OPTIMIZING A LINEAR FRACTIONAL FUNCTION OVER AN INTEGER EFFICIENT SET

Leila YOUNSI-ABBACI

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University of Bejaia, Department of Electrical Engineering, Faculty of Technology,
Research Unit LaMOS, 06000 Bejaia, Algeria
leila.abbaci@univ-bejaia.dz

Abstract

Over recent decades, significant advancements have been made in optimization over the efficient set. This paper introduces a novel exact algorithm designed to optimize a linear fractional objective function over the integer efficient set of a multi-objective linear programming problem (MOILP). Without enumerating all efficient solutions, our method employs a selection strategy to iteratively improve the primary objective while progressively refining the feasible region and excluding dominated points. By exploring edge connections within the truncated feasible space, the proposed algorithm ensures convergence to the global optimal value in a finite number of iterations. A numerical example demonstrates the algorithm's effectiveness and practical application. This approach addresses critical challenges in multi-objective integer programming, particularly the nonconvexity of the efficient set and the absence of explicit feasible set descriptions.

Keywords: multiple objective programming, integer programming, linear fractional programming, efficient solutions.

1. Introduction

Multi-objective integer programming (MOIP) is an important research area as many practical situations require discrete representations by integer variables and many decision makers have to deal with several objectives. Some note-worthy practical environments where the MOIP problems find their applications are supply chain design, logistics planning, scheduling and financial planning.

In the past two decades, researchers and practitioners have shown increased interest in the problem of optimizing a linear function on the efficient set of multiple objective linear programming problem (MOLP). Several methods and algorithmic ideas have been developed-in general, these approaches can be classified and grouped according to the methodological concepts-which include, among others, adjacent vertex search technique ([16, 9, 10], nonadjacent methods [7], dual approach [19], etc. An overview of these approaches can be found in Yamamoto [21].

In addition to the continuous case, few algorithms have been suggested for solving the problem involving discrete decision variables. For the first time in [15] made an attempt to optimize on the integer efficient set, where only an upper bound value for the main objective is proposed. Jorge [13] developed approach that defines a sequence of progressively more constrained single-objective integer problems that successively eliminates undesirable points.

Fractional programming is an optimization problem in which ratio of two linear functions is optimized subject to some constraints [5, 14]. Integer Linear Fractional Programming problem is an important class of problems arising in criteria Decision Making when some or all the model variables represent discrete decisions.

In preparing this paper, a special effort has been made do make certain that it is self-contained and that it is suitable both a as a text and as a reference. within we developed an algorithm that optimized linear fractional function ever the efficient set of a MOILP without explicitly having to enumerate all the efficient solutions. Given a Integer Linear Programming problem with Multiple Objective (MOILP):

$$(P_D)\begin{cases} \text{"}max\text{"} & Z_i = C_i x, \qquad\qquad i \in \{1,...,p\} \\ \text{s.t.} & x \in D \end{cases} \tag{1}$$

Where $C_i \in R^n$, for each $i \in \{1,...,p\}$, $A \in^{m\times n}$, $b \in^m$ and D is a polyhedral set of ^{n} defined as D=$\{x \in^n | Ax = b, x \geq 0, integer\}$. To avoid the technicality we assume throughout the paper that D is nonembounded.

The search of specific methods for solving (1) that provide the decision maker with his/her preferred efficient solution without having to explicitly determine the set of all efficient solutions of (1)denoted by $E(P_D)$, efficiency and non-dominance are defined as follows (see [17, 23, 24]) is doubtless a very difficult task that can be tackled in many different ways. One of such approaches, that has been studied successfully by Philip [16], in which an algorithm based on moving to adjacent efficient vertices is outlined when $\Phi(x)$ is a linear function, and lots of papers followed his work [22]. Our aim in this study is to provide one approach in the discrete case, consists of optimizing $\Phi(x)$ a Linear Fractional function representing the preferences of the decision-maker over the efficient set of (1). Formally, the problem under consideration can be defined as:

$$(P_E)\begin{cases} max & \Phi(x) = \dfrac{Ux+\alpha}{Vx+\beta} \\ \text{s.t.} & x \in (P_D) \end{cases} \tag{2}$$

Where α, β are scalars; $p, q \in R^n$.

The main difficulty of the problem arises from the nonconvexity of the efficient set ($E(P_D)$), which is the union of several faces of X. This problem was first considered by [16], in which an algorithm based on moving to adjacent efficient vertices is outlined when ¶ is a linear function, and lots of papers followed his work.

It is worth noting that solving (2) involves several difficulties since its feasible set, ($E(P_D)$), is not explicitly known, nor a convenient implicit description (say, e.g., integer linear) is available. As a consequence, (2) is a global optimization problem, frequently with multiple local (not necessarily global) optima [[22], [11]]. However, some particular instances of problem (1) can be solved straightforwardly, due to their special characteristics. More precisely, when the multi objective problem IP is completely efficient [2].

Generally, $E(P_D) \neq D$. Otherwise, if (D) is completely efficient, $E(P_D)$ can be substituted by D and, in such cases, solving (P_E) is equivalent to solving the following program:

$$(P_{E-relaxed})\begin{cases} max & \Phi(x) = \dfrac{Ux+\alpha}{Vx+\beta} \\ \text{s.t.} & x \in D \end{cases} \tag{3}$$

2. The main results

Definition 1. A point $x^0 \in D$ is said to be efficient of (1) if and only if there does not exist another point $x^1 \in D$ such that $Z_i(x^1) \geq Z_i(x^0)$ for all $i \in \{1,...,p\}$ and $Z_i(x^1) > Z_i(x^0)$ for at least one $i \in \{1,...,p\}$.

2.1. Testing Efficiency

The following result (see [12]) is used in various steps of the algorithm to test the efficiency of a given feasible solution of problem (1).

Theorem 1. Let x^* be an arbitrary element of the region D. $x^* \in E^{FF}$ if and only if the optimal value of the objective ψ is null in the following mixed integer linear programming problem:

$$(P_{x^*}) \begin{cases} \max & \psi(x) = \sum_{i=1}^{p} \Psi_i \\ \text{s.t.} & Cx - I\Psi = Cx^* \\ & x \in D, \Psi_i \in \mathbb{R}^+; \forall i \in \{1, ..., p\}. \end{cases} \tag{4}$$

C is a matrix (p, n) of which her $i^{ème}$ line corresponds to c^i, $i = 1, 2, ..., p$, I is the matrix identity (p, p) and $\Psi = (\Psi_i)_{i=1,...,p}$. The problem (P_{x^*}) .

Is often used to test the efficiency of a given point. (P_{x^*}) can be also used to generate an efficient point even x^* is not efficient ([9]).

2.2. Notation and Definitions

- $x_k = x_{k,j}$ is one optimal integer solution obtained in $D_k$at step k.
- B_k is the basis associated with solution x_k;
- $a_{k,j} \in R^{m_k \times 1}$ is the activity vector of $x_{k,j}$ with respect to the current truncated region D_k;
- $I_k = \{j \mid$ the vector $a_{k,j}$ is a column of the basis $B_k\}$ (indices of basic variables);
- $N_k = \{j \mid$ the vector $a_{k,j}$ is not a column of the basis $B_k\}$ (indices of non-basic variables);
- $y_{k,j} = (y_{k,ij}) = (B_k)^{-1} a_{k,j}$, where $y_{k,j} \in R^{m_k \times 1}$;
- $U_j =$ the j^{th} component of vector U;
- $V_j =$ the j^{th} component of vector V;
- $p_{k,j} = \sum_{i \in I_k} p_i y_{k,ij}$
- $q_{k,j} = \sum_{i \in I_k} q_i y_{k,ij}$
- $Z_1(x_k) = \dfrac{Z_{k,1}}{Z_{k,2}} = \dfrac{Ux_k + \alpha}{Vx_k + \beta}$
- $\gamma_{k,j} = Z_{k,2}(p_j - p_{k,j}) - Z_{k,1}(q_j - q_{k,j})$, the updated value of the j_{th} component of the reduce gradient vector $\bar{\gamma}_k$

Definition 2. Assume that $j_k \in N_k$ An edge E_{jk} incident to a solution X_k is defined as the set

$$E_{j_k} = \left\{ x_i \in R^{|I_k| + |N_k|} \;\middle|\; \begin{array}{l} x_i = x_{k,i} - \theta_{j_k} y_{k,ij_k} \; for \; i \in I_k \\ x_{j,k} = \theta_{j,k} \\ x_\alpha = 0 \; for \; all \; \alpha \in N_k \setminus \{j_k\} \end{array} \right\}$$

Where $0 < \theta \leq \min_{i \in I_k} \{ \dfrac{x_{k,i}}{y_{k,ij_k}} | y_{k,ij_k} > 0 \}$, θ_{j_k} is a positive integer and $\theta_{j_k} \times y_{k,ij_k}$ for $i \in I_k$ are integers for all $i \in I_k$ if such integer values exist.

Theorem 2. [14] Let X_1 be an optimal solution of problem (3) All integer feasible solutions of problem (3) alternate to X_1 on an edge E_{j_1} of region D (or truncated region D_1) emanating from it, in the direction of vector a_{1,j_1}, $j_1 \in J_1$ with $J_1 = \{j \in N_1 \mid \overline{\gamma}^1_{1,j} = 0\}$ lie in the open half space

$$\sum_{j \in N_1 \setminus \{j_1\}} x_j < 1$$

Theorem 3. [4] The point x_1 of D is an optimal solution of problem (3) if and only if the reduce gradient vector $\bar{\gamma} = \bar{\beta} p - \bar{\alpha} q$ is such that $\bar{\gamma}_j \leq 0$ for all $j \in_k$.

Theorem 4. [17] $x^* \in E(P_D$ if, and only if, $\{(x^* + C^{\geq}) \cap D = x^*\}$.

3. Development of the algorithm and theoretical results

The proposed algorithm provides a global optimal solution of (P_E) without specifying all efficient solutions of (P(D)).

Initially, we solve the relaxed problem (3) associated to problem (P_E). Obviously, only in a reduced number of special cases would the solution of (3) provide the optimal solution of (P_E). So if it were not the case, a new efficient solution dominating the previous one is then obtained. The efficient solution $\tilde{x}^l$ issued from the efficiency test is considered as a first efficient solution.

Assuming that all coefficients of matrix C are integers, at iteration k, the feasible set D is reduced gradually by eliminating all dominated solutions by $C(\tilde{x})^l$ (see Sylva and Crema, 2004, 2007). The resolution of the following problem enables us to perform this elimination:

$$(Rf_l) : max\{\frac{Ux + \alpha}{Vx + \beta} | x \in D - \cup_{s=1}^{l} D_s\} \tag{5}$$

$\{x^s; s = 1, ..., l-1\}$ are solutions of (P_D) obtained at iterations $1, 2, ..., l-1$ respectively. Where $D_s = \{x \in^n |Cx \leq Cx^s\}$ and $\{Cx^s\}_{s=1}^{l}$ is a subset of nondominated criteria vectors for problem (P_D).

$$D - \cup_{s=1}^{l} D_s = \left\{ \begin{array}{l} c^i x \geq (c^i x = 1) y_i^s + M_i(1 - y_i^s, \ i = 1, 2, ...p \ s = 1, 2, ..., l.) \\ \sum_{i=1}^{s} y_i^s \geq 1, \ s = 1, 2, ..., l \\ y_i^s \in \{0, 1\}, \ i = 1, 2, ...p \ s = 1, 2, ..., l \\ x \in D \end{array} \right\}$$

where M_i is a lower bound for any feasible value of the ith objective function. The associate variables y_i^s $i = 1, 2, ...p$ of $C\tilde{x}^s$ and additional constraints are added to impose an improvement on at least one objective function. Note that when $y_i^s = 0$, the constraint is not restrictive and when $y_i^s = 1$ a strict improvement is forced in the ith objective function evaluated at $C\tilde{x}^s$.

We start exploring all edges incident to $\tilde{x}^l$ corresponding to J_1 until an efficient solution is found to improve Φ_{opt}. We solve the problem (Rf_l). The optimal solution obtained, x^l, produces a minimum value of the criterion $\Phi(x)$ in the reduced domain. The process continue in this manner until the current feasible space becomes empty or $\Phi(x^l) > \Phi_{opt}$.

Proposition 1. [6] Let $\tilde{x}^1$ $\tilde{x}^2$,...,$\tilde{x}^l$ be efficient solutions to problem (P_D) and $D_s = \{x \in^n |Cx \leq Cx^s\}$. Let $\tilde{x}^*$ be an efficient solution to the multi-objective integer problem $P_k \equiv$ "max"$\{Cx, x \in D - \cup_{s=1}^{l} D_s\}$. Then $\tilde{x}^*$ is an efficient solution to the problem (P_D).

3.1. Theoretical Results

Proposition 2. Let $\tilde{x}^1$, $\tilde{x}^2$, ...,$\tilde{x}^l$ be efficient solutions to problem $P(D)$ and $D_s = \{x \in^n |Cx \leq Cx^s\}$.

Let $\ddot{x}^l$ be an alternative solution of x^l of the problem (Rf_l) with $\frac{Ux^{l+1} + \alpha}{Vx^{l+1} + \beta} > \max_{j \in 1, ..., l} \{\frac{U\tilde{x}^s + \alpha}{V\tilde{x}^s + \beta}\}$.

If $\ddot{x}^l$ is an efficient solution to problem $(P(D))$ then is an optimal solution of (P_E).

if problem (Rf_l) is unfeasible then $\{C\tilde{x}^s\}_{s=1}^l$ is the entire set of non-dominated criterion vectors for problem (P_D).

Proof. Suppose on the contrary that $\ddot{x}^l$ is not an optimal solution of (P_E). Then a feasible solution exists $\hat{x} \in E(P_D)$ such that with the value of the function main to the $\hat{x}$ point superior a $\frac{Ux^l+\alpha}{Vx^l+\beta}$. As $\ddot{x}^l$ is an alternative solution for x^l ($\Theta_{jl}^0 \neq 0$)to (Rf_l) because $\hat{x}^s$
$\frac{U\ddot{x}^l+\alpha}{V\ddot{x}^l+\beta} = \frac{Ux^l+\alpha}{Vx^l+\beta}$. Thus $\hat{x} \in \cup_{s=1}^l D_s$ therefore $\hat{x} \in D_s$ for some $s \in \{1, ..., l\}$ and, accordingly to the definition of D_s, $C\hat{x} \leq C\tilde{x}^s$. As $\hat{x} \in E(P_D$ we have that $\frac{U\hat{x}+\alpha}{V\hat{x}+\beta} < \frac{U\tilde{x}^s+\alpha}{V\tilde{x}^s+\beta}$.

consequently $\frac{U\ddot{x}^l+\alpha}{V\ddot{x}^l+\beta} = \frac{Ux^l+\alpha}{Vx^l+\beta} < \frac{U\hat{x}+\alpha}{V\hat{x}+\beta} < \frac{U\tilde{x}^s+\alpha}{V\tilde{x}^s+\beta}$ who is contradicting with the hypothesis $\frac{U\ddot{x}^l+\alpha}{V\ddot{x}^l+\beta} > \max_{j\in 1,...,l}\{\frac{U\tilde{x}^s+\alpha}{V\tilde{x}^s+\beta}\}$.

If(Rf_l) is unfeasible then $E(P_D) \subseteq \cup_{s=1}^l D_s$ and for any $x \in E(P_D)$ there exists an x^s such that $Cx \leq Cx^s$. In this case we must proceed as follows: let $\tilde{x} \in E(P_D)$ for the reason there is an $\exists s \in 1, ..., l$ with $Cx^s \geq C\tilde{x}$ then $Cx^s = C\tilde{x}$ (and Cx is a dominated vector).

■

3.2. Algorithm

The algorithm used to obtain an integer optimal solution to our main problem (P_E) is can be summarized as follows:

Algorithm 1: part 1

input :
$A_{(m\times n)}$: matrix of constraits,
$b_{(m\times n)}$,
RHS vector,
$C_{(p\times n)}$: matrix of criteria.
$U_{(1\times n)}$, $V_{(1\times n)}$: main criterion vector,
α, β: are scalars.
output :
X_{opt}:optimal solution of the problem (P_E),
Φ_{opt}:optimal value of the main criterion Φ
initialization:
for $i \leftarrow 1$ **to** p **do**
    solve M_i= $min\{C^i x, \quad x \in D\}$ set the lower bounds;
$\Phi_{opt} := -inf$,
$l := 1$,
$E_1 :=$, $\bar{D} := D$,
$optimal := false$,
$alternative := false$,
$explore := true$.

Algorithm 2: part 2

while *optimal:=false* **do**
  solve $P^l_{RF} \equiv max\{\frac{Ux+\alpha}{Vx+\beta}, \quad x \in \bar{D}\}$;
  if P^l_{RF} *is infeasible* **then**
    X_{opt} an optimal solution of (P_E); $optimal := true$,**Terminate**;
  else
    let x^l bean optimal solution of P^l_{RF};
    efficiency test: solve $(p(x^l))$, Ψ is the optimal solution criteria;
    if $\Psi = 0$ **then**
      x^l an efficient solution; X_{opt} an optimal solution of (P_E) and $\Phi_{opt} = \Phi(x^l)$;
    else
      x^l is not efficient solution, $\tilde{x}^l$ an optimal solution of $(p(x^l))$ is efficient;
      solve $Q(\tilde{x}^l) \equiv max\{\frac{Ux+\alpha}{Vx+\beta}, \quad x \in \bar{D}, \quad Cx = C\tilde{x}^l\}$;
      let $\hat{x}^l$ bean optimal solution of $Q(\tilde{x}^l)$
      if $\Phi(\hat{x}^l) > \Phi_{opt}$ **then**
        $X_{opt} = x^l$, $\Phi_{opt} = \frac{Ux^l+\alpha}{Vx^l+\beta}$, let $E_{l+1} = E_l \cup \{\hat{x}^l\}$;
      $l = l + 1$ and $\bar{D} := D \cup_{s=1}^{l-1} D_s$; $D_s = \{x \in Z^n / Cx \leq C\hat{x}^l, \quad \hat{x}^l \in E_{l-1}\}$
      solve$P_l \equiv \{max\frac{Ux+\alpha}{Vx+\beta}, \quad x \in \bar{D}\}$, let x^l an optimal of P_l;
      if $\bar{D} =$ *or* $\Phi(x^l) < \Phi_{opt}$ **then**
        X_{opt} an optimal solution of (P_E) and Φ_{opt} the optimal value of (P_E)
        $optimal := true$,**Terminate**;
      else
        optimal:=False ; solve $(P(x^l)$
        if $\Psi = 0$ **then**
          x^l an efficient solution $X_{opt} = x^l$; optimal:=True; $E_{l+1} = E_l \cup \{x^l\}$;
          Terminate;
        else
          $\bar{x}^l$ is an optimal solution of $P(x^l)$ is efficient $E_{l+1} = E_l \cup \{x^l\}$;
          construct the set $\Gamma_l = \{j \in N_l / \bar{\gamma}^l_j = 0\}$;
          if $\Gamma_l \neq$ **then**
            i:=1;
            while $\Gamma_l \neq$ *and explore:= true* **do**
              (search $\ddot{x}^l_1$ integer efficient solution for x^l) calculate $\Theta^0_{jl}$the integer part of $min_{i \in I_k}\{\frac{x^1_{l,i}}{x^1_{l,ij_l}} / y^1_{l,ij_l} > 0\}$;
              if $\Theta^0_{jl} = 0$ **then**
                $\Gamma_l = \Gamma_l \setminus \{J_l(i)\}$;
              else
                $\Theta := \Theta^0_{jl}$;
                while $\Theta > 0$ *and alternative:=False* **do**
                  searching for a efficient integer solution on edge E_{jl} corresponding to Θ^0_{jl} and test for efficiency, solve P(D)
                  if $\Psi = 0$ **then**
                    alternative:=true; $\Phi_{opt} := \Phi_{x_{expl}}$; optimal:=True;
                    $E_{l+1} = E_l \cup \{x_{expl}\}$; **Terminate**;
                  else
                    $\Theta := \Theta - 1$
              i:=i+1;
  l:=l+1;

Proposition 3. The algorithm terminates in a finite number of iterations.

Proof. By hypothesis provided D is non-empty and D is bounded, $\{Cx^s\}_{s=1}^{l}$ is finite. With the progression of to advance in the algorithm, the domain of feasibility becomes more and more is strictly reduced by the theorem (sylva [18],[6]) or Φ_{opt} strictly increases.

The theorem (3) guarantees that we can obtain an optimal solution integer of (Pf_R) if it exists, and the theorem (Testing efficiency 1 with [9]) one gets an optimal solution for the problem (2) having in mind that at least one new efficient solution is generated at each iteration since for an arbitrary l none of the previously generated efficient point is feasible, the proof is thus complete. ∎

4. Numerical illustration

To illustrate the use of this algorithm, we consider the following integer linear program with tow objectives:

$$(P_{(D)}) \begin{cases} max & Z_1 = x_1 - 2x_2 \\ max & Z_2 = -x_1 + 4x_2 \\ s.t. & -2x_1 + x_2 \leq 0, \\ & 6x_1 + x_2 \leq 21, \\ & -2x_1 + 4x_2 \leq 6, \\ & x_1, x_2 \in \end{cases} \quad (6)$$

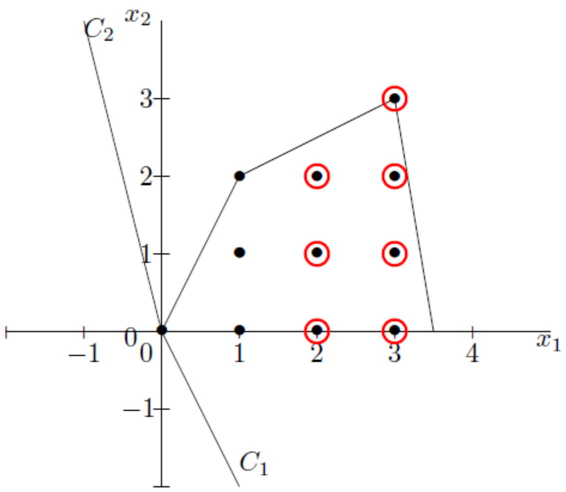

Figure 1: *Space of the decisions*

In this example, it is easy to see that D contains 11 feasible points (see Figure 1). Using the characterization of efficiency presented in Theorem (4), it can be shown that seven of them are efficient. Particularly, the efficient set $E(P_D)$ is given by: $^{FF} = \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(3,3)\}$. With the aim of illustrating how our algorithm works, we will solve the problem (2) given by

$$(Pf_{(E)}) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & x_1, x_2 \in_{FF} \end{cases} \quad (7)$$

step 0: Initialization We take $\Phi_{inf} = -\infty$, $\Phi_{sup} = +\infty$, $l = 1$.

After solving $\{\min C_i x,\ x \in D\}$ $i = 1, 2$, the lower bounds of the objective functions are $M_1 = -3$, $M_{12} = -3$

$$(Pf_{(R)}) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & -2x_1 + x_2 \leq 0, \\ & 6x_1 + x_2 \leq 21, \\ & -2x_1 + 4x_2 \leq 6, \\ & x_1, x_2 \in \end{cases} \tag{8}$$

(8) is solved, yielding the optimal solution $x^1 = (0, 0)$, Let $Z(x^1) = (0, 0)$.

- **Iteration 1.**
- **Step 1.** In order to test the efficiency of x^1 we solve the problem (9), that is:

$$(P_{(x^1)}) \begin{cases} max & \Theta = \Psi_1 + \Psi_2 \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 - \Psi_1 = 0 \\ & -x_1 + 4x_2 - \Psi_2 = 0 \\ & \Psi_i \geq 0, i = 1, 2. \end{cases} \tag{9}$$

The optimal value of (9) is 2 , which is achieved at the point $\hat{x}^1 = (2, 1)$. Thus, $\hat{x}^1 \in_{FF}$ and $x^1 \notin_{FF}$, since $C\hat{x}^1 \geq Cx^1$ We set $\Phi_{sup} = \Phi(x^1) = 1$

- **Step 2.** When (10) defined as:

$$(Tf_1) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 = 0 \\ & -x_1 + 4x_2 = 2 \end{cases} \tag{10}$$

is solved, $\tilde{x}^1 = \hat{x}^{1_1} = (2, 1)$ is obtained as the optimal solution. Let $\bar{z}^1 = C\tilde{x}^1 = (0, 2)$
$\Phi(\tilde{x}^1) = 1/5 > \Phi_{inf} = -\infty$, put $\Phi_{inf} = 1/5$ et $X_{opt} = \tilde{x}^1$
$\Phi_{inf} \neq \Phi_{sup}$, go to step 3

- **Step 3.** The optimal solution of (11)

$$(RF_1) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & x_1, x_2 \in D \\ & x_1 - 2x_2 \geq y_1^1 - 3(1 - y_1^1) \quad (1) \\ & -x_1 + 4x_2 \geq 3y_2^1 - 3(1 - y_2^1) \quad (2) \\ & y_1^1 + y_1^1 \geq 1, \quad y_1^1, y_2^1 \in \{0, 1\} \end{cases} \tag{11}$$

is $x^2 = (1, 2)$, $y = (0, 1)$, being $Z(x^2) = (-3, 7)$ and $\Psi = (0, 2)$. In order to test the efficiency of x^1 we solve the problem (9), that is:

$$(P_{(x^2)}) \begin{cases} max & \Theta = \Psi_1 + \Psi_2 \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 - \Psi_1 = -3 \\ & -x_1 + 4x_2 - \Psi_2 = 7 \\ & \Psi_i \geq 0, i = 1, 2. \end{cases} \tag{12}$$

The optimal value of (12) is 2 , which is achieved at the point $\hat{x}^2 = (3, 3)$; $\Psi = (0, 2)$. Thus, $\hat{x}^2 \in_{FF}$ and $x^2 \notin_{FF}$, We set $\Phi_{sup} = \Phi(x^2) = \dfrac{1}{3}$ go to step 4.

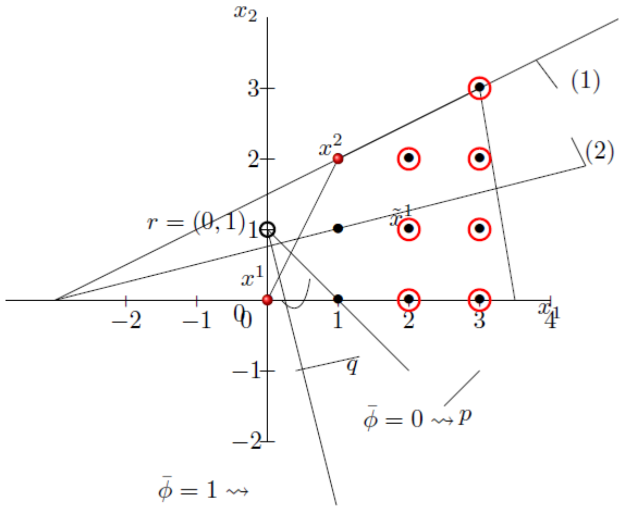

Figure 2: *The reduced region*D^1

- **Step 4.** $J_2 = \{j \in N_2 \mid \overline{\gamma}^2_{1,j} = 0\} = \emptyset$, go to step 2.

- **Iteration 2.**

- **Step 2.** When (10) defined as:

$$(Tf_1) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 = -3 \\ & -x_1 + 4x_2 = 9 \end{cases} \tag{13}$$

is solved, $\tilde{x}^2 = \hat{x}^2 = (3,3)$ is obtained as the optimal solution. Let $\bar{z}^1 = C\tilde{x}^1 = (0,2)$. $\Phi(\tilde{x}^1) = 5/17 > \Phi_{inf} = 1/5$, put $\Phi_{inf} = 5/17$ et $X_{opt} = \tilde{x}^2$
$\Phi_{inf} \neq \Phi_{sup}$, go to step 3

- **Step 3.**The optimal solution of (11)

$$(RF_1) \begin{cases} max & \Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1} \\ s.t. & (x_1, x_2) \in D \\ & x_1 - 2x_2 \geq y_1^1 - 3(1 - y_1^1) \quad (1) \\ & -x_1 + 4x_2 \geq 3y_2^1 - 3(1 - y_2^1) \quad (2) \\ & y_1^1 + y_1^1 \geq 1, \quad y_1^1, y_2^1 \in \{0,1\} \\ & x_1 - 2x_2 \geq -2y_1^2 - 3(1 - y_1^2) \quad (3) \\ & -x_1 + 4x_2 \geq 10y_2^2 - 3(1 - y_2^2) \quad (4) \\ & y_1^2 + y_2^2 \geq 1, \quad y_1^2, y_2^2 \in \{0,1\} \end{cases} \tag{14}$$

The problem (14))is note feasible. Terminate, $X_{opt} = x^2 = (3,3)$ is an optimal solution of (P_E) with $\Phi(x^2) = 5/17$.

The set of all solutions efficient of this problem is: $^{FF} = \{(2,0), (2,1), (2,2), (3,0), (3,1), (3,2), (3,3)\}$. However, our algorithm optimizes the linear fractional function $\Phi = \dfrac{x_1 + x_2 - 1}{5x_1 + x_2 - 1}$ without having to determine all these solutions but only $\{(2,1), (3,3)\}$.

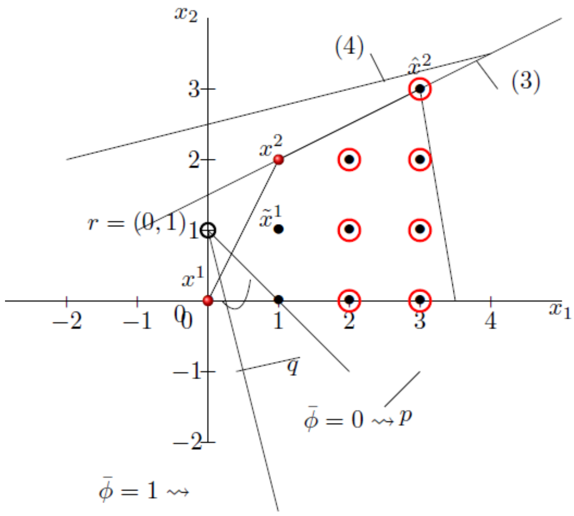

Figure 3: *The reduced region* D^2

5. Conclusion

The proposed algorithm optimizes a linear fractional function over the integer set of a multi-objective linear program (P_E) by using classical strategies of fractional programming and cutting plane techniques without having to enumerate all the efficient solutions. The main advantage of the proposed solution methodology is that no nonlinear optimization is required.

Although the research themes addressed is difficult, it is hoped that this article motivate the researchers to develop better solution procedures.

References

[1] Abbas, M. and Moulaï, M. (1999). An algorithm for mixed integer linear fractional programming problem. *Belgian Journal of Operations Research, Statistics and computer sciences*, Vol. 39/1 : 21–30.

[2] Benson, H. P. (1994). Optimization over the efficient set : four special cases. *Journal of Optimization Theory and Applications*, 80.

[3] Charnes, A. and Cooper, W. W. (1962). Programming with linear fractional functionals. *Naval Res. Logist. Quart.*, 9: 181–186.

[4] Cambini, A. and Martein, L. (1986). A modified version of Martos's algorithm for the linear fractional problem. *Methods of Oper. Res*, 53: 33–44.

[5] Charnes, A. and Cooper, N. (1961). management models and Industrial applications on linear Programming. *Joho Wiley, newyork*, Vol. 1.

[6] Crema, A. and Sylva, J. (2004). A method for finding the set of non-dominated vectors for multiple objective integer linear programs. *European Journal of Operational Research*, 158: 46–55.

[7] Dauer, J.P., Fosnaugh, T.A. (1995). Optimization over the efficient set. *Journal of Global Optimization*, 7, 3: 261–277.

[8] Ecker, J.G., Kouada, I.A. (1975). Finding Efficient Points for Multi-objectve Linear Programs. *Mathematical Programming*, 8: 375–9377.

[9] Ecker, J.G., Song, H.G. (1994). Optimizing a linear function over an efficient set. *Journal of Optimization Theory and Applications*, 83, 3: 541–563.

[10] Fül öp, J. (1994). Acutting planemethod for linear optimization over the efficient set. In:Komlosi, S., Rapcsak, T., Schaible, S. (eds.), Generalized Convexity. *Lecture Notes in Economics and Mathematics Systems, Springer, Berlin*, vol. 405: 374–385.

[11] Granot, D. and Granot, F. (1977). On integer and mixed integer fractional programming problems. *Ann. Discrete Math.*, 1: 221–231.

[12] Isermann, H. (1974). Proper efficiency and the linear vector maximization problem.*Operations Research*, 22: 189–191.

[13] Jorge, J.M. (2009). An algorithm for optimizing a linear function over an integer efficient set. *European Journal of Operational Research*, 195, 1: 98–103.

[14] Moulaï, M. and Abbas, M. (2002). Integer linear fractional programming with multiple objective. *Journal of the Italian Operations Research Society*, vol 1 N°1: 103–104,

[15] Nguyen, N.C.(1992). An algorithm for optimizing a linear function over the integer efficient set. *Konrad-Zuse-Zentrum fur Informationstechnik Berlin Preprint*, SC 92–23.

[16] Philip, J. (1972). Algorithms for the vector maximization problem. Mathematical Programming, 2: 207–229.

[17] Steuer, R. (1986). Multiple Criteria Optimization: Theory, Computation and Application. *John Wiley & Sons*.

[18] Sylva, J., Crema, A. (2007). A method for finding well-dispersed subsets of non-dominated vectors formultiple objective mixed integer linear programs. *Eur J Oper Res*, 180: 1011–91027.

[19] Thach, P.T., Konno, H., Yokota, D. (1996). Dual approach to minimization on the set of Pareto optimal solutions.*Journal of Optimization Theory and Applications*, 88, 3: 689–707.

[20] White, D.J. (1996). The maximization of a function over the efficient set via a penalty function approach. *European Journal of Operational Research*, 94, 1: 143–153.

[21] Yamamoto, Y. (2002). Optimization over the efficient set: Overview dedicated to Professor Reiner Horst on his 60th birthday. *Journal of Global Optimization*, 22, 1-4: 285–317.

[22] Younsi-Abbaci, L., Moulaï, M. (2021). Stochastic optimization over the Pareto front by the augmented weighted Tchebychev program. *Computational Technologies*, 26(3):86–9106. DOI:10.25743/ICT.26.3.006.

[23] Yu, P.L. (1985). Multiple Criteria Decision Making. *Plenum, New York*.

[24] Zionts, S. (1977). Integer programming with multiple objectives. *Annals of Discrete Mathematics*, 1 : 551–562.