UNRELIABLE M/G/1 QUEUE WITH GENERAL RETRIAL TIME, WORKING VACATION AND SETUP TIME

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Abstract

In the current article, a retrial queuing system with working vacations, interruptions, setup time, and perfect repair is analyzed. The scenario includes a server taking working vacations during empty periods without a complete halt of servicing customers; however, the rates of service remain reduced. Further, a setup time is included here, implying that if the server remains idle when a new customer enters, the state changes to inactive plus a setup duration before restarting operation. In this phase of setup, the setup failure happens and is replaced immediately before the server can proceed to normal operations. In addition to this, automatic power-off to conserve energy is there when no customer comes while the server is in vacation mode. Customers who find that the server cannot be accessed spend time waiting in retrial orbit instead of entering a normal queue. Here they're encouraged to try again for service after a random time. The steady state probability generating functions for system size and retrial group size are obtained by analyzing the system dynamics through the supplementary variable technique (SVT). Reliability and optimization analyses will be included in what will be studied from the system. Reliability concerns evaluating the chances of the server being available at different failure and repair sites while in the system, while optimization looks at the best configuration of system parameters that will work towards achieving greater efficiency and reduced delays. Explicit mathematical formulations can be obtained under ergodicity conditions describing the system size distribution and sojourn time and state probabilities. For a practical realization of the model, which numerically experiments would be carried out in Python, the theoretical results were validated .Such results therefore hold information on how direct retrials, setup times, service rates, and repair mechanisms affect overall system behavior. They also provide strong evidence for trade-offs between energy conservation on the one hand and reliability together with continuous service on the other. The proposed model together with practical implementation thus produces very significant inferences relevant to real service models in which the optimization of resources and efficiency of operation are critical.

Keywords: Retrial queue; working vacation; setup time; interruption; perfect repair ; general retrial times.

1. Introduction

Queueing theory is a fundamental branch of applied mathematics with widespread applications in various fields such as telecommunications, computer systems, transportation, and manufacturing. When servers are busy, customers often enter a retrial orbit, awaiting service availability according to predefined policies. The groundwork laid by Yang and Falin, Falin and Templeton [7], Yang et al. Extending by Artalejo, Gomez-Corral [8] , Arrar et al [2] [3] [4]. Queueing systems, particularly those involving retrials with vacations play a crucial role in balancing resource utilization and service efficiency. The queueing systems with server vacations introduced by Levy and Yechiali [14] are widely used in manufacturing systems, production systems, service systems, inventory systems, and other stochastic systems. In traditional scenarios, service stops when servers are on break, many researchers have worked on vacation interruption. Notably, Keilson and Servi [11] conducted significant research. One can refer also Li and Tian [15]. Takagi [21] contributed to the field by studying single-server queueing models with Bernoulli vacations. However, in many particular cases, an alternative approaches like working vacations provide service at a reduced rate during idle times, thereby enhancing the overall performance of the system. Numerous researchers have dedicated their efforts to develop models for queueing systems with working vacation concepts. Pioneering work on this subject was also carried out by Servi and Finn [18] considered an M/M/1 queue with working vacations, wherein vacation times are exponentially distributed. Most work on queueing models with working vacations can be found in Tian et al [19] and Chandrasekaran et al [6]. The M/M/1 retrial queue with working vacations was first studied by Do. Subsequently, Banik et al[5] analyzed a general working vacation queueing model. Arivudainambi et al [1] focused on analyzing a single-server retrial queue with working vacation dynamics. Zhang and Hou [20] studied M/G/1 queueing model with vacation interruption. And Gupta and Kumar [9] considered retrial queueing system with working vacation with breakdown and repair. Furthermore, the integration of setup time becomes essential, especially in the context of energy conservation. Setup time enables power-saving strategies by allowing server deactivation during periods of inactivity. Recognizing the importance of power conservation, several researchers have explored queueing models incorporating setup time. Phung-Duc [16] [17], for instance, integrated the notion of setup time into retrial queueing systems. Gupta and Kumar [10] analyzed retrial queue with feedback, setup time, working vacation perfect repair. For further insights into related research, readers may refer to Manoharan and Jeeva [12] [13]. And Pannom Gupta. The model under investigation is an M/G/1 retrial queue with a working vacation and setup time. This system represents a complex real-life case where customers may encounter delays due to server unavailability, requiring them to reattempt service after a waiting period. The setup times can significantly impact service efficiency and system performance. We extend the results obtained by considering the general law of inter-retrial times and service times. Our study aims to provide a detailed analysis of this type of complex system. We introduce the Markov chain to prove the stability condition of the studied model. Using the method of supplementary variables, we obtain the partial generating functions and the limiting distributions that this type of model can possess. We also present some performance measures of the studied model.

2. The model

In this paper, we consider an M/G/1 unreliable retrial queue with single working vacation , setup time. The model is described in great detail as follows:

1. The arrival of consumers follows a Poisson process with a rate $\lambda > 0$. If a consumer arrives and the server is idle, they begin service immediately . on the other hand, if he finds it busy, on working vacation, on setup, or broken then the consumer leaves the service area and enters a pool of blocked consumers called orbit in accordance with an First Come First Served discipline. While in orbit, the consumer waits a random amount of time before retrying. The inter-retrial times follow an arbitrary probability distribution function $N(x)$ with corresponding LST function $\mathcal{N}^*(\theta)$.
2. The server promptly initiates the normal service for the new or retrial consumers upon their arrival while in an idle state. The normal service time distributed with general distribution function $G(x)$ having LST $\mathcal{G}^*(\theta)$.
3. in the case of the orbit is empty.The server automatically begins a single working vacation, which follow an exponential distribution with rate ω .If a consumers comes up during the working vacation , the consumers are served at a reduced rate. when the working vacation is finished the server resumes normal service . while the working vacation period , The service time distributed with general distribution function $W_v(x)$ having LST $\mathcal{W}_v^*(\theta)$.

4. In the final of working vacation , if no consumers are waiting for their turn , the server is shut down directly to save power .
5. in the off-state of the server, if any client arrives he will wait for his turn in forward position of the server until it is activated (setup time). the setup time is assumed to follow general distribution with probability distribution function $T_s(x)$ with corresponding LST function $\mathcal{T}_s^*(\theta)$. The customers arriving in the setup state, have to join the orbit.
5. whene the server undergoes to the set-up state , this operation may fail with probability $\bar{\alpha} = (1-\alpha)$. Then the server is sent for repair and The repair time of the server has arbitrary probability distribution function $S_f(x)$ with corresponding LST function $\mathcal{S}_f^*(\theta)$.

2.1. Practical justifications of the suggested model

Consider a manufacturing system consisting of a paper recycling machine , a Foreman(server) and a worker(assistant) to operate the machine . The foreman will operate the machine if the waste paper (customer) is available and produce the products.if this last is not available due to transport issues , then the foreman may go on vacation .During the vacation period of the foreman, if waste paper becomes available then the worker will operate the machine , but the production will be relatively at a slow speed (working vacation) . When a batch of the product is completed, then the worker will call the foreman to resume the production at a higher speed (vacation interruption). In another situation, if the foreman's vacation period completes, he will return to the production to operate the machine . If the waste paper is available then he will manage the production at a higher speed otherwise, if the waste paper is not available, to save power, he may turn off the machine. Again, the availability of new waste paper will initiate the setup of the machine (setup time) and production starts again if setup occurs successfully otherwise the machine will be sent for repair, and during this period there will be no production.

3. Steady-state analysis

Let $\chi_1(t), \chi_2(t), \chi_3(t), \chi_4(t), \chi_5(t)$ be the elapsed time in retrial, time in regular service, working vacation time , repair time and setup time sequentially at time t.
Let supose that :
$N(0) = 0, N(\infty) = 1, G(0) = 0, G(\infty) = 1, W_v(0) = 0, W_v(\infty) = 1, T_s(0) = 0, T_s(\infty) = 1, S_f(0) = 0, S_f(\infty) = 1$ are continuous at $x = 0$.consequently, we specify the hazard rate functions $f(x), \mu_n(x), \mu_w(x), v(x), \delta(x)$, for retrial, normal service, lower rate service, delayed repair and repair, respectively.

$$f(x)dx = \frac{dN(x)}{1-N(x)}$$
$$\mu_n(x)dx = \frac{dG(x)}{1-G(x)}$$
$$\mu_w(x)dx = \frac{dW_v(x)}{1-W_v(x)}$$
$$v(x)dx = \frac{dS_f(x)}{1-S_f(x)}$$
$$\delta(x)dx = \frac{dT_s(x)}{1-T_s(x)}$$

The state of the system at time t can be defined by the Markov process $\{N(t); t \geq 0\} = \{D(t), X(t), \chi_1(t), \chi_2(t), \chi_3(t), \chi_4(t), \chi_5(t) t \geq 0\}$, where $X(t)$ denotes the number of customers in the orbit at time t and

$$\text{D(t)=}\begin{cases} 0, & \text{if the server is idle in a normal period} \\ 1, & \text{if the server is idle in a working vacation period} \\ 2, & \text{if the server is busy on normal service} \\ 3, & \text{if the server is busy on working vacation period} \\ 4, & \text{if the server is on repair} \\ 5, & \text{if the server is on setup state} \end{cases}$$

respectively. If $D(t) = 0$ and $X(t) > 0$, then $\chi_1(t)$ represents the elapsed retrial time, if $D(t) = 2$, then $\chi_2(t)$ represents the elapsed service time during normal busy period at time t, if $D(t) = 3$ and $X(t) \geq 0$ then $\chi_3(t)$ represents the elapsed working vacation time at time t and if $D(t) = 4$ and $X(t) \geq 1$ then $\chi_4(t)$ represents the elapsed repair time at time t and if $D(t) = 5$ and $X(t) \geq 1$ then $\chi_5(t)$ represents the elapsed setup time at time t.

3.1. Stability and ergodicity Condition

Let $\{t_n; n \in N\}$ be the sequence of epochs of either service completion times or vacation termination time. The sequence of random vectors $Z_n = \{D(t_n+), X(t_n+)\}$ form a Markov chain which is the embedded Markov chain for our queueing system. Its state space is $S = \{0, 1, 2, 3, 4 \text{ and } 5\} \times N$.

Theorem 1. The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if
$\rho = \lambda \left(\frac{1}{\mu_n} + \frac{1}{\mu_w} + \omega \mathcal{W}_v^{*\prime}(\omega) \right) < \mathcal{N}^*(\lambda)$.

3.2. Equations Governing The System

For the Markov process $\{N(t); t \geq 0\}$, we define the probability

$$P_{00}(t) = \{D(t) = 0, X(t) = 0\} Q_0(t) = \{D(t) = 1, X(t) = 0\}$$

and the probability densities

$$\begin{aligned} P_n(x,t)dx &= \{D(t) = 0, X(t) = n, x \leq \chi_1(t) < x + dx\}, x \geq 0, n \geq 1 \\ M_{n,b}(x,t)dx &= \{D(t) = 2, X(t) = n, x \leq \chi_2(t) < x + dx\}, x \geq 0, n \geq 0 \\ G_{n,v}(x,t)dx &= \{D(t) = 3, X(t) = n, x \leq \chi_3(t) < x + dx\}, x \geq 0, n \geq 0 \\ U_n(x,t)dx &= \{D(t) = 4, X(t) = n, x \leq \chi_4(t) < x + dx\}, x \geq 0, n \geq 1 \\ K_n(x,t)dx &= \{D(t) = 5, X(t) = n, x \leq \chi_5(t) < x + dx\}, x \geq 0, n \geq 1 \end{aligned}$$

We assume that the stability condition is fulfilled in the sequel and so that we can set $P_{00} = \lim_{t\to\infty} P_{00}(t)$ and $Q_0 = \lim_{t\to\infty} Q_0(t)$ limiting densities for $x > 0$ and $n \geq 0$

$$M_{n,b}(x) = \lim_{t\to\infty} M_{n,b}(x,t), P_n(x) = \lim_{t\to\infty} P_n(x,t) \text{ and } G_{n,v}(x) = \lim_{t\to\infty} G_{n,v}(x,t)$$

$$\text{and } U_n(x) = \lim_{t\to\infty} U_n(x,t) \text{ and } K_n(x) = \lim_{t\to\infty} K_n(x,t).$$

Based on the above assumptions and notations, our model is governed by the following set of differential difference equations,

$$\lambda P_{00} = \omega Q_0 \tag{1}$$

$$(\lambda + \omega) Q_0 = \int_0^\infty M_{0,b}(x) \mu_n(x) dx + \int_0^\infty G_{0,v}(x) \mu_w(x) dx \tag{2}$$

$$\frac{d}{dx} P_n(x) + (\lambda + f(x)) P_n(x) = 0, \quad x > 0, \quad n \geq 1 \tag{3}$$

$$\frac{d}{dx}M_{0,b}(x) + (\lambda + \mu_n(x))\, M_{0,b}(x) = 0, \quad x > 0 \tag{4}$$

$$\frac{d}{dx}M_{n,b}(x) + (\lambda + \mu_n(x))\, M_{n,b}(x) = \lambda M_{n-1,b}(x), \quad x > 0, \quad n \geq 1 \tag{5}$$

$$\frac{d}{dx}G_{0,v}(x) + (\lambda + \omega + \mu_w(x))\, G_{0,v}(x) = 0, \quad x > 0 \tag{6}$$

$$\frac{d}{dx}G_{n,v}(x) + (\lambda + \omega + \mu_w(x))\, G_{n,v}(x) = \lambda G_{n-1,v}(x), \quad x > 0, \quad n \geq 1 \tag{7}$$

$$\frac{d}{dx}U_0(x) + (\lambda + v(x))U_0(x) = 0, \quad x > 0 \tag{8}$$

$$\frac{d}{dx}U_n(x) + (\lambda + v(x))U_n(x) = \lambda U_{n-1}(x), \quad x > 0, \quad n \geq 1 \tag{9}$$

$$\frac{d}{dx}K_0(x) + (\lambda + \delta(x))K_0(x) = 0, \quad x > 0 \tag{10}$$

$$\frac{d}{dx}K_n(x) + (\lambda + \delta(x))K_n(x) = \lambda K_{n-1}(x), \quad x > 0, \quad n \geq 1 \tag{11}$$

The boundary conditions at $x = 0$ include $P_n(0), M_{0,b}(0), M_{n,b}(0), G_{0,v}(0), U_0(0), U_n(0), K_0(0)$. The description of these terms at $x = 0$ are described as follows:

$$P_n(0) = \int_0^\infty G_{n,v}(x)\mu_w(x)dx + \int_0^\infty M_{n,b}(x)\mu_n(x)dx, \quad n \geq 1 \tag{12}$$

$$\begin{aligned} M_{0,b}(0) = \alpha \int_0^\infty K_0(x)\delta(x)dx + \int_0^\infty P_1(x)f(x)dx \\ + \omega \int_0^\infty G_{0,v}(x)dx + \int_0^\infty U_0(x)v(x)dx \end{aligned} \tag{13}$$

$$\begin{aligned} M_{n,b}(0) = \alpha \int_0^\infty K_n(x)\delta(x)dx + \int_0^\infty P_{n+1}(x)f(x)dx + \omega \int_0^\infty G_{n,v}(x)dx \\ + \lambda \int_0^\infty P_n(x)dx + \int_0^\infty U_n(x)v(x)dx, \quad n \geq 1 \end{aligned} \tag{14}$$

$$G_{0,v}(0) = \lambda Q_0 \tag{15}$$

$$G_{n,v}(0) = 0, \quad n \geq 1 \tag{16}$$

$$K_0(0) = \lambda P_{00} \tag{17}$$

$$K_n(0) = 0, \quad n \geq 1 \tag{18}$$

$$U_0(0) = \bar{\alpha} \int_0^\infty K_0(x)\delta(x)dx \tag{19}$$

$$U_n(0) = \bar{\alpha} \int_0^\infty K_n(x)\delta(x)dx, \quad n \geq 1 \tag{20}$$

The normalization condition is given by

$$\begin{aligned} P_{00} + Q_0 + \sum_{n=1}^{\infty} \int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty} \int_0^\infty M_{n,b}(x)dx + \sum_{n=0}^{\infty} \int_0^\infty G_{n,v}(x)dx \\ + \sum_{n=0}^{\infty} \int_0^\infty U_n(x)dx + \sum_{n=0}^{\infty} \int_0^\infty K_n(x)dx = 1 \end{aligned}$$

In order to solve the above set of equations we define the generating functions as, $P(x,z) = \sum_{n=1}^{\infty} z^n P_n(x)$ for $|z| \leq 1$ and $x > 0, P(0,z) = \sum_{n=1}^{\infty} z^n P_n(0)$ for $|z| \leq 1, M_b(x,z) = \sum_{n=0}^{\infty} z^n M_{n,b}(x)$ for $|z| \leq 1$ and $x > 0, M_b(0,z) = \sum_{n=0}^{\infty} z^n M_{n,b}(0)$, $G_v(x,z) = \sum_{n=0}^{\infty} z^n G_{n,v}(x)$ for $|z| \leq 1$ and $x > 0, G_v(0,z) = \sum_{n=0}^{\infty} z^n G_{n,v}(0)$, $U(x,z) = \sum_{n=0}^{\infty} z^n U_n(x)$ for $|z| \leq 1$ and $U(0,z) = \sum_{n=0}^{\infty} z^n U_n(0)$ for $|z| \leq 1$, $K(x,z) = \sum_{n=0}^{\infty} z^n K_n(x)$ for $|z| \leq 1$ and $K(0,z) = \sum_{n=0}^{\infty} z^n K_n(0)$ for $|z| \leq 1$. Multiplying equations (2)-(11) by suitable powers of z and summing over n, we obtain the following set of partial differential equations :

$$\frac{\partial P(x,z)}{\partial x} + (\lambda + f(x))P(x,z) = 0, \tag{21}$$

$$\frac{\partial M_b(x,z)}{\partial x} + (\lambda - \lambda z + \mu_n(x))\, M_b(x,z) = 0, \tag{22}$$

$$\frac{\partial G_v(x,z)}{\partial x} + (\lambda - \lambda z + \omega + \mu_w(x))\, G_v(x,z) = 0, \tag{23}$$

$$\frac{\partial U(x,z)}{\partial x} + (\lambda - \lambda z + v(x))U(x,z) = 0, \tag{24}$$

$$\frac{\partial K(x,z)}{\partial x} + (\lambda - \lambda z + \delta(x))K(x,z) = 0. \tag{25}$$

Solving the above partial differential equations (22) to (26)

$$P(x,z) = P(0,z)[1 - N(x)]e^{-\lambda x}, \tag{26}$$

$$M_b(x,z) = M_b(0,z)\,[1 - G(x)]\, e^{-I(z)x}, \tag{27}$$

$$G_v(x,z) = G_v(0,z)\,[1 - W_v(x)]\, e^{-(I(z)+\eta)x}, \tag{28}$$

$$U(x,z) = U(0,z)\left[1 - S_f(x)\right] e^{-I(z)\cdot x}, \tag{29}$$

$$K(x,z) = K(0,z)\,[1 - T_s(x)]\, e^{-I(z)\cdot x}. \tag{30}$$

where $I(z) = \lambda(1 - z)$
Multiplying equation (12) by suitable powers of z, summing over n from 1 to ∞ and using equations (1) and (2) after some algebraic manipulations, we get

$$P(0,z) = \int_0^\infty G_v(x,z)\mu_w(x)dx + \int_0^\infty M_b(x,z)\mu_n(x)dx - \left(\frac{\lambda+\omega}{\omega}\right)\lambda P_{00} \tag{31}$$

Similarly, multiplying equations (14)-(21) by suitable powers of z, summing over n and after some algebraic manipulations, we obtain

$$M_b(0,z) = \frac{\alpha}{z}\int_0^\infty K(x,z)\delta(x)dx + \lambda\int_0^\infty P(x,z)dx + \frac{1}{z}\int_0^{+\infty} P(x,z)f(x)dx$$
$$+ \omega\int_0^{+\infty} G_v(x,z)dx + \int_0^{+\infty} U(x,z)v(x)dx, \tag{32}$$

$$G_v(0,z) = G_{0,v}(0), \tag{33}$$

$$U(0,z) = \bar{\alpha}\int_0^\infty K(x,z)\delta(x)dx, \tag{34}$$

$$K(0,z) = K_0(0). \tag{35}$$

inserting equations 28-29 in 32 we get :

$$P(0,z) = G_v(0,z)\mathcal{W}_v^*(I(z)+\eta) + M_b(0,z)G^*(I(z)) - \left(\frac{\lambda+\omega}{\omega}\right)\lambda P_{00} \tag{36}$$

in similary way inserting equations 27-31 in 33 we get :

$$M_b(0,z) = \frac{1}{z}P(0,z)\mathcal{N}^*(\lambda) + \alpha\mathcal{T}_s^*(I(z))K(0,z) + V(z)G_v(0,z) + P(0,z)(1-\mathcal{N}^*(\lambda))$$
$$+U(0,z)\mathcal{S}_f^*(I(z)) \tag{37}$$

where $V(z) = \frac{\omega}{I(z)+\omega}(1 - \mathcal{W}_v^*(I(z)+\omega))$ using equations 16 and 1 :

$$G_v(0,z) = \frac{\lambda^2}{\omega}P_{00} \tag{38}$$

using equation 18 :

$$K(0,z) = \lambda P_{00} \tag{39}$$

using equation 31 in 35 :

$$U(0,z) = \overline{\alpha}\mathcal{T}_s^*(I(z))K(0,z) \tag{40}$$

using equation 40 in 41 :

$$U(0,z) = \overline{\alpha}\lambda P_{00}\mathcal{T}_s^*(I(z)) \tag{41}$$

using equations 39 40 and 42 in 37 and 38 we get

$$P(0,z) = \frac{N_r(z)}{D_r(z)} \tag{42}$$

Where $N_r(z) = \lambda P_{00} z\mathcal{G}^*(I(z))\mathcal{T}_s^*(I(z))[(\alpha-1)\omega\mathcal{S}_f^*(I(z)) - \alpha\omega] + \omega$
$- \lambda\mathcal{G}^*(I(z))V(z) - \lambda(\mathcal{W}_v^*(I(z)+\omega) - 1)$
$D_r(z) = \omega\mathcal{G}^*(I(z))[z(1-\mathcal{N}^*(\lambda)) + \mathcal{N}^*(\lambda)] - z$

$$M_b(0,z) = \frac{-\lambda P_{00}}{D_r(z)}\left\{\begin{array}{c} \left(\omega z\left(-\alpha\mathcal{S}_f^*(I(z)) + \alpha + \mathcal{S}_f^*(I(z))\right)\mathcal{T}_s^*(I(z))\right)\right) \\ +\lambda zV(z) + z\left(\mathcal{N}^*(\lambda)-1\right)\left(\omega - \lambda\mathcal{W}_v^*(I(z)+\omega)+\lambda\right) \\ -\left(\omega - \lambda\mathcal{W}_v^*(I(z)+\omega)+\lambda\right)\mathcal{N}^*(\lambda) \end{array}\right\} \tag{43}$$

Substituting equations (39-44) in 27-31 .

4. Steady state results

If the system is in steady state condition $\rho < \mathcal{N}^*(\lambda)$, the PGFs are as follows:
(I) the number of customers in the orbit when the server is idle;

$$P(z) = \int_0^\infty P(x,z)dx = \frac{P(0,z)(1-\mathcal{N}^*(\lambda))}{\lambda} \tag{44}$$

(II) the number of customers in the orbit when the server is regularly busy;

$$M_b(z) = \int_0^\infty M_b(x,z)dx = \frac{M_b(0,z)(1-\mathcal{G}^*(I(z))}{I(z)} \tag{45}$$

(III) the number of customers in the orbit when the server is at a lower speed service;

$$G_v(z) = \int_0^\infty G_v(x,z)dx = \frac{G_v(0,z)(1-\mathcal{W}_v^*(I(z)+\omega)}{I(z)+\omega} \tag{46}$$

(IV) the number of customers in the orbit when the server is at repair time;

$$U(z) = \int_0^\infty U(x,z)dx = \frac{U(0,z)(1-\mathcal{S}_f^*(I(z))}{I(z)} \tag{47}$$

(V) the number of customers in the orbit when the server is at setup time;

$$K(z) = \int_0^\infty K(x,z)dx = \frac{K(0,z)(1-\mathcal{T}_s^*(I(z))}{I(z)} \tag{48}$$

From the above equations, the only unknown is P_{00} which can be obtained by using the normalization condition $P_{00} + Q_0 + P(1) + M_b(1) + G_v(1) + U(1) + K(1) = 1$ as

$$P_{00} = \frac{\omega^2(A^*(\lambda) - \lambda E(G))}{\left\{\begin{array}{c} \omega^2\left(\lambda\left(E(S_f)(1-\alpha) + E(T_s)\right) + \mathcal{N}^*(\lambda)\right)\right) \\ +\omega\left(\lambda^2\left\{2\mathcal{W}_v^{*'}(\omega)\left(\lambda E(N_b) - \mathcal{N}^*(\lambda)+1\right) - E(N_b)\mathcal{W}_v^*(\omega)\right\}\right) \\ +\lambda\mathcal{N}^*(\lambda) + \lambda^2\left(1-\mathcal{W}_v^*(\omega)\right) \end{array}\right\}} \tag{49}$$

Corollary 1. If the system satisfies the steady state condition, The PGF of the number of customers in the system $(K_s(z))$ is obtained using

$$K_s(z) = P_{00} + Q_0 + P(z) + zM_b(z) + zG_v(z) + U(z) + K(z). \tag{50}$$

The PGF of the number of customers in the orbit $(K_0(z))$ is obtained using

$$K_o(z) = P_{00} + Q_0 + P(z) + M_b(z) + G_v(z) + U(z) + K(z). \tag{51}$$

5. System performance measures

Our analysis is based on the following system characteristics of the retrial queueing system.

5.1. System state probabilities

1. The steady state probability that the server is idle during the retrial time is given by

$$\begin{aligned} I &= P(1) \\ &= \frac{-\lambda P_{00}\left(\mathcal{N}^*(\lambda)-1\right)}{\omega^2(\mathcal{N}^*(\lambda)-\lambda E\left(N_b\right))}\left\{\begin{array}{c} \omega\left(-E(N_b)\lambda\left(\mathcal{W}_v^*(\omega)-1\right)+\omega\left(-\alpha E(S_f)+E(N_b)+E(T_s)\right)\right) \\ +\omega E(S_f)+\lambda\omega\mathcal{W}_v^{*\prime}(\omega)+\lambda\left(\omega\mathcal{W}_v^{*\prime}(\omega)-\mathcal{W}_v^*(\omega)+1\right) \end{array}\right\} \end{aligned}$$

2. The steady state probability that the server is busy on normal service period is given by

$$U = M_b(1) = \frac{\lambda P_{00}E(N_b)}{\omega^2(\mathcal{N}^*(\lambda)-\lambda E\left(N_b\right))}\left\{\begin{array}{c} \omega^2\left(-\alpha E(S_f)\lambda+E(T_s)\lambda+E(S_f)\lambda+\mathcal{N}^*(\lambda)\right)\Big) \\ +\omega\left(2\lambda^2\mathcal{W}_v^{*\prime}(\omega)-\lambda\mathcal{W}_v^*(\omega)\mathcal{N}^*(\lambda)+\lambda\mathcal{N}^*(\lambda)\right) \\ -\lambda^2\mathcal{W}_v^*(\omega)+\lambda^2 \end{array}\right\}$$

3. The steady state probability that the server is busy on working vacation period is given by

$$V = G_v(1) = \frac{\lambda^2 P_{00}\cdot(1-\mathcal{W}_v^*(\omega))}{\omega^2}$$

4. The steady state probability that the server is under repair time is given by

$$S = U(1) = E(S_f)\lambda P_{00}\cdot(1-\alpha)$$

5.The steady state probability that the server is under setup time is given by

$$K = K(1) = E(T_s)\lambda P_{00}$$

5.2. Mean system size and orbit size

(i) The expected number of customers in the orbit $\left(L_q\right)$ is obtained by differentiating equation 52 with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z)$$

(ii) The expected number of customers in the system (L_s) is obtained by differentiating equation 51 with respect to z and evaluating at z = 1

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z)$$

(iii) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue $\left(W_q\right)$ are found using Little's formula

$$W_s = \frac{L_s}{\lambda} \quad \text{and} \quad W_q = \frac{L_q}{\lambda}.$$

6. Reliability Measures

In the retrial queueing system with unreliable server, the reliability measures provide the information, which is required for the improvement of the system.
(i) The steady state availability A_v, which is the probability that the server is either working for a positive customer or in an idle period such that the steady state availability of the server is given by

$$A_v = 1 - U(1) = 1 - E(S_f)\lambda P_{00} \cdot (1-\alpha) \tag{52}$$

(ii) Let F_f be the steady state probability of server failure,

$$F_f = \alpha K(1) = \alpha E(T_s)\lambda P_{00}$$

Theorem 2. Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions, we have

$$E(T_b) = \frac{1}{\omega^2(A^*(\lambda) - \lambda E(N_b))}\left\{\begin{array}{c} \omega^2\left(-\alpha E(S_f) + E(N_b) + E(T_s) + E(S_f)\right)\Big) \\ \omega\left(-E(N_b)\lambda \mathcal{W}_v^*(\omega) + 2\lambda \mathcal{W}_v^{*'}(\omega)\left(E(N_b)\lambda - \mathcal{N}^*(\lambda) + 1\right)\right. \\ +\mathcal{N}^*(\lambda) - \lambda \mathcal{W}_v^*(\omega) + \lambda \end{array}\right\} \tag{53}$$

$$E(T_c) = \frac{1}{\omega^2\lambda(\mathcal{N}^*(\lambda) - \lambda E(N_b))}\left\{\begin{array}{c} \omega^2\left(\lambda\left(E(S_f)(1-\alpha) + E(T_s)\right) + \mathcal{N}^*(\lambda)\right)\Big) \\ +\omega\left(\lambda^2\left\{2\mathcal{W}_v^{*'}(\omega)\left(\lambda E(N_b) - \mathcal{N}^*(\lambda) + 1\right)\right\}\right) \\ -\lambda^2\omega E(N_b)\mathcal{W}_v^*(\omega) + \lambda\mathcal{N}^*(\lambda) + \lambda^2\left(1 - \mathcal{W}_v^*(\omega)\right) \end{array}\right\} \tag{54}$$

Proof. The result follows directly by applying the argument of an alternating renewal process which leads to

$$P_{00} = \frac{E(T_0)}{E(T_b) + E(T_0)}; E(T_b) = \frac{1}{\lambda}\left(\frac{1}{P_{00}} - 1\right) \text{ and } E(T_c) = E(T_0) + E(T_b) \tag{55}$$

where T_0 is the time length that the system in empty state. Since the inter-arrival time between two customers follows exponential distribution with parameter λ, we have that $E(T_0) = (1/\lambda)$. Inserting equation 50 into 56 and by direct calculations, we can get 54 and 55.

7. Cost model

In practice, queue managers aim to minimize the operating cost per unit time. In this section of the paper, we begin by formulating a steady-state expected cost function per unit time, where the service rate μ_n is the decision variable. Our objective is to find the optimal value of μ_n to minimize the expected cost function. To reach this, We need to define cost elements as follows:
- $\mathcal{C}_1$: is the cost of each consomer in the system per unit of time.
- $\mathcal{C}_2$: represents the cost per unit time to leave the server functioning
- $\mathcal{C}_3$: Cost per service per unit time.
- $\mathcal{C}_4$: represents the cost per unit time needed to prepar starting up the server.

Let - $\mathcal{T}_c$ be the total expected cost per unit time of the system:

$$\mathcal{T}_c = \mathcal{C}_1 L_s + \mathcal{C}_2(1 - P_{00}) + \mathcal{C}_3\mu_n + \mathcal{C}_4 P_{00}.$$

7.1. Quadratic Fit Search Method

This part considers the cost optimization problem under a given cost structure via quadratic fit search method (QFSM), This method uses a 3-point pattern to fit a quadratic function that ensures a unique optimal solution., see [22] . So, We aim to optimize the service rate μ_n in various cases to minimize the expected cost function $\mathcal{T}_c$ denoted in this part by H. Assume that all system parameters have fixed values, and the only controlled parameter is the service rate μ_n. Thus, The optimization problem can be mathematically expressed as:

$$\text{Minimize}: H(\mu_n) = \mathcal{C}_1 L_s + \mathcal{C}_2(1 - P_{00}) + \mathcal{C}_3\mu_n + \mathcal{C}_4 P_{00}.$$

As it has been mentioned in Laxmi et al [23], given a 3-point pattern, we may fit a quadratic function via corresponding functional values that has a unique minimum, x^q, for the given objective function $H(x)$. the quadratic fit improves the current 3-point pattern by replacing one of its points with the optimum x^q , using this approximation. The unique optimum x^q of the quadratic function agreeing with $H(x)$ at 3-point operation $\left(x^l, x^m, x^u\right)$ is given as

$$x^q \cong \frac{1}{2}\left[\frac{H\left(x^l\right)\left((x^m)^2-(x^u)^2\right)+H\left(x^m\right)\left((x^u)^2-\left(x^l\right)^2\right)+H\left(x^u\right)\left(\left(x^l\right)^2-(x^m)^2\right)}{H\left(x^l\right)\left(x^m-x^u\right)+H\left(x^m\right)\left(x^u-x^l\right)+H\left(x^u\right)\left(x^l-x^m\right)}\right].$$

For the whole analysis in this numerical part, we fixe $\mathcal{C}_1 = 10, \mathcal{C}_2 = 350, \mathcal{C}_3 = 20, \mathcal{C}_4 = 120,$.

7.2. Optimization analysis

To conduct the numerical analysis for parameter optimization in the queueing system under consideration, we use the following default values for the parameters: $\alpha = 0.7, \lambda = 2, \delta = 8, \mu_w = 2, s = 5, r = 4$ and $\xi = 8$, and the tolerance of QFSM is $\epsilon = 10^{-6}$.

From **Figure** 1, The curves clearly show convexity, which means , the existence of a specific service rate μ_n that minimizes the total expected cost function for the given set of model parameters. By adopting QFSM and choosing the initial 3point pattern as $\left(\mu_n^l, \mu_n^m, \mu_n^u\right) = (3.05, 3.5, 3.75)$, and after finite iterations, we observe that the minimum expected operating cost per unit time converges to the solution $H = 372.29$ at $\mu_n^* = 3.282$,

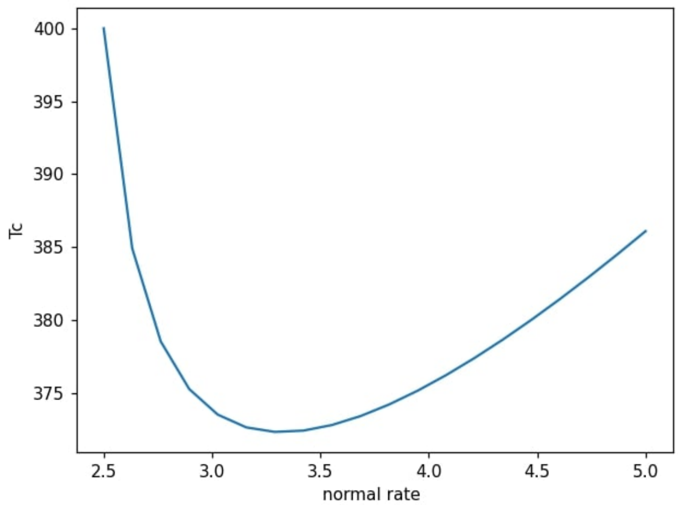

Figure 1: *The optimum service rate μ_n^**

Moreover, we examine the behavior of the expected cost function under different values of the cost parameters. System parameters are fixed as follows: $\alpha = 0.7, \lambda = 2, \delta = 8, \mu_v = 2, s = 5, r = 4$ and $\xi = 8$; Tables 1-3 illustrate the effects of $(\mathcal{C}_1, \mathcal{C}_2), (\mathcal{C}_2, \mathcal{C}_4)$ and $(\mathcal{C}_4, \mathcal{C}_3)$ on the expected cost function, respectively. It can be see that the expected cost function shows a linearly increasing trend with increasing cost parameters.

Table 1: *Effects of* $(\mathcal{C}_4, \mathcal{C}_3)$ *on the expected cost function* $\mathcal{T}_c$ *with* $\mathcal{C}_1 = 10$ *and* $\mathcal{C}_2 = 350$

$(\mathcal{C}_4, \mathcal{C}_3)$	(100,10)	(200,10)	(200,15)	(120,5)	(120,20)
$\mathcal{T}_c$	272.4873	313.8563	323.8563	270.7611	300.7611

Table 2: *Effects of* $(\mathcal{C}_2, \mathcal{C}_4)$ *on the expected cost function* $\mathcal{T}_c$

$(\mathcal{C}_2, \mathcal{C}_4)$	(350,150)	(350,200)	(250,200)	(150,120)	(100,120)
$\mathcal{T}_c$	293.1718	313.8563	255.2253	163.4991	134.1836

Table 3: *Effects of* $(\mathcal{C}_4, \mathcal{C}_3)$ *on the expected cost function* $\mathcal{T}_c$

$(\mathcal{C}_4, \mathcal{C}_3)$	(100,10)	(200,10)	(200,15)	(120,5)	(120,20)
$\mathcal{T}_c$	272.4873	313.8563	323.8563	270.7611	300.7611

8. Sensitivity Analysis and Numerical Examples

In this section, we provide numerical examples using Python to illustrate how different parameters affect system performance measures. We assume that retrial times, service times, lower-speed service times, vacation times, setup times, and repair times all follow exponential distributions. The parameter values are chosen to satisfy the system's stability conditions. The following tables present computed values for various model characteristics, such as the probability that the server is idle (P_{00}), the mean orbit size (L_q), probability that server in working vacation , probability that server in setup time, and probability that server in repair time . The exponential distribution is $k(x) = ve^{-vx}, x > 0$.

In **Figure** 2 , we examine the behavior of the idle probability (P_{00}) increases for increasing the value of the lower service rate (μ_n) and regular service rate (μ_w).

In **Figure** 3 , we examine the behavior of the mean orbit size (L_q) décreases for increasing the value of lower speed service rate μ_w and retrial rate ξ.

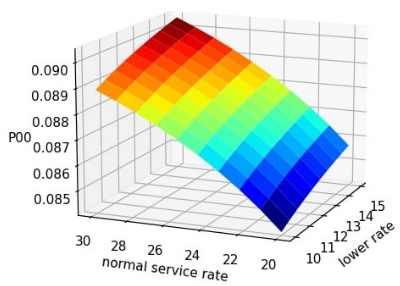

Figure 2: *Variation in* P_{00} *with* μ_n *and* μ_w

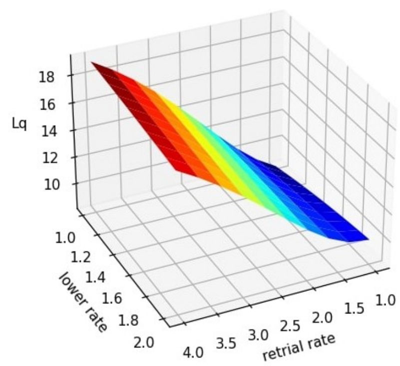

Figure 3: (L_q) *versus* ξ *and* μ_w

In **Figure** 4 , we see that the behavior of the mean orbit size (L_q) decreases as the values of the lower service rate μ_w and regular service rate μ_n increase.

In **Figure** 5 , we examine the behavior of the idle probability (P_{00}) increases with an increase in the setup rate, for a fixed value of repair rate.

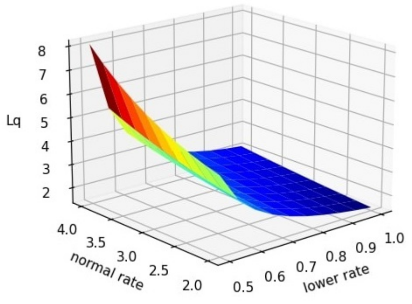

Figure 4: (L_q) *versus* μ_n *and* μ_w

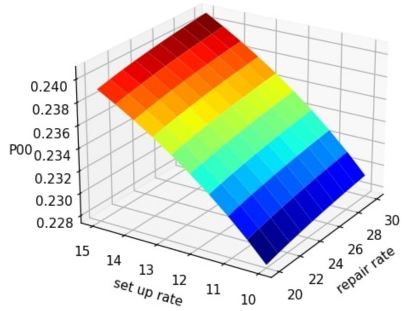

Figure 5: P_{00} *versus setup rate and repair rate*

Figure 6 depicts that with an increase in repair rate, the probability of the server being in repair state decreases.
Figure 7 depicts that with an increase in service rate μ_n , the probability of the server being in setup state increases ;this is due to faster activation of server with reduced setup time.

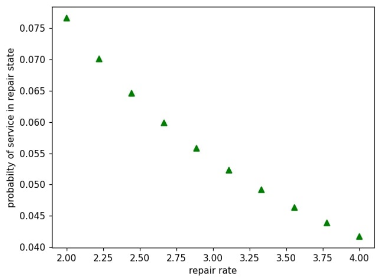

Figure 6: *Effect of repair rate on repair state probability of server*

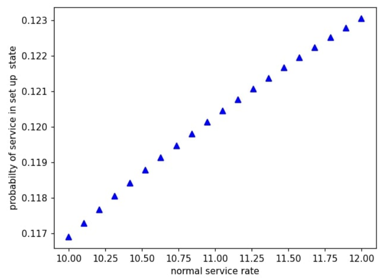

Figure 7: *Probability of server in set up versus* μ_n

We observe from **Figure** 8 that the probability of the server being in vacation state decreases with an increase in the rate of working vacation. The reason behind the observation is a decrease in the duration of vacation with an increase in the vacation rate.

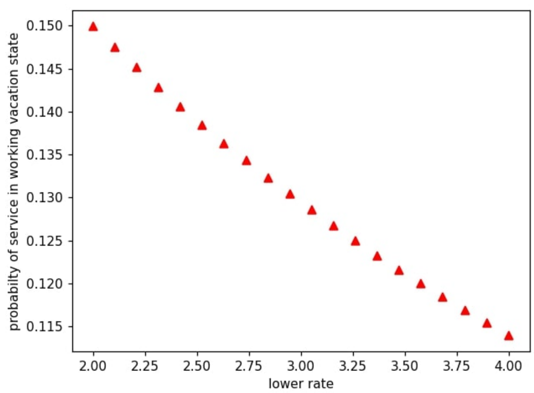

Figure 8: *Probability of server in vacation versus vacation rate* μ_w

9. Conclusion

In this article, we analyzed an Unreliable single-server retrial queue model with general retrial time, working vacation and setup time. if certain required and sufficient conditions are satisfied the system can be stabilized .Using he supplementary variable approach and the Probability Generating Function (PGF) approach to determine The PGF of the no. of clients in the system and its orbit . The performance of the model is illustrated using **PYTHON** . The operating cost of the queuing system is optimized by adjusting the service rate of the server.

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