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EVALUATION OF PARAMETRIC ESTIMATION METHODS FOR THE GAMMA DISTRIBUTION USING MAXIMUM LIKELIHOOD AND BAYESIAN APPROACHES IN A CENSORED LIFE-TESTING STRATEGY WITH MARKOV CHAIN MONTE CARLO SIMULATIONS

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Abstract

The goal of this study was to address the computational challenges associated with parametric estimation of the gamma distribution by evaluating the performance of the maximum likelihood and maximum a-posteriori estimation methods within the framework of Markov Chain Monte Carlo simulations. This was done by first assuming a censored life-testing strategy that terminates on the rth failure from a given sample of n electronic devices. Second, we obtained the joint distribution function of the first r-order statistic by arranging the r values in order of magnitude. Finally, we explored through the Markov Chain Monte Carlo framework using the maximum likelihood and maximum a-posteriori to estimate the gamma distribution parameters. The findings of this study suggest that both estimation methods were not significantly different from the actual hypothesized parameter values. Further, we observed that irrespective of the prior distribution used for the Bayesian maximum a-posteriori Markov Chain Monte Carlo estimation, the resulting parametric estimates of the gamma distribution remain the same, confirming the assertion that the Bayesian maximum a-posteriori Markov Chain Monte Carlo approach is a valuable tool for informative posterior analysis. The study's uniqueness lies in adopting a censored life-testing strategy centered on the joint distribution function of the first r-order statistic.

Keywords: bayesian inference, gamma distribution, maximum likelihood estimation, maximum a-posteriori, reliability analysis

I. Introduction

The Gamma distribution has been extensively studied and developed in statistical inference. It is preferred over other probability distributions for its superior applications in insurance and finance. Dickson [1] and Veazie et al. [2] project the two or three parameter Gamma distribution for its ability to model highly skewed positive and or negative data points. Gamma distribution application is most evident in studying random variables such as waiting times, claim size or frequency and investment returns. While every probability distribution offers some distinct advantage in specific contexts, Dey et al. [3] support the gamma distribution for its computational efficiency and memoryless property. Memoryless property means that the occurrences of past events do not influence the probability of future occurrences of the event (Noguchi & Robles [4]; Shore [5]; Tao [6]). In other words, the memoryless property of the gamma distribution makes it easy to study the probability of the occurrence of an event independently of the probability of future occurrences of the event. This property demonstrates the usefulness and versatility of the gamma distribution in several survival and reliability studies.

The use of the gamma distribution in reliability analysis extends to other fields such as engineering, manufacturing and biomedical research. For instance, in modeling time-to-failure of manufactured components, Elsayed [7] employed the gamma distribution alongside the Maximum Likelihood Estimation (MLE) method to estimate the parameters. Similarly, Shipes et al. [8] used the Poisson-Gamma Model in their survival analysis of time-to-event clinical trial data. These studies underscore the flexibility of the gamma distribution in capturing diverse event patterns. More importantly, the parametric estimation of the gamma distribution in these studies offers room for further exploration. Meeker and Escobar [9] explained parametric estimation in both reliability and survival analysis as fitting a specific probability distribution (e.g., gamma, exponential, Weibull) to the observed failure data and estimating its parameters (location, shape and scale parameters).

Literature abounds with different combinations of estimation and simulation methods to estimate the gamma distribution parameters. Several studies (Ghosh & Hamedani [10]; Junmei & Liqin [11]) seem to opt for the MLE method due to its optimal and consistent parameter estimators. Ghosh and Hamedani [10] provided detailed properties of the two-parameter gamma distribution using the MLE method to investigate its moments, hazard function and reliability parameters. When applied to a lifetime data set, the gamma distribution produced a superior fit compared to other models. While the gamma distribution is applauded for its flexibility in model fitting, Ozsoy, Unsal and Orkcu [12] warned of a potential computational complexity when MLE is used for its parameter estimation. They explained that the distribution function (or survival function) of the gamma distribution is not available in a closed form if the shape parameter is not an integer, thereby making the use of MLE a near futile exercise. This notwithstanding, studies (Hamada et al. [13]; Rubinstein & Kroese [14]) have adapted numerical methods to evaluate the parameters of the gamma distribution by exploring a combination of Bayesian estimation and simulation procedures. Hamada et al. [13] found the Bayesian estimation useful in their probabilistic framework for reliability estimation as it incorporates additional information about the distribution known as a prior. The Bayesian framework entails careful elicitations of prior expert information to enhance the data, leading to improved prediction of extreme cases (Coles & Tawn [15]). Hussain et al. [16] and Kohole et al. [17] proffer the Bayesian method, as it at least offers a way around the complexity of the root of the maximum likelihood equation known to exist in MLE. The Bayesian approach, therefore, appears more flexible and informative through its posterior analysis.

In recent decades, the surge in statistical applications has sparked a growing interest in Bayesian parametric simulation, giving rise to the efficient concept of Maximum a-Posteriori (MAP). Serving as the Bayesian counterpart to MLE, MAP estimation entails identifying parameter values that maximize the posterior distribution and act as estimates for the unknown parameters (Hesse et al., 2016). When there is a noninformative prior in the Bayesian analysis, the MAP estimate is the same as that of the MLE. Due to the computational intensity of MAP resulting from the incorporation of prior information, Hesse et al. [18] turned to Markov chain Monte Carlo (MCMC) to obtain samples from the posterior distribution, enabling the estimation of regression parameters. The concept of MCMC is popular in fields such as manufacturing, physics and finance, and it uses probability distributions to make selections (Benson & Kellner [19]). In reliability assessment, Naess, Leira and Batsevych [20]) noted that the MCMC can check failure criterion, regardless of the distribution or system complexity. Fauzi et al. [21]) relied on the MCMC algorithms for sampling from a posterior distribution, essentially, to simulate system behavior and estimate reliability metrics.

This study aims to tackle the computational challenges associated with parametric estimation of the gamma distribution by evaluating the performance of two estimation procedures: Maximum Likelihood Estimation (MLE) and Maximum a-Posteriori (MAP) estimation. These assessments will be conducted within the framework of Markov Chain Monte Carlo (MCMC) simulations. In this paper, the two MCMC-based estimation techniques are denoted as MLE_MCMC and MAP_MCMC. The study's uniqueness lies in the adoption of a censored life-testing strategy, terminating upon the occurrence of the r^{th} (where $r < n$) failure. This approach diverges from classical life testing, which requires the complete failure of all n samples. The study concentrates on the joint distribution function of the first r-order statistic, precisely the smallest r values, as an alternative to utilizing the complete dataset for estimation. Additionally, we look into the sensitivity of MAP_MCMC by applying various prior distributions. This study is relevant as it illustrates that the joint distribution function of the first r-order statistic proves more suitable for estimating the parameter(s) of the probability density function (pdf) of the time-to-failure random variable for any engineered device.

II. Methods

Consider n samples of manufactured components that were subjected to reliability life tests from a certain population of interest. The random variable T of interest is the time it takes until the component fails. Suppose the underlying failure times are $T_{(1)},\cdots,T_{(n)}$ where $T_{(i)} \le T_{(i+1)}$, $i = 1,\cdots,n-1$. And let $F_T(t)$ be the distribution function of T and let $f_T(t)$ be its probability density function (pdf). Assuming further that the reliability life tests conclude at the r^{th} failure, where r is less than or equal to n, the number of failures is treated as a fixed value, while the failure times are regarded as random variables. We employ the gamma distribution to model the time-to-failure random variables in this scenario of life testing, assuming that the failure rate is not constant. The gamma distribution is preferred in this instance because it exhibits a failure rate that follows a bathtub-shaped curve (decreasing failure rates at the initial phase and increasing failure rates at a later phase). Blanksby and Lyons [22] assert that the gamma distribution allows for flexibility in capturing diverse failure rate behaviours and is well-suited for scenarios where the hazard function varies over time. The next subsection presents a synopsis of the gamma distribution.

I. Gamma Time-to-Failure Random Variable

The continuous random variable T, is said to have the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by:

$$f_T(t) = \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)}, t > 0 \quad (1)$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function. The cumulative distribution function is the regularized gamma function:

$$F_T(t) = \frac{\gamma(\alpha,\ \beta t)}{\Gamma(\alpha)}, t > 0. \quad (2)$$

where $\gamma(\alpha,\ \beta t)$ is the lower incomplete gamma function, that is $\gamma(\alpha,\ \beta t) = \int_0^{\beta t} t^{\alpha-1}e^{-t}dt$. This gamma distribution is in the two-parameter family of continuous probability distributions. These are:

- Shape Parameter (α): This parameter determines the shape of the distribution. It is a positive real number.
- Scale Parameter (β): This parameter is associated with the rate of events. It is also a positive real number. The density and cumulative distribution functions are sometimes expressed in terms of the scale parameter, $\theta = 1/\beta$.

Table 1 shows some characteristics of the gamma distribution (Mann et al. [23]).

Table 1: *Properties of the gamma distribution*

Properties	
Measures	Properties
Mean	$\frac{\alpha}{\beta}$
Variance	$\frac{\alpha}{\beta^2}$
Median	No simple closed
Mode	$\frac{\alpha-1}{\beta}$ for $\alpha \geq 1$
Reliability function	$e^{-\beta t}\sum_{n=1}^{\alpha-1}\frac{(\beta t)^n}{n!}$

In the next subsection, we explore two approaches for estimating the parameters of the gamma distribution through the Markov Chain Monte Carlo (MCMC) simulation framework: (a) Maximum Likelihood Estimation (MLE) and (b) Maximum a-Posteriori (MAP).

II. Parametric Estimation of the Gamma Distribution

According to Ofosu and Hesse [24], the likelihood function L of the first r-order statistics, $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(r)}$, of the random variable of interest in this study can be specified as:

$$\begin{aligned}
L &= f_{T_{(1)},\cdots,T_{(r)}}(t_1,\cdots,t_r) \\
&= \frac{n!}{(n-r)!}[1-F_T(t_r)]^{n-r}\prod_{i=1}^{r} f_T(t_i) \\
&= \frac{n!}{(n-r)!}\left[1-\frac{\gamma(\alpha,\ \beta t_r)}{\Gamma(\alpha)}\right]^{n-r}\prod_{i=1}^{r}\left\{\frac{\beta^\alpha t_i^{\alpha-1}e^{-\beta t_i}}{\Gamma(\alpha)}\right\} \\
&= \frac{n!}{(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{r}\beta^{r\alpha}\prod_{i=1}^{r}\left\{t_i^{\alpha-1}e^{-\beta t_i}\right\} \\
&= \frac{n!}{(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{n}\beta^{r\alpha}\left(\prod_{i=1}^{r} t_i^{\alpha-1}\right)e^{-\beta\sum_{i=1}^{r}t_i}.
\end{aligned}$$

(3)

The natural logarithm of the likelihood function gives:

$$\ln L = \ln n! - \ln(n-r)! + (n-r)\ln[\Gamma(\alpha) - \gamma(\alpha,\ \beta t_r)] - n\ln\Gamma(\alpha) + r\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{r}\ln t_i - \beta\sum_{i=1}^{r} t_i. \tag{4}$$

This function yields the following logarithmic likelihood equations:

$$\frac{\partial \ln L}{\partial \beta} = \frac{r\alpha}{\beta} - \sum_{i=1}^{r}\ln t_i + \frac{\beta^{\alpha-1}(n-r)t_r^{\alpha}e^{-\beta t_r}}{[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]} = 0 \tag{5}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{(n-r)[\Gamma'(\alpha)-\Gamma'(\alpha,\ \beta t_r)]}{[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]} + \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + r\ln\beta + \sum_{i=1}^{r}\ln t_i = 0 \tag{6}$$

Solving Equations (5) and (6) is notably challenging. When a straightforward solution to the likelihood equations is elusive, various procedures are available for MLE. Common methods encompass Iterative Methods, the Expectation-Maximization (EM) Algorithm, Gradient Descent, Quasi-Newton Methods, Monte Carlo Methods, Profile Likelihood, Bootstrapping, and Numerical Optimization (Dempster et al.[25]; Gilks et al. [26]; Nocedal & Wright [27]; Press et al. [28]).

In cases where obtaining a solution to the log-likelihood equations proves to be difficult, we turn to Markov Chain Monte Carlo (MCMC) sampling techniques to generate samples from the likelihood function. Consequently, we applied the MCMC estimation technique and referred to it as MLE_MCMC. The primary objective is to determine parameter estimates that maximize the likelihood function given the sample data. The MLE_MCMC approach identifies the mode of the simulated MCMC sample from the bivariate likelihood function in Equation (3), representing the point estimate of the parameter vector $\theta = (\alpha,\ \beta)$. That is,

$$\theta_{MLE} = \arg\ max\left\{k[\Gamma(\alpha) - \gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^{n}\beta^{r\alpha}\left(\prod_{i=1}^{r} t_i^{\alpha-1}\right)e^{-\beta\sum_{i=1}^{r} t_i}\right\} \tag{7}$$

The following algorithm is the description for the multivariate Metropolis Hastings procedure (Steyvers [36]):

- Set $t = 1$
- Generate an initial value for $\beta \sim U(u_1,\ u_2)$.
- Repeat
  $t = t + 1$
  Do a MH step on α,
  Generate a proposal $\theta^* \sim N(\theta,\ \sigma^2)$;
  Evaluate the acceptance probability $a = min\left[1,\ \frac{L(\theta^*/x)}{L(\theta/x)}\right]$;
  Generate a u from a Uniform(0, 1) Distribution
  If $u \le a$, accept the proposal and set $\theta = \theta^*$
- Until $t = T$.

Maximum A Posteriori (MAP) estimation is the Bayesian counterpart to Maximum Likelihood Estimation (MLE), incorporating additional information through the prior distribution. Now, the joint pdf of $X_{(1)},\cdots,X_{(r)}$ and $\theta = (\alpha,\ \beta)$ is given by $g(x_1,\cdots,x_r,\ \theta) = f_{X_{(1)},\cdots,X_{(r)}}(x_1,\cdots,x_r|\theta)\pi(\theta)$, where $\pi(\theta)$ is the prior distribution of the parameter Θ. We assume α and β are independent and exponentially distributed with means a and b, respectively. Thus, $\pi(\theta) = \frac{1}{ab}e^{-(\alpha/a+\beta/b)}$, $\alpha > 0, \beta > 0$.

$$g(x_1,\cdots,x_r,\theta)=\frac{n!}{ab(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}. \quad (8)$$

Thus, the marginal p.d.f. of $X_{(1)},\cdots,X_{(r)}$ is

$$\begin{aligned}g_{X_{(1)},\cdots,X_{(r)}}(x_1,\cdots,x_r)&=\int_0^\infty\int_0^\infty g(x_1,\cdots,x_r,\theta)d\alpha d\beta\\&=\frac{n!}{ab(n-r)!}\int_0^\infty\int_0^\infty[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}d\alpha d\beta.\end{aligned} \quad (9)$$

The conditional p.d.f. of Θ, given $X_{(1)},\cdots,X_{(r)}$, is therefore defined by

$$\begin{aligned}\pi(\theta|x_1,\cdots,x_r)&=\frac{g(x_1,\cdots,x_r,\ \theta)}{g_{X_{(1)},\cdots,X_{(r)}}(x_1,\cdots,x_r)}\\&=\frac{\frac{n!}{ab(n-r)!}[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}}{\frac{n!}{ab(n-r)!}\int_0^\infty\int_0^\infty[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\prod_{i=1}^{r}\{t_i^{\alpha-1}e^{-\beta t_i}\}e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)}d\alpha d\beta}\\&=K[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\left(\prod_{i=1}^{r}t_i^{\alpha-1}\right)e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}+\beta\sum_{i=1}^{r}t_i\right)}.\end{aligned} \quad (10)$$

where K is independent of α and β. The typical approach in Bayesian estimation is to employ the posterior mean, $E(\theta|x_1,\cdots,x_r)$, as a point estimate for θ (Hesse et al. [18]). The Maximum a posteriori (MAP) estimator of θ is the value that maximizes the posterior distribution. This study uses the Markov Chain Monte Carlo (MCMC) sampling approach to draw samples from the posterior distribution. This specific method of estimation, denoted as MAP_MCMC for the purpose of this study, identifies the mode of the posterior distribution, representing the point estimate for the parameter θ. Thus,

$$\theta_{MLE}=\arg\ max\left\{=K[\Gamma(\alpha)-\gamma(\alpha,\ \beta t_r)]^{n-r}\left(\frac{1}{\Gamma(\alpha)}\right)^n\beta^{r\alpha}\left(\prod_{i=1}^{r}t_i^{\alpha-1}\right)e^{-\left(\frac{\alpha}{a}+\frac{\beta}{b}+\beta\sum_{i=1}^{r}t_i\right)}\right\}. \quad (11)$$

III. Results and Discussion

Consider a scenario where a company specializes in manufacturing a specific device, its components, or equipment. The company is dedicated to assessing the reliability of these items. In this context, reliability denotes the probability of a device successfully fulfilling its intended function. A sample of n such devices from a population of interest is placed in an environment closely resembling the conditions the items will encounter in actual use. One or more stresses of constant severity are then applied to simulate real-world scenarios. Given that, 200 of these devices are programmed to undergo reliability life testing with the test truncating when the 100th failed device was observed. The first 100 sampled time-to-failure units were ordered and fitted to the gamma distribution.

Given that T is a continuous random variable with a distribution function $F_T(t)$, then, according to the probability integral transformation concept, $U=F_T(t)$, follows the continuous uniform distribution over the interval (0, 1) (Ofosu & Hesse [24]). The inverse transform sampling method is employed subsequently to generate samples from the gamma distribution using the following steps:

- Generate a random number u from the standard uniform distribution in the interval (0, 1).
- Find the inverse of the desired cumulative density functions (CDF), denoted as $F^{-1}(u)$.
- Compute $t=F^{-1}(u)$. The computed values of the random variable X correspond to the desired distribution with probability density function $f_T(t)$.

There is no closed-form expression for the gamma distribution's inverse cumulative distribution function (CDF). However, various numerical methods and statistical software packages are available to calculate quantiles (inverse CDF values) for the gamma distribution. Commonly employed numerical methods include the Newton-Raphson Method and Brent's Method (Burden & Faires [29]; Kincaid & Cheney [30]; Dahlquist & Björck [31]). In addition, other statistical software packages like R, Python (SciPy), MATLAB and MS. Excel provide functions to compute quantiles for the gamma distribution (Eaton [32]; Hanselman & Littlefield [33]).

We, therefore, use the 'gaminv' function in MATLAB (MathWorks [34]) to calculate quantiles for a given probability, simulated from the uniform distribution over the interval (0, 1). Table 2 displays the first 100 out of 200 ordered data points simulated from the gamma distribution with parameters alpha (α) = 10 and beta (β) = 0.05, resulting in a mean (μ) of 200. It is assumed that these observations represent the outcomes of a reliability life test involving 200 devices until the failure of the 100th device.

Table 2: *Ordered data simulated from the gamma distribution*

73.230	76.643	81.308	85.709	88.578	90.103	91.545	91.884	95.091	97.130
98.564	99.319	101.788	105.722	107.333	107.880	110.057	111.884	113.007	115.471
117.521	121.779	125.079	125.766	128.059	128.199	130.834	132.564	134.648	135.505
135.923	136.019	138.582	141.671	142.111	143.420	144.675	146.128	147.288	147.939
152.135	153.413	153.475	155.564	155.716	156.338	156.353	157.133	157.848	157.903
159.273	159.895	159.900	160.563	160.563	161.096	162.658	162.837	168.294	168.322
169.501	169.539	170.437	170.723	170.831	172.767	173.089	173.164	173.700	175.040
175.844	176.285	176.485	177.148	177.296	177.426	177.463	178.349	178.942	179.484
181.027	181.703	181.777	182.658	182.677	182.807	185.919	188.967	189.609	189.759
189.848	190.031	191.716	192.004	192.051	192.169	192.448	194.546	194.670	195.500

On the MLE_MCMC parameter estimates, we deduce from Equation (3), given that n = 200, r = 100, $t_r = 195.500$ from Table 2, together with the Metropolis-Hastings algorithm, as described above, were employed to draw samples from the likelihood function. The MATLAB code for implementing the component-wise Metropolis sampler for the likelihood function is provided in Listings A1 and A2, in the appendix. The mode of this bivariate sample provides the maximum likelihood estimate for the parameters alpha (α) and beta (β) of the gamma distribution, which are given by $\alpha_{MLE} = 10.4169$ and $\beta_{MLE} = 0.0487$, respectively. Note that the maximum likelihood estimates are not significantly different from the actual value of the parameters, that is α = 10 and β = 0.05. Figure 1 shows the graph of the actual and the estimated gamma function with the given parameters.

The closeness of these estimated parameters of the gamma distribution is consistent with observations made by Saulo et al. [35], who observed that the generalized gamma distribution generated very close values of the log-likelihood function when compared to the Dagum distribution in an MLE procedure for censored remission times of cancer patients. The closeness of the values affirms that the MLE_MCMC approach is as trustworthy as the classical MLE.

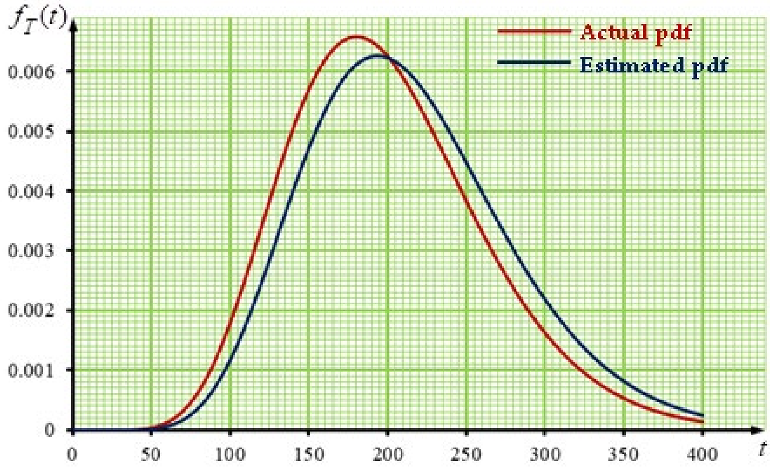

Figure 1: *pdf of the gamma distribution*

Similar to the MLE_MCMC estimates, we utilize the data from Table 2 and the Metropolis-Hastings algorithm to simulate a sample from the Bayesian posterior distribution. We further assumed that α and β are independent and exponentially distributed with means a = 15 and b = 0.08. The MATLAB code for implementing the Metropolis-Hastings sampler for the posterior distribution in Equation (9) is provided in Listings A3 and A4, in the appendix to obtain the MAP_MCMC parameter estimates of the gamma distribution.

The results reveal that the MAP_MCMC estimates of α and β are $\alpha_{MAP} = 10.4169$ and $\beta_{MAP} = 0.0487$ which precisely match the results obtained when sampling is done directly from the likelihood function. These results remain consistent even when varying the values of a and b, such as a = 20 and b = 1. In other words, the parameter estimates of the gamma distribution from the MAP_MCMC approach produce precisely the same estimates as the MLE_MCMC approach. The seemingly no difference between the two estimation methods confirms their flexibility in resolving estimation complexities associated with the gamma distribution for time-to-failure variables.

Further, we explore the sensitivity of the MAP_MCMC estimator to different prior distributions by conducting a repeated MCMC simulation assuming that the parameters α and β are independent and follow Lognormal, Pareto, Weibull, and Gumbel priors. The results of the MAP_MCMC estimate for α and β across these four prior distributions are consistent with those obtained using the exponential prior. The implication is that irrespective of the distribution used, the outcome for the two estimation procedures or methods produced the same parameter estimates. We, therefore, concur with previous studies (Hussain et al. [16]; Kohole et al. [17] to conclude that the Bayesian MCMC approach offers a more flexible and informative posterior analysis. In addition, our estimation procedure affirms Fauzi et al. [21] assertion that Bayesian MCMC algorithms for sampling from a posterior distribution essentially simulate system behavior and estimate reliability metrics.

IV. Conclusions

The applicability of the gamma distribution to study the distribution of random variables, such as time-to-failure of an event, has proven to be effective and versatile in many reliability studies. The preoccupation of these studies centers on the parametric estimation and derivation of other properties of the gamma distribution. The challenge, however, is the computational complexities encountered when the classical maximum likelihood estimation (MLE) method is used for the parameter estimation. It is contested that MLE does not provide a straightforward solution to the log-likelihood equations since the inverse cumulative distribution function (CDF) of the gamma distribution cannot be expressed in a closed form. Consequently, many studies have resorted to numerical methods to evaluate gamma distribution parameters by exploring a combination of Bayesian estimation and other simulation procedures.

Therefore, in this study, we relied on one of the Bayesian parametric simulation methods called Maximum a-Posteriori (MAP) to estimate the gamma distribution parameters. The MAP is considered the Bayesian counterpart to MLE. It entails identifying parameter values that maximize the posterior distribution and act as estimates for the unknown parameters. Subsequently, we used the Markov Chain Monte Carlo (MCMC) simulation procedure to obtain samples from the posterior distribution to obtain the gamma parameters. The MCMC was necessary because of the computational intensity the MAP requires to incorporate the prior information. Our objective was to evaluate the performance of the Maximum Likelihood Estimation (MLE) and Maximum a-Posteriori (MAP) estimation methods under the framework of Markov Chain Monte Carlo (MCMC) simulations. This evaluation was done by first assuming a censored life-testing strategy that terminates on the r^{th} (where $r < n$) failure from a given sample of n electronic devices. Second, we obtained the joint distribution function of the first r-order statistic by arranging the values of r in order of magnitude. Finally, we explored, through the MCMC framework, the parametric estimation of the gamma distribution using the MLE and the MAP.

The finding of the study suggests that both estimation methods yielded the exact parameter estimates of 10.4169 and 0.0487, respectively, for alpha (α) and beta (β) of the gamma distribution. These estimates were not significantly different from the actual hypothesized value of α =10 and β = 0.05. The seemingly no difference between the two estimation methods confirms their flexibility in resolving estimation complexities associated with the gamma distribution for time-to-failure variables. Further, we observed that irrespective of the prior distribution used for the MAP_MCMC estimation, the resulting parametric estimates of the gamma distribution remain the same (unchanged), confirming the assertion that the Bayesian MCMC approach is a valuable tool for informative posterior analysis.

Declarations

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Conflict of interest

The authors declare that they have no competing interests.

Availability of data and materials

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Appendix

Listing A1: *Likelihood function for* α *and* β

```
function y = MLE_gamma(alpha,beta,x,n,xr,r)
y=(gamma(alpha)-gammainc(alpha,beta*xr))^(n-r)*(1/gamma(alpha))^n*beta^
(r*alpha)*prod(x.^(alpha-1))*exp(-beta*sum(x));
```

Listing A2: *Implementation of Metropolis Hastings algorithm in MATLAB using the likelihood function*

```
% % Metropolis procedure to sample from the posterior distribution
% Component-wise updating. Use a normal proposal distribution
%% Set up the Import Options and import the data
opts = spreadsheetImportOptions("NumVariables", 1);
% Specify sheet and range
opts.Sheet = "Sheet1";
opts.DataRange = "A1:A100";
% Specify column names and types
opts.VariableNames = "x";
opts.VariableTypes = "double";
% Import the data
x = readtable("C:\Users\USER\OneDrive\New Papers\Life Testing\Gamma.xlsx", opts,
"UseExcel", false);
x=table2array(x);
r=length(x);
xr=x(100);
n=200;
% % Initialize the Metropolis sampler
T=5000;  % Set the maximum number of iteration
propsigma=[0.01,0.0001]; % standard deviation of proposal distribution
parametermin=[9,0.04];  % define minimum for alpha and beta
```

```
parametermax=[11,0.06];  % define maximum for alpha and beta
seed=1; rand( 'state' , seed ); randn('state',seed ); %#ok<RAND> % set the random seed
state=zeros(2,T);   % storage space for the state of the sampler
alpha=unifrnd(parametermin(1),parametermax(1)); % Start value for alpha
beta=unifrnd(parametermin(2),parametermax(2)); % Start value for beta
t=1;  % initialize iteration at 1
state(1,t)=alpha;  % save the current state
state(2,t)=beta;
% % Start sampling
while t<T  % Iterate until we have T samples
        t=t+1;
        % % Propose a new value for alpha
        new_alpha=normrnd(alpha,propsigma(1));
        pratio=MLE_gamma(new_alpha,beta,x,n,xr,r)/MLE_gamma(alpha,beta,x,n,xr,r);
        a=min([1 pratio]);  % Calculate the acceptance ratio
        u=rand;  % Draw a uniform deviate from [0 1]
        if u<a  % Do we accept this proposal?
        alpha=new_alpha;  % proposal becomes new value for alpha
        end
        % % Propose a new value for beta
        new_beta=normrnd(beta,propsigma(2));
        pratio=MLE_gamma(alpha,new_beta,x,n,xr,r)/MLE_gamma(alpha,beta,x,n,xr,r);
        a=min([1 pratio]);  % Calculate the acceptance ratio
        u=rand;  % Draw a uniform deviate from [0 1]
        if u<a  % Do we accept this proposal?
              beta=new_beta;  % proposal becomes new value for beta
        end
        % %  Save state
        state(1,t) = alpha;
        state(2,t) = beta;
end
Mean=mean(state,2)
Mode=mode(state,2)
```

Listing A3: *Likelihood function for α and β*

```
function y = MLE_gamma(alpha,beta,x,n,xr,r,a,b)
y=(gamma(alpha)-gammainc(alpha,beta*xr))^(n-r)*(1/gamma(alpha))^n*beta^(r*alpha)*prod(x.^(alpha-1))* exp(-beta*sum(x)-alpha/a-beta/b);
```

Listing A4: *Implementation of Metropolis Hastings algorithm in MATLAB using the likelihood function*

```
% % Metropolis procedure to sample from the posterior distribution
% Component-wise updating. Use a normal proposal distribution
%% Set up the Import Options and import the data
opts = spreadsheetImportOptions("NumVariables", 1);
% Specify sheet and range
opts.Sheet = "Sheet1";
opts.DataRange = "A1:A100";
% Specify column names and types
opts.VariableNames = "x";
opts.VariableTypes = "double";
```

```
% Import the data
x = readtable("C:\Users\USER\OneDrive\New Papers\Life Testing\Gamma.xlsx", opts, "UseExcel", false);
x=table2array(x);
r=length(x);
xr=x(100);
n=200;
a=15
b=0.08
% % Initialize the Metropolis sampler
T=5000;  % Set the maximum number of iteration
propsigma=[0.01,0.0001]; % standard deviation of proposal distribution
parametermin=[9,0.04];  % define minimum for alpha and beta
parametermax=[11,0.06];  % define maximum for alpha and beta
seed=1; rand( 'state' , seed ); randn('state',seed ); %#ok<RAND> % set the random seed
state=zeros(2,T);   % storage space for the state of the sampler
alpha=unifrnd(parametermin(1),parametermax(1)); % Start value for alpha
beta=unifrnd(parametermin(2),parametermax(2)); % Start value for beta
t=1;  % initialize iteration at 1
state(1,t)=alpha;  % save the current state
state(2,t)=beta;
% % Start sampling
while t<T  % Iterate until we have T samples
    t=t+1;
    % % Propose a new value for alpha
    new_alpha=normrnd(alpha,propsigma(1));
    pratio=MLE_gamma(new_alpha,beta, x,n,xr,r,a,b)/MLE_gamma(alpha,beta, x,n,xr,r,a,b);
    a=min([1 pratio]);  % Calculate the acceptance ratio
    u=rand;  % Draw a uniform deviate from [0 1]
    if u<a  % Do we accept this proposal?
    alpha=new_alpha;  % proposal becomes new value for alpha
    end
    % % Propose a new value for beta
    new_beta=normrnd(beta,propsigma(2));
    pratio=MLE_gamma(alpha,new_beta, x,n,xr,r,a,b)/MLE_gamma(alpha,beta, x,n,xr,r,a,b);
    a=min([1 pratio]);  % Calculate the acceptance ratio
    u=rand;  % Draw a uniform deviate from [0 1]
    if u<a  % Do we accept this proposal?
        beta=new_beta;  % proposal becomes the new value for beta
    end
    % %  Save state
    state(1,t) = alpha;
    state(2,t) = beta;
end
Mean=mean(state,2)
Mode=mode(state,2)
```