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MODELING RELIABILITY IN k-OUT-OF-m SYSTEMS WITH UNEQUAL LOAD SHARING USING PROPORTIONAL CONDITIONAL REVERSE HAZARD RATE

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Abstract

This paper explores a load-sharing model within a k-out-of-m system, where multiple components work together to handle a shared load. Such systems are prevalent in various engineering and industrial applications.While previous studies have focused on equal load-sharing rules, this research emphasizes systems operating under an unequal load-sharing rule, which has a significant impact on the system's reliability and performance. Specifically, the paper examines a k-out-of-m load-sharing system modeled using the proportional conditional reverse hazard rate model, incorporating unequal load sharing. We have derived expressions for the probability density function and cumulative distribution function of system failure. To illustrate the model, they use a 2-out-of-4 configuration with Weibull baseline distributions. The maximum likelihood estimation method is employed to estimate the model parameters, and the performance of these estimates is evaluated through a simulation study, assessing both bias and mean square errors. Additionally, the practical applicability of the model is demonstrated through the analysis of two real datasets.

Keywords: k-out-of-m system, Load sharing phenomenon, Order statistics, Proportional hazard rate, Reverse hazard rate, Unequal load share rule.

1. Introduction

A k-out-of-m system with m components fails when $(m - k + 1)$ or more of its components fail. This setup includes series systems (where $k = m$) and parallel systems (where $k = 1$) as specific instances. Generally, the failure times of the first $(m - k + 1)$ components are modeled as the first $(m - k + 1)$ order statistics from a set of m independent and identically distributed (i.i.d.) random variables. In k-out-of-m load-sharing systems, when one component fails, the load of the failed component is distributed on the remaining components. This creates a dependency among the lifetimes of the components, making these systems dynamic in terms of reliability. Failure of component may increase of release the load of reaming components. Examples of k-out-of-m load-sharing systems include fibrous composite materials, power plants, automobiles, and the two jet engines of an airplane. A similar pattern is observed in the human body, where the failure of one organ (e.g., a kidney) typically increases the failure rate of the surviving organ [1]. Conversely, in scenarios like food scarcity within a letter, the death of some offspring can improve the survival and growth of the remaining ones by increasing their food supply [10]. Similarly, in software development, detecting one bug can help in finding others, thereby reducing the detection time.

[12] and Kvam and Pena [13] have emphasized the importance of modeling the load-sharing phenomenon in various contexts. Besides these works, load-sharing systems have been explored extensively, beginning with Daniels [1], and continuing through the studies of Birnbaum and Saunders [2], Coleman [3], Rosen [4], [5], Singpurwalla [6], Hollander and Pena [7], Cramer and Kamps [8, 12], Lynch [11], Durham and Lynch [12], McCool [13], Pena [16], and Deshpande et al. [17], among others. A comprehensive review can be found in Dewan and Naik-Nimbalkar [18]. Then after Deshpande et al. [17], Jain and Gupta [19], Sutar and Naik-Nimbalkar [20], Sutar and Naik-Nimbalkar [21], Wang et al. [22], Zhao et al. [23], Xu et al. [24], Zhang et al. [25], Sutar and Naik-Nimbalkar [26], Choudhary et al. [27], Park et al. [28], Sutar [29], Zhang et al. [30], Rykov et al. [31], Pesch et al. [32], Sutar et al. [33], Biswas et al. [34], and Pesch et al. [35] all contributed to the study of load-sharing systems.

Sutar and Naik-Nimbalkar [21] explored load-sharing systems within the framework of a k-out-of-m system using a proportional conditional reverse hazard rate model. They concentrated on equal load sharing, particularly examining a two-component parallel system where the components' initial lifetimes followed Weibull and linear failure rate distributions as baseline models. Their research also extended to developing and applying inference techniques for analyzing this system. However, the assumption of equal load sharing does not always hold true, making it essential to create models that account for unequal load distribution. In these situations, it is necessary to develop models that accurately represent the varying loads each component bears to more effectively predict system reliability. An example of this is as follows.

Consider a power grid with multiple generators providing electricity to a city. If one generator fails, the remaining generators must absorb the additional load, but they may not equally distribute this increased demand due to differences in capacity and efficiency. In such a scenario, modeling the system with an unequal load-sharing approach is vital for accurately predicting the grid's reliability and preventing potential outages.

In this article, we present a model for the load-sharing phenomenon in a k-out-of-m system, utilizing proportional conditional reverse hazard rate (PCRHR) with an unequal load-sharing rule. We aim to capture the complexities of real-world systems where components do not equally share the load after failures occur. The structure of this article is as follows:

Section 2 introduces the model for a k-out-of-m unequal load-sharing system. Illustration of the model is given in Section 3. In Section 4, we examine parameter estimation for k-out-of-m and in particular 2-out-of-4, 2-out-of-3, and 1-out-of-4 load sharing systems, assuming that lifetimes are independently distributed according to a Weibull distribution. Section 5 reports a simulation study that assesses the proposed estimation method's performance based on bias and mean square error. Section 6 applies the model to a real dataset, and the final section summarizes the conclusions.

2. Proposed PCRHR based Load Sharing Model for k-out-of-$m$ystem

Let $U_1, U_2, \dots U_m$ are the components of a k-out-of-m system $(1 \leq k \leq m)$. Let use assume that the lifetimes of components are independent and identical distributed (i.i.d.) with baseline probability density function (pdf) $f(\cdot)$, Cumulative distribution function (cdf) $F(\cdot)$ and survival function (sf) $\bar{F}(\cdot)$ and reverse hazard rate function $r(\cdot)$. The system is to be functioning as long as $(m-k+1)$ components is not failed. Let $X_{(j)}$ be the time of j^{th} failure in the system, $j = 1, 2, \dots, (m-k+1)$. That is $X_{(j)}$ is minimum of the failure times of remaining $(m-j+1)$ surviving components and system failure is occurs at time point $X_{(m-k+1)}$. In the i.i.d. set up, the failure of a component does not affect the lifetime of the surviving component and the joint density of $X_{(j)}$ and $X_{(j+1)}$ can be written as

$$\begin{aligned} g_{X_{(j)}, X_{(j+1)}}\left(x_j, x_{j+1}\right) = & \frac{m!}{(j-1)!(m-j-1)!} \left(F(x_j)\right)^{j-1} \\ & \left(\bar{F}(x_{j+1})\right)^{m-j-1} f(x_j) f(x_{j+1}), \quad j = 1, 2, \dots, (m-k). \end{aligned} \tag{1}$$

The conditional density function of $X_{(j+1)}$ given $X_{(j)} = x_j$ is given by

$$g_{X_{(j+1)}|X_{(j)=x_j}}(x_{j+1}) = (m-j)\left\{\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right\}^{m-j}\frac{f(x_{j+1})}{\bar{F}(x_{j+1})},\ \ j = 1,2,\ldots,(m-k). \tag{2}$$

Thus the conditional distribution function of $X_{(j+1)}$ given $X_{(j)} = x_j$ is given by

$$G_{X_{(j+1)}|X_{(j)=x_j}}(x_{j+1}) = 1-\left(\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right)^{m-j},\ \ j = 1,2,\ldots,(m-k). \tag{3}$$

Assuming that there exists load sharing effect in model, then conditional distribution function with load sharing effect is given by

$$\begin{aligned} H_{X_{(j+1)}|X_{(j)=x_j}}(x_{j+1}) &= \left\{G_{X_{(j+1)}|X_{(j)=x_j}}(x_{j+1})\right\}^{\beta_j} \\ &= \left\{1-\left(\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right)^{m-j}\right\}^{\beta_j}, \end{aligned} \tag{4}$$

$$\beta_j > 0,\ \ x_{j+1} \geq x_j,\ \ j = 1,2,\ldots,(m-k).$$

Remark (1): From the above equation (4), we observe that if $\beta_j < 1$ then $H_{X_{(j+1)}|X_{(j)=x_j}}(x_{j+1}) > G_{X_{(j+1)}|X_{(j)=x_j}}(x_{j+1}), \forall\, j = 1,2,\ldots,(m-k)$, The lifetime of the surviving component under the distribution H is stochastically smaller than under the i.i.d. setup G. and if $\beta_j > 1$ then the residual life is stochastically larger. For $\beta_j = 1, \forall\, j = 1,2,\ldots,(m-k)$ means residual lifetimes are same as that under i.i.d. setup.
We also assume that the conditional distribution of the residual lifetime of a surviving component depends only on the last failure time, that is the failure epochs form a Markov process.

Theorem 1. If the conditional distribution of $X_{(j+1)}$ given $X_{(j)} = x_j$ is as given in equation (4), then we have the following

(i) The joint p.d.f. of $(X_{(1)}, X_{(2)}, ..., X_{(m-k+1)})$ is given by,

$$\begin{aligned} h(x_1, x_2, \ldots, x_{m-k+1}) =& \frac{m!}{(k-1)!}\prod_{j=1}^{(m-k)}\beta_j\left\{1-\left(\frac{\bar{F}(x_{j+1})}{\bar{F}(x_j)}\right)^{m-j}\right\}^{\beta_j-1} \\ & (\bar{F}(x_{m-k+1}))^{k-1}\prod_{j=1}^{m-k+1} f(x_j). \end{aligned} \tag{5}$$

$$0 < x_1 < x_2 < \ldots < x_{m-k+1},\ \beta_j > 0.$$

(ii) The marginal density of the system failure time X_{m-k+1}, is given by

$$\begin{aligned} h_{X_{(m-k+1)}}(x_{m-k+1}) =& m\prod_{j=1}^{m-k}\beta_j\,(\bar{F}(x_{m-k+1}))^{(m-1)} f(x_{m-k+1}) \\ & \int_{x_{m-k}=0}^{1-a_{m-k}} u_{m-k}^{\beta_{m-k}-1}(1-u_{m-k})^{\frac{-m}{k}}\cdots\int_{x_j=0}^{1-a_j} u_j^{\beta_j-1}(1-u_j)^{\frac{-m}{m-j}} \\ & \cdots\int_{x_1=0}^{1-a_1} u_1^{\beta_1-1}(1-u_1)^{\frac{-m}{m-1}}du_1du_2\cdots du_{(m-k)}. \end{aligned} \tag{6}$$

$0 < x_{m-k+1} < \infty,\ \beta_j > 0.\ \forall\, j = 1,2,\ldots,(m-k)$, where, $a_{m-k} = (\bar{F}(x_{m-k+1}))^k$ and

$$a_j = \frac{\left(\bar{F}\left(x_{m-k+1}\right)\right)^{m-j}}{(1-u_{j+1})^{\frac{m-j}{m-j-1}}(1-u_{j+2})^{\frac{m-j}{m-j-2}}\cdots(1-u_{m-k})^{\frac{m-j}{m-(m-k)}}}, \quad 1 \le j \le (m-k-1).$$

Proof. (a) Using (4), the conditional density of $X_{(j+1)}$ given $X_{(j)} = x_j$, is given by

$$h_{X_{(j+1)}|X_{(j)=x_j}}\left(x_{j+1}\right) = (m-j)\beta_j \left\{1 - \left(\frac{\bar{F}\left(x_{j+1}\right)}{\bar{F}\left(x_j\right)}\right)^{m-j}\right\}^{\beta_j - 1} \frac{f\left(x_{j+1}\right)\left(\bar{F}\left(x_{j+1}\right)\right)^{m-j-1}}{\left(\bar{F}\left(x_j\right)\right)^{m-j}}, \tag{7}$$

$$\beta_j > 0, \quad x_{j+1} \ge x_j, \quad j = 1, 2, \ldots, (m-k).$$

Using the Markov assumption, the joint density of $(X_{(1)}, X_{(2)}, ..., X_{(m-k+1)})$ can be written

$$h\left(x_1, x_2, \ldots, x_{m-k+1}\right) = \prod_{j=1}^{(m-k)} h_{X_{(j+1)}|X_{(j)=x_j}}\left(x_{j+1}\right) h(x_1), \; 0 < x_1 < x_2 < ... < x_{m-k+1}, \; \beta_j > 0.$$

Using expression (7) and the fact that $h(x_1) = m\left(\bar{F}(x_1)\right)^{m-1} f(x_1), \;\; x_1 > 0$, we get (5).

Proof. (b) The marginal density function of $X_{(m-k+1)}$ is obtained by integrating equation (5) with respect to $x_1, x_2, \ldots, x_{m-k}$ over th region defined by $0 \le x_1 \le x_2 \le \cdots \le x_{m-k+1}$.
That is,

$$\begin{aligned} h_{X_{(m-k+1)}}(x_{m-k+1}) &= \int_{x_{m-k}=0}^{x_{m-k+1}} \int_{x_{m-k-1}=0}^{x_{m-k}} \cdots \int_{x_1=0}^{x_2} h\left(x_1, x_2, \ldots, x_{m-k+1}\right) dx_1 dx_2 \cdots dx_{(m-k)} \\ &= \frac{m!}{(k-1)!}\left(\bar{F}(x_{m-k+1})\right)^{k-1} f(x_{m-k+1}) \int_{x_{m-k}=0}^{x_{m-k+1}} \int_{x_{m-k-1}=0}^{x_{m-k}} \cdots \\ &\quad \int_{x_1=0}^{x_2} \prod_{j=1}^{(m-k)} \left\{1 - \left(\frac{\bar{F}\left(x_{j+1}\right)}{\bar{F}\left(x_j\right)}\right)^{m-j}\right\}^{\beta_j - 1} f(x_j) dx_1 dx_2 \cdots dx_{(m-k)}. \end{aligned}$$

Let

$$I_1 = \int_{x_1=0}^{x_2} \left\{1 - \left(\frac{\bar{F}\left(x_2\right)}{\bar{F}\left(x_1\right)}\right)^{m-1}\right\}^{\beta_1 - 1} f(x_1) dx_1.$$

Putting $u_1 = 1 - \left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{m-1}$ in above expression and simplifying, we get

$$I_1 = \frac{\bar{F}(x_2)}{m-1} \int_{u_1=0}^{1-\left(\bar{F}(x_2)\right)^{(m-1)}} u_1^{\beta_1 - 1}(1-u_1)^{\frac{-m}{m-1}} du_1.$$

Similarly, let

$$\begin{aligned} I_2 =& \frac{1}{m-1} \int_{x_2=0}^{x_3} \left\{1 - \left(\frac{\bar{F}\left(x_3\right)}{\bar{F}\left(x_2\right)}\right)^{m-2}\right\}^{\beta_2 - 1} \bar{F}(x_2) f(x_2) \\ & \int_{u_1=0}^{1-\left(\bar{F}(x_2)\right)^{(m-1)}} u_1^{\beta_1 - 1}(1-u_1)^{\frac{-m}{m-1}} du_1 dx_2. \end{aligned}$$

After putting $u_2 = 1 - \left(\frac{\bar{F}(x_3)}{\bar{F}(x_2)}\right)^{m-2}$ and simplifying we get

$$\begin{aligned} I_2 =& \frac{\left(\bar{F}(x_2)\right)^2}{(m-1)(m-2)} \int_{u_1=0}^{1-\left(\bar{F}(x_3)\right)^{(m-2)}} u_2^{\beta_2 - 1}(1-u_2)^{\frac{-m}{m-2}} \\ & \int_{u_1=0}^{1-\left(\bar{F}(x_3)\right)^{(m-1)}(1-u_2)^{-\frac{(m-1)}{(m-2)}}} u_1^{\beta_1 - 1}(1-u_1)^{\frac{-m}{m-1}} du_1 du_2. \end{aligned}$$

Proceeding in the same manner and by letting $I_3, I_4, \cdots, I_{(m-k)}$, we get (6).
Remark 2. Under the model defined by (4), the conditional reversed hazard rate (CRHR) $r^*_{X_{(j+1)}|X_{(j)}=s}(t)$ of $X_{(j+1)}$ given $X_{(j)} = s$ is proportional to the CRHR $r_{X_{(j+1)}|X_{(j)}=s}(t)$ under the i.i.d. setup. That is,

$$r^*_{X_{(j+1)}|X_{(j)}=s}(t) = \beta_j r_{X_{(j+1)}|X_{(j)}=s}(t), \;\; \beta_j > 0, \; t \geq s, \; j = 1, 2, ..., (m-k). \tag{8}$$

Thus we refer to the model given in (5) as proportional conditional reversed hazard rate (PCRHR) model.

3. Illustration

3.1. The PCRHR based Load Sharing Model for 2-out-of-4 System

Let us consider the system involve four components with component lifetimes U_1, U_2, U_3, U_4 and are being iid with common lifetime distribution $f_{\underline{\theta}}(.)$, $\underline{\theta}$ may be scale or vector valued parameter. $f_{\underline{\theta}}(.)$ is called the baseline distribution and $\underline{\theta}$ is known as baseline parameter. Let $F_{\underline{\theta}}(.)$ and $\bar{F}_{\underline{\theta}}(.)$ are the distribution function (d.f.) and survival function (s.f.).

The 2-out-of-4 system will work until 3 components work and system fails after the tird failure. Let $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ be the first, second and third failure times. Therefore the $p.d.f.$ of $X_{(1)}$ is given by

$$h_{X_{(1)}}(x_1) = 4 f_{\underline{\theta}}(x_1) \left(\bar{F}_{\underline{\theta}}(x_1)\right)^3, \quad x_1 > 0.$$

From equation (7) the conditional density of $X_{(2)}$ given $X_{(1)} = x_1$ is given by

$$h_{X_{(2)}|X_{(1)}=x_1}(x_2) = 3\beta_1 \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_2)}{\bar{F}_{\underline{\theta}}(x_1)}\right)^3\right\}^{\beta_1 - 1} \frac{f_{\underline{\theta}}(x_2)\left(\bar{F}_{\underline{\theta}}(x_2)\right)^2}{\left(\bar{F}_{\underline{\theta}}(x_1)\right)^3},$$

$$0 \leq x_1 \leq x_2 \leq \infty, \;\; \beta_1 > 0,$$

and the conditional density of $X_{(3)}$ given $X_{(2)} = x_2$ is given by

$$h_{X_{(3)}|X_{(2)}=x_2}(x_3) = 2\beta_2 \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_3)}{\bar{F}_{\underline{\theta}}(x_2)}\right)^2\right\}^{\beta_2 - 1} \frac{f_{\underline{\theta}}(x_3)\bar{F}_{\underline{\theta}}(x_3)}{\left(\bar{F}_{\underline{\theta}}(x_2)\right)^2},$$

$$0 \leq x_2 \leq x_3 \leq \infty, \;\; \beta_2 > 0,$$

Therefore the joint distribution of $(X_{(1)}, X_{(2)}, X_{(3)})$ is given by

$$h(x_1, x_2, x_3) = 4!\beta_1\beta_2\bar{F}_{\underline{\theta}}(x_3) \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_2)}{\bar{F}_{\underline{\theta}}(x_1)}\right)^3\right\}^{\beta_1 - 1} \left\{1 - \left(\frac{\bar{F}_{\underline{\theta}}(x_3)}{\bar{F}_{\underline{\theta}}(x_2)}\right)^2\right\}^{\beta_2 - 1} \prod_{j=1}^{3} f_{\underline{\theta}}(x_j). \tag{9}$$

$$0 \leq x_1 \leq x_2 \leq x_3 \leq \infty, \;\; \beta_1, \beta_2 > 0$$

The marginal distribution of system failure, i.e. distribution of $X_{(3)}$ is given by

$$\begin{aligned} h_{X_{(3)}}(x_3) =& 4\beta_1\beta_2 f_{\underline{\theta}}(x_3)\left(\bar{F}_{\underline{\theta}}(x_3)\right)^3 \int_{u_1=0}^{1-(\bar{F}(x_3))^2} u_2^{\beta_2 - 1}(1-u_2)^{-2} \\ & \int_{u_1=0}^{1-(\bar{F}(x_3))^3(1-u_2)^{\frac{-3}{2}}} u_1^{\beta_1 - 1}(1-u_1)^{\frac{-4}{3}} du_1 du_2. \end{aligned} \tag{10}$$

$$0 \leq x_3 \leq \infty, \;\; \beta_1, \beta_2 > 0.$$

In the following subsection, we demonstrate the 2-out-of-4 load-sharing system using a Weibull baseline distribution.

3.2. The PCRHR based Load Sharing Model for 2-out-of-4 System with Weibull Baseline Distribution

Let us consider a 2-out-of-4 load sharing system, with the component lifetimes U_1, U_2, U_3 and U_4 being i.i.d. Weibull random variables. The p.d.f. of Weibull with shape parameter a and scale parameter b is

$$f(u_i) = \left(\frac{a}{b}\right)\left(\frac{u_i}{b}\right)^{a-1} \exp\left\{-\left(\frac{u_i}{b}\right)^a\right\}, \quad u_i > 0, \;\; a, b > 0, \;\; i = 1, 2, 3, 4.$$

Therefore the p.d.f of first failure $X_{(1)}$ is given by

$$h_{X_{(1)}}(x_1) = 4\left(\frac{a}{b}\right)\left(\frac{x_1}{b}\right)^{a-1} \exp\left\{-4\left(\frac{x_1}{b}\right)^a\right\}, \quad x_1 > 0, \;\; a, b > 0, \;\; i = 1, 2, 3, 4.$$

The conditional p.d.f of $X_{(2)}$ given $X_{(1)} = x_1$ is given by

$$\begin{aligned} h_{X_{(2)}|X_{(1)}=x_1}(x_2) =& 3\beta_1\left(\frac{a}{b}\right)\left(\frac{x_2}{b}\right)^{a-1} \exp\left\{-3\left[\left(\frac{x_2}{b}\right)^a - \left(\frac{x_1}{b}\right)^a\right]\right\} \\ & \left[1 - \exp\left\{-3\left[\left(\frac{x_2}{b}\right)^a - \left(\frac{x_1}{b}\right)^a\right]\right\}\right]^{\beta_1 - 1}, \end{aligned} \tag{11}$$

$$0 \le x_1 \le x_2 \le \infty, \;\; \beta_1 > 0, \;\; a, b > 0.$$

and conditional p.d.f of $X_{(3)}$ given $X_{(2)} = x_2$ is given by

$$\begin{aligned} h_{X_{(3)}|X_{(2)}=x_2}(x_3) =& 2\beta_2\left(\frac{a}{b}\right)\left(\frac{x_3}{b}\right)^{a-1} \exp\left\{-2\left[\left(\frac{x_3}{b}\right)^a - \left(\frac{x_2}{b}\right)^a\right]\right\} \\ & \left[1 - \exp\left\{-2\left[\left(\frac{x_3}{b}\right)^a - \left(\frac{x_2}{b}\right)^a\right]\right\}\right]^{\beta_2 - 1}, \end{aligned}$$

$$0 \le x_2 \le x_3 \le \infty, \;\; \beta_2 > 0, \;\; a, b > 0.$$

From (9) the joint density function of $(X_{(1)}, X_{(2)}, X_{(3)})$ is given by

$$\begin{aligned} h(x_1, x_2, x_3) =& 4!\beta_1\beta_2\left(\frac{a}{b}\right)^3\left(\frac{x_1}{b}\right)^{a-1}\left(\frac{x_2}{b}\right)^{a-1}\left(\frac{x_3}{b}\right)^{a-1} \exp\left\{-4\left(\frac{x_1}{b}\right)^a - 3\left[\left(\frac{x_2}{b}\right)^a - \left(\frac{x_1}{b}\right)^a\right]\right. \\ & \left.-2\left[\left(\frac{x_3}{b}\right)^a - \left(\frac{x_2}{b}\right)^a\right]\right\}\left[1 - \exp\left\{-2\left[\left(\frac{x_3}{b}\right)^a - \left(\frac{x_2}{b}\right)^a\right]\right\}\right]^{\beta_2 - 1} \\ & \left[1 - \exp\left\{-3\left[\left(\frac{x_2}{b}\right)^a - \left(\frac{x_1}{b}\right)^a\right]\right\}\right]^{\beta_1 - 1}, \end{aligned} \tag{12}$$

$$0 \le x_1 \le x_2 \le x_3 \le \infty, \;\; \beta_1, \beta_2 > 0, \;\; a, b > 0.$$

In the next subsequent section we discuss the parameter estimation procedures for the proposed model.

4. Parameter Estimation

Let $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)$ be the baseline parameters and $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_{(m-k)})$ be the load sharing parameters involved in the proposed PCRHR based load sharing model. The maximum likelihood estimation procedure is used to estimate the parameters.

4.1. For k-out-of-m Load Sharing System

Let $r = m - k + 1$, x_{ij} be the j^{th} failure $X_{(j)}$ of i^{th} k-out-of-m load sharing system and x_{ij+1} is the $(j+1)^{\text{th}}$ failure $X_{(j+1)}$ of i^{th} k-out-of-m load sharing system with $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m - k$.

The likelihood function of $(\underline{\theta}, \underline{\beta})$ based on n independent k-out-of-m load sharing systems is given by

$$L\left(\underline{\theta}, \underline{\beta}\right) = \left(\frac{m!}{(k-1)!}\right)^n \left(\prod_{j=1}^{(m-k)} \beta_j\right)^n \prod_{i=1}^{n} \bar{F}_{\underline{\theta}}\left(x_{ir}\right)^{k-1} \prod_{i=1}^{n}\prod_{j=1}^{r} f_{\underline{\theta}}\left(x_{ij}\right)$$
$$\prod_{i=1}^{n}\prod_{j=1}^{(m-k)} \left[1 - \left\{\frac{\bar{F}_{\underline{\theta}}\left(x_{ij+1}\right)}{\bar{F}_{\underline{\theta}}\left(x_{ij}\right)}\right\}^{m-j}\right]^{\beta_j - 1}.$$

The corresponding log-likelihood function can be written as,

$$\log L = n\log\left(\frac{m!}{(k-1)!}\right) + n\sum_{j=1}^{r}\log\beta_j + (k-1)\sum_{i=1}^{n}\log \bar{F}_{\underline{\theta}}\left(x_{ir}\right)$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{r}\log f_{\underline{\theta}}\left(x_{ij}\right) + \sum_{i=1}^{n}\sum_{j=1}^{m-k}(\beta_j - 1)\log\left[1 - \left\{\frac{\bar{F}_{\underline{\theta}}\left(x_{ij+1}\right)}{\bar{F}_{\underline{\theta}}\left(x_{ij}\right)}\right\}^{m-j}\right].$$

To obtain maximum likelihood estimates (MLEs) of unknowns parameters $\underline{\theta}$ and $\underline{\beta}$ differentiate the above log-likelihood partially w.r.t. unknown parameters and equate to zero, we have likelihood equations to estimate as

$$\frac{\partial \log L}{\partial \theta_l} = 0, \quad l = 1, 2, 3, \ldots, p, \tag{13}$$

$$\frac{\partial \log L}{\partial \beta_j} = 0, \quad j = 1, 2, \ldots, (m-k). \tag{14}$$

The above $p + (m-k)$ equations are not in closed form so are solved by iterative procedures. We could use the 'optim' function in R software to solve these equations.

4.2. For k-out-of-m Load Sharing System with Weibull Baseline

The likelihood function based on n i.i.d. k-out-of-m load systems with component lifetimes as Weibull (a, b) is given by

$$L(a, b, \underline{\beta}) = \left(\frac{m!}{(k-1)!}\right)^n \left(\frac{a}{b}\right)^{nr} \left(\prod_{j=1}^{m-k}\beta_j\right)^n \prod_{i=1}^{n}\prod_{j=1}^{r}\left(\frac{x_{ij}}{b}\right)^{a-1} \exp\left[-k\sum_{i=1}^{n}\left(\frac{x_{ir}}{b}\right)^a\right]$$
$$\exp\left[-\sum_{i=1}^{n}\sum_{j=1}^{(m-k)}\left(\frac{x_{ij}}{b}\right)^a\right]\prod_{i=1}^{n}\prod_{j=1}^{m-k}\left\{1 - \exp\left(-\frac{(m-j)}{b^a}\left[\left(x_{ij+1}\right)^a - \left(x_{ij}\right)^a\right]\right)\right\}^{\beta_j - 1}.$$

Hence, the log-likelihood function is given by

$$\log L(a, b, \underline{\beta}) = n\log\left(\frac{m!}{(k-1)!}\right) + nr\log\left(\frac{a}{b}\right) + n\sum_{j=1}^{(m-k)}\log\beta_j + (a-1)\sum_{i=1}^{n}\sum_{j=1}^{r}\log\left(\frac{x_{ij}}{b}\right) - k\sum_{i=1}^{n}\left(\frac{x_{ir}}{b}\right)^a$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{(m-k)}\left(\frac{x_{ij}}{b}\right)^a + \sum_{i=1}^{n}\sum_{j=1}^{m-k}(\beta_j - 1)\log\left\{1 - \exp\left(-\frac{(m-j)}{b^a}\left[\left(x_{ij+1}\right)^a - \left(x_{ij}\right)^a\right]\right)\right\}.$$

The MLEs of a, b and β_j, $j = 1, 2, \ldots, (m-k)$ are obtained by maximizing above log-likelihood function. The likelihood or log-likelihood function is maximized by differentiating it partially with respect to unknown parameters and equating to zero. The likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_j} = 0, \quad j = 1, 2, \ldots, (m-k).$$

We observed that likelihood equations are not in closed form, so iterative procedures are used to estimate the unknown parameters.

4.3. For 2-out-of-4 Load Sharing System with Weibull Baseline

The likelihood function for the 2-out-of-4 load sharing system with Weibull (a, b) baseline distribution is given by

$$L(a,b,\beta_1,\beta_2) = (4!)^n \left(\frac{a}{b}\right)^{3n} (\beta_1\beta_2)^n \prod_{i=1}^{n} \left(\frac{x_{i1}}{b}\frac{x_{i2}}{b}\frac{x_{i3}}{b}\right)^{(a-1)} \exp\left[-2\sum_{i=1}^{n}\left(\frac{x_{i3}}{b}\right)^a\right]$$
$$\exp\left[-\sum_{i=1}^{n}\left\{\left(\frac{x_{i1}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a\right\}\right] \prod_{i=1}^{n}\left\{1-\exp\left(-\frac{3}{b^a}\left[(x_{i2})^a-(x_{i1})^a\right]\right)\right\}^{\beta_1-1}$$
$$\prod_{i=1}^{n}\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i3})^a-(x_{i2})^a\right]\right)\right\}^{\beta_2-1}.$$

Thus the log-likelihood function is given by

$$\log L(a,b,\beta_1,\beta_2) = n\log(4!) + 3n\log\left(\frac{a}{b}\right) + n\log\beta_1 + n\log\beta_2 + (a-1)\sum_{i=1}^{n}\left(\log\left(\frac{x_{i1}}{b}\right)\right.$$
$$\left. + \log\left(\frac{x_{i2}}{b}\right) + \log\left(\frac{x_{i3}}{b}\right)\right) - \sum_{i=1}^{n}\left\{2\left(\frac{x_{i3}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\}$$
$$+ (\beta_1-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{3}{b^a}\left[(x_{i2})^a-(x_{i1})^a\right]\right)\right\}$$
$$+ (\beta_2-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i3})^a-(x_{i2})^a\right]\right)\right\},$$

and the likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_1} = 0, \quad \frac{\partial \log L}{\partial \beta_2} = 0.$$

These equation solved simultaneously to get MLEs of a, b, β_1 and β_2. It is seen that the likelihood equations not in closed form, iterative procedures are used.

4.4. For 2-out-of-3 Load Sharing System with Weibull Baseline

The likelihood function for the 2-out-of-3 load sharing model with Weibull (a, b) baseline distribution is given by

$$L(a,b,\beta_1) = (3!)^n \left(\frac{a}{b}\right)^{2n} \beta_1^n \prod_{i=1}^{n}\left(\frac{x_{i1}}{b}\frac{x_{i2}}{b}\right)^{(a-1)} \exp\left[-\sum_{i=1}^{n}\left\{\left(2\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\}\right]$$
$$\prod_{i=1}^{n}\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a-(x_{i1})^a\right]\right)\right\}^{\beta_1-1}.$$

Hence, the log-likelihood function is given by

$$\log L(a,b,\beta_1) = n\log(3!) + 2n\log\left(\frac{a}{b}\right) + n\log\beta_1 + (a-1)\sum_{i=1}^{n}\left\{\log\left(\frac{x_{i1}}{b}\right) + \log\left(\frac{x_{i2}}{b}\right)\right\}$$
$$-\sum_{i=1}^{n}\left\{2\left(\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\} + (\beta_1-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a-(x_{i1})^a\right]\right)\right\}.$$

The likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_1} = 0.$$

These equations are solved simultaneously to obtain the MLEs of a, b, and β_1. Since the likelihood equations do not have a closed-form solution, iterative procedures are required to solve them.

4.5. For 1-out-of-3 Parallel Load Sharing System with Weibull Baseline

Under the 1-out-of-3 load sharing model with Weibull (a, b) baseline Distribution the likelihood function is given by

$$L(a,b,\beta_1,\beta_2) = (3!)^n \left(\frac{a}{b}\right)^{3n} (\beta_1\beta_2)^n \prod_{i=1}^{n} \left(\frac{x_{i1}}{b}\frac{x_{i2}}{b}\frac{x_{i3}}{b}\right)^{(a-1)} \exp\left[-\sum_{i=1}^{n}\left\{\left(\frac{x_{i3}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a\right\}\right] \prod_{i=1}^{n}\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\}^{\beta_1-1} \prod_{i=1}^{n}\left\{1-\exp\left(-\frac{1}{b^a}\left[(x_{i3})^a - (x_{i2})^a\right]\right)\right\}^{\beta_2-1}.$$

The log-likelihood function is given by

$$\begin{aligned}\log L(a,b,\beta_1,\beta_2) =& n\log(3!) + 3n\log\left(\frac{a}{b}\right) + n\log\beta_1 + n\log\beta_2 + (a-1)\sum_{i=1}^{n}\left\{\log\left(\frac{x_{i1}}{b}\right)\right. \\ &\left. + \log\left(\frac{x_{i2}}{b}\right) + \log\left(\frac{x_{i3}}{b}\right)\right\} - \sum_{i=1}^{n}\left\{\left(\frac{x_{i3}}{b}\right)^a + \left(\frac{x_{i2}}{b}\right)^a + \left(\frac{x_{i1}}{b}\right)^a\right\} \\ &+ (\beta_1-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{2}{b^a}\left[(x_{i2})^a - (x_{i1})^a\right]\right)\right\} \\ &+ (\beta_2-1)\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{1}{b^a}\left[(x_{i3})^a - (x_{i2})^a\right]\right)\right\}\end{aligned}$$

The likelihood equations are

$$\frac{\partial \log L}{\partial a} = 0, \quad \frac{\partial \log L}{\partial b} = 0, \quad \frac{\partial \log L}{\partial \beta_1} = 0, \quad \frac{\partial \log L}{\partial \beta_2} = 0.$$

These equation solved simultaneously to get MLEs of a, b, β_1 and β_2. It is seen that the likelihood equations not in closed form, iterative procedures are used.

5. Simulation Study

In this section, we carry out the simulation study for 2-out-of-4 load sharing model with Weibull (a, b) baseline distribution. The MLEs, Bias and Mean Square Estimates (MSE) are obtained for various combination of sample size, baseline parameters and load sharing parameters. The performance of estimates are accessed to bias and MSE. We consider sample sizes $30, 50, 100$ and 200, the baseline parameters are considered to be $a = 1, 1.5, 2$ and $b = 0.5, 0.7, 1, 2$, and load share parameters $\beta_j = 1, 1.5, 2, \quad j = 1, 2$. The parameters are estimated using 1000 samples for each sample size and parameter combinations. The results, including the MLEs, bias, and MSE, are detailed in Tables 1 to 4 and illustrated in Figures 1 to 4. These tables and figures provide a comprehensive overview of the estimation accuracy and performance metrics associated with the model.

Table 1: *MLE's Bias and MSE's for 2-out-of-4 system with n=30*

n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
30	1	0.5	1	1	1.0268	0.4931	1.0589	1.0666	0.0748	0.05	0.1792	0.1813	0.0092	0.0039	0.06	0.0647
30	1	0.5	1	1.5	1.0201	0.495	1.046	1.5878	0.0711	0.0484	0.1694	0.2763	0.0081	0.0038	0.0499	0.1323
30	1	0.5	1	2	1.0108	0.4889	1.0613	2.1738	0.0645	0.0481	0.1742	0.4075	0.0065	0.0036	0.0557	0.3271
30	1	0.5	1.5	1	1.0178	0.4912	1.5951	1.0784	0.0698	0.05	0.2743	0.1909	0.0079	0.0038	0.134	0.0678
30	1	0.5	1.5	1.5	1.018	0.4913	1.6154	1.6238	0.0676	0.047	0.2773	0.2865	0.0076	0.0035	0.1397	0.1599
30	1	0.5	1.5	2	1.0154	0.491	1.6017	2.2001	0.0645	0.046	0.2854	0.4067	0.0066	0.0033	0.1505	0.3141
30	1	0.5	2	1	1.0205	0.4917	2.1517	1.0744	0.0698	0.0472	0.3935	0.1758	0.0077	0.0035	0.3023	0.054
30	1	0.5	2	1.5	1.0144	0.4894	2.1618	1.624	0.0669	0.0461	0.3895	0.2769	0.0072	0.0033	0.2934	0.1457
30	1	0.5	2	2	1.0148	0.4931	2.1526	2.1655	0.0661	0.0461	0.3874	0.404	0.0071	0.0033	0.2859	0.3107
30	1	0.7	1	1	1.0146	0.6869	1.0624	1.0735	0.0724	0.0671	0.1725	0.184	0.0083	0.0073	0.052	0.0621
30	1	0.7	1	1.5	1.023	0.6904	1.045	1.6113	0.0694	0.0679	0.1664	0.2789	0.0081	0.0074	0.0487	0.15
30	1	0.7	1	2	1.0167	0.6893	1.0636	2.1653	0.0654	0.0696	0.1756	0.4029	0.0074	0.0075	0.0566	0.2987
30	1	0.7	1.5	1	1.0181	0.6894	1.5874	1.0652	0.0696	0.0662	0.2766	0.1751	0.008	0.0068	0.1353	0.057
30	1	0.7	1.5	1.5	1.0154	0.6907	1.6015	1.5991	0.0667	0.0685	0.2911	0.2789	0.0072	0.0072	0.1428	0.1463
30	1	0.7	1.5	2	1.0118	0.6914	1.5864	2.1605	0.0646	0.065	0.2757	0.39	0.007	0.0068	0.1341	0.2808
30	1	0.7	2	1	1.0169	0.6898	2.1319	1.0763	0.0677	0.0683	0.3823	0.1817	0.0073	0.0074	0.2684	0.0593
30	1	0.7	2	1.5	1.0086	0.6908	2.1502	1.5954	0.0639	0.0659	0.393	0.2727	0.0064	0.0068	0.2875	0.1303
30	1	0.7	2	2	1.0132	0.6875	2.121	2.1564	0.0644	0.0625	0.3673	0.3911	0.0067	0.006	0.237	0.2704
30	1	1	1	1	1.0164	0.9878	1.0532	1.0703	0.0705	0.0994	0.1676	0.1735	0.0082	0.0152	0.0526	0.0552
30	1	1	1	1.5	1.0234	0.9811	1.0652	1.6203	0.0703	0.0974	0.1778	0.283	0.0083	0.0147	0.0575	0.142
30	1	1	1	2	1.0166	0.9886	1.0675	2.2175	0.0665	0.0982	0.1734	0.4385	0.0075	0.0148	0.0551	0.3774
30	1	1	1.5	1	1.0144	0.9875	1.6024	1.062	0.0685	0.0969	0.2817	0.1739	0.0076	0.0146	0.146	0.0547
30	1	1	1.5	1.5	1.0118	0.9774	1.5964	1.6388	0.0694	0.0934	0.2795	0.2952	0.0078	0.0137	0.1357	0.1639
30	1	1	1.5	2	1.0138	0.9764	1.6034	2.1774	0.0659	0.093	0.2766	0.3923	0.0069	0.0135	0.1408	0.3079
30	1	1	2	1	1.0158	0.9817	2.1508	1.0642	0.0709	0.0964	0.3795	0.1763	0.0083	0.0141	0.2644	0.0553
30	1	1	2	1.5	1.0149	0.975	2.1838	1.6278	0.067	0.0931	0.3964	0.2881	0.0073	0.0137	0.3241	0.1543
30	1	1	2	2	1.0162	0.9871	2.1375	2.1845	0.0677	0.0948	0.3796	0.4103	0.0073	0.0138	0.2725	0.3222
30	1	2	1	1	1.0136	1.9619	1.0629	1.0621	0.0715	0.2029	0.1802	0.1736	0.0082	0.0636	0.0591	0.0545
30	1	2	1	1.5	1.0176	1.9727	1.0535	1.6167	0.0722	0.1984	0.1709	0.288	0.0085	0.0605	0.0525	0.1493
30	1	2	1	2	1.0167	1.9602	1.0503	2.1844	0.0686	0.1968	0.1629	0.4129	0.0075	0.0593	0.0485	0.314
30	1	2	1.5	1	1.0161	1.9704	1.6143	1.0541	0.0715	0.1895	0.285	0.1701	0.0085	0.0544	0.1511	0.0507
30	1	2	1.5	1.5	1.0152	1.9627	1.6089	1.6237	0.0676	0.1843	0.282	0.2824	0.0076	0.0544	0.1519	0.1466
30	1	2	1.5	2	1.0165	1.9822	1.597	2.1522	0.0672	0.1937	0.2893	0.3881	0.0073	0.0577	0.1505	0.2959
30	1	2	2	1	1.0157	1.9759	2.1385	1.0594	0.0685	0.1868	0.3799	0.1748	0.0075	0.0554	0.2744	0.0576
30	1	2	2	1.5	1.0169	1.9666	2.1583	1.636	0.0686	0.1845	0.4054	0.2894	0.0073	0.0535	0.3028	0.1555
30	1	2	2	2	1.0134	1.9694	2.1574	2.1781	0.0645	0.1853	0.395	0.3899	0.0069	0.0533	0.2934	0.2832
30	1.5	0.5	1	1	1.5264	0.4921	1.0647	1.0691	0.1092	0.0339	0.1774	0.1764	0.0198	0.0018	0.0546	0.0572
30	1.5	0.5	1	1.5	1.5228	0.4931	1.0622	1.6364	0.1022	0.0326	0.1748	0.2978	0.017	0.0016	0.0545	0.1737
30	1.5	0.5	1	2	1.526	0.4931	1.0446	2.1664	0.1049	0.0315	0.1651	0.4004	0.0173	0.0015	0.0468	0.2951
30	1.5	0.5	1.5	1	1.5337	0.4953	1.5892	1.0754	0.1088	0.0318	0.2717	0.1832	0.0187	0.0015	0.1356	0.0606
30	1.5	0.5	1.5	1.5	1.519	0.4931	1.5938	1.597	0.1032	0.0322	0.2758	0.2807	0.0178	0.0016	0.1465	0.149
30	1.5	0.5	1.5	2	1.525	0.495	1.5924	2.1785	0.1013	0.0305	0.2674	0.4083	0.016	0.0014	0.1241	0.3183
30	1.5	0.5	2	1	1.525	0.4926	2.1701	1.0778	0.1044	0.0317	0.405	0.1787	0.0174	0.0016	0.2946	0.0578
30	1.5	0.5	2	1.5	1.5221	0.4948	2.1594	1.6119	0.0999	0.0321	0.4105	0.2809	0.0162	0.0016	0.3054	0.1384
30	1.5	0.5	2	2	1.5308	0.4959	2.1456	2.1656	0.0988	0.0305	0.4034	0.4034	0.0155	0.0015	0.2903	0.3198
30	1.5	0.7	1	1	1.5217	0.6894	1.0594	1.0626	0.1104	0.0457	0.1739	0.1783	0.0194	0.0033	0.0547	0.0586
30	1.5	0.7	1	1.5	1.5298	0.6894	1.0705	1.6159	0.1068	0.0435	0.1799	0.2843	0.0184	0.0029	0.0577	0.153
30	1.5	0.7	1	2	1.524	0.6894	1.0697	2.202	0.1023	0.0458	0.1801	0.4265	0.0168	0.0033	0.0608	0.3443
30	1.5	0.7	1.5	1	1.5227	0.6913	1.5908	1.0733	0.1028	0.0463	0.2727	0.1866	0.0172	0.0033	0.1341	0.0641
30	1.5	0.7	1.5	1.5	1.5244	0.6923	1.591	1.6344	0.1068	0.0432	0.2738	0.2965	0.018	0.003	0.1312	0.1661
30	1.5	0.7	1.5	2	1.5311	0.6923	1.5875	2.1931	0.0975	0.0415	0.2755	0.4173	0.0156	0.0028	0.137	0.3263
30	1.5	0.7	2	1	1.5215	0.6915	2.1586	1.0702	0.1028	0.0459	0.3918	0.1805	0.0167	0.0033	0.2861	0.0579
30	1.5	0.7	2	1.5	1.5286	0.6932	2.1551	1.6275	0.0967	0.0437	0.3857	0.2958	0.0157	0.003	0.3045	0.1644
30	1.5	0.7	2	2	1.5196	0.6891	2.1493	2.2065	0.0961	0.0433	0.3904	0.4354	0.0155	0.003	0.2791	0.3526
30	1.5	1	1	1	1.5298	0.9896	1.0581	1.0668	0.1123	0.0656	0.1701	0.178	0.0207	0.0067	0.0505	0.0594
30	1.5	1	1	1.5	1.524	0.9882	1.0607	1.5935	0.1043	0.0633	0.1753	0.279	0.0176	0.0064	0.0541	0.1455
30	1.5	1	1	2	1.528	0.9902	1.0533	2.1783	0.1013	0.065	0.1682	0.4156	0.0164	0.0066	0.0501	0.3123
30	1.5	1	1.5	1	1.5226	0.9889	1.5983	1.0665	0.1021	0.0651	0.2743	0.178	0.0172	0.0066	0.1441	0.059
30	1.5	1	1.5	1.5	1.5191	0.989	1.6092	1.5996	0.0958	0.0643	0.2735	0.2878	0.0149	0.0065	0.1428	0.1494
30	1.5	1	1.5	2	1.5205	0.9901	1.6032	2.1649	0.0994	0.0634	0.28	0.4113	0.016	0.0064	0.1512	0.3202
30	1.5	1	2	1	1.5289	0.9884	2.16	1.0743	0.1068	0.0637	0.3926	0.1881	0.0185	0.0064	0.2772	0.0653
30	1.5	1	2	1.5	1.5216	0.9879	2.1723	1.6159	0.0969	0.0657	0.414	0.281	0.0152	0.0067	0.3224	0.1448
30	1.5	1	2	2	1.5173	0.9848	2.1531	2.1837	0.0952	0.0635	0.4059	0.4079	0.0148	0.0064	0.2972	0.3037
30	1.5	2	1	1	1.524	1.976	1.0745	1.0682	0.1097	0.134	0.1843	0.177	0.0202	0.0273	0.0596	0.0563
30	1.5	2	1	1.5	1.5267	1.9729	1.0566	1.6222	0.1066	0.1262	0.182	0.2857	0.0184	0.0249	0.0576	0.162
30	1.5	2	1	2	1.5246	1.9786	1.0604	2.1575	0.099	0.1265	0.1821	0.3777	0.0159	0.0248	0.0551	0.2765
30	1.5	2	1.5	1	1.52	1.9707	1.5948	1.0654	0.1082	0.1281	0.2678	0.1775	0.0189	0.0255	0.1306	0.0586
30	1.5	2	1.5	1.5	1.5285	1.9681	1.6067	1.6337	0.1027	0.1315	0.2797	0.3078	0.0173	0.0269	0.143	0.1856
30	1.5	2	1.5	2	1.5258	1.9811	1.5839	2.1726	0.1013	0.131	0.2762	0.4167	0.0163	0.0268	0.1423	0.3165
30	1.5	2	2	1	1.5203	1.97	2.1675	1.0626	0.1016	0.1237	0.4025	0.1739	0.0165	0.0244	0.2863	0.0544
30	1.5	2	2	1.5	1.5213	1.983	2.1255	1.609	0.0998	0.1229	0.3853	0.2927	0.0157	0.0236	0.2591	0.1622
30	1.5	2	2	2	1.5247	1.9748	2.1301	2.1947	0.0963	0.1185	0.3689	0.4202	0.0151	0.0228	0.2574	0.3442
30	2	0.5	1	1	2.0398	0.4955	1.0594	1.081	0.1454	0.0243	0.1744	0.1852	0.0346	0.001	0.0549	0.0667
30	2	0.5	1	1.5	2.0421	0.4963	1.0407	1.604	0.1407	0.0233	0.1621	0.2868	0.0324	0.0009	0.0487	0.1608
30	2	0.5	1	2	2.0297	0.4948	1.0567	2.1828	0.1325	0.024	0.1721	0.4023	0.028	0.0009	0.0573	0.3122
30	2	0.5	1.5	1	2.0302	0.4943	1.6357	1.0626	0.143	0.0241	0.3003	0.175	0.0333	0.0009	0.1702	0.0574
30	2	0.5	1.5	1.5	2.0275	0.4943	1.6087	1.6032	0.1424	0.0232	0.2747	0.2727	0.0326	0.0008	0.1413	0.1399
30	2	0.5	1.5	2	2.0409	0.4951	1.6065	2.1743	0.1377	0.0226	0.2706	0.392	0.0301	0.0008	0.1429	0.2928
30	2	0.5	2	1	2.0252	0.4953	2.1554	1.0713	0.1359	0.0239	0.3936	0.1839	0.029	0.0009	0.2879	0.0619
30	2	0.5	2	1.5	2.0293	0.4942	2.1786	1.6413	0.1339	0.0236	0.3977	0.2919	0.029	0.0009	0.3124	0.1589
30	2	0.5	2	2	2.0367	0.4961	2.1458	2.1796	0.1277	0.0242	0.3861	0.4068	0.0268	0.0009	0.2804	0.3141
30	2	0.7	1	1	2.0289	0.6957	1.0527	1.0514	0.1532	0.0345	0.1728	0.1735	0.0378	0.0019	0.0533	0.0527
30	2	0.7	1	1.5	2.0371	0.6936	1.0569	1.6267	0.1423	0.0342	0.1781	0.2863	0.0335	0.0018	0.0564	0.1603
30	2	0.7	1	2	2.0348	0.6913	1.0651	2.1986	0.1355	0.0344	0.1761	0.4282	0.0305	0.0019	0.0567	0.3639
30	2	0.7	1.5	1	2.0335	0.6924	1.6073	1.078	0.1455	0.0341	0.2667	0.1911	0.0342	0.0018	0.1298	0.0671
30	2	0.7	1.5	1.5	2.0282	0.6925	1.6076	1.6236	0.1301	0.033	0.2864	0.2876	0.0278	0.0017	0.1462	0.155
30	2	0.7	1.5	2	2.0305	0.6937	1.5913	2.166	0.1357	0.0315	0.2788	0.3829	0.0294	0.0016	0.1491	0.2747
30	2	0.7	2	1	2.0296	0.6901	2.1985	1.0755	0.1414	0.0331	0.426	0.1885	0.0322	0.0017	0.3442	0.0631
30	2	0.7	2	1.5	2.0212	0.6922	2.1931	1.6337	0.1286	0.0339	0.4167	0.2974	0.0265	0.0018	0.3243	0.1741
30	2	0.7	2	2	2.0286	0.6948	2.136	2.2045	0.1285	0.0321	0.39	0.4388	0.0265	0.0016	0.2715	0.365
30	2	1	1	1	2.0441	0.9905	1.0588	1.0774	0.1479	0.0514	0.17	0.1893	0.0356	0.0041	0.052	0.0667
30	2	1	1	1.5	2.0291	0.9889	1.0631	1.6235	0.1418	0.0474	0.1777	0.2989	0.0325	0.0035	0.0565	0.162
30	2	1	1	2	2.0295	0.9892	1.0647	2.1925	0.1346	0.0478	0.1737	0.4314	0.0286	0.0035	0.0531	0.3347
30	2	1	1.5	1	2.0325	0.9866	1.6182	1.0814	0.1411	0.0493	0.2832	0.1857	0.031	0.0037	0.1516	0.0637
30	2	1	1.5	1.5	2.0386	0.989	1.6044	1.6333	0.1351	0.0468	0.2801	0.2951	0.0296	0.0034	0.1405	0.1698
30	2	1	1.5	2	2.0247	0.9934	1.5908	2.1767	0.1282	0.0474	0.2733	0.4004	0.026	0.0035	0.1386	0.3143
30	2	1	2	1	2.0329	0.9903	2.1331	1.0593	0.1414	0.0494	0.3966	0.1762	0.0313	0.0038	0.2839	0.0568
30	2	1	2	1.5	2.0321	0.9914	2.1564	1.6206	0.1377	0.0471	0.3945	0.2914	0.0307	0.0035	0.3034	0.1549
30	2	1	2	2	2.0209	0.9906	2.1356	2.1618	0.1276	0.0463	0.4067	0.3998	0.0261	0.0033	0.3058	0.3029
30	2	2	1	1	2.0334	1.9786	1.0532	1.0659	0.1456	0.1039	0.1776	0.1771	0.0339	0.0166	0.0557	0.0555
30	2	2	1	1.5	2.0334	1.9744	1.0604	1.6298	0.1433	0.1022	0.1735	0.3014	0.0323	0.0161	0.0541	0.1643
30	2	2	1	2	2.0278	1.981	1.0616	2.1682	0.1331	0.0978	0.1802	0.4014	0.0276	0.015	0.057	0.3047
30	2	2	1.5	1	2.0278	1.9821	1.614	1.0695	0.1386	0.0971	0.2774	0.1794	0.0314	0.0144	0.1404	0.0555
30	2	2	1.5	1.5	2.0368	1.9795	1.6109	1.6258	0.1379	0.0944	0.2919	0.2974	0.0313	0.0139	0.1594	0.1691
30	2	2	1.5	2	2.0209	1.985	1.6262	2.1672	0.1287	0.0926	0.2862	0.4002	0.0267	0.0134	0.1548	0.3059
30	2	2	2	1	2.0328	1.9779	2.1479	1.0774	0.145	0.0954	0.3935	0.1776	0.0344	0.014	0.3015	0.059
30	2	2	2	1.5	2.0299	1.982	2.1556	1.587	0.136	0.0973	0.3988	0.2725	0.0289	0.0147	0.2886	0.1319
30	2	2	2	2	2.0312	1.9853	2.1534	2.1765	0.128	0.0926	0.3944	0.398	0.0263	0.0136	0.2958	0.3017

Table 2: *MLE's Bias and MSE's for* 2*-out-of-*4 *system with n=50*

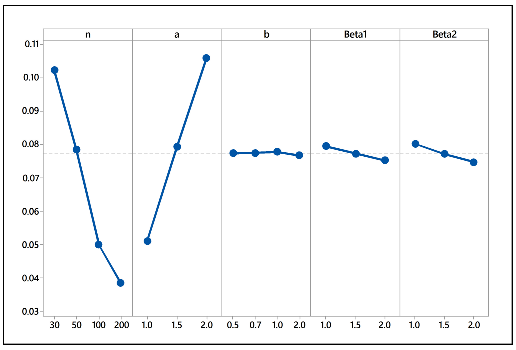
n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
50	1	0.5	1	1	1.0133	0.4936	1.0305	1.0455	0.0542	0.0378	0.1267	0.1337	0.0046	0.0022	0.0279	0.0311
50	1	0.5	1	1.5	1.0129	0.4941	1.0436	1.5596	0.0548	0.0368	0.1295	0.2088	0.005	0.0021	0.03	0.0739
50	1	0.5	1	2	1.0081	0.497	1.0249	2.1013	0.051	0.0377	0.1286	0.2897	0.004	0.0022	0.0276	0.1463
50	1	0.5	1.5	1	1.0105	0.4947	1.5642	1.0469	0.0542	0.0381	0.2101	0.1347	0.0047	0.0022	0.0742	0.0306
50	1	0.5	1.5	1.5	1.0085	0.4974	1.5487	1.5621	0.0527	0.0368	0.1996	0.2146	0.0045	0.0021	0.067	0.0787
50	1	0.5	1.5	2	1.0093	0.4951	1.5611	2.1001	0.0507	0.0357	0.2009	0.2956	0.0042	0.002	0.0663	0.1423
50	1	0.5	2	1	1.0094	0.4957	2.0719	1.037	0.053	0.0363	0.28	0.1364	0.0045	0.0021	0.1361	0.0298
50	1	0.5	2	1.5	1.0063	0.4962	2.088	1.5732	0.0512	0.0367	0.2904	0.2161	0.0042	0.0021	0.148	0.0828
50	1	0.5	2	2	1.0061	0.4955	2.0713	2.0885	0.0484	0.0358	0.2811	0.2937	0.0037	0.002	0.1384	0.1443
50	1	0.7	1	1	1.012	0.6937	1.0278	1.0383	0.058	0.0523	0.1298	0.1273	0.0054	0.0043	0.0283	0.0276
50	1	0.7	1	1.5	1.0087	0.691	1.0408	1.5579	0.0541	0.053	0.1313	0.2155	0.0047	0.0044	0.0298	0.0801
50	1	0.7	1	2	1.0093	0.6935	1.035	2.097	0.0509	0.0519	0.1288	0.2921	0.0042	0.0042	0.028	0.1498
50	1	0.7	1.5	1	1.0111	0.6931	1.5656	1.0337	0.0547	0.0529	0.2093	0.1327	0.0047	0.0043	0.0743	0.0306
50	1	0.7	1.5	1.5	1.012	0.6956	1.5627	1.568	0.0509	0.052	0.2078	0.214	0.0041	0.0043	0.0775	0.0783
50	1	0.7	1.5	2	1.0103	0.6928	1.5507	2.1132	0.0519	0.0508	0.1977	0.3028	0.0043	0.004	0.0682	0.156
50	1	0.7	2	1	1.007	0.6912	2.0763	1.0376	0.0511	0.0543	0.2792	0.1366	0.0041	0.0045	0.1368	0.0303
50	1	0.7	2	1.5	1.01	0.6933	2.1062	1.5658	0.0515	0.0505	0.3049	0.2062	0.0042	0.004	0.1633	0.0752
50	1	0.7	2	2	1.0072	0.695	2.0728	2.0819	0.0489	0.0511	0.2902	0.2899	0.0038	0.0041	0.1397	0.1471
50	1	1	1	1	1.0112	0.986	1.0442	1.0458	0.0553	0.0768	0.1349	0.1348	0.0048	0.0094	0.0317	0.0318
50	1	1	1	1.5	1.0095	0.9889	1.039	1.5793	0.0516	0.0735	0.1301	0.2141	0.0043	0.0083	0.0295	0.0816
50	1	1	1	2	1.0081	0.9916	1.0303	2.106	0.0506	0.0717	0.1284	0.2981	0.0041	0.0083	0.0282	0.1549
50	1	1	1.5	1	1.0079	0.9909	1.5515	1.0354	0.0547	0.0745	0.2043	0.1305	0.0047	0.0087	0.069	0.0286
50	1	1	1.5	1.5	1.0126	0.9894	1.562	1.5688	0.0537	0.072	0.2109	0.2052	0.0046	0.0081	0.0792	0.0739
50	1	1	1.5	2	1.008	0.9909	1.5716	2.1045	0.0507	0.0725	0.2174	0.2934	0.0041	0.0082	0.0813	0.1501
50	1	1	2	1	1.0134	0.9964	2.0557	1.0397	0.0535	0.0735	0.278	0.1334	0.0047	0.0086	0.1374	0.0305
50	1	1	2	1.5	1.0081	0.9909	2.0845	1.5651	0.0501	0.075	0.2845	0.2085	0.0041	0.0088	0.1368	0.0734
50	1	1	2	2	1.0068	0.9896	2.0872	2.1097	0.0499	0.0729	0.2919	0.2972	0.004	0.0083	0.1499	0.1513
50	1	2	1	1	1.0132	1.9823	1.0231	1.0364	0.0559	0.1562	0.1299	0.1287	0.0049	0.0378	0.0275	0.0278
50	1	2	1	1.5	1.008	1.9865	1.0232	1.5618	0.0537	0.1525	0.132	0.2167	0.0045	0.0365	0.0287	0.0784
50	1	2	1	2	1.0111	1.9849	1.0334	2.0933	0.0497	0.1464	0.1321	0.2926	0.004	0.0327	0.0296	0.1519
50	1	2	1.5	1	1.0069	1.9834	1.5697	1.0356	0.0535	0.1523	0.2151	0.135	0.0045	0.036	0.08	0.0327
50	1	2	1.5	1.5	1.0099	1.9783	1.5591	1.5738	0.0505	0.1425	0.2041	0.2142	0.004	0.0314	0.0694	0.0793
50	1	2	1.5	2	1.0105	1.9774	1.5507	2.1054	0.051	0.1493	0.2059	0.3047	0.0041	0.034	0.0715	0.1562
50	1	2	2	1	1.0118	1.9925	2.0623	1.0397	0.0538	0.1478	0.2863	0.1347	0.0046	0.0352	0.1448	0.0316
50	1	2	2	1.5	1.007	1.9793	2.0973	1.5526	0.0492	0.1427	0.2886	0.2071	0.0038	0.0319	0.1453	0.0746
50	1	2	2	2	1.0094	1.9829	2.0889	2.0993	0.0499	0.1406	0.2874	0.2916	0.004	0.0313	0.1437	0.1455
50	1.5	0.5	1	1	1.5198	0.4964	1.0407	1.0461	0.0846	0.0252	0.1333	0.1363	0.0108	0.001	0.0308	0.0321
50	1.5	0.5	1	1.5	1.5147	0.496	1.0255	1.5712	0.0814	0.0249	0.1231	0.2151	0.0102	0.001	0.0249	0.0783
50	1.5	0.5	1	2	1.5142	0.4959	1.0389	2.0926	0.0772	0.0243	0.1358	0.2913	0.0098	0.0009	0.031	0.1526
50	1.5	0.5	1.5	1	1.5119	0.4964	1.5453	1.0413	0.0818	0.0251	0.2027	0.1341	0.0105	0.001	0.0723	0.031
50	1.5	0.5	1.5	1.5	1.5201	0.4962	1.5629	1.5678	0.0778	0.0258	0.2106	0.2053	0.0101	0.001	0.075	0.0737
50	1.5	0.5	1.5	2	1.516	0.4974	1.5642	2.0999	0.0741	0.0245	0.2135	0.2967	0.0088	0.0009	0.0771	0.1614
50	1.5	0.5	2	1	1.5162	0.4975	2.0883	1.0427	0.082	0.0242	0.2908	0.1296	0.0108	0.0009	0.1467	0.0289
50	1.5	0.5	2	1.5	1.514	0.4961	2.1021	1.5932	0.0759	0.0246	0.2941	0.2171	0.0092	0.0009	0.1501	0.085
50	1.5	0.5	2	2	1.5129	0.4966	2.0801	2.0923	0.0742	0.0235	0.2879	0.283	0.0088	0.0009	0.1433	0.1422
50	1.5	0.7	1	1	1.5149	0.6948	1.0342	1.0343	0.0861	0.0361	0.1305	0.1286	0.0115	0.002	0.0294	0.0284
50	1.5	0.7	1	1.5	1.5193	0.6935	1.0351	1.5778	0.0843	0.034	0.1314	0.2178	0.0113	0.0018	0.0304	0.0827
50	1.5	0.7	1	2	1.5188	0.6927	1.0378	2.1083	0.0785	0.0347	0.1326	0.3028	0.0099	0.0019	0.0298	0.1586
50	1.5	0.7	1.5	1	1.5204	0.695	1.547	1.0448	0.0816	0.036	0.2015	0.1354	0.0105	0.002	0.0708	0.0316
50	1.5	0.7	1.5	1.5	1.5117	0.6947	1.5663	1.5638	0.0775	0.0343	0.2113	0.2127	0.0095	0.0018	0.0766	0.08
50	1.5	0.7	1.5	2	1.5124	0.694	1.5551	2.0891	0.0765	0.0341	0.203	0.3042	0.0097	0.0018	0.0705	0.1648
50	1.5	0.7	2	1	1.5145	0.6929	2.0955	1.0447	0.0803	0.0351	0.303	0.1275	0.01	0.0019	0.1641	0.0278
50	1.5	0.7	2	1.5	1.5099	0.6929	2.1013	1.5583	0.0754	0.0352	0.3049	0.2111	0.009	0.0019	0.1614	0.0787
50	1.5	0.7	2	2	1.5111	0.6942	2.0835	2.099	0.0715	0.0329	0.2897	0.291	0.0084	0.0017	0.1453	0.1513
50	1.5	1	1	1	1.5214	0.9894	1.0494	1.0557	0.0815	0.0515	0.1344	0.1299	0.0112	0.0041	0.0314	0.0309
50	1.5	1	1	1.5	1.5116	0.9915	1.0326	1.5616	0.0818	0.0479	0.1291	0.2025	0.0108	0.0036	0.0267	0.075
50	1.5	1	1	2	1.5156	0.9924	1.0299	2.1147	0.0791	0.0504	0.1254	0.3049	0.0101	0.004	0.0261	0.1585
50	1.5	1	1.5	1	1.515	0.9918	1.56	1.0331	0.0775	0.0504	0.2066	0.1287	0.01	0.0039	0.0721	0.0274
50	1.5	1	1.5	1.5	1.5206	0.9938	1.5535	1.5658	0.0815	0.0492	0.2022	0.2067	0.0104	0.0038	0.0692	0.0729
50	1.5	1	1.5	2	1.5101	0.9933	1.5457	2.0914	0.0735	0.0492	0.2109	0.2805	0.0087	0.0037	0.0767	0.1392
50	1.5	1	2	1	1.5131	0.9937	2.0884	1.0412	0.082	0.0505	0.2935	0.1361	0.0107	0.0041	0.1482	0.0317
50	1.5	1	2	1.5	1.5079	0.9932	2.0776	1.5625	0.0717	0.0482	0.2755	0.2124	0.0085	0.0036	0.1284	0.0753
50	1.5	1	2	2	1.5086	0.9905	2.0892	2.1054	0.0727	0.0467	0.2893	0.294	0.0085	0.0035	0.1456	0.1478
50	1.5	2	1	1	1.5154	1.9786	1.0374	1.0383	0.0831	0.1027	0.1292	0.1348	0.0108	0.0168	0.0289	0.0312
50	1.5	2	1	1.5	1.5193	1.9805	1.0358	1.5872	0.0822	0.1043	0.136	0.2312	0.0106	0.0167	0.0306	0.0916
50	1.5	2	1	2	1.515	1.9795	1.0415	2.1138	0.0787	0.0993	0.1313	0.297	0.0103	0.0152	0.0291	0.1511
50	1.5	2	1.5	1	1.5139	1.9771	1.5613	1.0467	0.0838	0.0993	0.209	0.1323	0.0113	0.0158	0.0739	0.0309
50	1.5	2	1.5	1.5	1.5155	1.9836	1.5592	1.5715	0.0778	0.0958	0.2011	0.2147	0.0098	0.0142	0.0689	0.0789
50	1.5	2	1.5	2	1.5123	1.9803	1.5583	2.1148	0.0759	0.0965	0.1998	0.2913	0.0093	0.0144	0.0696	0.1496
50	1.5	2	2	1	1.5105	1.9804	2.0973	1.0419	0.0789	0.1016	0.3075	0.1346	0.0096	0.0159	0.1648	0.0313
50	1.5	2	2	1.5	1.5157	1.9873	2.0769	1.5705	0.0754	0.0963	0.2893	0.2111	0.0088	0.0147	0.1404	0.0795
50	1.5	2	2	2	1.5135	1.9834	2.0869	2.0957	0.0758	0.0988	0.2874	0.2906	0.009	0.015	0.1358	0.152
50	2	0.5	1	1	2.025	0.497	1.0331	1.0456	0.1121	0.0192	0.1269	0.1378	0.0197	0.0006	0.0275	0.0323
50	2	0.5	1	1.5	2.0184	0.4977	1.0279	1.5588	0.1041	0.0186	0.1341	0.2098	0.0178	0.0006	0.0293	0.08
50	2	0.5	1	2	2.0134	0.4967	1.0338	2.1003	0.0996	0.0183	0.1288	0.2976	0.0158	0.0005	0.028	0.156
50	2	0.5	1.5	1	2.0223	0.4963	1.5636	1.0285	0.1113	0.0195	0.2192	0.1313	0.0197	0.0006	0.0854	0.0301
50	2	0.5	1.5	1.5	2.0148	0.4975	1.5511	1.5597	0.0994	0.0192	0.2079	0.2126	0.0158	0.0006	0.0722	0.0797
50	2	0.5	1.5	2	2.0222	0.4961	1.5579	2.1214	0.1085	0.0184	0.2007	0.3121	0.0185	0.0005	0.0698	0.1623
50	2	0.5	2	1	2.0148	0.4973	2.0912	1.0352	0.105	0.0176	0.2831	0.131	0.0176	0.0005	0.1428	0.0297
50	2	0.5	2	1.5	2.0138	0.497	2.0865	1.5705	0.1012	0.0184	0.2919	0.2091	0.0163	0.0005	0.1415	0.078
50	2	0.5	2	2	2.0158	0.4972	2.0773	2.1211	0.0971	0.0172	0.2756	0.3005	0.0153	0.0005	0.1293	0.1557
50	2	0.7	1	1	2.0187	0.6943	1.0462	1.0351	0.1072	0.0276	0.1346	0.132	0.0185	0.0012	0.0315	0.0291
50	2	0.7	1	1.5	2.0193	0.6957	1.0372	1.5715	0.1018	0.0263	0.1341	0.2153	0.0173	0.0011	0.0313	0.0815
50	2	0.7	1	2	2.0182	0.6954	1.0359	2.1108	0.1045	0.0258	0.1245	0.295	0.017	0.001	0.027	0.1539
50	2	0.7	1.5	1	2.0308	0.6962	1.567	1.0466	0.1121	0.0254	0.2106	0.1367	0.0203	0.001	0.0732	0.0326
50	2	0.7	1.5	1.5	2.0167	0.6954	1.5625	1.559	0.1032	0.0261	0.209	0.2052	0.0172	0.001	0.078	0.0745
50	2	0.7	1.5	2	2.0188	0.6963	1.5583	2.1124	0.1013	0.0253	0.2204	0.3077	0.0162	0.001	0.0783	0.1661
50	2	0.7	2	1	2.0179	0.6978	2.0763	1.0375	0.103	0.0252	0.284	0.1322	0.0167	0.001	0.1509	0.0299
50	2	0.7	2	1.5	2.0216	0.696	2.1024	1.5755	0.0986	0.025	0.2877	0.2149	0.0164	0.001	0.1394	0.079
50	2	0.7	2	2	2.0188	0.6966	2.0867	2.1149	0.0983	0.0254	0.293	0.2952	0.0157	0.001	0.1522	0.1604
50	2	1	1	1	2.0203	0.9935	1.0323	1.0448	0.1105	0.0401	0.1267	0.1341	0.0197	0.0025	0.0278	0.031
50	2	1	1	1.5	2.0158	0.9951	1.0312	1.5555	0.1063	0.0373	0.129	0.2037	0.0181	0.0022	0.0272	0.0692
50	2	1	1	2	2.0165	0.9939	1.0186	2.0898	0.1066	0.037	0.1244	0.2801	0.0177	0.0021	0.0258	0.1434
50	2	1	1.5	1	2.0242	0.9942	1.5634	1.0327	0.1114	0.0384	0.2046	0.1344	0.0201	0.0023	0.0742	0.0306
50	2	1	1.5	1.5	2.0181	0.9952	1.5371	1.5745	0.1017	0.0373	0.2081	0.2118	0.0163	0.0022	0.0712	0.0763
50	2	1	1.5	2	2.0147	0.9933	1.5585	2.1206	0.1019	0.0381	0.1995	0.3049	0.0168	0.0022	0.0671	0.1626
50	2	1	2	1	2.0123	0.9938	2.0907	1.0306	0.1063	0.0366	0.2901	0.131	0.0179	0.002	0.1436	0.0293
50	2	1	2	1.5	2.0245	0.9942	2.093	1.5647	0.1002	0.0375	0.2923	0.21	0.0161	0.0022	0.1483	0.0749
50	2	1	2	2	2.0087	0.9971	2.0817	2.0914	0.0983	0.0357	0.2847	0.2955	0.0155	0.002	0.1385	0.1516
50	2	2	1	1	2.0192	1.9871	1.0295	1.033	0.1095	0.076	0.1304	0.1291	0.019	0.0092	0.0295	0.0297
50	2	2	1	1.5	2.0214	1.9898	1.0322	1.5646	0.1056	0.0723	0.1313	0.2108	0.0179	0.0084	0.0299	0.0811
50	2	2	1	2	2.0167	1.9836	1.0375	2.1223	0.1021	0.0788	0.1349	0.3135	0.0173	0.0098	0.0309	0.1749
50	2	2	1.5	1	2.0238	1.9965	1.5528	1.0331	0.1068	0.0721	0.2093	0.128	0.0183	0.0083	0.0734	0.0289
50	2	2	1.5	1.5	2.0159	1.9857	1.5642	1.5644	0.1023	0.0716	0.2124	0.2094	0.0167	0.0083	0.0773	0.0712
50	2	2	1.5	2	2.0197	1.9879	1.5591	2.1024	0.1031	0.0693	0.202	0.2964	0.0165	0.0077	0.0677	0.1602
50	2	2	2	1	2.0227	1.9887	2.0887	1.0379	0.1045	0.0719	0.2903	0.1392	0.0173	0.0084	0.1439	0.0326
50	2	2	2	1.5	2.0133	1.9911	2.0853	1.5666	0.0991	0.0747	0.2876	0.2182	0.0161	0.0088	0.1411	0.0817
50	2	2	2	2	2.0151	1.9819	2.117	2.1113	0.1004	0.0715	0.3016	0.2958	0.0158	0.0083	0.1586	0.1529

Table 3: *MLE's Bias and MSE's for* 2*-out-of-*4 *system with n=100*

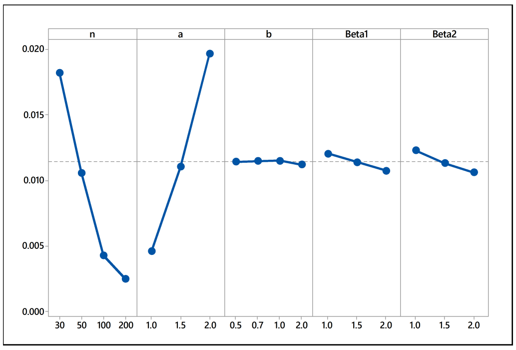
n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
100	1	0.5	1	1	1.0044	0.4961	1.019	1.0199	0.0384	0.0266	0.089	0.0893	0.0023	0.0011	0.0131	0.013
100	1	0.5	1	1.5	1.0037	0.4973	1.0171	1.5267	0.0369	0.0266	0.0917	0.1416	0.0021	0.0011	0.0138	0.033
100	1	0.5	1	2	1.0027	0.4974	1.0165	2.0491	0.0362	0.0257	0.0904	0.1946	0.0021	0.001	0.0135	0.063
100	1	0.5	1.5	1	1.0066	0.4971	1.5287	1.0194	0.0373	0.0268	0.1406	0.0868	0.0022	0.0011	0.0319	0.0122
100	1	0.5	1.5	1.5	1.0024	0.4969	1.5289	1.5294	0.0363	0.0255	0.1377	0.1453	0.0021	0.001	0.0313	0.0348
100	1	0.5	1.5	2	1.0024	0.4986	1.5277	2.0465	0.0345	0.026	0.1458	0.2003	0.0019	0.0011	0.034	0.067
100	1	0.5	2	1	1.0052	0.4984	2.0357	1.0181	0.037	0.0244	0.1977	0.0919	0.0022	0.0009	0.0647	0.0135
100	1	0.5	2	1.5	1.0052	0.5009	2.0343	1.5121	0.0369	0.0253	0.1944	0.1392	0.0021	0.001	0.0609	0.0307
100	1	0.5	2	2	1.0042	0.4962	2.0524	2.0673	0.0355	0.0248	0.1989	0.2018	0.002	0.001	0.0648	0.069
100	1	0.7	1	1	1.0046	0.6969	1.017	1.0186	0.0382	0.0382	0.0891	0.0906	0.0023	0.0023	0.0127	0.0138
100	1	0.7	1	1.5	1.0042	0.6967	1.0152	1.5289	0.0361	0.0366	0.0885	0.141	0.002	0.0021	0.0135	0.0331
100	1	0.7	1	2	1.0069	0.6971	1.012	2.0511	0.0366	0.0369	0.0857	0.2052	0.0021	0.0021	0.0115	0.07
100	1	0.7	1.5	1	1.0056	0.6967	1.5262	1.0187	0.0373	0.0381	0.1428	0.0892	0.0022	0.0023	0.0332	0.0131
100	1	0.7	1.5	1.5	1.0072	0.6938	1.5352	1.5361	0.0366	0.0376	0.1429	0.1446	0.0022	0.0022	0.034	0.0334
100	1	0.7	1.5	2	1.0036	0.6957	1.5298	2.055	0.0353	0.0356	0.1346	0.1964	0.002	0.002	0.0306	0.0659
100	1	0.7	2	1	1.0049	0.6977	2.0355	1.0158	0.0375	0.0372	0.1961	0.0885	0.0022	0.0022	0.0648	0.0131
100	1	0.7	2	1.5	1.0029	0.6978	2.056	1.5329	0.0345	0.0358	0.2051	0.1377	0.002	0.002	0.0694	0.0323
100	1	0.7	2	2	1.0036	0.6993	2.0392	2.0338	0.035	0.0357	0.1912	0.1983	0.0019	0.002	0.0601	0.0634
100	1	1	1	1	1.006	0.9934	1.0127	1.0244	0.0397	0.0544	0.0876	0.0903	0.0025	0.0046	0.0122	0.0134
100	1	1	1	1.5	1.0052	0.9949	1.0117	1.5402	0.0379	0.0532	0.0886	0.1473	0.0023	0.0046	0.0127	0.0376
100	1	1	1	2	1.004	0.9943	1.0189	2.0531	0.0348	0.051	0.0877	0.2018	0.002	0.004	0.0127	0.0661
100	1	1	1.5	1	1.0042	0.9948	1.5375	1.0241	0.0379	0.0543	0.1461	0.0904	0.0023	0.0046	0.0356	0.0133
100	1	1	1.5	1.5	1.0046	0.9979	1.5177	1.5281	0.0363	0.0525	0.1427	0.1394	0.002	0.0043	0.0342	0.0326
100	1	1	1.5	2	1.0044	0.9981	1.5142	2.0407	0.035	0.0526	0.1434	0.1971	0.0019	0.0043	0.0341	0.0633
100	1	1	2	1	1.0041	0.9982	2.042	1.0192	0.0373	0.0517	0.1971	0.0902	0.0022	0.0041	0.064	0.0134
100	1	1	2	1.5	1.0052	0.9944	2.0545	1.5383	0.0358	0.0508	0.2045	0.1384	0.002	0.004	0.0684	0.0319
100	1	1	2	2	1.0035	0.9937	2.0391	2.0546	0.0332	0.0514	0.2028	0.2072	0.0017	0.0042	0.0671	0.0713
100	1	2	1	1	1.0083	1.9845	1.0233	1.0236	0.0391	0.111	0.0934	0.0931	0.0025	0.0193	0.0147	0.0144
100	1	2	1	1.5	1.0037	1.9841	1.0181	1.5355	0.0352	0.1049	0.0897	0.1459	0.002	0.0175	0.0134	0.0352
100	1	2	1	2	1.0052	1.9914	1.0134	2.0471	0.0368	0.1053	0.092	0.2053	0.0021	0.0175	0.0138	0.0731
100	1	2	1.5	1	1.005	1.9951	1.5276	1.0174	0.0389	0.1051	0.138	0.089	0.0024	0.0173	0.031	0.013
100	1	2	1.5	1.5	1.0036	1.9884	1.5306	1.5302	0.035	0.102	0.1382	0.1466	0.002	0.0164	0.0319	0.0354
100	1	2	1.5	2	1.0044	1.9939	1.5344	2.0389	0.0355	0.1033	0.1425	0.1874	0.002	0.0166	0.034	0.0572
100	1	2	2	1	1.0069	1.9917	2.0367	1.0205	0.0368	0.1042	0.1978	0.0892	0.0021	0.0175	0.0642	0.0132
100	1	2	2	1.5	1.0052	1.9941	2.0546	1.528	0.0343	0.1055	0.2003	0.1421	0.0019	0.0171	0.0688	0.033
100	1	2	2	2	1.0063	1.9906	2.0382	2.0515	0.0344	0.101	0.1995	0.1929	0.0019	0.0159	0.0653	0.0647
100	1.5	0.5	1	1	1.5084	0.4974	1.0138	1.0204	0.056	0.0172	0.0881	0.0939	0.005	0.0005	0.0126	0.0141
100	1.5	0.5	1	1.5	1.5074	0.498	1.015	1.534	0.056	0.0173	0.0881	0.1401	0.0049	0.0005	0.0125	0.0315
100	1.5	0.5	1	2	1.5083	0.4987	1.0183	2.0514	0.0545	0.0172	0.0883	0.1969	0.0048	0.0005	0.0128	0.0659
100	1.5	0.5	1.5	1	1.5122	0.4983	1.532	1.0214	0.0571	0.0177	0.1411	0.0916	0.0052	0.0005	0.0335	0.0138
100	1.5	0.5	1.5	1.5	1.5055	0.4978	1.5322	1.5367	0.057	0.0179	0.1402	0.1463	0.0051	0.0005	0.0316	0.0345
100	1.5	0.5	1.5	2	1.5099	0.4974	1.5321	2.0439	0.053	0.0172	0.1432	0.1989	0.0045	0.0005	0.0334	0.0675
100	1.5	0.5	2	1	1.5108	0.4982	2.0424	1.0207	0.0554	0.0176	0.1963	0.092	0.005	0.0005	0.0655	0.0137
100	1.5	0.5	2	1.5	1.51	0.5	2.0346	1.5222	0.053	0.0163	0.1921	0.1382	0.0046	0.0004	0.0619	0.0318
100	1.5	0.5	2	2	1.5041	0.4973	2.051	2.0532	0.0533	0.0166	0.1948	0.2024	0.0044	0.0004	0.0622	0.0694
100	1.5	0.7	1	1	1.5058	0.6972	1.0194	1.0193	0.0589	0.0255	0.0893	0.0907	0.0056	0.001	0.0134	0.0133
100	1.5	0.7	1	1.5	1.5062	0.6965	1.0161	1.5375	0.0549	0.0248	0.0879	0.1433	0.0047	0.001	0.0124	0.0351
100	1.5	0.7	1	2	1.5052	0.6979	1.0151	2.0436	0.0531	0.0248	0.0885	0.2023	0.0045	0.001	0.0127	0.0662
100	1.5	0.7	1.5	1	1.5086	0.6971	1.5275	1.0162	0.0556	0.0248	0.1413	0.0858	0.005	0.001	0.0321	0.0122
100	1.5	0.7	1.5	1.5	1.5029	0.6956	1.5433	1.5349	0.0543	0.0254	0.149	0.1483	0.0045	0.001	0.037	0.0365
100	1.5	0.7	1.5	2	1.5075	0.6976	1.5269	2.042	0.0527	0.0241	0.1445	0.1921	0.0044	0.0009	0.0335	0.0625
100	1.5	0.7	2	1	1.5056	0.6969	2.0374	1.0249	0.0524	0.0249	0.1911	0.0931	0.0045	0.001	0.062	0.0147
100	1.5	0.7	2	1.5	1.5041	0.6966	2.0505	1.5324	0.0529	0.0234	0.1951	0.1369	0.0045	0.0008	0.0632	0.0308
100	1.5	0.7	2	2	1.5058	0.697	2.0354	2.0473	0.0517	0.0229	0.1881	0.1955	0.0044	0.0008	0.0572	0.061
100	1.5	1	1	1	1.508	0.9958	1.0204	1.0119	0.058	0.0383	0.0914	0.0876	0.0053	0.0023	0.0138	0.0124
100	1.5	1	1	1.5	1.5124	0.9984	1.0177	1.534	0.0565	0.0374	0.0928	0.1438	0.005	0.0022	0.0142	0.0338
100	1.5	1	1	2	1.5033	0.9941	1.0216	2.0522	0.0533	0.0359	0.0898	0.1963	0.0046	0.0019	0.0135	0.0637
100	1.5	1	1.5	1	1.5068	0.9977	1.5296	1.0188	0.0563	0.0357	0.1428	0.0903	0.0051	0.002	0.0333	0.0127
100	1.5	1	1.5	1.5	1.508	0.9969	1.5257	1.5383	0.0543	0.0346	0.1432	0.1492	0.0048	0.0019	0.0332	0.0365
100	1.5	1	1.5	2	1.5081	0.996	1.5329	2.0603	0.0525	0.0334	0.1398	0.2053	0.0044	0.0017	0.0313	0.0714
100	1.5	1	2	1	1.5026	0.9973	2.0422	1.0135	0.0534	0.0341	0.198	0.0897	0.0046	0.0018	0.0633	0.013
100	1.5	1	2	1.5	1.5065	0.9927	2.0546	1.5428	0.0545	0.0343	0.1979	0.1417	0.0047	0.0018	0.0653	0.0342
100	1.5	1	2	2	1.507	0.9971	2.0402	2.0543	0.0533	0.0335	0.2013	0.2121	0.0043	0.0018	0.0659	0.0734
100	1.5	2	1	1	1.5089	1.9884	1.0215	1.0222	0.0582	0.0727	0.0903	0.091	0.0053	0.0084	0.013	0.0141
100	1.5	2	1	1.5	1.5093	1.9923	1.0162	1.5404	0.0575	0.0717	0.0912	0.1444	0.0052	0.0082	0.0137	0.0356
100	1.5	2	1	2	1.51	1.9924	1.0163	2.0383	0.0549	0.0701	0.0887	0.1991	0.0047	0.008	0.0127	0.0652
100	1.5	2	1.5	1	1.5079	1.9908	1.5317	1.0182	0.0554	0.0694	0.1402	0.0974	0.005	0.0076	0.0318	0.0154
100	1.5	2	1.5	1.5	1.5063	1.9885	1.5387	1.5344	0.0561	0.0681	0.1382	0.1353	0.0048	0.0073	0.0327	0.0304
100	1.5	2	1.5	2	1.5079	1.9883	1.5328	2.0683	0.0526	0.0687	0.1458	0.206	0.0044	0.0073	0.0341	0.0692
100	1.5	2	2	1	1.5074	1.9913	2.0529	1.0206	0.0564	0.0683	0.1963	0.0919	0.0051	0.0073	0.0654	0.0135
100	1.5	2	2	1.5	1.5079	1.9955	2.0392	1.527	0.0534	0.0694	0.1973	0.1408	0.0047	0.0074	0.0646	0.0327
100	1.5	2	2	2	1.5076	1.9938	2.0398	2.0397	0.0518	0.0662	0.1958	0.2042	0.0042	0.007	0.0649	0.0679
100	2	0.5	1	1	2.0141	0.4981	1.0176	1.02	0.0793	0.0129	0.0911	0.0878	0.0101	0.0003	0.0135	0.0131
100	2	0.5	1	1.5	2.0077	0.4977	1.0197	1.5334	0.074	0.0135	0.0865	0.1436	0.0087	0.0003	0.0122	0.0337
100	2	0.5	1	2	2.0084	0.4986	1.0156	2.0431	0.0733	0.0127	0.0907	0.2034	0.0084	0.0003	0.0137	0.0667
100	2	0.5	1.5	1	2.0169	0.4983	1.5341	1.0237	0.0749	0.0134	0.145	0.089	0.009	0.0003	0.0352	0.0131
100	2	0.5	1.5	1.5	2.0065	0.4973	1.5277	1.533	0.0729	0.0126	0.1452	0.1429	0.0085	0.0002	0.035	0.0337
100	2	0.5	1.5	2	2.0103	0.4986	1.528	2.045	0.0722	0.0126	0.1424	0.2048	0.0081	0.0002	0.0343	0.0697
100	2	0.5	2	1	2.0089	0.4977	2.0321	1.0244	0.0719	0.0132	0.1961	0.0938	0.0082	0.0003	0.0625	0.0146
100	2	0.5	2	1.5	2.0079	0.4983	2.0534	1.5229	0.0717	0.0133	0.1993	0.141	0.0081	0.0003	0.0638	0.0327
100	2	0.5	2	2	2.0097	0.4982	2.0336	2.0464	0.0663	0.0134	0.1928	0.2018	0.0069	0.0003	0.0621	0.0654
100	2	0.7	1	1	2.01	0.6978	1.0153	1.0186	0.0778	0.0192	0.0928	0.0932	0.0097	0.0006	0.0142	0.0145
100	2	0.7	1	1.5	2.0043	0.6982	1.014	1.5226	0.0731	0.0175	0.0913	0.1345	0.0085	0.0005	0.0138	0.0293
100	2	0.7	1	2	2.0095	0.7003	1.0153	2.0401	0.0738	0.0183	0.0855	0.1979	0.0081	0.0005	0.0123	0.0645
100	2	0.7	1.5	1	2.006	0.6968	1.5426	1.0249	0.0727	0.0185	0.1466	0.0906	0.0087	0.0005	0.0345	0.0136
100	2	0.7	1.5	1.5	2.0061	0.6973	1.5409	1.5424	0.0698	0.0182	0.1462	0.1486	0.0077	0.0005	0.0358	0.037
100	2	0.7	1.5	2	2.0102	0.6986	1.5165	2.0377	0.0721	0.0172	0.1356	0.2004	0.0086	0.0005	0.0296	0.0673
100	2	0.7	2	1	2.0079	0.6983	2.0437	1.0188	0.0743	0.0176	0.19	0.0924	0.0087	0.0005	0.059	0.0139
100	2	0.7	2	1.5	2.0054	0.6976	2.0369	1.5228	0.0693	0.0176	0.1892	0.1408	0.0077	0.0005	0.0576	0.0322
100	2	0.7	2	2	2.0115	0.6986	2.0441	2.0439	0.0682	0.0176	0.1936	0.1982	0.0074	0.0005	0.0608	0.0645
100	2	1	1	1	2.0089	0.9964	1.0171	1.0196	0.0803	0.0272	0.0906	0.0905	0.0102	0.0012	0.0136	0.0133
100	2	1	1	1.5	2.008	0.9964	1.0172	1.5272	0.0738	0.0262	0.088	0.1422	0.0086	0.0011	0.0127	0.0349
100	2	1	1	2	2.0103	0.9969	1.0189	2.0424	0.0755	0.027	0.0906	0.2073	0.0087	0.0011	0.0134	0.068
100	2	1	1.5	1	2.0031	0.9953	1.5348	1.029	0.0743	0.0269	0.1429	0.0896	0.0089	0.0011	0.033	0.0136
100	2	1	1.5	1.5	2.0064	0.9977	1.5238	1.5306	0.0728	0.0252	0.1405	0.1368	0.0083	0.001	0.0318	0.0316
100	2	1	1.5	2	2.0086	0.9968	1.5263	2.0525	0.0696	0.0253	0.1372	0.2008	0.0078	0.001	0.0319	0.0676
100	2	1	2	1	2.0108	0.999	2.0305	1.0136	0.0724	0.0253	0.1974	0.0868	0.0084	0.001	0.0633	0.0121
100	2	1	2	1.5	2.0055	0.9948	2.0442	1.5397	0.0681	0.0257	0.2014	0.1419	0.0075	0.001	0.067	0.0319
100	2	1	2	2	2.0083	0.9994	2.0303	2.0373	0.0691	0.0244	0.1893	0.1894	0.0072	0.001	0.0572	0.0581
100	2	2	1	1	2.007	1.9939	1.024	1.0144	0.0785	0.0538	0.0957	0.0873	0.0094	0.0045	0.0151	0.0124
100	2	2	1	1.5	2.005	1.9916	1.0117	1.5323	0.0755	0.0558	0.0879	0.1444	0.0089	0.0048	0.0123	0.0353
100	2	2	1	2	2.0071	1.9969	1.0151	2.041	0.0692	0.0494	0.085	0.1913	0.0079	0.0038	0.0116	0.0604
100	2	2	1.5	1	2.0135	1.9936	1.5266	1.0227	0.0762	0.0523	0.1312	0.0884	0.0092	0.0042	0.028	0.013
100	2	2	1.5	1.5	2.006	1.9939	1.5276	1.535	0.0717	0.0516	0.1448	0.1453	0.0082	0.0041	0.0333	0.0341
100	2	2	1.5	2	2.0068	1.9928	1.5265	2.0507	0.0666	0.0501	0.1401	0.2026	0.0071	0.0039	0.0322	0.0664
100	2	2	2	1	2.0091	1.9951	2.0468	1.0185	0.0738	0.0528	0.2041	0.0912	0.0084	0.0045	0.0707	0.0135
100	2	2	2	1.5	2.0059	1.9921	2.0596	1.5272	0.0751	0.0519	0.1969	0.136	0.0087	0.0041	0.0676	0.0303
100	2	2	2	2	2.0058	1.9966	2.0459	2.0429	0.0686	0.0505	0.1928	0.1952	0.0074	0.0039	0.0612	0.0631

Table 4: *MLE's Bias and MSE's for 2-out-of-4 system with n=200*

n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\beta_1}$)	Bias($\hat{\beta_2}$)	MSE($\hat{a}$)	MSE($\hat{b}$)	MSE($\hat{\beta_1}$)	MSE($\hat{\beta_2}$)
200	1	0.5	1	1	1.0019	0.4982	1.0096	1.0066	0.0278	0.0191	0.0616	0.062	0.0012	0.0006	0.0062	0.0063
200	1	0.5	1	1.5	1.0048	0.4996	1.0062	1.5151	0.0279	0.019	0.0631	0.1	0.0012	0.0006	0.0064	0.0157
200	1	0.5	1	2	1.0008	0.4982	1.0095	2.0232	0.0255	0.0179	0.0637	0.1363	0.001	0.0005	0.0064	0.0316
200	1	0.5	1.5	1	1.0022	0.4985	1.5094	1.011	0.0268	0.0191	0.0981	0.0626	0.0011	0.0006	0.015	0.0062
200	1	0.5	1.5	1.5	1	0.4987	1.512	1.5104	0.0252	0.0183	0.0971	0.0969	0.001	0.0005	0.0149	0.0153
200	1	0.5	1.5	2	1.0031	0.499	1.5127	2.0196	0.0245	0.0178	0.0982	0.1408	0.0009	0.0005	0.0153	0.0318
200	1	0.5	2	1	1.0034	0.4996	2.0098	1.0098	0.0261	0.0189	0.1372	0.0649	0.0011	0.0006	0.0306	0.0068
200	1	0.5	2	1.5	1.0013	0.4986	2.019	1.5198	0.0249	0.0177	0.133	0.1024	0.001	0.0005	0.028	0.0165
200	1	0.5	2	2	1.0009	0.4993	2.0244	2.0149	0.024	0.0182	0.1422	0.1311	0.0009	0.0005	0.0316	0.0282
200	1	0.7	1	1	1.0024	0.6975	1.0076	1.0108	0.0272	0.0272	0.0629	0.0631	0.0012	0.0012	0.0063	0.0065
200	1	0.7	1	1.5	1.0035	0.6992	1.0064	1.5128	0.0262	0.0276	0.0638	0.0991	0.0011	0.0012	0.0066	0.0156
200	1	0.7	1	2	1.0016	0.6981	1.0068	2.0196	0.0261	0.0252	0.0641	0.1421	0.0011	0.001	0.0064	0.0322
200	1	0.7	1.5	1	1.0011	0.6989	1.5137	1.0095	0.0264	0.0263	0.0989	0.0621	0.0011	0.0011	0.0157	0.0061
200	1	0.7	1.5	1.5	1.0019	0.6982	1.5185	1.5191	0.0253	0.0269	0.0981	0.1014	0.001	0.0011	0.015	0.0168
200	1	0.7	1.5	2	1.0028	0.6985	1.5105	2.0229	0.0248	0.0248	0.0952	0.1394	0.001	0.001	0.0146	0.0306
200	1	0.7	2	1	1.0037	0.6988	2.0135	1.0096	0.0265	0.0254	0.1332	0.0636	0.0011	0.001	0.0284	0.0066
200	1	0.7	2	1.5	1.0026	0.6974	2.0215	1.5158	0.0245	0.0259	0.135	0.0978	0.0009	0.001	0.0296	0.0154
200	1	0.7	2	2	1.0007	0.6991	2.0202	2.0166	0.0253	0.0263	0.1356	0.1332	0.001	0.0011	0.0297	0.0292
200	1	1	1	1	1.0026	0.9974	1.0072	1.0103	0.0271	0.0383	0.0608	0.063	0.0012	0.0023	0.0061	0.0062
200	1	1	1	1.5	1.0021	0.9956	1.0139	1.5282	0.0265	0.0387	0.0618	0.1002	0.0011	0.0024	0.0063	0.0163
200	1	1	1	2	1.0018	0.9954	1.0107	2.0352	0.0265	0.036	0.0626	0.1382	0.0011	0.0021	0.0063	0.032
200	1	1	1.5	1	1.0027	0.994	1.5196	1.0107	0.0271	0.0374	0.1033	0.061	0.0012	0.0022	0.0169	0.006
200	1	1	1.5	1.5	1.0038	0.9984	1.5109	1.5169	0.027	0.0352	0.0996	0.0988	0.0011	0.002	0.0156	0.016
200	1	1	1.5	2	1.0014	0.998	1.5124	2.015	0.025	0.037	0.1017	0.1396	0.001	0.0021	0.0164	0.0323
200	1	1	2	1	1.0023	0.9984	2.0106	1.0072	0.0259	0.0357	0.1371	0.0621	0.0011	0.002	0.0297	0.0061
200	1	1	2	1.5	1.0022	0.9991	2.0154	1.5085	0.0251	0.0365	0.1388	0.0975	0.001	0.0021	0.0308	0.0156
200	1	1	2	2	1.0013	0.9973	2.0261	2.0268	0.0243	0.035	0.1364	0.1367	0.0009	0.002	0.03	0.0302
200	1	2	1	1	1.004	1.9956	1.0095	1.0094	0.0277	0.0789	0.0632	0.065	0.0012	0.0097	0.0063	0.0069
200	1	2	1	1.5	1.0034	1.9946	1.0065	1.5194	0.0255	0.0742	0.0626	0.0984	0.001	0.0088	0.0061	0.0164
200	1	2	1	2	1.0005	1.9899	1.0057	2.0317	0.0254	0.0738	0.0602	0.1385	0.001	0.0086	0.0056	0.0312
200	1	2	1.5	1	1.0043	1.9973	1.5143	1.0122	0.0258	0.0748	0.1018	0.0613	0.001	0.0088	0.0169	0.006
200	1	2	1.5	1.5	1.0043	1.9974	1.5121	1.5184	0.0251	0.0763	0.0996	0.1005	0.001	0.0092	0.0161	0.0163
200	1	2	1.5	2	1.0034	2.001	1.5053	2.0175	0.0245	0.0764	0.099	0.1356	0.001	0.009	0.0157	0.0297
200	1	2	2	1	1.0015	1.9973	2.0171	1.0105	0.025	0.0708	0.1322	0.0631	0.001	0.0081	0.0288	0.0063
200	1	2	2	1.5	1.0009	1.9963	2.0176	1.5114	0.0255	0.0745	0.1364	0.098	0.001	0.0085	0.0303	0.0148
200	1	2	2	2	1.002	1.9915	2.0262	2.0187	0.0252	0.0694	0.1347	0.1346	0.001	0.0077	0.0289	0.0305
200	1.5	0.5	1	1	1.5056	0.4992	1.0053	1.0091	0.0418	0.0123	0.0616	0.0627	0.0027	0.0002	0.0058	0.0064
200	1.5	0.5	1	1.5	1.5009	0.5002	1.0085	1.5077	0.0374	0.0127	0.0595	0.102	0.0022	0.0003	0.0057	0.0163
200	1.5	0.5	1	2	1.5058	0.499	1.0099	2.0262	0.0395	0.0123	0.0618	0.1436	0.0024	0.0002	0.0061	0.0325
200	1.5	0.5	1.5	1	1.505	0.4999	1.5109	1.0086	0.0416	0.0123	0.1015	0.0672	0.0026	0.0002	0.016	0.0071
200	1.5	0.5	1.5	1.5	1.503	0.4994	1.512	1.5208	0.0391	0.0121	0.0992	0.0962	0.0024	0.0002	0.0157	0.0153
200	1.5	0.5	1.5	2	1.5027	0.4999	1.5143	2.0128	0.0354	0.0115	0.1	0.1403	0.002	0.0002	0.0159	0.0317
200	1.5	0.5	2	1	1.5034	0.4994	2.0198	1.0096	0.0394	0.0127	0.1404	0.0642	0.0025	0.0002	0.0305	0.0064
200	1.5	0.5	2	1.5	1.5033	0.4992	2.0186	1.5211	0.0374	0.0124	0.1407	0.0997	0.0023	0.0002	0.0307	0.017
200	1.5	0.5	2	2	1.5042	0.5003	2.0139	2.0168	0.0359	0.0121	0.1321	0.1324	0.002	0.0002	0.0282	0.028
200	1.5	0.7	1	1	1.5026	0.699	1.0051	1.0051	0.0411	0.0182	0.0647	0.0618	0.0028	0.0005	0.0066	0.006
200	1.5	0.7	1	1.5	1.502	0.6982	1.0112	1.5224	0.0395	0.0174	0.0631	0.0968	0.0024	0.0005	0.0065	0.0155
200	1.5	0.7	1	2	1.5033	0.6993	1.0088	2.0169	0.0382	0.0174	0.0639	0.1279	0.0023	0.0005	0.0064	0.0266
200	1.5	0.7	1.5	1	1.5045	0.6995	1.5062	1.0094	0.0413	0.017	0.1003	0.0634	0.0027	0.0005	0.0155	0.0063
200	1.5	0.7	1.5	1.5	1.503	0.6981	1.5139	1.5115	0.0374	0.0171	0.0964	0.0995	0.0022	0.0005	0.0149	0.0158
200	1.5	0.7	1.5	2	1.5007	0.6971	1.5184	2.0358	0.0379	0.0172	0.101	0.1344	0.0022	0.0005	0.0164	0.0295
200	1.5	0.7	2	1	1.5036	0.6984	2.0257	1.0058	0.0385	0.0166	0.1413	0.0629	0.0024	0.0004	0.0317	0.0064
200	1.5	0.7	2	1.5	1.5028	0.6986	2.0217	1.5143	0.0398	0.0167	0.1333	0.0971	0.0025	0.0004	0.0285	0.015
200	1.5	0.7	2	2	1.5033	0.6988	2.0145	2.0163	0.0365	0.0173	0.1379	0.1433	0.0022	0.0005	0.0306	0.0336
200	1.5	1	1	1	1.5035	0.9984	1.0074	1.007	0.041	0.0253	0.0612	0.0612	0.0027	0.001	0.0061	0.006
200	1.5	1	1	1.5	1.5015	0.9994	1.0069	1.5124	0.0381	0.0242	0.0605	0.0991	0.0023	0.0009	0.0057	0.0155
200	1.5	1	1	2	1.5024	0.9984	1.0034	2.0262	0.0384	0.0241	0.0603	0.1381	0.0022	0.0009	0.0059	0.0311
200	1.5	1	1.5	1	1.5046	0.9973	1.5193	1.0103	0.0391	0.0256	0.0989	0.0643	0.0024	0.001	0.0156	0.0066
200	1.5	1	1.5	1.5	1.5016	0.9999	1.5145	1.5096	0.0393	0.0251	0.1005	0.0986	0.0024	0.001	0.0164	0.0154
200	1.5	1	1.5	2	1.5027	0.9991	1.5119	2.0197	0.0374	0.0242	0.0946	0.1393	0.0021	0.0009	0.0145	0.0315
200	1.5	1	2	1	1.5032	0.9987	2.0159	1.0099	0.0387	0.0243	0.1407	0.0612	0.0025	0.0009	0.0314	0.006
200	1.5	1	2	1.5	1.5032	0.9968	2.0193	1.5122	0.0386	0.0247	0.1348	0.1013	0.0023	0.0009	0.029	0.0166
200	1.5	1	2	2	1.5053	0.9986	2.0253	2.0268	0.0364	0.0238	0.1455	0.1334	0.0021	0.0009	0.0337	0.0284
200	1.5	2	1	1	1.5049	1.9978	1.0084	1.0126	0.0407	0.0514	0.0611	0.0649	0.0026	0.0042	0.0061	0.0067
200	1.5	2	1	1.5	1.5019	1.9955	1.0077	1.5205	0.0407	0.0509	0.0622	0.1018	0.0025	0.004	0.0063	0.0166
200	1.5	2	1	2	1.5036	1.9971	1.0069	2.0193	0.0369	0.0484	0.0631	0.142	0.0021	0.0037	0.0062	0.0327
200	1.5	2	1.5	1	1.5037	1.9974	1.5135	1.0117	0.0375	0.0504	0.0988	0.0632	0.0023	0.004	0.016	0.0065
200	1.5	2	1.5	1.5	1.5043	1.9974	1.5161	1.5141	0.0397	0.0483	0.0991	0.0967	0.0025	0.0037	0.0158	0.0149
200	1.5	2	1.5	2	1.5028	1.9978	1.5091	2.0191	0.0388	0.0509	0.098	0.1445	0.0023	0.0039	0.0155	0.0324
200	1.5	2	2	1	1.503	1.9972	2.0199	1.0102	0.0382	0.0512	0.1408	0.0656	0.0023	0.0041	0.0319	0.007
200	1.5	2	2	1.5	1.5038	1.998	2.0252	1.5157	0.0373	0.0467	0.1399	0.0976	0.0022	0.0035	0.0314	0.0153
200	1.5	2	2	2	1.5046	1.9946	2.0122	2.0302	0.0378	0.0496	0.1382	0.1402	0.0022	0.0039	0.0304	0.0319
200	2	0.5	1	1	2.01	0.4986	1.0122	1.0118	0.0579	0.0096	0.065	0.0637	0.0052	0.0001	0.0069	0.0067
200	2	0.5	1	1.5	2.0073	0.4991	1.0115	1.5159	0.0516	0.0091	0.0637	0.1004	0.0043	0.0001	0.0066	0.0159
200	2	0.5	1	2	2.0092	0.4993	1.0045	2.0287	0.0509	0.0093	0.0629	0.1453	0.0042	0.0001	0.0062	0.0345
200	2	0.5	1.5	1	2.0012	0.4991	1.52	1.012	0.0549	0.0091	0.0988	0.0602	0.0047	0.0001	0.0157	0.0058
200	2	0.5	1.5	1.5	2.0003	0.4997	1.5111	1.5048	0.0512	0.0092	0.0977	0.0915	0.0041	0.0001	0.0157	0.0138
200	2	0.5	1.5	2	2.0035	0.4999	1.5126	2.0175	0.0486	0.0095	0.0978	0.1391	0.0038	0.0001	0.015	0.0316
200	2	0.5	2	1	2.0037	0.4993	2.0246	1.0109	0.0516	0.0094	0.1402	0.0623	0.0042	0.0001	0.0313	0.0063
200	2	0.5	2	1.5	2.0094	0.4993	2.0184	1.5125	0.0508	0.0089	0.1369	0.0998	0.004	0.0001	0.0307	0.0156
200	2	0.5	2	2	2.0016	0.499	2.0237	2.0294	0.0482	0.0091	0.1405	0.1352	0.0037	0.0001	0.0319	0.0301
200	2	0.7	1	1	2.0049	0.6995	1.0044	1.0081	0.055	0.0131	0.0609	0.0608	0.0047	0.0003	0.0059	0.0061
200	2	0.7	1	1.5	2.0037	0.6985	1.0078	1.5173	0.0526	0.0135	0.0645	0.0989	0.0044	0.0003	0.0068	0.0163
200	2	0.7	1	2	2.0014	0.6983	1.0111	2.0267	0.0506	0.0132	0.0652	0.1429	0.0041	0.0003	0.0068	0.0331
200	2	0.7	1.5	1	2.0047	0.6988	1.5087	1.0087	0.0527	0.0131	0.0964	0.0627	0.0045	0.0003	0.0151	0.0063
200	2	0.7	1.5	1.5	2.0072	0.699	1.5174	1.5104	0.0536	0.0127	0.1	0.0941	0.0044	0.0002	0.0162	0.0142
200	2	0.7	1.5	2	2.0023	0.6988	1.5131	2.0248	0.0491	0.0129	0.0973	0.1428	0.0038	0.0003	0.0154	0.0319
200	2	0.7	2	1	2.0041	0.6993	2.0308	1.008	0.0524	0.0127	0.1384	0.0577	0.0043	0.0003	0.031	0.0055
200	2	0.7	2	1.5	2.0048	0.6996	2.0201	1.5219	0.0504	0.013	0.1456	0.1005	0.0041	0.0003	0.0341	0.0164
200	2	0.7	2	2	2.004	0.699	2.0207	2.0247	0.0476	0.0121	0.136	0.1361	0.0035	0.0002	0.0292	0.0299
200	2	1	1	1	2.0087	0.9977	1.0098	1.0078	0.0568	0.0195	0.0626	0.0617	0.005	0.0006	0.0062	0.0061
200	2	1	1	1.5	2.0069	0.9986	1.0121	1.522	0.0543	0.019	0.0641	0.0999	0.0046	0.0006	0.0065	0.0162
200	2	1	1	2	2.0065	0.9995	1.0059	2.0222	0.0502	0.0183	0.0609	0.1306	0.004	0.0005	0.006	0.0279
200	2	1	1.5	1	2.0036	0.9978	1.5105	1.0115	0.0538	0.0186	0.099	0.0627	0.0045	0.0006	0.016	0.0063
200	2	1	1.5	1.5	2.0022	0.9992	1.5109	1.5208	0.0511	0.0186	0.097	0.1027	0.0041	0.0006	0.0153	0.0168
200	2	1	1.5	2	2.0021	0.9986	1.5168	2.0172	0.048	0.0182	0.098	0.1391	0.0037	0.0005	0.0154	0.0314
200	2	1	2	1	2.0017	0.9977	2.0237	1.0111	0.0512	0.0179	0.1393	0.0633	0.0041	0.0005	0.0311	0.0064
200	2	1	2	1.5	2.0039	0.9984	2.0227	1.5191	0.0515	0.0183	0.1431	0.1	0.0041	0.0005	0.0323	0.0168
200	2	1	2	2	2.0023	0.9987	2.0221	2.0268	0.0478	0.0177	0.1371	0.1377	0.0036	0.0005	0.0299	0.0306
200	2	2	1	1	2.0045	1.9973	1.0047	1.008	0.0574	0.0374	0.0623	0.0618	0.0051	0.0023	0.0063	0.0061
200	2	2	1	1.5	2.0062	1.9942	1.0054	1.5211	0.052	0.0361	0.0594	0.0975	0.0042	0.002	0.0056	0.0161
200	2	2	1	2	2.0078	1.9969	1.008	2.0209	0.0505	0.0356	0.0652	0.1419	0.0041	0.002	0.0065	0.0319
200	2	2	1.5	1	2.0074	1.9985	1.5165	1.0108	0.0521	0.0354	0.1026	0.0638	0.0043	0.002	0.0171	0.0064
200	2	2	1.5	1.5	2.0076	1.9951	1.5133	1.5237	0.0513	0.0376	0.1048	0.1063	0.0042	0.0022	0.017	0.0183
200	2	2	1.5	2	2.0048	1.9972	1.5109	2.0162	0.0497	0.0368	0.0977	0.1428	0.0038	0.0021	0.0155	0.0323
200	2	2	2	1	2.0063	1.9966	2.02	1.0097	0.052	0.0382	0.1404	0.0605	0.0042	0.0022	0.0314	0.0058
200	2	2	2	1.5	2.0064	1.9975	2.027	1.5173	0.049	0.036	0.1396	0.0993	0.0039	0.002	0.0312	0.016
200	2	2	2	2	2.0042	1.9964	2.0243	2.0203	0.0493	0.0353	0.133	0.1374	0.0038	0.002	0.0284	0.0306

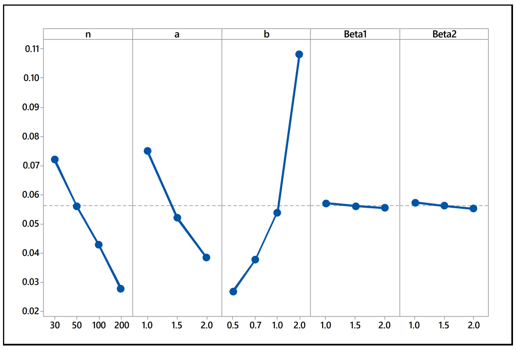


(a)

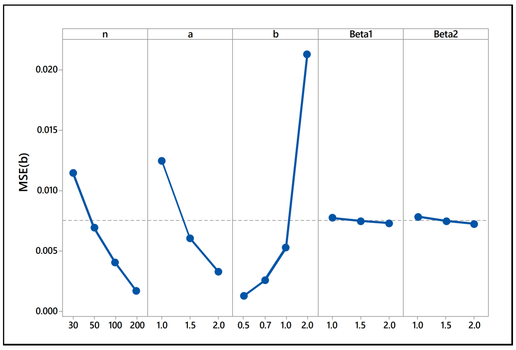


(b)

Figure 1: *Main Effect Plots of and MSE of shape parameter : (a) Bias($\hat{a}$) (b) MSE($\hat{a}$)*



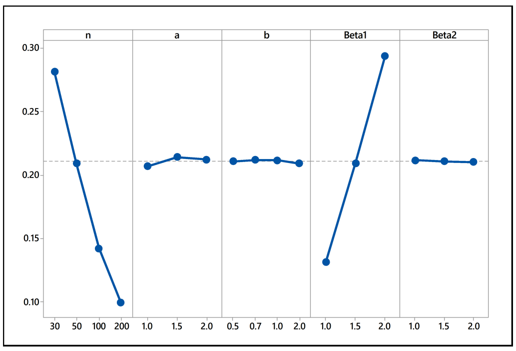
(a)



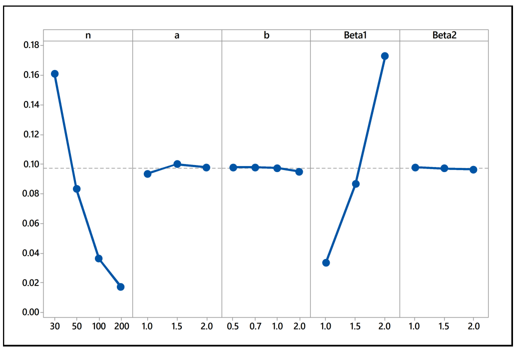
(b)

Figure 2: *Main Effect Plots of and MSE of scale parameter: (a)Bias($\hat{b}$) (b) MSE($\hat{b}$)*

Figure 1 is the main effect plot for the bias and MSE of shape parameter a for various sample sizes (n), shape parameters(a), scale parameters (b), and load share parameters (β_1 and β_2). It is observed that as n increases both the bias and MSE decreases. Bias and MSE increase as its own value is increases. Scale parameter (b) is not affecting on the bias and MSE of shape parameter (a). Bias in (a) slightly decreases as load share parameters β_1 and β_2 increases. Figure 2 is the main effect plot for the bias and MSE of scale parameter b for various values of different parameters. From figure 2 observed that bias as well as MSE is decreases as a and n increases and it increases as its own value increases while is not affected by changes in load share parameters.

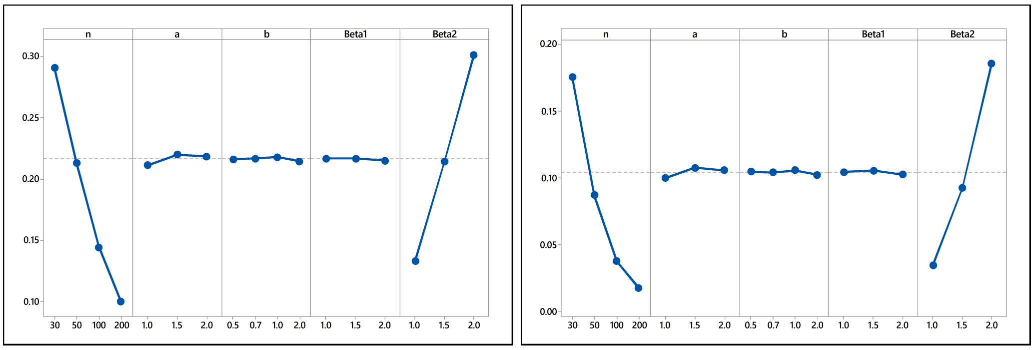


(a)



(b)

Figure 3: *Main Effect Plots of bias of different parameters: (a) Bias($\hat{\beta}_1$) (b) MSE($\hat{\beta}_1$)*



(a) (b)

Figure 4: *Main Effect Plots of MSE of different parameters: (a) Bias($\hat{\beta}_2$) (b) MSE($\hat{\beta}_2$)*

Figure 3 and 4 is the main effect plot for the bias and MSE of load share parameters β_1 and β_2 for various sample sizes (n), shape parameters(a), scale parameters (b), and load share parameters (β_1 and β_2). It is observed that as n increases both the bias and MSE decreases. Bias and MSE increase as its own value is increases. Changes in scale parameter (b) is not affecting on the bias as well as MSE. Also, the bias and MSE of load share parameter in not affected by the changes in other load share parameter.

6. Data Analysis

To illustrate the practical application of the proposed method, we have analyzed two datasets taken from Kim and Kvam [13]. Each datasets consists of 20 observations, specifically capturing the first three failure times of the systems under consideration. We interpret these datasets in the context of two load-sharing configurations: a three-component parallel (1-out-of-3) load-sharing system and a 2-out-of-3 load-sharing system. In our analysis, we utilize Weibull distributions as the baseline distributions for modeling

The failure time data for both datasets are summarized in Table 5, showing the values of the first three failure times (x_1, x_2, x_3) for each dataset.

Table 5: *Failure Time Data*

Data Set 1			Data Set 2		
x_1	x_2	x_3	x_1	x_2	x_3
0.37	1.94	6.93	0.36	3.85	6.49
0.06	2.42	7.44	0.14	0.32	7.57
0.14	0.2	0.2	0.12	5.98	8.29
1.62	2.14	2.34	0.86	3.43	6.12
1.91	1.96	5.7	1.19	2.42	6
2.25	4.6	8.23	0.2	1.53	6.26
0.09	1.4	2.5	1.01	1.18	2.5
0.79	2.44	7.27	1.19	1.3	9.13
0.06	0.12	0.92	2.08	3.62	4.32
0.73	0.79	8.61	0.49	2.89	6.28
1.38	2.78	7.22	3.25	3.88	6.22
0.85	2.81	5.05	0.03	4.18	17.87
0.52	4.13	8.5	0.46	8.99	27.63
1.11	5.67	12.93	2.17	4.08	15.02
0.96	3.54	4.46	1.93	6.81	10.18
2.38	3.5	7.16	0.37	2.7	5.04
0.32	1.89	19.59	0.34	0.97	2.47
1.54	4.98	7.32	2.64	5.16	5.43
2.58	8.61	10.29	0.1	2.38	4.03
1.22	1.73	2.22	0.16	2.26	3.98

The Table 6 summarizes the estimated parameters, along with the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for the models applied to the given datasets.

Table 6: *Estimates of Parameters, AIC, and BIC for Different Load Sharing Models*

Data	Model	Estimate				AIC	BIC
		$\hat{a}$	$\hat{b}$	$\hat{\beta_1}$	$\hat{\beta_2}$		
Dataset 1	1 out of 3	0.948	3.597	0.915	0.931	-194.09	-191.1
	2 out of 3	1.001	3.484	0.926	–	-100.227	-97.25
Dataset 2	1 out of 3	0.978	3.419	1.41	1.35	-209.04	-206.05
	2 out of 3	0.928	3.492	1.395	–	-108.72	-105.74

The table above presents the estimated parameters, along with the AIC and BIC values, for different load-sharing models applied to two datasets. The analysis compares the performance of "1-out-of-3" and "2-out-of-3" load-sharing systems, with parameters $\hat{a}$, $\hat{b}$, $\hat{\beta_1}$, and $\hat{\beta_2}$ estimated for each model. Notably, the "2-out-of-3" model is simpler, as it omits the $\hat{\beta_2}$ parameter.

The comparison of AIC and BIC values reveals that the "1-out-of-3" model consistently outperforms the "2-out-of-3" model in both datasets, as indicated by its lower AIC and BIC values. This suggests that the "1-out-of-3" model provides a better fit, despite its increased complexity. The variability in the estimates of $\hat{\beta_1}$ and $\hat{\beta_2}$ across the models further highlights the different behaviors of the systems under the respective load-sharing rules.

Overall, these findings suggest that while the "2-out-of-3" model is simpler, the "1-out-of-3" model may more accurately represent the underlying load-sharing systems in these datasets. This makes it potentially more effective and practical for engineering and industrial applications where precise modeling is crucial.

7. Conclusions

In this paper, we developed a load-sharing model for k-out-of-m systems using the Proportional Conditional Reverse Hazard Rate approach with an unequal load-sharing rule. Our model addresses the complexities of real-world systems where components share the load unevenly after failures occur. To illustrate the model, a 2-out-of-4 configuration with Weibull baseline distributions is employed. Maximum likelihood estimation is used to estimate the model parameters, and the accuracy of these estimates is assessed through a simulation study that evaluates bias and mean square error. Furthermore, the model's practical applicability is demonstrated by analyzing two real datasets.

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