We generate 10000 posterior samples with a burn period of 2000 sample are used.
6: Confidence interval based on observed Fisher information matrix and Highest Posterior Density (HPD) credible interval are computed for 95 %.

Table 1: *Estimate of MLE's and MSE (in parenthesis) for α ,σ along with confidence interval when $\alpha = 2$, σ=1*

m	$\hat{\alpha}_{\text{MLE}}$	$\hat{\sigma}_{\text{MLE}}$	$\text{CI}_{\hat{\alpha}}$	$\text{LCI}_{\hat{\alpha}}$	$\text{CI}_{\hat{\sigma}}$	$\text{LCI}_{\hat{\sigma}}$
6	2.3136(7.3692)	0.8780 (0.8532)	(0.3515,7.1391)	6.7875	(0.4649,2.4442)	1.9793
7	1.9778 (5.0171)	0.7579 (0.7859)	(0.1058,6.0209)	5.9151	(0.4580,2.3812)	1.9232
8	1.8063 (3.4510)	0.7925 (0.7225)	(0.3828,5.2598)	4.8770	(0.4475,2.3299)	1.8824
9	1.7981 (2.4550)	0.7931(0.6868)	(0.5546,4.7534)	4.1988	(0.4396,2.2933)	1.8537

Table 2: *Estimate of MLE and MSE (in parenthesis) for α, σ along with confidence interval when $\alpha = 1$ and $\sigma = 1$*

m	$\hat{\alpha}_{\text{MLE}}$	$\hat{\sigma}_{\text{MLE}}$	$\text{CI}_{\hat{\alpha}}$	$\text{LCI}_{\hat{\alpha}}$	$\text{CI}_{\hat{\sigma}}$	$\text{LCI}_{\hat{\sigma}}$
6	1.2427(1.9978)	0.9327 (4.3135)	(0.0464,3.1658)	3.1194	(0.1935,3.7162)	3.5227
7	0.8527 (1.1597)	0.7835 (4.0323)	(0.2089,2.6796)	2.4707	(0.1698,3.6325)	3.4627
8	0.9565 (0.7039)	0.8269 (3.8289)	(0.3002,2.3759)	2.0757	(0.1467,3.5706)	3.4239
9	0.9590(0.4689)	0.8293(3.6965)	(0.3552,2.1821)	1.8269	(0.1303,3.5289)	3.3986

Table 3: *MSE's(in parentheses) of TK and MCMC Bayes estimates of parameter values based on record values for prior (0.2,0.2) at (1,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	1.1039(5.6130)	0.7300(1.5430)	1.7841(1.0749)	1.1316(0.1894)	(0.5900,3.1165)	2.5265	(0.3773,1.9681)	1.5908
7	0.7510(1.9584)	0.5738(1.4414)	1.1619(0.1904)	0.9902(0.1598)	(0.4396,2.9633)	2.5237	(0.2812,1.7995)	1.5183
8	0.8711(0.6620)	0.6420(1.3094)	1.2595(0.2350)	0.9966(0.1554)	(0.5422,2.0560)	1.5138	(0.2486,1.7505)	1.5019
9	0.8762(0.3964)	0.6482(1.2667)	1.2064(0.1860)	0.9843(0.1860)	(0.4841,1.9342)	1.4501	(0.2622,1.6971)	1.4349

Table 4: *MSE's(in parentheses) of TK and MCMC Bayes estimates of paramter values based on record values for prior (0.5,0.5) at (1,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	1.1430(3.9937)	0.7701(0.8875)	1.7934(1.0677)	1.1166(0.1723)	(0.6164,3.0766)	2.4602	(0.3224,1.8573)	1.5349
7	0.7953(1.4146)	0.6489(0.8168)	1.1442(0.1622)	0.9646(0.1494)	(0.4872,2.8896)	2.4024	(0.2576,1.7041)	1.4465
8	0.8985(0.5799)	0.6943(0.7549)	1.2675(0.2363)	1.0064(0.1492)	(0.5259,2.0663)	1.5404	(0.2751,1.7057)	1.4306
9	0.8984(0.3504)	0.6963(0.7253)	1.2220(0.1992)	1.0021(0.1420)	(0.5107,1.9473)	1.4366	(0.2617,1.6775)	1.4158

Table 5: *MSE's(in parentheses) of TK and MCMC Bayes estimates of paramter values based on record values for prior (0.2,0.2) at (2,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	2.1127(6.8287)	0.7480(0.4823)	2.5948(0.7996)	1.0486(0.0960)	(1.2007,6.6411)	5.4404	(0.4850,1.6617)	1.1767
7	1.9563(4.4619)	0.6261(0.4425)	2.1602(0.6114)	0.8843(0.09384)	(0.8237,4.6751)	3.8514	(0.3691,1.4662)	1.0971
8	1.6683(3.0084)	0.6797(0.4030)	2.4131(0.8801)	0.9173(0.0931)	(0.8816,4.0792)	3.1976	(0.3311,1.4743)	1.1432
9	1.6834(2.0903)	0.6862(0.3803)	2.2680(0.6123)	0.9261(0.0916)	(0.9144,3.7480)	2.8336	(0.3390,1.320)	0.9810

Table 6: *MSEs(in parentheses) of TK and MCMC Bayes estimates of paramter values based on record values for prior (0.5,0.5) at (2,1).*

	TK		MCMC		HPD Interval		HPD Interval	
m	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	Length	$\hat{\sigma}$	Length
6	2.1524(6.5179)	0.7685(0.4027)	2.1645(5.1493)	1.0567(0.0959)	(0.9770,6.7309)	5.7539	(0.4583,1.6389)	1.1806
7	1.4706(4.2555)	0.6536(0.3725)	2.1839(0.9530)	0.8829(0.0951)	(0.8348,3.7032)	2.8684	(0.3062,1.4533)	1.1471
8	1.6937(2.8772)	0.7007(0.3397)	2.1308(0.8968)	0.9114(0.0949)	(0.8590,3.0922)	2.2332	(0.3320,1.4685)	1.1365
9	1.7044(1.9899)	0.7054(0.3215)	2.1052(0.8002)	0.9261(0.0674)	(0.9150,2.8668)	1.9518	(0.3858,1.4851)	1.0993

From Tables 1, 2, 3, 4, 5 , and 6 the following conclusions are given as:
In cases where the sample size increases, the mean square error of the maximum likelihood decreases, and the length of the asymptotic confidence interval also decreases in all cases. Bayes estimates are better than the maximum likelihood function as compared to MSEs. In the case of Bayesian estimation, MCMC methods are better than T-K approximation methods. The length of HPD intervals also decreases as the sample size increases.

Therefore, in situations where prior knowledge about parameters is known or where non-informative priors are being used, we advise utilizing the Bayes estimators. In other circumstances, ML estimators could be utilized to get an immediate outcome.

6. Application

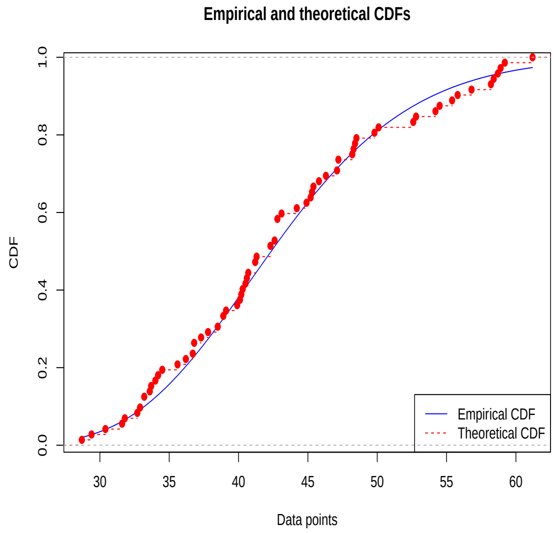
ALAF Industry, a part of the Safal Group, is a leading producer of steel roofing in Tanzania. The Safal Group is renowned for its trusted steel roofing brand and operates in 11 countries across Eastern and Southern Africa. The group has introduced advanced coating technology to Africa, with four coating mills located in Kenya, Uganda, Tanzania, and South Africa. ALAF Industry, as one of Safal Group's coating mills, focuses on enhancing the quality of steel roofing.

One crucial process in improving steel roofing quality is the coating process, where ALAF Industry utilizes aluminum-zinc galvanization technology. Two datasets were analyzed to demonstrate the effectiveness of the coating process. The first dataset comprises 72 observations on coating weight using chemical methods on the top center side (Tcs), while the second dataset consists of 72 observations on coating weight using chemical methods on the bottom center side (Bcs), the two Data sets are given as:
Data set1(Tcs):36.8 47.2 35.6 36.7 55.8 58.7 42.3 37.8 55.4 45.2 31.8 48.3 45.3 48.5 52.8 45.4 49.8 48.2 54.5 50.1 48.4 44.2 41.2 47.2 39.1 40.7 40.3 41.2 30.4 42.8 38.9 34.0 33.2 56.8 52.6 40.5 40.6 45.8 58.9 28.7 37.3 36.8 40.2 58.2 59.2 42.8 46.3 61.2 58.4 38.5 34.2 41.3 42.6 43.1 42.3 54.2 44.9 42.8 47.1 38.9 42.8 29.4 32.7 40.1 33.2 31.6 36.2 33.6 32.9 34.5 33.7 39.9.

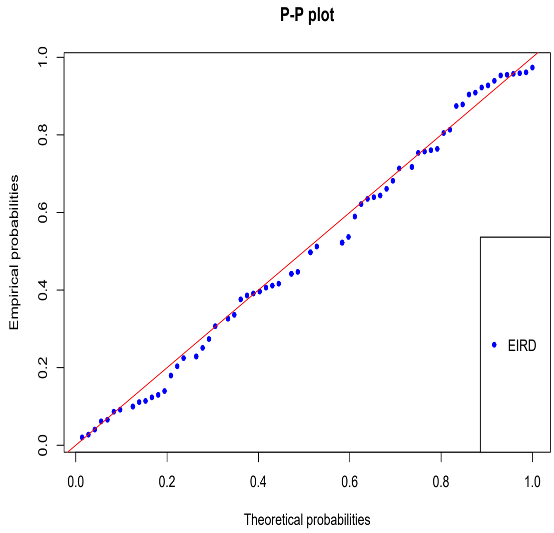
Data Set2(Bcs):45.5 37.5 44.3 43.6 47.1 52.9 53.6 42.9 40.6 34.1 42.6 38.9 35.2 40.8 41.8 49.3 38.2 48.2 44.0 30.4 62.3 39.5 39.6 32.8 48.1 56.0 47.9 39.6 44.0 30.9 36.6 40.2 50.3 34.3 54.6 52.7 44.2 38.9 31.5 39.6 43.9 41.8 42.8 33.8 40.2 41.8 39.6 24.8 28.9 54.1 44.1 52.7 51.5 54.2 53.1 43.9 40.8 55.9 57.2 58.9 40.8 44.7 52.4 43.8 44.2 40.7 44.0 46.3 41.9 43.6 44.9 53.6

To check whether the data set follows the EIRD, the K-S test, emperical cdf, and P-P plot are applied to the test. Data set I supports the EIRD for alpha and beta, with a K-S distance of 0.0523 and p values of 0.8325. Similarly, data set II also supports the EIRD determination with a K-S distance of 0.0731 and p-value of 0.7602. Furthermore, the empirical and theoretical CDFs, as well as the P-P plot (probability-probability plot) displayed in Figure 1, confirm that the EIRD provides a good fit for both the data sets.

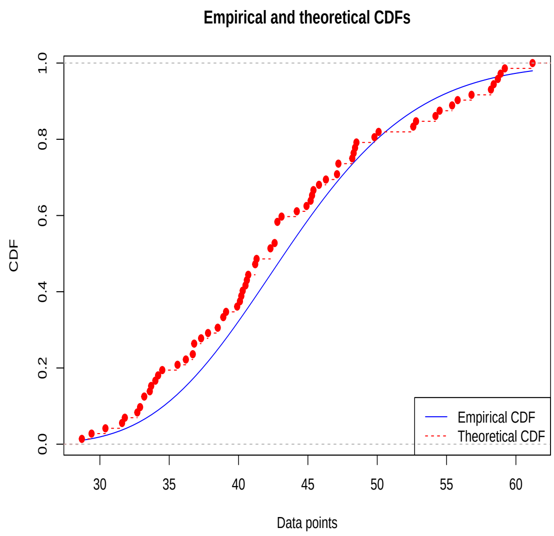
Overall, based on the statistical analysis and visual inspection of the data, it can be concluded that the EIRD is suitable for analyzing the both the Data sets. Now the upper record values generated from the data sets I and II are (36.8,47.2,55.8,58.7,58.9,59.2,61.2) and (45.5,52.9,53.6,62.3), respectively.



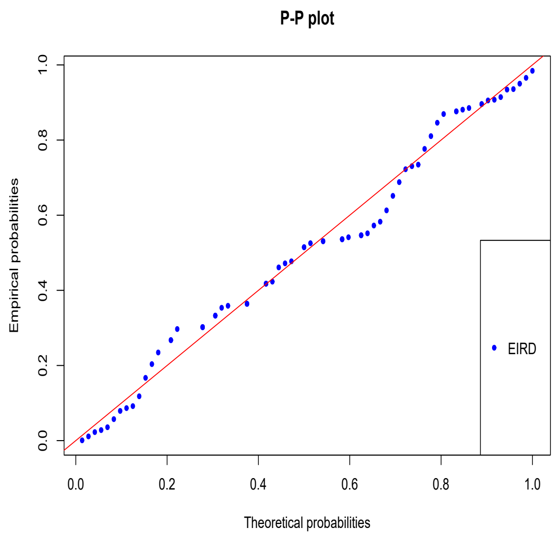
1



2



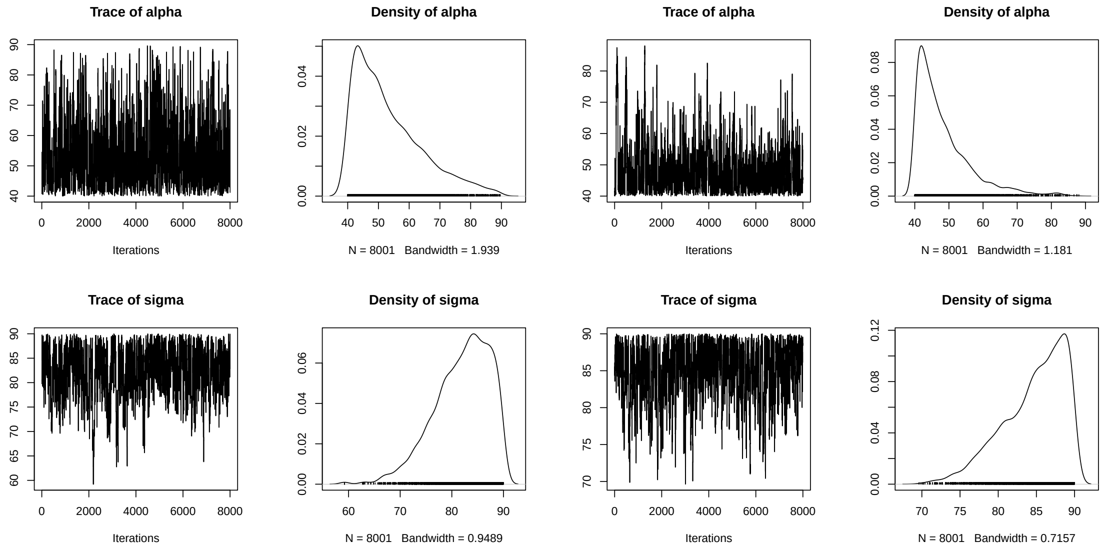
3



4

Figure 1. The emperical and theoratical CDF plot and P-P plot for the real data set.

The article describes obtaining maximum likelihood (ML) estimates and Bayesian estimates using a square error loss function, following the procedure outlined earlier. The results are presented in Table 7. Additionally, trace plots and posterior density plots for the parameters α and σ are depicted in Figures 2 .



Data set I Data set II

Figure 2. The iteration plot and posterior sample plot of Data set I and Data set II.

Table 7: *MLE's, MCMC and TK approximation methods to estimates of parameters for real data set.*

	MLE		MCMC		TK	
Data	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$
Data set I	64.5901	92.3745	52.8539	81.9270	58.3177,	83.6670
Data set II	42.7885	97.2838	49.3734	84.84805	52.4409	93.4215

From these Figure 2, it's concluded that the Markov Chain Monte Carlo (MCMC) samples exhibit good mixing, indicating effective exploration of the parameter space. Moreover, the skewed posterior density suggests a preference for higher parameter values. This observation supports the conclusion that the MCMC chain is stationary, meaning that it has reached a stable distribution.

7. Conclusions

In this study, we have examined the EIRD in a situation where the data are available as upper record values. We follow on the task of estimating the unknown parameter of the EIRD distribution and obtain the maximum likelihood estimators and corresponding confidence intervals for the distribution parameters. In the simulation study, we noticed that the behaviour of estimations in terms of mean square error improved with an increase in the sample size of record values. Additionally, the true value and estimate values are contained in the asymptotic confidence interval. Next, we discussed the problems of computing Bayes estimates under the square error loss function using the TK and MCMC methods in Bayesian estimation. We discovered that the MCMC approach performs better than TK in our simulation study. Still, HPD interval estimates were computed with the help of the MCMC approach.We have used real data sets to demonstrate each of the suggested estimation techniques.

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