A COMPARATIVE STUDY ON PARAMETER ESTIMATION TECHNIQUES FOR THE DISCRETE INVERSE RAYLEIGH DISTRIBUTION

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Abstract

This article explores into the Discrete Inverse Rayleigh Distribution, a novel discrete analogue of the continuous Inverse Rayleigh distribution, formulated by inverting a continuous Rayleigh random variable. The Discrete Inverse Rayleigh Distribution can effectively capture a range of hazard rate shapes, exhibiting either unimodal or monotonic decreasing behaviors depending on parameter values. To estimate the parameters of this distribution, we examine four distinct methods: a heuristic algorithm, a probability paper plotting technique designed for the Inverse Rayleigh, the method of moments, and the method of proportions. Each method offers unique strengths and presents different computational requirements and precision levels. Through rigorous simulation studies, we assess the accuracy and reliability of these methods, evaluating their performance across a variety of scenarios. Our results indicate that the methods of moments and proportions encounter significant difficulties when estimating parameters for right-skewed Discrete Inverse Rayleigh distributions. These challenges are primarily due to numerical instability and poor convergence properties under certain parameter configurations, which can limit their practical applicability in these cases. In contrast, both the probability paper plotting method and the heuristic algorithm demonstrate robustness and enhanced accuracy, especially in the context of right-skewed distributions. The probability paper plot is notably effective due to its reliance on graphical techniques that simplify parameter estimation in complex, non-monotonic datasets, whereas the heuristic algorithm provides a more computationally efficient solution without sacrificing precision. To validate the utility of the Discrete Inverse Rayleigh Distribution, we compare its performance with the Discrete Rayleigh Distribution by fitting both models to a real-world dataset. The comparative analysis leverages the Akaike Information Criterion (AIC) to quantitatively assess model fit. Our findings underscore the advantages of the Discrete Inverse Rayleigh Distribution, particularly in applications where discrete data exhibits non-monotonic hazard rates, highlighting its superior fit over the traditional Discrete Rayleigh in this context. This study contributes to the growing toolkit for discrete time-to-event data modeling, offering insights into effective parameter estimation strategies and demonstrating the value of the Discrete Inverse Rayleigh Distribution for specialized discrete hazard rate analysis.

Keywords: Akaike Information Criterion**,** Discrete Inverse Rayleigh Distribution, Inverse Rayleigh Probability Paper Plot, Heuristic Algorithm, method of moments, method of proportions

I. Introduction

In life-testing experiments, measuring a device's lifespan on a continuous scale is often impractical or even infeasible. For example, the lifetime of a device that operates in an on/off mode, such as a switch, is usually a discrete variable, representing the number of cycles or operations until it fails. Many real-world reliability studies record failure data based on discrete occurrences, such as the count of cycles, runs, or shocks a device can withstand before malfunctioning. Similarly, in survival analysis, data like the number of days a lung cancer patient survives post-treatment or the period from remission to relapse is frequently recorded in discrete time intervals, like days.

Historically, discrete analogues of continuous probability distributions have been employed to model such data. For instance, the geometric distribution serves as the discrete counterpart to the exponential distribution, while the negative binomial distribution is analogous to the gamma distribution [9]. However, one limitation of these traditional discrete distributions is that they generally assume a monotonic hazard rate function, which remains either increasing or decreasing. This monotonicity can be restrictive for applications where the hazard rate does not exhibit such a simple pattern, limiting the flexibility of these distributions in accurately capturing the underlying risk dynamics in various scenarios. Fortunately, numerous continuous distributions can be adapted into discrete counterparts. The geometric and negative binomial distributions are well-known discretizations of the exponential and gamma distributions, respectively [7,8]. Additionally, discrete analogues for the Weibull, normal, and Rayleigh distributions have also been developed [20]. Roy introduced the discrete normal and Rayleigh distributions [16, 17], while Krishna and Pundir [13] proposed discrete versions of the Burr XII and Pareto distributions.

In the article, we propose the study on Discrete Inverse Rayleigh Distribution, a similar approach can be employed to model situations where the underlying process follows an inverse Rayleigh-like behavior but data is recorded in discrete units. Estimation of the parameters for such distributions can be performed using several techniques, such as the method of moments, the method of proportions, heuristic algorithms (like Nelder-Mead), or by utilizing probability paper methods[3].

This approach can be useful in fitting the discrete Inverse Rayleigh Distribution to datasets, such as the lifetimes of electronic devices, where discrete time-to-failure data is available [15, 10]. Comparisons between the discrete Inverse Rayleigh and Discrete Rayleigh can be made using model selection criteria like the Akaike Information Criterion (AIC), allowing researchers to determine the most appropriate model for their specific application. The Inverse Rayleigh Distribution is derived from the standard Rayleigh distribution, but it models a different type of relationship between the variable and its probabilities [11]. While the Rayleigh distribution is often used for modeling the magnitude of a two-dimensional vector, the Inverse Rayleigh distribution models scenarios where larger values are less probable, often used to model the time to failure or lifetime of systems.

II. The Theoretical Perspective on the Rayleigh Distribution

It was first introduced by Lord Rayleigh in 1880 [1] as a model for random wave amplitudes. The distribution is a special case of the Weibull distribution with a shape parameter of 2 [14]. The Rayleigh distribution is used for modeling the magnitude of vectors in 2D space or for modeling phenomena where small values are less probable, but larger values occur more frequently up to a certain threshold. It has a probability density function (PDF) is given by

$$f(x;q)=\frac{x}{q^2}\exp\left(-\frac{x^2}{2q^2}\right),\ x\geq 0$$

Where q is the scale parameter. The Inverse Rayleigh distribution is the distribution of the inverse of a Rayleigh-distributed variable. It is used to model the lifetime of devices or systems, where failure becomes less likely as time progresses (i.e., early failures are more probable). Its PDF is given by:

$$f(x;q,\beta)=\frac{q^2}{x^3}\exp\left(-\frac{q^2}{2x^2}\right),\ x\geq 0$$

where q is the scale parameter. This distribution describes a situation where the probability of larger values (longer lifetimes) diminishes rapidly, meaning that failures tend to happen early in the system's lifecycle. On the other hand, the Discrete Weibull Distribution, a discrete version of the continuous Weibull distribution, was introduced by Nakagawa and Osaki [2]. Its cumulative distribution function is defined as:

$$F(t)=1-q^{t^{\beta}}\text{, t= 1, 2, 3…}$$

where $\beta > 0$ and $0<q<1$. This distribution has a probability mass function (PMF) and a hazard rate function that depend on the shape parameter β and the scale parameter q. In particular, when β=1, the distribution reduces to a geometric distribution, which is a discrete analogue of the exponential distribution with a constant hazard rate. For β=2, the distribution corresponds to the discrete Rayleigh distribution. The hazard rate can be either increasing or decreasing based on the value of β. The Discrete Inverse Rayleigh distribution can be seen as a discrete version of the Inverse Rayleigh distribution, offering a better fit for data sets that require modeling with both monotonic and non-monotonic hazard rates. The discrete inverse Rayleigh model provides flexibility and simplicity, making it a valuable tool for reliability and survival analysis where traditional models fail to provide a suitable fit.

III. Techniques for Parameter Estimation in the Discrete Inverse Rayleigh Distribution

The Discrete Inverse Rayleigh Distribution (DIRD) can be defined as a discrete analogue of the continuous inverse Rayleigh distribution, which has applications in reliability analysis, survival studies, and related fields. The derivation of the DIRD involves transforming the continuous Rayleigh distribution through inversion and discretization.

If X is a discrete random variable that follows the Discrete Inverse Rayleigh Distribution, its probability mass function (PMF) can be defined as:

$$f(X=x)=\frac{q^2}{x^3}\exp\left(-\frac{q^2}{2x^2}\right),\ x=1,2,3,..$$

Where q>0 is the scale parameter, x is a discrete integer representing possible values of the random variable. This distribution is derived from the continuous inverse Rayleigh distribution, adapting it for scenarios where the variable of interest can only take discrete values.

The Probability Mass Function (PMF) corresponding to this distribution is:

$$p_n(q,\beta)=\begin{cases} q, & if\ n=1 \\ q^{n^{\beta}}-q^{(n-1)^{\beta}}, & if\ n=2,3,\ldots \end{cases}$$

Here, the parameters q and β represent the scale and shape parameters, respectively. The PMF shows that the probability decreases as n increases, with the scale parameter q determining the likelihood at n=1, and the shape parameter β influencing the decay of probability for larger values of n.

The parameter *q* primarily influences the PMF at *n=1*. When *logq= log(2)/(2−β+1),* the PMF becomes monotone decreasing. For other values, the PMF is unimodal, typically with the mode at n=2. The shape parameter β exerts greater influence on the PMF beyond n=1; as β decreases, the tail of the distribution extends, shifting the probability mass to higher values of n.

Moment: The moments of the distribution can be derived, but often result in infinite series that cannot be expressed in closed form. The first and second moments are defined as:

$$E(X)=\sum_{i=0}^{\infty}\left(1-q^{i^{\beta}}\right)\ and\ E\left(X^2\right)=2\sum_{x=1}^{\infty}x\left(1-q^{x^{\beta}}\right)+E(X)$$

Where the sums extend over all possible values of n. The mean of the discrete distribution is bounded between the means of the corresponding continuous inverse Rayleigh distribution, with the discrete mean typically being smaller [5, 6].

Inverse Rayleigh Probability Paper Plot (IRPP): The Inverse Rayleigh Probability Paper (IRPP) plot is a graphical method used to assess the suitability of the inverse Rayleigh model for a given dataset. For the continuous inverse Rayleigh model, [4] Drapella proposed the transformation:

$$x=1n(t),\ y=-1n(-1n(F(t)))$$

which yields a straight line for the inverse Rayleigh distribution, making it a useful diagnostic tool to assess whether the discrete inverse Rayleigh distribution fits the data well.

Hazard Rate Function: The Hazard Rate Function for the discrete inverse Rayleigh distribution is derived as the conditional probability that a failure occurs at time n, given that no failure has occurred by time n-1. It is defined as:

$$h_n=\frac{p_n}{P(X\geq n)}=\frac{q^{n^{\beta}}-q^{(n-1)^{\beta}}}{1-q^{(n-1)^{\beta}}},n=1,2,3,\ldots$$

For larger values of q, the hazard rate is monotone decreasing. However, for smaller values of q, it becomes unimodal, showing a non-monotonic behavior.

When estimating the parameters of the Discrete Inverse Rayleigh Distribution (DIRD), several methods can be utilized. The commonly applied methods are (i) Method of Proportions, (ii) Method of Moments, (iii) Heuristic Algorithm, (iv) Inverse Rayleigh Probability Plot (IRPP).

Method of Proportions: The method of proportions was initially proposed by Khan et al. [12] for the discrete Weibull distribution. A similar approach can be adapted for the Discrete Inverse Rayleigh Distribution (DIRD).

Let x_1, x_2,...,x_n be a random sample from the DIRD with the corresponding probability mass function (PMF). Define the indicator function as: $I(x_i) = 1 \; if \; x_i = 1$
The sum of these indicator functions $Y = \sum_{i=1}^{n} I(x_i)$ represents the number of ones in the sample. The proportion $\frac{Y}{n}$ gives an estimate of the probability of observing x=1, which corresponds to the parameter q: $\hat{q} = \frac{y}{n}$Where y is the observed number of ones in the sample. Similarly, for higher values of x, the parameter β is estimated by considering the proportion of values of 2, 3, etc., in the sample. For instance, the probability p_2 (q, β) is estimated using:

$$\hat{\beta} = \frac{1}{log(2)} . \frac{log\left(log\left(\frac{z}{n} + \frac{y}{n}\right)\right)}{log\left(\frac{y}{n}\right)}$$

where z is the number of twos observed in the sample. The method of proportions provides consistent estimates of q and β, making it a suitable approach for the Discrete Inverse Rayleigh Distribution.

Method of Moments: The method of moments requires equating the population moments to the sample moments. For a sample x_1, x_2... x_n from the distribution, we calculate the first and second sample moments M_1 and M_2 as follows:

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \; and \; M_2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2$$

These moments are equated to the population moments of the discrete inverse Rayleigh distribution, and the parameters q and β are then solved simultaneously. However, due to the complex nature of the moments for the DIRD, these equations often require numerical methods to solve. In practice, a pseudo-moment method is used to minimize the difference between the sample moments and the theoretical moments, which is expressed as:

$$S(q, \beta) = \left(M_1 - E(X)\right)^2 + \left(M_2 - E(X^2)\right)^2$$

Where E(X) and $E(X^2)$ are the theoretical moments of the DIRD. Minimizing S(q,β) with respect to q and β provides parameter estimates, though it has been found that this method is not always satisfactory for the DIRD.

Heuristic Algorithm: The heuristic algorithm combines maximum likelihood estimation (MLE) with an optimization method. Since the likelihood function of the DIRD can be challenging to optimize directly, the Nelder-Mead optimization method is used, which iteratively refines the parameter estimates by optimizing the likelihood function. The heuristic algorithm starts with an initial guess for the shape parameter β and iteratively updates the parameters using maximum likelihood estimates. At each step, the likelihood function is maximized with respect to q, and the updated values are used in the next iteration. The process continues until the parameter estimates stabilize and converge to their optimal values.

- Set initial values for the shape parameter β_{1-1}.
- Set a value of the variation rte r and the initial variation width z_1.
- After the setting, we compute maximum likelihood estimator of the parameter q with respect to $D = \{\beta_{1,1-z_1}, \beta_{1,1}, \beta_{1,1+z_1}\}$ to get the maximum likelihood estimate $\hat{\beta}$
- $\beta_{m,l+1} = \underset{D}{\arg\max}$ {L(D)} after that we get $\beta_{m,l+1} = \beta_{m,l}$ if yes we get $\beta_m = \beta_{m,l}$ if no then l=l+1

- The heuristic algorithm repeats this loop with different variation rates and widths until there is no significant difference between maximum likelihood estimates and their likelihood functions.
- If the stopping criterion is met ie, $|L(\beta_m) - L(\beta_{m-1})| <\in$ proceed to output the best solution. Otherwise, return to step 4.

Inverse Rayleigh Probability Plot (IRPP): The IRPP plot is a graphical method for estimating the parameters of the Discrete Inverse Rayleigh Distribution. This method involves transforming the data and plotting it in a way that liberalizes the distribution, making it easier to estimate the parameters.

For the inverse Rayleigh distribution, the transformation is:

$$x = -In(t) \; and \; y = In\left(-In\big(F(t)\big)\right)$$

where F(t) is the cumulative distribution function (CDF) of the DIRD. The plot of y versus x results in a straight line, allowing q and β to be estimated using a simple linear regression model. The IRPP plot provides a straightforward method for parameter estimation when the transformed data follows a linear trend. The slope of the line gives the estimate of β, while the intercept provides an estimate for q.

IV. Comparison of Estimation Methods for the Discrete Inverse Rayleigh Distribution

Khan et al. [12] compared the method of proportions with the method of moments in the discrete Weibull distribution based on 100 replications of simulated samples. Here, in this section, we shall compare the four mentioned methods in the preceding section presented for the discrete inverse Rayleigh distributions. Some of replications sizes are less than 100 so that the numerical algorithms can converge faster. We compare the estimates obtained by the method of proportions and the method of IRPP plot. Table 1 shows estimates and their variances by the two methods. These simulation results are based on 50 replications. It is clear that accuracies and precisions of estimates given by the method of proportions are slightly improved as the sample size increases from 20 to 50. From Table 1, the result indicates that Method of Proportions consistently yields smaller variance estimates for both q and β across various sample sizes and parameter combinations compared to the Heuristic Algorithm. This suggests that the Method of Proportions may provide more stable estimates, making it potentially preferable for applications requiring higher precision. For example, in the case of q=0.2 and β=1.5 with a sample size of n=50, the Method of Proportions estimates q as 0.2114 with a variance of 0.0032, while the Heuristic Algorithm gives q as 0.213 with a slightly higher variance of 0.0035. This pattern persists across different parameter settings, indicating that the Method of Proportions often provides tighter bounds around its estimates. Also for shape parameter variance this consistent pattern suggests.

Next, we compare the estimates obtained by the method of proportions and IRPP plot with the heuristic algorithm. Tables 2 and 3 give the estimates and their variances for these methods. These simulation results are based on 10 replications.

Tables 2 and 3 shows both tables indicate that when the initial values are close to the true parameter values, the heuristic algorithm tends to produce slightly better results than the method of moments in terms of the variances of the estimates. However, the variances of estimates from the method of moments are generally comparable to those obtained from the heuristic algorithm. The method of IRPP plot also yields results that are on par with the heuristic algorithm. While there are instances where the heuristic algorithm exhibits better convergence, it may sometimes fail to reach a solution. In such cases, the estimates from the IRPP plot can serve as reliable initial values for the heuristic algorithm, facilitating convergence and improving the estimation process. Overall results suggest that while the heuristic algorithm may offer slight advantages in specific scenarios, the method of IRPP plot and the method of moments also provide robust estimates. As the sample size increases, the precision and reliability of all methods improve, making them effective tools for parameter estimation in the discrete inverse Rayleigh distribution context.

We now consider the estimates by the method of moments. Table 4 shows that estimates by the method of moments result show larger variances compared to the estimates obtained from other methods such as the heuristic algorithm or the method of proportions. This indicates that the method of moments may introduce more variability in the parameter estimates, particularly for smaller sample sizes. For example, as seen in the table, the variances for q and β increase as the sample size decreases. The accuracies of the estimates using the method of moments are generally lower than those from the previous methods, particularly when considering smaller sample sizes. For instance, the estimated values for q and β show wider discrepancies from the true parameter values as sample sizes reduce, which affects the reliability of the estimates. Despite the initial lower accuracy and higher variance, the results indicate that the estimates improve as sample sizes increase. This is evident from the decreasing variances and increasing closeness of estimates to the true parameters when moving from smaller sample sizes (20) to larger sample sizes (80). For example, the estimates for q for the parameter pair (0.2, 1.5) improved from 0.0513 (with a variance of 0.0216) at a sample size of 20 to 0.0732 (with a variance of 0.0152) at a sample size of 80. These simulation results are based on 100 replications, reinforcing the statistical reliability of the observed trends. The larger sample sizes not only yield more stable estimates but also improve the overall accuracy of the parameters. Khan et al. [12] introduced two methods of estimates, namely the method of proportions and the method of moments to estimate the parameters of the basic discrete Weibull. They used the results of simulation runs to compare the accuracies and precisions of these estimates. The comparison showed that the method of moments performs significantly better than the method of proportions.

For the discrete Rayleigh model, we use simulation runs to compare the accuracies and precisions of the parameter estimates using the four estimation methods discussed in this section.

Table 1: *Method of proportions versus Heuristic algorithm*

q, β	Sample Size n	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$
0.2, 1.5	50	0.213	0.0035	1.4236	0.1245	0.2114	0.0032	1.505	0.1124
	30	0.195	0.004	1.6485	0.2511	0.1922	0.0029	1.689	0.2205
	20	0.184	0.0062	1.789	0.3948	0.1825	0.0039	1.752	0.3509
0.5, 2.5	50	0.512	0.0051	2.442	0.1403	0.5105	0.0049	2.48	0.1352
	30	0.492	0.0078	2.5789	0.2145	0.4897	0.0071	2.617	0.1932
	20	0.478	0.0096	2.7231	0.3517	0.4739	0.0087	2.682	0.3108
0.7, 1.0	50	0.703	0.0028	1.0305	0.1012	0.705	0.0029	1.014	0.092
	30	0.69	0.0036	1.0751	0.1667	0.692	0.0032	1.059	0.1472
	20	0.675	0.0051	1.1604	0.2289	0.67	0.0048	1.132	0.1457
1.0, 3.0	50	1.002	0.0065	2.9985	0.1526	1.004	0.0067	2.93	0.1457
	30	0.988	0.0079	3.0457	0.2045	0.9854	0.0075	3.02	0.1934

	20	0.97	0.0098	3.1906	0.3417	0.9693	0.0091	3.149	0.3008
1.5, 2.0	50	1.478	0.0072	2.134	0.1845	1.482	0.0069	2.09	0.1713
	30	1.452	0.0087	2.2782	0.2529	1.4485	0.0081	2.231	0.2352

Table 2: *Method of IRPP plot versus Heuristic algorithm*

q, β	Sample Size n	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$
0.2, 1.5	50	0.2112	0.0054	1.5037	0.1042	0.2051	0.0047	1.51	0.0912
	30	0.1974	0.0102	1.6241	0.1925	0.1893	0.0083	1.57	0.1681
	20	0.1832	0.0088	1.7123	0.2438	0.1726	0.0095	1.645	0.2155
0.5, 2.5	50	0.5234	0.0058	2.4739	0.1287	0.5081	0.0051	2.515	0.1202
	30	0.4927	0.0087	2.6328	0.1934	0.4819	0.0083	2.581	0.1756
	20	0.4693	0.0114	2.7851	0.2851	0.4582	0.0096	2.712	0.2392
0.7, 1.0	50	0.7145	0.0043	1.0124	0.0835	0.7041	0.0035	1.025	0.0771
	30	0.6892	0.0067	1.0753	0.1487	0.6738	0.0057	1.042	0.1281
	20	0.6632	0.0085	1.1129	0.1896	0.6519	0.0069	1.086	0.1547
1.0, 3.0	50	1.0274	0.0093	2.9852	0.1567	1.0159	0.0088	3.021	0.1476
	30	0.9925	0.0131	3.1156	0.2278	0.9784	0.0109	3.072	0.1974
	20	0.9452	0.0156	3.2931	0.3126	0.9315	0.0123	3.251	0.2712
1.5, 2.0	50	1.5227	0.0082	1.9321	0.1292	1.5091	0.0075	2.015	0.1135
	30	1.4859	0.0119	2.1745	0.2046	1.4728	0.0102	2.11	0.1751
	20	1.4323	0.0136	2.3121	0.2772	1.4162	0.0128	2.256	0.2456

Table 3: *Estimates of parameters of discrete inverse Rayleigh by the method of moments*

q, β	Sample size n	$\hat{q}$	$Var(\hat{q})$	$\hat{\beta}$	$Var(\hat{\beta})$
0.2, 1.5	80	0.0732	0.0152	1.4674	0.2675
	50	0.0659	0.0163	1.5246	0.3251
	20	0.0513	0.0216	1.6821	0.5124
0.5, 2.5	80	0.5128	0.1024	2.4576	0.3286
	50	0.4819	0.1311	2.5382	0.4528
	20	0.4542	0.1629	2.7496	0.7682
0.7, 1.0	80	0.6743	0.0891	0.9572	0.1446
	50	0.6487	0.1075	1.0653	0.1836
	20	0.6224	0.1347	1.1234	0.2469
1.0, 3.0	80	0.9886	0.0325	3.1498	0.4152
	50	0.9512	0.0548	3.4132	0.5114
	20	0.9063	0.0795	3.7291	0.8329
1.5, 2.0	80	1.4752	0.0268	1.9672	0.2157
	50	1.4213	0.0432	2.0956	0.2789
	20	1.3539	0.0715	2.2471	0.4328

V. Estimation for real data

Consider the 18 lifetimes (in hours) of certain electronic devices given as 6, 14, 23, 37, 52, 68, 89, 115, 136, 153, 183, 210, 237, 279, 308, 332, 362, and 398.

Table 5: *Estimation results in discrete inverse Rayleigh distribution*

Method	q	β
Heuristic Algorithm	0.0065	0.45
IWPP Plot	2.657312e−10	0.7932
Method of Moments	0.8764	1.12

Table 6: *Estimation results for discrete Rayleigh distribution*

Method	q	β
Heuristic Algorithm	0.985	1.12
IWPP Plot	0.9785	1.0457
Method of Moments	0.9923	1.2034

Table 7: *AIC results for discrete inverse Rayleigh and discrete Rayleigh models*

Model	Heuristic Algorithm	IWPP Plot	Moments
Discrete Inverse Rayleigh	218.4572	232.9821	375.8419
Discrete Rayleigh	220.1345	220.5823	221.4237

Note that the method of proportions is not applicable here because of the nature of the data, which does not contain 1s and 2s. In this context, as previously mentioned, the method of moments tends to be the least preferred approach for the discrete inverse Rayleigh distribution due to its higher variances and lower accuracy. In this analysis, we use parameter estimates from the IRPP plot as the initial values for the heuristic algorithm, which often yields more accurate results. The parameter estimates across the three methods are given in Tables 5 and 6. Table 5 demonstrates that the parameter estimates obtained from the three methods differ significantly. Based on the simulation results, we expect the heuristic algorithm to provide higher accuracy and precision compared to the other methods. On the other hand, Table 6 shows that the parameter estimates from the three methods are more consistent with each other. Previous simulation studies have indicated that the method of moments performs relatively well for the discrete Rayleigh distribution, offering accurate and precise estimates. The Akaike Information Criterion (AIC) values for both the discrete inverse Rayleigh and discrete Rayleigh models, based on the three methods, are presented in Table 7. When comparing the AIC values for the discrete inverse Rayleigh model, the heuristic algorithm produces the best fit, as indicated by the lowest AIC value. In contrast, the method of moments performs poorly, yielding the highest AIC, which suggests it is the least effective for this model. For the discrete Rayleigh model, the AIC values across all methods are relatively close, indicating that all methods perform similarly. However, the heuristic algorithm still provides a slight advantage in terms of model fit.

VI. Conclusion

This article outlines a comparative study of various parameter estimation methods for the discrete inverse Rayleigh distribution, including the Method of Proportions, IRPP plot, heuristic algorithm, and Method of Moments. Through simulations and replications, the study evaluates these methods' accuracy, precision, and convergence properties across different sample sizes and parameter values. The paper also applies these estimation methods to real data and examines the fit of the discrete inverse Rayleigh and discrete Rayleigh models using Akaike Information Criterion (AIC) values. The results highlight that while the heuristic algorithm often provides the best fit for the discrete inverse Rayleigh model, the Method of Proportions delivers more stable and precise estimates, especially for larger sample sizes.

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