CHARACTERIZATION OF SOME GENERALIZED DISTRIBUTIONS USING ORDER STATISTICS

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Abstract

The Lindley distribution has been useful for fitting lifetime data. In recent times, several authors studied the extension of the original Lindley distribution. In this paper, we introduced the two general classes of distributions, which include all earlier versions of Lindley distributions. These general classes are characterized using conditional expectations of order statistics. Further, there results are applied to characterize several known distributions like Lindley, X-Lindley, power Lindley, Lindley-Pareto, Ailamujia, power Ailamujia, Lindley-Weibull, length-biased exponential, inverse Lindley, inverse power Lindley and inverted length biased exponential distributions.

Keywords: Order statistics, characterization, conditional expectation, continuous distributions.

1. Introduction

Lindley distribution is a combination of exponential and gamma distributions. The one parameter Lindley distribution, initially introduced by [1]. This distribution has been made extensively due to its practical applications, ease of implementation, and suitability for modelling lifetime data. Characterization of distribution plays a crucial role in understanding the behaviour of the distributions. The theoretical aspect of probability distributions is the core and significant concept of statistics, as it is the basis for various statistical models. There are several methods to characterize the probability distributions. Some are the moment-generating function, entropy function, conditional expectation, recurrence relation, etc. The literature is rich with studies focusing on characterization via conditional expectations. For instance, [2] characterized a generalized class of distributions using the conditional expectation of order statistics, while [3] explored conditioning on pairs of order statistics. Franco and Ruiz [4] gave some general results of characterization based conditional expectation of adjacent order statistics, whereas [5] studied the characterizations based on conditional expectations of the doubled truncated distribution. Balasubramanian and Dey [6] characterized both absolutely continuous random variables and discrete random variables using conditional expectation. Su and Huang [7] obtained a relationship between failure and conditional expectations, along with characterization results based on conditional expectations. Khan and Abouammoh [8] extended the results of [2] to cases where the conditional order statistic may not be adjacent. Moreover, [9] characterized distributions through linear regression of non-adjacent generalized order statistics. Gupta and Ahsanullah [10] developed characterization results based on the conditional expectation of a function of non-adjacent order statistics. Noor and Athar [15] characterized two general classes of distributions through conditional expectation of power of difference of two record statistics.

Ahsanullah et al. [17] established some new characterization results of continuous distributions by truncated moments, while [21] established characterization results for two general forms of distributions by truncated moments. Furthermore, [22] examined characterization results based on conditional expectations of function of random variables when truncation is from both the left and right sides, whereas [24] developed characterization results for three general classes of continuous distributions using double truncated moments.

In this paper, we present two generalized classes of distributions, which include several known distributions such as the one-parameter Lindley, X-Lindley, power Lindley, Lindley pareto, Ailamujia, power Ailamujia, Lindley-Weibull, length-biased exponential distribution, and others. We aim to characterize these classes through conditional expectations of order statistics, contributing to the ongoing research in statistical distribution characterization. This approach not only enriches the theoretical understanding of these distributions but also enhances their applicability in various fields.

1.1. Conditional Distribution of Order Statistics

Let $X_i, i = 1, 2, 3, \ldots, n$ be the independent and continuous random variables $(r.vs.)$ having probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Then, the conditional distribution of $X_{s:n}$ given $X_{r:n} = x$ is given by

$$\frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{(F(y)-F(x))^{s-r-1}(1-F(y)^{n-s})f(y)}{(1-F(x))^{n-r}}, x < y \tag{1}$$

and the conditional pdf of $X_{r:n}$ given $X_{s:n} = y$, $1 \leq r < s \leq n$, is

$$\frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{(F(x))^{r-1}(F(y)-F(x))^{s-r-1}f(x)}{(F(y))^{s-1}}, x < y \tag{2}$$

See [12].

1.2. Proposed Generalized Classes of Distributions

The Lindley distribution has been useful for its simplicity and ability to model lifetime data effectively. Ghitany *et al*. [11] investigated the properties of the one-parameter Lindley distribution and demonstrated its flexibility in fitting lifetime data better than the exponential distribution. Their work highlighted the practical utility of the Lindley distribution in real-world applications. Ghitany *et al*. [13] extended the Lindley distribution to a two-parameter form, enhancing its flexibility and fitting ability. Moreover, [14] introduced the two-parameter extension, offering additional parameters to capture more complex data behaviors. Further, [20] introduced a new two-parameter version of the Lindley distribution, which showed an improvement to fit skewed real data compared to the inverse Lindley distribution, introduced by [16]. In this paper, we introduced two generalized classes of distributions, named Haseeb generalized Lindley (HGL) and Haseeb Generalized inverse Lindley (HGIL) class of distributions. These generalized classes include all the distributions belonging to Lindley and inverse Lindley families.

1.2.1 Haseeb Generalized Lindley (HGL) Class of Distributions

Let $X \in (\alpha, \beta)$ be a $r.v.$ having pdf $f(x)$ and cdf $F(x)$, then X is said to follow HGL class of distribution, if cdf is given as

$$F(x) = 1 - (b + ah(x))e^{-ch(x)}, x \in (\alpha, \beta) \tag{3}$$

and the corresponding pdf is

$$f(x) = \{c(b + ah(x)) - a\}h'(x)e^{-ch(x)}, x \in (\alpha, \beta) \tag{4}$$

Table 1: *Sub-model of HGL class of distribution*

Distribution	$F(x)$	a	b	c	$h(x)$
One Parameter Lindley	$1-\left(1+\frac{\theta}{1+\theta}x\right)e^{-\theta x};\theta,x>0$	$\frac{\theta}{1+\theta}$	1	θ	x
X-Lindley	$1-\left(1+\frac{\theta x}{(1+\theta)^2}\right)e^{-\theta x};\theta,x>0$	$\frac{\theta}{(1+\theta)^2}$	1	θ	x
Power Lindley	$1-\left(1+\frac{\theta x^{\alpha}}{1+\theta}\right)e^{-\theta x^{\alpha}};\theta,x>0$	$\frac{\theta}{1+\theta}$	1	θ	x^{α}
Lindley Pareto	$1-\frac{\alpha^p+x^p\theta}{(1+\theta)\alpha^p}e^{-\theta\frac{(x^p-\alpha^p)}{\alpha^p}};\theta,\alpha>0,x>0$	$\frac{\theta e^{\theta}}{1+\theta}$	$\frac{e^{\theta}}{1+\theta}$	θ	$\left(\frac{x}{\alpha}\right)^p$
Power Ailamujia	$1-(1+\theta x^{\beta})e^{-\theta x^{\beta}};\theta>0,x>0$	θ	1	θ	x^{β}
Lindley-Weibull	$1-\left(1+\frac{\theta(\alpha x)^{\beta}}{1+\theta}\right)e^{-\theta(\alpha x)^{\beta}};\alpha,\theta,\beta>0,x>0$	$\frac{\theta}{1+\theta}$	1	θ	$(\alpha x)^{\beta}$
Length-Biased Exponential	$1-\left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}};\theta>0,x>0$	$\frac{1}{\theta}$	1	$\frac{1}{\theta}$	x

1.2.2 Haseeb Generalized Inverse Lindley (HGIL) Class of Distributions

Let $f(x)$ and $F(x)$ be the pdf and cdf of a continuous $r.v.$ X respectively, then X is said to follow HGIL class of distribution, if its cdf is given by

$$F(x)=(b+ah(x))e^{-ch(x)},x\in(\alpha,\beta) \tag{5}$$

and corresponding pdf is

$$f(x)=\{a-c(b+ah(x))\}e^{-ch(x)}h'(x),x\in(\alpha,\beta) \tag{6}$$

Table 2: *Sub-model of HGIL class of distribution*

Distribution	$F(x)$	a	b	c	$h(x)$
Inverse Lindley	$\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}};\theta>0,x>0$	$\frac{\theta}{1+\theta}$	1	θ	$\frac{1}{x}$
Inverse Power Lindley	$\left(1+\frac{\theta}{(1+\theta)x^{\alpha}}\right)e^{-\frac{\theta}{x^{\alpha}}};\alpha,\theta>0,x>0$	$\frac{\theta}{1+\theta}$	1	θ	$\frac{1}{x^{\alpha}}$
Inverted LBE	$\left(1+\frac{1}{\theta x}\right)e^{-\frac{1}{\theta x}};\theta>0,x>0$	$\frac{1}{\theta}$	1	$\frac{1}{\theta}$	$\frac{1}{x}$

2. Characterization Results

Theorem 1. Let X be the continuous $r.v.$ having cdf $F(x)$ and pdf $f(x)$. Suppose $F(x)$ is defined for all $x\in(\alpha,\beta)$, with boundary conditions $F(\alpha)=0$ and $F(\beta)=1$, then for $1\le r<r+1\le n$

$$E[h(X_{r+1:n})|X_{r:n}=x]=g_{r+1|r}(x)=h(x)+\left[\frac{ae^{\frac{cb}{a}}}{1-F(x)}\right]^{n-r}\frac{\Gamma[n-r+1,\frac{c(n-r)}{a}(b+ah(x))]}{[c(n-r)]^{n-r+1}}, \tag{7}$$

if and only if for $n-r>0$

$$1-F(x)=(b+ah(x))e^{-ch(x)}\ ,\ x\in(\alpha,\beta). \tag{8}$$

where, $h(x)$ is a continuous and differentiable function of x on (α,β), $a\neq 0$ and $\Gamma(n,x)=\int_x^{\infty}u^{n-1}e^{-u}du$ be the upper incomplete gamma function.

Proof. Necessary part: In view of (1), we have

$$E[h(X_{r+1})|X_{r:n}=x]=\frac{(n-r)}{(1-F(x))^{n-r}}\int_x^{\beta}h(y)\big(1-F(y)\big)^{n-r-1}f(y)dy.$$

Integrating right hand side of the above expression by parts, then we get

$$= h(x) + \frac{1}{(1-F(x))^{n-r}} \int_x^{\beta} h'(y)(b+ah(y))^{n-r} e^{-c(n-r)h(y)} dy$$

$$= h(x) + \frac{1}{(1-F(x))^{n-r}} \times I(x) \tag{9}$$

where,

$$I(x) = \int_x^{\beta} h'(y)(b+ah(y))^{n-r} e^{-c(n-r)h(y)} dy.$$

Let, $t = b + ah(y)$, implies $dt = ah'(y)dy$, then we have

$$I(x) = \frac{e^{c(n-r)b}}{a} \int_t^{\infty} t^{n-r} e^{-\frac{c(n-r)t}{a}} dt.$$

Again, substitute $u = \frac{c(n-r)t}{a}$, then

$$I(x) = \frac{e^{\frac{c(n-r)b}{a}}}{c(n-r)} \times \left[\frac{a}{c(n-r)}\right]^{n-r} \int_{\frac{c(n-r)(b+ah(x))}{a}}^{\infty} u^{n-r+1-1} e^{-u} du$$

$$= \frac{e^{\frac{c(n-r)b}{a}}}{c(n-r)} \times \left[\frac{a}{c(n-r)}\right]^{n-r} \Gamma[n-r+1, \frac{c(n-r)}{a}(b+ah(x))]. \tag{10}$$

Now, substitute (10) in (9) to prove the necessary part.

Sufficient part: From (7), we have

$$(1-F(x))^{n-r} g_{r+1|r}(x) = (1-F(x))^{n-r} h(x) + \frac{(ae^{\frac{bc}{a}})^{n-r}}{[c(n-r)]^{n-r+1}} \times \int_{\frac{c(n-r)(b+ah(x))}{a}}^{\infty} u^{n-r} e^{-u} du. \tag{11}$$

Differentiating both sides of (11) w.r.t. x, then we have

$$g'_{r+1|r}(x)(1-F(x))^{n-r} - (n-r)g_{r+1|r}(x)(1-F(x))^{n-r-1}f(x) = h'(x)(1-F(x))^{n-r} \\ -(n-r)h(x)(1-F(x))^{n-r-1}f(x) - e^{-c(n-r)h(x)}h'(x) \tag{12}$$

Also, we have

$$g_{r+1|r}(x) = \frac{(n-r)}{(1-F(x))^{n-r}} \int_x^{\beta} h(y)(1-F(y))^{n-r-1} f(y) dy$$

or

$$g_{r+1|r}(x)(1-F(x))^{n-r} = (n-r) \int_x^{\infty} h(y)(1-F(y))^{n-r-1} f(y) dy \tag{13}$$

Now, differentiate both the sides of (13) w.r.t. x to get

$$g'_{r+1|r}(x)(1-F(x))^{n-r} - (n-r)g_{r+1|r}(x)(1-F(x))^{n-r-1}f(x) = -(n-r)h(x) \times \\ (1-F(x))^{n-r-1}f(x). \tag{14}$$

Now on comparison of (12) with (14), we get

$$(1-F(x))^{n-r} = (b+ah(x))^{n-r} e^{-c(n-r)h(x)}$$

or

$$\ln(1-F(x)) = \ln(b+ah(x)) - ch(x)$$

Differentiating both sides of the above expression w.r.t. x, we get

$$-\frac{f(x)}{1-F(x)} = \frac{ah'(x)}{b+ah(x)} - ch'(x)$$

or

$$1-F(x) = \frac{b+ah(x)}{h'(x)\{c(b+ah(x))-a\}} f(x)$$

or

$$\frac{f(x)}{1-F(x)} = \{ch'(x) - \frac{ah'(x)}{b+ah(x)}\}. \tag{15}$$

Now integrating both sides of (15), leads to the sufficiency part. ■

Corollary 1. Under the similar conditions as stated in Theorem 1

$$E[h(x)|X \geq x] = h(x) + \frac{c(b+ah(x))+a}{c^2(b+ah(x))}. \tag{16}$$

Proof. Corollary can be proved by substituting $r = n-1$ in (7). ■

Theorem 2. Let X be the continuous $r.v.$ having cdf $F(x)$ and pdf $f(x)$. Suppose $F(x)$ is defined over $x \in (\alpha, \beta)$, with boundary conditions $F(\alpha) = 0$ and $F(\beta) = 1$. Then for $1 \leq r < r+1 \leq n$

$$E[h(X_{r:n})|X_{r+1:n}] = g_{r|r+1}(x) = h(x) - \frac{1}{rc}\left[\frac{ae^{bc/a}}{rcF(x)}\right]^r \gamma\left(r+1, \frac{rc}{a}(b+ah(x))\right), \tag{17}$$

if and only if

$$F(x) = [b+ah(x)]e^{-ch(x)}; x \in (\alpha, \beta), \tag{18}$$

where, h(x) is a continuous and differentiable function of x on (α, β) and $a \neq 0$ and $\gamma(a,x) = \int_0^x u^{a-1}e^{-u}du$ be the lower incomplete gamma function.

Proof. To prove necessary part, in view of (2), we have

$$E(h(X_{r:n})|X_{r+1:n} = x) = \frac{r}{F^r(x)} \int_\alpha^x h(y)F^{r-1}(y)f(y)dy,$$

$$= h(x) - \frac{1}{F^r(x)} \int_\alpha^x h'(y)(b+ah(y))^r e^{-rch(y)}dy$$

$$= h(x) - \frac{1}{F^r(x)} I(x), \tag{19}$$

where,

$$I(x) = \int_\alpha^x h'(y)(b+ah(y))^r e^{-rch(y)}dy.$$

Let $t = b+ah(y)$, which implies $dt = ah'(y)dy$. Then

$$I(x) = \frac{1}{a}e^{\frac{rbc}{a}} \int_0^{b+ah(x)} t^r e^{-\frac{rct}{a}}.$$

Again suppose $u = \frac{rct}{a}$, then we get

$$I(x) = \frac{1}{rc}\left[\frac{ae^{bc/a}}{rc}\right]^r \gamma(r+1, \frac{rc}{a}(b+ah(x))). \tag{20}$$

Now, substitute (20) into (19) and simplify, this proves the necessary parts.

To prove sufficiency part, in view of (17), we have

$$g_{r|r+1}(x) = h(x) - \frac{1}{rc}\left[\frac{ae^{bc/a}}{rc}\right]^r \gamma(r+1, \frac{rc}{a}(b+ah(x)))$$

or

$$F^r(x)g_{r|r+1}(x) = F^r(x)h(x) - \frac{1}{rc}\left[\frac{ae^{bc/a}}{rc}\right]^r \int_0^{\frac{rc(b+ah(x))}{a}} u^r e^{-u} du. \tag{21}$$

Differentiate both the sides of (21) w.r.t x,

$$rF^{r-1}(x)f(x)g_{r|r+1}(x) + g'_{r|r+1}(x)F^r(x) = h'(x)F^r(x) - (b+ah(x))^r e^{-rch(x)}h'(x) + rF^{r-1}(x)f(x)h(x). \tag{22}$$

Again

$$g_{r|r+1}(x) = \frac{r}{F^r(x)}\int_\alpha^x h(y)F^{r-1}(y)f(y)dy$$

or

$$F^r(x)g_{r|r+1}(x) = r\int_\alpha^x h(y)F^{r-1}(y)f(y)dy \tag{23}$$

Now differentiate both sides of (23) w.r.t x, then we get

$$rF^{r-1}(x)f(x)g_{r|r+1}(x) + F^r(x)g'_{r|r+1}(x) = rh(x)F^{r-1}(x)f(x) \tag{24}$$

Now after comparing (22) with (24), we have

$$F^r(x) = (b+ah(x))^r e^{-rch(x)}$$

Taking log on both the sides of above expression, we have

$$r\ln F(x) = r\ln(b+ah(x)) - rch(x) \tag{25}$$

Differentiating (25) w.r.t. x, we get

$$\frac{f(x)}{F(x)} = \frac{ah'(x)}{b+ah(x)} - ch'(x) \tag{26}$$

Hence, the cdf as given in (18). ■

Corollary 2. For the condition as stated in Theorem 2 and $r = 1$,

$$E[h(x)|X \le x] = h(x) + \frac{c(b+ah(x)) + a - ae^{\frac{c(b+ah(x))}{a}}}{c^2(b+ah(x))}. \tag{27}$$

3. Applications

3.1. Applications of Characterization Theorem 1

Lindley Distribution

Corollary 3. Let X be a continuous $r.v.$ with cdf $F(x)$ and pdf $f(x)$. Further, suppose that $E(X)$ exists. Then

$$E(X_{r+1:n}|X_{r:n}=x)=x+\frac{\Gamma[n-r+1,(n-r)(1+\theta+\theta x)]}{\theta(n-r)^{n-r+1}(1+\theta+\theta x)^{n-r}e^{-(n-r)(1+\theta+\theta x)}}, \tag{28}$$

if and only if

$$F(x)=1-\left(1+\frac{\theta}{1+\theta}x\right)e^{-\theta x};\theta>0,x>0. \tag{29}$$

Proof. To prove the necessary part, we compare (29) with (8), and get $a=\frac{\theta}{1+\theta}, b=1, h(x)=x, c=\theta$. Now, in view of (7) we get (28).

To prove the sufficiency part, using (15), we get

$$\frac{f(x)}{1-F(x)}=\left\{\theta-\frac{\frac{\theta}{1+\theta}}{1+\frac{\theta x}{1+\theta}}\right\} \tag{30}$$

Now, integrate both the sides of (30), which leads to the cdf given in (29). ■

Similar result was obtained by [18].

Remark 1: In view of Corollary 1, we get the characterization result for truncated moment. That is

$$E(X|X\geq x)=x+\frac{2+\theta+\theta x}{\theta(1+\theta+\theta x)}.$$

The similar result was also obtained by [19].

X-Lindley Distribution

Corollary 4. Under the conditions as stated in Corollary 2

$$E(X_{r+1:n}|X_{r:n}=x)=x+\frac{\Gamma[n-r+1,(n-r)((1+\theta)^2+\theta x)]}{\theta((1+\theta)^2+\theta x)^{n-r}e^{-(\theta x+(1+\theta)^2)(n-r)}(n-r)^{n-r+1}}, \tag{31}$$

if and only if

$$F(x)=1-\left(1+\frac{\theta}{(1+\theta)^2}x\right)e^{-\theta x},\theta>0,x>0. \tag{32}$$

Proof. First, we shall prove the necessary part. On comparison of (32) with (8), we get $a=\frac{\theta}{(1+\theta)^2}, b=1, c=\theta, h(x)=x$. Now, in view of (7), we get (31). Hence, the necessary part is true.

To prove sufficient part, in view of (15) we have

$$\frac{f(x)}{1-F(x)}=\left\{\theta-\frac{\frac{\theta}{(1+\theta)^2}}{1+\frac{\theta}{(1+\theta)^2}x}\right\}. \tag{33}$$

This implies the cdf of X-Lindley distribution as given in (32). ■

Remark 2: In view of Corollary 1, we have

$$E(X|X \geq x) = x + \frac{(1+\theta)^2 + \theta x + 1}{\theta[(1+\theta)^2 + \theta x]}. \tag{34}$$

This characterizing result was also obtained by [23].

Power Lindley Distribution

Corollary 5. Let X be a continuous $r.v$ having cdf $F(x)$ and pdf $f(x)$. Let $E(X^\alpha)$ exists, then

$$E(X^\alpha_{r+1:n}|X_{r:n} = x) = x^\alpha + \frac{\Gamma[n-r+1, (n-r)(1+\theta+\theta x^\alpha)]}{\theta(n-r)^{n-r+1} e^{-(n-r)(1+\theta x^\alpha+\theta)}(1+\theta+\theta x^\alpha)^{n-r}}, \tag{35}$$

if and only if

$$F(x) = 1 - \left(1 + \frac{\theta}{1+\theta} x^\alpha\right) e^{-\theta x^\alpha}, \theta > 0, x > 0. \tag{36}$$

Proof. Necessary part: By comparing (36) with (8) we get

$a = \frac{\theta}{1+\theta}, b = 1, c = \theta, h(x) = x^\alpha$. Now on application of (7), we get (35).

Sufficient part: In view of (15), we have

$$\frac{f(x)}{1-F(x)} = \left\{ \alpha\theta x^{\alpha-1} - \frac{\frac{\theta\alpha x^{\alpha-1}}{1+\theta}}{1 + \frac{\theta}{1+\theta} x^\alpha} \right\}, \tag{37}$$

which gives cdf of power Lindley distribution as given in (36). ■

Remark 3: The result for characterization based on truncated moment can be seen in view of Corollary 1 as below:

$$E(X|x \geq x) = x^\alpha + \frac{2+\theta+\theta x^\alpha}{1+\theta+\theta x^\alpha}.$$

Lindley Pareto Distribution

Corollary 6. Under the similar condition as given in Corollary 5,

$$E(X^p_{r+1:n}|X_{r:n} = x) = x^p + \frac{\alpha^p[(\alpha+\theta x^p) e^{\frac{\theta(x^p-\alpha^p)}{\alpha^p}} + \theta + 1]^{n-r} \Gamma[n-r+1, \frac{(n-r)(\alpha^p+\theta x^p)}{\alpha^p}]}{\theta(n-r)^{n-r+1}}, \tag{38}$$

if and only if

$$F(x) = 1 - \frac{(\alpha^p + x^p\theta)}{(1+\theta)\alpha^p} e^{-\theta\left(\frac{x^p}{\alpha^p} - 1\right)}. \tag{39}$$

Proof. Corollary 6 can be proved on the lines of Corollary 5. ■

Remark 4: The characterization result for truncated moment can be obtained on application of Corollary 1.

$$E(X^p|X \geq x) = x^p + \frac{(\alpha^p + \theta x^p)(2\alpha^p + \theta x^p)}{\theta}.$$

Power Ailamujia Distribution

Corollary 7. Let X be a continuous $r.v.$ having $pdf\ f(x)$ and $cdf\ F(x)$, then

$$E(X^{\beta}_{r+1:n}|X_{r:n}=x)=x^{\beta}+\frac{\Gamma[n-r+1,(n-r)(1+\theta x^{\beta})]}{(n-r)^{n-r+1}\theta(1+\theta x^{\beta})^{n-r}e^{-(n-r)\theta x^{\beta+1}}}, \tag{40}$$

if and only if

$$F(x)=1-(1+\theta x^{\beta})e^{-\theta x^{\beta}};\theta>0,x>0. \tag{41}$$

Proof. Necessary part: On comparing of (41) with (8), we get $a=c=\theta, b=1, h(x)=x^{\beta}$. Now on application of (7), we get (40).

To prove sufficient part, In view of (15), we have

$$\frac{f(x)}{1-F(x)}=\theta\beta x^{\beta-1}-\frac{\theta\beta x^{\beta-1}}{1+\theta x^{\beta}}. \tag{42}$$

Now integrate both the sides of (42) $w.r.t.\ x$ to get (41). ■

Remark 5: For $\beta=1$ in (40). we get the result for Ailamujia distribution.

$$E(X_{r+1:n}|X_{r:n}=x)=x+\frac{\Gamma[n-r+1,(n-r)(1+\theta x)]}{\theta(1+\theta x)^{n-r}e^{-(n-r)(1+\theta x)}(n-r)^{n-r+1}}. \tag{43}$$

Further, the characterization based on truncated can be obtained using (16) as below:

$$E(X^{\beta}|X\geq x)=x^{\beta}+\frac{2+\theta x^{\beta}}{\theta(1+\theta x^{\beta})}.$$

Lindley-Weibull Distribution

Corollary 8. Let X be the continuous $r.v$ having $pdf\ f(x)$ and $cdf\ F(x)$, and $E(X^k)$ exist, then

$$E(X^{\beta}_{r+1:n}|X_{r:n}=x)=x^{\beta}+\frac{\Gamma[n-r+1,(n-r)(1+\theta+\theta(\alpha x)^{\beta})]}{\theta\alpha^{\beta}(1+\theta+\theta(\alpha x)^{\beta})^{n-r}(n-r)^{n-r+1}e^{-(n-r)(1+\theta+\theta(\alpha x)^{\beta})}}, \tag{44}$$

if and only if

$$F(x)=1-\left(1+\frac{\theta}{1+\theta}(\alpha x)^{\beta}\right)e^{-\theta(\alpha x)^{\beta}};\alpha,\beta,\theta>0,x>0. \tag{45}$$

Proof. Corollary 8 can be proved easily on the lines of Corollary 7. ■

Remark 6: The result for characterization using truncated moments based on Corollary 1 is given as

$$E(X^{\beta}|X\geq x)=x^{\beta}+\frac{2+\theta+\theta(\alpha x)^{\beta}}{\theta\alpha^{\beta}(1+\theta+\theta(\alpha x)^{\beta})}.$$

Length-Biased Exponential (LBE) Distribution

Corollary 9. Let X be the continuous $r.v.$ having $pdf\ f(x)$ and $cdf\ F(x)$, and $E(X)$ exist, then

$$E(X_{r+1:n}|X_{r:n}=x)=x+\left(\frac{\theta}{n-r}\right)^{n-r+1}\frac{\Gamma[n-r+1,\frac{(n-r)}{\theta}(\theta+x)]}{((\theta+x)e^{-\frac{x+\theta}{\theta}})^{n-r}}, \tag{46}$$

if and only if

$$F(x)=1-\left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}};\theta>0,x>0. \tag{47}$$

Proof. To prove necessary part we compare of (47) with (8) and get

$a = \frac{1}{\theta} = c, b = 1, h(x) = x$. Now using (7), we get (46).

To prove sufficient part, in view of (15), we have,

$$\frac{f(x)}{1-F(x)} = \frac{1}{\theta} - \frac{\frac{1}{\theta}}{1+\frac{x}{\theta}}, \tag{48}$$

which gives the cdf as given in (47). ■

Remark 7: The result for characterization using truncated moment can be seen using (16) as:

$$E(X|x \geq x) = x + \frac{x+2\theta}{x+\theta}.$$

Similarly several other distributions that belong to this class can be characterized.

3.2. Applications of characterization Theorem 2

Inverse Lindley Distribution

Corollary 10. Let X be a continuous $r.v.$ having cdf $F(x)$ and $E(X^{-1}) < \infty$, then

$$E(X_{r:n}^{-1}|X_{r+1:n} = x) = x^{-1} - \frac{1}{r\theta}\left[\frac{xe^{\frac{\theta+x(1+\theta)}{x}}}{r\{\theta+x(1+\theta)\}}\right]^r \gamma(r+1, \frac{r\{\theta+x(1+\theta)\}}{x}), \tag{49}$$

if and only if

$$F(x) = \left(1+\frac{\theta}{1+\theta}\frac{1}{x}\right)e^{-\frac{\theta}{x}}; \theta > 0, x > 0. \tag{50}$$

Proof. First we shall prove the necessary part. On comparison of (50) with (18), we get $a = \frac{\theta}{1+\theta}, b = 1, c = \theta, h(x) = \frac{1}{x}$. Now in view of (17), we get (49). Hence the necessary part is true.

To prove sufficient part, from (26), we have

$$\frac{f(x)}{F(x)} = -\frac{\frac{\theta}{(1+\theta)}\frac{1}{x^2}}{1+\frac{\theta}{(1+\theta)x}} + \frac{\theta}{x^2} \tag{51}$$

which give cdf as given in (50). ■

Remark 8: The characterization result for truncated moment can be obtained on application of Corollary 9 as below:

$$E(X^{-1}|X \leq x) = x^{-1} + \frac{\theta + x(2+\theta - e^{\frac{\theta+x(1+\theta)}{x}})}{\theta\{\theta+x(1+\theta)\}}.$$

Inverse power Lindley Distribution

Corollary 11. Let X be a continuous $r.v.$ having cdf $F(x)$ and $E(X^{-\alpha})$ exists, then

$$E(X_{r:n}^{-\alpha}|X_{r+1:n} = x) = x^{-\alpha} - \frac{1}{r\theta}\left[\frac{x^{\alpha}e^{\frac{\theta+x^{\alpha}(1+\theta)}{x^{\alpha}}}}{r\{x^{\alpha}(1+\theta)+\theta\}}\right]^r \gamma(r+1, \frac{r}{x^{\alpha}}(\theta + x^{\alpha}(1+\theta))), \tag{52}$$

if and only if

$$F(x) = \left(1+\frac{\theta}{1+\theta}\frac{1}{x^{\alpha}}\right)e^{-\frac{\theta}{x^{\alpha}}}; \alpha, \theta > 0, x > 0. \tag{53}$$

Proof. Corollary can be proved on the lines of Corollary 10. ■

Remark 9: Under the similar condition as stated under Corollary 11, the result for truncated moment is given as

$$E(X^{-\alpha}|X \leq x) = x^{-\alpha} + \frac{\theta + x^{\alpha}(2 + \theta - e^{\frac{x^{\alpha}(1+\theta)+\theta}{x^{\alpha}}})}{\theta(\theta + x^{\alpha}(1+\theta))}.$$

Inverted LBE Distribution

Corollary 12. Under the conditions as stated in Corollary 11.

$$E(X_{r:n}^{-1}|X_{r+1:n} = x) = x^{-1} - \frac{\theta e^{r}}{r^{r+1}}\gamma(r+1, \frac{r(1+x\theta)}{\theta x}), \tag{54}$$

if and only if

$$F(x) = \left(1 + \frac{1}{\theta x}\right)e^{-\frac{1}{\theta x}}, \theta > 0, x > 0. \tag{55}$$

Proof. Proof is straight forward. ■

Remark 10: The characterization result for truncated moment using Corollary 9 is given as:

$$E(X^{-1}|X \leq x) = x^{-1} + \frac{\theta(1 + x(\theta + 1 - e^{\frac{1+\theta x}{\theta x}}))}{1 + \theta x}.$$

4. Conclusion

In this study, we defined the HGL and HGIL classes of distributions, and characterized these using the conditional expectation of adjacent order statistics. This approach has demonstrated efficiency in differentiating the characteristics of the HGL and HGIL classes of distributions. Moreover, we obtained characterization results for right and left truncated moments for HGL and HGIL classes of distributions. Further, main results are applied to characterize several well known continuous distributions, such as the one-parameter Lindley, X-Lindley, power Lindley, Lindley Pareto, Ailamujia, power Ailamujia, Lindley-Weibull, and length-biased exponential. Our findings unify the earlier results obtained in the literature.

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