ANALYSIS OF AN ENCOURAGED ARRIVAL MARKOVIAN QUEUE WITH SINGLE WORKING VACATION, IMPATIENCE AND RENEGING OF CUSTOMERS

V. Narmadha[1], P. Rajendran[2,*]

•

[1,*] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology,
Vellore, Tamil Nadu 632014, India.
[1]narmadha.v2020a@vitstudent.ac.in, [2,*]prajendran@vit.ac.in

Abstract

In this paper, we analyze a single server markovian queueing model with encouraged arrivals that undergoes a single working vacation. Additionally, we consider the impatience and reneging behavior of customers in the queue during the working vacation period. Customers arrive at the system following a Poisson distribution. The server goes on vacation when the system is empty and stays on vacation for a random period that follows an exponential distribution. During the working vacation period, the server continues to provide service at a slower rate. After the vacation, the server returns to the regular service period and continues providing service at the regular busy period rate if there are one or more customers in the system, or it remains idle until a new customer arrives. During the working vacation, customers in the queue become impatient and renege from the system, with the reneging time assumed to follow an exponential distribution. The system is characterised as a quasi-birth-death process, and the stationary probabilities are derived using the probability generating function method. Some numerical analysis is also carried out to show the effect of encouraged arrivals on performance measures.

Keywords: Encouraged arrivals, impatience, reneging, working vacation, probability generating function(PGF).

I. Introduction

Since the 1970s, numerous researchers have studied the mathematical modelling and implementation of queueing models that undergoes server vacations. Congestion issues in a variety of research domains could be readily represented by vacation queueing models that undergoes server vacations. Several studies have been conducted on queues with vacations in [1, 2]. A single server finite source markovian queueing model with server vacations, baling and reneging behaviour of customers are analysed in [3] using the solution of steadystate probabilities in the matrix form. For a variety of real-world scenarios, including computer networks, digital communication, and production/inventory systems benefits from the generalisation of queueing models [4, 5]. It is assumed that in these investigations, the service is completely terminated when on the server is on vacation. This kind of vacation denotes classical vacation model. A working vacation (wv) is when the server continues to offer service while on vacation, but at a reduced service rate. This type of wv is first introduced in [6], which also examined a markovian queue with several working vacation policies on a single server. In [7], the matrix-geometric approach is used to study an M/M/1 queue with numerous working vacations and derive precise formulas for the performance metrics. Using the same method , analysis of a single server queue with single working vacation(swv) is carried out in [8]. The investigation by [6] was expanded to an M/G/1/WV queue by [9, 10, 11]. In [12], the work of [6] is extended to a GI/M/1 queue with a general arrival process and several working vacations using the matrix-geometric solution method. The GI/M/1 queue with a swv was further examined by [11].

Clients are frequently seen waiting in line for assistance in today's busy environment. Clients experience impatience while the server is on vacation, At present, queueing system analysis with impatient customers is becoming steadily more popular. There are several related studies which are explained in [12], [13]. A comprehensive analysis of queues with vacation and client impatience for single and multiserver systems are given in [14]. customers are drawn to the business by the discounts and offers. In [15], such customers are known as Encouraged Arrivals(ea). The concept of customer movement explained in [16], which states that a system can draw in new customers by looking at its substantial client base. The variation in percentage of customers depends on ea brought about by sales and discount . A finite capacity ea queue with multiple servers and reverse reneging is carried out in [17].

In this paper we analyse an encouraged arrival single server queue with swv, impatience and the reneging behaviour of customers due to impatience during the working vacation session. The introduction of the paper is given in section 1. Section 2 comprises of the model description. The stationary analysis the of model with ea, swv, impatience and reneging of impatient customers are provided in section 3. Section 4 deals with the performance measures of the model. The numerical analysis is given in section 5. The conclusion is given in section 6.

II. Model description

We consider a single server markovian queueing model with ea, swv, impatience and reneging behaviour of impatient clients during working vacation session. The arrivals follow a poisson distribution with parameter $\lambda(1+\Omega)$, where " Ω " denotes the percentahe variation in the total count of clients estimated from observed data. For instance, if a firm previously offered discounts and a percentage change in the total count of clients was noticed of +10%, +30% or +50%, then $\Omega = 0.1$, 0.3 or 0.5, respectively. The server operation follows an exponential distribution with parameters μ and α during busy hours and working vacations respectively, where ($\alpha < \mu$). The server takes a swv when the system is empty, and the duration of this vacation is distributed exponentially with parameter ψ. If there are clients in the system at the end of the vacation, the server returns to its actual service rate. Otherwise, it will remain idle until a new client shows up. Clients who wait for his turn to get service, may become impatient and choose to leave the queue. The reneging behaviour of impatient clients follows an exponential distribution with parameter β.

III. Steady state analysis of the queue with encouraged arrivals, single working vacation, impatience and reneging of impatient clients during WV:

Let the number of clients in the system is given by N and the state of the system is given by S. Then the markov process is given as $\{(N,S), t\geq 0\}$. The state space is given by $\theta = \{(n,s), n = 0,1,2,\ldots.,$
$s = 0,1\}$ where s = 0 denoted the swv and s = 1 denotes the regular busy session.

The following are the differential-difference equations governing the quasi-birth-death process in the steady state:

$$(\lambda(1+\Omega)+\varphi)\mathrm{P}_{0,0}=\alpha P_{1,0}+\mu P_{1,1} \qquad (1)$$

$$(\lambda(1+\Omega)+\alpha+\varphi+(n-1)\beta)\mathrm{P}_{n,0}=\lambda P_{n-1,0}+(\alpha+n\beta)\mathrm{P}_{n+1,0}\,, n\geq 1 \qquad (2)$$

$$\lambda(1+\Omega)\mathrm{P}_{0,1}=\varphi\mathrm{P}_{0,0} \qquad (3)$$

$$(\lambda(1+\Omega)+\mu)\mathrm{P}_{n,1}=(\lambda(1+\Omega)\mathrm{P}_{n-1,1}+\mu P_{n+1,1}+\varphi\mathrm{P}_{n,0},, n\geq 1 \qquad (4)$$

The PGF are defined as follows:

$$G_0(y)=\sum_{n=0}^{\infty}y^n\,\mathrm{P}_{n,0}\,,\ G_1(y)=\sum_{n=0}^{\infty}y^n\,\mathrm{P}_{n,1} \text{ and } G_0'(y)=\sum_{n=0}^{\infty}y^{n-1}\,\mathrm{P}_{n,0}$$

for $0\leq y\leq 1$ (5)

Equations (1) and (2) are multiplied by 1 and y^n respectively. Summing them for all possible values of n, we get

$$\beta y(1-y)G_0'(y)+[\lambda(1+\Omega)y^2-(\lambda(1+\Omega)+\varphi+\alpha-\beta)y+(\alpha-\beta)]G_0(y)=(1-y)(\alpha-\beta)\mathrm{P}_{0,0}-\mu yP_{1,1} \qquad (6)$$

Similarly (3) and (4) are multiplied by 1 and y^n and are added over all possible values of n, we get

$$(1-y)(\,(\lambda(1+\Omega)y-\mu)G_1(y)=\varphi yG_0(y)-\mu(1-y)P_{0,1}-\mu yP_{1,1} \qquad (7)$$

Rewriting (6) for $y\neq 0$ and $y\neq 1$, we have

$$G_0'(y)-\left(\frac{\lambda(1+\Omega)}{\beta}+\frac{(\varphi+\alpha-\beta)}{\beta(1-y)}-\frac{\alpha-\beta}{\beta y(1-y)}\right)G_0(y)\quad=\frac{(\alpha-\beta)}{\beta y}\mathrm{P}_{0,0}-\frac{\mu}{\beta(1-y)}P_{1,1} \qquad (8)$$

Multiplying (8) with $e^{\frac{-\lambda(1+\Omega)y}{\beta}}(1-y)^{\frac{\varphi}{\beta}}y^{\frac{(\alpha-\beta)}{\beta}}$ on both the sides, we have

$$G_0(y)=\frac{e^{\frac{\lambda(1+\Omega)y}{\beta}}}{(1-y)^{\frac{\varphi}{\beta}}y^{\frac{(\alpha-\beta)}{\beta}}}\left[\frac{(\alpha-\beta)}{\beta y}F_1(\mathrm{y})\mathrm{P}_{0,0}-\frac{\mu}{\beta(1-y)}F_2(\mathrm{y})P_{1,1}\ \right] \qquad (9)$$

Where

$$F_1(\mathrm{y})=\int_0^y e^{\frac{-\lambda(1+\Omega)u}{\beta}}(1-u)^{\frac{\varphi}{\beta}}u^{\frac{(\alpha-\beta)}{\beta}-1}\mathrm{du}$$

$$F_2(\mathrm{y})=\int_0^y e^{\frac{-\lambda(1+\Omega)u}{\beta}}(1-u)^{\frac{\varphi}{\beta}-1}u^{\frac{(\alpha-\beta)}{\beta}}\mathrm{du}$$

Since $0\leq G_0(1)=\sum_{n=0}^{\infty}\mathrm{P}_{n,0}\leq 1$ and $\lim\limits_{y\to 0}(1-y)^{\frac{\varphi}{\beta}}=0$ it must be

$$\frac{(\alpha-\beta)}{\beta}F_1(1)\mathrm{P}_{0,0}-\frac{\mu}{\beta}F_2(1)P_{1,1}\ =0$$

Which in turn gives

$$P_{1,1}=\frac{(\alpha-\beta)}{\mu}\,\frac{F_1(1)}{F_2(1)}\,\mathrm{P}_{0,0} \qquad (10)$$

By solving (6) at y=1 and by using (10), we have

$$\varphi G_0(1)=\mu yP_{1,1}=\frac{(\alpha-\beta)F_1(1)}{F_2(1)}\,\mathrm{P}_{0,0} \qquad (11)$$

Using (10), equation (9) becomes

$$G_0(y)=\frac{(\alpha-\beta)e^{\frac{\lambda(1+\Omega)y}{\beta}}}{\beta(1-y)^{\frac{\varphi}{\beta}}y^{\frac{(\alpha-\beta)}{\beta}}}\left[F_1(\mathrm{y})-\frac{F_1(1)}{F_2(1)}\,F_2(\mathrm{y})\right]\mathrm{P}_{0,0} \qquad (12)$$

From (6), we obtain for $y\neq 0$ and $y\neq 1$

$$G_0'(y)=\frac{(1-y)(\alpha-\beta)\mathrm{P}_{0,0}-[\lambda(1+\Omega)y^2-(\lambda(1+\Omega)+\varphi+\alpha-\beta)y+(\alpha-\beta)]G_0(y)-\mu yP_{1,1}}{\beta y(1-y)} \qquad (13)$$

we get $G_0'(1)$ by applying L'hospital's rule on (13),

$$G_0'(1)=\frac{(\,\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi} \qquad (14)$$

From (7) we have for $y\neq 1$

$$G_1(y)=\frac{\varphi yG_0(y)-\mu(1-y)\mathrm{P}_{0,1}-\mu yP_{1,1}}{(1-y)(\lambda y-\mu)} \qquad (15)$$

We get $G_1(1)$ by applying L'hospital's rule on (15)

$$G_1(1)=\frac{\varphi G_0'(1)+\mu P_{0,1}}{\mu\lambda(1+\Omega)} \qquad (16)$$

From (3), we obtain

$$P_{0,1}=\frac{\varphi\mathrm{P}_{0,0}}{\lambda(1+\Omega)} \qquad (17)$$

Using normalization condition , we have

$$G_0(1) + G_1(1) = \sum_{n=0}^{\infty} \mathrm{P}_{\mathrm{n},0} + \sum_{n=0}^{\infty} \mathrm{P}_{\mathrm{n},1} = 1$$

Using equations (11), (14), (16) and (17), we obtain the following

$$\mathrm{P}_{0,0} = \left\{\frac{(\alpha-\beta)F_1(1)}{\varphi F_2(1)} + \frac{(\lambda(1+\Omega)-(\alpha-\beta))(\alpha-\beta)F_1(1)}{(\beta+\varphi)(\mu-\lambda(1+\Omega)F_2(1)} + \frac{\varphi(\alpha-\beta)}{(\beta+\varphi)(\mu-\lambda(1+\Omega))} + \frac{\mu\varphi}{\lambda(1+\Omega)}\right\}^{-1} \tag{18}$$

IV. Performance measures

- Expected number of clients in the system during swv is given by

$$E(N_{swv}) = G_0'(1) = \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi} \tag{19}$$

- Expected number of clients in the system during regular busy session is given by

$$E(N_{rb}) = G_1'(1) = \frac{\varphi G_0''(1)}{2(\mu-\lambda(1+\Omega))} + \frac{\mu\varphi G_0'(1)}{(\mu-\lambda(1+\Omega))^2} + \frac{\mu\varphi \mathrm{P}_{0,0}}{(\mu-\lambda(1+\Omega))^2} \tag{20}$$

Where

$$G_0''(1) = \frac{2(\lambda(1+\Omega)-\varphi-\alpha)G_0'(1)+2\lambda(1+\Omega)G_0(1)}{2\alpha+\varphi} \tag{21}$$

- The total expected number of clients in the system is given as

$$E(N) = E(N_{swv}) + E(N_{rb})$$

Therefore

$$E(N) = \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi} + \frac{(\lambda(1+\Omega)-(\alpha-\beta))G_0(1)+(\alpha-\beta)\mathrm{P}_{0,0}}{\beta+\varphi}$$

- The expected rate of reneging is given as follows

$$E(R) = \sum_{n=1}^{\infty} \beta(n-1)\mathrm{P}_{\mathrm{n},0} = \beta(G_0'(1) - G_0(1) + \mathrm{P}_{0,0})$$

V. Numerical analysis

The numerical analysis shows the impact of parameters on system's performance measures. We consider the following parameters for numerical computation λ=2, μ=5, α=3, ψ=3 and β=0.7

Table 1: *Evaluation of performance measures with respect to varying arrival rate*

Performance measures	λ=2	$\lambda(1+\Omega)$ $\Omega = 10\%$	$\lambda(1+\Omega)$ $\Omega = 20\%$	$\lambda(1+\Omega)$ $\Omega = 30\%$
$E(N_{swv})$	0.12503	0.13081	0.14353	0.15442
$E(N_{rb})$	0.61054	0.73431	0.86898	1.02835
$E(N)$	0.73548	0.86424	1.00343	1.17371
$E(R)$	0.02030	0.02557	0.02981	0.03321
$\mathrm{P}_{0,0}$	0.2366	0.2238	0.21657	0.2785
$\mathrm{P}_{0,1}$	0.3424	0.30402	0.2785	0.2468

From table 1. We observe that the performance measures increases with increase in arrival rate. In other words, as the number of clients joining the firm increases the probability of system in swv and the probability of firm being in regular busy session decreases.

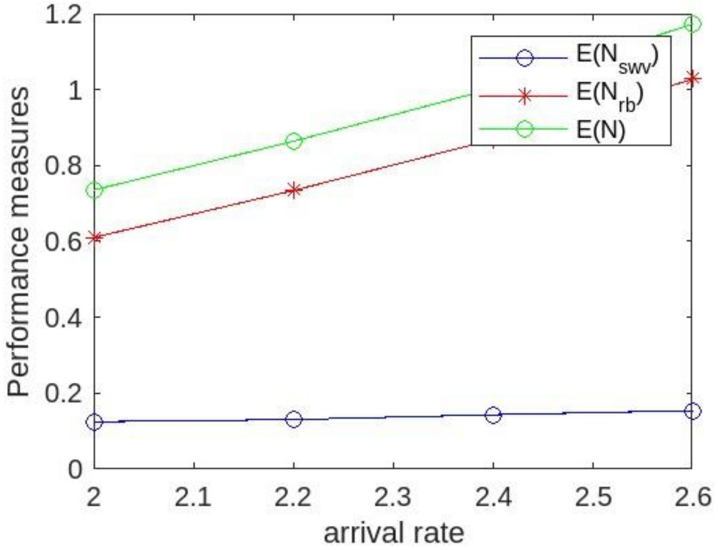

Figure 1. *Variation in performance measures with respect to arrival rate*

Table 2: *Evaluation of performance measures with respect to varying service rate during swv*

α	$E(N_{swv})$	$E(N_{rb})$	$E(N)$	$E(R)$	$P_{0,0}$	$P_{0,1}$
3	0.12503	0.61054	0.73548	0.02030	0.2366	0.3412
3.2	0.1125	0.6141	0.7184	0.0282	0.2383	0.3448
3.4	0.1180	0.60011	0.7182	0.0273	0.2314	0.3462
3.6	0.1148	0.6853	0.7012	0.01742	0.2323	0.3504
3.8	0.1037	0.6018	0.7837	0.0174	0.2346	0.3514
4	0.1003	0.5970	0.6974	0.01469	0.2358	0.3535

From table 2. We observe that the performance measures decreases with increase in service rate during swv.

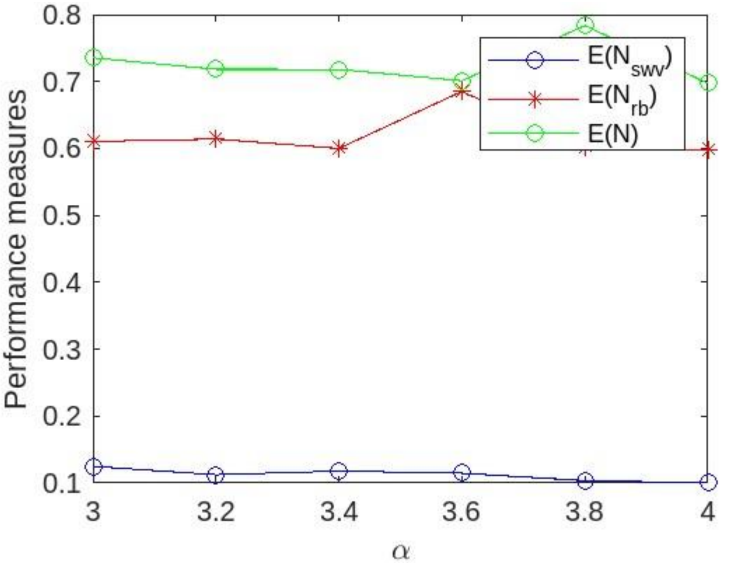

Figure 2. *Variation in performance measures with respect to α*

VI. Conclusion

In this paper, we consider a single server markovian queueing model with encouraged arrival, single working vacation, impatient clients and reneging of such impatient clients during working vacation period. We derived the performance measures using the probability generating function of the system's steady state probabilities. The numerical analysis shows the impact of encouraged arrivals on the performance measures. As the arrival rate increases, the performance measures increases which benefits the firm .

References

[1] Ke, J.C., Wu, C.H. and Zhang, Z.G. (2010). Recent Developments in Vacation Queueing Models: A Short Survey, *International Journal of Operations Research*, 7(4):3-8.

[2] Tian, N. and Zhang, G. (2006). Vacation queueing models: Theory applications, *Springer-Verlag*, New York.

[3] Yue, D., Zhang, Y. and Yue, W. (2006). Optimal performance analysis of an M/M/1/N queue system with balking, reneging and server vacation, *International Journal of Pure and Applied Mathematics*, 28:101-115.

[4] Doshi, B.T. (1986). Queueing Systems with Vacations, a Survey. *Queueing Systems*, 1:29-66

[5] Takagi, H. (1991). Queueing Analysis: A Foundation of Performance Analysis, *Vacation and Priority Systems*, 1(1).

[6] Servi, L.D. and Finn, S.G. (2002). M/M/1 Queue with Working Vacations (M/M/1/WV), *Performance Evaluation*, 50:41-52.

[7] Liu W. Y., Xu, X. L., and Tian, N. S. (2007). Stochastic decompositions in the M/M/1 queue with working vacations, *Operations Research Letters*, 35(5):595–600.

[8] Tian, N., Zhao, X. and Wang, K. (2008). The M/M/1 queue with single working vacation, *International Journal of Information and Management Sciences*, 4:621-634.

[9] Kim, J., Choi, D. and Chae, K. (2003). Analysis of queue-length distribution of the M/G/1 queue with working vacations, *International Conference on Statistics and Related Fields*, Hawaii.

[10] Wu, D. and Takagi, H. (2006). M/G/1 Queue with Multiple Working Vacation. *Performance Evaluation*, 63:654-681.

[11] Li, J., Tian, N., Zhang, Z.G. and Luh, H.P. (2011). Analysis of the M/G/1 Queue with exponentially working vacations-a matrix analytic approach. *Queueing Systems*, 61:139–166.

[12] Baba, Y. (2005). Analysis of a GI/M/1 queue with multiple working vacations, *Operation Research Letters*, 33:201–209.

[13] Gans, N., Koole, G. and Mandelbaum, A. (2003). Telephone call centers: Tutotial, review, research prospects, *Manufacturing and Service Operations Management*. 5: 79-141.

[14] Benjaafar, S., Gayon, J. and Tepe, S. (2010). Optimal control of a production-inventory system with customer impatience, *Operations Research Letters*. 38: 267-272.

[15] Altman, E. and Yechiali, U. (2006). Analysis of customer's impatience in queues with server vacations, *Queueing Systems*. 52:261-279.

[16] Som, B. K. and Seth, S. (2017). An M/M/1/N Queuing system with Encouraged Arrivals, *Global Journal of Pure and Applied Mathematics*, 17:3443-3453.

[17] Som, B.K. (2020). Multi-server Finite Waiting-space Encouraged Arrival Queuing System with Reverse Reneging, *Jagannath University Research Journal*, 1(1):2582-6263.

[18] Jain, N.K., Kumar, R. and Som, B.K. (2014). An M/M/1/N Queuing system with reverse balking, *American Journal of Operational Research*, 4(2):17-20.

[19] Donald Gross, John F. Shortle, James M. Thompson: Fundamentals of Queueing Theory(fifth edition),Wiley Series in Probability and Statistics, (2018).

[20] Medhi J: Stochastic models in Queueing theory(second edition) , (2003).

[21] Veerarajan T: Probability, statistics and random processes with Queueing Theory and Queueing Networks, (2009)

[22] Hamdy A.Taha: Operations Research An Introduction(eighth edition), (2007).