ON SOME PROPERTIES AND APPLICATIONS OF THE MODI-FRÉCHET DISTRIBUTION

Akhila P.[1], Girish Babu M.[2]

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Department of Statistics, Government Arts and Science College, Kozhikode, Kerala, India
akhila@gasckkd.ac.in[1], giristat@gmail.com [2]

Abstract

In this paper we introduce a novel expansion of Fréchet distribution from Modi family of probability distributions. The important statistical properties like moments, stochastic ordering, and entropy are studied in this paper. Two distinct characterizations of the proposed distribution are derived through the hazard rate function and truncated moments. The statistical inference about the parameters of the new distribution is studied using the method of maximum likelihood estimation. To study the flexibility and practical utility of the distribution, two real-life data sets from the reliability sector and from the biomedical field were analyzed. An extensive simulation study is also conducted to validate the accuracy and consistency of the estimation techniques.

Keywords: Characterization, Entropy, Fréchet distribution, Hazard rate function, Maximum Likelihood Estimation, Statistical modelling.

1. Introduction

The study of statistical distributions is crucial across various disciplines, like economics, engineering, and particularly in reliability analysis. The reliability sector focuses on modeling and understanding failure rates in systems, components, and products over time. These necessitating distributions are robust and versatile to capture the inherent complexities of these processes. This paper introduces a new distribution meticulously designed to meet these demands and to offer enhanced adaptability for reliability analysis.

In modern industries, accurate and reliable models are essential for predicting critical systems, machinery, and equipment lifespan and failure patterns. Traditional distributions, such as the Weibull distribution (see, [1] & [2]) and the exponential distribution (see [3]), have long been utilized in reliability studies due to their simplicity and ease of use. However, these models often fall short when modeling complex or non-standard failure rates. For instance, while the Weibull distribution is well-suited for systems with increasing or decreasing failure rates, it struggles with scenarios involving bathtub-shaped failure rates, which are common in electronic systems. Similarly, the exponential distribution assumes a constant failure rate, making it inadequate for mechanical systems that experience wear-out failures over time. Our proposed distribution overcomes these limitations by providing a more flexible framework that can adapt to a broader range of reliability scenarios, including those with non-monotonic hazard functions.

Moreover, this distribution has been applied to the biomedical field, specifically in analyzing infant mortality rates, where the precise modeling of survival times and risk factors is crucial.

Traditional statistical models can struggle with the intricacies of biomedical data, particularly in capturing the variability and heterogeneity inherent in patient population. By offering a more adaptable structure, our distribution enhances the accuracy and reliability of statistical modeling in both reliability and biomedical contexts, making it a valuable tool for researchers and practitioners alike.

René Fréchet developed the Fréchet distribution [4], recognized as the maximum value distribution, a concept further explored by Fisher and Tippet [5] and Gumbel [6]. This distribution has become widely used and studied across various fields due to experimental research from multiple disciplines. It is particularly significant in survival analysis and reliability studies, finding applications in engineering, social, physical, environmental, and life sciences. The cumulative distribution function (cdf) and probability density function (pdf) of Fréchet distribution are, respectively,

$$G_{\sigma,\lambda}(x) = e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, x > 0, \tag{1}$$

and

$$g_{\sigma,\lambda}(x) = \lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, x > 0, \tag{2}$$

where $\sigma > 0$ is the scale parameter and $\lambda > 0$ is the shape parameter.

For further reading, see Kotz and Nadarajah [7] and Mubarak [8]. The Fréchet distribution has been extensively generalized in the literature. Recent developments are; Slash-Exponential-Fréchet distribution by Gmez *et al.* [9], Cosine Fréchet Loss distribution by Abonongo *et al.* [10], Marshall-Olkin exponentiated Fréchet distribution [11], the inverted Gompertz-Fréchet distribution [12], Yun-Fréchet distribution [13], cubic transmuted Fréchet distribution [14], the generalized odd log-logistic Fréchet distribution [15], the novel Kumaraswamy power Fréchet distribution [16] and generalization of Fréchet distribution[17]. Harlow [18] demonstrated that the Fréchet distribution is crucial for modeling the statistical behavior of material properties in various engineering applications.

Modi *et al.* [19] proposed the Modi family of distributions with cdf $T(x)$ and pdf $t(x)$ as follows:

$$T(x) = \frac{(1+\alpha^{\beta})S(x)}{\alpha^{\beta}+S(x)}, \qquad x > 0, \alpha > 0, \beta > 0, \tag{3}$$

$$t(x) = \frac{(1+\alpha^{\beta})(\alpha^{\beta}s(x))}{\left(\alpha^{\beta}+S(x)\right)^{2}}, \qquad x > 0, \alpha > 0, \beta > 0, \tag{4}$$

where S(x) is an arbitrary cdf of a continuous univariate distribution and s(x) is the corresponding pdf. Recent contributions to this family of distributions include Modi Exponential Distribution [19], Modi Weibull [20] and Modi Exponentiated Exponential Distribution [21]. In this paper we introduce a new distribution developed from this family of distributions, utilizing the Fréchet distribution as the baseline distribution. Named the Modi-Fréchet Distribution, this four-parameter distribution offers a superior fit compared to other competitive lifetime distributions.

The present paper is organized as follows: In Section 2 the model construction and basic statistical properties such as moments, stochastic ordering, and entropy are studied. Section 3 is devoted to characterizations of the distribution based on hazard function and truncated moments. In Section 4 parameters of the new distribution are derived using the maximum likelihood estimation method. A simulation study has been carried out in Section 5. The flexibility and utility of the proposed model are studied in Section 6 and conclusions are given in Section 7.

2. Modi Fréchet Distribution

In this section, we develop a special distribution from Modi family, based on the Fréchet distribution. The cdf and pdf of Modi Fréchet distribution (MFD) are;

$$F(x) = \frac{(1+\alpha^{\beta})e^{-(\frac{\sigma}{x})^{\lambda}}}{\alpha^{\beta}+e^{-(\frac{\sigma}{x})^{\lambda}}}, \qquad x > 0, \alpha, \beta, \sigma, \lambda > 0. \tag{5}$$

The corresponding pdf is given by;

$$f(x) = \frac{(1+\alpha^{\beta})\left(\lambda\alpha^{\beta}\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha^{\beta}+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^{2}}, \qquad x > 0, \alpha, \beta, \sigma, \lambda > 0. \tag{6}$$

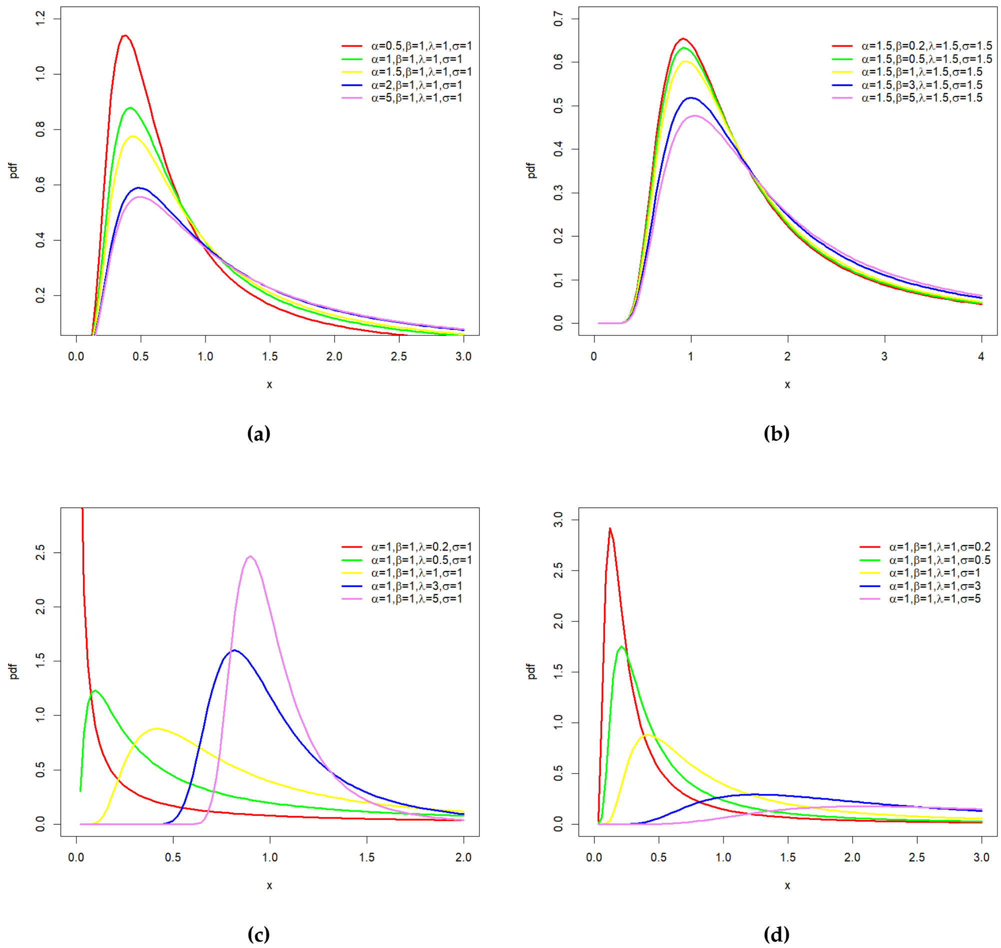


Figure 1: *Plots of the pdf of the MFD for various parameter values.*

Fig. 1. shows the pdf can be unimodal, approximately normal, increasing-decreasing, and right-skewed.
The hazard function of MFD is;

$$h(x) = \frac{(1+\alpha^{\beta})\left(\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha^{\beta}+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}, \qquad x > 0, \alpha, \beta, \sigma, \lambda > 0. \tag{7}$$

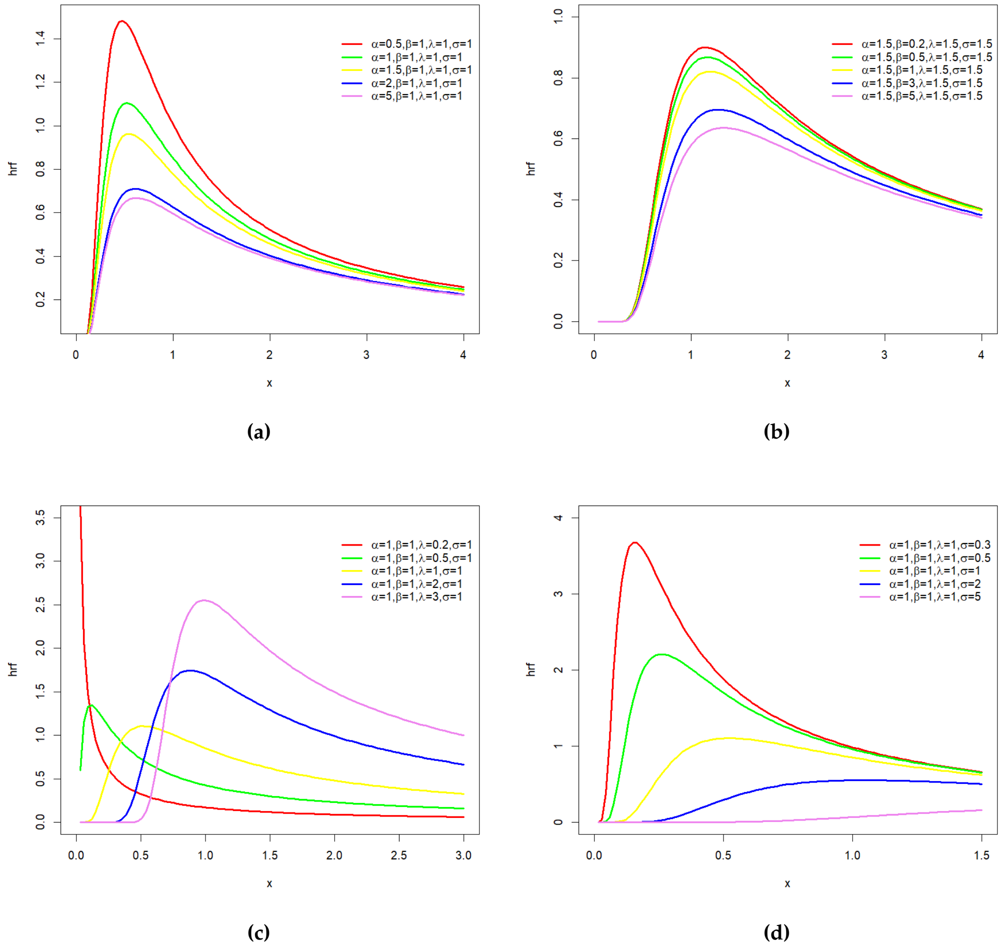


Figure 2: *Plots of the hrf of the MFD for various parameter values.*

Fig 2. shows decreasing, increasing-decreasing, constant, and unimodal behaviour of hazard function.

We derived the quantile function of MFD. The quantile function obtained using the inversion method is given as;

$$F^{-1}(y) = \frac{\sigma}{\left(\log\left(1+\alpha^{\beta}-y\right)-\log\left(y\alpha^{\beta}\right)\right)^{1/\lambda}}, \qquad y\epsilon[0,1] \tag{8}$$

2.1. Moments

The mean, standard deviation, variance, skewness, and kurtosis for the MFD are computed using the raw moments. With the help of R software, we computed them using the standard definitions.

Table 1: *Moment characteristics of the MFD for various parameter values.*

Parameters	$\alpha \rightarrow$	0.6	1	2	5
$\beta = 9$	Mean	3.0120	4.2253	4.6556	4.6569
$\sigma = 4$	Variance	0.1283	1.4521	2.1379	2.1402
$\lambda = 5$	Skewness	6.8163	3.9213	3.5360	3.5351
	Kurtosis	210.25	57.6110	48.1140	48.0920
$\beta = 5.5$	Mean	2.2716	2.6963	2.8949	2.9019
$\sigma = 2.6$	Variance	0.0909	0.3086	0.4320	0.4365
$\lambda = 6.5$	Skewness	4.2027	2.9117	2.5991	2.5899
	Kurtosis	49.9690	24.4780	20.4440	20.3350
$\beta = 2.5$	Mean	3.1501	3.4990	3.7779	3.8626
$\sigma = 3.3$	Variance	0.6722	0.1080	1.5023	1.6292
$\lambda = 4.8$	Skewness	4.7686	4.1751	3.8589	3.7811
	Kurtosis	93.3690	72.3430	62.8610	60.7000
$\beta = 1.2$	Mean	0.8987	0.9206	0.9431	0.9596
$\sigma = 0.9$	Variance	0.0143	0.0170	0.0198	0.0218
$\lambda = 9$	Skewness	2.4785	2.3161	2.1727	2.0808
	Kurtosis	16.4210	14.7680	13.4440	12.6580

The calculated values are presented in Table 1. It shows that the MFD is suitable for under-dispersed data. The skewness and kurtosis values show positive skewness and leptokurtic behaviour. As α increases both mean and variance are increasing while skewness and kurtosis values decreasing.

2.2. Stochastic Ordering

Stochastic ordering is a powerful tool to demonstrate the comparison of random variables in terms of statistical functions of distribution theory. Different types of orderings can also be defined based on the hazard rate, reverse hazard rate, or by applying transformations to the random variables, as discussed in [22]. Let X_1 and X_2 be two random variables with parameters $\alpha_1, \beta, \sigma, \lambda$ and $\alpha_2, \beta, \sigma, \lambda$, their respective density functions $f_1(x)$ and $f_2(x)$, the reliability functions be $\bar{F}_1(x)$ and $\bar{F}_2(x)$, then we say X_1 is smaller than X_2 if

- $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all x,$\Longrightarrow X_1 \leq_{st} X_2$ (Stochastic order).
- $\frac{f_1(x)}{\bar{F}_1(x)} \geq \frac{f_2(x)}{\bar{F}_2(x)}$ for all x,$\Longrightarrow X_1 \leq_{hr} X_2$ (Hazard rate order).
- $\frac{f_1(x)}{F_1(x)} \geq \frac{f_2(x)}{F_2(x)}$ for all x,$\Longrightarrow X_1 \leq_{rh} X_2$ (Reversed hazard rate order).
- $\frac{f_1(x)}{f_2(x)}$ is a monotonic decreasing function for all x,$\Longrightarrow X_1 \leq_{lr} X_2$ (Likelihood ratio order).

Suppose the densities of X_1 and X_2 be

$$f_1(x) = \frac{\left(1+\alpha_1^{\beta}\right)\left(\lambda \alpha_1^{\beta} \sigma^{\lambda} x^{-(\lambda+1)} e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha_1^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}, \qquad x > 0, \text{and}$$

$$f_2(x) = \frac{\left(1+\alpha_2^{\beta}\right)\left(\lambda \alpha_2^{\beta} \sigma^{\lambda} x^{-(\lambda+1)} e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha_2^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}, \qquad x > 0.$$

respectively. Then,
case (i): When α is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha_1^\beta}{\alpha_2^\beta}\frac{\left(1+\alpha_1^\beta\right)\left(\alpha_2^\beta + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}{\left(1+\alpha_2^\beta\right)\left(\alpha_1^\beta + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}.$$

For $\alpha_1 < \alpha_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.
case (ii): When β is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha^{\beta_1}}{\alpha^{\beta_2}}\frac{\left(1+\alpha^{\beta_1}\right)\left(\alpha^{\beta_2} + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}{\left(1+\alpha^{\beta_2}\right)\left(\alpha^{\beta_1} + e^{-(\frac{\sigma}{x})^\lambda}\right)^2}.$$

For $\beta_1 < \beta_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.
case (iii): When σ is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\sigma_1^\lambda}{\sigma_2^\lambda}\frac{e^{-(\frac{\sigma_1}{x})^\lambda}}{e^{-(\frac{\sigma_2}{x})^\lambda}}\frac{\left(\alpha^{\beta} + e^{-(\frac{\sigma_2}{x})^\lambda}\right)^2}{\left(\alpha^{\beta} + e^{-(\frac{\sigma_1}{x})^\lambda}\right)^2}.$$

For $\sigma_1 < \sigma_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.
case (iv): When λ is different.

$$\frac{f_1(x)}{f_2(x)} = \frac{\lambda_1}{\lambda_2}\frac{\sigma_1^\lambda}{\sigma_2^\lambda}\frac{e^{-(\frac{\sigma}{x})^{\lambda_1}}}{e^{-(\frac{\sigma}{x})^{\lambda_2}}}\frac{\left(\alpha^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda_2}}\right)^2}{\left(\alpha^{\beta} + e^{-(\frac{\sigma}{x})^{\lambda_1}}\right)^2}.$$

For $\lambda_1 < \lambda_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$ which satisfies $X_1 \leq_{lr} X_2$.

2.3. Entropy

Every statistical distribution inherently possesses some degree of uncertainty, and entropy serve as a quantifiable measure of this uncertainty. In modern statistical analysis, information measures like entropy plays a crucial role in addressing and understanding such uncertainties, making them vital tools for statisticians.

If X is a non-negative continuous random variable with pdf $f(x)$, and cdf $F(x)$ then the Renyi Entropy is defined by,

$$H_\theta(x) = \frac{1}{1-\theta}\log\int_0^\infty [f(x)]^\theta\, dx. \tag{9}$$

The Shannon entropy of X is defined as

$$S(x) = -\int_0^\infty f(x)\ln[f(x)]\, dx. \tag{10}$$

Using the pdf of MFD, we can write;

$$[f(x)]^\theta = \left(1+\alpha^\beta\right)^\theta \left(\lambda\alpha^\beta\sigma^\lambda\right)^\theta \frac{\left(x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^\lambda}\right)^\theta}{\left(\alpha^\beta + e^{-(\frac{\sigma}{x})^\lambda}\right)^{2\theta}}. \tag{11}$$

Varentropy, the variance of Shannon information associated with a random variable X, was introduced by Song [23] as a measure of distribution shape, offering an alternative to kurtosis. This concept captures the variability of information content, also known as information varentropy, as discussed by Bobkov and Madiman [24]. Varentropy is significant in fields like information theory, computer science, and statistics, providing valuable insights into how information is distributed around the entropy of X.

Consider X as a continuous random variable with a density function $f(x)$. The Shannon varentropy of X is then defined as follows:

$$V = V(X) := Var[h(X)] = \int_S f(x)\,[\ln f(x)]^2\,dx - \left[\int_S f(x)\ln f(x)dx\right]^2 \tag{12}$$

The calculated entropy values presented in Table 2 provide a detailed comparison of Shannon entropy, Rényi entropy, and varentropy across different parameter settings. As shown in the table, the Shannon entropy values are consistently negative, indicating the uncertainty associated with each parameter set. In contrast, Rényi entropy exhibits both positive and negative values, reflecting the variation in the information content under different parameter configurations. Varentropy values, which measure the dispersion of information content around the entropy, are consistently positive, with the magnitude decreasing as the parameter shape parameters β and λ increase. This comprehensive comparison highlights how each entropy measure captures distinct aspects of the information content and its variability.

Table 2: *Entropy measures for different parameters*

Parameters	Shannon Entropy			Renyi Entropy			Varentropy		
β	0.3	0.9	2.5	0.3	0.9	2.5	0.3	0.9	2.5
λ	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
$\alpha = 0.9$, $\sigma = 0.2$	-44.2040	-12.9194	-7.8138	37.8881	25.5873	20.5589	40.1693	8.5762	4.3618
$\alpha = 1.2$, $\sigma = 0.8$	-45.2709	-13.9355	-9.3877	-19.0633	-13.1722	-11.1970	40.6525	8.7924	4.5635
$\alpha = 3$, $\sigma = 1.2$	-48.6200	-16.6447	-11.7576	-1.9420	-1.4060	-1.2323	42.0571	9.2918	4.7959

3. Characterization Results

Accurately characterizing probability distributions is pivotal across diverse fields, as it facilitates profound insights into complex phenomena. The characterization of continuous probability distributions has been extensively investigated, with seminal contributions from researchers including Glänzel [25, 26] and Hamedani [27], who have pioneered various techniques. In this section, we have rigorously established the characterizations of the MFD by examining its truncated moments and hazard function.

3.1. Characterization based on truncated moments

The characterization of the probability distributions through truncated moments was initially pioneered by Galambos and Kotz [28]. Building on this foundational work, numerous scholars have made significant contributions to the field. Among the most notable are Kotz and Shanbag [29], as well as Glänzel *et al.* [30] with further advancements by Glänzel [25, 31]. The characterization of the MFD using truncated moments is an extension of these efforts, specifically developed in accordance with Theorem 3.1 from [25] which is stated as follows,

Theorem 3.1. *Let (Ω, Σ, P) be a given probability space, and let $D = [\alpha, \beta]$ be an interval for some $a < b$ ($\alpha = \infty, \beta = -\infty$ might as well be allowed). Let $X : \Omega \to D$ be a continuous random variable with* *distribution function $G(x)$ and let κ_1 and κ_2 be two real functions defined on D such that*

$$E[\kappa_1(X)|X \geq x] = E[\kappa_2(X)|X \geq x]\zeta(x), x \in D$$

is defined with some real function ζ. Assume that $\kappa_1, \kappa_2 \in C_1(D), \zeta \in C_2(D)$, and $G(x)$ is a twice continuously differentiable and strictly monotone function on the set D. Finally, assume that the equation $\kappa_2\zeta = \kappa_1$ has no real solution in the interior of D. Then G is uniquely determined by the functions κ_1, κ_2 and ζ. In particular,

$$G(x) = \int_a^x C \left| \frac{\zeta'(\nu)}{\zeta(u)\kappa_2(u) - \kappa_1(u)} \right| e^{-\tau(\nu)}$$

where the function τ is a solution of the differential equation $\tau' = \frac{\zeta'\kappa_2}{\zeta\kappa_2 - \kappa_1}$ and C is a constant chosen to make $\int_D dG = 1$.

The above theorem has the advantage that the cdf G is not required to have a closed form and is given in terms of an integral whose integrand depends on the solution of a first-order differential equation, which can serve as a bridge between probability and differential equation.

Proposition 3.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable, and let $\kappa_2(x) = \left(\alpha^\beta + e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^2$ and $\kappa_1(x) = \kappa_2(x)e^{-\left(\frac{\sigma}{x}\right)^\lambda}$ for $x > 0$. The pdf of X is Eq.6 if and only if the function ζ defined in Theorem 3.1 has the form*

$$\zeta(x) = \frac{1}{2}e^{-\left(\frac{\sigma}{x}\right)^\lambda}, x > 0. \tag{13}$$

Proof. Let X have pdf Eq.6, then

$$(1 - G(x))E[\kappa_2(X)|X \geq x] = \left(1 + \alpha^\beta\right)\alpha^\beta e^{-\left(\frac{\sigma}{x}\right)^\lambda}, \qquad x > 0,$$

$$(1 - G(x))E[\kappa_2(X)|X \geq x] = \frac{\left(1 + \alpha^\beta\right)\alpha^\beta}{2}e^{-2\left(\frac{\sigma}{x}\right)^\lambda}, \qquad x > 0,$$

and then

$$\zeta(x)\kappa_2(x) - \kappa_1(x) = -\frac{1}{2}e^{-\left(\frac{\sigma}{x}\right)^\lambda}\left(\alpha^\beta + e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^2 < 0, \quad for \quad x > 0.$$

Conversely, if ζ is given as Eq.12, then

$$\tau'(x) = \frac{\zeta'(x)\kappa_2(x)}{\zeta(x)\kappa_2(x) - \kappa_1(x)} = -\lambda\sigma^\lambda x^{-(\lambda+1)}, x > 0,$$

and hence,

$$\tau(x) = \left(\frac{\sigma}{x}\right)^\lambda$$

or

$$e^{-\tau(x)} = e^{-\left(\frac{\sigma}{x}\right)^\lambda}.$$

Now, using Theorem 3.1, X has the pdf Eq.6. ■

3.2. Characterization based on hazard function

The hrf $h(x)$ of a twice differentiable distribution function $F(x)$ and its corresponding pdf $f(x)$ satisfy the first-order differential equation:

$$\frac{f'(x)}{f(x)} = \frac{h'(x)}{h(x)} - h(x). \tag{14}$$

For many univariate continuous distributions, this is the sole characterization expressible in terms of the hazard function. Hamedani and Ahsanullah [32] provided characterizations of certain widely recognized distributions grounded in the hazard function. The following characterization introduces a non-trivial distintion for the MFD when $\beta = 1$, diverging from the aforementioned trivial form.

Proposition 3.2. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is Eq.6 if and only if its hazard function $h(x)$ satisfies the differential equation*

$$x^{\lambda+1}h'(x) + (\lambda+1)x^{\lambda}h(x) = \frac{d}{dx}\left[\frac{(1+\alpha)\lambda\sigma^{\lambda}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(\alpha+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}})(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}\right]. \tag{15}$$

Proof. When $\beta = 1$, the pdf $f(x)$ and hrf $h(x)$ of X are respectively

$$f(x) = \frac{(1+\alpha)\left(\lambda\alpha\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}, \qquad x > 0, \alpha, \sigma, \lambda > 0. \tag{16}$$

and

$$h(x) = \frac{(1+\alpha)\left(\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}, \qquad x > 0, \alpha, \sigma, \lambda > 0. \tag{17}$$

Then we have

$$\frac{f'(x)}{f(x)} = -\frac{(\lambda+1)}{x} + \lambda\sigma^{\lambda}x^{-(\lambda+1)} - \frac{2\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)}. \tag{18}$$

Using Eq.14 we can write,

$$\begin{aligned} h'(x) + h(x)\frac{(\lambda+1)}{x} &= \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)} + \frac{(1+\alpha)^2\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2} \\ &\quad - \frac{2(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}\left(e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}, \end{aligned}$$

which implies,

$$\begin{aligned} x^{\lambda+1}h'(x) + (\lambda+1)x^{\lambda}h(x) &= \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)} + \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-(\lambda+1)}\left(e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2} \\ &\quad - \frac{(1+\alpha)\lambda^2\sigma^{2\lambda}x^{-(\lambda+1)}\left(e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)^2\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}. \end{aligned}$$

Now, Eq.15 holds, then

$$\frac{d}{dx}\left[x^{\lambda+1}h(x)\right]=\frac{d}{dx}\left[\frac{(1+\alpha)\,\lambda\sigma^{\lambda}e^{-(\frac{\sigma}{x})^{\lambda}}}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)}\right]$$

from which we obtain

$$h(x)=\frac{(1+\alpha)\left(\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\right)}{\left(\alpha+e^{-(\frac{\sigma}{x})^{\lambda}}\right)\left(1-e^{-(\frac{\sigma}{x})^{\lambda}}\right)},$$

which is the hrf of MFD when $\beta=1$. ■

4. Maximum Likelihood Estimation

This section provides the parameter estimates for the MFD derived through the maximum likelihood method. This method is widely recognized as the predominant approach in statistical inference. The log-likelihood for $\theta=(\alpha,\beta,\sigma,\lambda)^T$ based on a given sample is given by;

$$\begin{aligned}\log L(\alpha,\beta,\lambda,\sigma)=&n\log(1+\alpha^{\beta})+n\log(\lambda)+n\beta\log(\alpha)+n\lambda\log(\sigma)-\\&(\lambda+1)\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda}-2\sum_{i=1}^{n}\log\left[\alpha^{\beta}+e^{-(\frac{\sigma}{x_i})^{\lambda}}\right].\end{aligned}\tag{19}$$

To obtain the maximum likelihood estimators (MLE) of the MFD, we maximize the log-likelihood function. This is accomplished by taking the first derivative of the Eq.19 with respect to parameters α,β,λ and σ.

$$\frac{\partial\log L(\alpha,\beta,\lambda,\sigma)}{\partial\alpha}=\frac{n\beta\alpha^{\beta-1}}{1+\alpha^{\beta}}+\frac{n\beta}{\alpha}-2\sum_{i=1}^{n}\left(\frac{\beta\alpha^{\beta-1}}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right),$$

$$\frac{\partial\log L(\alpha,\beta,\lambda,\sigma)}{\partial\beta}=\frac{n\alpha^{\beta}log\alpha}{1+\alpha^{\beta}}+nlog\alpha-2\sum_{i=1}^{n}\left(\frac{\alpha^{\beta}log\alpha}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right),$$

$$\begin{aligned}\frac{\partial\log L(\alpha,\beta,\lambda,\sigma)}{\partial\sigma}=&\frac{n}{\lambda}+n\log\sigma-\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\left(\frac{\lambda}{\sigma}\right)\left(\frac{\sigma}{x_i}\right)^{\lambda}-\\&2\left(\frac{\lambda}{\sigma}\right)\sum_{i=1}^{n}\left(\frac{\left(\frac{\sigma}{x_i}\right)^{\lambda}e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\log\left(\frac{\sigma}{x_i}\right)}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right).\end{aligned}$$

and

$$\frac{\partial\log L(\alpha,\beta,\lambda,\sigma)}{\partial\lambda}=\frac{n\lambda}{\sigma}+\sum_{i=1}^{n}\left(\frac{\lambda}{\sigma}\right)\left(\frac{\sigma}{x_i}\right)^{\lambda}-2\left(\frac{\lambda}{\sigma}\right)\sum_{i=1}^{n}\left(\frac{\left(\frac{\sigma}{x_i}\right)^{\lambda}e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}{\alpha^{\beta}+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right),$$

MLE $\hat{\theta}=(\hat{\alpha},\hat{\beta},\hat{\sigma},\hat{\lambda})$ of $\theta=(\alpha,\beta,\sigma,\lambda)$ can be obtained by solving simultaneously the following normal equations.

$$\frac{\partial\log L}{\partial\alpha}=0;\qquad\frac{\partial\log L}{\partial\beta}=0;\qquad\frac{\partial\log L}{\partial\sigma}=0;\qquad\frac{\partial\log L}{\partial\lambda}=0.$$

Table 3: *Simulation results.*

True value	n	Average Value	MSE	Bias
	50	10.0852	1041.897	-4.0852
	100	8.4113	188.1615	-2.4113
$\alpha = 6$	200	7.1525	52.4647	-1.1525
	300	6.8824	42.855	-0.8824
	500	6.4090	6.5330	-0.4090
	50	3.4438	18.3209	-0.4438
	100	3.4103	10.7163	-0.4103
$\beta = 3$	200	3.5286	30.133	-0.5286
	300	3.4228	10.4221	-0.4228
	500	3.3492	5.8894	-0.3492
	50	1.7126	4.7213	-0.7126
	100	1.3902	1.7355	-0.3902
$\sigma = 1$	200	1.2486	0.6024	-0.2486
	300	1.1863	0.3508	-0.1863
	500	1.1088	0.1051	-0.1088
	50	0.9450	0.0316	0.0550
	100	0.9521	0.0186	0.0479
$\lambda = 1$	200	0.9621	0.0113	0.0379
	300	0.9654	0.0082	0.0346
	500	0.9777	0.0043	0.0223

5. Simulation Study

In this section, we assess the accuracy of parametric estimation through Monte Carlo simulation. Using the quantile function of MFD given in Eq.8, we generate samples of observations for sizes $n = 50, 100, 200, 300$ and 500 with $N = 1000$ replications. Two sets of parameter values are considered; $\alpha = 6, \beta = 3, \sigma = 1, \lambda = 1$ and $\alpha = 1.2, \beta = 2.5, \sigma = 0.2, \lambda = 0.5$.

The numerical outcomes are evaluated using the R statistical programming language, leveraging the widely used optimization package 'optim'. The Average Value, Mean Square Error (MSE), and Average Bias are computed and displayed in Tables 3 and 4. The results indicate that as the sample size increases, the MSE decreases and the Average Value of each parameter converges to the initial parameter values. These findings demonstrate the accuracy and consistency of the estimation methods.

6. Applications

In this section, we fit the MFD model to a reliability data set to check the model's flexibility. The MFD was compared to that of Modi Exponentiated distribution (MED) by [19], Modi Exponentiated Exponential distribution (MEED) by [21] and Modi Weibull distribution (MWD) by [20]. The maximum likelihood method is employed to estimate the parameters for the candidate models. We evaluated different goodness-of-fit measures to illustrate the flexibility of the model. Specifically, $-logL$(negative log-likelihood function), W (Cramér-von Mises Statistic), A (Anderson-Darling Statistic) K_1S (Kolmogorov"Smirnov Statistic), AIC (Akaike Information Criterion), $CAIC$ (Akaike Information Criterion with correction), BIC (Bayesian Information Criterion) and $HQIC$ (Hannan"Quinn Information Criterion).
Where

Table 4: *Simulation results.*

True value	n	Average Value	MSE	Bias
$\alpha = 1.2$	50	4.2749	84.5601	-3.0749
	100	3.6350	69.6801	-2.4350
	200	2.9488	58.9493	-1.7480
	300	2.8059	100.5479	-1.6059
	500	2.2570	19.7147	-1.0570
$\beta = 2.5$	50	5.9501	103.8486	-3.4501
	100	4.8420	37.1668	-2.3420
	200	4.2936	36.3745	-1.7936
	300	3.9396	19.2506	-1.4396
	500	3.5822	10.1569	-1.0822
$\sigma = 0.2$	50	1.3159	45.1542	-1.1159
	100	0.8870	12.4818	-0.6870
	200	0.5636	6.9543	-0.3636
	300	0.3382	0.4328	-0.1382
	500	0.3496	1.0409	-0.1496
$\lambda = 0.5$	50	0.5014	0.0112	-0.0014
	100	0.4897	0.0086	0.0103
	200	0.4960	0.0056	0.0039
	300	0.4972	0.0039	0.0028
	500	0.4971	0.0034	0.0029

$$
\begin{aligned}
AIC &= -2logL + 2k, \\
CAIC &= -2logL + \frac{2kn}{(n-k-1)}, \\
BIC &= -2logL + klog(n), \\
HQIC &= -2logL + 2klog(log(n))
\end{aligned}
$$

where L is the likelihood function, k is the number of parameters of the model and n is the sample size. By respecting the standards in the field, the best model corresponds to smaller $-logL, K1S, AIC, CAIC, BIC, HQIC$, and greater p-value. Here, we used the "AdequacyModel" package in R programming language to obtain the MLEs and goodness-of-fit tests of the given data sets.

Data Set I: This data represents the total time on test plot analysis for mechanical components of the RSG-GAS reactor [33]

2.160 0.746 0.402 0.954 0.491 6.560 4.992 0.347 0.150 0.358 0.101 1.359 3.465 1.060 0.614 1.921 4.082 0.199 0.605 0.273 0.070 0.062 5.320.

Data Set II: data set is the information of the infant mortality rate per 1,000 live births for a few chosen nations in 2021, as reported by `https://data.worldbank.org/indicator/SP.DYN.IMRT.IN`

56 10 22 3 69 6 7 11 4 4 19 13 7 27 12 3 4 11 84 27 25 6 35 14 11 12 6

Table 5: *Basic statistical description of the dataset.*

Size (n)	Min.	Max.	Mean	Median	SD	Skewness	Kurtosis
23	0.06	6.56	1.58	0.61	1.93	1.36	3.54
27	3	84	18.81	11	20.51	1.95	3.05

Table 5 displays basic descriptive statistics of the datasets. Here, the distribution of the dataset shows a positive skewness and leptokurtic behaviour, which goes with the moment properties of this distribution. Figure 3 shows the boxplots and Figure 4 shows the TTT plots of the data set and it goes with the features of hrf of MFD.

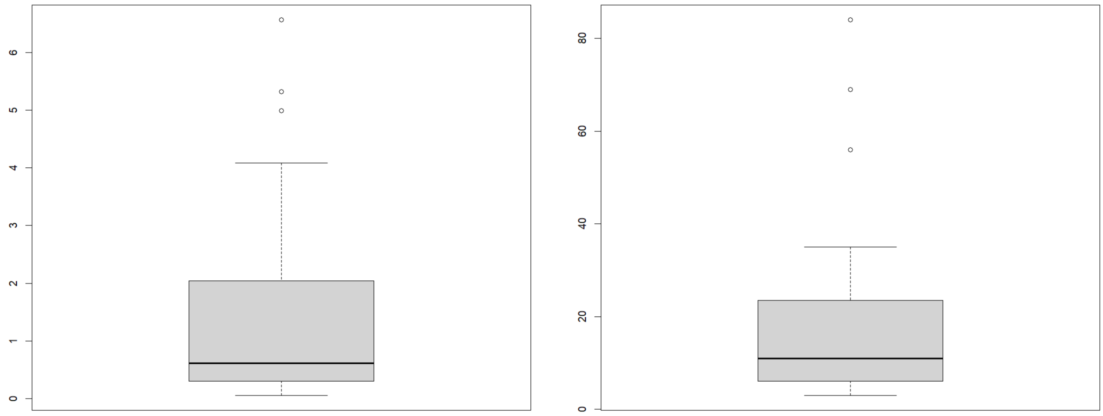


Figure 3: *The box plots of the first data set (left) and the second data set(right).*

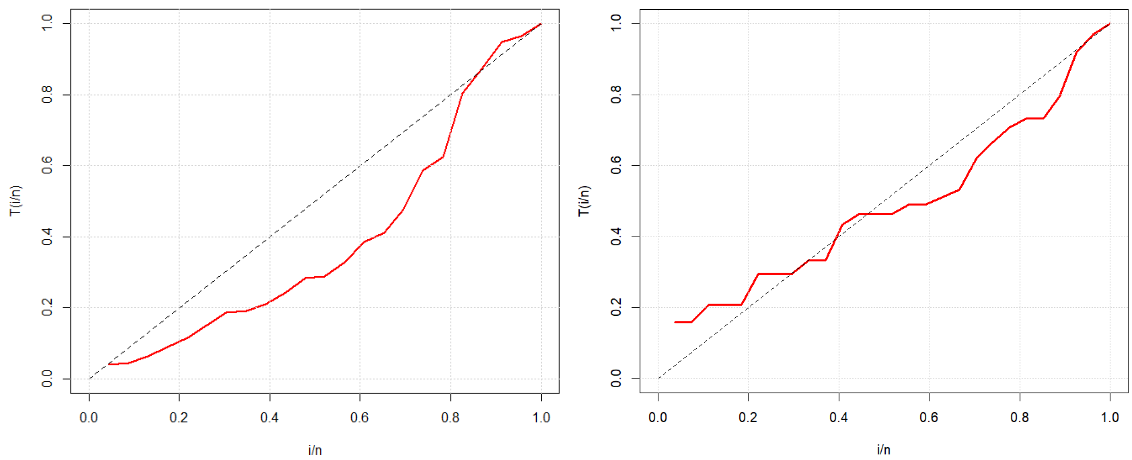


Figure 4: *The TTT plots of the first data set (left) and the second data set(right).*

Table 6: *The MLEs of the first data set.*

Model	MLEs	-log L
MFD	$\hat{\alpha} = 3.1367, \hat{\beta} = 5.9797, \hat{\sigma} = 0.8032, \hat{\lambda} = 0.3838$	33.0133
MWD	$\hat{\alpha} = 3.3513, \hat{\beta} = 0.9966, \hat{\sigma} = 0.7085, \hat{\lambda} = 0.9824$	34.2910
MEED	$\hat{\alpha} = 4.4807, \hat{\beta} = 3.2477, \hat{\sigma} = 0.4016, \hat{\lambda} = 0.5374$	33.4931
MED	$\hat{\alpha} = 5.8525, \hat{\sigma} = 9.6862, \hat{\lambda} = 0.8800$	34.8765

Table 7: *The goodness of fit statistics for the first data set.*

Model	W	A	AIC	BIC	CAIC	HQIC	K-S	p value
MFD	0.0471	0.3874	74.0266	78.5686	76.2488	75.1689	0.0971	0.9670
MWD	0.0544	0.3702	76.5819	81.1239	78.8041	77.7242	0.1827	0.3799
MEED	0.0864	0.5451	75.7530	79.1595	77.0161	76.6097	0.1700	0.4687
MED	0.0795	0.5052	74.9862	79.5282	77.2085	76.1285	0.1374	0.7273

Table 6 shows the results of the MLEs and negative log-likelihood values. From Table 7 we can conclude that MFD provides the lowest W, A, AIC, BIC, CAIC, HQIC, K-S values, and the largest p-value. Therefore, MFD is chosen as the best fit for the data.

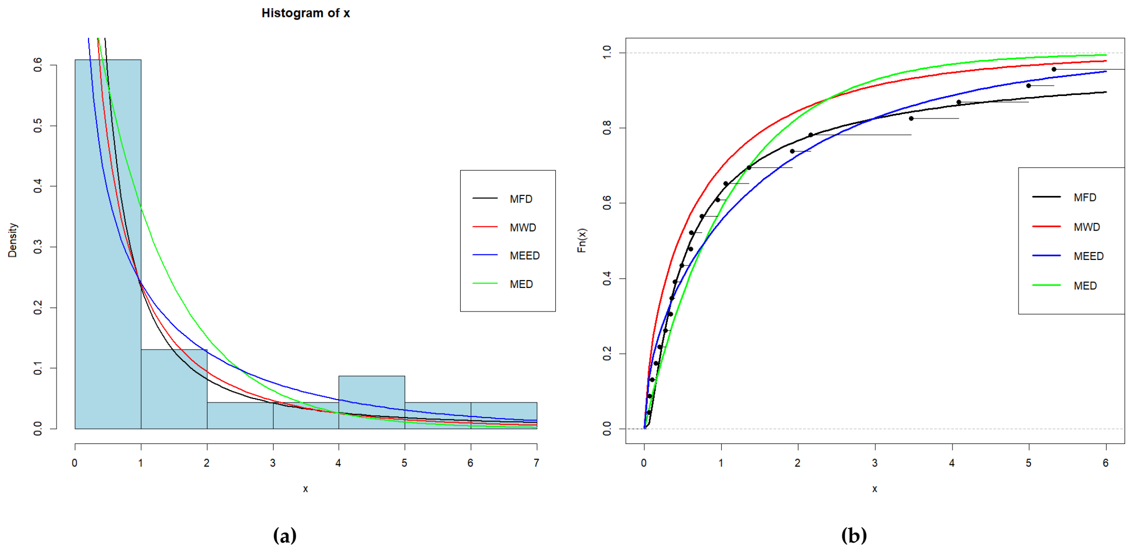


Figure 5: *Fitted pdf (a) and cdf (b) of distributions to the first data set.*

Figure 5 illustrates the fitted pdfs overlaid on the histogram and the corresponding cdfs for the dataset. The histogram indicates that the data distribution is unimodal and exhibits a pronounced positive skewness. The comparison of theoretical and empirical cdfs reveals that the MFD provides the closest fit to the empirical cdf, outperforming other distributions in terms of accuracy. To verify that the log-likelihood function behaves properly and that a distinct optimum has been attained, we plot the profiles of the log-likelihood function for the MF distribution under the first dataset and displayed in Figure 6.

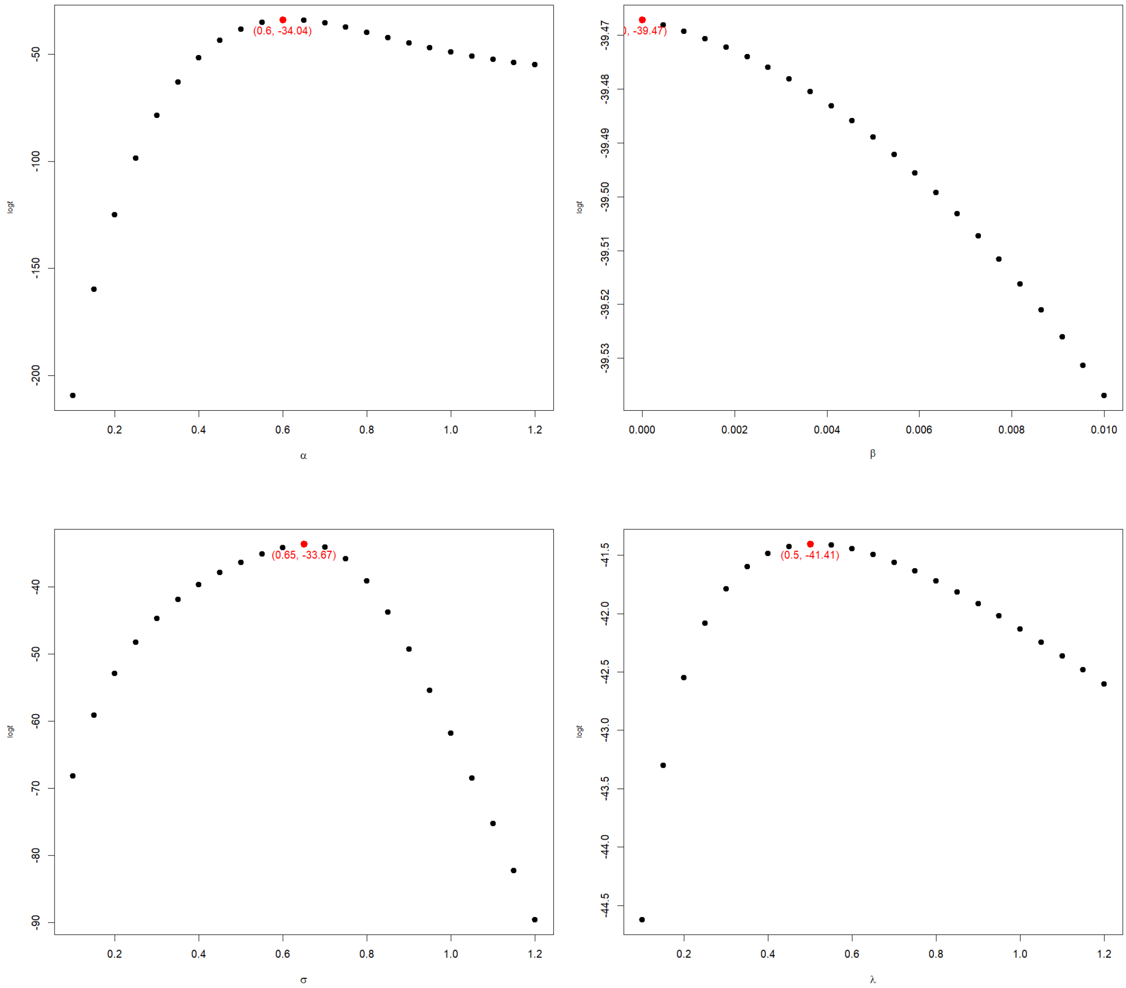


Figure 6: *Fitted profile of the log-likelihood function for the MLEs from the MFD based on the first data set.*

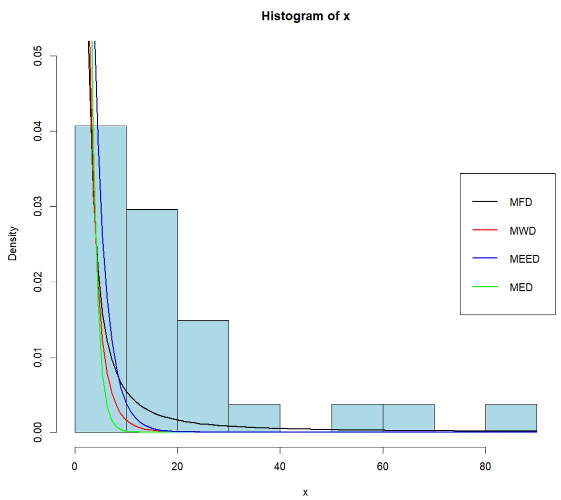
The performance of MFD for the second data was also compared to that of MWD, MEED and MED. The MLEs and goodness-of-fit statistics for the second data set are presented in Tables 8 and 9.

Table 8: *The MLEs of the second data set.*

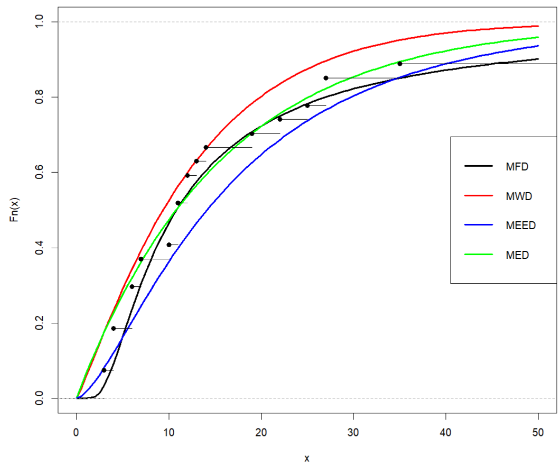
Model	MLEs	-log L
MFD	$\hat{\alpha} = 2.3680, \hat{\beta} = 9.4772, \hat{\sigma} = 1.2422, \hat{\lambda} = 8.0659$	102.7194
MWD	$\hat{\alpha} = 5.5851, \hat{\beta} = 11.4085, \hat{\sigma} = 1.1231, \hat{\lambda} = 12.9991$	109.7501
MEED	$\hat{\alpha} = 1.5881, \hat{\beta} = 0.7937, \hat{\sigma} = 0.0544, \hat{\lambda} = 1.5894$	104.7283
MED	$\hat{\alpha} = 9.4307, \hat{\sigma} = 16.8944, \hat{\lambda} = 0.0642$	106.7475

Table 9: *The goodness of fit statistics for the second data set.*

Model	W	A	AIC	BIC	CAIC	HQIC	K-S	p value
MFD	0.0459	0.3064	213.4387	218.6221	215.2569	214.9800	0.0992	0.9532
MWD	0.1477	0.9498	227.5002	232.6835	229.3183	229.0414	0.1752	0.3783
MEED	0.1274	0.8277	217.4565	222.6399	219.2747	218.9978	0.1706	0.4121
MED	0.1273	0.8277	219.4949	223.3825	220.5384	220.6509	0.1752	0.3790



(a)



(b)

Figure 7: *Fitted pdf (a) and cdf (b) of distributions to the second data set.*

Figure 7 presents the fitted pdfs and corresponding cdfs for the second dataset. The histogram reveals a unimodal distribution with notable positive skewness. The plot exhibits that the cdf of MFD is very closer to the empirical cdf than others. To further validate the model, we examine the behavior of the log-likelihood function for the MFD. The profiles of the log-likelihood function are plotted for the second dataset, (see Fig.8) confirming the proper behavior of the function and the attainment of a distinct optimum.

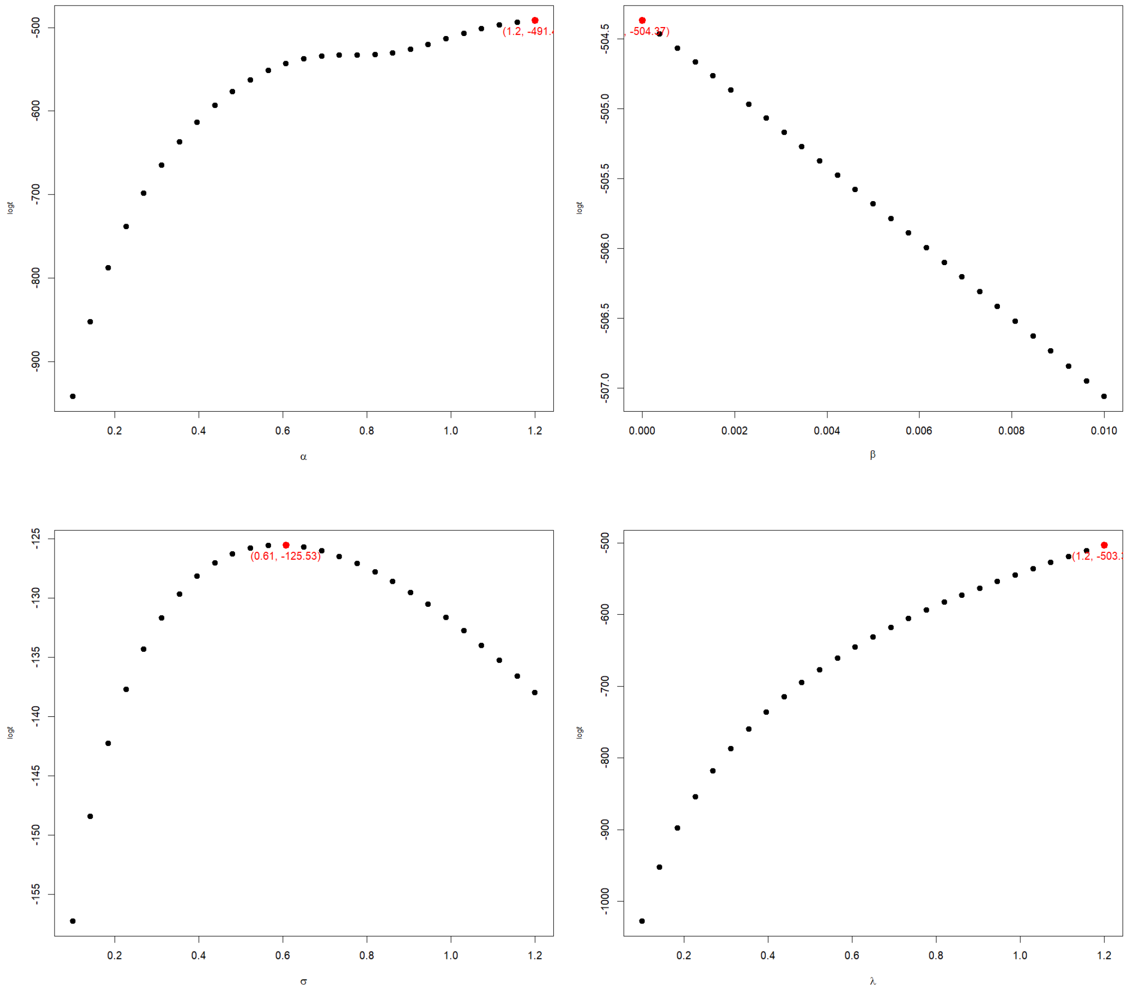


Figure 8: *Fitted profile of the log-likelihood function for the MLEs from the MFD based on the second data set.*

7. Conclusion

In this article, we proposed a new distribution based on the Modi family, namely MFD. Several statistical properties of the proposed distribution, such as moments, skewness, kurtosis, stochastic ordering, and entropy are evaluated. Two characterizations of the distribution are obtained using the hazard rate function and truncated moments. The simulation study showed the accuracy and consistency of the maximum likelihood estimation method. Two real-world data sets one from the reliability sector and the other from biomedical sector were used to demonstrate the flexibility of the proposed model. The MFD provided the best fit for the data compared to other sub-models in the family.

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