A NEW FAMILY OF LINDLEY DISTRIBUTIONS FEATURING BIMODAL CASES

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Abstract

Several lifetime distributions have been developed in literature to handle different real-world scenario. Most of these distributions were developed to model a unimodal (symmetric or asymmetric) data. Only a hand-full of these distributions exhibits a bimodal property. This paper explores a new family of Lindley distributions featuring a bimodal property. We introduce five different sub-families of the T-Power Lindley$\{Y\}$ family based on the quantile function of the uniform, exponential, Frechet, log-logistic and logistic distributions. Useful mathematical properties of the proposed T-Power Lindley$\{Y\}$ family of distributions are derived and sub-models were the random variable T follows the one-parameter Topp-Leone, exponential, exponentiated exponential, Weibull and Gumbel distributions are introduced. From the graphical representation of the density function of these sub-models, we observe that the shape of the density function accommodates a decreasing (reversed-J), left-skewed, right-skewed, symmetric, as well as a bimodal shape. In order to illustrate the usefulness and performance of the proposed T-Power Lindley$\{Y\}$ family of distributions, the Gumbel Power Lindley (GPL) distribution belonging to the proposed family of distribution was employed to fit a bimodal data set alongside with the beta-Normal distribution. Result obtained from the analysis revealed that the Gumbel Power Lindley (GPL) distribution compares favourably better than the beta-Normal distribution. The density fits of the distributions for the data set was also investigated to support the claim.

Keywords: Lindley distribution; power Lindley distribution; quantiles; bimodality.

1. Introduction

Lifetime distributions are parametric models that seeks to analyze time-to-event data. Many lifetime models such as exponential, Weibull, gamma, beta, Gumbel distributions etc, have been studied and applied in literature to analyze lifetime data. Obviously, to increase the flexibility of these classical models remains the strong reason for developing new ones, thus many researchers have proposed generalized forms of these classical lifetime distribution.

The classical one-parameter Lindley distribution introduced by [11] has its density function defined by

$$f(x) = \frac{\beta^2}{\beta+1}(1+x)e^{-\beta x}, \quad x > 0, \beta > 0, \tag{1}$$

and cumulative distribution function as

$$F(x) = 1 - \left(\frac{\beta+1+\beta x}{\beta+1}\right) e^{-\beta x}, \quad x > 0, \beta > 0. \tag{2}$$

In order to increase the flexibility of the classical one-parameter Lindley distribution in analyzing lifetime data, [7] introduced a two parameter Power Lindley distribution by considering the power transformation $T = X^{\frac{1}{\alpha}}$, with the density function defined by

$$f(x) = \frac{\alpha\beta^2}{(1+\beta)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}, \quad x > 0, \beta, \alpha > 0, \tag{3}$$

and cumulative distribution function defined by,

$$F(x) = 1 - \left(\frac{\beta + 1 + \beta x^\alpha}{\beta + 1}\right)e^{-\beta x^\alpha}, \quad x > 0, \alpha, \beta > 0. \tag{4}$$

A competing risks model when the causes of failure follow the one-parameter Lindley distribution was studied by [12] and was applied to a data set representing the lifetime of 194 patients with squamous cell carcinoma reported in [9]. Their result shows that the Lindley competing risks model provides a better fit to the data set under study than the exponential and the Weibull distributions. Nonetheless, due to the monotonic property of the one parameter Lindley distribution, there are situations where the distribution will fail to provide good fit in modeling lifetime data.

Several methods of generating new classes of probability distributions have been established in literature. The Kumaraswamy Lindley distribution and the Kumaraswamy Power Lindley distribution have been introduced, respectively, by [13] and [14] using the Kumaraswamy-G family of distributions proposed by [4]. A wider family of distributions called the "$T - X$ family of distributions" was introduced by [2]. The CDF of the $T - X$ family of distributions is defined as

$$G(x) = \int_0^{W(F(x))} r(t)dt, \quad = \quad R[W(F(x))], \tag{5}$$

where $R(t)$ is the CDF of the random variable T and $W(F(x))$ is a continuous and monotonic function of the CDF of a random variable X. Using this framework, [10] proposed the Lindley-X family of distribution and considered a special case of Lindley-Pareto distribution. For a random variable T following the density function of the Lindley distribution, [16] proposed the Odd Lindley-G family of distributions with cumulative distribution function defined as

$$F(x, \theta, \xi) = \frac{\theta^2}{\theta + 1}\int_0^{\frac{G(x,\xi)}{1-G(x,\xi)}}(1+t)e^{-\theta t}dt, \tag{6}$$

where $G(x, \xi)$ is the CDF of the random variable X, depending on a parameter vector ξ.

Undoubtedly, these generalizations have addressed some major drawbacks of the classical one-parameter Lindley distribution. However, their flexibility is limited to handling unimodal lifetime data. The need for developing a generalized Lindley distribution which can appropriately model bimodal lifetime data forms the motivation of this paper and the T-Power Lindley$\{Y\}$ Family of distributions is one of such. The remaining sections of this paper are organized as follows: Section 2 defines some sub-families of the T-Power Lindley$\{Y\}$ based on different quantile functions of a random variable Y. Section 3 gives some general mathematical properties of the T-Power Lindley$\{Y\}$ distribution. In Section 4, some new distributions belonging to the T-Power Lindley$\{Y\}$ family and some of their properties are discussed. Section 5 presents an application of the T-Power Lindley$\{Y\}$ family of distributions to a bimodal data set. Finally, Section 6 presents the conclusion.

2. Sub-families of the T-Power Lindley$\{Y\}$ Distribution Based on different Quantile Functions

Let T, R and Y be random variables from a known probability distribution with the cumulative distribution function defined by $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$, and $F_Y(x) = P(Y \leq x)$, respectively. Let the corresponding quantile functions be given as $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$. If the density function exists, then we can denote them by $f_T(p)$, $f_R(p)$ and $f_Y(p)$ respectively.

A unified definition of the random variables in [1] was given by [3]. The authors defined the cumulative distribution function of a random variable X as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P\left\{T \leq Q_Y(F_R(x))\right\} = F_T[Q_Y(F_R(x))], \tag{7}$$

and the corresponding density function defined as

$$f_X(x) = \frac{f_R(x)}{f_Y\left\{Q_Y(F_R(x))\right\}} f_T\left\{Q_Y(F_R(x))\right\}. \tag{8}$$

Table 1: *Quantile Function of Some Known Distributions*

Distributions	Quantile Function $Q_Y(p)$	Support of Y
Uniform	p	$[0,1]$
Exponential	$-log(1-p)$	$(0,\infty)$
Frechet	$[-log(p)]^{-1}$	$(0,\infty)$
Log-logistic	$\frac{p}{1-p}$	$(0,\infty)$
Logistic	$log\left(\frac{p}{1-p}\right)$	$(-\infty,\infty)$

If R be a random variable following the power Lindley distribution with PDF $f_R(x)$ and CDF $F_R(x)$ defined in (3) and (4), respectively, then the sub-families of the T-Power Lindley$\{Y\}$ distribution can be generated based on different quantiles defined in Table 1.

2.1. T-Power Lindley$\{Uniform\}$ Distribution

This family of distributions is generated by using the quantile function of the uniform distribution in Table 1, with the support of $T \in [0,1]$. From (7), the cumulative distribution function of the T-Power Lindley$\{Uniform\}$ distribution is defined by

$$\begin{aligned} F_X(x) = \quad & F_T[Q_Y(F_R(x))] = \quad F_T(F_R(x)), \\ & = F_T\left\{1 - \left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}, \end{aligned} \tag{9}$$

and the corresponding density function is given by

$$\begin{aligned} f_X(x) &= f_T(x) \times f_T\left\{F_R(x)\right\}, \\ &= \frac{\alpha\beta^2}{(1+\beta)}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}} f_T\left\{1 - \left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}. \end{aligned} \tag{10}$$

The Kumaraswamy Power Lindley distribution proposed by [14] and the Kumaraswamy Lindley distribution proposed by [13] are members of this family.

2.2. T-Power Lindley$\{Exponential\}$ Distribution

This family of distributions is generated by using the quantile function of the exponential distribution in Table 1, with the support of $T \in (0, \infty)$. From (7), the cumulative distribution function of the T-Power Lindley$\{exponential\}$ distribution is defined by

$$F_X(x) = F_T[Q_Y(F_R(x))] = F_T\{-log(1 - F_R(x))\},$$
$$= F_T\left\{-log\left[\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]\right\}, \tag{11}$$

and the corresponding density function is given by

$$f_X(x) = \frac{f_R(x)}{1 - F_R(x)} \times f_T\{-log(1 - F_R(x))\},$$
$$= \frac{\alpha\beta^2(1+x^{\alpha})x^{\alpha-1}}{(1+\beta+\beta x^{\alpha})} f_T\left\{-log\left[\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]\right\}. \tag{12}$$

2.3. T-Power Lindley$\{Frechét\}$ Distribution

This family of distributions is generated by using the quantile function of the $frechét$ distribution in Table 1, with the support of $T \in (0, \infty)$. The cumulative distribution function of the T-Power Lindley$\{Frechét\}$ distribution is defined from (7) as

$$F_X(x) = F_T\{Q_Y(F_R(x))\} = F_T\left\{[-\log(F_R(x))]^{-1}\right\},$$
$$= F_T\left\{\left\{-\log\left(1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right)\right\}^{-1}\right\}, \tag{13}$$

and the corresponding density function is given by

$$f_X(x) = \frac{f_R(x)}{F_R(x)[-\log(F_R(x))]^2} \times f_T\left\{[-\log(F_R(x))]^{-1}\right\},$$
$$= \frac{\alpha\beta^2(1+x^{\alpha})x^{\alpha-1} f_T\left\{\left\{-\log\left(1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right)\right\}^{-1}\right\}}{((1+\beta)e^{\beta x^{\alpha}} - (1+\beta+\beta x^{\alpha}))\left[\log\left(1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right)\right]^2}. \tag{14}$$

2.4. T-Power Lindley$\{log - logistic\}$ Distribution

This family of distributions is generated by using the quantile function of the exponential distribution in Table 1, with the support of $T \in (0, \infty)$. From (7), the cumulative distribution function of the T-Power Lindley$\{log - logistic\}$ distribution is defined by

$$F_X(x) = F_T\{Q_Y(F_R(x))\} = F_T\left\{\frac{F_R(x)}{(1 - F_R(x))}\right\},$$
$$= F_T\left\{\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right\}, \tag{15}$$

and the corresponding density function is given by

$$f_X(x) = \frac{f_R(x)}{(1 - F_R(x))^2} \times f_T\left\{\frac{F_R(x)}{(1 - F_R(x))}\right\},$$
$$= \frac{\alpha\beta^2(1+\beta)(1+x^{\alpha})x^{\alpha-1}e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})^2} f_T\left\{\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right\}. \tag{16}$$

2.5. T-Power Lindley$\{logistic\}$ Distribution

This family of distributions is generated by using the quantile function of the logistic distribution in Table 1, with the support of $T \in (-\infty, \infty)$. From (7), the cumulative distribution function of the T-Power Lindley$\{logistic\}$ distribution is defined by

$$F_X(x) = F_T\left\{Q_Y(F_R(x))\right\} = F_T\left\{\log\left(\frac{F_R(x)}{(1-F_R(x))}\right)\right\},$$

$$= F_T\left\{\log\left(\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right)\right\}, \tag{17}$$

and the corresponding density function is given by

$$f_X(x) = \frac{f_R(x)}{F_R(x)(1-F_R(x))} \times f_T\left\{\log\left(\frac{F_R(x)}{(1-F_R(x))}\right)\right\},$$

$$= \frac{\alpha\beta^2(1+\beta)(1+x^{\alpha})x^{\alpha-1} f_T\left\{\log\left(\frac{(1+\beta)e^{\beta x^{\alpha}}}{(1+\beta+\beta x^{\alpha})} - 1\right)\right\}}{(1+\beta+\beta x^{\alpha})\left[(1+\beta)-(1+\beta+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right]}. \tag{18}$$

Clearly, we observe that the support of the random variable T follows the support of Y, and the support of the proposed random variable X follows the support of the random variable R.

3. Some Mathematical Properties of the T-Power Lindley$\{Y\}$ Families of Distributions

3.1. Transformation and Quantile Function

Lemma 1 presents a mathematical relationship between the random variable X following the T-Power Lindley$\{Y\}$ Distribution and the generator random variable T. The random variable Y follows the Uniform, Exponential, Frechét, log-logistic and logistic distribution.
Lemma 1:

Let T be a random variable with pdf $f_T(x)$,

(a) if T is defined on the interval [0,1], then the random variable

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1-T)(1+\beta)}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Uniform\}$ Family of Distributions;

(b) if T is defined on the interval $(0, \infty)$, then the random variable

(i)

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{e^{T+\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Exponential\}$ Family of Distributions;

(ii)

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)(1-e^{-T^{-1}})}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Frechét\}$ Family of Distributions;

(iii)

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+T)e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Log-logistic\}$ Family of Distributions;

(c) if T is defined on the interval $(-\infty, \infty)$, then the random variable

$$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+e^{T})e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$$

belongs to the T-Power Lindley$\{Logistic\}$ Family of Distributions. Where $W_{-1}(.)$ is the negative branch of the Lambert W function.

Proof:

The proof follows from a simple transformation between the random variables X and T as defined in (9), (11), (13), (15) and (17), respectively. From these relationships, random samples for X can be generated by using T, that is, random samples for X following the T-Power Lindley$\{Uniform\}$ distribution can be generated by first generating random samples for T from the pdf $f_T(x)$ and then compute

$X = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1-T)(1+\beta)}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}$, which has the cdf $F_X(x)$.

Lemma 2:

The quantile function for the (i) T-Power Lindley$\{Uniform\}$, (ii) T-Power Lindley$\{Exponential\}$, (iii) T-Power Lindley$\{Frechét\}$, (iv) T-Power Lindley$\{Log-logistic\}$, and (v) T-Power Lindley$\{Logistic\}$ families of distribution are, respectively, given by

(i) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1-Q_T(p))(1+\beta)}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(ii) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{e^{Q_T(p)+\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(iii) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)(1-e^{-[Q_T(p)]^{-1}})}{e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(iv) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+e^{Q_T(p)})e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}},$$

(v) $$Q_X(p) = \left\{-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-\frac{(1+\beta)}{(1+Q_T(p))e^{\beta+1}}\right]\right\}^{\frac{1}{\alpha}}.$$

Proof:

The proofs follow directly by solving $F_X(Q_X(p)) = p$, for $Q_X(p)$, Where $F_X(.)$ is the cdf given by (9), (11), (13), (15) and (17), respectively.

3.2. The Mode(s) of T-Power Lindley$\{Y\}$ families of distribution

Lemma 3:

The mode(s) of the (i) T-Power Lindley$\{Uniform\}$, (ii) T-Power Lindley$\{Exponential\}$, (iii) T-Power Lindley$\{Frechét\}$, (iv) T-Power Lindley$\{Log-logistic\}$, and (v) T-Power Lindley$\{Logistic\}$ distributions, respectively, are the solutions of (19), (20), (21), (22), and (23).

$$\Psi(x) = \frac{-f_T^{'}[F_R(x)]\bar{F}_R(x)}{f_T[F_R(x)]}, \tag{19}$$

$$\Psi(x) = \frac{-f_T'[P_1(x)]}{f_T[P_1(x)]} - 1, \tag{20}$$

$$\Psi(x) = \frac{\bar{F}_R(x)}{F_R(x)[\log F_R(x)]}\left\{\frac{f_T'[P_2(x)]}{f_T[P_2(x)]}P_2(x) + \log F_R(x) + 2\right\}, \tag{21}$$

$$\Psi(x) = \frac{-f_T'[P_3(x)]}{\bar{F}_R(x) f_T[P_3(x)]} - 2, \tag{22}$$

$$\Psi(x) = \frac{1}{F_R(x)}\left\{\frac{-f_T'[P_4(x)]}{f_T[P_4(x)]} - 2F_R(x) + 1\right\}, \tag{23}$$

where $P_1(x) = \{-\log(1 - F_R(x))\}$, $P_2(x) = \{-\log F_R(x)\}^{-1}$,

$P_3(x) = \dfrac{F_R(x)}{(1 - F_R(x))}$, $P_4(x) = \log\left\{\dfrac{F_R(x)}{(1 - F_R(x))}\right\}$, and
$\Psi(x) = \Big\{(1 + \{(1 + x^\alpha)\beta)\}^{-1})(-1 + (\alpha - 1)(\alpha\beta x^\alpha)^{-1} + (1 + x^\alpha)\beta)\Big\}$

Proof:

We need to first show that the first derivative of the density of the Power Lindley Distribution is expressed as

$$f_R'(x) = \Psi(x)\frac{f_R^2(x)}{\bar{F}_R(x)} \tag{24}$$

where $\bar{F}_R(x) = \dfrac{\{1 + (1 + x^\alpha)\beta\}}{1 + \beta} e^{-\beta x^\alpha}$ is the survival function of the Power Lindley distribution.

Also, the derivative of (10) can be expressed as

$$f_X'(x) = f_R^2(x) f_T'\{F_R(x)\} + f_R'(x)\{F_R(x)\}. \tag{25}$$

Substituting (24) into (25), we have

$$f_X'(x) = f_R^2(x) m(x), \tag{26}$$

where $m(x) = f_T'\{F_R(x)\} + \dfrac{f_T\{F_R(x)\}}{\bar{F}_R(x)}\Psi(x)$.

Solving the system of equation $m(x) = 0$, gives the result in (19). The results in (20)-(23) follow using similar approach.

4. Some new distribution arising from the T-Power Lindley$\{Y\}$ family of distributions

In this Section, we present some distribution belonging to the T-Power Lindley$\{Y\}$ family of distributions with different T-distribution. Details of the T-distribution is given in Table 2.

Table 2: *Some Known Distributions*

Distributions	PDF	CDF
Topp Leone, $T \in [0,1]$	$2\alpha(1-t)\left[1-(1-t)^2\right]^{\alpha-1}$	$\left[1-(1-t)^2\right]^{\alpha}$
Exponentiated exponential, $T \in (0,\infty)$	$\frac{\theta}{\lambda}\left\{1-exp\left(-\frac{t}{\lambda}\right)\right\}^{\theta-1}exp\left(-\frac{t}{\lambda}\right)$	$\left\{1-exp\left(-\frac{t}{\lambda}\right)\right\}^{\theta}$
Exponential, $T \in (0,\infty)$	$\frac{1}{\lambda}exp\left(-\frac{t}{\lambda}\right)$	$1-exp\left(-\frac{t}{\lambda}\right)$
Weibull, $T \in (0,\infty)$	$\frac{\theta}{\lambda}\left(\frac{t}{\lambda}\right)^{\theta-1}exp\left\{-\left(\frac{t}{\lambda}\right)^{\theta}\right\}$	$1-exp\left\{-\left(\frac{t}{\lambda}\right)^{\theta}\right\}$
Gumbel, $T \in (-\infty,\infty)$	$\frac{\gamma}{\sigma}exp\left(-\frac{t}{\sigma}\right)exp\left\{-\gamma exp\left(-\frac{t}{\sigma}\right)\right\}$	$exp\left\{-\gamma exp\left(-\frac{t}{\sigma}\right)\right\}$

4.1. Topp Leone Power Lindley$\{Uniform\}$ Distribution (TLPLD)

Suppose the random variable T follows the one-parameter Topp-Leone distribution with bounded support $[0,1]$ reported in [17], then the density function of the Topp-Leone Power Lindley distribution is define as

$$f(x) = \frac{2\lambda\alpha\beta^2}{(1+\beta)}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left\{\bar{G}(x)\right\}\left\{1-\left(\bar{G}(x)\right)^2\right\}^{\lambda-1}, \qquad x>0, \alpha, \beta, \lambda > 0, \tag{27}$$

and the corresponding cumulative distribution function is given by

$$F(x) = \left\{1-\left(\bar{G}(x)\right)^2\right\}^{\lambda}, \qquad x>0, \alpha, \beta, \lambda > 0, \tag{28}$$

where $\bar{G}(x) = \left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}$ is the survival function of the Power Lindley distribution.

Other useful mathematical properties of this TLPL distribution has been studied in [15]. Figure 1 displays the plots of the density function of the Topp-Leone Power Lindley distribution (TLPLD) at various choices of the parameters. The plots indicates that the TLPLD can be left skewed, right skewed, monotonically decreasing (reversed J-shape), and symmetric.

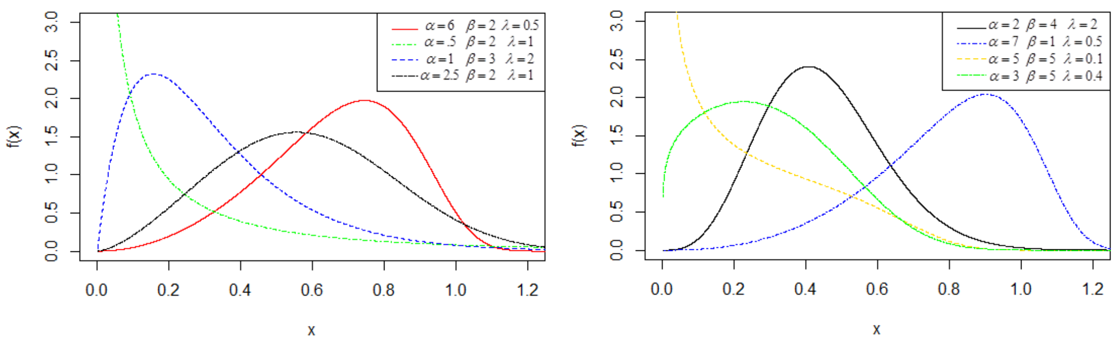

Figure 1: *Density function of the TLPLD for different choices of the parameters*

4.2. Exponentiated Exponential Power Lindley$\{Exponential\}$ Distribution

Let the random variable T follows the Exponentiated Exponential distribution introduced by [8], then the density function of the Exponentiated Exponential Power Lindley distribution (EEPLD) is define as

$$f(x) = \frac{\theta\alpha\beta^2(1+x^\alpha)x^{\alpha-1}}{\lambda(1+\beta+\beta x^\alpha)} \left\{1-(\bar{G}(x))^{\frac{1}{\lambda}}\right\}^{\theta-1} \left\{\bar{G}(x)\right\}^{\frac{1}{\lambda}}, \qquad x>0, \theta, \alpha, \beta, \lambda > 0, \tag{29}$$

and the corresponding cumulative distribution function is given by

$$F(x) = \left\{1-(\bar{G}(x))^{\frac{1}{\lambda}}\right\}^{\theta}, \qquad x>0, \theta, \alpha, \beta, \lambda > 0. \tag{30}$$

The plots of the probability density function of the Exponentiated Exponential Power Lindley distribution (EEPLD) for different values of the parameters is shown in Figure 2. It shows that the shape of the EEPLD can be left skewed, right skewed, monotonically decreasing (reversed J-shape), and symmetric.

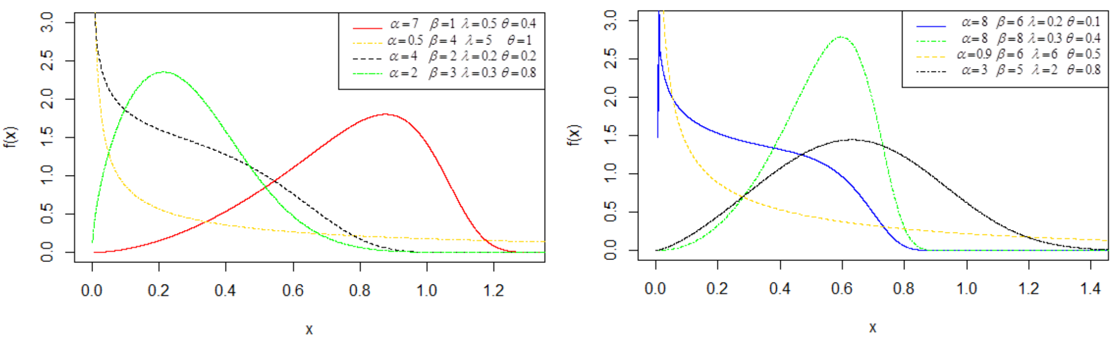

Figure 2: *Density function of the EEPLD for different values of the parameters*

4.3. Exponential Power Lindley{*Frechét*} Distribution (EPLD)

Let the random variable T follows the exponential distribution, then the density function of the Exponential Power Lindley distribution is define as

$$f(x) = \frac{\alpha\beta^2(1+x^\alpha)x^{\alpha-1}exp\left\{\{\theta\log(G(x))\}^{-1}\right\}}{\{\theta(1+\beta)e^{\beta x^\alpha}-(1+\beta+\beta x^\alpha)\}\,[\log(G(x))]^2}, \qquad x>0, \theta, \alpha, \beta > 0, \tag{31}$$

and the corresponding cumulative distribution function is given by

$$F(x) = 1 - exp\{\theta\log(G(x))\}^{-1}, \qquad x>0, \theta, \alpha, \beta > 0. \tag{32}$$

The plots of the probability density function of the Exponential Power Lindley distribution (EPLD) for different values of the parameters is shown in Figure 3. It indicates that the shape of the EPLD can be left skewed, right skewed, monotonically decreasing (reversed J-shape), modified unimodal.

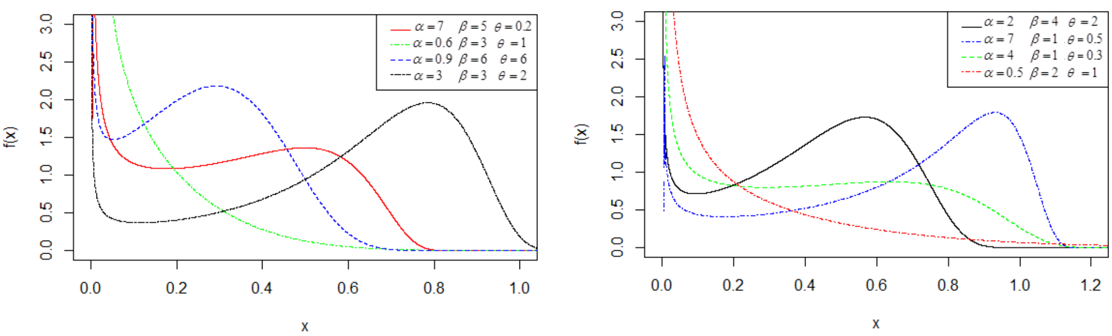

Figure 3: *Density function of the EPLD for different values of the parameters*

4.4. Weibull Power Lindley$\{log - logistic\}$ Distribution (WPLD)

Let the random variable T follows the Weibull distribution, then the density function of the Weibull Power Lindley distribution is define as

$$f(x) = \frac{\alpha\theta\beta^2(1+\beta)(1+x^\alpha)x^{\alpha-1}}{\lambda(1+\beta+\beta x^\alpha)^2} \left\{\frac{\varphi(x)}{\lambda}\right\}^{\theta-1} exp\left\{\beta x^\alpha - \left\{\frac{\varphi(x)}{\lambda}\right\}^{\theta}\right\}, \tag{33}$$

and the corresponding cumulative distribution function is given by

$$F(x) = 1 - exp\left\{-\left\{\frac{\varphi(x)}{\lambda}\right\}^{\theta}\right\}, \qquad x > 0, \theta, \alpha, \beta, \lambda > 0, \tag{34}$$

where $\varphi(x) = \dfrac{\frac{1}{\lambda}(1+\beta)e^{\beta x^\alpha}}{(1+\beta+\beta x^\alpha)} - 1$.

Figure 4 gives the graph of the density function of the Weibull Power Lindley distribution (WPLD) for different values of the parameters. Figure 4 clearly shows that the shape of the density function of WPLD can be monotonically decreasing (reversed J-shape), left skewed, right skewed, symmetric and bimodal.

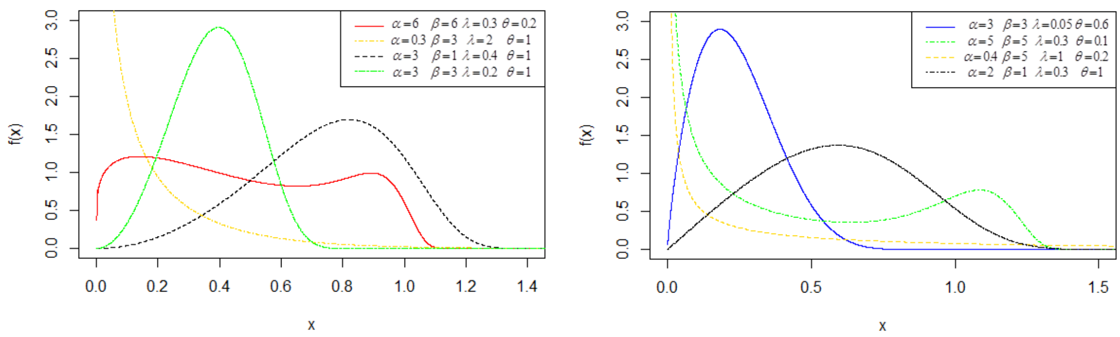

Figure 4: *Density function of the WPLD for different values of the parameters*

4.5. Gumbel Power Lindley$\{logistic\}$ Distribution

Let the random variable T follows the Gumbel distribution, then the density function of the Gumbel Power Lindley distribution (GPLD) is define as

$$f(x) = \frac{\alpha\gamma\beta^2(1+\beta)(1+x^\alpha)x^{\alpha-1}\{\varphi(x)\}^{-\frac{1}{\sigma}} exp\left\{-\gamma\{\varphi(x)\}^{-\frac{1}{\sigma}}\right\}}{\sigma(1+\beta+\beta x^\alpha)\left[1+\beta-(1+\beta+\beta x^\alpha)e^{-\beta x^\alpha}\right]}, \tag{35}$$

with the cumulative distribution function given by

$$F(x) = exp\left\{-\gamma\{\varphi(x)\}^{-\frac{1}{\sigma}}\right\}, \qquad x > 0, \alpha, \beta, \sigma > 0, \gamma = e^{\frac{\mu}{\sigma}}. \tag{36}$$

Figure 5 shows the plots of the density function of the Gumbel Power Lindley distribution for various choices of the parameters. The plots indicate that the GPLD exhibits a monotonically decreasing (reversed J-shape), left skewed, right skewed, symmetric and bimodal shape.

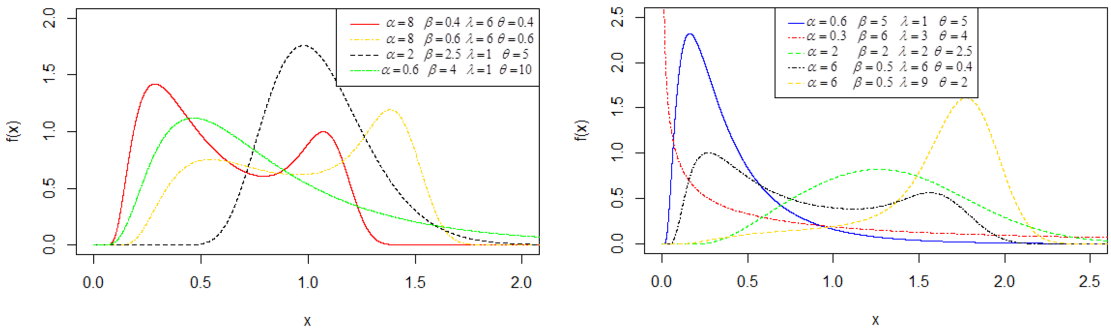

Figure 5: *Density function of the GPLD for different values of the parameters*

5. Application of the T-Power Lindley$\{Y\}$ family of distributions to a bimodal data set

To illustrate the flexibility of the T-Power Lindley$\{Y\}$ family of distributions in fitting real world data, we employ the Gumbel Power Lindley Distribution belonging to the T-Power Lindley$\{Y\}$ family of distributions to fit the egg size distribution data set reported in [5]. The data set consists of 88 asteroid species divided into three types; 35 planktotrophic larvae, 36 lecithotrophic larvae and 17 brooding larvae. [6] considered the logarithm of the asteroid data set which exhibits a bimodal shape and applied it to the beta-normal distribution. The descriptive statistics of the asteroid data is shown in Table 3, while Figure 6 provides the total time on test (TTT) and boxplot plot of the asteroid data.

Table 3: *Descriptive Statistics of the Asteroid Data*

Min.	1st Qu.	Median	3rd Qu.	Mean	Skewness	Kurtosis
4.605	5.134	6.126	6.869	6.070	0.1378	1.8217

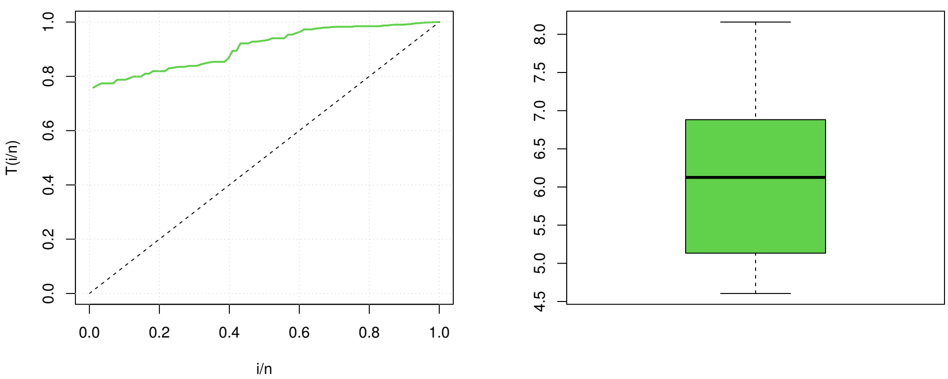

Figure 6: *TTT plot and Boxplot for the Asteroid Data*

Table 3 indicates that the data set is skewed to the right, whereas, Figure 6 shows that the data set exhibits an increasing hazard rate property.

Here, we apply the proposed Gumbel-Power Lindley distribution (GPLD) alongside with the beta-normal distribution (Beta-Norm) due to [6] to fit the bimodal data set. For the purpose of model comparison, the fits of the distributions were compared based on the maximized log-likelihood (Log-Lik), Aikaike Information Criterion (AIC), Corrected Aikaike Information

Criterion (AICc) and Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC).

Table 4: *Summary Statistics for the Asteroid Data*

Distributions	Estimates	Log-lik	AIC	$AICc$	BIC	HQIC
GPLD	$\alpha = 0.0046$	-109.1930	226.3861	226.8680	236.2954	230.3782
	$\beta = 0.0026$					
	$\lambda = 5.7755$					
	$\theta = 0.0465$					
*Beta-Norm	$\alpha = 0.0126$	-109.5108	227.0215	227.5034	236.9309	231.0138
	$\beta = 0.0064$					
	$\mu = 5.7109$					
	$\sigma = 0.0651$					

The Estimates and log-lik value of (*) were obtained from [6]

Figure 7 shows the graphical illustration of the density fit of the distributions for the Asteroid data set.

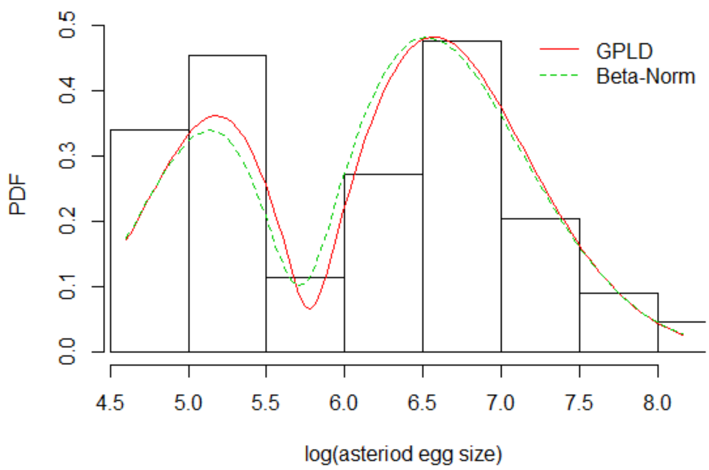

Figure 7: *Density Fit for the Asteroid Data*

5.1. Discussion of Result

A suitable model for analyzing lifetime data set can be investigated among several distributions by examining the model with the maximized log-likelihood value and the least value in terms of AIC, AICc, BIC, and HQIC. Table 4 reveals that the Gumbel-Power Lindley distribution which belongs to the T-Power Lindley$\{Y\}$ family of Distributions outperformed the beta-normal distribution in analyzing the data set and thus, can be employed as a better alternative to the beta-normal distribution in fitting real-life data exhibiting a bimodal property. This is evidently clear as the Gumbel-Power Lindley distribution has the maximized log-likelihood value and the least value in terms of the AIC, AICc, BIC, and HQIC as shown in Table 4.

6. Conclusion

A new class of generalized Lindley family of distributions with bimodal property is introduced. Sub-families of the T-Power Lindley$\{Y\}$ family based on the quantile function of the uniform, expenential, frechet, log-logistic and logistic distributions as well as some mathematical properties were derived. A bimodal data set was used to illustrate the applicability of the T-Power

Lindley$\{Y\}$ family of distributions and result obtained revealed that the GPL distribution from the proposed T-Power Lindley$\{Y\}$ family of distributions can be used as an alternative model to other existing distributions in modelling lifetime data sets.

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