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EXPLORING AN EXTENDED RAYLEIGH DISTRIBUTION: MODELING AND APPLICATIONS IN REAL LIFE SCENARIOS

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Abstract

In this manuscript, we propose a new extension of the Rayleigh distribution, named as Ratio Transformation Rayleigh Distribution (RTRD), which offers superior fits compared to the Rayleigh distribution and several of its known generalizations. We derive various properties of the proposed distribution, including moments, moment generating function, hazard rate, conditional moments, Bonferroni and Lorenz curves, mean residual life, mean waiting time, Renyi entropy and order statistics. The unknown parameters are estimated using the maximum likelihood estimation procedure. An extensive simulation study is conducted to illustrate the behavior of the maximum likelihood estimators (MLEs) based on Mean Square Errors. The flexibility of the new distribution is demonstrated by applying it to two real data sets. Comparative analysis with the Rayleigh distribution, Weighted Rayleigh distribution, Exponentiated Rayleigh distribution and Transmuted Rayleigh distribution reveals that RTRD outperforms these competing distributions based on Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion Corrected (AICC) and other goodness of fit measures.

Keywords: Ratio transformation, Rayleigh distribution , Moments, Conditional moments , Renyi entropy, Maximum Likelihood estimation

1. Introduction

Probability distributions are vital for statistical inference and data analysis, enabling meaningful interpretations and informed decision-making. While classical distributions are widely used across various fields, they often struggle to accurately model real-world data. Consequently, researchers have focused on extending classical distributions to improve their fit and adaptability in data modeling.

The Rayleigh distribution, originally introduced by Rayleigh [22] in the context of acoustics, has been extensively studied in the statistical literature. Several extensions and applications of the Rayleigh distribution (RD) have been proposed over time. Siddiqui [24] explored its genesis and origin, while Howlader and Hossain [14] examined its Bayesian estimation under type-II censored data. Lalitha and Mishra [17] discussed modified maximum likelihood estimation for the Rayleigh distribution. Surles and Padgett [25] introduced the two-parameter Burr type X distribution, referring to it as the exponentiated Rayleigh distribution (ERD) or generalized Rayleigh distribution. Kundu and Raqab [16] investigated parameter estimation techniques for the generalized Rayleigh distribution. Abd Elfattah et al. [1] studied the efficiency of maximum likelihood estimators under different censored sampling schemes. Dey and Tanujit [13] explored

Bayesian estimation of the scale parameter, while Ahmed et al. [4] employed the square error loss function and Al-Bayyati's loss function for Bayesian analysis of the Rayleigh distribution.

Ajami and Jhansi [5] focused on parameter estimation for the weighted Rayleigh distribution, while Ahmad et al. [3] introduced the Weibull-Rayleigh distribution, characterizing and estimating its parameters using the transformed transformer technique. Ardianti [7] applied classical and Bayesian methods to estimate Rayleigh distribution parameters. Bhat and Ahmad [12] proposed a novel extension of the exponentiated Rayleigh distribution, studied its properties, and demonstrated its applicability using various datasets. The same authors [11] investigated the mixture of Gamma and Rayleigh distributions. Kilai et al. [15] developed a versatile modification of the Rayleigh distribution for modeling COVID-19 mortality rates. Bhat et al. [9] proposed a new extension of the odd Lindley power Rayleigh distribution, analyzing its properties and parameter estimation using classical and Bayesian approaches. Bhat and Ahmad [10] introduced a generalization of the Rayleigh distribution using a power transformation technique, while Mir and Ahmad [20] proposed the sine power Rayleigh distribution, examining its properties and applications. Abdelall and Yassmen [2] studied the Marshall-Olkin power Rayleigh distribution with properties and engineering applications. Anis et al. [6] reviewed the Rayleigh distribution, discussing its properties, estimation techniques, and application to COVID-19 data.

This manuscript aims to present and analyze a new lifetime model, termed the Ratio Transformation Rayleigh Distribution (RTRD), developed using the Ratio Transformation (RT) method. A notable advantage of the RTRD is its additional parameter, which imparts desirable properties and enhances the flexibility of its density and hazard rate functions. Furthermore, the model demonstrates superior performance compared to several established distributions when applied to real-world datasets.

The structure of the paper is as follows: Section 2 introduces the RT method. Section 3 outlines the formulation of the RTRD, while Section 4 discusses its statistical properties in detail. The maximum likelihood approach for parameter estimation is addressed in Section 5. Sections 6 and 7 present the results of an extensive simulation study and demonstrate the model's practical applicability, respectively. Finally, Section 8 provides concluding remarks.

2. Ratio Transformation Method

The CDF and PDF of the Ratio Transformation (RT) Method proposed by [18] are defined by the following equations:

$$F_{RT}(x) = \frac{F(x)}{1+\alpha-\alpha^{F(x)}}\ ; \qquad \alpha > 0. \tag{1}$$

$$f_{RT}(x) = f(x)\frac{\left(1+\alpha-\alpha^{F(x)}\left(1-F(x)log\alpha\right)\right)}{\left(1+\alpha-\alpha^{F(x)}\right)^2}\ ; \qquad \alpha > 0. \tag{2}$$

Where $F(x)$ and $f(x)$ in Eq. (1) and Eq. (2) above are the CDF and PDF of the base line distribution respectively.
Rasool and Ahmad [21] explored the Ratio Transformation Lomax distribution and its applications.

3. Ratio Transformation Rayleigh Distribution (RTRD)

The Rayleigh distribution (RD), named after Lord Rayleigh [22] is prominent lifetime probability model concerned with describing skewed data. The probability density function (PDF) associated with random variable $x > 0$ having RD with scale parameter θ is given by

$$f(x;\theta) = \frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\ ; \quad x > 0, \theta > 0 \tag{3}$$

and the corresponding cumulative distribution function (CDF) is given as

$$F(x;\theta) = 1 - e^{-\frac{x^2}{2\theta^2}}\ ; \quad x > 0, \theta > 0 \tag{4}$$

Here we introduce, RT method. Considering F(x; θ) be the CDF of Rayleigh distribution. Then the CDF of RTRD can be obtained by inserting Eq. (4) in Eq.(1) and is given by

$$F(x;\alpha,\theta) = \begin{cases} \dfrac{1-e^{-\frac{x^2}{2\theta^2}}}{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}} & ;\alpha \neq 1, \alpha, \theta > 0 \\ 1 - e^{-\frac{x^2}{2\theta^2}} & ;\alpha = 1, \theta > 0 \end{cases} \tag{5}$$

The corresponding PDF of RTRD is obtained as

$$f(x;\alpha,\theta) = \begin{cases} \dfrac{\frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x^2}{2\theta^2}}\right)\log\alpha\right)\right)}{\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^2} & ,\alpha \neq 1, \alpha, \theta > 0 \\ \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & ,\alpha = 1, \theta > 0 \end{cases} \tag{6}$$

Figure 1 illustrates the probability density function (PDF) of the RTRD for various parameter combinations of α and θ.

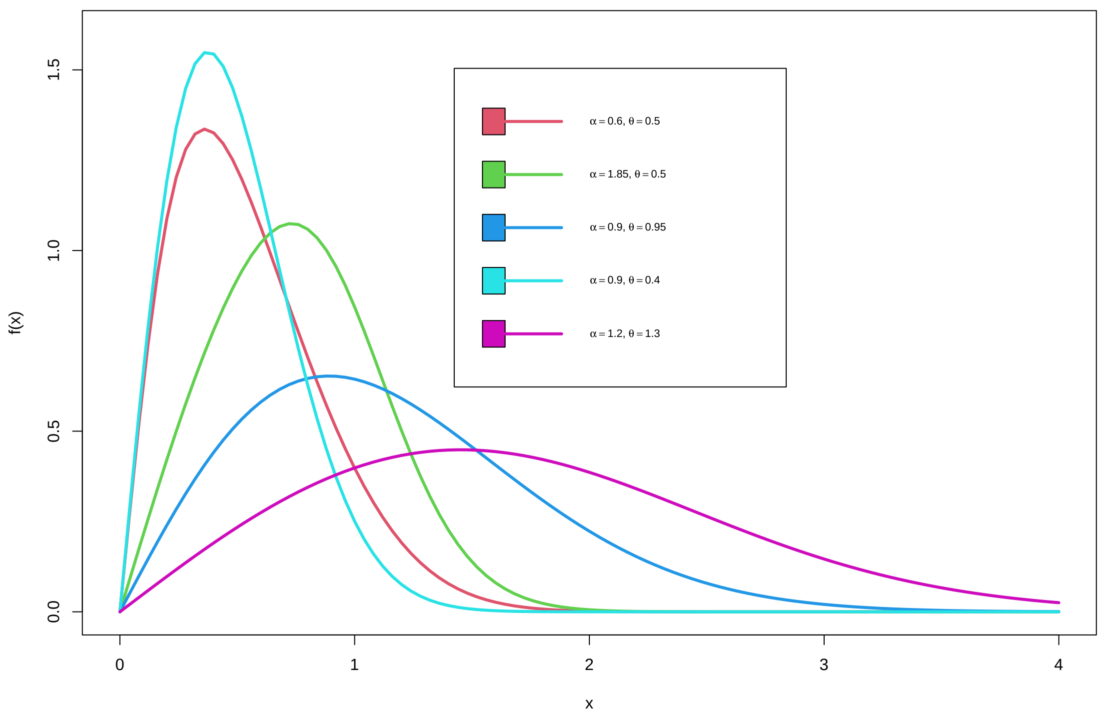

Figure 1: *Plots of the pdf of the RTRD*

3.1. Survival function

The survival function for the RTRD is given as

$$R_{RTRD}(x;\alpha,\theta)=\frac{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}}{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}\;;\;\alpha,\theta>0 \tag{7}$$

3.2. Hazard Rate

The hazard rate for RTRD is obtained as

$$h_{RTRD}(x)==\frac{\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x^2}{2\theta^2}}\right)\log\alpha\right)\right)}{\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)\left(\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}\right)}\;;\;\alpha,\theta>0 \tag{8}$$

Figure 2 depicts graphs of the hazard rate of the RTRD for different parameter values. Figure 2 suggests that the proposed distribution is quite flexible in nature and can exhibit variety of shapes over the parameter space.

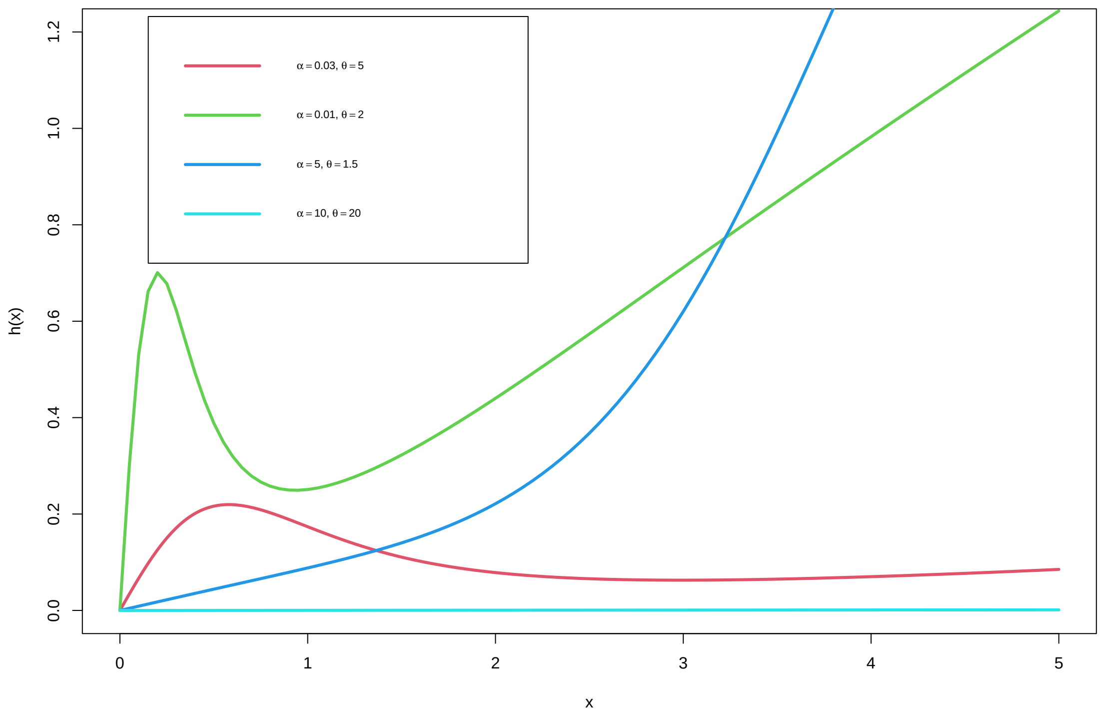

Figure 2: *Plots of the hazard rate of the model*

3.3. Reverse Hazard function

The reverse hazard rate for RTRD is obtained as

$$h_r(x;\alpha,\theta)=\frac{\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x^2}{2\theta^2}}\right)\log\alpha\right)\right)}{\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)\left(1-e^{-\frac{x^2}{2\theta^2}}\right)}\ ;\ \alpha,\theta>0 \tag{9}$$

3.4. Cumulative Hazard function

The cumulative hazard function for the RTRD is defined as

$$\Lambda_{RTRD}(x;\alpha,\theta)=\log\left\{\frac{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}}\right\}$$

3.5. Mills Ratio

The mills ratio for RTRD is given by

$$M.R=\frac{1-e^{-\frac{x^2}{2\theta^2}}}{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}} \tag{10}$$

4. Statistical Properties of RTRD

This section focuses on deriving several key mathematical properties, such as the r^{th} moment, moment generating function, conditional moments and their associated measures, entropy and order statistics.

4.1. Moments

The r^{th} moment of X can be obtained as

$$\begin{aligned}E(X^r)&=\int_0^\infty x^r f(x;\alpha,\theta)dx\\&=\int_0^\infty x^r\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-(1-e^{-\frac{x^2}{2\theta^2}})\log\alpha\right)\right)\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^{-2}dx.\end{aligned} \tag{11}$$

By substituting $1-e^{-\frac{x^2}{2\theta^2}}=y$ in (11), we get

$$E(X^r)=\sum_{j=0}^{\infty}\frac{1}{(1+\alpha)^{j+1}}\left(\int_0^1\left(-2\theta^2 log(1-y)\right)^{\frac{r}{2}}\left(\alpha^{yj}+\frac{\alpha^{(j+1)y}(j+1)log\alpha}{1+\alpha}y\right)dy\right). \tag{12}$$

Again, substituting $-2\theta^2 log(1-y) = x$ in (12), we get the final expression as

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{r}{2}+1)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\ log\alpha\ (j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{r}{2}+1}} - \frac{1}{(m+2)^{\frac{r}{2}+1}}\right)\right\}. \tag{13}$$

setting $r = 1$ in Eq. (13) the mean of the model is computed as

$$E(X) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{1}{2}}\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{3}{2})\left\{\frac{j^m}{(m+1)^{\frac{3}{2}}} + \frac{\alpha\ log\alpha\ (j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{3}{2}}} - \frac{1}{(m+2)^{\frac{3}{2}}}\right)\right\} \tag{14}$$

Similarly for r = 2, 3 and 4 in Eq. (13) , the second, third and fourth moment about origin are respectively.

4.2. Moment Generating function of RTRD

The following theorem provides the MGF for the RTRD .

Theorem 1. Let X follow the RTRD (α, θ) , then the moment generating function, $M_X(t)$, is

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^r}{r!}\frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma\left(\frac{r}{2}+1\right)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^3} - \frac{1}{(m+2)^3}\right)\right\} \tag{15}$$

Proof: The moment generating function of the RTRD is defined as

$$M_X(t) = \int_0^{\infty} e^{tx} f(x;\alpha,\theta)dx \tag{16}$$

Using the series representation of e^{tx}, we have

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r) \tag{17}$$

Substituting the value of Eq. (13) in Eq. (17) , we get

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^r}{r!}\frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma\left(\frac{r}{2}+1\right)\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^3} - \frac{1}{(m+2)^3}\right)\right\} \tag{18}$$

4.3. Conditional moments and associated measures

In this sub section, the expression for conditional moments is acquired. But first we will introduce an important lemma which will be applied in the next sub section.
Lemma 1. Let us suppose a random variable X follows RTRD (α, θ) with PDF given in Eq. (6) and let $\varphi_r(z) = \int_0^z x^r f(x)\,dx$ denotes the r^{th} incomplete moment, then we have

$$\begin{aligned}\varphi_r(z) = &\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right)\right.\\ &\left. + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)z^2, \frac{r}{2}+1\right)\right]\right\}\end{aligned} \tag{19}$$

where $\gamma(a,b) = \int_0^b z^{a-1}e^{-z}\,dz$ denotes the lower incomplete gamma function.

Proof. Using the PDF of RTRD given in Eq. (6), we have

$$\varphi_r(z) = \int_0^z x^r f(x;\alpha,\theta)dx \tag{20}$$

$$= \int_0^z x^r \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\left(1-(1-e^{-\frac{x^2}{2\theta^2}})\log\alpha\right)\right)\left(1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^{-2} dx. \tag{21}$$

By using the same procedure as in the Eq. (13) above, we get

$$\varphi_r(z) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left\{\frac{j^m}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{r}{2}+1\right)\right.$$
$$\left.+\frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2,\frac{r}{2}+1\right)-\frac{1}{(m+2)^{\frac{r}{2}+1}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)z^2,\frac{r}{2}+1\right)\right]\right\} \tag{22}$$

Setting $r=1$ in Eq. (22) will yield first incomplete moment as given by

$$\varphi_1(z) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{1}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\left\{\frac{j^m}{(m+1)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2, \frac{3}{2}\right)\right.$$
$$\left.+\frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha}\left[\frac{1}{(m+1)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)z^2,\frac{3}{2}\right)-\frac{1}{(m+2)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+2}{2\theta^2}\right)z^2,\frac{3}{2}\right)\right]\right\} \tag{23}$$

4.3.1 Lorenz and Bonferroni inequality Curves

The Lorenz and Bonferroni inequality curves represent significant applications of the first incomplete moment. For a given probability distribution, they are defined as follows.

$$L_p = \frac{1}{E(X)}\int_0^t x f(x;\alpha,\theta)\,dx = \frac{\varphi_1(t)}{E(X)}$$

$$L_p = \frac{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\{A_2+B_2\}}{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{3}{2})\{A_1+B_1\}}$$

Similarly,

$$B_P = \frac{1}{pE(X)}\int_0^t x f(x;\alpha,\theta)\,dx = \frac{\varphi_1(t)}{pE(X)}$$

$$B_P = \frac{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!}\{A_2+B_2\}}{p\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^j(-log\alpha)^m}{(1+\alpha)^{j+1}m!}\Gamma(\frac{3}{2})\{A_1+B_1\}}$$

Where,

$$A_1 = \frac{j^m}{(m+1)^{\frac{3}{2}}}\ ,$$

and

$$B_1 = \frac{\alpha\ log\alpha\ (j+1)^{m+1}}{1+\alpha}\left(\frac{1}{(m+1)^{\frac{3}{2}}}-\frac{1}{(m+2)^{\frac{3}{2}}}\right).$$

$$A_2 = \frac{j^m}{(m+1)^{\frac{3}{2}}}\gamma\left(\left(\frac{m+1}{2\theta^2}\right)t^2,\frac{3}{2}\right)\ ,$$

and

$$B_2 = \frac{\alpha \log \alpha (j+1)^{m+1}}{1+\alpha} \left[\frac{1}{(m+1)^{\frac{3}{2}}} \gamma \left(\left(\frac{m+1}{2\theta^2} \right) t^2, \frac{3}{2} \right) - \frac{1}{(m+2)^{\frac{3}{2}}} \gamma \left(\left(\frac{m+2}{2\theta^2} \right) t^2, \frac{3}{2} \right) \right].$$

4.3.2 r-th Conditional Moment and r-th Reversed Conditional Moment of RTRD

The r^{th} conditional moment of the RTRD is calculated by

$$E\left[X^r | x > t\right] = \frac{1}{R(t)} \int_t^\infty x^r f(x;\alpha,\theta)\, dx = \frac{1}{R(t)} \left[E(X^r) - \varphi_r(t)\right]$$

where $R(t)$ is the reliability of RTRD at time t. Inserting the value of Eq.s (7), (13) and (22), we obtain

$$E\left[X^r \mid x > t\right] = \frac{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}{\alpha\left(1-\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)+e^{-\frac{x^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!} \Gamma\left(\frac{r}{2}+1\right) \left\{ \frac{j^m}{(m+1)^{\frac{r}{2}+1}} + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha} \left(\frac{1}{(m+1)^{\frac{r}{2}+1}} - \frac{1}{(m+2)^{\frac{r}{2}+1}} \right) \right\} - \left(\frac{j^m}{(m+1)^{\frac{r}{2}+1}} \gamma\left(\frac{(m+1)t^2}{2\theta^2}, \frac{r}{2}+1\right) + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha} \left[\frac{1}{(m+1)^{\frac{r}{2}+1}} \gamma\left(\frac{(m+1)t^2}{2\theta^2}, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}} \gamma\left(\frac{(m+2)t^2}{2\theta^2}, \frac{r}{2}+1\right) \right] \right).$$

Similarly, the r^{th} reversed conditional moment of the RTRD is defined by

$$E\left[X^r | x \leq t\right] = \frac{1+\alpha-\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}}{1-e^{-\frac{x^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta^2)^{\frac{r}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!} \left\{ \frac{j^m}{(m+1)^{\frac{r}{2}+1}} \gamma\left(\left(\frac{m+1}{2\theta^2}\right)t^2, \frac{r}{2}+1\right) + \frac{\alpha\log\alpha(j+1)^{m+1}}{1+\alpha} \left[\frac{1}{(m+1)^{\frac{r}{2}+1}} \gamma\left(\left(\frac{m+1}{2\theta^2}\right)t^2, \frac{r}{2}+1\right) - \frac{1}{(m+2)^{\frac{r}{2}+1}} \gamma\left(\left(\frac{m+2}{2\theta^2}\right)t^2, \frac{r}{2}+1\right) \right] \right\}.$$

4.3.3 Mean Residual Life (MRL) and Mean Waiting Time (MWT)

Mean Residual Life (MRL) is the expected remaining lifetime of an item that has already survived up to a certain time t. It provides a measure of the average future life expectancy of an item given that it has lasted until time t. The MRL is defined as

$$\mu(t) = \frac{1}{R(t)} \left[E(t) - \int_0^t x f(x;\alpha,\theta)\, dx\right] - t = \frac{1}{R(t)} \left[E(t) - \varphi_1(t)\right] - t$$

After inserting the value of Eq. (7), Eq. (14) and Eq. (23), we obtain the required expression for MRL as

$$\mu(t) = \frac{1+\alpha-\alpha^{1-e^{-\frac{t^2}{2\theta^2}}}}{\alpha\left(1-\alpha^{-e^{-\frac{t^2}{2\theta^2}}}\right)+e^{-\frac{t^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2\theta)^{\frac{1}{2}}\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!} \left(\Gamma(\frac{3}{2}) \left\{A_1+B_1\right\} - \left\{A_2+B_2\right\} \right) - t$$

The mean waiting time is crucial for analyzing the actual time of failure of an item that has already failed. It represents the elapsed time since the failure, assuming the failure happened within the interval $[0,t]$. This mean waiting time, denoted as $\overline{\mu}(t)$, is defined as

$$\overline{\mu}(t) = t - \frac{1}{F(t)} \int_0^t x f(x;\alpha,\theta)\, dx = t - \frac{\varphi_1(t)}{F(t)}$$

$$\overline{\mu}(t) = t - \frac{1+\alpha-\alpha^{1-e^{-\frac{t^2}{2\theta^2}}}}{1-e^{-\frac{t^2}{2\theta^2}}} \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^j(-\log\alpha)^m}{(1+\alpha)^{j+1}m!} \left\{A_2+B_2\right\}$$

4.4. Renyi entropy

Theorem 2. If $X \sim \text{RTRD}(\alpha, \theta)$, then the Renyi entropy of the RTRD is given as

$$R_\eta = \frac{1}{1-\eta} \log \left[\left(\frac{1}{\theta^2}\right)^{\eta-1} \left(\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^j (-\log \alpha)^m}{(1+\alpha)^{j+1} m!} \right)^\eta \left(2\theta^2\right)^{\frac{\eta+1}{2}-1} \Gamma\left(\frac{\eta+1}{2}\right) \right.$$
$$\left. \times \left\{ \frac{j^{\eta m}}{(\eta(m+1))^{\frac{\eta+1}{2}}} + \left(\frac{\alpha \ \log \alpha \ (j+1)^{m+1}}{1+\alpha} \right)^\eta \left(\frac{1}{(\eta(m+1))^{\frac{\eta+1}{2}}} - \frac{1}{(\eta(m+2))^{\frac{\eta+1}{2}}} \right) \right\} \right]$$

Proof: The Renyi entropy, which Alfred Renyi introduced [23] and generalises Shannon's measure of information, is defined as

$$R_\eta = \frac{1}{1-\eta} \log \int_{-\infty}^{\infty} f^\eta(x; \alpha, \theta)\, dx, \quad \eta > 0, \quad \eta \neq 1$$

By using the same procedure as in the Eq. (13) , we get the final expression for Renyi entropy as

$$R_\eta = \frac{1}{1-\eta} \log \left[\left(\frac{1}{\theta^2}\right)^{\eta-1} \left(\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^j (-\log \alpha)^m}{(1+\alpha)^{j+1} m!} \right)^\eta \left(2\theta^2\right)^{\frac{\eta+1}{2}-1} \Gamma\left(\frac{\eta+1}{2}\right) \right.$$
$$\left. \times \left\{ \frac{j^{\eta m}}{(\eta(m+1))^{\frac{\eta+1}{2}}} + \left(\frac{\alpha \ \log \alpha \ (j+1)^{m+1}}{1+\alpha} \right)^\eta \left(\frac{1}{(\eta(m+1))^{\frac{\eta+1}{2}}} - \frac{1}{(\eta(m+2))^{\frac{\eta+1}{2}}} \right) \right\} \right]$$

4.5. Order Statistics of RTRD

Let $x_{(r;n)}$ be the r^{th} order statistics with the random sample $x_{(1)}, x_{(2)}, x_{(3)}, ...x_{(n)}$ derived from the RTRD having the PDF $f(x; , \alpha, \theta)$ and CDF $F(x; \alpha, \theta)$. Therefore, the PDF and CDF of $x_{(r;n)}$ say $f_{(r;n)}(x)$ and $F_{(r;n)}(x)$ are respectively defined as

$$f_{(r;n)}(x) = \frac{1}{B(n, n-r+1)} \left[F(x;\alpha,\theta)\right]^{r-1} \left[1 - F(x;\alpha,\theta)\right]^{n-r} f(x;\alpha,\theta) \tag{24}$$

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[F(x;\alpha,\theta)\right]^j \left[1 - F(x;\alpha,\theta)\right]^{n-j} \tag{25}$$

Using Eq. (6) and Eq. (5) in Eq. (24) and Eq. (25), the PDF and CDF of r^{th} ordered statistics for the RTRD are derived and are expressed as

$$f_{r:n}(x) = \frac{\frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}} \left(1 - \left(1 - e^{-\frac{x^2}{2\theta^2}}\right) \log \alpha\right)\right)}{B(r, n-r+1) \left(1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^{n+1}} \left(1 - e^{-\lambda x^\beta}\right)^{r-1} \left(\alpha \left(1 - \alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right) + e^{-\frac{x^2}{2\theta^2}}\right)^{n-r}.$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[\frac{1 - e^{-\frac{x^2}{2\theta^2}}}{1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}}} \right]^j \left[\frac{\alpha \left(1 - \alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right) + e^{-\frac{x^2}{2\theta^2}}}{1 + \alpha - \alpha^{1-e^{-\frac{x^2}{2\theta^2}}}} \right]^{n-j}$$

5. Estimation

This section covers the maximum likelihood estimation method for determining the unknown parameters, α and θ, of the RTRD.

5.1. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from RTRD with parameters $\alpha, \theta > 0$. Then, the logarithm of the likelihood function of RTRD is given by

$$l = \sum_{i=1}^{n} \log x_i - 2n\log\theta - \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n}\log\left(1+\alpha-\alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}\right) + \sum_{i=1}^{n}\log\left[1+\alpha-\alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}\left(1-\log\alpha\left(1-e^{-\frac{x_i^2}{2\theta^2}}\right)\right)\right] \tag{26}$$

The MLEs of α and θ are obtained by partially differentiating equation (26) with respect to the corresponding parameters and equating to zero. We have:

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n} \frac{1+\left(1-e^{-\frac{x_i^2}{2\theta^2}}\right)^2 \alpha^{-e^{-\frac{x_i^2}{2\theta^2}}}\log\alpha}{1+\alpha-\alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x_i^2}{2\theta^2}}\right)\log\alpha\right)} - 2\sum_{i=1}^{n}\frac{1-\left(1-e^{-\frac{x_i^2}{2\theta^2}}\right)\alpha^{-e^{-\frac{x_i^2}{2\theta^2}}}}{1+\alpha-\alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}} \tag{27}$$

$$\frac{\partial l}{\partial \theta} = \frac{1}{\theta^3}\sum_{i=1}^{n} x_i^2 - \frac{2n}{\theta} + \frac{\alpha\log\alpha}{\theta^3}\sum_{i=1}^{n} x_i^2 e^{-\frac{x_i^2}{2\theta^2}} \alpha^{-e^{-\frac{x_i^2}{2\theta^2}}}\left[\frac{\left(1-e^{-\frac{x_i^2}{2\theta^2}}\right)\log\alpha}{1+\alpha-\alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}\left(1-\left(1-e^{-\frac{x_i^2}{2\theta^2}}\right)\log\alpha\right)} - \frac{2}{1+\alpha-\alpha^{1-e^{-\frac{x_i^2}{2\theta^2}}}}\right] \tag{28}$$

The expressions in equations (27) and (28) do not possess a closed-form representation, posing a challenge for obtaining analytical solutions. Consequently, determining the parameter estimates for α and θ becomes intricate. Despite this complexity, numerical methods using R software can be employed to derive these estimates effectively.

6. Simulation Study

In this section, we carry out a simulation study using R software to examine the behavior of the MLEs for various sample sizes. We generate random samples of sizes 25, 75, 150, 300, and 500 from the RTRD and repeat the process 1000 times in R software. Various combinations of parameters are chosen as $(1.8, 2.2)$ and $(3.0, 3.5)$ in relation to the standard order (α, θ). The average MLE values, biases and related empirical mean squared errors (MSEs) were determined for each scenario. The results are presented in Tables 1 and 2. The estimates are stable and close to the true parameter values, as shown in Tables 1 and 2. Furthermore, in all scenarios, the MSE decreases as the sample size increases.

Table 1: *MLE, Bias, and MSE for the parameters α and θ*

Sample size	Parameters		MLE		Bias		MSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	1.8	2.2	2.42229	2.07894	0.62229	-0.12106	1.67886	0.08581
75			2.03232	2.14810	0.23232	-0.05189	0.38731	0.03327
150			1.90745	2.18024	0.10745	-0.01975	0.15921	0.01795
300			1.84770	2.19133	0.04770	-0.00866	0.08808	0.01115
500			1.83268	2.19420	0.03268	-0.00579	0.05256	0.00668

Table 2: *MLE, Bias, and MSE for the parameters α and θ*

Sample size	Parameters		MLE		Bias		MSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	3.0	3.5	3.87207	3.40580	0.87207	-0.09419	4.96014	0.21477
75			3.25054	3.48193	0.25054	-0.01806	1.01548	0.08117
150			3.11418	3.50042	0.11418	0.00042	0.48716	0.04540
300			3.03796	3.50412	0.03796	0.00412	0.21861	0.02087
500			3.01584	3.50210	0.01584	0.00210	0.12086	0.01244

7. Applications to Real Life Data

This section focuses on the application of the proposed model to real-life data sets. The potential of the proposed model is assessed by comparing its performance with several other models, namely Weighted Rayleigh Distribution (WRD) [5], Transmuted Rayleigh Distribution (TRD) [19], Exponentiated Rayleigh Distribution (ERD) [25] and Rayleigh Distribution (RD) [22]. Using two actual data sets, we demonstrate the utility of the RTRD in this section.

Data Set 1: The first data set pertains to the breaking stress of carbon fibers of 50 mm length (GPa). This data has been previously used by [10].

Data Set 2: The second data set represents the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at a gauge length of 20 mm. This data was originally reported by [8]. For illustrative purposes, we consider the same transformed data set as used by [16].

The results presented in Tables 5 and 6 reveal that the RTRD achieves the smallest values of AIC, BIC, and AICC compared to the other competing models. This demonstrates that the RTRD outperforms the base Rayleigh distribution as well as the mentioned competing models. Moreover, its strong performance across two engineering datasets underscores its practical utility and effectiveness in accurately modeling complex data patterns. The results are further supported by Figures 3 and 4.

Table 3: *MLEs of RTRD and competitive models with corresponding SE (given in parenthesis) for Data set 1*

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\eta}$
RTRD	6.9048	1.2180	-	-
	(2.2212)	(0.08174)		
WRD	-	1.3551	2.5727	-
		(0.1234)	(0.7452)	
TRD	-	1.6956	-	-0.9587
		(0.0824)		(0.0929)
ERD	2.3483	0.1919	-	-
	(0.4311)	(0.0245)		
RD	-	2.0491	-	-
		(0.1261)		

Table 4: *MLEs of RTRD and competitive models with corresponding SE (given in parenthesis) for Data set 2*

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\eta}$
RTRD	5.2366	0.6858	-	-
	(1.6048)	(0.0488)		
WRD	-	0.7457	2.2209	-
		(0.0667)	(0.6696)	
TRD	-	0.89478	-	-0.9610
		(0.0443)		(0.1193)
ERD	2.1746	0.6621	-	-
	(0.3875)	(0.0847)		
RD	-	1.0833	-	-
		(0.0652)		

Table 5: *Comparison of RTRD and competitive models for Data set 1*

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
RTRD	170.1694	174.1694	178.5487	174.3599	0.0635	0.9528
WRD	175.7107	179.7107	184.0900	179.9012	0.1104	0.3963
TRD	177.7488	181.7488	186.1282	181.9393	0.1410	0.1446
ERD	177.2735	181.2735	185.6528	181.4640	0.1205	0.2930
RD	196.4168	198.4168	200.6065	198.4793	0.2265	0.0022

Table 6: *Comparison of RTRD and competitive models for Data set 2*

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
RTRD	98.4043	102.4043	106.8725	102.5861	0.0599	0.9654
WRD	100.6399	104.6399	109.1081	104.8217	0.0664	0.9206
TRD	101.9050	105.9050	110.3732	106.0868	0.0887	0.6494
ERD	101.8098	105.8098	110.2780	105.9916	0.0752	0.8293
RD	118.8367	120.8367	123.0708	120.8964	0.1999	0.0080

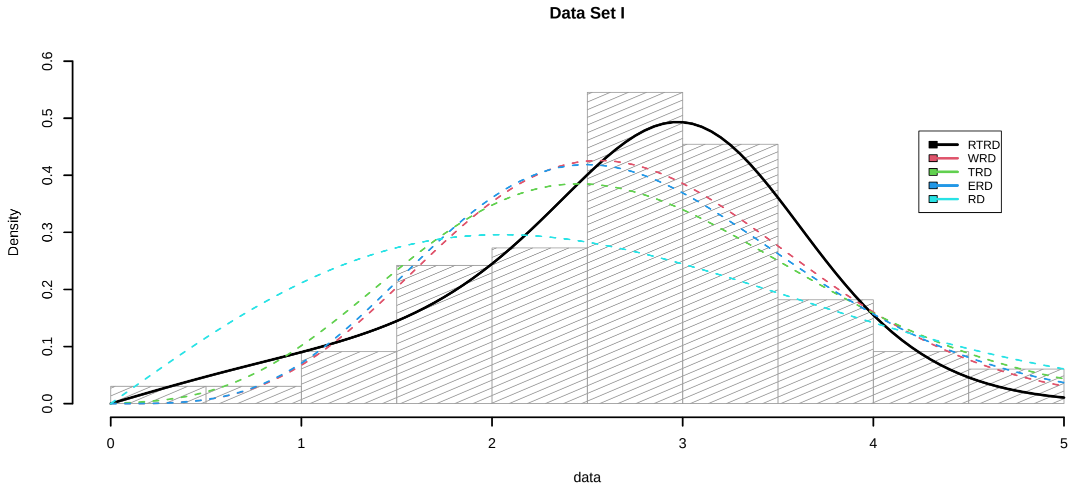

Figure 3: *Fitted density plots for data set 1*

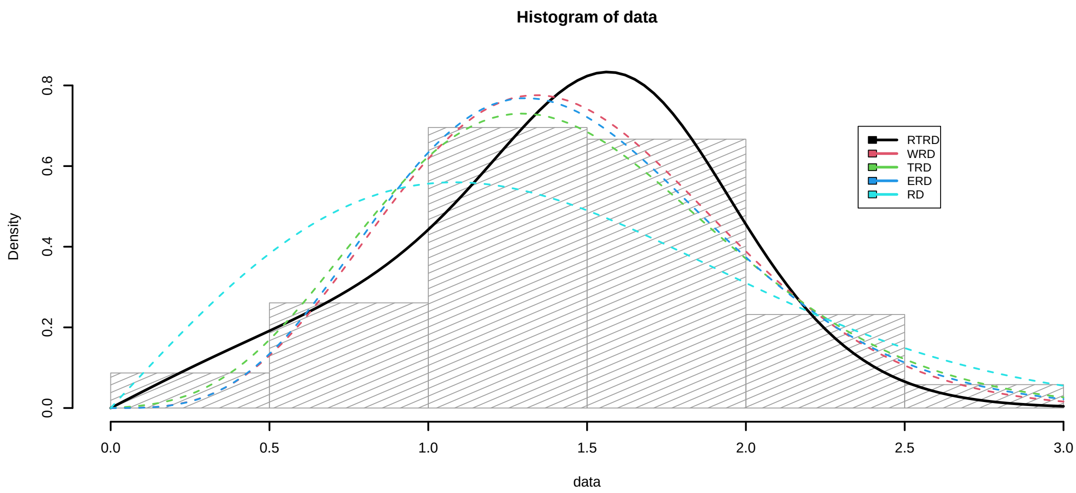

Figure 4: *Fitted density plots for data set 2*

8. Conclusion

In this manuscript, we introduce the Ratio Transformation Rayleigh Distribution (RTRD), a new model that extends the Rayleigh distribution for analyzing data with real support. The motivation behind this generalization is to enhance the flexibility of the standard distribution, thereby improving its ability to model real-world data. We derive key statistical properties of the proposed model and examine several reliability measures. The RTRD showcases greater flexibility, with its hazard rate function exhibiting a variety of complex shapes. A simulation study was conducted to assess the performance of the maximum likelihood estimate, demonstrating both its consistency and precision. Parameter estimation is performed using maximum likelihood estimation. Furthermore, we analyze two real datasets, showing that RTRD provides a superior fit compared to other competitive distributions.The RTRD model was applied in the engineering field to analyze material properties such as breaking stress and tensile strength, successfully capturing complex data patterns. We foresee the RTRD's broad applicability across statistics and other domains, with future research focused on extending the model to multidimensional frameworks.

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