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BAYESIAN ESTIMATION OF INVERSE AILAMUJIA DISTRIBUTION USING DIFFERENT LOSS FUNCTIONS

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Abstract

This paper focuses on the Bayesian estimation of the parameter of the inverse Ailamujia distribution, employing advanced prior structures and diverse loss functions. Specifically, the extended Jeffreys' prior and gamma prior are utilized to derive the Bayesian estimators. Estimation is performed under various loss functions, including squared error, entropy, precautionary, and Linex loss functions, ensuring a comprehensive analysis. To demonstrate the practical applicability and comparative performance of these estimators, an empirical investigation is conducted using a real dataset. The findings highlight the adaptability and effectiveness of the proposed Bayesian approach across different estimation scenarios.

Key words: Bayesian analysis, priors, maximum likelihood estimator, different loss functions.

1. Introduction

In statistical literature, the Ailamujia distribution, introduced by Lv et al. [5], represents a novel probability distribution with significant versatility and practical relevance. This distribution has gained attention due to its ability to model various types of real-world data effectively. Its unique structural properties make it particularly suitable for applications in engineering and related disciplines. By accommodating a wide range of data patterns, the Ailamujia distribution has proven to be a valuable tool for analyzing reliability, survival times, and other stochastic phenomena. Its mathematical flexibility and applicability have inspired ongoing research into its properties, extensions, and potential for broader utilization across diverse fields. They have expounded its various distributional properties which includes moments, moment generating function, mode, median, order statistics. They have derived and discussed various reliability functions. The probability density function and cumulative distribution function of Ailamujia distribution are respectively given as

$$f(y,\alpha) = 4\alpha^2 \, y \, e^{-2\alpha y} \, ; y > 0 \, , \alpha > 0$$

$$F(y,\alpha) = 1 - (1 + 2\alpha y)e^{-2\alpha y} \, , \alpha > 0, y > 0$$

In recent past decade authors have proposed several extensions of Ailamujia distribution. Pan et al [7] has worked on Ailamujia distribution for interval estimation and hypothesis testing based on small sample size. Long [6] has obtained its Bayesian estimation under type II censoring on the basis of conjugate prior, Jeffrey's prior and no informative prior distribution. Yu et al [10] proposed a new method by applying Ailamujia distribution to solve the problem in the production and distribution of battle field injury in campaign macrocosm. Recently Ahmad et al [1] developed the inverse analogue of Ailamujia distribution and examine its usefulness through two real life time data sets.

Suppose Y is a random variable follows inverse Ailamujia distribution. Then its probability density function (p.d.f), is given by

$$f(y,\alpha) = 4\alpha^2 \frac{1}{y^3} \, e^{-\frac{2\alpha}{y}} \quad , y > 0 \, , \alpha > 0 \tag{1}$$

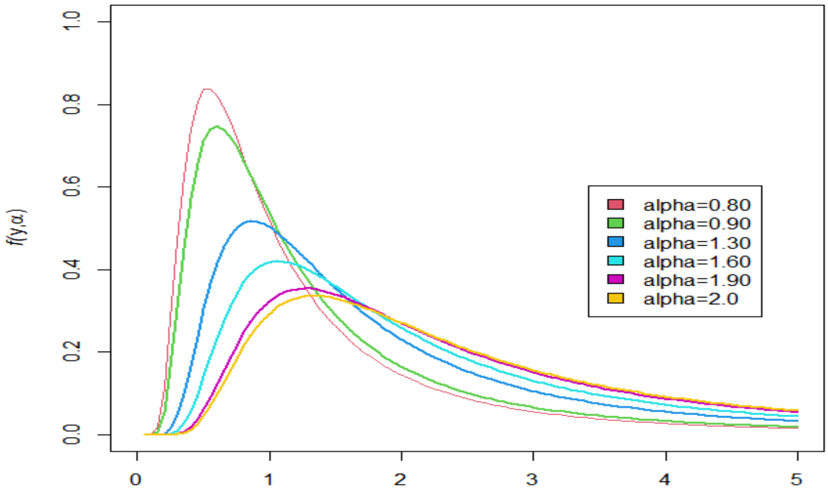

Fig. 1: *pdf plot of IAD under different values of parameters*

Figure 1, illustrates several possible shapes of the probability density function (pdf) for different parameter values, showcasing the flexibility and versatility of the proposed distribution. As the parameters vary, the shape of the pdf adapts to exhibit diverse behaviour's such as unimodal, skewed, or near-uniform profiles, depending on the parameter configuration. This graphical representation provides insight into how the distribution can be tailored to model a wide range of real-world phenomena

Figure 2, presents the cumulative distribution function (cdf) for the same parameter values as Figure 1, offering a complementary view of the proposed distribution. The cdf curves demonstrate the accumulation of probability across the range of the variable, reflecting the gradual transition from 0 to 1 as the variable increases. This graphical representation emphasizes the smoothness and consistency of the cdf, which is critical for probabilistic interpretation and applications such as reliability analysis and quantile estimation.

The corresponding cumulative distribution function (c.d.f), is given by

$$F(Y) = \frac{(2\alpha+y)}{y}\, e^{-\frac{2\alpha}{y}} \quad , y > 0\,, \alpha > 0 \tag{2}$$

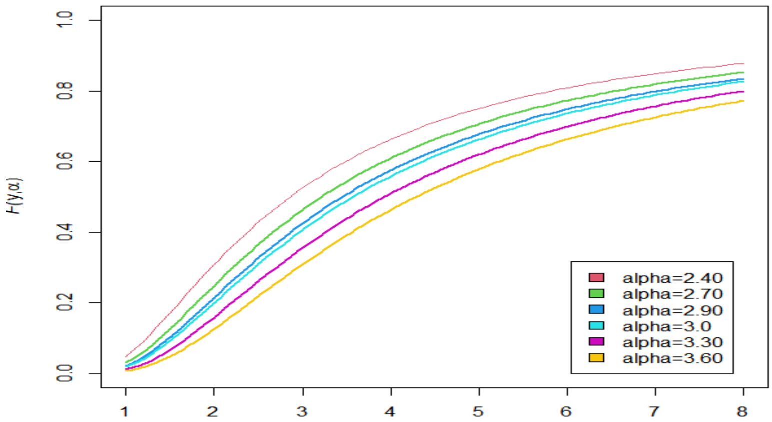

Fig. 1: *pdf plot of IAD under different values of parameters*

2. Maximum Likelihood Estimation

Let $Y_1, Y_2 \ldots Y_n$ be random samples from the inverse Ailamujia distribution. Then the likelihood function of inverse Ailamujia distribution is given as

$$l = \prod_{i=1}^{n} f(y_i, \alpha)$$

$$= \prod_{i=1}^{n} 4\alpha^2 \frac{1}{y_i^3}\, e^{-\frac{2\alpha}{y_i}} = (4\alpha^2)^n \prod_{i=1}^{n} \frac{1}{y_i^3}\; e^{-2\alpha \sum_{i=1}^{n} \frac{1}{y_i}}$$

Taking log we get log likelihood function as

$$\log l = \; 2n \log 2\alpha - 3 \sum_{i=1}^{n} \log y_i - 2\alpha \sum_{i=1}^{n} \frac{1}{y_i}$$

Differentiating w.r.t, we get

$$\frac{\partial \log l}{\partial \alpha} = 2n \frac{1}{2\alpha} - 2 \sum_{i=1}^{n} \frac{1}{y_i}$$

Now equating $\frac{\partial \log l}{\partial \alpha} = 0$, we get

$$\hat{\alpha} = \frac{n}{2S}$$

Where $S = \sum_{i=1}^{n} y_i^{-1}$

3. Bayesian Estimation of Inverse Ailamujia Distribution

Bayesian estimation procedure is a remarkable way to estimate the parameters of the distribution model. This estimation provides a posterior distribution of an existing life time distribution by considering prior information. From Bayesian point of view there can't be put the lid on selecting prior(s) by considering one's prior(s) is more suitable than others. In case of meager interpretative information about the unknown parameter it is preferable to select non informative prior. However, if one has sufficient information about the parameter(s) it is better to select informative prior. The aim of present study is to obtain a Bayesian estimation of parameter α of inverse Ailamujia distribution by using extended Jeffrey's and gamma prior. In recent past years several research papers have been published in this direction. Afaq et al [2] estimation of parameters of two parameter exponentiated gamma distribution. Mudasir et al [9] studied the Bayesian estimation of weighted Erlang distribution. Raqab and Madi [8] studied Bayesian estimation for exponentiated Rayleigh distribution. Fatima Bi and Afaq Ahmad [4], B. Singh et al. [11], Ahmad et al. [12] and again Ahmad et al. [13] studied different estimations of different distribution. In this paper our goal is to find the Bayesian estimators of the parameters of inverse Ailamujia distribution using extended Jeffery's prior and gamma prior under different loss functions.

3.1: Bayesian Estimation of Inverse Ailamujia Distribution Under the Assumption of Extended Jeffery's Prior

We assume the prior distribution of α to be extended Jeffrey's prior i.e $g(\alpha) \propto \frac{1}{\alpha^{2c}}$

Under the assumption of extended Jeffrey's prior. The posterior distribution of α can be obtained as

$$\pi(\alpha|y) \propto l(y|\alpha) g(\alpha)$$

$$\Rightarrow \pi(\alpha|y) \propto \left(4^n \prod_{i}^{n} \frac{1}{y_i^3}\right) \alpha^{2n}\, e^{-2\alpha \sum_i^n \frac{1}{y_i}} \frac{1}{\alpha^{2c}}$$

$$\Rightarrow \pi(\alpha|y) = k\, \alpha^{2(n-c)} e^{-2\alpha \sum_i^n \frac{1}{y_i}}$$

Where k is independent of α and

$$k^{-1} = \int_0^\infty \alpha^{2(n-c)} e^{-2\alpha \sum_i^n \frac{1}{y_i}}\, d\alpha$$

$$k^{-1} = \frac{\Gamma(2n - 2c + 1\,)}{\left(2 \sum_i^\infty \frac{1}{y_i}\right)^{2n-c+1}}$$

So that $$k = \frac{\left(2 \sum_i^\infty \frac{1}{y_i}\right)^{2(n-c)+1}}{\Gamma(2n-2c+1\,)} = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1\,)}$$

Where $S = \sum_i^\infty \frac{1}{y_i}$

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Hence the posterior distribution of α is given as

$$\pi(\alpha|y) = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)} \alpha^{2(n-c)} e^{-2S\alpha}$$

Where $S = \sum_i^\infty \frac{1}{y_i}$

3.1.1: Estimation Under Squared Error Loss Function (SELF)

The squared error loss function is defined as $l(\hat{\alpha},\alpha) = c_1(\hat{\alpha}-\alpha)^2$ for some constantant c_1 the risk function is given as

$$R(\hat{\alpha},\alpha) = E[I(\hat{\alpha},\alpha)]$$

$$= \int_0^\infty c_1(\hat{\alpha}-\alpha)^2 \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)} \alpha^{2(n-c)} e^{-2S\alpha} d\alpha$$

$$= c_1 \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)} \left[\hat{\alpha} \int_0^\infty \alpha^{2(n-c)} e^{-2S\alpha} d\alpha + \int_0^\infty \alpha^{2(n-c)+2} e^{-2S\alpha} d\alpha - 2\hat{\alpha} \int_0^\infty \alpha^{2(n-c)+1} e^{-2S\alpha} d\alpha\right]$$

After solving the integral, we obtain

$$= c_1 \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)} \left\lfloor \frac{\hat{\alpha}\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + \frac{(2n-2c+2)(2n-2c+1)\Gamma(2n-2c+1)}{(2S)^{2(n-c)+3}} - \frac{(2n-2c+1)\Gamma(2n-2c+1)}{(2S)^{2(n-c)+2}} \right\rfloor$$

$$R(\hat{\alpha},\alpha) = c_1\left[\hat{\alpha}^2 + \frac{(2n-2c+2)(2n-2c+1)}{(2S)^2} - \frac{\hat{\alpha}(2n-2c+1)}{(2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\hat{\alpha}_s = \frac{(2n-2c+1)}{4S}$$

Where $s = \sum_i^\infty \frac{1}{y_i}$

3.1.2: Estimation Under Entropy Loss Function

The entropy loss function is defined as $L(\delta) = b[\delta - \log(\delta) - 1]; b > 0, \delta = \frac{\hat{\alpha}}{\alpha}$ the risk functions given as

$$R(\hat{\alpha},\alpha) = \int_0^\infty b[\delta - \log(\delta) - 1] \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)} \alpha^{2(n-c)} e^{-2S\alpha} d\alpha$$

$$R(\hat{\alpha},\alpha) = b \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)} \int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\hat{\alpha} + \log\alpha - 1\right] \alpha^{2(n-c)} e^{-2S\alpha} d\alpha$$

$$= b\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\int_0^\infty \alpha^{2(n-c)-1}e^{-2S\alpha}d\alpha - \log\hat{\alpha}\int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha + \int_0^\infty (\log\alpha)\,\alpha^{2(n-c)}e^{-2S\alpha}d\alpha - \int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha\right]$$

After solving the integral, we obtain

$$= b\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\frac{\Gamma(2n-2c)}{(2S)^{2(n-c)}} - \log\hat{\alpha}\frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + \frac{\Gamma'(2n-2c+1)}{(2S)^{2(n-c)+1}} - \frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}}\right]$$

$$= b\left[\frac{\hat{\alpha}(S)}{(n-c)} - \log\hat{\alpha} + \frac{\Gamma'(2n-2c+1)}{\Gamma(2n-2c+1)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_e = \frac{n-c}{S}$$

Where $s = \sum_i^\infty \frac{1}{y_i}$

3.1.3: Estimation Under Precautionary Loss Function

The precautionary loss function is defined as $(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}$, the risk function is given as

$$R(\hat{\alpha},\alpha) = \int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\alpha^{2(n-c)}\,e^{-2S\alpha}d\alpha$$

$$R(\hat{\alpha},\alpha) = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)}\int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\alpha^{2(n-c)}\,e^{-2S\alpha}d\alpha$$

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\int_0^\infty \alpha^{2(n-c)}e^{-2S\alpha}d\alpha + \frac{1}{\hat{\alpha}}\int_0^\infty \alpha^{2(n-c)+2}e^{-2S\alpha}d\alpha - 2\int_0^\infty \alpha^{2(n-c)+1}e^{-2S\alpha}d\alpha\right]$$

After solving the integral, we obtain

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)}\left[\hat{\alpha}\frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + \frac{1}{\hat{\alpha}}\frac{\Gamma(2n-2c+3)}{(2S)^{2(n-c)+3}} - 2\frac{\Gamma(2n-2c+2)}{(2S)^{2(n-c)+2}}\right]$$

$$= \left[\hat{\alpha} + \frac{(2n-2c+2)(2n-2c+1)}{\hat{\alpha}\,(2S)^2} - \frac{2(2n-2c+1)}{(2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_p = \frac{[(n-c+1)(2n-2c+1)]^{\frac{1}{2}}}{(S)}$$

Where $S = \sum_i^\infty \frac{1}{y_i}$

3.1.4: Estimation Under Linex Loss Function

The linex loss function is defined as $L(\hat{\alpha},\alpha) = exp\{b_1(\hat{\alpha}-\alpha)\} - b_1(\hat{\alpha}-\alpha) - 1$, the risk function is given as

$$l(\hat{\alpha},\alpha) = \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)} \int_0^\infty \{e^{\left(b_1(\hat{\alpha}-\alpha)\right)} - b_1(\hat{\alpha}-\alpha) - 1\}\alpha^{2(n-c)}\, e^{-2S\alpha} d\alpha$$

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-c+1)} \left[e^{b_1\hat{\alpha}} \int_0^\infty \alpha^{2(n-c)} e^{-\alpha(b_1+2S)} d\alpha - b_1\hat{\alpha} \int_0^\infty \alpha^{2(n-c)} e^{-2S\alpha} d\alpha \right.$$
$$\left. + b_1 \int_0^\infty \alpha^{2(n-c)+1} e^{-2S\alpha} d\alpha - \int_0^\infty \alpha^{2(n-c)} e^{-2S\alpha} d\alpha \right]$$

$$= \frac{(2S)^{2(n-c)+1}}{\Gamma(2n-2c+1)} \left[e^{b_1\hat{\alpha}} \frac{\Gamma(2n-2c+1)}{(b_1+2S)^{2(n-c)+1}} - b_1\hat{\alpha} \frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} + b_1 \frac{\Gamma(2n-2c+2)}{(2S)^{2(n-c)+2}} - \frac{\Gamma(2n-2c+1)}{(2S)^{2(n-c)+1}} \right]$$

$$= \left[e^{b_1\hat{\alpha}} \left(\frac{2S}{b_1+2S}\right)^{2(n-c)+1} - b_1\hat{\alpha} + b_1 \frac{(2n-2c+1)}{(2S)} - 1 \right]$$

Now solving $\frac{\partial l(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\hat{\alpha_l} = \frac{1}{b_1} \log\left(\frac{b_1+2S}{2S}\right)^{2(n-c)+1}$$

4. Bayesian Estimation of Inverse Ailamujia Distribution Under the Assumption of Gamma Distribution

We assume the prior distribution of α to be gamma distribution i.e $g(\alpha) \propto \frac{a^b}{\Gamma(b)} e^{-a\alpha}\, \alpha^{b-1}$

Now under the assumption of gamma prior. The posterior distribution of α can be obtained as

$$\pi(\alpha|y) \propto l(y|\alpha)g(\alpha)$$

$$\Rightarrow\ \pi(\alpha|y) \propto \left(4^n \prod_i^n \frac{1}{y_i^3}\right) \alpha^{2n}\, e^{-2\alpha \sum_i^n \frac{1}{y_i}}\, \frac{a^b}{\Gamma(b)} e^{-a\alpha}\, \alpha^{b-1}$$

$$\Rightarrow\ \pi(\alpha|y) = k\, \alpha^{2n+b-1} e^{-\alpha\left(a+2\sum_i^n \frac{1}{y_i}\right)}$$

Where k is independent of α and

$$k^{-1} = \int_0^\infty \alpha^{2n+b-1} e^{-\alpha\left(a+2\sum_i^n \frac{1}{y_i}\right)} d\alpha$$

$$= \frac{\Gamma(2n+b)}{\left(a+2\sum_i^n \frac{1}{y_i}\right)^{2n+b}}$$

So that

$$k = \frac{\left(a + 2\sum_i^n \frac{1}{y_i}\right)^{2n+b}}{\Gamma(2n+b)} = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}$$

Where $S = \sum_i^n \frac{1}{y_i}$

Hence the posterior distribution of α is given as

$$\pi(\alpha|y) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\ \alpha^{2n+b-1} e^{-\alpha(a+2S)}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.1: Estimation Under Squared Error Loss Function

The squared error loss function is defined as $l(\hat{\alpha},\alpha) = c_1(\hat{\alpha}-\alpha)^2$ for some constantant c_1 the risk function is given as

$$R(\hat{\alpha},\alpha) = E[I(\hat{\alpha},\alpha)]$$

$$= \int_0^\infty c_1(\hat{\alpha}-\alpha)^2 \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\ \alpha^{2n+b-1} e^{-\alpha(a+2S)} d\alpha$$

$$= c_1 \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)} \int_0^\infty (\hat{\alpha}-\alpha)^2\ \alpha^{2n+b-1} e^{-\alpha(a+2S)} d\alpha$$

After solving the integral, we obtain

$$R(\hat{\alpha},\alpha) = c_1 \left[\hat{\alpha}^2 + \frac{(2n+b)(2n+b+1)}{(a+2S)^2} - 2\hat{\alpha}\frac{(2n+b)}{(a+2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\hat{\alpha}_s = \frac{2n+b}{a+2S}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.2: Estimation Under Entropy Loss Function

The entropy loss function is defined as $L(\delta) = b[\delta - \log(\delta) - 1]; b > 0, \delta = \frac{\hat{\alpha}}{\alpha}$ the risk functions given as

$$R(\hat{\alpha},\alpha) = \int_0^\infty b[\delta - \log(\delta) - 1] \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\ \alpha^{2n+b-1} e^{-\alpha(a+2S)}\, d\alpha$$

$$= b \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)} \int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\hat{\alpha} + \log\alpha - 1\right] \alpha^{2n+b-1} e^{-\alpha(a+2S)} d\alpha$$

After solving the integral, we obtain

$$R(\hat{\alpha},\alpha) = b\left[\hat{\alpha}\frac{(a+2S)}{(2n+b-1)} - \log\hat{\alpha} + \frac{\Gamma'(2n+b)}{\Gamma(2n+b)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_e = \frac{2n+b-1}{a+2S}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.3: Estimation Under Precautionary Loss Function

The precautionary loss function is defined as$l(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}$, the risk function is given as

$$R(\hat{\alpha},\alpha) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\alpha^{2n+b-1}e^{-\alpha(a+2S)}d\alpha$$

After solving the integral, we get

$$= \left[\hat{\alpha} + \frac{(2n+b)(2n+b-1)}{\hat{\alpha}(a+2S)^2} - 2\frac{(2n+b)}{(a+2S)}\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\hat{\alpha}_p = \frac{[(2n+b)(2n+b-1)]^{\frac{1}{2}}}{(a+2S)}$$

Where $S = \sum_i^n \frac{1}{y_i}$

4.4: Estimation Under Linex Loss Function

The linex loss function is defined as $L(\hat{\alpha},\alpha) = exp\{b_1(\hat{\alpha}-\alpha)\} - b_1(\hat{\alpha}-\alpha) - 1$, the risk function is given as

$$R(\hat{\alpha},\alpha) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\int_0^\infty \{e^{(b_1(\hat{\alpha}-\alpha))} - b_1(\hat{\alpha}-\alpha) - 1\}\alpha^{2n+b-1}\,e^{-\alpha(a+2S)}d\alpha$$

$$= \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\left[e^{b_1\hat{\alpha}}\int_0^\infty \alpha^{2n+b-1}\,e^{-\alpha(a+b_1+2S)}d\alpha \right.$$
$$\left. - b_1\hat{\alpha}\int_0^\infty \alpha^{2n+b-1}\,e^{-\alpha(a+2S)}d\alpha + b_1\int_0^\infty \alpha^{2n+b}\,e^{-\alpha(a+2S)}d\alpha - \int_0^\infty \alpha^{2n+b-1}\,e^{-\alpha(a+2S)}d\alpha\right]$$

After solving the integrals, we obtain

$$R(\hat{\alpha},\alpha) = \frac{(a+2S)^{2n+b}}{\Gamma(2n+b)}\left[e^{b_1\hat{\alpha}}\frac{\Gamma(2n+b)}{(a+b_1+2S)^{2n+b}} - b_1\hat{\alpha}\frac{\Gamma(2n+b)}{(a+2S)^{2n+b}} + b_1\frac{\Gamma(2n+b+1)}{(a+2S)^{2n+b+1}} - \frac{\Gamma(2n+b)}{(a+2S)^{2n+b}}\right]$$

$$= \left[e^{b_1\hat{\alpha}}\left(\frac{a+2S}{a+b_1+2S}\right)^{2n+b} - b_1\hat{\alpha} + b_1\frac{(2n+b)}{(a+2S)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\hat{\alpha}_l = \frac{1}{b_1} \log \left(\frac{a + b_1 + 2S}{a + 2S} \right)^{2n+b}$$

Where $S = \sum_i^n \frac{1}{y_i}$

5. Application

In this section we provide an application through which the performance of the estimators and posterior risk of different loss function has been obtained. The data set are follows:

Data set 1: The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bekker et al. [3]. The data are follows

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16,
2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

By using different loss functions, the Bayes estimates and posterior risks of the posterior distribution through both priors are as follows where posterior risk are in parenthesis.

Table 1: *Bayes Estimation and Posterior Risks Using Jeffery's Prior*

α	C	$\hat{\alpha}$	$\hat{\alpha}_S$	$\hat{\alpha}_E$	$\hat{\alpha}_P$	$\hat{\alpha}_L$	
						$b_1 = 0.01$	$b_1 = 0.05$
1.0	0.5	0.5583	0.5583 (1.260)	1.109 (4.862)	2.241 (17.97)	1.116 (0.0111)	1.116 (0.0558)
	1.0	0.5583	0.5545 (1.247)	1.101 (4.862)	2.225 (17.85)	1.108 (0.0110)	1.108 (0.0554)
	1.5	0.5583	0.5506 (1.234)	1.093 (4.862)	2.210 (17.73)	1.1012 (0.0110)	1.1010 (0.0550)
2.0	0.5	0.5583	0.5583 (1.260)	1.1090 (4.862)	2.2413 (17.97)	1.1167 (0.0111)	1.1165 (0.0558)
	1.0	0.5583	0.5545 (1.247)	1.1012 (4.862)	2.2258 (17.85)	1.1089 (0.0110)	1.1088 (0.0554)
	1.5	0.5583	0.5506 (1.234)	1.093 (4.862)	2.2102 (17.73)	1.1012 (0.0110)	1.1010 (0.0550)
3.0	0.5	0.5583	0.5583 (1.260)	1.1090 (4.862)	2.2413 (17.97)	1.1167 (0.0111)	1.1165 (0.0558)
	1.0	0.5583	0.5545 (1.247)	1.1012 (4.862)	2.2258 (17.85)	1.1089 (0.0110)	1.1088 (0.0554)
	1.5	0.5583	0.5506 (1.234)	1.0935 (4.862)	2.2102 (17.73)	1.1012 (0.0110)	1.1010 (05506)

$\hat{\alpha}$ = MLE, $\hat{\alpha}_S$ = Estimation under SELF, $\hat{\alpha}_E$ = Estimation under Entropy,

$\hat{\alpha}_P$ = Estimation under Precautionary, $\hat{\alpha}_L$ = Estimation under LINEX

Table 2: *Bayes Estimation and Posterior Risks Using Gamma Prior*

α	a	b	$\hat{\alpha}$	$\hat{\alpha}_S$	$\hat{\alpha}_E$	$\hat{\alpha}_P$	$\hat{\alpha}_L$	
							$b_1 = 0.01$	$b_1 = 0.05$
1.0	0.5	0.5	0.5583	1.1163 (0.0086)	1.1240 (4.8667)	1.1124 (1.1085)	1.1162 (0.0111)	1.1161 (0.0558)
	0.5	1.0	0.5583	1.1201 (0.0086)	1.1279 (4.8666)	1.1163 (1.1124)	1.1201 (0.0112)	1.1199 (0.0560)
	1.0	0.5	0.5583	1.1120 (0.0085)	1.119 (4.8705)	1.1081 (1.1043)	1.1119 (0.0111)	1.1118 (0.0556)
2.0	0.5	0.5	0.5583	1.1163 (0.0086)	1.1240 (4.8667)	1.1124 (1.1085)	1.1162 (0.0111)	1.1161 (0.0558)
	0.5	1.0	0.5583	1.1201 (0.0086)	1.1279 (4.8666)	1.1163 (1.1124)	1.1201 (0.0112)	1.1199 (0.0560)
	1.0	0.5	0.5583	1.1120 (0.0085)	1.1197 (4.8705)	1.1081 (1.1043)	1.1119 (0.0111)	1.1118 (0.0556)
3.0	0.5	0.5	0.5583	1.1163 (0.0086)	1.1240 (4.8667)	1.1124 (1.1085)	1.1162 (0.0111)	1.1161 (0.0558)
	0.5	1.0	0.5583	1.1201 (0.0086)	1.1279 (4.8666)	1.1163 (1.1124)	1.1201 (0.0112)	1.1199 (0.05600)
	1.0	0.5	0.5583	1.1120 (0.0085)	1.1197 (4.8705)	1.1081 (1.1043)	1.1119 (0.0111)	1.1118 (0.0556)

Among other loss functions, it is evident from Table 1 and Table 2. That the Linex loss function shows smaller Bayes posterior risk under the both assumptions (extended Jeffery's prior and gamma prior). According to decision rule of less Bayes posterior risk, we accomplish that Linex loss function is more useful than others.

6. Conclusion

In this study, we derived the Bayes posterior distribution and parameter estimation for the inverse Ailamujia distribution using both informative and non-informative priors. We explored various loss functions to assess their impact on the estimation process, with a specific focus on the Linex loss function. The results, presented in Table 1 and Table 2, clearly demonstrate that the Linex loss function yields the smallest Bayes posterior risk under both the extended Jeffery's prior and the gamma prior assumptions. This comparative analysis highlights the superior performance of the Linex loss function, indicating its effectiveness in minimizing the Bayes posterior risk.

By applying the decision rule of minimizing the Bayes posterior risk, we conclude that the Linex loss function is the most useful among the considered alternatives. The performance of the estimators was evaluated through practical applications, and the results underscore the flexibility and robustness of the inverse Ailamujia distribution in Bayesian estimation. The findings also emphasize the utility of the Linex loss function in enhancing the precision of parameter estimation across various contexts. This work contributes to the growing body of literature on Bayesian methods, offering valuable insights into the application of different loss functions for parameter estimation. It provides a clear advantage of using the Linex loss function in terms of minimizing posterior risk, which can be applied to diverse statistical modelling scenarios. The study reinforces the importance of selecting appropriate loss functions for effective Bayesian estimation, ensuring better model performance and more reliable results.

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