SAMPLING INSPECTION SCHEMES WITH SWITCHING RULES FOR LIFE TESTS BASED ON EXPONENTIAL DISTRIBUTION

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Abstract

A life test is a random experiment which is performed on manufactured items such as electric and electronic components in order to estimate their lifetime by selecting the items randomly from the production process. The lifetime / lifespan of the product is a random variable that follows a specific continuous-type probability distribution, called the lifetime distribution. Reliability sampling, which is one among the classifications of product control techniques, deals with inspection procedures for sentencing one or more lots or batches of items submitted for inspection. An acceptance sampling scheme is a combination of sampling inspection plans with switching rules for changing from one plan to another. A switching rule is an instruction within a sampling scheme for changing from one sampling plan to another of greater or lesser severity of sampling based on the demonstrated quality history. In this paper, the concept of sampling schemes for life tests with a switching rule involving two samples under the assumption that the lifetime random variable follows an exponential distribution is introduced. A procedure is developed for designing the optimum sampling schemes with minimum sample sizes when two points on the desired operating characteristic curve are prescribed providing protection to the producer and the consumer.

Keywords: Consumer's risk, Exponential distribution, operating characteristic function, producer's risk, Reliability sampling, Sampling system.

1. Introduction

An acceptance sampling scheme is a combination of acceptance sampling plans with switching rules for changing from one plan to another. A switching rule is an instruction within a sampling scheme for changing from one acceptance sampling plan to another of greater or lesser severity of sampling based on the demonstrated quality history. The sampling plans having greater severity are called tightened plans and those having lesser severity are termed as normal plans. Severity of sampling plans can be defined either in terms of sample sizes or in terms of acceptance numbers. While the acceptance number is fixed, the sample size under a tightened sampling plan would, generally, be larger than the sample size under a normal plan. Similarly, while the sample size is fixed, acceptance number under a tightened plan is smaller than the acceptance number under a normal plan. The procedure for switching between tightened and normal plans is essential to exert pressure on the producer to take corrective action when quality falls below the prescribed levels and to provide incentives, in terms of reduced sample size, for quality improvement. Quick switching systems (QSS) proposed by [6], tightened - normal - tightened (TNT) schemes of [4] and [8] are the examples of sampling schemes

QSS utilizes two single sampling plans with the same sample size and different acceptance numbers, together with a switching criterion of switching between normal inspection and tightened inspection. TNT sampling scheme involves a criterion for switching between two zero acceptance number single sampling plans, having different sample sizes, n_1 and n_2, where n1 is the sample size of a tightened plan and n_2 ($< n_1$) is the sample size of a normal plan. The salient feature of TNT scheme is that while the *OC* curve of the scheme is in a desirable shape at the upper portion providing greater protection to the producer against the producer's quality, the switching rules have no real effect on the consumer quality, which would remain essentially that of the tightened plan. This is due to the change in the sample sizes rather than the acceptance number. Following this feature, QSS can also be defined with a common acceptance number, but with different sample sizes under tightened and normal inspections.

In the context of life testing sampling plans or schemes, the acceptance number is defined as the allowable number of failures in the sample. In the following subsection, the concept of sampling inspection schemes for life tests involving a switching rule with two different sample sizes and zero failures is devised. The procedure for designing such sampling schemes for the prescribed acceptable mean (or median or reliability) and unacceptable mean (or median or reliability) life ensuring protection to the producer and consumer with reduced risks is also discussed.

2. Sampling Inspection Schemes for Life Tests

A sampling inspection scheme for life tests in reliability sampling involves two single sampling plans defined with common zero acceptance number, called zero failures, but with different sample sizes n_1 and n_2 and a switching rule. The single sampling plans, denoted by (n_1,0) and (n_2,0), are termed as normal and tightened inspection plans, where $n_2 > n_1$. With a provision to switching between the normal and tightened plans, the operating procedure of a specific sampling inspection scheme is described as follows:

Step 1: Start with normal inspection, drawing a random sample of n_1 items from a current lot. If no failures are observed, accept the lot. If one or more failures occur, reject the lot and switch to tightened inspection (given in Step 2) from the subsequent lot.

Step 2: Under tightened inspection, draw a random sample of n_2 items from a lot. If no failures are observed, accept the lot and switch to normal inspection (given in Step 1) from the subsequent lot.

Thus, the sampling inspection scheme for life tests is designated as *SIS*-(n_1,n_2:0), where n_1 is the sample size under a normal plan and n_2 is the sample size under a tightened plan.

It can be noted that as the provision of switching to tightened inspection when a lot is rejected under normal inspection and switching to normal inspection when a lot is accepted under tightened inspection is given within the system or scheme, the scheme is also termed as quick switching system, devised by [6]. Construction to the study of Quick Switching System (QSS) and its applications are presented by [1]. The exponential distribution, which is a special case of gamma family of distributions as demonstrated by [5], has a wider application in the fields of queueing theory, reliability theory and engineering, and hydrology. It is used to model the performance of components that have a constant failure rate and is applied to the cases involving items that do not degrade with time or do not result in wear out failures. Examples include components of high-quality integrated circuits, such as diodes, transistors, resistors, and capacitors. The exponential distribution is considered as a perfect model for the long and constant period of low failure risk that characterizes the useful life of the product and represents the intrinsic failure phase in the field of reliability.

The application of exponential distribution in the fields of actuarial, biological and engineering sciences. One may refer to [2], [3], [7], and [9] for more details. While the exponential distribution is appropriate for modeling the lifetime of an item, it is commonly applied for the inferential aspect of utilizing life information. Hence, as a member of the lifetime continuous probability distributions, the exponential distribution can be considered as an apt probability model to adopt in real life situations. Application of the exponential distribution in reliability sampling is now considered for sampling plans with switching rules when the lots are formed from the items resulted from a continuous stream of production. In the following sampling plans with switching rules, the description of exponential distribution as the probability model for the lifetime quality characteristic, the operating characteristics of the plans and the procedures for the selection of sampling plans for life tests under the assumption of the exponential distribution are presented.

3. Exponential Distribution

Let T be a random variable representing the lifetime of the components. Assume that T follows an exponential distribution with a scale parameter θ. The probability density function and the cumulative distribution function of T are, respectively, defined as follows:

$$f(t;\theta) = \frac{1}{\theta}exp\left(-\frac{t}{\theta}\right), 0 \leq t < \infty; \theta > 0 \tag{1}$$

$$F(t;\theta) = 1 - exp\left(-\frac{t}{\theta}\right) \tag{2}$$

The mean life time, the median life time, the reliability function and hazard function for specified time t under the exponential distribution are, respectively, given below:

$$\mu = E(t) = \theta \tag{3}$$

$$\mu_d = \theta In(2) \tag{4}$$

$$R(t;\theta) = exp\left(-\frac{t}{\theta}\right) \tag{5}$$

$$z(t;\theta) = \frac{1}{\theta}, 0 \leq t < \infty \tag{6}$$

It is known that the reliable life is the life beyond which some specified proportion of items in the lot will survive. Associated with the exponential distribution, it is defined by

$$\rho = -\theta In(R) \tag{7}$$

where R is the proportion of items surviving to time ρ. The proportion, p of product failing before time t, is defined by the cumulative probability distribution of T.
It is expressed as

$$p = F(t;\theta) = 1 - exp\left(-\frac{t}{\theta}\right) \tag{8}$$

4. Operating Characteristic Function of $SIS - (n_1, n_2 : 0)$

The operating characteristic function of $SIS - (n_1, n_2 : 0)$ is defined from [6] as

$$P_a(p) = \frac{P_T}{1 - P_N - P_T} \tag{9}$$

Where P_N=$P(d = 0|n_1)$ is the probability of accepting the lot when using a normal inspection plan, $P_T = P(d = 0|n_2)$ is the probability of accepting the lot when using a tightened inspection plan, and d is the number of failures observed in the sample.

Under the conditions for the application of binomial and Poisson distributions, the expressions for $P_a(p)$ are, respectively, given by

$$P_a(p) = \frac{(1-p)^{n_2}}{1-(1-p)^{n_1}+(1-p)^{n_2}} \tag{10}$$

and

$$P_a(p) = \frac{e^{-n_2 p}}{1-e^{-n_1 p}+e^{-n_2 p}} \tag{11}$$

Under a sampling plan for life tests, the failure probability, p, is the proportion of products failing before time t. When the lifetime random variable, T, follows an exponential distribution, p is defined from equation 8. It can be noted that a specific value of p is associated with a unique value of t/θ. As the mean life is defined by $\mu = \theta$, the value of p can be related to t/μ. Similarly, associated with any specific value of p are the values of t/μ_d or t/ρ, where μ_d is the median life is and ρ is the reliability life. Similarly, for a specified value of t/μ or t/μ_d or t/ρ, the value of p could be obtained. Thus, when p is associated with t/μ, the operating characteristic function of a life test sampling plan can be considered as a function of t/μ_d , rather than p, and the *OC* curve of the plan could be obtained by plotting the acceptance probabilities against the values of t/μ_d. If the median life and reliability life are considered for the operating characteristics of the desired plan, p can be associated with t/μ_d=$(\frac{t}{\theta})(ln(2))^{-1}$ and t/p=-$(\frac{t}{\theta})[ln(R)]^{-1}$.

Associated with each specific value of p, a unique value of t/μ or t/μ_d or t/ρ can be determined using the following simple procedure:

Step 1: Specify p.

Step 2: Obtain t/θ from the cumulative distribution function using (2) and (8).

Step 3: Using the value of t/θ obtained in Step 2, determine t/μ , t/μ_d and t/ρ from (3), (4) and (7), respectively.

Step 4: Define the following dimensionless ratios:

$$\frac{\mu}{\mu_0}=\frac{Actual\ Mean\ Life}{Assumed\ Mean\ Life}, \ \frac{\mu_d}{\mu_{d_0}}=\frac{Actual\ Mean\ Life}{Assumed\ Mean\ Life}, \ \frac{\rho}{\rho_0}=\frac{Actual\ Reliable\ Life}{Assumed\ Reliable\ Life}$$

Given the assumed mean life μ_0, the median life μ_{d_0} and the reliable life ρ_0 the ratios can be obtained using (3), (4) and (7) along with (2).

Step 5: Determine $\frac{\mu}{\mu_0}$, $\frac{\mu_d}{\mu_{d_0}}$ and $\frac{\rho}{\rho_0}$ corresponding to each specified value of p.

Step 6: Find the probability of acceptance using (2) under the conditions of binomial model or using (3) under the conditions of Poisson model corresponding to each specified value of p or $\frac{\mu}{\mu_0}$ or $\frac{\mu_d}{\mu_{d_0}}$ or $\frac{\rho}{\rho_0}$.

Step 7: Plot the probability of acceptance against each value of $\frac{\mu}{\mu_0}$ or $\frac{\mu_d}{\mu_{d_0}}$ or $\frac{\rho}{\rho_0}$ in order to obtain the required *OC* curve of $SIS-(n_1, n_2 : 0)$ for life tests under the assumption of exponential distribution for the lifetime quality characteristic. In a similar way, for a specified value of $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$, the value of p could be obtained following the above procedure in the reverse order. As p is associated with $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$, the operating characteristic function of $SIS-(n_1, n_2 : 0)$ for life tests can be considered as a function of $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$ rather than p. Thus, the *OC* curve of the sampling scheme could be obtained by plotting the acceptance probabilities against the values of $\frac{t}{\mu}$ or $\frac{t}{\mu_d}$ or$\frac{t}{\rho}$.

5. Procedure for the Selection of $SIS - (n_1, n_2 : 0)$ for Life Tests Indexed by Acceptable and Unacceptable Mean (or Median or Reliable) Life

It is the usual practice in selecting a sampling scheme or a system to fix the operating characteristic curve in accordance with the desired degree of discrimination. The *OC* curve is, in turn, fixed by suitably chosen parameters, *viz.*,$(p_0, 1-\alpha)$ and (p_1, β) or equivalently by $(\mu_0, 1-\alpha)$ and (μ_1, β) or $(\frac{t}{\mu_0}, 1-\alpha)$ and $(\frac{t}{\mu_1}, \beta)$, where p_0 is the producer's quality level, p_1 is the consumer's quality level, μ_0 is the acceptable mean life, μ_1 is the unacceptable mean life, α is the producer's risk and β is the consumer's risk. Similarly, $(\mu_{d_0}, 1-\alpha)$ and (μ_{d_1}, β), $(\rho_0, 1-\alpha)$ and (ρ_1, β) can be used to find the sampling schemes based on median life and reliable life criteria, respectively.

Now, an optimum $SIS - (n_1, n_2 : 0)$ based on mean life can be obtained satisfying the following two conditions which would also ensure protection to the producer and the consumer:

$$P_a(p_0) \geq 1-\alpha \tag{12}$$

and

$$P_a(p_1) \leq \beta \tag{13}$$

Equivalently, one can also fix the conditions as given below:

$$P_a(\mu_0) \geq 1-\alpha \tag{14}$$

and

$$P_a(\mu_1) \leq \beta \tag{15}$$

Or

$$P_a\left(\frac{t}{\mu_0}\right) \geq 1-\alpha \tag{16}$$

and

$$P_a\left(\frac{t}{\mu_1}\right) \leq \beta \tag{17}$$

Assuming that the *OC* curves of $SIS - (n_1, n_2 : 0)$ and the *OC* curves of tightened single sampling plans $(n_2, 0)$ coincide at $\frac{t}{\mu}$=$\frac{t}{\mu_1}$ i.e., at $p = p_1$, the sample sizes, n_1 and n_2 of $SIS - (n_1, n_2 : 0)$ can be determined easily as given below:

For tightened single sampling plan, $(n_2, 0)$, the expression for the OC function with the quality level, p_1, and the probability of acceptance, β, under the conditions for application of binomial model, is given by

$$\beta = (1-p_1)^{n_2} \tag{18}$$

From which the solution for n2 can be shown to be

$$n_2 = \frac{ln(\beta)}{ln(1-p_1)} \tag{19}$$

by taking natural logarithm and on simplification. If $\frac{ln(\beta)}{ln(1-p_1)}$ is not an integer, the solution for n_2 will be obtained as

$$n_2 = int\left[\frac{ln(\beta)}{ln(1-p_1)}\right] + 1 \tag{20}$$

If the OC curve of $SIS - (n_1, n_2 : 0)$ is required to pass through $(p_1, 1-\aleph)$, the sample size, n_1 can be determined as follows: From (12), consider

$$1\text{-}\alpha = \frac{(1-p_0)^{n_2}}{1-(1-p_0)^{n_1}+(1-p_0)^{n_2}}$$

which would result in

$$n_1 = \frac{In(1-\alpha-\alpha(1-p_0)^{n_2} - In(1-\alpha)}{In(1-p_0)}$$

The solution for in terms of an integer can be determined as

$$n_1 = int\left[\frac{In(1-\alpha-\alpha(1-p_0)^{n_2} - In(1-\alpha)}{In(1-p_0)}\right]+1 \tag{21}$$

In order to determine an optimum $SIS-(n_1,n_2:0)$ based on mean life satisfying the specified requirements under the assumption of the exponential distribution and to implement the plan in practical situations, the following procedure is developed:

Step 1: Specify the values of $\frac{t}{\mu_0}$ and $\frac{t}{\mu_1}$ with α = 0.05 and β = 0.10 respectively.

Step 2: Find p_0 and p_1 corresponding to $\frac{t}{\mu_0}$ and $\frac{t}{\mu_1}$ using the relationship existing between p and $\frac{t}{\mu}$.

Step 3: Obtain the optimum values of and for the specified strength $(\mu_0, 1-\alpha)$ and (μ_1, β) from (20) and (19) with the values of p_0 and p_1.

Step 4: Begin the normal inspection drawing a random sample of n_1 items from a current lot.

Step 5: Perform the life test on each of the selected sample items considering t as the test termination time and μ as the expected mean life and observe the number, d, of failures.

Step 6: If no failures are observed in the sample or if no failures occurred until the termination time is reached, accept the current lot and continue with normal inspection. If one or more failures are observed before reaching the test termination time, reject the lot and switch to tightened inspection.

Step 7: Under tightened inspection, draw a random sample of n_2 items from the subsequent lot and perform the life test on each of the selected items. If no failures are found in the entire sample of n_2 items or if no failures occurred until the test termination time is reached, accept the lot and return to normal inspection from the subsequent lot. If at least one failure is observed before reaching the test termination, reject the lot and continue with tightened inspection.

The above procedure can also be followed for determining optimum $SIS-(n_1,n_2:0)$ when the median and reliability life criteria are desired. Tables 1, 2 and 3 are constructed utilizing the first three steps of the procedure described above specifying three sets of wide range of values, *viz.*, (i) $\frac{t}{\mu_0}$ and R_μ, (ii)$\frac{t}{\mu_{d_0}}$ and R_{μ_0}, and (iii) $\frac{t}{p_0}$ and R_p where R_μ=$\frac{\mu_0}{\mu_1}$, R_{μ_d}=$\frac{\mu_{d_0}}{\mu_{d_1}}$, and R_ρ=$\frac{\rho_0}{\rho_1}$ are the operating ratios (dimensionless), which are the measures of discrimination. The tables, respectively, provide the parameters $n_1 and n_2$ of $SIS-(n_1,n_2:0)$ for life tests indexed by mean life, median life and reliable life. Each of the plans listed in the tables would ensure the conditions that the maximum producer's and consumer's risks are restricted to 5 percent (*i.e.*, α = 0.05) and 10 percent (*i.e.*, β = 0.10), respectively.

5.1. Numerical Illustration 1

Let us consider the case that the lifetime of an automobile voltage regulator is a random variable following an exponential distribution with parameter θ. An industrial practitioner is interested to adopt a sampling inspection scheme which should have a provision of switching between two single sampling plans of different sample sizes, and insists that no failures should be allowed until the termination time t is reached. Based on the past history, the acceptable mean life and unacceptable mean life of the regulators were estimated as μ_0 = 30500 minutes and μ_1 = 2200 minutes, respectively. In order to make the decisions on sentencing the lots coming from the industrial process which produce automobile voltage regulators, the practitioner wishes to implement $SIS-(n_1,n_2:0)$. It is assumed that the risk of rejecting the lots having acceptable mean life and the lots having unacceptable mean life should not exceed 5 percent and 10 percent, respectively. The test termination time is fixed as t_0 = 45 minutes. For the specified requirements, the measure of discrimination is found to be $R_\mu = \frac{\mu_0}{\mu_1} = \frac{30500}{2200} = 13.86 \approx 14$ and $\frac{t}{\mu_0} = \frac{45}{30500} = 0.00148 \approx 0.0015$.

Therefore, $R_\mu = 14$ and $\frac{t}{\mu_0} = 0.0015$ based on the given information, one finds the optimum sample sizes for normal and tightened inspection under $SIS - (n_1, n_2 : 0)$ as $n_1 = 28$ and $n_2 = 144$, respectively. Thus, the optimum sampling inspection scheme is implemented as given below:

1. Select a random sample of 28 items from the current lot under normal inspection.
2. Conduct the life tests on each of the 28 sampled items and observe the number of failures until the termination time t = 45 minutes is reached. If no failures are observed, accept the lot and continue with normal inspection. If at least one failure occurs at the test termination time or before reaching the test termination time, reject the current lot and switch to tightened inspection from the subsequent lot.
3. Draw a random sample of 144 items from the lot and conduct the life test on each of the 144 sampled items. If no failures are observed while testing all the sampled items or until reaching t = 45 minutes, accept the lot and switch to normal inspection from the subsequent lot; if at least one failure occurs before reaching t = 45 minutes, reject the lot and continue with the tightened inspection from the subsequent lot.

Figures 1 displays the operating characteristic curve of the optimum sampling inspection scheme obtained in the above illustration. It is observed that the *OC* curve passes through the respective prescribed points with the reduced producer's risk of 4.91 percent (less than 5 percent) and the reduced consumer's risk of 9.86 percent (less than 10 percent) and ensure that the prescribed conditions (4) and (5) are satisfied.

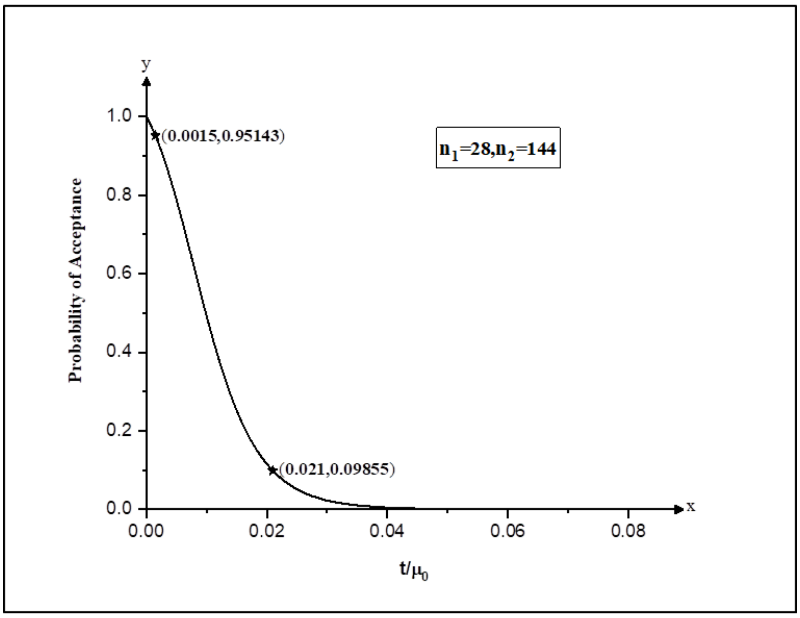

Figure 1: *OC Curve of $SIS - (n_1, n_2 : 0)$ for Life Tests Based on Exponential Distribution Indexed by Mean Life Ratio when n_1 = 28 and n_2 = 144.*

5.2. Numerical Illustration 2

A production engineer assessed from his experience that the time to failure of semiconductors can be modeled by an exponential distribution with parameter θ. The assembled semiconductors are resulted from a continuous stream of production. The acceptable median life and unacceptable median life of the semiconductors are respectively, specified as 48000 hours and 2600 hours, respectively. It is desired that the maximum risk of rejecting the lots when the acceptable median life is given as 48000 hours and the maximum risk of accepting lots when the unacceptable median life is specified as 2600 hours are fixed at 5 percent and 10 percent, respectively. It is assumed that the total time duration of life test is fixed at t_0 = 120 hours. For the given requirements, $R_{\mu_d} = \frac{\mu_{d_0}}{\mu_{d_1}}$ = 48000/2600 = 18.46 is the measure of discrimination and $\frac{t}{\mu_{d_0}}$ is 0.0025. Based on these values of R_{μ_d} and $\frac{t}{\mu_0}$, the optimum sampling scheme is determined by the sample size $n_1 = 26$ for normal inspection and the sample size $n_2 = 87$ for tightened inspection.

It can be noted from the above illustrations that the sample size required for tightened inspection is larger than the sample size required for normal inspection. Similar to Numerical Illustrations 1 and 2, suitable illustrations can be given.

6. Conclusion

Sampling plans with switching rules are considered and procedures for designing the optimum plans to provide protection to the producer and the consumer are discussed based on exponential distribution as the lifetime probability distribution. A procedure for selection of sampling Inspection Schemes for life tests ensuring protection to the producer and consumer is described. The tables yielding the parameters of the optimum $SIS - (n_1, n_2 : 0)$ indexed by acceptable mean life (or median or reliable) and unacceptable mean life (or median or reliable), respectively, are constructed for a specified set of values of the parameter of the Exponential distribution.

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