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LEVERAGING RANK SET SAMPLING FOR ENHANCED STRESS-STRENGTH ESTIMATION IN THE CONTEXT OF NAKAGAMI DISTRIBUTION

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Abstract

This study addresses the estimation of the stress-strength reliability model, where stress and strength both following the Nakagami distribution. While conventional approaches have relied on simple random sampling (SRS) for estimating reliability models, recent research suggests that ranked set sampling (RSS) offers a more efficient alternative. RSS yields more informative samples compared to SRS, potentially enhancing the accuracy of reliability estimations. Our investigation focuses on deriving maximum likelihood estimators (MLEs) for stress-strength under both SRS and RSS methodologies. To evaluate the comparative efficacy of these sampling techniques, we conduct a comprehensive Monte Carlo simulation study. The results of this analysis provide compelling evidence that RSS-based estimators outperform their SRS counterparts in terms of efficiency and precision. This research contributes to the growing body of literature supporting the adoption of RSS in reliability engineering. By demonstrating the superior performance of RSS in the context of Nakagami-distributed stress-strength models, we offer valuable insights for researchers and practitioners seeking to optimize their estimation procedures in reliability analysis.

Keywords*:* Stress–strength reliability, simple random sampling, ranked set sampling, Nakagami distribution, maximum likelihood estimation.

1. Introduction

The concept of stress-strength reliability plays a pivotal role in engineering decision-making, design optimization, and risk evaluation, particularly where safety, performance, and longevity are paramount. This analytical approach is indispensable for ensuring that engineered systems, structures, and components not only meet functional requirements but also withstand the challenges posed by fluctuating loads, environmental influences, and operational dynamics. At the core of reliability engineering and statistics lies the stress-strength model, which primarily aims to quantify the probability of system success or failure when both stress and strength are subject to random variations. This methodology finds applications across diverse sectors, including engineering, materials science, quality control, and even finance. Within the framework of the stress-strength paradigm, the expression "$P_r(Y < X)$" represents the likelihood that a system's stress remains below its inherent strength. Essentially, this metric gauges the probability of system survival in the stress-strength model. Conversely, system failure occurs when the applied stress exceeds the material or component strength.

In the literature, the work on stress-strength model was first done by Birnbaum [2] and Birnbaum and McCarty [3]. The word stress-strength was first used by Church and Harris [4] in their research article, and they done a remarkable work under parametric and non-parametric inference. After that various authors choose different probabilistic models for estimating the stress-strength models. Some of these choices were summarised by Johnson [5]. A summary of all approaches and findings on the stress-strength model during the previous four decades was published by Kotz et al. [6]. The situation where X and Y are independent Type XII Burr random variables was examined by Awad and Gharraf [7]. For the recent development on this topic, one may refer to Chaturvedi and Kumar [8], Kundu and Gupta [9], Kundu and Raqab [10], Krishnamoorthy and Lin [11], Lio and Tsai [12], Barbiero [13], Chaturvedi and Kumari [14]. In all the above studies the authors have used the simple random sampling technique.

The ranked set sampling introduced by McIntyre [1][1], gained importance when Halls and Dell [15] applied ranked set sampling to estimate forage yields under pine-hardwood forest. Takahashi and Wakimoto [16], Dell and Clutter [17], David and Levine [18] focused on the efficiency of the estimators based on RSS and they established that RSS outperforms its counterpart simple random sampling with an identical sample size. Expanding the horizons of RSS, Yu and Lam [19] and Chen [20] explored regression estimation based on this methodology, providing notable examples and results. Additionally, studies on the estimation of distribution functions under various RSS techniques were conducted by Stokes and Sager [21], Kvam and Samaniego [22], and Chen [23]. Zamanzade and Vock [24], Zhang et al. [25] and Ozturk [26] have yields insights into inferential procedures reliant on ranked set sampling.

To delve deeper into this specialized data collection technique, one may refer to the review papers of Kaur et al. [27], Bai and Chen [28], and Wolfe [29]. These review papers include all pertinent references on RSS, including historical development, current status and future research direction. Hassan et al. [30] obtained the point and interval estimators of $P = P_r(Y < X)$ for the case of independent Gompertz random variables with common scale and different shape parameters based on RSS.

Here, we have consider the estimation of $P = P_r(Y < X)$ with a focus on situation where the random stress Y and random strength X are two independent Nakagami random variables with shape parameters (α_1, α_2) and scale parameters (λ, λ), respectively. The point estimator of $P = P_r(Y < X)$, is obtained using the maximum likelihood method based on both SRS and RSS, and the efficiency of this method based on SRS and RSS is compared. In Section 2, we present a brief overview about the Nakagami distribution and its relationship with other probability distributions. Point estimation of the parameters is given in Section 3. Section 4 and Section 5 comprises the point estimation of stress-strength model under SRS and under RSS, respectively. A simulation study employing the Monte Carlo method is discussed in Section 6. Section 7 details an empirical data analysis, and lastly Section 8 provides concluding remarks for the paper.

2. Preliminary

Consider a random variable X that adheres to the Nakagami distribution, denoted as NAD (α, λ). In this distribution, α represents the shape parameter, which is bounded by the condition $\alpha > 0.5$, while λ symbolizes the scale parameter, constrained to be strictly positive ($\lambda > 0$). For this distribution, the probability density function (PDF) and cumulative distribution function (CDF) are characterized as follows:

$$f(x;\alpha,\lambda)=\frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha}x^{(2\alpha-1)}exp\left(-\frac{\alpha}{\lambda}x^2\right);x>0,\alpha>0.5,\lambda>0 \tag{1}$$

and

$$F(x)=\frac{1}{\Gamma\alpha}\gamma\left(\alpha,\ \frac{\alpha}{\lambda}x^2\right);x>0,\alpha>0.5,\lambda>0 \tag{2}$$

Where, $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.

The reliability function of NAD (α,λ) is

$$R(t)=1-\frac{1}{\Gamma\alpha}\gamma\left(\alpha,\ \frac{\alpha}{\lambda}t^2\right);t>0,\alpha>0.5,\lambda>0 \tag{3}$$

The hazard rate of NAD (α,λ) is.

$$h(t)=\frac{\frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha}t^{(2\alpha-1)}exp\left(-\frac{\alpha}{\lambda}t^2\right)}{1-\frac{1}{\Gamma\alpha}\gamma\left(\alpha,\ \frac{\alpha}{\lambda}t^2\right)};t>0,\alpha>0.5,\lambda>0 \tag{4}$$

Other distribution relationships

1) If α = 0.5, then Nakagami distribution (α,λ) becomes Half Normal Distribution.
2) For α = 1, then Nakagami distribution (α,λ) reduces to Rayleigh Distribution.
3) If random variable Y ~ Gamma (α,λ) where α is shape parameter and λ is scale parameter, then $\sqrt{Y}$ ~ NAD $(\alpha,\alpha\lambda)$.
4) If Z ~ chi-square (2α) and then $\sqrt{\frac{\lambda}{2\alpha}Z}$ ~ NAD (α,λ) where 2α is integer-valued.

3. Point estimation of the parameters

Let us draw a random sample $X_1,X_2,\dots,X_n$ from the NAD (α,λ) of size n. The likelihood function of the Nakagami distribution NAD (α,λ) is given by

$$L\ (x,\alpha,\lambda)=\frac{(2\alpha^{\alpha})^n}{(\Gamma\alpha)^n(\lambda)^{n\alpha}}\prod_{i=1}^{n}x_i^{2\alpha-1}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{n}x_i^2\right)$$

Theorem 1. The Maximum Likelihood Estimator of scale parameter λ is

$$\hat{\lambda}=\frac{\sum_{i=1}^{n}x_i^{\ 2}}{n}$$

Theorem 2. The Maximum Likelihood Estimator of shape parameter α is

$$\hat{\alpha} = \frac{0.5}{\log\left(\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)}$$

Proof. If we suppose that λ is known, then the likelihood function for the parameter α is given as

$$L(\alpha|\underline{x}) = \frac{(2\alpha^{\alpha})^n}{(\Gamma\alpha)^n(\lambda)^{n\alpha}}\prod_{i=1}^{n} x_i^{2\alpha-1}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{n} x_i^2\right)$$

The log likelihood function is

$$\log L = n\log 2 - n\log(\Gamma\alpha) + n\alpha\log(\alpha) - n\alpha\log(\lambda) + \sum_{i=1}^{n}(2\alpha-1)\log(x_i) - \frac{\alpha}{\lambda}\sum_{i=1}^{n} x_i^{\,2}$$

Partially differentiating with respect to α, and equating it equal to zero, we get

$$\hat{\alpha} = \frac{0.5}{\log\hat{\lambda} - \frac{2}{n}\left(\sum_{i=1}^{n}\log x_i\right) + \frac{1}{\hat{\lambda}}\left(\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}\right) - 1}$$

From theorem 1

$$\hat{\alpha} = \frac{0.5}{\log\left(\frac{\sum_{i=1}^{n} x_i^{\,2}}{n}\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)}$$

4. Point estimation of $P = P_r(Y < X)$ in case of simple random sampling

To derive the stress-strength reliability model $P = P_r(Y < X)$, here we assumed that X is the strength variable and Y is the stress variable, both are following the Nakagami distribution with common scale parameter $\lambda > 0$ and different shape parameters $\alpha_1 > 0.5$ and $\alpha_2 > 0.5$, respectively. By notation $X \sim NAD\ (\alpha_1, \lambda)$ and $Y \sim NAD\ (\alpha_2, \lambda)$, then

$$P = \int_0^{\infty} P\ (Y < X) f(x) dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma\alpha_2}\gamma\left(\alpha_2, \frac{\alpha_2}{\lambda}x^2\right)\frac{2}{\Gamma\alpha_1}\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1} x^{(2\alpha_1-1)} exp\left(-\frac{\alpha_1}{\lambda}x^2\right) dx$$

where $\gamma(n,x) = \Gamma n\left(1 - e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}\right)$ is the lower incomplete gamma function.

$$= 2\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1}\frac{1}{\Gamma\alpha_1}\left(\int_0^\infty x^{(2\alpha_1-1)}exp\left(-\frac{\alpha_1}{\lambda}x^2\right)dx - \int_0^\infty x^{(2\alpha_1-1)}exp\left(-\frac{\alpha_1}{\lambda}x^2\right)exp\left(-\frac{\alpha_2}{\lambda}x^2\right)\sum_{m=0}^{\alpha_2-1}\frac{\left(\alpha_2 x^2/\lambda\right)^m}{m!}dx\right)$$

$$= 2\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1}\frac{1}{\Gamma\alpha_1}\left(\frac{1}{2}\left(\frac{\lambda}{\alpha_1}\right)^{\alpha_1}\Gamma\alpha_1 - \sum_{m=0}^{\alpha_2-1}\frac{{\alpha_2}^m\,\lambda^{\alpha_1}}{2m!}\left(\frac{\Gamma(\alpha_1+m)}{(\alpha_1+\alpha_2)^{\alpha_1+m}}\right)\right)$$

$$P = 1 - \frac{(\alpha_1)^{\alpha_1}}{\Gamma\alpha_1}\sum_{m=0}^{\alpha_2-1}\frac{(\alpha_2)^m}{m!}\frac{\Gamma(\alpha_1+m)}{(\alpha_1+\alpha_2)^{\alpha_1+\mathrm{m}}} \tag{5}$$

Let two independent random samples X and Y of size n and m are drawn from Nakagami distribution with parameters (α_1, λ) and (α_2, λ), respectively. For known λ, the invariance characteristic of the maximum likelihood estimator provides the maximum likelihood estimator for P. The maximum likelihood estimators of α_1 and α_2 are

$$\hat{\alpha}_{1SRS} = \frac{0.5}{\log\left(\frac{\sum_{i=1}^{n} {x_i}^2}{n}\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)} \quad \text{and} \quad \hat{\alpha}_{2SRS} = \frac{0.5}{\log\left(\frac{\sum_{k=1}^{m} {y_k}^2}{m}\right) - 2\left(\frac{1}{m}\sum_{k=1}^{m}\log y_k\right)}$$

Maximum likelihood estimator of P in case of simple random sampling is given by

$$\hat{P}_{SRS} = 1 - \frac{(\hat{\alpha}_{1SRS})^{\hat{\alpha}_{1SRS}}}{\Gamma\hat{\alpha}_{1SRS}}\sum_{m=0}^{\hat{\alpha}_{1SRS}-1}\frac{(\hat{\alpha}_{2SRS})^m}{m!}\frac{\Gamma(\hat{\alpha}_{2SRS}+m)}{(\hat{\alpha}_{1SRS}+\hat{\alpha}_{2SRS})^{\hat{\alpha}_{1SRS}+\mathrm{m}}}$$

5. Point estimation of $P = P_r(Y < X)$ in case of ranked set sampling

1. Standard ranked set sampling

Ranked set sampling (RSS) represents a cutting-edge approach in statistical sampling, designed to boost the accuracy of parameter estimation, particularly in scenarios where resources are scarce or data collection costs are prohibitive. This method diverges from traditional random sampling by utilizing the ranked order or order statistics of sampled observations, thereby enhancing the quality and efficiency of estimations. The concept of RSS, initially proposed in the mid-20th century, has since gained traction across diverse fields such as environmental science, forestry, and ecology. Its popularity stems from its ability to yield robust statistical insights even when comprehensive population surveys are unfeasible. By offering a pragmatic and economical alternative to conventional sampling techniques, RSS has become an invaluable asset for researchers and statisticians aiming to refine their sampling strategies. The implementation of RSS to generate a sample of size n = r*m involves a series of structured steps, where m denotes the number of sample units selected in each cycle (of fixed size) and r represents the total number of cycles. These steps are executed sequentially as follows:

1. A random subset of the population consisting of m^2 units is selected.
2. The m^2 units are then divided arbitrarily into m sets, each containing m units.
3. The units within each set are ranked based on either professional judgment or correlation

with the variable of interest.

4. An individual quantile sample is constructed by taking the lowest ranked unit from the first set, the second lowest ranked unit from the second set, and continuing in this fashion.
5. To obtain a larger sample of size n = r*m, steps 1 through 4 can be repeated for r cycles.

The ranked set sampling (RSS) method takes only one observation from each set in each cycle. In the first cycle, it chooses the lowest observation $X_{(11)r}$. In later cycles, it independently selects the second lowest $X_{(22)r}$ from a different set of m observations and the highest $X_{(mm)r}$ from the final set of m. Let $X_{(ii)k},\ i = 1,2,\dots,m;\ k = 1,2,\dots,r$, be a ranked sample set with set size m and r cycles. For convenience, this paper will use the notation $X_{(i)r}$ in place of the full description.

2. The maximum likelihood estimation of $P = P_r(Y < X)$ in case of RSS

Let $X_{(ij)}, i = 1,2,\dots,r_1;\ j = 1,2,\dots,m_1$, denote the ranked set sample of size $n_1 = r_1 m_1$ from Nakagami distribution with parameter (α_1, λ), where m_1 is the set size and r_1 is the number of cycles and $Y_{(kl)},\ k = 1,2,\dots,r_2;\ l = 1,2,\dots,m_2$, denote the ranked set sample of size $n_2 = r_2 m_2$ from Nakagami distribution with parameter (α_2, λ), where m_2 is the set size and r_2 is the number of cycles. Then the PDF of $X_{(ij)}$ and $Y_{(kl)}$ are given by

$$f_i(x_{ij}) = \frac{m_1!}{(i-1)!(m_1-i)!}[F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i}f(x_{ij}) \tag{6}$$

$$g_k(y_{kl}) = \frac{m_2!}{(k-1)!(m_2-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k}g(y_{kl}) \tag{7}$$

Now the likelihood function is given as

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k(y_{kl})$$

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} \frac{m_1!}{(i-1)!\,(m_1-i)!}[F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i}f(x_{ij})$$
$$\prod_{k=1}^{r_2}\prod_{l=1}^{m_2} \frac{m_2!}{(k-1)!\,(m_2-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k}g(y_{kl}) \tag{8}$$

Let $u = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} \frac{m_1!}{(i-1)!\,(m_1-i)!}$, $v = \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} \frac{m_2!}{(k-1)!\,(m_2-k)!}$

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} u[F_X(x_{ij})]^{i-1}[1-F_X(x_{ij})]^{m_1-i}f(x_{ij})$$
$$\prod_{k=1}^{r_2}\prod_{l=1}^{m_2} v[F_Y(y_{kl})]^{k-1}[1-F_Y(y_{kl})]^{m_2-k}g(y_{kl})$$

$$L = u\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left(\frac{1}{\Gamma\alpha_1}\right)^{m_1-1}\left[\gamma\left(\alpha_1,\frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{i-1}\left[\Gamma\alpha_1-\gamma\left(\alpha_1,\frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{m_1-i}$$
$$\left(\frac{2}{\Gamma\alpha_1}\right)\left(\frac{\alpha_1}{\lambda}\right)^{\alpha_1}{x_{ij}}^{2\alpha_1-1}\exp\left(-\frac{\alpha_1}{\lambda}{x_{ij}}^2\right)\ v\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left(\frac{1}{\Gamma\alpha_2}\right)^{m_2-1}\left[\gamma\left(\alpha_2,\frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{k-1}$$
$$\left[\Gamma\alpha_2-\gamma\left(\alpha_2,\frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{m_2-k}\left(\frac{2}{\Gamma\alpha_2}\right)\left(\frac{\alpha_2}{\lambda}\right)^{\alpha_2}{y_{kl}}^{2\alpha_2-1}\exp\left(-\frac{\alpha_2}{\lambda}{y_{kl}}^2\right) \tag{9}$$

sum of lower incomplete gamma function and upper incomplete gamma function is a gamma function, which implies

$$\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right) + \Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right) = \Gamma\alpha_1$$

gives

$$\Gamma\alpha_1 - \gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right) = \Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)$$

Thus,

$$L = \left(\frac{1}{\Gamma\alpha_1}\right)^{n_1(m_1-1)} \left(\frac{2}{\Gamma\alpha_1}\right)^{n_1} \left(\frac{\alpha_1}{\lambda}\right)^{n_1\alpha_1} \left(\frac{1}{\Gamma\alpha_2}\right)^{n_2(m_2-1)} \left(\frac{2}{\Gamma\alpha_2}\right)^{n_2} \left(\frac{\alpha_2}{\lambda}\right)^{n_2\alpha_2} uv$$
$$\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{i-1} \left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{m_1-i} {x_{ij}}^{2\alpha_1-1} exp\left(-\frac{\alpha_1}{\lambda}{x_{ij}}^2\right)$$
$$\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{k-1} \left[\Gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{m_2-k} {y_{kl}}^{2\alpha_2-1} exp\left(-\frac{\alpha_2}{\lambda}{y_{kl}}^2\right) \tag{10}$$

Taking log on both sides

$$\log L = \log L_1 + \log L_2 \tag{11}$$

where

$$L_1 = u\left(\frac{1}{\Gamma\alpha_1}\right)^{n_1(m_1-1)} \left(\frac{2}{\Gamma\alpha_1}\right)^{n_1} \left(\frac{\alpha_1}{\lambda}\right)^{n_1\alpha_1} \prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{i-1}$$
$$\left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]^{m_1-i} {x_{ij}}^{2\alpha_1-1} exp\left(-\frac{\alpha_1}{\lambda}{x_{ij}}^2\right) \tag{12}$$

and

$$L_2 = v\left(\frac{1}{\Gamma\alpha_2}\right)^{n_2(m_2-1)} \left(\frac{2}{\Gamma\alpha_2}\right)^{n_2} \left(\frac{\alpha_2}{\lambda}\right)^{n_2\alpha_2} \prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{k-1}$$
$$\left[\Gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right]^{m_2-k} {y_{kl}}^{2\alpha_2-1} exp\left(-\frac{\alpha_2}{\lambda}{y_{kl}}^2\right) \tag{13}$$

This implies,

$$\log L_1 = \log u + n_1(m_1-1)(-\log\Gamma\alpha_1) + n_1(\log 2 - \log\Gamma\alpha_1) + n_1\alpha_1(\log\alpha_1 - \log\lambda)$$
$$+ \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\log\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\log\left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right]$$
$$+ (2\alpha_1-1)\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij} - \frac{\alpha_1}{\lambda}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}^2 \tag{14}$$

Differentiating Eq.(5.2.9) with respect to α_1 and α_2 respectively, we get

$$\frac{\partial \log L_1}{\partial \alpha_1} = -m_1 \frac{\partial}{\partial \alpha_1} \log \Gamma \alpha_1 + n_1(\log \alpha_1 + 1) - n_1 \log \lambda + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial \alpha_1}\log\left[\gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1 - i)\frac{\partial}{\partial \alpha_1}\log\left[\Gamma\left(\alpha_1, \frac{\alpha_1 {x_{ij}}^2}{\lambda}\right)\right] + 2\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij} - \frac{1}{\lambda}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1} x_{ij}^2 \tag{15}$$

and

$$\frac{\partial \log L_2}{\partial \alpha_2} = -m_2 \frac{\partial}{\partial \alpha_2} \log \Gamma \alpha_2 + n_2(\log \alpha_2 + 1) - n_2 \log \lambda + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\frac{\partial}{\partial \alpha_2}\log\left[\gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right] + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2 - k)\frac{\partial}{\partial \alpha_2}\log\left[\Gamma\left(\alpha_2, \frac{\alpha_2 {y_{kl}}^2}{\lambda}\right)\right] + 2\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\log y_{kl} - \frac{1}{\lambda}\sum_{k=1}^{r_2}\sum_{l=1}^{m_2} {y_{kl}}^2 \tag{16}$$

Differentiating Eq.(5.2.6) with respect to λ, we get

$$\frac{\partial logL}{\partial \lambda} = -\frac{n_1\alpha_1}{\lambda} + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial \lambda}log\left[\gamma\left(\alpha_1, \frac{\alpha_1 x^2}{\lambda}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1 - i)\frac{\partial}{\partial \lambda}log\left[\Gamma\left(\alpha_1, \frac{\alpha_1 x^2}{\lambda}\right)\right] + \frac{\alpha_1}{\lambda}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1} x_{ij}^2 - \frac{n_2\alpha_2}{\lambda} + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\frac{\partial}{\partial \lambda}log\left[\gamma\left(\alpha_2, \frac{\alpha_2 y^2}{\lambda}\right)\right] + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2 - k)\frac{\partial}{\partial \lambda}log\left[\Gamma\left(\alpha_2, \frac{\alpha_2 y^2}{\lambda}\right)\right] + \frac{\alpha_2}{\lambda^2}\sum_{k=1}^{r_2}\sum_{k=1}^{m_2} y_{kl}^2 \tag{17}$$

A numerical approach is utilized to obtain the maximum likelihood estimates for α_1 and α_2, denoted as by $\hat{\alpha}_{1RSS}$ and $\hat{\alpha}_{2RSS}$, from equations 5.2.10 and 5.2.11, respectively using the ranked set sampling method. Applying the invariance property of maximum likelihood estimators, the maximum likelihood estimate of the reliability parameter P based on RSS, denoted $\hat{P}_{RSS}$, can then be derived as

$$\hat{P}_{RSS} = 1 - \frac{(\hat{\alpha}_{1RSS})^{\hat{\alpha}_{1RSS}}}{\Gamma\hat{\alpha}_{1RSS}} \sum_{m=0}^{\hat{\alpha}_{1RSS}-1} \frac{(\hat{\alpha}_{2RSS})^m}{m!} \frac{\Gamma(\hat{\alpha}_{2RSS} + m)}{(\hat{\alpha}_{1RSS} + \hat{\alpha}_{2RSS})^{\hat{\alpha}_{1RSS}+\mathrm{m}}}$$

6. Simulation study

In this section we carried out a simulation study. Bias and mean square error (MSE) for P are provided by $Bias(\hat{P}) = E(\hat{P} - P)$ and $MSE(\hat{P}) = E(\hat{P} - P)^2$, respectively to compare our suggested reliability estimator P based on ranked set sampling RSS with the conventional reliability estimator of P based on SRS. The formula for calculating the relative efficiency RE of the estimator of P is $\frac{MSE(\hat{P}_{SRS})}{MSE(\hat{P}_{RSS})}$. Relative efficiency values greater than one suggest that the $\hat{P}_{RSS}$ is more efficient than the $\hat{P}_{SRS}$. All computations are performed using the R programming language. The simulation study is explained in the following steps.

Step 1: We generate 1000 simple random samples of $X_1, X_2, \dots, X_{n_1}$, and $Y_1, Y_2, \dots, Y_{n_2}$ from Nakagami distribution with the sample sizes of (n_1, n_2) = (15, 15), (15, 20), (15, 25), (20, 20), (20, 25), (25, 25) in Case 1 and (20, 20), (20, 30), (20, 40), (30, 30), (30, 40), (40, 40) in Case 2.

Step 2: We generate 1000 ranked set samples of $X_{11}, \dots, X_{m_1 r_1}$ and $Y_{11}, \dots, Y_{m_2 r_2}$ from Nakagami distribution for the first case when the number of cycles is taken as $r_1 = r_2 = 5$ with set sizes $m_1 = m_2 = 3, 4, 5$ and for the second case when the number of cycles is taken as $r_1 = r_2 = 10$ with set sizes $m_1 = m_2 = 2, 3, 4$.

Step 3: To generate the simple random samples and ranked set samples for Nakagami distribution, we consider the true value of the common scale parameter $\lambda = 3$ and the true values of the shape parameter α_x and α_y are (0.5, 0.9), (0.7, 1.2) and (0.9, 1.5), respectively for the strength variable X and the stress variable Y, respectively. For these values, the true value of stress-strength model P is 0.40238, 0.50290 and 0.58635, respectively.

Step 4: The Biases, MSES and relative efficiency are presented in the Table 1.

Table 1: *Biases, MSES and RE of P under SRS and RSS when the common scale parameter λ = 3*

	Case-1				SRS			RSS		
						$r_1 = r_2 = 5$				
(α_1, α_2)	(n_1, n_2)	(m_1, m_2)	P_{True}	$\hat{P}_{SRS}$	Bias	MSE	$\hat{P}_{RSS}$	Bias	MSE	RE
(0.5,0.9)	(15,15)	(3,3)	0.40238	0.38205	-0.02033	0.007075	0.37479	-0.02759	0.005877	1.2037
	(15,20)	(3,4)		0.37976	-0.02262	0.006309	0.36312	-0.03926	0.005434	1.1609
	(15,25)	(3,5)		0.36858	-0.0338	0.005457	0.35188	-0.05050	0.005283	1.0328
	(20,20)	(4,4)		0.36992	-0.03247	0.005543	0.36500	-0.03738	0.004656	1.1903
	(20,25)	(4,5)		0.36951	-0.03287	0.004886	0.36047	-0.04190	0.004239	1.1522
	(25,25)	(5,5)		0.36761	-0.03477	0.004822	0.35752	-0.04486	0.004346	1.1095
(0.7,1.2)	(15,15)	(3,3)	0.50290	0.48768	-0.01521	0.0081549	0.48285	-0.02004	0.007379	1.1051
	(15,20)	(3,4)		0.48781	-0.01509	0.0071741	0.47443	-0.02846	0.006434	1.1149
	(15,25)	(3,5)		0.48497	-0.01792	0.0064565	0.46760	-0.03529	0.005642	1.1442
	(20,20)	(4,4)		0.48402	-0.01887	0.0065401	0.47210	-0.03079	0.005534	1.1816
	(20,25)	(4,5)		0.48118	-0.02171	0.0058446	0.46763	-0.03527	0.004607	1.2684
	(25,25)	(5,5)		0.47852	-0.02437	0.0056632	0.46007	-0.04282	0.005261	1.0762
(0.9,1.5)	(15,15)	(3,3)	0.58635	0.58608	-0.00026	0.009415	0.56558	-0.02759	0.008915	1.0560
	(15,20)	(3,4)		0.57906	-0.00729	0.008066	0.56648	-0.03926	0.006912	1.1669
	(15,25)	(3,5)		0.57434	0.57434	0.006819	0.55942	-0.05050	0.005731	1.1897
	(20,20)	(4,4)		0.57625	-0.01010	0.007531	0.55986	-0.03738	0.006197	1.2151
	(20,25)	(4,5)		0.56799	-0.01836	0.006249	0.54742	-0.04190	0.005974	1.0460
	(25,25)	(5,5)		0.56607	-0.02027	0.005941	0.55713	-0.04486	0.004836	1.2282

	Case-2					$r_1 = r_2 = 10$				
(α_1, α_2)	(n_1, n_2)	(m_1, m_2)	P_{True}	$\hat{P}_{SRS}$	Bias	MSE	$\hat{P}_{RSS}$	Bias	MSE	RE
(0.5,0.9)	(20,20)	(2,2)	0.40238	0.37497	-0.02741	0.005234	0.37116	-0.03121	0.005073	1.0316
	(20,30)	(2,3)		0.36741	-0.03497	0.004678	0.36359	-0.03879	0.004379	1.0682
	(20,40)	(2,4)		0.36257	-0.03980	0.004350	0.36089	-0.04149	0.003852	1.1290
	(30,30)	(3,3)		0.36183	-0.04055	0.004494	0.36021	-0.04216	0.004249	1.0574
	(30,40)	(3,4)		0.36209	-0.04028	0.004011	0.35657	-0.04581	0.003841	1.0444
	(40,40)	(4,4)		0.35891	-0.04346	0.004020	0.35649	-0.04589	0.003606	1.1146
(0.7,1.2)	(20,20)	(2,2)	0.50290	0.48040	-0.02249	0.007313	0.48090	-0.02200	0.006477	1.1290
	(20,30)	(2,3)		0.47824	-0.02465	0.005291	0.47161	-0.03129	0.004850	1.0908
	(20,40)	(2,4)		0.47323	-0.02966	0.004726	0.46726	-0.03563	0.004282	1.1037
	(30,30)	(3,3)		0.47157	-0.03132	0.004867	0.46328	-0.03961	0.004715	1.0322
	(30,40)	(3,4)		0.46866	-0.03423	0.004614	0.46346	-0.03943	0.004210	1.0958
	(40,40)	(4,4)		0.46736	-0.03553	0.004181	0.46037	-0.04253	0.003984	1.0494
(0.9,1.5)	(20,20)	(2,2)	0.58635	0.57266	-0.01369	0.007225	0.56863	-0.01771	0.006831	1.0576
	(20,30)	(2,3)		0.56358	-0.02276	0.006070	0.55271	-0.03364	0.005913	1.0265
	(20,40)	(2,4)		0.56086	-0.02548	0.004649	0.55686	-0.02948	0.004561	1.0192
	(30,30)	(3,3)		0.55881	-0.02753	0.005109	0.55648	-0.02986	0.005039	1.0137
	(30,40)	(3,4)		0.55972	-0.02662	0.004832	0.55016	-0.03619	0.004452	1.0853
	(40,40)	(4,4)		0.55734	-0.02901	0.004548	0.55338	-0.03296	0.003982	1.1421

It is evident from the Table 1 that the relative efficiency is greater than one in every case; so, we can say that the ranked set sampling is showing more efficient results in comparison to simple random sampling in estimating the stress-strength reliability.

7. Real data application

In order to comprehend and provide a broad illustration of the processes covered in the preceding sections, we now take two real data sets. The first data set is used for the strength variable X and second data set is used for the stress variable Y in the stress-strength model $P = P_r(Y < X)$.

7.1 First Data Set

Lawless (2003, pp. 267) is the source of the data set. The first report on this was published in 1987 by Schat, Staton, Mandel, and Shott. The hours to failure of 59 conductors with a length of 400 micrometres are represented by this data. The specimens are tested at the same temperature and current density, and at a specific high temperature and current density, they all failed. The MLES of the parameters α and λ for this dataset is $\hat{\alpha}_x$= 4.6731 and $\hat{\lambda}_x$ = 51.2823

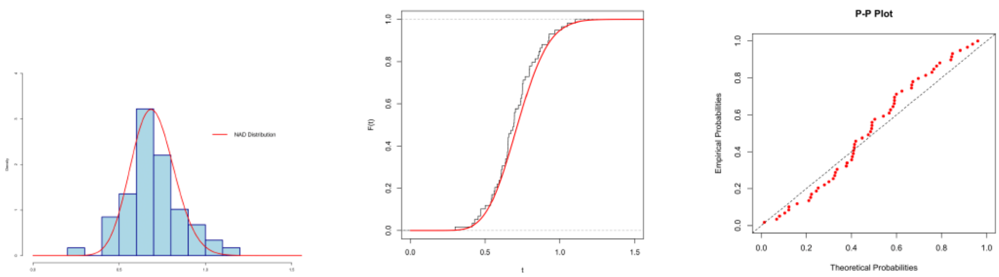
7.2 Second Data Set

The second data set is taken from Murthy et al. (2004, pp.180). This data represents 50 items that are put on use at time t = 0 and failure times are recorded (in weeks). The MLES for the parameters α and λ for this dataset is $\hat{\alpha}_y$= 0.1924 and $\hat{\lambda}_y$ = 144.2292. Both the datasets are shown in Table 2.

Table 2: *Dataset – 1 supposed to be X - Population and Dataset-2 supposed to be Y – Population*

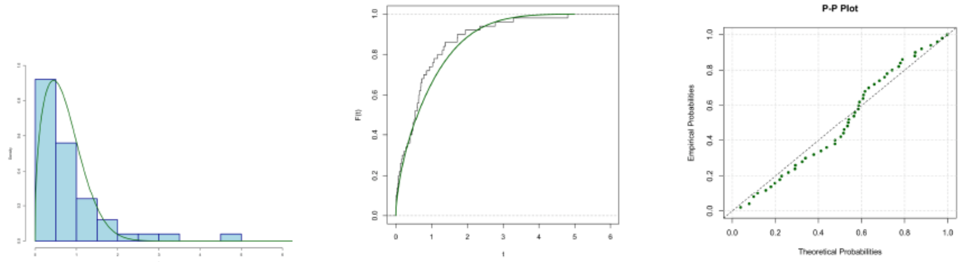
Dataset-1 : X-Population			
6.545	6.522	7.945	7.224
9.289	4.137	6.869	7.365
7.543	7.459	6.352	6.923
6.956	7.495	4.7	5.64
6.492	6.573	6.948	5.434
5.459	6.538	9.254	7.937
8.12	5.589	5.009	6.515
4.706	6.087	7.489	6.476
8.687	5.807	7.398	6.071
2.997	6.725	6.033	10.491
8.591	8.532	10.092	5.923
6.129	9.663	7.496	
11.038	6.369	4.531	
5.381	7.024	7.974	
6.958	8.336	8.799	
4.288	9.218	7.683	

Dataset-2 : Y-Population			
0.013	2.838	7.291	32.795
0.065	3.269	7.087	48.105
0.111	3.977	7.787	
0.111	3.981	8.596	
0.163	4.52	9.388	
0.309	4.789	10.261	
0.426	4.849	10.713	
0.535	5.202	11.658	
0.684	5.291	13.006	
0.747	5.349	13.388	
0.997	5.911	13.842	
1.284	6.018	17.152	
1.304	6.427	17.283	
1.647	6.456	19.418	
1.829	6.572	23.471	
2.336	7.023	27.777	



(a) PDF Plot (b) CDF Plot (c) P-P Plot

Figure 1: *The PDF, CDF and P-P Plots of the Nakagami distribution for First dataset*



(a) PDF Plot (b) CDF Plot (c) P-P Plot

Figure 2: *The PDF, CDF and P-P Plots of the Nakagami distribution for Second dataset*

Before we dive into the core of our investigation, it's crucial to thoroughly examine the key characteristics of our data. To validate the strength of our results, we employ a powerful statistical instrument: the Kolmogorov-Smirnov (K-S) test, along with its corresponding P-value (P-V). This approach enables us to measure how well our empirical observations align with theoretical expectations.

Our analysis yields promising outcomes. For the first dataset, we calculate a K-S distance of 0.06779 and a P-V of 0.99940. The second dataset produces similar results, with a K-S distance of 0.12 and a P-V of 0.86428. These metrics provide compelling evidence that our model closely matches the observed data.

To enhance our understanding and provide visual context, we have created a series of graphical representations. These illustrations, found in the accompanying Figure 1 and 2, offer a comprehensive view of our statistical findings. They include probability-probability (PP) plots, as well as visualizations of the estimated probability density function (PDF) and cumulative distribution function (CDF) for both datasets. These visual aids serve to reinforce and clarify the numerical results of our analysis

We consider these two datasets as our random strength X and random stress Y, respectively. The MLES for α and λ i.e. $\hat{\alpha}_x$= 4.6731, $\hat{\lambda}_x$= 51.2823 and $\hat{\alpha}_y$= 0.1924 and $\hat{\lambda}_y$= 144.2292 is taken as the true value of the parameters for this study. Now if $\hat{\alpha}_x = \alpha_1$ = 4.6731 and $\hat{\alpha}_y = \alpha_2$ = 0.1924 then the true value of the stress-strength model from Eq.(4.1) is P = 0.1718.

In this analysis, we draw simple random samples of size 10 from each dataset and estimate the MLES for α and λ, respectively. The simple random samples and MLES are presented in Table 3 and Table 4, respectively.

Table 3: *MLES of α and λ for each random sample of X-Population*

Simple Random Samples from X-Population												
											$\hat{\alpha}_x$	$\hat{\lambda}_x$
Sample 1	7.489	5.589	7.495	4.137	6.492	8.687	6.538	8.532	7.683	7.459	6.7031	50.8427
Sample 2	4.531	6.956	6.923	10.491	7.937	8.12	4.137	4.7	5.589	6.129	3.1983	46.3841
Sample 3	8.532	6.087	7.489	6.958	6.522	6.545	7.398	8.591	6.923	7.496	7.9133	42.4977
Sample 4	6.033	6.352	8.799	2.997	5.64	6.956	11.038	6.522	7.024	5.009	2.6220	48.2145
Sample 5	5.807	7.398	5.923	7.945	10.092	4.706	11.038	6.369	7.459	5.589	3.7265	55.9926
Sample 6	10.491	6.071	8.591	7.024	7.495	4.706	4.137	4.531	6.369	6.352	3.2477	46.6938
Sample 7	2.997	4.137	4.7	9.254	7.683	6.923	7.974	8.687	6.573	6.545	2.6515	46.5869
Sample 8	5.923	9.663	6.515	5.589	10.491	6.948	7.495	11.038	4.531	6.492	3.3188	60.1161
Sample 9	5.807	6.956	7.459	7.496	8.687	5.64	5.434	4.288	11.038	5.381	3.5093	49.9915
Sample10	6.948	6.538	5.434	7.365	5.589	7.543	4.706	4.7	7.489	9.218	5.7396	44.8218

Table 4: *MLES of α and λ for each random sample of Y-Population*

Simple Random Samples from Y-Population												
											$\hat{\alpha}_y$	$\hat{\lambda}_y$
Sample 1	7.787	6.456	7.087	5.202	5.911	7.023	0.684	23.471	19.418	7.291	0.4860	124.543
Sample 2	5.202	7.023	1.284	6.018	9.388	1.647	4.789	0.684	10.713	0.163	0.3309	34.329
Sample 3	13.842	0.997	5.911	13.006	23.471	0.535	1.284	0.684	5.349	7.787	0.2469	103.923
Sample 4	11.658	17.152	23.471	7.787	8.596	0.163	6.456	13.006	4.789	0.013	0.2078	134.931
Sample 5	5.291	1.284	13.006	8.596	0.426	11.658	9.388	4.789	17.283	7.787	0.4596	87.918
Sample 6	0.426	8.596	2.838	1.647	6.572	0.111	7.291	0.111	1.829	13.388	0.2006	36.379
Sample 7	0.997	3.981	7.087	0.426	6.456	8.596	13.388	6.427	13.006	4.789	0.4675	59.545
Sample 8	23.471	2.336	3.269	6.427	0.163	4.849	4.52	7.787	5.291	5.349	0.3125	76.955
Sample 9	9.388	6.456	17.152	17.283	24.777	6.018	13.388	4.789	0.163	10.713	0.4078	168.979
Sample10	0.747	2.838	0.309	8.596	5.202	10.713	13.006	19.418	23.471	1.647	0.2632	132.424

Now, we draw 10 samples using ranked set sampling technique. We run two cycle (r = 2) of set size m = 5 to get a ranked set sample of size n = r * m = 10.

Table 5: *MLES of α and λ for each ranked set sample of X-Population*

Ranked Set Samples from X-Population												
											$\hat{\alpha}_x$	$\hat{\lambda}_x$
Sample 1	4.137	6.476	6.369	6.515	8.799	5.381	6.948	5.807	8.532	8.687	5.3408	47.8695
Sample 2	4.288	6.492	6.725	6.071	10.491	4.288	6.369	8.591	7.024	9.254	3.3715	52.0406
Sample 3	4.288	6.869	6.071	7.496	11.038	5.009	6.071	7.495	7.495	11.038	3.0263	57.6589
Sample 4	7.683	7.489	6.515	9.218	6.476	6.958	4.288	6.492	7.543	7.543	8.1074	50.7209
Sample 5	4.137	6.958	6.545	6.956	11.038	4.706	6.476	8.12	7.543	9.289	3.5078	55.1790
Sample 6	4.531	6.522	7.683	7.024	10.092	4.288	6.033	7.489	7.937	7.398	4.5176	50.1875
Sample 7	4.137	6.515	6.129	6.129	8.591	2.997	6.869	5.807	7.489	9.218	3.0288	43.9437
Sample 8	4.137	5.807	6.948	7.224	10.092	4.531	4.706	9.289	7.945	10.491	2.5873	55.5291
Sample 9	6.071	6.948	7.224	6.948	9.289	4.531	6.492	5.009	7.398	11.038	3.9824	53.6212
Sample 10	5.589	5.589	6.087	7.945	9.218	5.009	4.531	6.545	7.459	8.799	4.8530	46.9136

Table 6: *MLES of α and λ for each ranked set sample of Y-Population*

Ranked Set Samples from Y-Population												
											$\hat{\alpha}_y$	$\hat{\lambda}_y$
Sample 1	5.291	0.997	4.52	17.152	9.388	0.065	1.284	1.647	13.842	11.658	0.24212	76.3620
Sample 2	0.013	7.787	2.336	6.456	17.152	3.977	0.747	11.658	19.418	17.283	0.21862	123.001
Sample 3	0.535	4.52	2.336	9.388	13.388	0.747	0.065	4.849	13.006	23.471	0.20209	103.7666
Sample 4	3.981	0.426	4.52	4.789	23.471	0.684	4.789	7.023	7.023	48.105	0.19412	304.6421
Sample 5	1.284	3.269	11.658	8.596	10.713	0.065	4.789	6.572	6.456	13.842	0.34995	63.6314
Sample 6	1.647	1.647	5.911	13.842	7.023	0.111	4.789	5.291	7.291	11.658	0.36021	52.1297
Sample 7	0.111	1.829	3.977	23.471	7.787	0.065	6.427	4.789	4.849	48.105	0.14340	303.2548
Sample 8	0.111	3.977	6.018	4.52	17.283	0.163	0.065	3.977	7.023	32.795	0.14298	151.186
Sample 9	0.111	5.202	0.535	4.52	7.023	2.336	0.426	2.336	13.006	5.911	0.25672	31.2303
Sample10	1.304	4.789	8.596	7.023	32.795	0.684	0.997	6.456	32.795	19.418	0.23438	271.9073

To obtain 10 samples of size 10, we conducted 20 cycles. Every pair of consecutive cycles makes up one sample of size 10. The 20 cycles we performed to get the 10 ranked samples from Population X. The ranked set samples from Population X, along with the corresponding maximum likelihood estimates of α_x and λ_x, are shown in Table 5. Similarly, we run the 20 cycles to draw ranked set samples from Population Y and the ranked set samples with maximum likelihood estimates of α_y and λ_y for Y Population are shown in Table 6.

Table 7: *Bias, MSE and Relative efficiency of MLE of stress-strength model P in case of SRS and RSS*

	SRS					RSS					
	$\hat{\alpha}_x$	$\hat{\alpha}_y$	$\hat{P}_{SRS}$	Bias	MSE	$\hat{\alpha}_x$	$\hat{\alpha}_y$	$\hat{P}_{RSS}$	Bias	MSE	RE (%)
Sample 1	6.7031	0.4860	0.2725	0.1007	0.01494	5.3408	0.2421	0.2016	0.0298	0.00359	415.43%
Sample 2	3.1983	0.3309				3.3715	0.2186				
Sample 3	7.9133	0.2469				3.0263	0.2020				
Sample 4	2.6220	0.2078				8.1074	0.1941				
Sample 5	3.7265	0.4596				3.5078	0.3499				
Sample 6	3.2477	0.2006				4.5176	0.3602				
Sample 7	2.6515	0.4675				3.0288	0.1434				
Sample 8	3.3188	0.3125				2.5873	0.1429				
Sample 9	3.5093	0.4078				3.9824	0.2567				
Sample10	5.7396	0.2632				4.8530	0.2343				

An analysis of the statistical outcomes presented in the Table 7. This summary reveals a notable difference in the Mean Square Error (MSE) of the stress-strength model P between two sampling techniques. The Ranked Set Sampling (RSS) method demonstrates a significantly lower MSE compared to that obtained through Simple Random Sampling (SRS). Quantitatively, the relative efficiency (RE) of RSS surpasses SRS by a remarkable 415.43%. This substantial improvement in efficiency underscores the superior performance of RSS in practical applications. The findings strongly suggest that RSS offers more reliable and accurate results in real-world scenarios, outperforming the conventional SRS approach in the context of stress-strength modeling.

8. Conclusion

Delving into the realm of reliability engineering, this study sheds new light on the estimation of stress-strength models, with a particular focus on the intriguing $P_r(Y < X)$ paradigm. Here, we explore the behavior of independent random variables Y and X, both dancing to the tune of the Nakagami distribution. While conventional wisdom has long favored simple random sampling, our research unveils a game-changing approach: ranked set sampling. By deriving maximum likelihood estimators for P under both sampling regimes, we set the stage for a riveting comparison.

Our simulation studies paint a vivid picture of ranked set sampling's superiority, showcasing its ability to outperform its traditional counterpart in efficiency. But we don't stop at theoretical musings - we put our findings to the test in the crucible of real-world data, where ranked set sampling continues to shine brightly.

As we draw the curtain on this investigation, one conclusion stands tall: in the arena of Nakagami stress-strength model estimation, ranked set sampling emerges as the undisputed champion over simple random sampling. Yet, this is not the end of our journey. The horizon beckons with tantalizing possibilities, as we set our sights on exploring the potential of other ranked set sampling methods in this critical field of study. The quest for ever-more efficient estimation techniques in stress-strength modeling continues, promising exciting developments in the future of reliability engineering.

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References

[1] McIntyre, G.A. (1952). A method for unbiased selective sampling, using ranked sets. *Australian journal of agricultural research*. ; 3(4), 385-390.

[2] Birnbaum, Z.W. (1956). On a use of the Mann-Whitney statistic. In Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics, University of California Press. Jan; Vol. 3, pp. 13-18.

[3] Birnbaum, Z.W. and McCarty, R.C. (1958). A Distribution-Free Upper Confidence Bound for \Pr{Y<X\}, Based on Independent Samples of X and Y. *The Annals of Mathematical Statistics*. Jun; 29(2):558–62.

[4] Church J.D. and Harris, B. (1970). The estimation of reliability from stress-strength relationships. *Technometrics;* 12(1), 49-54.

[5] Johnson, R.A. (1988). Stress-strength models for reliability. Handbook of Statistics. Volume 7: Quality Control and Reliability, (PR Krishnaiah and CR Rao, eds.)

[6] Kotz ,S., Lumelskii, Y., Pensky, M. (2003). The stress–strength model and its generalizations: theory and applications. World Scientific Publishing Co Pte Ltd.

[7] Awad, A.M. and Gharraf, M.K. (1986). Estimation of P (Y< X) in the Burr case: A comparative study. *Communications in Statistics-Simulation and Computation*; 15(2), 389-403

[8] Chaturvedi, A. and Kumar, S. (1999). Further remarks on estimating the reliability function of exponential distribution under type I and type II censorings. *Brazilian Journal of Probability and Statistics*.; 29-39

[9] Kundu, D. and Gupta, R.D. (2006). Estimation of P [Y< X] for Weibull distributions. *IEEE Trans. Reliab*; 55(2), 270-280.

[10] Kundu, D. and Raqab, MZ. (2009). Estimation of R = P (Y < X) for three-parameter Weibull distribution. *Statistics & Probability Letters*; 79(17), 1839-1846.

[11] Krishnamoorthy, K. and Lin, Y. (2010). Confidence limits for stress–strength reliability involving Weibull models. *Journal of Statistical Planning and Inference*; 140(7), 1754-1764.

[12] Lio, Y.L. and Tsai, T.R. (2012). Estimation of δ= P (X< Y) for Burr XII distribution based on the progressively first failure-censored samples. *Journal of Applied Statistics*; 39(2), 309-322

[13] Barbiero, A. (2013). Inference on Reliability of Stress-Strength Models for Poisson Data. *Journal of Quality and Reliability Engineering*; 2013:1–8.

[14] Chaturvedi, A and Kumari, T. (2018). Estimation and comparison of the stress-strength models with more than two states under Weibull distribution and type II censoring scheme. *Communications in Statistics - Theory and Methods*. Jan 5; 48(3):537–48.

[15] Halls, L.K. and Dell, T.R. (1966). Trial of ranked-set sampling for forage yields. *Forest Science;* 12(1), 22-26.

[16] Takahasi, K. and Wakimoto, K. (1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering. *Annals of the institute of statistical mathematics*; 20(1), 1-31.

[17] Dell, T.R. and Clutter, J.L. (1972). Ranked set sampling theory with order statistics background. *Biometrics*; 545-555.

[18] David, H.A. and Levine, D.N. (1972). Appendix to ranked set sampling theory with order tatistics background. *Biometrics;* 28, 553-555

[19] Yu, P. L. H. and Lam. K. Erratum. (1998). Regression estimator in ranked set sampling. *Biometrics*; 53 (1070-1080).

[20] Chen, Z. (2011). Ranked-set sampling with regression-type estimators. *Journal of Statistical Planning and Inference*; 92(1-2), 181-192

[21] Stokes, S.L. and Sager, T.W. (1988). Characterization of a ranked-set sample with application to estimating distribution functions. *Journal of the American Statistical Association*; 83(402), 374-381

[22] Kvam, P.H. and Samaniego, F.J. (1993). On the inadmissibility of empirical averages as estimators in ranked set sampling. *Journal of Statistical Planning and Inference*; 36(1), 39-55.

[23] Chen, Z. (2000). The efficiency of ranked-set sampling relative to simple random sampling under multi-parameter families. *Statistica Sinica*; 247-263

[24] Zamanzade, E. and Vock, M. (2015). Variance estimation in ranked set sampling using a concomitant variable. *Statistics & Probability Letters*; 105, 1-5

[25] Zhang, L., Dong, X., Xu X., and Cui L., (2014). Weighted estimation of quantiles using unbalanced ranked set sampling. *Quality Technology & Quantitative Management*; 11(3), 281-295.

[26] Ozturk, O. and Kavlak, K.B. (2018). Model based inference using ranked set samples. *Survey Methodology*; 44(1), 1-17.

[27] Kaur, A., Patil, G. P., Sinha, A. K., & Taillie, C. (1995). Ranked set sampling: an annotated bibliography. *Environmental and Ecological Statistics*, 2(1), 25-54.

[28] Bai, Z., and Chen, Z. (2003). On the theory of ranked-set sampling and its ramifications. *Journal of Statistical Planning and Inference*, 109(1-2), 81-99.

[29] Wolfe, D. A. (2012). Ranked set sampling: its relevance and impact on statistical inference. *International scholarly research notices*, 2012(1), 568385.

[30] Hassan, M. K., Alohali, M. I., and Alojail, F. A. (2021). Point and Interval Estimators of R= P [Y< X] Based on Gompertz Distribution and Ranked Set Sampling Data. *Thailand Statistician*, 19(4), 784-796.