PROBABILISTIC INVENTORY MODEL FOR DETERIORATING ITEMS WITH UNCERTAIN DEMAND UNDER PENTAGONAL FUZZY ENVIRONMENT

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Abstract

Using a pentagonal fuzzy framework, this research presents a probabilistic inventory model for deteriorating items under an uncertain demand. Degeneration of items puts a company's financial ability to meet its objectives at risk. Few models have synchronized optimization over this whole scenario with all components, according to a survey of the literature. It deals with the difficulties of inventory control in situations where demand is represented by fuzzy sets but is not precisely known. The model offers a clearer and more useful understanding of demand uncertainty by defuzzifying pentagonal fuzzy numbers using the Graded Mean Integration Representation (GMIR) approach. The goal of the study is to optimize inventory levels in order to minimize total costs, which include holding, degradation, shortage, and purchase. These components are included into a mathematical model, and numerical scenarios are shown to compare the both potential strategies. The sensitivity of the solution and decision variables with respect to different inventory characteristics is examined in both crisp and fuzzy settings. Fuzzy logic is integrated into the model to provide a strong framework for making decisions when dealing with ambiguous demand and the complications that come with deteriorating inventory. The paper includes numerical examples and sensitivity analyses to demonstrate the model™s effectiveness and practical relevance. These findings provide valuable guidance for inventory managers aiming to improve decision-making and operational efficiency in contexts with fuzzy demand and deteriorating products. At the optimal position, the total cost is relatively inelastic to an increase in base deterioration rate and more elastic to a decrease in it. Although the crisp example is marginally less efficient per unit cost, total costs are lower than in the fuzzy case, which is to be expected given the fuzzy case's potential for superior results.

Keywords: Probabilistic Inventory Model, Deteriorating Items, Uncertain Demand, Pentagonal Fuzzy Environment, Graded mean integration representation (GMIR).

1. Introduction

In the current situation, Showrooming and time-sensitive processes are closely related. Due to the coronavirus incident in this case, unique protocols necessitated significant modifications to the stock structure. The global health crisis has compelled businesses to rethink their current and upcoming marketing strategies in order to sustain a steady stream of revenue. Short-term effort could focus on multiple goals, such as more advanced plans or improved consumer perception. Potential benefits could also include increased employee inspiration and an increase in in-store visitors. The objective is to increase revenue and get rid of excess goods. Thus, inventory control is essential to every sophisticated, modern business. There are several benefits to having well-managed inventories, including direct profits and devoted customers. Furthermore, the intricate connections between these several business objectives uphold the astounding significance of inventory management. Because of its ability to address a wide range of problems and its mathematical methodology, the continuous review has garnered more attention than the periodic review.

Disintegration is a character that arises from natural problems during caching and is represented by degradation, harm, decay, hurt, or other changes in item quality. Items such as batteries, semiconductor chips, food assortments, unstable fluids, and therapeutic items such as blood face degeneration and gradually lose potential are a few instances. Managing and remaining cognizant of the stock framework's decomposing goods inventories is a major concern. The aim of inventory management is to increase business profits by reducing wasteful inventory and deteriorating items are a hindrance to this goal. One way to think of the rate at which products deteriorate is as a dependent variable that can be managed with protection innovation. Businesses are aware that they have to control degradation losses to the letter. Enhancing and modernizing storage procedures is one of the typical control strategies. Through feasible capital input along these channels, retailers can slow down the rate at which things deteriorate, avoiding unnecessary waste, limiting financial losses, and improving business efficiency. These degradation control models have received a great deal of attention and are more in line with the real inventory conditions. In today's volatile marketplaces, precise inventory control is especially important for perishable products. For instance, the retailer's reputation and goodwill will suffer due to food deterioration. Weakening increases the associations cost and therefore reduces the advantage, which is a major cause of stock misfortune. The weakening of interactions caused by oxidation, chemicals, and microbes frequently depends on ecological factors like environment, temperature, and stickiness. Temperature regulation is necessary to maintain product quality as it has a substantial impact on deterioration. Innovative protective measures, such temperature-regulating equipment and creative bundling, can affect the rate of weakening and postpone the crumbling cycle.

At that time, when management introduces a new product, they don't fully understand the market and other aspects of the product. The analysis of experts is trusted by the management. Fuzzy principles can be used to represent demand or other elements connected to the expert opinion when the counsel is imprecise. The resulting environment is referred to as a fuzzy environment. This work orders a new way to improve demand forecasting, which is one of the main challenges of a continuous review inventory model. The impact of fuzzy demand across an infinite time horizon is further investigated in this work. For a continuous review inventory system, the best operating strategy is looked for in order to minimize the overall layout in a fuzzy setup. It is believed that the full-backordering method will balance the loss component during the inventory shortage. The optimal policy is analyzed and both crisp and fuzzy examples of a continuous review inventory system are numerically solved. The findings provided can greatly benefit decision makers' methods in situations of uncertain demand and lower total inventory costs.

The approach is complicated by practical concerns like attenuated deterioration, demand affected by both fuzzy base and promotional upscaling. It has been suggested that analytical convexity be taken into account without deterioration factor approximation. Because of the infinite Time horizon, the continuous domain for cycle length leads to continuous time duration Variables. Given these factors, the study's uniqueness and significance are that it contrasts an inventory model for ongoing evaluation, it enables an inventory model to perform a thorough exponential depreciation attenuation analysis. With a pentagonal fuzzy foundation and a term that scales with promotion efforts, demand uncertainty is modelled in a more realistic scheme for which the retention approach is intended.

Forecasting demand patterns is a prerequisite for the a priori planning decision-making process for retailers and sellers. Flexibility in resource and operation management can enhance the arrangement's overall performance. The first model, which follows, is a useful mathematical prototype and employs the pure deterministic situation in which the decision-maker has access to precise demand information. The second model, which employs a fuzzy formulation for the imprecisely forecasted demand, goes back to the first claim of flexibility in addressing uncertainty. When combined, the two models give the researcher a clear and succinct understanding of the mathematical process and the financial rationale for using fuzziness to address uncertainty and degradation, respectively.

The current article's organization follows the following structure: Section 2 has the literature review, which reviews past research that provides context for this work. Section 3 includes the research question and the presumptions used to help plan the model's layout. The utilized notations are tallied. The modelling approach and solutions for the environments that are both crisp and fuzzy are covered in Section 4.In the following, numerical examples are used to illustrate how this paradigm can be applied in a real-world scenario Section 5. Section 6 contains the administrative architecture and sensitivity analysis for the inventory systems. In Section 7, the research's conclusions and future directions are examined.

2. Literature review

2.1. Deterioration

Inventory management has extensively studied Deterioration. Food rotting due to oxidation or microorganisms is a common occurrence. Storage of electronic items must take into consideration contamination, moisture, and electrostatic discharge damage. Pervin et al. [1] established an EOQ model for perishable commodities, taking into account time-dependent holding costs and demand that fluctuated with stock level. Pervin et al. [2] created a multi-item inventory model that considered constant rate of deterioration, on-demand, and trade credits. Barman et al. [3] examined an economic production quantity (EPQ) model in a fuzzy environment with shortages and inflation, with time-dependent demand and a fixed rate of deterioration. Roy et al. [4] created a probabilistic system for decaying items with two warehouses, two credit levels. Roy et al. [5] suggested a credit strategy for a deteriorating product and an imperfect production system with a partial backlog. Khan et al. [6] examined a system that had a variable demand pattern, constant deterioration, and delayed payment. Currently, Shah et al. [7] examined the situation where products demand and deteriorate varies according to selling price while taking the greening effect into account. Yadav et al. [56] optimize an inventory model for deteriorating items using a two-warehouse system, highlighting the need to balance cost and efficiency in managing perishable goods. Yadav, K.K., Yadav, A.S., and Bansal, S. [57] employ an interval number technique to enhance two-warehouse inventory management while considering preservation technology investments, demonstrating the advantages of resource allocation for cost savings and efficiency. Yadav, A.S., Kumar, A., and Yadav, K.K. [58] present a model that incorporates carbon emissions and time-sensitive demand in optimizing inventory for deteriorating items, focusing on sustainable management practices. Mahata and Debnath [59] tackle a profit-maximizing problem in single-item inventory management by considering price-dependent demand and preservation technologies, illustrating the synergy between preservation strategies and demand dynamics to improve profitability and efficiency.

Researchers have been interested in the difficult task of managing inventory items that are naturally decaying for decades. Any industry that experiences Deterioration suffers financial losses as a result of this phenomena. It is, nevertheless, a normal and inevitable procedure. Therefore, during operations management, strategic choices to avoid the aforementioned loss and its impact have generated a lot of attention in a variety of real-life situations. Ghare and Schrader [8] conducted the first study using degrading objects for exponentially deteriorating items. Subsequently, using the discounted cash flow (DCF) method, Jaggi and Aggarwal [9] investigated the best ordering strategy for deteriorating goods while assumption trade credit. Aggarwal and Jaggi [10] investigated the best ordering policy for deteriorating commodities based on the allowable payment delay. The optimum credit policies for degrading things were recently shown by Jaggi et al. [11] under the presumptions of faulty items, rapidly expanding demand and partial backlog. Additionally, Mandal et al. [12] postulated an inventory model based on geometric programming that included deteriorating items. Afterwards, Panda et al. [13] developed an inventory model for a seasonal product with ramp-type demand. Bakker et al. [14] provides a thorough analysis of inventory models with deteriorating items. Khanna et al. [15] have produced some excellent work on deterioration that is worthy of attention in this context. Additionally, Jaggi et al. [16] developed the best course of action for defective and deteriorating items while taking into account a two-warehouse situation. Controllable probabilistic deterioration with shortages was examined by Mishra [17]. Jaggi et al. [18] also studied at price-dependent demand and two warehouses in non-instantaneous deterioration. A replenishing scenario for a deteriorating item was examined by Pervin et al. [19] under the presumptions of time-dependent holding costs, time-dependent demand, and shortages. A sustainable three-tier inventory model for decaying products was studied by Daryanto et al. [20]. Shaw et al. [21] studied an integrated model that took into account multi-stage inspection, carbon emissions, and deterioration while putting single setup multi delivery (SSMD) policy into practice for the delivery of high-quality products.

2.2. Probabilistic demand

Given the current state of the market, it is becoming more difficult to precisely estimate client preferences for a product; therefore, a probabilistic demand method is a better fit for handling uncertainty. Shah, Nita H. [22] developed a probabilistic inventory model with allowable payment delays, thereby spearheading the development of such models. On the basis of this, Shah, Nita H., and Y. K. Shah. [23] Expanded the model to include trade credit policy and declining products over a specific time interval. Several scholars, such as Shah, Nita H. [24] and Shah, Nita H. [25], developed inventory models that included trade credit finance, probabilistic demand, and shortages. Federgruen, Awi and Aliza Heching [26] Looked into simultaneous inventory and pricing decisions under probabilistic demand. Petruzzi, Nicholas C. and Maqbool Dada [27] developed a price-sensitive inventory model and perishable goods into account to determine the best pricing in the newsvendor scenario. Chen, Xin, and David Simchi-Levi [28] examined a periodic review model for an infinite planning horizon in order to determine the best pricing and inventory strategies with probabilistic demand. Under probabilistic demand, Khedlekar et.al. [29] Examined the optimal replenishment choices taking into account pricing, promotion tactics, and inventory. Chao, Xiuli, Baimei Yang, and Yifan Xu [30] to maximize pricing, a capacitated probabilistic inventory system was proposed. Maihami, Reza, and Behrooz Karimi [31] developed an optimal replenishment plan that takes into account promotional efforts for non-instantaneously deteriorating products with price-sensitive probabilistic demand. Inventory control methods under probabilistic demand were provided by Roy et al. [32] and AlDurgam et al. [33] with a range of parameters and assumptions. Probability distributions are typically used to reflect demand uncertainty. With price-sensitive probabilistic demand and non-instantaneous deteriorating products, they created an inventory model for profit maximization that took additive promotional activities into account. The bulk of research is done on probabilistic demand functions that are sensitive to price. Shah, Nita H., et al. [34] for a full view, consider a demand rate that is uniformly distributed and is influenced by price, inventory level, and advertisement.

2.3. Pentagonal Fuzzy Number

There are several varieties of polygon fuzzy numbers in classical fuzzy theory, including trapezoidal, pentagonal, and triangular fuzzy numbers, among others. Srinivasan [35] provided a method for solving TP using generalized pentagonal and hexagonal fuzzy numbers throughout the literature. Additionally, Karthikeyan [36] provided a method for using pentagonal fuzzy numbers to solve transportation problems (TP). Maheswari and Ganesan [37] provide a technique that uses pentagonal fuzzy numbers to solve completely fuzzy TP. Chakraborty [38] has investigated representations of pentagonal fuzzy numbers. Barazandeh and Ghazanfari have tackled the ranking approach for generalized fuzzy numbers [39]. Membership functions for symmetric and asymmetric hexagonal fuzzy numbers: an overview was extended to non-linear membership functions by Khan and Mondal [40]. Mondal [41] used the average technique to find arithmetic operations and provided representations for a variety of non-linear membership functions. Arora, Aparna, Rashmi Gupta, and Ratnesh Rajan Saxena [42] Asymmetric Pentagonal Fuzzy Numbers as a representation of costs (APFN).

2.4. Fuzzy modeling of uncertainty

Demand uncertainty arises from several unknown components of inventory models, aside from preservation technologies, deteriorating commodities, and promotional efforts. However, in practical circumstances, the uncertain parameters such as lead time, preservation cost, demand, and other pertinent expenses may be more likely to deviate from the exact value, which could result in a situation where the uncertain parameters are not distributed according to any probability. Originally, the fuzzy set concept was developed by Zadeh [43]. Following that, a number of trailblazing scholars developed several fuzzy inventory models to capture the impreciseness, including Yao et al. [44], Glock et al. [45], and Shah and Soni [46]. By taking into account trapezoidal fuzzy numbers, the model examined by Garai et al. [47] had holding costs that scaled with price-dependent and time demand. Shah and Patel [48] created an inventory model and employed preservation technologies to lower the rate of spoiling under a cloud hazy prescription. In a fuzzy setting, Yadav et al. [49] examined a flexible manufacturing system with a changeable pollution control. De and Mahata [50] used a learning environment for dense fuzzy demand when there was an order overlap with rework batches. Kumar and Paikray [51] modelled the time-varying demand for decaying commodities using crisp and fuzzy formulations with three distinct scenarios under total backlog. To effectively address a fuzzy inflationary model, Sarkar et al. [52] used a multithreaded neural network. Fuzzy logic, specifically graded mean integration-representation distance, is used by this similarity function (GMIR). We can incorporate more flexible data agglomeration strategies thanks to fuzzy logic [53]. This model used a pentagonal fuzzy number with the GMIR difuzyfication method.

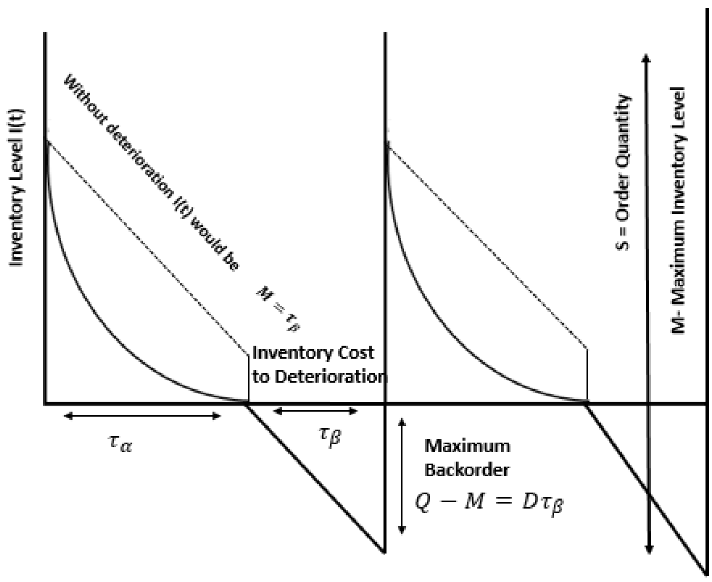

Figure 1: *Inventory Model with backorder and deterioration*

3. Assumptions and notations

3.1. Assumptions

1. The inventory system is examined using a single item.
2. An infinite planning horizon is taken into consideration.
3. Additive price-sensitive probabilistic demand $D = D_0 + \epsilon$ where D_0 is fixed base demand while ϵ continuous random variable with expected value μ.
4. Shortages are allowed, and lead time is zero.
5. The rate of production exceeding the rate of demand.

3.2. Notations

Table 1: *Symbols and Description*

Symbol	Description
S	Order quantity per unit time
D	Rate of demand per unit of time
A	Ordering cost per order ($)
K	Holding cost per order ($)
L	Deteriorating cost per order ($)
m	Shortage cost per order ($)
α_0	constant rate of inventory item deterioration $(0 < \alpha_0 < 1)$
α	Effective deterioration rate, dampened by use of $(\alpha = \alpha_0 e^{-\mu})$
μ	Expected value of Continuous random variable ϵ
M	Maximum positive inventory level´ at time $t = 0$
$I(t)$	Inventory level at the time
τ_α	The duration of time it takes for inventory to zero following replenishing
τ_β	Order backlog occurs in the interval between having zero inventory and replenishing it (represented by negative inventory level)
$TAC(\tau_\alpha, \tau_\beta)$	Total average cost of inventory (Model-1)($/per unit time)
$\widetilde{TAC}(\tau_\alpha, \tau_\beta)$	Total average cost for fuzzy environment (Model-2)($/per unit time)

4. Mathematical Model

This section lays out the models and their approach for solving them. The total cost function is obtained by setting up and solving the governing differential equations. This objective function is subjected to necessary and sufficient criteria for convexity and global optimality. When fuzzy parameters are utilized, defuzzification is applied.

4.1. A model for continuous review inventory that has constant deterioration rate and crisp demand (Model-1)

A continual evaluation Based on the aforementioned presumptions, the EOQ setup is created. An immediate restocking initially the cycle at $t = 0$ and the inventory level surges to its highest point, $M = I(0)$. $I(t)$ decreases in time interval $[0, \tau_\alpha]$ as certain components are lost to degradation and others are consumed according to demand. Every unit has been utilized at $t = \tau_\alpha$ So $I(\tau_\alpha) = 0$. Backorders are maintained for the duration of $[\tau_\alpha, \tau_{\beta+\alpha}]$ this must be satisfied from the upcoming replenishment due to their complete backlogged. These presumptions allow the differential equations controlling the subsequent cases to be determined:

Case 1 $(0 \leq t \leq \tau_\alpha)$:Inventory is depleted by loss from degradation and consumption brought on by demand; consequently, the inventory level equation $I(t)$:

$$\frac{dI(t)}{dt} + \alpha I(t) = -D; \quad 0 \leq t \leq \tau_\alpha \tag{1}$$

The deterioration term $\alpha(t)$ is in parallel to the current inventory that is on hand and $\alpha = \alpha_0$ is constant in Model 1. On the differential equation (1), the boundary condition $I(\alpha_0) = 0$is used to determine the inventory level.

$$\frac{dI(t)}{dt} = \frac{D}{\alpha_0}(e^{\alpha_0(\tau_\alpha - t)} - 1); \quad 0 \leq t \leq \tau_\alpha \tag{2}$$

Using equation (2) the topest inventory level M is at $t = 0$ in the following manner:

$$M = I(0) = \frac{D}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - 1) \tag{3}$$

Now, equation (3) is distinct from the absence of deterioration where $M = D\tau_\alpha$. The difference between the two provides the amount of inventory lost as a result of deterioration, in the buyer's inventory model that was examined in Wee et al. [54], given by

$$M - D\tau_\alpha = \frac{D}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1) \tag{4}$$

Case 2 $(\tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta)$: Backorders resulting from a negative inventory level $I(t)$, as well as shortages during the period, should be taken into account. There is only one term, as orders are wholly backlogged, which is due to demand. Consequently, the differential equations that follow

$$\frac{dI(t)}{dt} = -D; \quad \tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta \tag{5}$$

Now, applying boundary condition $I(\tau_\alpha) = 0$; equation (5) results in the following inventory level expression:

$$I(t) = -D(t - \tau_\alpha); \quad \tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta \tag{6}$$

In fact, this negative inventory level suggests that the backorder at that point in time t in $(\tau_\alpha \leq t \leq \tau_\alpha + \tau_\beta)$ is $D(t - \tau_\alpha)$. Here $I(\tau_\alpha + \tau_\beta) = -D\tau_\beta$ is called lowest inventory level and the maximum backorder is $D\tau_\beta$. The maximum inventory level is provided by the leftover inventory after ordered amount S completes this backorder first.

$$S - D\tau_\beta = M \Rightarrow S = D\tau_\beta + \frac{D}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - 1) \tag{7}$$

In equation (7), the order quantity is higher than in the traditional backorder approach and does not deterioration $D(\tau_\alpha + \tau_\beta)$, since it must meet demand in addition to replacing products lost to deterioration. Consequently, the components of the total inventory cost are as follows:

Ordering Cost (OC)

$$OC = A$$

Holding Cost (HC):

$$HC = \int_0^{\tau_\alpha} \frac{DK}{\alpha_0}(e^{\alpha_0(\tau_\alpha - t)} - 1)dt = \frac{DK}{\alpha_0^2}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)$$

Shortage Cost (SC):

$$SC = \int_{\tau_\alpha}^{\tau_\alpha + \tau_\beta} (-I(t))sdt = \int_{\tau_\alpha}^{\tau_\alpha + \tau_\beta} D(t - \tau_\alpha)sdt = \frac{\tau_\beta^2 Dm}{2}$$

Cost as a result of the deteriorated goods (w is the cost of deterioration per unit) (DC):

$$DC = (M - D\tau_\alpha)w = \frac{DL}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)$$

As a result, the total inventory cost every cycle, taking into account expenditures associated with deterioration but not preservation, is as follows:

$$TAC(\tau_\alpha, \tau_\beta) = \frac{1}{\tau_\beta + \tau_\alpha}[OC + HC + SC + DC]$$

$$TAC(\tau_\alpha, \tau_\beta) = \frac{1}{\tau_\beta + \tau_\alpha}\left[A + \frac{DK}{\alpha_0^2}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1) + \frac{\tau_\beta^2 Dm}{2} + \frac{DL}{\alpha_0}(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)\right]$$

$$TAC(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2(\tau_\beta + \tau_\alpha)} + \frac{\tau_\beta^2 Dm}{2(\tau_\beta + \tau_\alpha)} \tag{8}$$

5. Optimization methodology

The decision variable values are obtained at the lowest total cost per cycle ($) through the classical optimization process. There are two components to the computational process.

Step 1: Obtain critical point $(\tau_\alpha^*, \tau_\beta^*)$ satisfying $\frac{\partial TAC}{\partial \tau_\alpha} = 0$ and $\frac{\partial TAC}{\partial \tau_\beta} = 0$.

Step 2: Verify the convexity of $TAC(\tau_\alpha, \tau_\beta)$ by proving that (in the feasible region)

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} > 0$$

and

$$\frac{\partial^2 TAC}{\partial \tau_\alpha^2} . \frac{\partial^2 TAC}{\partial \tau_\beta^2} - \left[\frac{\partial^2 TAC}{\partial \tau_\alpha \partial \tau_\beta}\right]^2 > 0$$

The factors that do not contain the decision variables in equation (8) are gathered and categorized for the sake of algebraic efficiency.

$$TAC(\tau_\alpha, \tau_\beta) = \frac{A_1}{\tau_\beta + \tau_\alpha} + \frac{B_1 \tau_\beta^2}{(\tau_\beta + \tau_\alpha)} + \frac{C_1(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{(\tau_\beta + \tau_\alpha)} \tag{9}$$

where, $A_1 = A$, $B_1 = \frac{Dm}{2}$, $C_1 = \frac{D(K+\theta_0 L)}{\alpha_0^2}$ The positive inventory time τ_α and negative inventory time τ_β cause a continuous fluctuation in the total cost per time unit. The decision variables τ_α and τ_β are employed to minimize this objective function. The first-order partial derivatives of $TAC(\tau_\alpha, \tau_\beta)$ are determined by equation (9).

$$\frac{\partial TAC}{\partial \tau_\alpha} = \frac{\alpha_0 C_1(e^{\alpha_0 \tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)} - \frac{TAC}{\tau_\beta + \tau_\alpha} \tag{10}$$

$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{2B_1 \tau_\beta}{(\tau_\beta + \tau_\alpha)} - \frac{TAC}{\tau_\beta + \tau_\alpha} \tag{11}$$

The second-order partial derivatives of $TAC(\tau_\alpha, \tau_\beta)$ are

$$\frac{\partial^2 TAC}{\partial \tau_\alpha^2} = \frac{2TAC}{(\tau_\beta + \tau_\alpha)^2} - \frac{2\alpha_0 C_1(e^{\alpha_0 \tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)^2} + \frac{\alpha_0^2 C_1 e^{\alpha_0 \tau_\alpha}}{\tau_\beta + \tau_\alpha} \tag{12}$$

$$\frac{\partial^2 TAC}{\partial \tau_\alpha \tau_\beta} = \frac{2TAC}{(\tau_\beta + \tau_\alpha)^2} - \frac{\theta_0 C_1(e^{\alpha_0 \tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)^2} - \frac{2\beta_1 \tau_\beta}{(\tau_\beta + \tau_\alpha)^2} \tag{13}$$

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} = \frac{2TAC}{(\tau_\beta + \tau_\alpha)^2} - \frac{4B_1 \tau_\beta}{(\tau_\beta + \tau_\alpha)^2} + \frac{2B_1}{\tau_\beta + \tau_\alpha} \tag{14}$$

As $e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1 > 0 \; \forall \; \alpha_0 \tau_\alpha > 0$,
Hence equation (9) gives

$$TAC(\tau_\alpha, \tau_\beta) > \frac{B_1 \tau_\beta^2}{\tau_\beta + \tau_\alpha} \; \forall \; \alpha_0 \tau_\alpha > 0$$

Using this inequality in equation (14), which becomes

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} > \frac{2B_1 \tau_\beta^2}{(\tau_\beta + \tau_\alpha)^3} + \frac{2B_1(\tau_\alpha - \tau_\beta)}{(\tau_\beta + \tau_\alpha)^2} = \frac{2B_1 \tau_\alpha^2}{(\tau_\beta + \tau_\alpha)^3} > 0 \; \forall \; \alpha_0 \tau_\alpha > 0 \tag{15}$$

To be necessary for the objective function to achieve minimum cost, the first order partial derivative must be zero (Step-1above). The optimal solution is achieved by setting these partial derivatives to zero when the sufficient conditions (Step-2 above) are satisfied.

$$\frac{\partial TAC}{\partial \tau_\alpha} = 0$$

and

$$\frac{\partial TAC}{\partial \tau_\beta} = 0$$

Additionally, sufficient circumstances must be fulfilled for certain optimality. From now on, the equivalent fundamental minors ought to be in the positive definite. The Hessian determinant is

$$H(\tau_\alpha, \tau_\beta) = \frac{\partial^2 TAC}{\partial \tau_\alpha^2} . \frac{\partial^2 TAC}{\partial \tau_\beta^2} - \left[\frac{\partial^2 TAC}{\partial \tau_\alpha \partial \tau_\beta} \right]^2$$

From equations (12), (13) and (14), we get

$$(\tau_\beta + \tau_\alpha)^4 H(\tau_\alpha, \tau_\beta) = 2A_1(2B_1 + C_1 \alpha_0^2 e^{\alpha_0 \tau_\alpha}) + 2C_1 B_1 (e^{\alpha_0 \tau_\alpha}(1 - \alpha_0 \tau_\alpha)^2 + e^{\alpha_0 \tau_\alpha} - 2) + \alpha_0^2 C_1^2 (e^{2\alpha_0 \tau_\alpha} - 2\alpha_0 \tau_\alpha e^{\alpha_0 \tau_\alpha} - 1) \tag{16}$$

simplifying $e^{\alpha_0 \tau_\alpha}(1 - \alpha_0 \tau_\alpha)^2 + e^{\alpha_0 \tau_\alpha} - 2$ and $e^{2\alpha_0 \tau_\alpha} - 2\alpha_0 \tau_\alpha e^{\alpha_0 \tau_\alpha} - 1$.

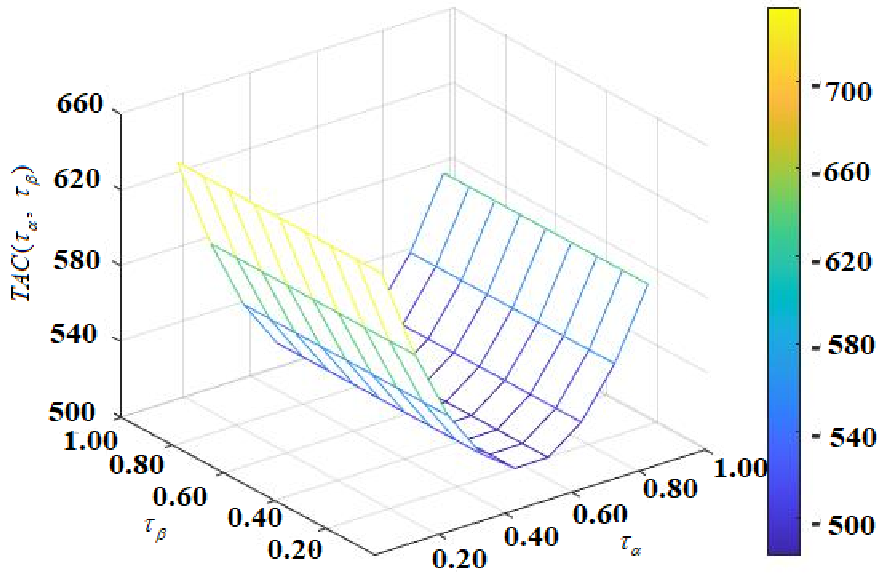

Figure 2: *Convexity objective function (Model-1)*

$$(\tau_\beta + \tau_\alpha)^4 H(\tau_\alpha, \tau_\beta) > 2A_1(2B_1 + C_1\alpha_0^2 e^{\alpha_0\tau_\alpha}) + C_1B_1(\alpha_0\tau_\alpha)^4 + \frac{C_1^2\alpha_0^2(\alpha_0\tau_\alpha)^4}{4}$$

$$\rightarrow H(\tau_\alpha, \tau_\beta) > 0 \forall \alpha_0\tau_\alpha > 0 \tag{17}$$

The Hessian implied by equations (15) and (17) is positive definite. Consequently, the cost function $TAC(\tau_\alpha, \tau_\beta)$ is convex and Figure 2 provides more dramatic evidence of this tendency. From equation (10) and (11), within the feasible region, it possesses a unique global minima. at $(\tau_\alpha^*, \tau_\beta^*)$ fulfilling (Critical point)

$$\frac{\partial TAC}{\partial \tau_\alpha} = \frac{\alpha_0 C_1(e^{\alpha_0\tau_\alpha} - 1)}{(\tau_\beta + \tau_\alpha)} - \frac{TAC}{\tau_\beta + \tau_\alpha} = 0 \tag{18}$$

$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{2B_1\tau_\beta}{(\tau_\beta + \tau_\alpha)} - \frac{TAC}{\tau_\beta + \tau_\alpha} = 0 \tag{19}$$

equation (18) and (19)

$$TAC(\tau_\alpha^*, \tau_\beta^*) = \alpha_0 C_1(e^{\alpha_0\tau_\alpha^*} - 1) = 2B_1\tau_\beta^* \tag{20}$$

Hence, equations (9) and (20) give

$$2B_1\tau_\beta^* = \frac{A_1}{\tau_\beta^* + \tau_\alpha^*} + \frac{2B_1\tau_\beta^{2*}}{\tau_\beta^* + \tau_\alpha^*} + \frac{C_1(e^{\alpha_0\tau_\alpha^*} - \alpha_0\tau_\alpha^* - 1)}{\tau_\beta^* + \tau_\alpha^*} \tag{21}$$

Simplifying equation (21), one can get

$$\tau_\beta^* = \sqrt{\tau_\alpha^{*2} + \frac{A_1 + C_1(e^{\alpha_0\tau_\alpha^*} - \alpha_0\tau_\alpha^* - 1)}{B_1}} - \tau_\alpha^* \tag{22}$$

Equation (20) implies

$$\tau_\alpha^* = \frac{1}{\alpha_0} log\left\{\frac{2B_1\tau_\beta^*}{\alpha_0 C_1} + 1\right\} \tag{23}$$

Equations (22) and (23) are numerically solved iteratively to produce the appropriate values of τ_β^* and τ_α^* for optimization.The economic order quantity and the minimum total cost per unit time are found using equations (20) and (7).

$$TAC(\tau_\alpha^*, \tau_\beta^*) = 2B_1\tau_\beta^*, S^* = D\tau_\beta^* + \frac{D(e^{\alpha_0\tau_\alpha^*} - \alpha_0\tau_\alpha^* - 1)}{\alpha_0} \tag{24}$$

Equation (4) yields the effective rate of loss, which is the average deterioration per unit of time across a cycle.$= \frac{D(e^{\alpha_0\tau_\alpha^*} - \alpha_0\tau_\alpha^* - 1)}{\alpha_0(\tau_\beta + \tau_\alpha)} = \frac{S^*}{\tau_\beta + \tau_\alpha} - D$.

6. A continuous review inventory model with fuzzy environment (Model-2)

Demand has been taken in the form $D = D_0 + \epsilon$. This subsection examines fuzzy demand where D_0 a pentagonal fuzzy number, i.e. $\widetilde{D_0} = (D_0 - \delta_2, D_0 - \delta_1, D_0^{'}, D_0 + v_1, D_0 + v_2)$. This improves the modelling of real-world scenarios' flexibility.
The function principle in this article and Graded Mean Integration Representation (GMIR) method are considered. Currently, the membership function of $\widetilde{D_0}$ is the following:

$$\eta_{\widetilde{D_0}}(x) = \begin{cases} 0 & for\ x < D_0 - \delta_1,\ a_5 \leq x \\ \frac{x-(D_0-\delta_1)}{(D_0-\delta_2)-(D_0-\delta_1)} & for\ D_0 - \delta_1 \leq x \leq D_0 - \delta_2 \\ 1 & for\ D_0 - \delta_2 \leq x \leq D_0^{'} \\ \frac{(D_0+v_1)-x}{(D_0+v_1)-D_0^{'}} & for\ D_0^{'} \leq x \leq D_0 + v_1 \\ \frac{(D_0+v_2)-x}{(D_0+v_2)-(D_0+v_1)} & for\ D_0 + v_1 \leq x \leq D_0 + v_2 \end{cases} \tag{25}$$

This fuzzy demand changes equation (8) to

$$\widetilde{TAC}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{\widetilde{D_0}}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \tag{26}$$

Next, using the function principle (Mahata and Goswami [55]), given that the demand's basic value is PeFN. The total cost per unit time becomes a PeFN, as seen in Figure, according to the preceding expression.

Real valued functions themselves make up this PeFN's parameters. Regarding any feasible values of $(\tau_\alpha, \tau_\beta)$ the following is true:

$$TAC_{\delta 1}(\tau_\alpha, \tau_\beta) \leq TAC_{\delta 2}(\tau_\alpha, \tau_\beta) \leq TAC_{\widetilde{D_0}}(\tau_\alpha, \tau_\beta) \leq TAC_{v1}(\tau_\alpha, \tau_\beta) \leq TAC_{v2}(\tau_\alpha, \tau_\beta)$$

$$\widetilde{TAC}(\tau_\alpha, \tau_\beta) = PeFN\Big(TAC_{\delta 1}, TAC_{\delta 2}, TAC_{\widetilde{D_0}}, TAC_{v1}, TAC_{v2}\Big)$$

where

$$TAC_{\delta_1}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - \delta_1}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{\delta_2}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - \delta_2}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{\widetilde{D_0}}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{\widetilde{D_0}}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{v_1}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - v}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

$$TAC_{v_2}(\tau_\alpha, \tau_\beta) = \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0 - v_2}{(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right]$$

The total cost per unit time in Model-2 is estimated by using the median calculation.

$$Median(\widetilde{TAC}(\tau_\alpha, \tau_\beta)) = \frac{1}{5}\Big[TAC_{\delta_1}(\tau_\alpha, \tau_\beta) + TAC_{\delta_2}(\tau_\alpha, \tau_\beta) + TAC_{\widetilde{D_0}}(\tau_\alpha, \tau_\beta) + TAC_{v_1}(\tau_\alpha, \tau_\beta) + TAC_{v_2}(\tau_\alpha, \tau_\beta)\Big]$$

$$\begin{aligned} Median(\widetilde{TAC}(\tau_\alpha, \tau_\beta)) = {} & \frac{A}{\tau_\beta + \tau_\alpha} + \frac{D_0}{4(\tau_\beta + \tau_\alpha)}\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \\ & + \left(\frac{D_0^{'}}{(\tau_\beta + \tau_\alpha)} + \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)}\right)\left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2}\right] \end{aligned} \tag{27}$$

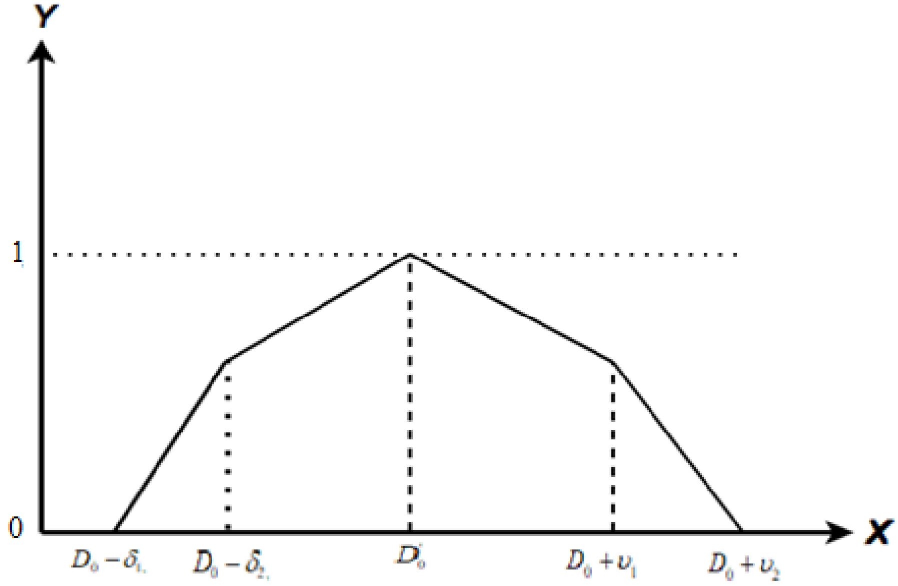

Figure 3: *Demand parameter and objective function*

As a result, the total of the two components is the median estimate of the total cost per unit time for the fuzzy demand model. The first term is the same as Model's total cost per unit time (crisp demand).

$$Median(\widetilde{TAC}(\tau_\alpha, \tau_\beta)) = TAC(\tau_\alpha, \tau_\beta) + FC(\tau_\alpha, \tau_\beta)$$

Where

$$FC(\tau_\alpha, \tau_\beta) = \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2} \right] \tag{28}$$

7. Optimization methodology

The convexity of $FC(\tau_\alpha, \tau_\beta)$ as it is defined by equation (28) is examined.
The first-order partial derivatives of $FC(\tau_\alpha, \tau_\beta)$ with respect to τ_α and τ_β are as follows:

$$\frac{\partial FC}{\partial \tau_\alpha} = \frac{-(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2} \right] + \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - 1)}{\alpha_0} \right]$$

$$\frac{\partial FC}{\partial \tau_\beta} = \frac{-(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2} \right] + \frac{2\tau_\beta m(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)}$$

The second-order partial derivatives of $FC(\tau_\alpha, \tau_\beta)$ with respect to τ_α and τ_β are as follows:

$$\begin{aligned} \frac{\partial^2 FC}{\partial \tau_\alpha^2} &= \frac{2(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^3} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - \alpha_0 \tau_\alpha - 1)}{\alpha_0^2} + \frac{\tau_\beta^2 Dm}{2} \right] \\ &- \frac{3(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - 1)}{\alpha_0} \right] - \frac{(v_1 - \delta_1 + v_2 - \delta_2)}{5(\tau_\beta + \tau_\alpha)^2} \left[\frac{(K + \alpha_0 L)(e^{\alpha_0 \tau_\alpha} - 1)}{\alpha_0} \right] \end{aligned} \tag{29}$$

$$\frac{\partial^2 FC}{\partial\tau_\alpha\partial\tau_\beta}=\frac{2(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^3}\left[\frac{(K+\alpha_0L)(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)}{\alpha_0^2}+\frac{\tau_\beta^2Dm}{2}\right]-\frac{(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^2}\left[\frac{(K+\alpha_0L)(e^{\alpha_0\tau_\alpha}-1)}{\alpha_0}+\tau_\beta m\right] \quad (30)$$

$$\frac{\partial^2 FC}{\partial\tau_\beta^2}=\frac{2(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^3}\left[\frac{(K+\alpha_0L)(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)}{\alpha_0^2}+\frac{\tau_\beta^2Dm}{2}\right]+\frac{m(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)}\left[2\tau_\beta-\frac{\tau_\beta}{(\tau_\beta+\tau_\alpha)}+1\right] \quad (31)$$

The ineqality $(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)>0\ \forall\ \alpha_0\tau_\alpha>0$ gives
So equation (31)

$$\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\beta^2}>\frac{2(v_1-\delta_1+v_2-\delta_2)}{5(\tau_\beta+\tau_\alpha)^3}\left[\frac{(K+\alpha_0L)(e^{\alpha_0\tau_\alpha}-\alpha_0\tau_\alpha-1)}{\alpha_0^2}+\frac{\tau_\beta^2Dm}{2}\right]>0 \quad (32)$$

The Hessian determinant of $FC(\tau_\alpha,\tau_\beta)$ is $H(\tau_\alpha,\tau_\beta)=\left(\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\alpha^2}\right).\left(\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\beta^2}\right)-\left(\frac{\partial^2 FC(\tau_\alpha,\tau_\beta)}{\partial\tau_\alpha\partial\tau_\beta}\right)^2$
Substituting the values for the second order partial derivatives from equations (29), (30), and (31)

$$(\tau_\beta+\tau_\alpha)^6H(\tau_\alpha,\tau_\beta)=2(e^{2\alpha_0\tau_\alpha}+\alpha_0^2\tau_\alpha^2+1-2\alpha_0\tau_\alpha e^{\alpha_0\tau_\alpha}+\alpha_0\tau_\alpha-2e^{\alpha_0\tau_\alpha})-(e^{2\alpha_0\tau_\alpha}-2e^{\alpha_0\tau_\alpha}+1) \quad (33)$$

The iteration scheme for optimization of $Median(\widetilde{TAC}(\tau_\alpha,\tau_\beta))$ is as follows, derived from these three equations:

$$\tau_{f\alpha}=\frac{1}{\theta_f}log\left\{\frac{\theta_f\tau_{f\beta}m}{K+L\theta_f}+1\right\} \quad (34)$$

$$\tau_{f\beta}=\left[\tau_{f\alpha}^2+\frac{10A}{m(5D+(v_1-\delta_1+v_2-\delta_2))}+\frac{2(K+\alpha_fL)(e^{\alpha_f\tau_{f\alpha}}-\alpha_f\tau_{f\alpha}-1)}{m\alpha_f^2}\right]^{\frac{1}{2}}-\tau_{f\alpha} \quad (35)$$

The Model-2 optimal outcomes. The optimal TAC and S for Model-2 are

$$S_f^*=\left(D+\frac{(v_1-\delta_1+v_2-\delta_2)}{4}\right)\left[\tau_{f\beta}^*+\frac{(e^{\alpha_f^*\tau_f^*}-1)}{\alpha_f^*}\right] \quad (36)$$

$$Median(\widetilde{TAC}_f^*)=m\tau_{f\beta}^*\left(D+\frac{(v_1-\delta_1+v_2-\delta_2)}{4}\right) \quad (37)$$

8. Numerical Analysis

The models built are demonstrated with a numerical example. By examining the outcomes, a decision-maker can gain insightful information. The parameters' numerical values are given below: $A=75$, $K==0.4$, $D_0=950$, $\epsilon=10$, $L=5.5$, $m=3$, $\delta_1=100$,$\delta_2=75$, $v_1=150$, $v_2=200$ and $\alpha_0=0.16$. The deterioration rate as $(\alpha=\alpha_0e^{-\mu})$ where the positive parameter μ is an Expected value of Continuous random variable ϵ. And value of $\mu=0.03$ in this article. The decision variables' values include the amount of time it takes for inventory to reach zero following replenishment τ_α, the interval between having no inventory and having a fully back-logged replenishment $\tau_\beta$for Model-1 and Model-2 and table 2 tabulates the corresponding order quantity and total cost at the optimal position. The column $\frac{S^*}{\tau_\alpha^*+\tau_\beta^*}-D$ provides the effective rate of deterioration as the number of units lost per unit time across a cycle. The cost per unit and time per unit at the optimal point are shown in Table 3 is $\frac{TAC^*}{S^*}=0.886, 0.723$ for Models-1 and

2, respectively. This demonstrates that, although having a little higher TAC^* than model-1, the fuzzy formulation (Model-3) is the most efficient on a per unit basis.
Figure 4 represents the variation in TAC^* for numerous cycle lengths $\tau_\alpha^* + \tau_\beta^*$. It displays two separate stages. For very short cycle durations in the first phase, the influence of deterioration is not significant, and the findings produced by both models are comparable. The effects of deterioration and the economics of preservation become evident as cycle length grows. In the second stage, Model-1's overall cost increases much over its global minimum at $(\tau_\alpha^* + \tau_\beta^* = 0.52)$ but at longer cycle lengths around $(\tau_\alpha^* + \tau_\beta^* = 0.71)$ the other model it achieves lower global optimal costs. As cycle length increases, the change in TAC^* Model-2 is much more gradual.

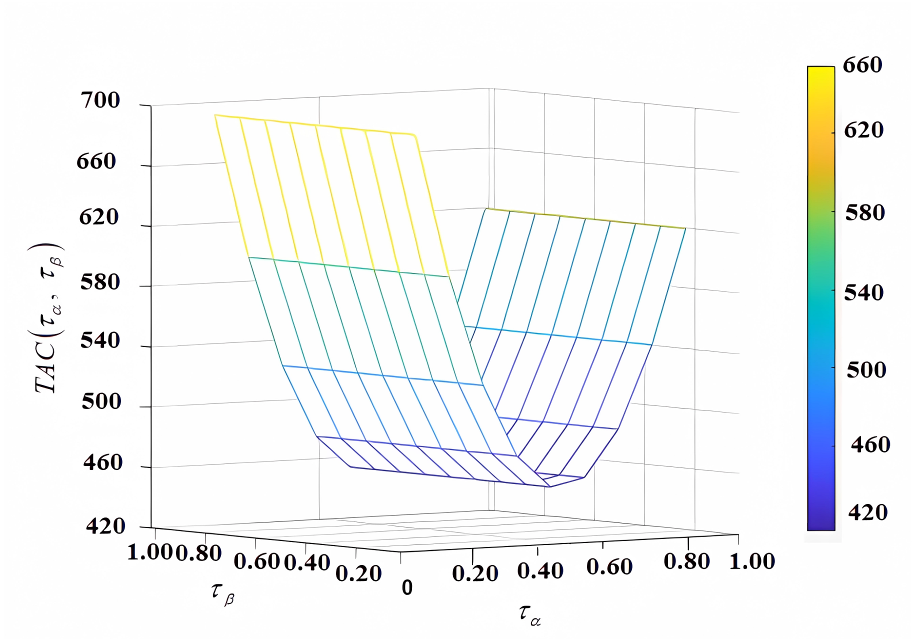

Figure 4: *Objective function convexity (Model-2)*

Table 2: Optimal solutions of Model-1 and Model-2.

Model	τ_α^*	τ_β^*	S^*	$\frac{S^*}{\tau_\alpha^*+\tau_\beta^*} - D$	$TAC(\tau_\alpha^*, \tau_\beta^*)$
Model-1	0.36	0.15	520.87	61.31	516.38
Model-2	0.61	0.14	705.56	33.74	451.37

9. Sensitivity analysis

1. **Effect of unit shortage cost (m):** With a decrease in shortage cost m, the backorder phase τ_α lasts longer and TAC is slightly reduced. Maintaining a larger optimal backorder during a cycle becomes advantageous. The movement of the crisp values is mirrored by the fuzzy values. A slight rise in EOQ balances out the fuzziness of demand without having a significant effect on TAC.

2. **Impact of unit holding cost (k):** A holding cost decrease has a proportionately bigger effect compared to an increase of the same magnitude. Since this τ_α time is more influenced, K complement m in the sensitivity computation. A displacement that is consistent with the decision-sharp maker's bounds is shown by the median fuzzy output.

3. **Impact of the cost of unit deterioration (l):** In order to benefit from decreased and even greater item loss, both the positive inventory time and the EOQ grow. The TAC is reduced in Model-1. Model-2 is somewhat less affected in this situation than Model-1.

4. **Ordering cost impact (A):** In both models, the ordering cost A consistently has an effect. The change in positive inventory time and EOQ has increased little but significantly, whereas the change in backorder time τ_β and TAC has decreased. Both these movements are marginally less in Model-2.

5. **Effect of constant deterioration rate (α_0):** The TAC rises with increasing but falls more sharply with decreasing this parameter. In Model-1, higher EOQ and τ_α are optimal for minimal deterioration, and greater backorders are recommended when deterioration grows.

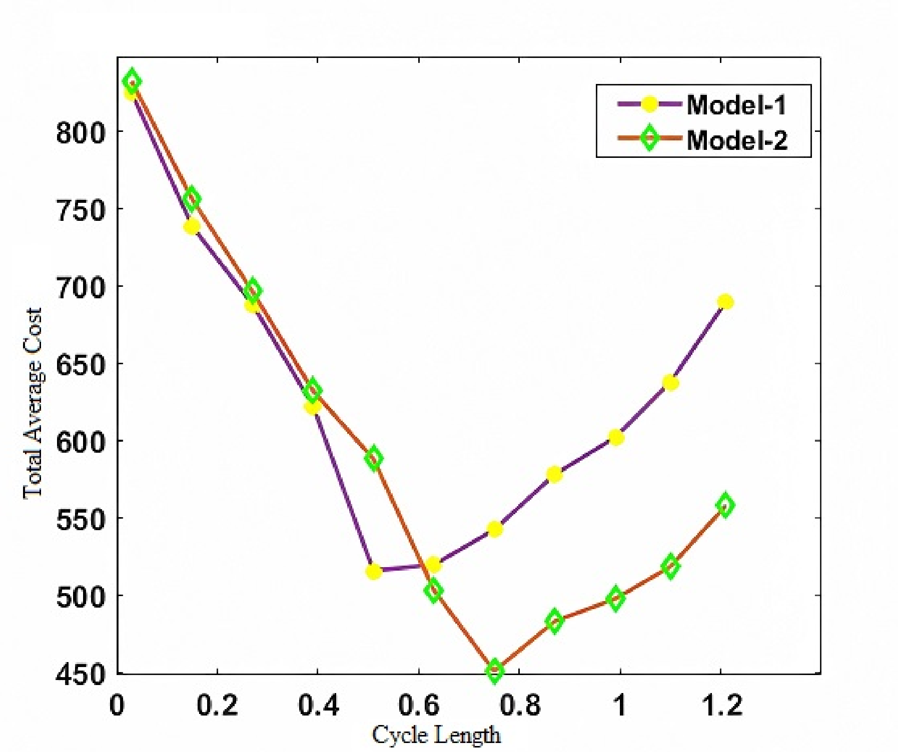

Figure 5: *Variation optimum total cost with cycle length.*

10. Conclusion

This study improved upon earlier EOQ models and techniques for managing deteriorating inventory levels. The expected total average cost was minimized while the per unit time cost was assessed with the length of time with stock-out condition and the duration of on-hand inventory in a reorder sequence to determine the proper reorder size and cycle length. Deterioration caused parts to be lost, which raised the total cost. Expanded to produce Model-2 by providing modelling uncertainty in demand. The optimal parameters for the reorder procedure were found by the development of a helpful formulation. Decision-makers were able to calculate the optimal investment for this aim by using this model, which demonstrated the impact of controlling spoiling. The Graded mean integration representation (GMIR) was used to defuzzify the fuzzy cost function in the model with uncertain demand. . From the analytical results of Model-2, an algorithm was developed to determine the optimal solutions.

The created models were validated using a numerical setup, and their sensitivity to important parameters was examined to ascertain the precise impact on the model. According to the traditional inventory model, holding costs and shortages had a connected impact on positive inventory duration and shortfall time, respectively. The total cost was more sensitive to a decline in the rate of degradation than to an increase in it. The following aspects can be expanded for future work. Numerous opportunities exist to further this research, such as time-limited replenishment, reworking or substituting damaged goods, uncertainty and randomization in other aspects, an expiration date-dependent deterioration rate a multi-item inventory and learning effects. An alternative model to recover part of the impact of degradation could be offered by the theory that uses animal fat waste as degraded goods to produce renewable energy .

11. Future Work

This model can be expanded to take policies like carbon caps and CO_2 quotas that aim to minimize emissions. The amount of carbon emissions may have a sensitive effect on some of the cost parameters. With such additions, it is possible to research how environmental deterioration affects ecosystems and conservation methods for sustainable supply chains. Some potential expansions of the models described include the analysis of the implications of time-dependent and non-linear operating costs, incomplete goods, prepayments, trade credit, and inflation in economic policy.

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