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THE MARSHALL-OLKIN EXTENDED SHANKER DISTRIBUTION AND ITS APPLICATIONS

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Abstract

In this paper, we introduce the Marshall–Olkin extended Shanker distribution, as an extension of the Shanker distribution, using the Marshall-Olkin approach. Several important properties of the new distribution, such as the hazard rate function, moments, incomplete moments, mean deviations, Lorenz and Bonferroni curves, and Rényi entropy are explored. The estimation of the parameters is discussed with the help of the maximum likelihood method. The performance of the estimators is evaluated using a simulation study. Two real data applications are developed in order to assess the flexibility and power of the new distribution. The goodness of fit criteria reveal that the new model may provide a better fit than the Shanker distribution and other competing models that belong to the Marshall-Olkin G family of distributions.

Keywords: Shanker distribution, Hazard rate function, Moments, Mean deviations, Bonferroni and Lorenz curves, Rényi entropy, Estimation of parameter

1. Introduction

Marshall and Olkin [17] introduced an interesting method of adding a new parameter to an existing distribution. Let $F(x)$ and $\bar{F}(x) = 1 - F(x)$ be the CDF and survival function of the baseline distribution, respectively. Then, using the above-mentioned method, the survival function of the new distribution takes the following form

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}, \quad -\infty < x < \infty, \tag{1}$$

where $\alpha > 0$, and , $\bar{\alpha} = 1 - \alpha$. The corresponding PDF of (1) is given by

$$g(x) = \frac{\alpha f(x)}{[1 - \bar{\alpha}\bar{F}(x)]^2}, \quad -\infty < x < \infty, \tag{2}$$

We note resulting new distribution admits an additional shape parameter, which can affect the behavior of the hazard rate function of the new distribution. The PDF in (2) is called the Marshall-Olkin extended G (MOE-G for short) distribution. Note that for $\alpha = 1$, $F(x) = G(x)$ and thus the new family includes the baseline distribution as its special case. Marshall and Olkin [17] discussed two special cases of (2), which are the MOE exponential and MOE Weibull distributions. Since then many authors implemented the above-mentioned method to obtain a new family of distributions from an existing baseline distribution. For example, Ghitany et al. [10] introduced the MOE Lindley distribution, MirMostafaee et al. [19] proposed the MOE generalized Rayleigh distribution, and Benkhelifa [6] defined the MOE generalized Lindley distribution. Examples of more recent studies include the Marshall-Olkin inverse Maxwell distribution (Yadav et al. [24]), the Marshall-Olkin Sujatha distribution (Ikechukwu and Eghwerido [14]), the Marshall-Olkin two-parameter Lindley distribution (Gillariose and Tomy [11]), the Marshall-Olkin length biased weighted generalized uniform distribution (Mathew [18]), the Marshall-Olkin alpha power inverse Rayleigh distribution (Adegbite et al. [2]). Some general results and mathematical properties of the MOE family of distributions have been discussed in detail by Barreto-Souza et al. [5], and Cordeiro et al. [8].

Shanker [22] introduced a new lifetime distribution, called the *Shanker distribution*, and showed that the new distribution can give closer fits to lifetime data sets than both exponential and Lindley distributions. The Shanker distribution possesses the following probability density function (PDF)

$$f(x,\theta) = \frac{\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}, \quad x>0, \ \ \theta>0. \tag{3}$$

The corresponding cumulative distribution function (CDF) is also given by

$$F(x,\theta) = 1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}\mathrm{e}^{-\theta x}, \qquad x>0, \ \ \theta>0. \tag{4}$$

Shanker [22] showed that the PDF of the Shanker distribution is a mixture of an exponential distribution and a gamma distribution, and then discussed many mathematical properties of this distribution. Both Shanker and Lindley distributions involve increasing hazard rate functions (HRFs). There are several generalizations of the Shanker distribution in the literature, for example, Shanker and Shukla [23] presented the power Shanker distribution, Abdollahi Nanvapisheh et al. [1] and Jayakumar et al. [15] introduced the exponentiated Shanker distribution, Alzoubi et al. [3] proposed the transmuted Shanker distribution, Helal et al. [13] worked on the weighted Shanker distribution, and Ganaei et al. [9] suggested the weighted power Shanker distribution.

In this paper, we intend to introduce a new extension of the Shanker distribution using the method developed by Marshall and Olkin [17]. The new model is called the Marshall-Olkin extended Shaker (MOE-Sh for short) distribution. The MOE-Sh distribution involves increasing, increasing-decreasing-increasing and decreasing-increasing HRFs so that it can be a very flexible model in lifetime experiments. The new distribution can work better than some other lifetime distribution in a fitting data problem. The rest of the paper is organized as follows: The new distribution is defined in Section 2. The HRF of the new distribution is discussed in Section 3. Several mathematical properties of the new distribution are investigated in Section 4. Section 5 is devoted to the maximum likelihood (ML) estimation of the parameters. A Monte Carlo simulation is developed in Section 6. Two real data applications are given in Section 7. The paper ends with some remarks in Section 8.

2. The New Distribution

If we let $\bar{F}(x,\theta) = \frac{(\theta^2+1)+\theta x}{\theta^2+1}\mathrm{e}^{-\theta x}, x>0$, i.e. the survival function (SF) of the Shanker distribution, in equation (1), we arrive at the following SF

$$\bar{G}(x,\alpha,\theta) = \frac{\alpha(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}}, \quad x>0, \ \ \alpha,\theta>0, \ \ \bar{\alpha}=1-\alpha, \tag{5}$$

which is the SF of the MOE-Sh distribution. If a random variable X possesses the SF (5) with parameters α and θ, then we write $X \sim$ MOE-Sh(α,θ). The PDF of the MOE-Sh distribution with parameters α and θ is given by

$$g(x,\alpha,\theta) = \frac{\alpha\theta^2(\theta+x)(\theta^2+1)\mathrm{e}^{-\theta x}}{(\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x})^2} \ ,x>0, \ \ \alpha,\theta>0, \ \ \bar{\alpha}=1-\alpha. \tag{6}$$

The graphs of the PDF of the MOE-Sh distribution for selected values of α and θ are given in Figure 1.

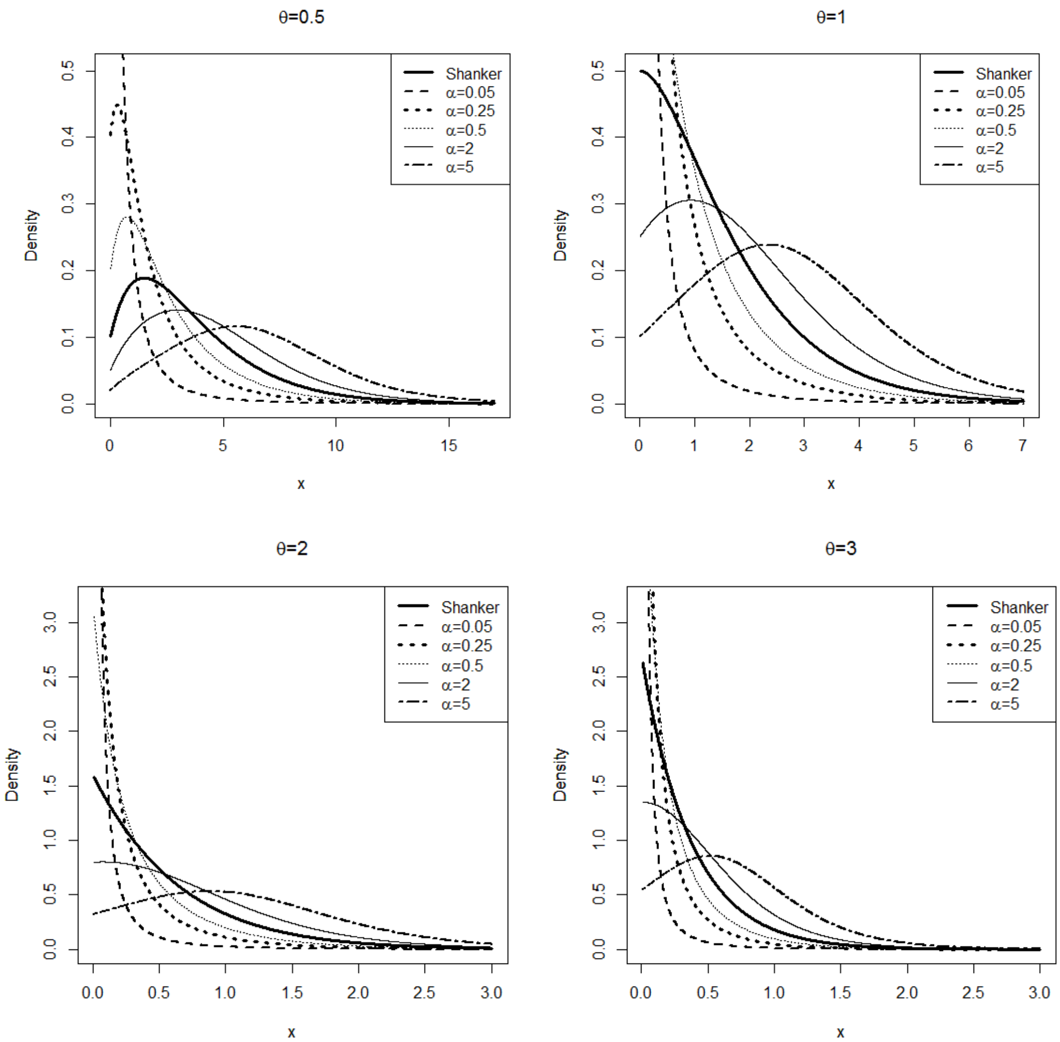

Figure 1: *PDFs of MOE-Sh(α,θ) distribution for selected values of α and θ.*

From Figure 1, we observe that the PDF of the MOE-Sh distribution can be decreasing or unimodal depending on the values of parameters.

The CDF of $X \sim$ MOE-Sh(α,θ) is also given by

$$G(x,\alpha,\theta)=\frac{\theta^2+1-(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}},\quad x>0,\ \ \alpha,\theta>0,\ \ \bar{\alpha}=1-\alpha. \tag{7}$$

3. Hazard Rate Function

The HRF of the MOE-Sh distribution with parameters α and θ is given by

$$h(x)=\frac{\theta^3(\theta^2+1+\theta x)+\theta^2 x}{(\theta^2+1+\theta x)[\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta x}]}.$$

We see that

$$h(0)=\frac{\theta^3}{\alpha(\theta^2+1)},\qquad \text{and} \qquad \lim_{x\to\infty} h(x)=\theta.$$

Therefore, the HRF of the MOE-Sh distribution is bounded. The graphs of the HRF of MOE-Sh distribution for selected values of α and θ are displayed in Figure 2. From Figure 2, we observe that the HRF of the new distribution can be increasing, decreasing-increasing, or increasing-decreasing-increasing depending on the values of parameters. Note that for example for the case when α =0.5 and θ =3, we see that the HRF decreases and then increases very slowly after it attains its minimum.

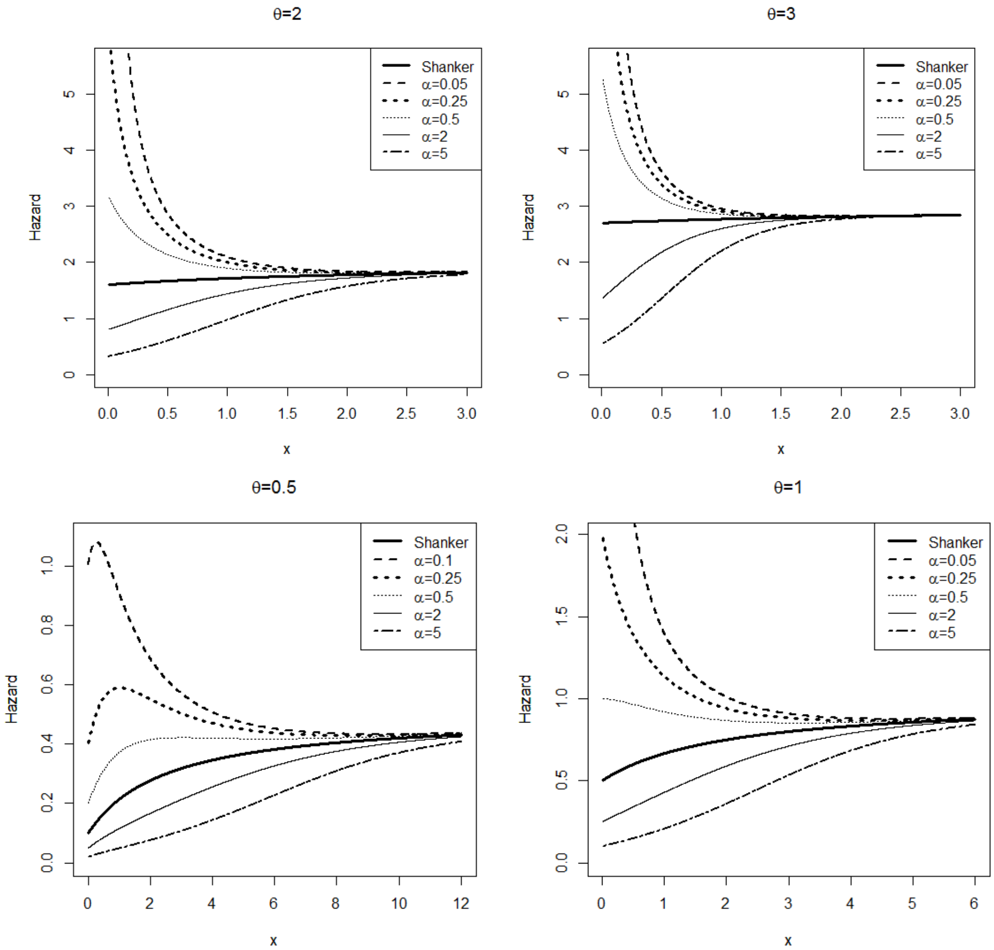

Figure 2. *HRFs of MOE-Sh(α,θ) distribution for selected values of α and θ.*

4. Several Mathematical Properties of the New Distribution

In this section, we discuss some mathematical properties of the new distribution such as the moment generating function, moments, incomplete moment, the mean deviation from the mean and the mean deviation from the median, Bonferroni and Lorenz curves, and Rényi entropy. First, we obtain an expansion for the density of the new distribution, which will be used to obtain general properties of this distribution in the next discussions.

For $|z| < 1$ and $\rho > 0$, we have

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho) j!} z^j, \tag{8}$$

where $\Gamma(.)$ is the gamma function. Applying (8) to (6), for $0 < \alpha < 1$, gives

$$g(x,\alpha,\theta) = \frac{\alpha\theta^2(\theta+x)e^{-\theta x}}{\theta^2+1} \sum_{j=0}^{\infty} \frac{\Gamma(2+j)}{\Gamma(2) j!} (1-\alpha)^j \left[\left(1+\frac{\theta x}{\theta^2+1}\right)e^{-\theta x}\right]^j. \tag{9}$$

Upon applying the binomial expansion to (9), for $x, \theta > 0$ and $0 < \alpha < 1$, we get

$$g(x,\alpha,\theta) = \sum_{j=0}^{\infty}\sum_{m=0}^{j} \alpha(j+1)(1-\alpha)^j \binom{j}{m} \frac{\theta^{2+m}(\theta+x)}{(\theta^2+1)^{m+1}} x^m e^{-(j+1)\theta x}. \tag{10}$$

We can rewrite (2) as follows

$$g(x) = \frac{f(x)}{\alpha\left(1-\frac{(\alpha-1)}{\alpha}F(x)\right)^2}. \tag{11}$$

Therefore, using (8) and (11), and then using the binomial expansion for two times, we arrive at the following expansion for the density when $\alpha > 1$

$$g(x,\alpha,\theta) = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m} \binom{j}{m}\binom{m}{k} \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^j (j+1)(-1)^m \frac{\theta^{2+k}(\theta+x)}{(\theta^2+1)^{k+1}} x^k e^{-\theta(m+1)x}. \tag{12}$$

4.1. Moment Generating Function

Using (10), the moment generating function of the MOE-Sh distribution with parameters α and θ, denoted by $M_X(t)$, for $0 < \alpha < 1$ is given by

$$M_X(t) = \sum_{j=0}^{\infty}\sum_{m=0}^{j} \binom{j}{m} \frac{\alpha(1-\alpha)^j (j+1)\theta^{2+m}\Gamma(m+1)}{(\theta^2+1)^{m+1}(\theta(j+1)-t)^{m+2}} \left(\theta\left((j+1)\theta - t\right) + m + 1\right), \quad t < \theta.$$

For $\alpha > 1$, using (12), we find

$$M_X(t) = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m} \binom{j}{m}\binom{m}{k} \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^j \frac{(j+1)(-1)^m\theta^{2+k}\Gamma(k+1)}{(\theta^2+1)^{k+1}(\theta(m+1)-t)^{k+2}} \left(\theta(\theta(m+1)-t)+k+1\right), \quad t < \theta.$$

4.2. Moments and Related Measures

Some of the most important features and characteristics of a distribution can be studied through its moments. Using (10), the r-th moment of $X \sim$ MOE-Sh(α,θ) for $\alpha \in (0,1)$ has been obtained as

$$\mu_r = \sum_{j=0}^{\infty}\sum_{m=0}^{j} \alpha(1-\alpha)^j \binom{j}{m} \frac{\Gamma(r+m+1)}{(\theta^2+1)^{m+1}(j+1)^{r+m}\theta^r}\left(\theta^2 + \frac{r+m+1}{j+1}\right).$$

Besides, using (12), the r-th moment of $X \sim$ MOE-Sh(α,θ) for $\alpha > 1$ is given by

$$\mu_r = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m} \binom{j}{m}\binom{m}{k}\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^j \frac{(j+1)(-1)^m\Gamma(r+k+1)}{(\theta^2+1)^{k+1}(m+1)^{r+k+1}\theta^r}\left(\theta^2+\frac{r+k+1}{m+1}\right).$$

Thus, the mean of the new distribution for $0 < \alpha < 1$ can be expressed as

$$\mu = \mu_1 = E(X) = \sum_{j=0}^{\infty}\sum_{m=0}^{j} \alpha(1-\alpha)^j \binom{j}{m} \frac{(m+1)!}{(\theta^2+1)^{m+1}(j+1)^{m+1}\theta}\left(\theta^2 + \frac{m+2}{j+1}\right).$$

Moreover, the mean of the new model for $\alpha > 1$ is given by

$$\mu = \mu_1 = E(X) = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m} \binom{j}{m}\binom{m}{k}\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^j \frac{(j+1)(-1)^m(k+1)!}{(\theta^2+1)^{k+1}(m+1)^{k+2}\theta}\left(\theta^2+\frac{k+2}{m+1}\right).$$

Now, the skewness and kurtosis of the new distribution can be obtained with the help of the following equations, respectively

$$Skewness = \frac{E([X-E(X)]^3)}{\left(E([X-E(X)]^2)\right)^{3/2}} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{[\mu_2-\mu_1^2]^{3/2}},$$

and

$$Kurtosis = \frac{E([X-E(X)]^4)}{\left(E([X-E(X)]^2)\right)^{2}} = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{[\mu_2-\mu_1^2]^{2}},$$

where μ_r denotes the r-th moment of $X \sim$ MOE-Sh(α,θ).

Next, we work on finding an expression for the incomplete moment of $X \sim$ MOE-Sh(α,θ). Using (10), for $0 < \alpha < 1$, the incomplete moment of X is given by

$$\int_0^z xg(x,\alpha,\theta)\mathrm{d}x = \sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{\binom{j}{m}\alpha(1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}}\left(\theta^2\Gamma(m+2\,,(j+1)\theta z) + \frac{\Gamma(m+3\,,(j+1)\theta z)}{j+1}\right), \quad (13)$$

where $\Gamma(a\,,z) = \int_0^z x^{a-1}\mathrm{e}^{-x}\mathrm{d}x$ is the incomplete gamma function.

Besides, using (12), for $\alpha > 1$, the incomplete moment of X is obtained to be

$$\int_0^z xg(x,\alpha,\theta)\mathrm{d}x = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m} \frac{\left(1-\frac{1}{\alpha}\right)^j\binom{j}{m}\binom{m}{k}(j+1)(-1)^m\left(\theta^2\Gamma(k+2\,,\theta(m+1)z) + \frac{\Gamma(k+3\,,\theta(m+1)z)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}}. \quad (14)$$

4.3. Mean Deviations and Bonferroni and Lorenz Curves

The mean deviations from the mean and the mean deviation from the median are defined as

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\mathrm{d}x, \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x-M| f(x)\mathrm{d}x,$$

respectively, where $f(x)$ is the density of X, $\mu = E\,(X)$ and M denotes the median of X.

The measures $\delta_1(X)$ and $\delta_2(X)$ can be computed using the following expressions

$$\delta_1(X) = 2\mu F(\mu) - 2\int_0^\mu x f(x)\mathrm{d}x, \tag{15}$$

and

$$\delta_2(X) = \mu - 2\int_0^M x f(x)\mathrm{d}x, \tag{16}$$

respectively, where $F(x)$ is the CDF of X.

Let of $X \sim$ MOE-Sh(α,θ). Then, for $0<\alpha<1$, from (7), (13) and (15), the mean deviation from the mean becomes

$$\delta_1(X) = \frac{2\mu\left(\theta^2+1-(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}\right)}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}} - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{\binom{j}{m}\alpha(1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}}\left(\theta^2\Gamma(m+2\,,(j+1)\theta\mu)+\frac{\Gamma(m+3\,,(j+1)\theta\mu)}{j+1}\right).$$

Besides, for $\alpha>1$, from (7), (14) and (15), the mean deviation from the mean is given by

$$\delta_1(X) = \frac{2\mu\left(\theta^2+1-(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}\right)}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x)\mathrm{e}^{-\theta\mu}} - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\left(1-\frac{1}{\alpha}\right)^j\binom{j}{m}\binom{m}{k}(j+1)(-1)^m\left(\theta^2\Gamma(k+2\,,\theta(m+1)\mu)+\frac{\Gamma(k+3\,,\theta(m+1)\mu)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}}.$$

Moreover, for $0<\alpha<1$, from (13) and (16), the mean deviation from the median becomes

$$\delta_2(X) = \mu - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{\binom{j}{m}\alpha(1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}}\left(\theta^2\Gamma(m+2\,,(j+1)\theta M)+\frac{\Gamma(m+3\,,(j+1)\theta M)}{j+1}\right).$$

Besides, for $\alpha>1$, from (14) and (16), the mean deviation from the median is given by

$$\delta_2(X) = \mu - 2\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\left(1-\frac{1}{\alpha}\right)^j\binom{j}{m}\binom{m}{k}(j+1)(-1)^m\left(\theta^2\Gamma(k+2\,,\theta(m+1)M)+\frac{\Gamma(k+3\,,\theta(m+1)M)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}}.$$

Next, we focus on the formulas of the Bonferroni and Lorenz curves, which are important tools in various fields such as economics, reliability, medicine and insurance. The Bonferroni and Lorenz curves are defined as

$$B_F(F(x)) = \frac{1}{\mu F(x)} \int_0^x u f(u) \mathrm{d}u, \tag{17}$$

and

$$L_F(F(x)) = F(x) B_F(F(x)) = \frac{1}{\mu} \int_0^x u f(u) \mathrm{d}u, \tag{18}$$

respectively, where $F(x)$ is the CDF of X, $f(x)$ is the density of X, and $\mu = E\,(X)$.

Let of $X \sim$ MOE-Sh(α, θ). Then, for $0 < \alpha < 1$, from (7), (13), (17) and (18), the Bonferroni and Lorenz curves are given by

$$B_F(F(x)) = \frac{\theta^2 + 1 - \bar{\alpha}(\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x}}{\mu(\theta^2 + 1 - (\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x})} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{\binom{j}{m} \alpha (1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}} \left(\theta^2 \Gamma(m+2\,,(j+1)\theta x) + \frac{\Gamma(m+3\,,(j+1)\theta x)}{j+1}\right),$$

and

$$L_F(F(x)) = \frac{1}{\mu} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{\binom{j}{m} \alpha (1-\alpha)^j}{\theta(\theta^2+1)^{m+1}(j+1)^{m+1}} \left(\theta^2 \Gamma(m+2\,,(j+1)\theta x) + \frac{\Gamma(m+3\,,(j+1)\theta x)}{j+1}\right),$$

respectively.

Besides, for $\alpha > 1$, from (7), (14), (17) and (18), the Bonferroni and Lorenz curves are given by

$$B_F(F(x)) = \frac{\theta^2 + 1 - \bar{\alpha}(\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x}}{\mu(\theta^2 + 1 - (\theta^2 + 1 + \theta x)\mathrm{e}^{-\theta x})} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \frac{\left(1 - \frac{1}{\alpha}\right)^j \binom{j}{m}\binom{m}{k}(j+1)(-1)^m \left(\theta^2 \Gamma(k+2\,,\theta(m+1)x) + \frac{\Gamma(k+3\,,\theta(m+1)x)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}},$$

and

$$L_F(F(x)) = \frac{1}{\mu} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{k=0}^{m} \frac{\left(1 - \frac{1}{\alpha}\right)^j \binom{j}{m}\binom{m}{k}(j+1)(-1)^m \left(\theta^2 \Gamma(k+2\,,\theta(m+1)x) + \frac{\Gamma(k+3\,,\theta(m+1)x)}{m+1}\right)}{\alpha\theta(\theta^2+1)^{k+1}(m+1)^{k+2}},$$

respectively.

4.4. Rényi Entropy

The entropy of a random variable X is the measure of variation of uncertainty. If X is a continuous random variable having PDF $f(x)$, then the Rényi entropy is defined as

$$T_R(q) = \frac{1}{1-q} \log\left\{\int f^q(x) \mathrm{d}x\right\}, \tag{19}$$

where $q > 0$ and $q \neq 1$.

Let of $X \sim$ MOE-Sh(α,θ). Then, from (6) and (8) and using the binomial expansion, for $0<\alpha<1$, we have

$$g(x,\alpha,\theta)^q=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{\Gamma(2q+j)}{\Gamma(2q)j!}\alpha^q(1-\alpha)^j\binom{j}{m}\frac{\theta^{2q+m}(\theta+x)^q}{(\theta^2+1)^{m+q}}x^m\mathrm{e}^{-\theta(q+j)x}.$$

Therefore, the Rényi entropy is given by

$$T_R(q)=\frac{1}{1-q}\log\left\{\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\Gamma(2q+j)}{\Gamma(2q)j!}\binom{j}{m}\binom{m}{k}\frac{\alpha^q(1-\alpha)^j(-1)^{m-k}\mathrm{e}^{\theta^2(q+j)}\gamma\left(q+k+1,\theta^2(q+j)\right)}{\theta^{2k+1-q-2m}\left(\theta^2+1\right)^{m+q}(j+q)^{q+k+1}}\right\},$$

where $\gamma(a\,,z)=\int_z^\infty x^{a-1}\mathrm{e}^{-x}\mathrm{d}x=\Gamma(a)-\Gamma(a\,,z)$.

From (6) and (8) and using the binomial expansion for two times, for $\alpha>1$, we have

$$g(x,\alpha,\theta)^q=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\frac{\Gamma(2q+j)}{\alpha^q\Gamma(2q)j!}\binom{j}{m}\binom{m}{k}\left(1-\frac{1}{\alpha}\right)^j(-1)^m\frac{\theta^{k+2q}(\theta+x)^q}{(\theta^2+1)^{q+k}}x^k\mathrm{e}^{-\theta(q+m)x}.$$

Thus, the Rényi entropy becomes

$$T_R(q)=\frac{1}{1-q}\log\left\{\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{k=0}^{m}\sum_{i=0}^{k}\frac{\Gamma(2q+j)}{\alpha^q\Gamma(2q)j!}\binom{j}{m}\binom{m}{k}\binom{k}{i}\frac{\left(1-\frac{1}{\alpha}\right)^j(-1)^{m+k-i}\mathrm{e}^{\theta^2(q+m)}\gamma\left(q+i+1,\theta^2(q+m)\right)}{\theta^{2i+1-q-2k}(\theta^2+1)^{q+k}(q+m)^{q+i+1}}\right\}.$$

5. Maximum Likelihood Estimation

Let $\boldsymbol{x}=(x_1,x_2,x_3,\dots,x_n)$ be an observed random sample of size n from the MOE-Sh distribution with parameters α and θ. Then, the likelihood function of the parameters given $\boldsymbol{x}$ is given by

$$\mathcal{L}(\alpha,\theta|\boldsymbol{x})=\frac{\alpha^n\theta^{2n}(\theta^2+1)^n\mathrm{e}^{-\theta\sum_{i=1}^n x_i}\prod_{i=1}^n(\theta+x_i)}{\prod_{i=1}^n(\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i})^2}.$$

Thus, the log-likelihood function takes the following form

$$\ell(\alpha,\theta|\boldsymbol{x})=n\ln\alpha+2n\ln\theta+n\ln(\theta^2+1)+\sum_{i=1}^n\ln(\theta+x_i)-\theta\sum_{i=1}^n x_i-2\sum_{i=1}^n\ln\left(\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}\right).$$

Upon taking the derivatives from the log-likelihood function with respect to (w.r.t.) the parameters, we obtain the following equations that might help us to find the ML estimates of the unknown parameters

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}-2\sum_{i=1}^n\frac{(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}}=0,$$
$$\frac{\partial\ln L}{\partial\theta}=\frac{2n}{\theta}+\frac{2n\theta}{\theta^2+1}+\sum_{i=1}^n\frac{1}{\theta+x_i}-\sum_{i=1}^n x_i-2\theta\sum_{i=1}^n\frac{2-\bar{\alpha}(2-x_i{}^2-\theta x_i)\mathrm{e}^{-\theta x_i}}{\theta^2+1-\bar{\alpha}(\theta^2+1+\theta x_i)\mathrm{e}^{-\theta x_i}}=0.$$

Numerical procedures such as the Newton-Raphson may be implemented to solve the above nonlinear equations.

6. A Simulation Study

In this section, we evaluate the performance of the ML estimators of the parameters of the MOE-Sh distribution by means of a simulation study. The inverse transform algorithm is used to generate random data from the MOE-Sh distribution. We generated $N = 10000$ samples of sizes $n = 50$, 150, 300 from the MOE-Sh distribution with the parameter combinations: $(\alpha, \theta) = (0.5, 0.5)$, (0.5, 4), (3, 2), and (2, 0.5). The performance of the ML estimators is assessed by means of the estimated bias (bias for short), the estimated mean squared error (EMSE), and the estimated mean relative error (EMRE). Let $\hat{\alpha}$ be the ML estimator of α and $\hat{\alpha}_i$ be the ML estimator of α that is obtained in the i-th iteration, then the estimated bias, EMSE, and EMRE of $\hat{\alpha}$ can be obtained using the following equations

$$bias(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha), \qquad EMSE(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2, \qquad \text{and} \qquad EMRE(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\hat{\alpha}_i}{\alpha}\right),$$

respectively. We can obtain the estimated bias, MSE, and MRE of $\hat{\theta}$ (the ML estimator of θ) similarly. The numerical results of the simulation are given in Table 1. It is clear from Table 1 that the estimated biases and estimated MSEs decrease when the sample size n increases. Besides, the estimated MREs of all parameters are close to one and approach this nominal value when the sample size increases.

7. Applications

In this section, we provide two real data applications in order to demonstrate the flexibility of the MOE-Sh distribution. We check how well the MOESH distribution fits the data compared to several other lifetime distributions which are

1. The Shanker distribution with the following PDF

$$f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x}, \quad x > 0, \ \theta > 0.$$

2. The Marshall-Olkin Sujatha (MOS) [14] distribution with the following PDF

$$f(x;\alpha,\theta) = \frac{\alpha\,\theta^3 e^{-\theta x}(\theta^2+\theta+2)(1+x+x^2)}{[(\theta^2+\theta+2)-(1-\alpha)((\theta^2+\theta+2)+\theta x(\theta x+\theta+2))e^{-\theta x}]^2}, \qquad x > 0, \qquad \alpha,\theta > 0.$$

3. The Marshall-Olkin extended Lindley (MOE-L) [10] distribution with the following PDF

$$f(x;\alpha,\theta) = \frac{\alpha\,\theta^2(\theta+1)(1+x)e^{-\theta x}}{[\theta+1-(1-\alpha)(\theta+1+\theta x)e^{-\theta x}]^2}, \quad x > 0, \quad \alpha,\theta > 0.$$

The fitness performance of the considered distributions is investigated using the Akaike information criteria (AIC), Bayesian information criteria (BIC), and Kolmogorov-Smirnov (K-S) along with its p-value. The distribution with the smallest K-S, AIC and BIC values and the highest p-value is considered to possess the best fit to the data sets.

The first real data set, denoted by D I, reported by Chinedu et al. [7] is related to the infant mortality rate per 1000 live births for a few selected nations in 2021, see https://data.worldbank.org/indicator/SP.DYN.IMRT.IN (accessed on 2021). The data are

56, 10, 22, 3, 69, 6, 7, 11, 4, 4, 19, 13, 7, 27, 12, 3, 4, 11, 84, 27, 25, 6, 35, 14, 11, 12, 6

Table 1: *The simulation results*

	$\alpha = 0.5$	and	$\theta = 4$	
n	Parameters	bias	EMSE	EMRE
50	α	0.135	0.148	1.269
	θ	0.485	2.449	1.121
150	α	0.041	0.032	1.083
	θ	0.153	0.664	1.038
300	α	0.019	0.014	1.039
	θ	0.072	0.311	1.018
	$\alpha = 3$	and	$\theta = 2$	
n	Parameters	bias	EMSE	EMRE
50	α	0.726	5.057	1.242
	θ	0.079	0.143	1.039
150	α	0.211	1.012	1.070
	θ	0.025	0.042	1.012
300	α	0.105	0.447	1.035
	θ	0.012	0.021	1.006
	$\alpha = 0.5$	and	$\theta = 0.5$	
n	Parameters	bias	EMSE	EMRE
50	α	0.270	0.543	1.540
	θ	0.038	0.040	1.077
150	α	0.079	0.088	1.159
	θ	0.014	0.011	1.029
300	α	0.039	0.036	1.077
	θ	0.007	0.005	1.014
	$\alpha = 2$	and	$\theta = 0.5$	
n	Parameters	bias	EMSE	EMRE
50	α	0.678	4.143	1.339
	θ	0.021	0.012	1.042
150	α	0.189	0.685	1.094
	θ	0.006	0.004	1.013
300	α	0.084	0.287	1.042
	θ	0.003	0.002	1.005

The second data set, denoted by D II, was originally taken from Aydin [4]. This data set is related to the average daily wind speed collected in 2015 from meteorological Turkish services, see also Salahuddin et al. [21]. The data are

2.8, 1.8, 3.2, 5.0, 2.4, 4.8, 2.9, 2.9, 2.3, 3.2, 2.3, 2.0, 1.9, 3.3, 4.4, 6.7,
4.3, 1.9, 2.2, 3.3, 2.1, 4.0, 2.0, 3.1, 3.8, 3.1, 3.2, 3.4, 2.8, 2.1, 3.1

We compute the ML estimates of the parameters for the considered distributions. We also use the Kolmogorov-Smirnov (K-S) test, the Akaike information criterion (AIC), and the Bayesian information criterion (BIC) for the purpose of comparing the fits of the distributions. We know that ties should not be present for the K-S test, when we analyze continuous data. However, ties may arise due to rounding numbers. Here, to avoid this problem, we added (and also subtracted when there are 3 equal numbers) a too small number, which is $z = 10^{-14}$, to one of the equal numbers, when we want to calculate K-S test statistics. For example, D II has been changed to the following data in this regard

$$2.8,\ 1.8,\ 3.2{-}z,\ 5.0,\ 2.4,\ 4.8,\ 2.9{+}z,\ 2.9,\ 2.3{+}z,\ 3.2,\ 2.3,\ 2.0{+}z,\ 1.9,\ 3.3,\ 4.4,\ 6.7,$$

$$4.3,\ 1.9{+}z,\ 2.2,\ 3.3{+}z,\ 2.1{+}z,\ 4.0,\ 2.0,\ 3.1{+}z,\ 3.8,\ 3.1,\ 3.2{+}z,\ 3.4,\ 2.8{+}z,\ 2.1,\ 3.1{-}z$$

The computed ML estimates, K-S test statistics along with their corresponding p-values, and the values of AIC and BIC for both data sets are given in Table 2. We note that the smaller values of AIC, BIC and K-S test statistics (and equivalently the larger p-values) indicate a better fit to a data set. Table 2 reveals that the MOE-Sh distribution possesses the best fits for both data sets among the considered distributions. Figures 3 and 4 include the probability-probability (P-P) plots for D I and D II, respectively. From Figures 3 and 4, we might conclude the superiority of the MOE-Sh distribution over the other considered models.

Table 2: *The ML estimates of the parameters, K-S test statistics along with their corresponding p-values, and the values of AIC and BIC for D I and D II*

Data set	Models	α	θ	AIC	BIC	K-S	p-value
D I	Shanker		0.10577	217.2489	218.5447	0.22721	0.1046
	MOS	0.04691	0.06489	214.1856	216.7773	0.13897	0.6245
	MOE-L	0.08771	0.03662	211.8659	214.4576	0.09468	0.9500
	MOE-Sh	0.07497	0.03690	**210.8241**	**213.4158**	**0.08978**	**0.9678**
D II	Shanker		0.54730	120.4348	121.8688	0.34356	0.0009
	MOS	68.37348	2.13868	95.0517	97.9196	0.12053	0.7139
	MOE-L	100.0365	1.90636	94.4033	97.2713	0.11641	0.7518
	MOE-Sh	123.2082	1.88271	**94.2711**	**97.1391**	**0.11478**	**0.7666**

12. Concluding Remarks

In this paper, we follow the Marshall-Olkin strategy of developing more flexible models to introduce a new two-parameter lifetime distribution, called the Marshall-Olkin extended Shanker (MOE-Sh) distribution. Several useful properties of the new distribution are discussed. A simulation study has been conducted to examine the performance of the ML estimators of the proposed MOE-Sh distribution. Two real data applications have been analyzed to illustrate the flexibility of the new distribution in comparison with several competitive distributions. The data analyses indicate that the MOE-Sh has the potential power to model real data quite well and it can be useful in the study of real-life phenomena. Still, there exist some other characteristics of the new distribution such as the reliability parameter, stochastic ordering, order statistics and so on that have not been investigated in this paper. Moreover, some inferential subjects for the new distribution such as the Bayesian estimation of the parameters, prediction of future observations and so on may be considered to be studied in the future.

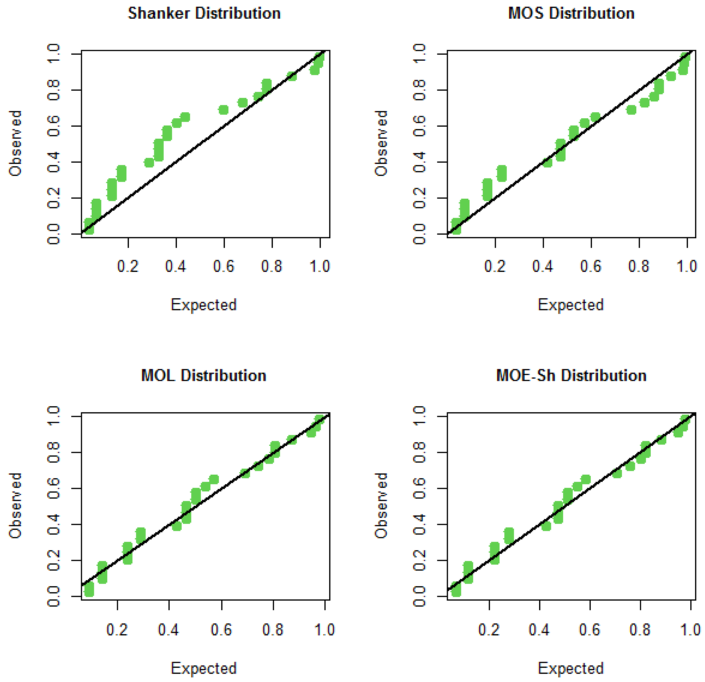

Figure 3. *P-P plots for D I.*

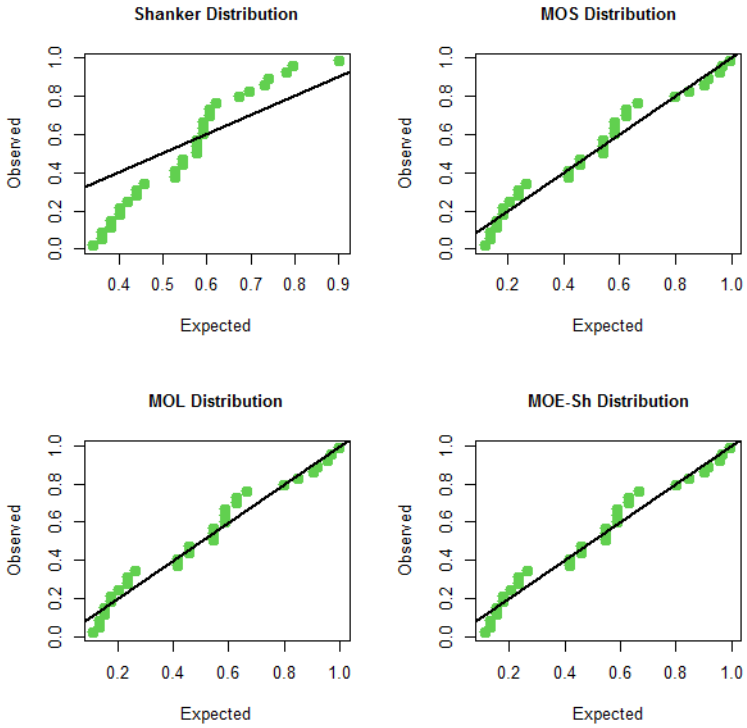

Figure 4. *P-P plots for D II.*

All the computations of the paper were performed using the statistical software R (R Core Team [20]) and the packages *nleqslv* (Hasselman [12]) and *AdequacyModel* (Marinho et al. [16]) therein.

Data Availability Statement

The data sets used in this paper are provided in the manuscript.

Declaration of Conflicting Interests

The Authors declare that there is no conflict of interests.

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