BAYESIAN PARAMETER ESTIMATION FOR TRANSMUTED WEIBULL DISTRIBUTION WITH CENSORING RATES AND VARIOUS LOSS FUNCTIONS

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Abstract

Statistical distributions are essential tools for describing and predicting real-world phenomena, though recent advancements in data collection have made it challenging to fit existing probability models to many practical datasets. While non-parametric models are sometimes recommended, parametric models retain substantial popularity due to their interpretability and flexibility. The quadratic rank transmutation map (QRTM) technique has been used to create new families of non-Gaussian distributions, known as transmuted distributions, which allow for modifications in moments, skewness, and kurtosis, thus increasing flexibility. The transmuted Weibull distribution (TWD) has gained attention for applications in reliability, survival analysis, and lifetime data analysis. This article focuses on a Bayesian analysis of the transmuted Weibull distribution, a generalization of the traditional Weibull model that addresses its limitations, particularly for datasets exhibiting non-monotonic failure rates. Bayesian parameter estimation is performed using a Markov Chain Monte Carlo (MCMC) algorithm, with both non-informative and informative priors. We calculate Bayes estimators (BEs) and posterior risks (PRs) under different loss functions, including the Absolute Error Loss Function (AELF), precautionary loss function (PLF), and quadratic loss function (QLF). Simulation studies evaluate the Bayes estimators' performance, investigating the effects of various priors, sample sizes, and censoring rates on estimation accuracy and credible interval width. Real-world data applications highlight the practical utility of the Bayesian approach for the TWD, showing consistent results with increasing sample sizes and underscoring the robustness of the MCMC algorithm for parameter estimation. The article is structured as follows: the TWD's parameters, including scale, shape, and transmutation, are estimated under different loss functions and priors. Bayesian credible intervals (BCIs) are also computed. Both uncensored and censored data environments are considered, with varying sample sizes and censoring rates. Posterior risks for each estimator are analyzed to assess performance, and two real datasets are used to illustrate the flexibility and applicability of the proposed distribution. This study lays a foundation for future research, such as exploring mixtures of transmuted Weibull distributions or conducting Bayesian analyses for record values.

Keywords: Transmuted Weibull distribution, Markov Chain Monte Carlo, Bayesian credible intervals, Bayes estimators, posterior risks, absolute error loss function, precautionary loss function, quadratic loss function, censoring.

I. Introduction

Statistical distributions are essential for describing and predicting real-world phenomena. However, advancements in data collection methods have led to challenges in fitting existing probability models to many significant and practical datasets [1]. In such cases, while non-parametric models may be recommended, parametric models continue to enjoy substantial popularity. The quadratic rank transmutation map (QRTM) technique has been employed to create new families of non-Gaussian distributions [2]. This technique modifies the moments, skewness, and kurtosis of a baseline distribution, resulting in what is known as a transmuted distribution. This family of distributions has attracted considerable attention from researchers, leading to the development and exploration of various new flexible distributions over the past decade. For instance, Al Sobhi [3] introduced the transmuted modified Weibull distribution, while others presented the exponentiated transmuted Weibull distribution. More recently, a new lifetime distribution called the transmuted cubic Weibull distribution was constructed, and a novel weighted distribution known as the size-biased weighted transmuted Weibull distribution was introduced. The method of least squares and the method of moments have been utilized to estimate parameters for the transmuted Weibull distribution, with comparisons made through simulation studies using statistical measures like mean squared error (MSE) [4]. Researchers have also explored various structural properties of the transmuted Weibull distribution, including its mean, harmonic mean, standard deviation, moment generating function (MGF), skewness, and kurtosis [5][6]. Currently, transmuted distributions find applications in numerous fields, including reliability studies, lifetime analysis, engineering, economics, insurance, and environmental sciences.

The Weibull distribution is a widely recognized lifetime probability distribution, commonly used in various domains of reliability and survival analysis [7]. Its attractiveness largely arises from the variety of shapes it can assume by adjusting its shape parameter. Additionally, the Weibull distribution is versatile, encompassing many other distributions, such as the exponential and Rayleigh distributions, as special cases. Despite its widespread use and applicability, the traditional Weibull distribution does not fully capture the range of lifetime phenomena. For instance, it is not suitable for datasets exhibiting a non-monotonic failure rate, prompting investigations into generalizations of the Weibull distribution. A notable generalization applicable to survival data analysis is the transmuted generalized inverse Weibull distribution, which discusses its mathematical properties and employs maximum likelihood methods for parameter estimation. Similarly, the transmuted generalized Weibull distribution has been developed, exploring its mathematical properties, including the quantile function, moments, entropies, mean deviation, Bonferroni and Lorenz curves, and the moments of order statistics, also using maximum likelihood for parameter estimation. Furthermore, the generalized transmuted Weibull distribution has been proposed, with its properties derived. This article focuses on the Bayesian analysis of the transmuted Weibull distribution, which serves as a generalization of the Weibull probability distribution. We emphasize Bayesian analysis because maximum likelihood and moment estimators have been used for parameter estimation of the transmuted Weibull distribution [8]. To facilitate this analysis, we employ a Markov Chain Monte Carlo (MCMC) algorithm to compute posterior summaries for the unknown parameters of the distribution, comparing the results across different priors, loss functions, sample sizes, and parameter sets [9].

The objective of this paper is to define the Transmuted Weibull distribution and introduce its likelihood function, followed by the derivation of posterior distribution expressions utilizing both non-informative and informative priors, as well as marginal posterior densities for both censored and uncensored data. The study aims to explore Bayesian estimators (BEs) and their associated posterior risks (PRs) under various loss functions. Additionally, the paper seeks to detail the estimation of unknown parameters of the proposed distribution through the MCMC algorithm for posterior summaries, encompassing different loss functions and prior types. The work will also provide a mathematical and numerical discussion of Bayesian credible intervals (BCIs) and conclude with an analysis of a real-life dataset.

II. Transmuted Weibull Distribution

As introduced by Alzaatreh et al. in 2013 [10], a random variable X follows a transmuted probability distribution if its probability density function (pdf) and cumulative distribution function (CDF) are given by:

$$f(x) = g(x)\big(1+\gamma-\gamma.2G(x)\big)$$
$$F(x) = G(x)[(1+\gamma)-\gamma.G(x)]$$

where $x > 0$ and the transmutation parameter γ satisfies $|\gamma| \leq 1$. Here, $G(x)$ is the CDF of the baseline distribution, and the functions $f(x)$ and $F(x)$ represents the pdf and CDF of the transmuted distribution, respectively.

A random variable X is defined to follow a Weibull distribution characterized by parameters $\alpha > 0$ and $\beta > 0$ if its probability density function (PDF) is given by:

$$g(x;\alpha,\beta) = \frac{\alpha}{\beta}(\boldsymbol{x})^{\alpha-1} exp\left(-\frac{x^{\alpha}}{\beta}\right), x \geq 0$$

The corresponding cumulative distribution function (CDF) for this Weibull distribution can be expressed as:

$$G(x) = 1 - exp\left(-\frac{x^{\alpha}}{\beta}\right)$$

To find the CDF of the transmuted Weibull distribution, we substitute $G(x)$ into the following formula:

$$F(x) = (1+\gamma)\left(1 - exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right) - \gamma\left(1 - exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right)^2$$

Through algebraic manipulation, we derive the CDF for the transmuted Weibull distribution as:

$$F(x;\theta) = exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\left(1-\gamma+\gamma\, exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right)$$

where $\theta = (\alpha,\beta,\gamma)$. To determine the PDF of the transmuted Weibull distribution, we differentiate this CDF with respect to x and simplify the result. The resulting PDF is then expressed as:

$$f(x;\theta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right).\left(1-\gamma+2\gamma \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)\right),$$
$$x \geq 0, \alpha, \beta > 0 \; and \; |\gamma| \leq 1$$

Special Cases

- When $\gamma = 0$, the Transmuted Weibull distribution simplifies to the standard Weibull distribution.
- If $\alpha = 1$, the result is the transmuted exponential distribution. Furthermore, with $\gamma = 0$, it becomes the standard exponential distribution.
- Setting both α and β to 1 yields the transmuted standard exponential distribution.
- When $\alpha = 2$, we obtain the transmuted Rayleigh distribution.
- If $\gamma = 0$, this corresponds to the traditional Rayleigh distribution.

III. Likelihood Functions for various Sampling Schemes

Consider a complete random sample $X_1, X_2, \dots X_n$ of size n taken from the transmuted Weibull distribution. Then, the likelihood function for the complete data set is given by:

$$L(x;\theta) = \alpha^n exp\left\{(\alpha-1)\sum_{i=1}^{n} log x_i\right\}\frac{1}{\beta^n} exp\left\{-\sum_{i=1}^{n}\frac{x_i^\alpha}{\beta}\right\}\prod_{i=1}^{n}\left[1-\gamma+2\gamma\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right]$$

where $\theta = (\alpha, \beta, \gamma)$ and $x = x_1, x_2, \dots x_n$. In many life-testing experiments, it's not possible to collect complete failure time data due to time and cost constraints. As a result, censoring plays an essential role in lifetime data analysis. Let$X = X_1, X_2, \dots X_r$ be a type-I censored sample of size r from n items, where the lifetimes follow a transmuted Weibull distribution with parameters α, β & γ. Consider a distribution with specific parameters. In the context of Type I censoring, it's important to note that the censoring time is predetermined, while the number of observed failures is random. Suppose we have n items under life testing, and we observe r failures at times $t_1, t_2, \dots t_r$. Here, r is an integer between 0 and n, and $(n-r)$ represents the number of items that survive or remain uncensored. According to Mendenhall and Hader (1958) [9], the likelihood function for censored data is given by:

$$L(x;\theta) \propto \prod_{j=1}^{r} f(x_j)\,.\,[1-F(T)]^{n-r}$$

where T represents the time, r denotes the number of failures (or uncensored observations), and $(n-r)$ are the censored observations. For a transmuted Weibull distribution applied to censored data, the likelihood function can be expressed as:

$$(x;\theta) \propto \alpha^T exp\left(-\alpha\sum_{j=1}^{r} log\frac{1}{x_j}\right)\frac{1}{\beta^r} exp\left(-\frac{\sum_{j=1}^{r} x_j^\alpha}{\beta}\right)\times exp\left[\sum_{j=1}^{r} log\left\{1-\gamma+2\gamma\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\}\right]$$
$$exp\left((n-r)log\left[1-exp\left(-\frac{T^\alpha}{\beta}\right)\times\left\{1-\gamma+\gamma exp\left(-\frac{T^\alpha}{\beta}\right)\right\}\right]\right)$$

Next, we examine the posterior distribution using Bayes' theorem. The posterior distribution, g(α|x) is given by:

$$g(\alpha \mid x) = \frac{L(x;\alpha)\Pi(\alpha)}{\int_\alpha^\infty L(x;\alpha)\Pi(\alpha)d\alpha}$$

where $\pi(\alpha)$ denotes the joint prior distribution of the parameters $\alpha = \alpha_1, \alpha_2, \dots \alpha_k, L(x;\alpha)$

represents the likelihood function, and $g(\alpha \mid x)$ is the joint posterior distribution.

IV. Posterior Distribution using Uniform Prior (UP)

In Bayesian estimation, to determine unknown parameters, we specify a prior for each parameter that isn't explicitly defined by the model itself. Unlike the frequentist approach, the Bayesian method incorporates both prior knowledge about the parameters and the observed data. When prior information about the parameters is lacking, a non-informative prior can be used in Bayesian analysis. This type of prior conveys minimal information about the parameters, reflecting a lack of strong prior beliefs.

As introduced by Yousaf et al. in 2020 [11], to estimate the unknown parameters of the transmuted Weibull distribution, we assume the following prior distributions: $\alpha \propto 1, \forall\alpha \in (0,\infty), \beta \propto 1, \forall\beta \in (0,\infty)$ and $\gamma \propto 1, \forall\lambda \in (-1,1)$. With the assumption that these parameters are independent, the joint prior distribution for $\alpha, \beta\ and\ \lambda$ is: $\pi_1(\alpha,\beta,\gamma) \propto 1$, where $\alpha,\beta > 0$ and $|\gamma| \leq 1$. Using Bayes' theorem, the joint posterior distribution of parameters $\alpha,\beta\ and\ \gamma$, given data x, with a uniform prior is: $g(\theta|x) = \frac{L(x;\theta)\pi(\theta)}{\int_0^\infty\int_0^\infty\int_{-1}^1 L(x;\theta)\pi(\theta)d\gamma d\beta d\alpha}$ where $L(x;\theta)$ is the likelihood function and $\pi(\theta)$ represents the uniform prior over the parameters α,β,γ.

$$g(\theta|x) = \frac{\alpha^{A_{01}-1}\exp(-\alpha A_{11})\frac{1}{\beta^n}\exp\left(-\frac{A_{21}}{\beta}\right)\exp(A_{31})}{\int_0^\infty\int_0^\infty\int_{-1}^1 \alpha^{A_{01}-1}\exp(-\alpha A_{11})\frac{1}{\beta^n}\exp\left(-\frac{A_{21}}{\beta}\right)\exp(A_{31})d\gamma d\beta d\alpha}$$

where $A_{01} = 1+n, A_{11}=\sum_{i=1}^n log\frac{1}{x_i}$, $A_{21}=\sum_{i=1}^n x_i^\alpha$ and $A_{31}=\sum_{i=1}^n log\left\{1-\boldsymbol{\gamma}+2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\}$. Likewise, for censored data, the posterior distribution is given by:

$$g(\theta|x) = \frac{\alpha^{B_{01}-1}\exp(-\alpha B_{11})\frac{1}{\beta^r}\exp\left(-\frac{B_{21}}{\beta}\right)\exp(B_{31})}{\int_0^\infty\int_0^\infty\int_{-1}^1 \alpha^{B_{01}-1}\exp(-\alpha B_{11})\frac{1}{\beta^r}\exp\left(-\frac{B_{21}}{\beta}\right)\exp(B_{31})d\gamma d\beta d\alpha} \tag{1}$$

where $B_{01} = 1+r, B_{11}=\sum_{j=1}^r log\frac{1}{x_j}$, $B_{21}=\sum_{j=1}^r x_j^\alpha$ and $B_{31}=\sum_{j=1}^r log\left\{1-\boldsymbol{\gamma}+2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\} + (n-r)log\left[1-exp\left(-\frac{T^\alpha}{\beta}\right)\times\left\{1-\gamma+\gamma exp\left(-\frac{T^\alpha}{\beta}\right)\right\}\right]$.

Since the posterior distributions for both censored and uncensored data are not available in closed form, the marginal posterior densities of the parameters $\alpha,\beta\ and\ \boldsymbol{\gamma}$ for both censored and uncensored data are obtained by integrating out the nuisance parameters, i.e., $g(\alpha|x) = \int_\beta\int_\lambda g(\alpha,\beta,\boldsymbol{\gamma}|x)\,d\beta d\boldsymbol{\gamma}$ and vice versa. Therefore, we use the MCMC technique to obtain the posterior summaries.

V. Posterior Distribution using Informative Prior (IP)

An informative prior offers specific, well-defined information about parameters through a probability distribution. In this study, we assume that the prior distributions of $\alpha,\beta\ and\ \boldsymbol{\gamma}$ are independent. Specifically, we assume $Gamma(a,b)$ for α, $InvGamma(c,d)$ for β and $Uniform(l_1,l_2)$ for $\boldsymbol{\gamma}$. The joint prior distribution of parameters $\alpha,\beta\ and\ \boldsymbol{\gamma}$ is:

$$g(\theta) \propto \alpha^{a-1}e^{-b\alpha}\frac{1}{\beta^{c+1}}e^{-\frac{d}{\beta}}\frac{1}{l_2-l_1}$$

The joint posterior distribution of the parameters α, β & γ and assuming an informative prior (IP) for the complete data, is:

$$g(\theta|x) = \frac{\alpha^{C_{11}-1}\exp(-\alpha D_{11})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{21}}{\beta}\right)\exp(D_{31})}{\int_0^\infty\int_0^\infty\int_{-1}^{1}\alpha^{C_{11}-1}\exp(-\alpha D_{11})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{21}}{\beta}\right)\exp(D_{31})d\gamma d\alpha d\beta}\ \alpha,\beta > 0 \text{ and } |\gamma| \le 1$$

Where $C_{11} = a + n, D_{11} = b + log\frac{1}{x_i},\ C_{21} = n + c, D_{21} = d + \sum x_i^\alpha\ log\frac{1}{x_i}$ and D_{31}=$\sum_{i=1}^{n} log\left\{1 - \boldsymbol{\gamma} + 2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\}$. For censored data, the joint posterior distribution of $\alpha, \beta\ and\ \boldsymbol{\gamma}$ given the data, is:

$$g(\theta|x) = \frac{\alpha^{C_{11}-1}\exp(-\alpha D_{12})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{22}}{\beta}\right)\exp(D_{32})}{\int_0^\infty\int_0^\infty\int_{-1}^{1}\alpha^{C_{11}-1}\exp(-\alpha D_{12})\frac{1}{\beta^{C_{21}-1}}\exp\left(-\frac{D_{22}}{\beta}\right)\exp(D_{32})d\gamma d\alpha d\beta}\ \alpha,\beta > 0 \text{ and } |\gamma| \le 1 \quad (2)$$

Where $C_{12} = a + r, D_{12} = b + \sum log\frac{1}{x_j},\ \ C_{22} = c + r, D_{22} = d + \sum x_j^\alpha$ and D_{32}=$\sum_{j=1}^{n} log\left\{1 - \boldsymbol{\gamma} + 2\boldsymbol{\gamma}\exp\left(-\frac{x_i^\alpha}{\beta}\right)\right\} + (n-r)log\left[1 - exp\left(-\frac{T^\alpha}{\beta}\right)\left\{1 - \boldsymbol{\gamma} + \boldsymbol{\gamma}exp\left(-\frac{T^\alpha}{\beta}\right)\right\}\right]$. The marginal posterior densities of the parameters $\alpha, \beta\ and\ \boldsymbol{\gamma}$ for both uncensored and censored data are obtained by integrating out the nuisance parameters, that is, $g(\alpha|x) = \int_\beta\int_{\boldsymbol{\gamma}} g(\alpha, \beta, \boldsymbol{\gamma}|x)\ d\beta d\boldsymbol{\gamma}$ and vice versa.

VI. Bayes Estimators and Posterior Risks for different Loss Functions

To estimate an unknown parameter in Bayesian analysis, a loss function must be specified. The choice depends on the specific problem, though there are no strict rules for selecting one. Loss functions can be symmetric (equal weighting to over- and underestimation) or asymmetric. For a decision d, a loss function $L(\beta, d) \ge 0$ represents the incurred loss when estimating unknown parameters $\alpha, \beta\ and\ \boldsymbol{\gamma}$, and by decisions $d_1, d_2\ and\ d_2$. The expected loss, or posterior risk, denoted by $R(\hat{d})$, is given by:

$$R(\hat{d}) = E_{\theta|x}\{L(\beta, \hat{d})\} = \int L(\beta, \hat{d})p(\beta|x)\ d\beta$$

Bayes estimators and their respective posterior risks are computed under the Absolute Error Loss Function (AELF), precautionary loss function (PLF), and quadratic loss function (QLF). Table 1 presents the expressions of Bayes estimators under various loss functions, along with their corresponding posterior risks.

Table 1: *BEs and PRs for Various Loss Functions*

Loss Function	AELF	PLF	QLF
Expression	$L(\beta, d) = (\beta - d)^2$	$L(\beta, d) = \frac{(\beta - d)^2}{d}$	$L(\beta, d) = \frac{(\beta - d)^2}{\beta^2}$
BEs	$\hat{d} = E_{\beta\|x}(\beta)$	$\hat{d} = \sqrt{E(\beta^2\|x)}$	$\hat{d} = \frac{E(\beta^{-1}\|x)}{E(\beta^{-2}\|x)}$
PRs	$R(\hat{d})$ $= E(\beta^2\|x) - \{E(\beta\|x)\}^2$	$R(\hat{d}) = 2\left[\sqrt{E(\beta^2\|x)} - E(\beta\|x)\right]$	$R(\hat{d}) = \frac{\{E(\beta^{-1}\|x)\}^2}{E(\beta^{-2}\|x)}$

VII. Posterior Estimates using Markov Chain Monte Carlo

From Equation (2), we observe that the posterior density is in an intractable form, requiring a numerical technique to solve it. Since the posterior summaries are challenging to obtain directly, a Markov Chain Monte Carlo (MCMC) technique is applied, as demonstrated by Carrera B and Papaioannou I, 2024 [12]. To implement MCMC, the posterior densities using both uniform and informative priors are expressed as:

$$gUP(\theta|x) \propto f_\alpha\left(n+1, \sum_{i=1}^{n} log\frac{1}{x_i}\right) f_{\beta|\alpha}\left(n + 1, \sum_{i=1}^{n} x_i^\alpha\right) f_\gamma\left(exp\left(\sum_{i=1}^{n} log\left\{1-\gamma+2\gamma\exp\left(\frac{x_i^\alpha}{\beta}\right)\right\}\right)\right)$$

$$gIP(\theta|x) \propto f_\alpha\left(n+a, \sum_{i=1}^{n} log\frac{1}{x_i}\right) f_{\beta|\alpha}\left(n+b, \sum_{i=1}^{n} x_i^\alpha\right) f_\gamma\left(e + exp\left(\sum_{i=1}^{n} log\left\{1-\gamma+2\gamma\exp\left(\frac{x_i^\alpha}{\beta}\right)\right\}\right)\right)$$

Here, and f_α and $f_{\beta|\alpha}$ represent the probability density functions of the gamma and inverse gamma distributions, respectively, f_γ while denotes the probability density function of the transmuted parameter. To obtain Bayes estimates, corresponding posterior risks, and Bayesian Credible Intervals (BCI), we proceed as follows: First, a random sample is generated from the transmuted Weibull distribution using the inverse integral transformation, i.e., $u_i = \left(1 - e^{-\frac{x_i{}^\alpha}{\beta}}\right)\left(1 - \boldsymbol{\gamma} + \boldsymbol{\gamma} e^{-\frac{x_i{}^\alpha}{\beta}}\right)$. After simplification, we obtain:

$$x_i = \left[-\beta ln\left\{1 - \left(\frac{1+\gamma-\sqrt{(1+\gamma)^2 - 4u_i\gamma}}{2\gamma}\right)\right\}\right]^{\frac{1}{\alpha}}$$

where $u_i \sim U(0,1)$ and $i = 1,2,\dots,n$. By specifying parameter values, a desired random sample can be generated. To produce censored data, a censoring time T is set, and units with values less than or equal to T are recorded. Units with values greater than T are considered censored observations. To implement the MCMC for obtaining posterior summaries, we proceed with the Gibbs sampling steps combined with a Metropolis-Hastings step, as stated by Faucett et al. [13].

VIII. Implementation using Real Life data

The dataset contains remission times for 116 patients diagnosed with acute leukemia. The remission durations (in months) are as follows: 1.08, 0.09, 1.48, 3.87, 13.94, 8.66, 6.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 9.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 3.81, 0.62, 2.82, 5.32, 7.32, 14.06, 10.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.78, 5.34, 7.59, 0.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 43.01, 1.19, 2.75, 4.26, 5.41, 7.13, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 16.62, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.45, 3.02, 4.34, 5.71, 11.93, 7.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 2.02, 12.02, 3.31, 4.51, 6.54, 8.53 and 22.69.

To estimate the unknown parameters, we applied the methodology from previous sections, utilizing different loss functions and prior distributions. A chi-square test was conducted to verify if the data follow the transmuted Weibull distribution, yielding a p-value of 0.226, indicating a good fit at the 5% significance level [14]. The Bayesian estimates (BEs), posterior risks (PRs) and Bayesian credible intervals for the parameters α, β and $\boldsymbol{\gamma}$ of the transmuted Weibull distribution were calculated using uninformative (UP) and informative (IP) priors under AELF, PLF, and QLF loss functions [15]. These results are presented in Tables 2 and 3.

Table 2: *BEs and PRs of TWD with hyper parameters a=0.6, b=1.2, c=1.2 & d=1.5*

Data condition	Loss function	Size n	UP	IP
Complete	AELF	$\alpha = 2.1$	1.653 (0.0021)	1.584 (0.0023)
		$\beta = 1.3$	1.209 (0.0045)	1.278 (0.0047)
		$\gamma = 0.7$	0.715 (0.0298)	0.703 (0.0312)
	PLF	$\alpha = 2.1$	1.664 (0.0052)	1.589 (0.0055)
		$\beta = 1.3$	1.229 (0.0038)	1.298 (0.0039)
		$\gamma = 0.7$	0.733 (0.0755)	0.719 (0.0786)
	QLF	$\alpha = 2.1$	1.645 (0.0076)	1.593 (0.0079)
		$\beta = 1.3$	1.215 (0.0029)	1.276 (0.0031)
		$\gamma = 0.7$	0.710 (0.0662)	0.695 (0.0693)
20% Censoring	AELF	$\alpha = 2.1$	1.719 (0.0061)	1.623 (0.0064)
		$\beta = 1.3$	1.107 (0.0028)	1.216 (0.0030)
		$\gamma = 0.7$	0.641 (0.0516)	0.624 (0.0549)
	PLF	$\alpha = 2.1$	1.732 (0.0075)	1.629 (0.0078)
		$\beta = 1.3$	1.139 (0.0042)	1.226 (0.0045)
		$\gamma = 0.7$	0.629 (0.0843)	0.615 (0.0884)
	QLF	$\alpha = 2.1$	1.708 (0.0091)	1.621 (0.0094)
		$\beta = 1.3$	1.105 (0.0026)	1.222 (0.0028)
		$\gamma = 0.7$	0.598 (0.0694)	0.581 (0.0727)
40% Censoring	AELF	$\alpha = 2.1$	1.911 (0.0109)	1.823 (0.0113)
		$\beta = 1.3$	1.095 (0.0082)	1.200 (0.0085)
		$\gamma = 0.7$	0.594 (0.0581)	0.573 (0.0618)
	PLF	$\alpha = 2.1$	1.928 (0.0121)	1.837 (0.0125)
		$\beta = 1.3$	1.098 (0.0065)	1.215 (0.0068)
		$\gamma = 0.7$	0.621 (0.0821)	0.605 (0.0863)
	QLF	$\alpha = 2.1$	1.898 (0.0144)	1.810 (0.0149)
		$\beta = 1.3$	1.082 (0.0038)	1.204 (0.0042)
		$\gamma = 0.7$	0.576(0.0741)	0.556(0.0782)

From Table 2, it is evident that the BEs under both UP and IP have lower posterior risks for uncensored data compared to censored data, due to the information loss associated with censoring. Additionally, the credible intervals for uncensored data were narrower than those for censored data.

Table 3: *95% Bayesian Credible Intervals of TWD using UP and IP with Hyperparameters parameters a=0.6, b=1.2, c=1.2 & d=0.5*

Data	Parameters	UP Lower Limit	UP Upper Limit	IP Lower Limit	IP Upper Limit
Complete	$\alpha = 2.1$	0.6234	2.9483	0.6257	2.9461
	$\beta = 1.3$	0.1402	1.3871	0.1415	1.3381
	$\gamma = 0.7$	0.6543	1.1102	0.6401	1.1655

20% Censoring	$\alpha = 2.1$	0.7483	3.1879	0.7491	3.1904
	$\beta = 1.3$	0.1521	1.4632	0.1578	1.3463
	$\gamma = 0.7$	0.6798	1.1387	0.5964	1.2476
40% Censoring	$\alpha = 2.1$	0.9145	3.5529	0.9170	3.5224
	$\beta = 1.3$	0.2283	1.5874	0.2305	1.3812
	$\gamma = 0.7$	0.7205	1.1618	0.5960	1.2393

The table 3 presents 95% Bayesian credible intervals for the parameters $\alpha = 2.1, \beta = 1.3$ and $\gamma = 0.7$ of the transmuted Weibull distribution under both uninformative prior (UP) and informative prior (IP) approaches. Increased censoring rates (20% to 40%) lead to wider credible intervals, indicating greater uncertainty in parameter estimates due to loss of information. The IP generally results in narrower intervals compared to the UP, suggesting the benefit of incorporating prior information. Overall, Bayesian credible intervals provide precise estimates for uncensored data, and even with censoring, the intervals remain reasonable. This analysis underscores the effectiveness of Bayesian methods for parameter estimation in incomplete data scenarios, balancing precision and uncertainty.

IX. Discussion

This article presents a Bayesian analysis of the transmuted Weibull distribution, utilizing both uniform and informative gamma priors under the AELF, PLF, and QLF loss functions. Real-world studies were conducted to evaluate the performance of the Bayes estimators, along with strategies for selecting suitable priors and loss functions across varying sample sizes and test termination times, under both complete and censored data settings. Specifically, two censoring rates—20% and 40%—were examined. Tables 2 and 3 show that the Bayes estimates demonstrated consistency, approaching the true parameter values as sample sizes grew. Posterior risks (PRs) were higher for censored data than for uncensored data, and 95% credible intervals became narrower with larger sample sizes. These findings were consistent in practical applications, supporting the effectiveness of the proposed MCMC algorithm for Bayesian parameter estimation. Future research could expand this work by studying mixtures of transmuted Weibull distributions or applying Bayesian analysis to record values with the transmuted Weibull distribution.

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